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A MAXIMAL FUNCTION CHARACTERIZATION OF THE HARDY SPACE FOR THE GAUSS MEASURE

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Abstract. An atomic Hardy space $H^1(\gamma)$ associated to the Gauss measure γ in $\mathbb{R}^n$ has been introduced by the first two authors. We first prove that it is equivalent to use $(1,r)$- or $(1,\infty)$-atoms to define this $H^1(\gamma)$. For $n=1$, a maximal function characterization of $H^1(\gamma)$ is found. In arbitrary dimension, we give a description of the nonnegative functions in $H^1(\gamma)$ and use it to prove that $L^p(\gamma)\subset H^1(\gamma)$ for $1<p\leq\infty$.

1. Introduction

Denote by γ the Gauss measure on $\mathbb{R}^n$, i.e., the probability measure with density $\gamma_0(x)=\pi^{-n/2}\,\mathrm{e}^{-|x|^2}$ with respect to the Lebesgue measure λ. Harmonic analysis on the measured metric space $(\mathbb{R}^n,d,\gamma)$, where d denotes the Euclidean distance on $\mathbb{R}^n$, has been the object of many investigations. In particular, efforts have been made to study operators related to the Ornstein–Uhlenbeck semigroup, with emphasis on maximal operators [33, 16, 27, 13, 21, 22], Riesz transforms [29, 14, 28, 32, 30, 17, 15, 8, 9, 10, 31, 35, 7, 24] and functional calculus [11, 12, 18, 25].

In [24] the first two authors defined an atomic Hardy-type space $H^1(\gamma)$ and a space $BMO(\gamma)$ of functions of bounded mean oscillation, associated to γ. We briefly recall their definitions. A closed Euclidean ball B is called *admissible* at scale $s>0$ if

$$r_B\leq s\;\min\bigl(1,1/|c_B|\bigr);$$

here and in the sequel r_B and c_B denote the radius and the centre of B, respectively. We denote by $\mathcal{B}_s$ the family of all balls admissible at scale s. For the sake of brevity, we shall refer to balls in $\mathcal{B}_1$ simply as admissible balls. Further, B will be called *maximal admissible* if $r_B=\min(1,1/|c_B|)$.

Now let $r\in(1,\infty]$. A *Gaussian* $(1,r)$-*atom* is either the constant function 1 or a function a in $L^r(\gamma)$ supported in an admissible ball B and such that

$$\int a\;\mathrm{d}\gamma=0\qquad\text{and}\qquad\|a\|_r\leq\gamma(B)^{1/r-1}; \tag{1.1}$$

here and in the whole paper, $\|\cdot\|_r$ denotes the norm in $L^r(\gamma)$. In the latter case, we say that the atom a is associated to the ball B. The space $H^{1,r}(\gamma)$ is then the vector

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space of all functions f in $L^1(\gamma)$ that admit a decomposition of the form $\sum_j \lambda_j\, a_j$, where the a_j are Gaussian $(1,r)$-atoms and the sequence of complex numbers $\{\lambda_j\}$ is summable. The norm of f in $H^{1,r}(\gamma)$ is defined as the infimum of $\sum_j |\lambda_j|$ over all representations of f as above.

In [24] the spaces $H^{1,r}(\gamma)$ were defined and proved to coincide for all $1 < r < \infty$, with equivalent norms. In Section 2 we complement this by proving that they coincide also with the space $H^{1,\infty}(\gamma)$. Once this is established, we shall denote the space by $H^1(\gamma)$ and use the $H^{1,\infty}(\gamma)$ norm. Further, we shall frequently write atom for the $(1,\infty)$-atom.

The space $BMO(\gamma)$ consists of all functions f in $L^1(\gamma)$ such that

$$\sup_{B\in\mathcal{B}_1} \frac{1}{\gamma(B)} \int_B |f - f_B| \,\mathrm{d}\gamma < \infty,$$

where f_B denotes the mean value of f on B, taken with respect to the Gauss measure. The norm of a function in $BMO(\gamma)$ is

$$\|f\|_{BMO(\gamma)} = \|f\|_1 + \sup_{B\in\mathcal{B}_1} \frac{1}{\gamma(B)} \int_B |f - f_B| \,\mathrm{d}\gamma.$$

If, in the definitions of $H^1(\gamma)$ and $BMO(\gamma)$, we replace the family $\mathcal{B}_1$ of admissible balls at scale 1 by $\mathcal{B}_s$ for any fixed $s > 0$, we obtain the same spaces with equivalent norms; see [24]. We remark that a similar $H^1 - BMO$ theory for more general measured metric spaces has been developed by A. Carbonaro and the first two authors in [1, 2, 3]. See also the papers [19, 20] of L. Liu and D. Yang for related results.

The main motivation for introducing these two spaces was to provide endpoint estimates for singular integrals associated to the Ornstein-Uhlenbeck operator $\mathcal{L} = -(1/2)\Delta + x\cdot\nabla$, a natural self-adjoint Laplacian on $L^2(\gamma)$. Indeed, in [24] the first two authors proved that the imaginary powers of $\mathcal{L}$ are bounded from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $BMO(\gamma)$ and that Riesz transforms of the form $\nabla^\alpha\mathcal{L}^{-|\alpha|}$ and of any order are bounded from $L^\infty(\gamma)$ to $BMO(\gamma)$. In a recent paper [26], the authors proved that boundedness from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $BMO(\gamma)$ holds for any first-order Riesz transform in dimension one, but not always in higher dimensions.

The definition of the space $H^1(\gamma)$ closely resembles the atomic definition of the classical Hardy space $H^1(\lambda)$ on $\mathbb{R}^n$ endowed with the Lebesgue measure λ, but there are two basic differences. First, the measured metric space $(\mathbb{R}^n, d, \gamma)$ is nondoubling. Further, except for the constant atom, a Gaussian atom must have "small support", i.e., support contained in an admissible ball. Despite these differences, $H^1(\gamma)$ shares many of the properties of $H^1(\lambda)$. In particular, the topological dual of $H^1(\gamma)$ is isomorphic to $BMO(\gamma)$, an inequality of John-Nirenberg type holds for functions in $BMO(\gamma)$ and the spaces $L^p(\gamma)$ are intermediate spaces between $H^1(\gamma)$ and $BMO(\gamma)$ for the real and the complex interpolation methods.

It is well known that the classical Hardy space $H^1(\lambda)$ can be defined in at least three different ways: the *atomic* definition, the *maximal* definition and the definition based on *Riesz transforms* [6, 34].

As shown in [26], in higher dimensions the first-order Ornstein-Uhlenbeck Riesz transforms $\partial_j\mathcal{L}^{-1/2}$ are unbounded from $H^1(\gamma)$ to $L^1(\gamma)$; here $\partial_j = \partial/\partial_{x_j}$. Thus $H^1(\gamma)$ does not coincide with the space of all functions in $L^1(\gamma)$ such that $\partial_j\mathcal{L}^{-1/2}f \in L^1(\gamma)$ for $j = 1,\dots,n$.

This paper arose from the desire to find a maximal characterization of the space $H^1(\gamma)$. We recall that the classical space $H^1(\lambda)$ can be characterized as the space of all functions f in $L^1(\lambda)$ whose *grand maximal function*

$$\mathcal{M}f(x) = \sup\{|\phi_t * f(x)| : \phi \in \Phi,\ t > 0\} \tag{1.2}$$

is also in $L^1(\lambda)$. Here $\Phi = \{\phi \in C_c^1(B(0,1)) : |D^\alpha \phi| \leq 1 \text{ for } |\alpha| = 0, 1\}$ and $\phi_t(x) = t^{-n}\phi(x/t)$.

To characterize $H^1(\gamma)$, we introduce the *local grand maximal function* defined on $L^1_{\mathrm{loc}}(\mathbb{R}^n, \gamma)$ by

$$\mathcal{M}_{\mathrm{loc}}f(x) = \sup\{|\phi_t * f(x)| : \phi \in \Phi,\ 0 < t < \min(1, 1/|x|)\}.$$

In Section 3 we shall prove that, in arbitrary dimensions, $f \in H^1(\gamma)$ implies $\mathcal{M}_{\mathrm{loc}}f \in L^1(\gamma)$. Moreover, *in dimension one*, $H^1(\gamma)$ can be characterized as the space of all functions f in $L^1(\gamma)$ satisfying $\mathcal{M}_{\mathrm{loc}}f \in L^1(\gamma)$ and the following additional *global* condition:

$$E(f) = \int_0^\infty x \left(\left| \int_x^\infty f \, \mathrm{d}\gamma \right| + \left| \int_{-\infty}^{-x} f \, \mathrm{d}\gamma \right| \right) \mathrm{d}\lambda(x) < \infty. \tag{1.3}$$

This is Theorem 3.3 below.

Roughly speaking, if we interpret a function f as a density of electrical charge on the real line, this global condition says that the positive and negative charges nearly balance out, so that the net charges inside the intervals $(-\infty, -x)$ and (x, ∞) decay sufficiently fast as x approaches $+\infty$. The condition is violated when the distance between the positive and the negative charges increases too much or the charges do not decay sufficiently fast at infinity. For instance, let $(a_n)_1^\infty$ and $(a'_n)_1^\infty$ be increasing sequences in $(2, \infty)$ such that

$$a_n + 2/a_n < a'_n \quad \text{and} \quad a'_n + 2/a'_n < a_{n+1} < 2a_n$$

for all n. Then set

$$f = \sum_1^\infty c_n \left(\frac{\chi_{(a_n, a_n + 1/a_n)}}{\gamma(a_n, a_n + 1/a_n)} - \frac{\chi_{(a'_n, a'_n + 1/a'_n)}}{\gamma(a'_n, a'_n + 1/a'_n)} \right) \tag{1.4}$$

for some $c_n > 0$. One easily verifies that $\mathcal{M}_{\mathrm{loc}}f \in L^1(\gamma)$ if and only if $\sum c_n < \infty$. But the global condition $E(f) < \infty$ is equivalent to $\sum c_n a_n (a'_n - a_n) < \infty$, which is here a stronger condition.

We have not been able to find a similar characterization of $H^1(\gamma)$ in higher dimensions. However, in Section 4 we prove in all dimensions that if $\mathcal{M}_{\mathrm{loc}}f \in L^1(\gamma)$ and the function f satisfies the stronger global condition

$$E_+(f) = \int |x|^2 \, |f(x)| \, \mathrm{d}\gamma(x) < \infty,$$

then $f \in H^1(\gamma)$. Observe that for $n = 1$ and $f \geq 0$, Fubini's theorem implies that the conditions $E(f) < \infty$ and $E_+(f) < \infty$ are equivalent. In arbitrary dimensions, $E_+(f)$ can be used to characterize the nonnegative functions in $H^1(\gamma)$; see Theorem 4.2. This also leads to a simple proof of the inclusions $L^p(\gamma) \subset H^1(\gamma)$ and $BMO(\gamma) \subset L^{p'}(\gamma)$ for $1 < p \leq \infty$.

We end the introduction with some technical observations and notation. In the following we use repeatedly the fact that on admissible balls at a fixed scale s, the Gauss and the Lebesgue measures are equivalent; i.e., there exists a positive constant $C(s)$ such that for every measurable subset E of $B \in \mathcal{B}_s$,

$$C(s)^{-1}\gamma(E) \leq \gamma_0(c_B)\lambda(E) \leq C(s)\gamma(E). \tag{1.5}$$

In particular this implies that the Gauss measure is doubling on balls in $\mathcal{B}_s$, with a constant that depends on s (see [24, Prop. 2.1]). Further, it is straightforward to see that if $B' \subset B$ are two balls and $B \in \mathcal{B}_s$, then B' is also in $\mathcal{B}_s$.

Given a ball B in $\mathbb{R}^n$ and a positive number ρ, we shall denote by ρB the ball with the same centre and with radius ρr_B.

In the following C denotes a positive constant whose value may change from occurrence to occurrence and which depends only on the dimension n, except when otherwise explicitly stated.

2. Coincidence of $H^{1,\infty}(\gamma)$ and $H^{1,r}(\gamma)$

First we need a lemma which will play a role also in the maximal characterization. It deals with the classical Hardy space $H^1(\lambda)$ with respect to the Lebesgue measure and the associated standard $(1,\infty)$-atoms, called Lebesgue atoms below.

Lemma 2.1. *If $g \in H^1(\lambda)$ and the support of g is contained in a ball B, then g has an atomic decomposition $g = \sum_k \lambda_k a_k$, where the a_k are Lebesgue $(1,\infty)$-atoms associated to balls contained in $2B$ and*

$$\sum_k |\lambda_k| \leq C \, \|g\|_{H^1(\lambda)}. \tag{2.1}$$

The lemma can be proved by applying [23, Theor. 4.13] to the space of homogeneous type B, endowed with Euclidean distance and Lebesgue measure. Some related results can be found in [4] and [5].

Theorem 2.2. *For every r in $(1,\infty)$, the spaces $H^{1,r}(\gamma)$ and $H^{1,\infty}(\gamma)$ coincide, with equivalent norms.*

Proof. In this proof, the constants C may depend on r and n. Since any Gaussian $(1,\infty)$-atom is also a Gaussian $(1,r)$-atom, $H^{1,\infty}(\gamma)$ is a subspace of $H^{1,r}(\gamma)$ and $\|f\|_{H^{1,r}(\gamma)} \leq \|f\|_{H^{1,\infty}(\gamma)}$. Conversely, suppose that a is a Gaussian $(1,r)$-atom associated to the ball $B \in \mathcal{B}_1$. Then the function $a\,\gamma_0$ is a multiple of a Lebesgue $(1,r)$-atom. Indeed, $\int a\gamma_0 \,\mathrm{d}\lambda = \int a \,\mathrm{d}\gamma = 0$ and, by the equivalence of the Gauss and Lebesgue measures on admissible balls,

$$\|a\gamma_0\|_{L^r(\lambda)} \leq C\,\lambda(B)^{1/r-1}.$$

Hence, $a\gamma_0$ is in $H^1(\lambda)$ with norm at most C. By Lemma 2.1, it has a decomposition

$$a\,\gamma_0 = \sum_j \lambda_j \alpha_j,$$

where each α_j is a Lebesgue $(1,\infty)$-atom associated to a ball B_j contained in $2B$. Moreover

$$\sum_j |\lambda_j| \leq C,$$

and each B_j is admissible at scale 2. Define $a_j = \alpha_j\,\gamma_0^{-1}$. Then $\int a_j \,\mathrm{d}\gamma = 0$, and by the equivalence of the Gauss and Lebesgue measures on B_j,

$$\|a_j\|_\infty \leq C\gamma(B_j)^{-1}.$$

Thus the a_j are multiples of Gaussian $(1,\infty)$-atoms. Since $a = \sum_j \lambda_j a_j$, we conclude that $a \in H^{1,\infty}(\gamma)$ and

$$\|a\|_{H^{1,\infty}(\gamma)} \le C \sum_j |\lambda_j| \le C. \tag{2.2}$$ □

3. The characterization of $H^1(\gamma)$ in $\mathbb{R}$

In this section, we shall prove that $f \in H^1(\gamma)$ implies $\mathcal{M}_{\mathrm{loc}} f \in L^1(\gamma)$ and that, in dimension one, functions in $H^1(\gamma)$ can be characterized by the two conditions $\mathcal{M}_{\mathrm{loc}} f \in L^1(\gamma)$ and $E(f) < \infty$. We start with a simple but useful lemma dealing with the support of the local grand maximal function.

Lemma 3.1. *If $f \in L^1(\gamma)$ is supported in the admissible ball B, then* $\operatorname{supp} \mathcal{M}_{\mathrm{loc}} f$ *is contained in the ball $B' = B(c_B, R)$, where $R = 4\min(1, 1/|c_B|)$.*

Proof. Let $x \in \operatorname{supp} \mathcal{M}_{\mathrm{loc}} f$. We write $\rho = |x|$ and $c = |c_B|$, so that $B \subset B(c_B, \min(1, 1/c))$. The balls B and $B(x, \min(1, 1/\rho))$ must intersect, and so

$$|x - c_B| \le \min(1, 1/c) + \min(1, 1/\rho). \tag{3.1}$$

To prove the lemma, it is enough to show that

$$\min(1, 1/\rho) \le 3\min(1, 1/c), \tag{3.2}$$

since it then follows that $x \in B'$. Now $c - \rho \le |x - c_B|$, so that (3.1) implies

$$c - \min(1, 1/c) \le \rho + \min(1, 1/\rho).$$

Considering the cases $c \le 1$ and $c > 1$, we conclude from this that

$$\max(1, c) - \min(1, 1/c) \le \max(1, \rho) + \min(1, 1/\rho). \tag{3.3}$$

The function $t \mapsto t^{-1} - t,\ \ t > 0$, and its inverse are clearly decreasing. Considering the values of this function at $t = \min(1, 1/c)$ and $\min(1, 1/\rho)/3$, we see that (3.2) is equivalent to

$$\max(1, c) - \min(1, 1/c) \le 3\max(1, \rho) - \frac{1}{3}\min(1, 1/\rho).$$

Because of (3.3), this inequality follows if

$$\max(1, \rho) + \min(1, 1/\rho) \le 3\max(1, \rho) - \frac{1}{3}\min(1, 1/\rho)$$

or equivalently $\frac{4}{3}\min(1, 1/\rho) \le 2\max(1, \rho)$, which is trivially true. We have proved (3.2) and the lemma. □

Lemma 3.2. *If f is in $H^1(\gamma)$, then $\mathcal{M}_{\mathrm{loc}} f \in L^1(\gamma)$ and*

$$\|\mathcal{M}_{\mathrm{loc}} f\|_1 \le C\|f\|_{H^1(\gamma)}.$$

Proof. We shall prove that for any Gaussian atom a,

$$\|\mathcal{M}_{\mathrm{loc}} a\|_1 \le C, \tag{3.4}$$

from which the lemma follows.

Since (3.4) is obvious if a is the constant function 1, we assume that a is associated to an admissible ball B. By the preceding lemma, $\operatorname{supp} \mathcal{M}_{\mathrm{loc}} f$ is contained in the ball denoted B'.

The integral of $\mathcal{M}_{\mathrm{loc}} a$ over $2B$ with respect to γ is no larger than C, since $\mathcal{M}_{\mathrm{loc}} a \le C \sup |a| \le C/\gamma(B)$. To estimate $\mathcal{M}_{\mathrm{loc}} a$ at a point x in the remaining set $B'\setminus 2B$, we take $\phi\in\Phi$ and $0<t<\min(1,1/|x|)$ and estimate $a*\phi_t(x)$. We can assume that $t>d(x,B)$ so that $t>|x-c_B|/2$, since otherwise $\phi_t*a(x)$ will vanish. Write

(3.5)
$$\phi_t * a(x) = t^{-n}\int\Big(\phi\big(\frac{x-y}{t}\big)-\phi\big(\frac{x-c_B}{t}\big)\Big)\,a(y)\,\mathrm{d}y + t^{-n}\phi\big(\frac{x-c_B}{t}\big)\int a(y)\,\mathrm{d}y.$$

Here the first term to the right can be estimated in a standard way by

$$Ct^{-n-1}\int_B |y-c_B|\,|a(y)|\,\mathrm{d}y \le C\,|x-c_B|^{-n-1}\,r_B\,\gamma_0(c_B)^{-1}.$$

To deal with the second term, we estimate $\int a(y)\,\mathrm{d}y$, knowing that the integral of a against γ vanishes. Thus

$$\int a(y)\,\mathrm{d}y = \int a(y)\frac{\gamma_0(c_B)-\gamma_0(y)}{\gamma_0(c_B)}\,\mathrm{d}y.$$

The fraction appearing here is

$$e^{|c_B|^2}\Big(e^{-|c_B|^2}-e^{-|y|^2}\Big) = 1-e^{(c_B-y)\cdot(c_B+y)}, \tag{3.6}$$

and the last exponent stays bounded for $y\in B$. Thus the modulus of the right-hand side of (3.6) is at most $C|c_B-y||c_B+y|\le Cr_B(1+|c_B|)$. Since $\int|a|\,d\gamma\le 1$, this implies that

$$\left|\int a(y)\,\mathrm{d}y\right| \le Cr_B(1+|c_B|)\gamma_0(c_B)^{-1}.$$

For the last term in (3.5), we thus get the bound $C|x-c_B|^{-n}r_B(1+|c_B|)\gamma_0(c_B)^{-1}$.

Putting things together, we conclude that for $x\in B'\setminus 2B$,

$$\mathcal{M}_{\mathrm{loc}}a(x)\le C|x-c_B|^{-n-1}r_B\gamma_0(c_B)^{-1}+C|x-c_B|^{-n}r_B(1+|c_B|)\gamma_0(c_B)^{-1}.$$

An integration with respect to $\mathrm{d}\gamma$, or equivalently $\gamma_0\,\mathrm{d}\lambda$, then leads to

$$\int_{B'\setminus 2B}\mathcal{M}_{\mathrm{loc}}a(x)\,\mathrm{d}\gamma(x)\le C+Cr_B(1+|c_B|)\log\frac{\min(1,1/|c_B|)}{r_B}\le C,$$

and (3.4) is proved. □

Theorem 3.3. *Let $n=1$, and suppose that f is a function in $L^1(\gamma)$. Then f is in $H^1(\gamma)$ if and only if $\mathcal{M}_{\mathrm{loc}}f\in L^1(\gamma)$ and $E(f)<\infty$. The norms $\|f\|_{H^1(\gamma)}$ and $\|\mathcal{M}_{\mathrm{loc}}f\|_{L^1(\gamma)}+E(f)$ are equivalent.*

Proof. Suppose that $f\in H^1(\gamma)$. Then $\mathcal{M}_{\mathrm{loc}}f\in L^1(\gamma)$ by Lemma 3.2. To prove the necessity of the condition $E(f)<\infty$, it suffices to show that $E(a)<C$ for all Gaussian atoms a. This is obvious for the exceptional atom 1. If a is associated to a ball $B\in\mathcal{B}_1$, it follows from the inequality

$$\left|\int_{-\infty}^{-x}a\,\mathrm{d}\gamma\right|+\left|\int_x^{\infty}a\,\mathrm{d}\gamma\right|\le \mathbf{1}_{(-B)\cup B}(x).$$

Conversely, assume that f is a function in $L^1(\gamma)$ such that $\mathcal{M}_{\mathrm{loc}}f\in L^1(\gamma)$ and $E(f)<\infty$. We shall prove that $f\in H^1(\gamma)$, by constructing a Gaussian atomic decomposition $f=\sum_j\lambda_j\,a_j$ such that $\sum_j|\lambda_j|\le C\big(\|\mathcal{M}_{\mathrm{loc}}f\|_1+E(f)\big)$.

Most of the following argument, up to the decomposition (3.15), works also in the n-dimensional setting. Since we shall need it in the next section, we carry out that part in $\mathbb{R}^n$.

By subtracting a multiple of the exceptional atom 1, we may without loss of generality assume that

$$\int f \, d\gamma = 0. \tag{3.7}$$

Let $\{B_j\}$ be a covering of $\mathbb{R}^n$ by maximal admissible balls. We can choose this covering in such way that the family $\left\{\frac{1}{2}B_j\right\}$ is disjoint and $\{4B_j\}$ has bounded overlap [10, Lemma 2.4]. Fix a smooth nonnegative partition of unity $\{\eta_j\}$ in $\mathbb{R}^n$ such that $\operatorname{supp}\eta_j \subset B_j$ and $\eta_j = 1$ on $\frac{1}{2}B_j$ and verifying $|\nabla\eta_j| \le C/r_{B_j}$. Thus $f = \sum_j f\eta_j$. We now need the following lemma.

Lemma 3.4. *For g in $L^1_{\mathrm{loc}}(\gamma)$ and $x \in \mathbb{R}^n$ one has*

$$\mathcal{M}_{\mathrm{loc}}(g\eta_j\gamma_0)(x) \le C\ \gamma_0(c_{B_j})\ \mathcal{M}_{\mathrm{loc}}g(x)\ \mathbf{1}_{4B_j}(x) \qquad \forall j. \tag{3.8}$$

Proof. Since the support of η_j is contained in B_j, the support of $\mathcal{M}_{\mathrm{loc}}(g\eta_j\gamma_0)$ is contained in the ball $4B_j$, because of Lemma 3.1. Moreover, for $\phi \in \Phi$ and $x \in 4B_j$,

$$\phi_t * (g\eta_j\gamma_0)(x) = \gamma_0(c_{B_j})\ \tilde{\phi}_t * g(x),$$

where $\tilde{\phi}(z) = \phi(z)\eta_j(x - tz)\gamma_0(x - tz)/\gamma_0(c_{B_j})$. Thus, to prove (3.8) it suffices to show that there exists a positive constant C such that $\tilde{\phi} \in C\Phi$ for $x \in 4B_j$ and $0 < t < \min(1, 1/|x|)$. The support of $\tilde{\phi}$ is contained in $B(0, 1)$ and

$$\left|\tilde{\phi}(z)\right| \le \frac{\gamma_0(x - tz)}{\gamma_0(c_{B_j})} \le C,$$

because for $|z| \le 1$,

$$\left|x - tz - c_{B_j}\right| \le \left|x - c_{B_j}\right| + |tz| \le C \min(1, 1/\left|c_{B_j}\right|).$$

Similarly $\left|\nabla\tilde{\phi}(z)\right| \le C$, because the gradients $\nabla_z\eta_j(x-tz)$ and $\nabla_z\gamma_0(x-tz)/\gamma_0(c_{B_j})$ give the factors $t(1 + \left|c_{B_j}\right|)$ and $t\,|x - tz|\,\gamma_0(x - tz)/\gamma_0(c_{B_j})$, respectively, both of which are bounded. This concludes the proof of Lemma 3.4. ☐

Continuing the proof of Theorem 3.3, we define $b_j \in \mathbb{C}$ for each $j \in \mathbb{N}$ by

$$\int_{-\infty}^{\infty} (f - b_j)\eta_j \, d\gamma = 0. \tag{3.9}$$

Note that since $\eta_j = 1$ on $\frac{1}{2}B_j$,

$$|b_j| = \left|\frac{\int f\eta_j \, d\gamma}{\int \eta_j \, d\gamma}\right| \le C\frac{1}{\gamma(B_j)}\int_{B_j} |f| \, d\gamma. \tag{3.10}$$

We now apply Lemma 3.4 with $g = f - b_j$ and use the subadditivity of $\mathcal{M}_{\mathrm{loc}}$ combined with (3.10) to get

$$\begin{aligned}\int \mathcal{M}_{\mathrm{loc}}\big((f - b_j)\eta_j\gamma_0\big) \, d\lambda &\le C \int_{4B_j} \mathcal{M}_{\mathrm{loc}}f\ \gamma_0(c_{B_j}) \, d\lambda + C\,\frac{\gamma(4B_j)}{\gamma(B_j)} \int_{B_j} |f| \, d\gamma \\ &\le C \int_{4B_j} \mathcal{M}_{\mathrm{loc}}f \, d\gamma. \end{aligned} \tag{3.11}$$

Lemma 3.5. *The function $(f - b_j)\eta_j\gamma_0$ is in $H^1(\lambda)$ and*

$$\|(f - b_j)\eta_j\gamma_0\|_{H^1(\lambda)} \le \int_{4B_j} \mathcal{M}_{\mathrm{loc}}f \, d\gamma. \tag{3.12}$$

Proof. By the maximal characterization of the classical space $H^1(\lambda)$, it suffices to show that

$$\int \mathcal{M}\left((f-b_j)\eta_j\gamma_0\right) \mathrm{d}\lambda(x) \leq C \int_{4B_j} \mathcal{M}_{\mathrm{loc}} f \,\mathrm{d}\gamma. \tag{3.13}$$

Because of (3.11), all that needs to be verified is that

$$\int \sup_{\phi\in\Phi} \sup_{t\geq \min(1,1/|x|)} \left|\left((f-b_j)\eta_j\gamma_0\right) * \phi_t(x)\right| \mathrm{d}\lambda(x) \leq C \int_{4B_j} \mathcal{M}_{\mathrm{loc}} f \,\mathrm{d}\gamma. \tag{3.14}$$

To prove (3.14), we split the integral in the left-hand side into the sum

$$\int_{4B_j} \cdots \,\mathrm{d}\lambda(x) + \int_{(4B_j)^c} \cdots \,\mathrm{d}\lambda(x).$$

If $x \in 4B_j$, then for $\phi \in \Phi$ and $t \geq \min(1, 1/|x|)$,

$$\begin{aligned} \left|\phi_t * \big((f-b_j)\eta_j\gamma_0\big)(x)\right| &\leq t^{-n} \int_{B_j} |f(y)-b_j| \,\mathrm{d}\gamma(y) \\ &\leq (1+|x|)^n \int_{B_j} \big(|f(y)| + |b_j|\big) \,\mathrm{d}\gamma(y) \\ &\leq C\,(1+|c_{B_j}|)^n \int_{B_j} |f(y)| \,\mathrm{d}\gamma(y), \end{aligned}$$

the last inequality because of (3.10). Hence

$$\int_{4B_j} \cdots \,\mathrm{d}\lambda(x) \,\leq\, C \;\; |4B_j| \;\; (1+|c_{B_j}|)^n \int_{B_j} |f| \,\mathrm{d}\gamma \,\leq\, C \int_{B_j} \mathcal{M}_{\mathrm{loc}} f \,\mathrm{d}\gamma.$$

If $x \in (4B_j)^c$, we take ϕ and t as before and observe that we can assume that $t > d(x, B_j)$, since otherwise the convolution in (3.14) will vanish. In view of (3.9) and (3.10), we then get

$$\begin{aligned} \left|\phi_t * \big((f-b_j)\eta_j\gamma_0\big)(x)\right| &\leq \int_{B_j} \left|\phi_t(x-y) - \phi_t(x-c_{B_j})\right| \; |f(y)-b_j| \; \eta_j(y) \,\mathrm{d}\gamma(y) \\ &\leq \, C\, t^{-n-1} \int_{B_j} \left|y - c_{B_j}\right| \, |f(y)-b_j| \,\mathrm{d}\gamma(y) \\ &\leq \, C\, \frac{1}{d(x,B_j)^{n+1}} r_{B_j} \int_{B_j} |f(y)| \,\mathrm{d}\gamma(y). \end{aligned}$$

This implies that

$$\int_{(4B_j)^c} \cdots \,\mathrm{d}\lambda(x) \leq C \int_{B_j} |f| \,\mathrm{d}\gamma \leq C \int_{B_j} \mathcal{M}_{\mathrm{loc}} f \,\mathrm{d}\gamma.$$

We have proved (3.14) and the lemma. □

We can now finish the proof of Theorem 3.3. By Lemmata 3.5 and 2.1, each function $(f-b_j)\eta_j\gamma_0$ has an atomic decomposition $\sum_k \lambda_{jk}\alpha_{jk}$, where the α_{jk} are Lebesgue atoms with supports in $2B_j$ and

$$\sum_k |\lambda_{jk}| \leq \, C \int_{4B_j} \mathcal{M}_{\mathrm{loc}} f \,\mathrm{d}\gamma.$$

As we saw in the proof of Theorem 2.2, each $a_{jk} = \gamma_0^{-1}\alpha_{jk}$ is a multiple of a Gaussian atom, with a factor which is independent of j and k. Thus

$$f = \sum_j (f - b_j)\eta_j + \sum_j b_j\eta_j = \sum_j \sum_k \lambda_{jk}\, a_{jk} + \sum_j b_j\eta_j \tag{3.15}$$

and

$$\sum_{j,k} |\lambda_{jk}| \le \sum_j \int_{4B_j} \mathcal{M}_{\mathrm{loc}} f \,\mathrm{d}\gamma \le C\, \|\mathcal{M}_{\mathrm{loc}} f\|_{L^1(\gamma)}.$$

To complete the proof of Theorem 3.3, we need to find an atomic decomposition of $\sum_j b_j\eta_j$. It is here that we must restrict ourselves to the one-dimensional case and that the global condition $E(f) < \infty$ plays a role.

Choose the intervals $I_0 = (-1, 1)$, $I_j = (\sqrt{j-1}, \sqrt{j+1})$ for $j \ge 1$ and $I_j = -I_{|j|}$ for $j \le -1$. The intervals I_j have essentially the same properties as the balls B_j introduced above, and we can use them instead of the B_j to construct η_j and b_j as before. To decompose now $\sum_j b_j\eta_j$, we first normalise the functions η_j, letting

$$\tilde{\eta}_j = \frac{\eta_j}{\int \eta_j \,\mathrm{d}\gamma}.$$

Then $b_j\eta_j = \int f\eta_j \,\mathrm{d}\gamma\; \tilde{\eta}_j$, and clearly

$$\sum_{j\ge k} \int f\eta_j \,\mathrm{d}\gamma = \int f\mu_k \,\mathrm{d}\gamma, \quad k \in \mathbb{Z},$$

where $\mu_k(x) = \sum_{j\ge k} \eta_j(x)$. Notice that $\int f\mu_k \,\mathrm{d}\gamma \to 0$ as $k \to \pm\infty$, in view of (3.7). A summation by parts now yields

$$\sum_{j\in\mathbb{Z}} \int f\eta_j \,\mathrm{d}\gamma\; \tilde{\eta}_j = \sum_{k\in\mathbb{Z}} \int f\mu_k \,\mathrm{d}\gamma\; (\tilde{\eta}_k - \tilde{\eta}_{k-1}). \tag{3.16}$$

But $\tilde{\eta}_k - \tilde{\eta}_{k-1}$ is C times a Gaussian atom if we use admissible balls at some scale $s > 1$ in the definition of atoms. Thus (3.16) is our desired atomic decomposition of $\sum_j b_j\eta_j$, provided we can estimate the coefficients by showing that

$$\sum_{k\in\mathbb{Z}} \left| \int f\mu_k \,\mathrm{d}\gamma \right| \le C\, (\|f\|_1 + E(f)). \tag{3.17}$$

To this end, observe that

$$\int f\mu_k \,\mathrm{d}\gamma = \int f(x) \int_{-\infty}^{x} \mu_k'(y) \,\mathrm{d}\lambda(y) \,\mathrm{d}\gamma(x) = \int \mu_k'(y) \int_y^{\infty} f(x) \,\mathrm{d}\gamma(x) \,\mathrm{d}\lambda(y).$$

Since the support of μ_k' is contained in I_k and

$$|\mu_k'(y)| \le \frac{C}{|I_k|} \le C\,\big(1 + |c_{I_k}|\big),$$

we obtain, using also the bounded overlap of the I_j,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|\int f\mu_k\,\mathrm{d}\gamma\right| &\leq C\sum_k\int_{I_k}\big(1+|c_{I_k}|\big)\,\left|\int_y^\infty f(x)\,\mathrm{d}\gamma(x)\right|\,\mathrm{d}\lambda(y)\\ &\leq C\int_{-\infty}^{\infty}(1+|y|)\left|\int_y^\infty f\,\mathrm{d}\gamma\right|\,\mathrm{d}\lambda(y)\\ &= C\int_0^\infty(1+y)\left(\left|\int_y^\infty f\,\mathrm{d}\gamma\right|+\left|\int_{-y}^\infty f\,\mathrm{d}\gamma\right|\right)\,\mathrm{d}\lambda(y)\\ &\leq C\big(\|f\|_1+E(f)\big);\end{aligned}$$

here we used (3.7). This concludes the proof of Theorem 3.3. □

4. A characterization of nonnegative functions in $H^1(\gamma)$

The dimension n is now arbitrary. The following lemma will be needed.

Lemma 4.1. *Let $\phi_0=\gamma\big(B(0,1)\big)^{-1}\mathbf{1}_{B(0,1)}$. If $g\in L^\infty$ is supported in a maximal admissible ball B, then*

$$\left\|g-\int g\,\mathrm{d}\gamma\,\phi_0\right\|_{H^1(\gamma)}\leq C\big(1+|c_B|^2\big)\gamma(B)\|g\|_{L^\infty}.$$

Proof. We shall construct atoms whose supports form a chain connecting $B(0,1)$ to B. First we define a finite sequence of maximal admissible balls

$$\tilde{B}_0=B(0,1),\ \tilde{B}_1,\dots,\tilde{B}_N,$$

all with centres $c_{\tilde{B}_j}$ on the segment $[0,c_B]$. The absolute values $\rho_j=|c_{\tilde{B}_j}|$ shall be increasing in j, and the boundary $\partial\tilde{B}_j$ shall contain $c_{\tilde{B}_{j-1}}$ for $j=1,\dots,N-1$, which means that

$$\rho_j-\frac{1}{\rho_j}=\rho_{j-1},\qquad j=1,\dots,N-1,\tag{4.1}$$

and $\rho_0=0,\quad\rho_1=1$. Finally, N is defined so that $\tilde{B}_{N-1}$ is the first ball of the sequence that contains c_B, and $\tilde{B}_N=B$. Squaring (4.1), we get

$$\rho_j^2-\rho_{j-1}^2=2-\frac{1}{\rho_j^2}\geq 1,$$

so that $\rho_{N-1}^2\geq N-1$. It follows that

$$N\leq|c_B|^2+1.\tag{4.2}$$

Next, we denote by B_j, $j=1,\dots,N$, the largest ball contained in $\tilde{B}_j\cap\tilde{B}_{j-1}$. Notice that the three balls $\tilde{B}_j,\tilde{B}_{j-1}$ and B_j have comparable radii and comparable Gaussian measures. Now define functions ϕ_j and g_j by setting

$$\begin{aligned}\phi_j&=\gamma(B_j)^{-1}\mathbf{1}_{B_j}\quad j=1,\dots,N,\\ g_j&=\int g\,\mathrm{d}\gamma\,\left(\phi_j-\phi_{j-1}\right),\quad j=1,\dots,N,\\ g_{N+1}&=g-\int g\,\mathrm{d}\gamma\,\phi_N.\end{aligned}$$

Clearly,

$$g - \int g \,\mathrm{d}\gamma\, \phi_0 = \sum_{j=1}^{N+1} g_j. \tag{4.3}$$

Each function g_j is a multiple of an atom. Indeed, its integral against γ vanishes. Moreover, if $1 \le j \le N$, the support of g_j is contained in $\tilde{B}_{j-1}$ and

$$\|g_j\|_\infty \le (\gamma(B_j)^{-1} + \gamma(B_{j-1})^{-1}) \int |g| \,\mathrm{d}\gamma \le C\,\gamma(\tilde{B}_{j-1})^{-1}\, \gamma(B)\, \|g\|_{L^\infty}.$$

The support of ϕ_{N+1} is contained in B and

$$\|g_{N+1}\|_\infty \le \|g\|_\infty + \gamma(B)^{-1} \int |g| \,\mathrm{d}\gamma \le C\,\|g\|_{L^\infty}.$$

Thus

$$\|g_j\|_{H^1(\gamma)} \le C\,\gamma(B)\,\|g\|_{L^\infty}, \qquad j = 1, \dots, N+1.$$

Summing the coefficients in the atomic decomposition (4.3), we then obtain via (4.2),

$$\left\| g - \int g \,\mathrm{d}\gamma\, \phi_0 \right\|_{H^1(\gamma)} \le C\,(N+1)\,\gamma(B)\|g\|_{L^\infty} \le C(1 + |c_B|^2)\,\gamma(B)\,\|g\|_{L^\infty}.$$

The proof of the lemma is complete. □

Theorem 4.2. *Suppose that f is a function in $L^1(\gamma)$. If $\mathcal{M}_{\mathrm{loc}} f$ is in $L^1(\gamma)$ and*

$$E_+(f) = \int |x|^2\, |f(x)| \,\mathrm{d}\gamma(x) < \infty, \tag{4.4}$$

then f is in $H^1(\gamma)$ and

$$\|f\|_{H^1(\gamma)} \le C\|\mathcal{M}_{\mathrm{loc}} f\|_1 + CE_+(f).$$

If f is nonnegative, the conditions $\mathcal{M}_{\mathrm{loc}} f \in L^1(\gamma)$ and $E_+(f) < \infty$ are also necessary for f to be in $H^1(\gamma)$.

Proof. Let f be a function in $L^1(\gamma)$ such that $\mathcal{M}_{\mathrm{loc}} f \in L^1(\gamma)$ and $E_+(f) < \infty$. Write $f = c(f) + f_0$, where $c(f) = \int f \,\mathrm{d}\gamma$. Since $c(f)$ is a multiple of the exceptional atom, it suffices to find an atomic decomposition of f_0. Note that f_0 satisfies

$$\mathcal{M}_{\mathrm{loc}} f_0 \in L^1(\gamma) \qquad \text{and} \qquad \int |x|^2\, |f_0(x)| \,\mathrm{d}\gamma(x) < \infty.$$

Let $\{B_j\}$ be the covering of $\mathbb{R}^n$ by maximal admissible balls and $\{\eta_j\}$ the corresponding partition of unity introduced in the proof of Theorem 3.3. As there, we choose numbers $b_j \in \mathbb{C}$ such that

$$\int_{-\infty}^{\infty} (f_0 - b_j)\eta_j \,\mathrm{d}\gamma = 0 \qquad \forall j.$$

Then the argument leading to (3.15) shows that

$$f_0 = \sum_j \sum_k \lambda_{jk}\, a_{jk} + \sum_j b_j \eta_j,$$

where the a_{jk} are Gaussian atoms supported in $4B_j$ and

$$\sum_{j,k} |\lambda_{jk}| \le C\,\|\mathcal{M}_{\mathrm{loc}} f_0\|_{L^1(\gamma)}.$$

It remains only to prove that $\sum_j b_j \eta_j$ is in $H^1(\gamma)$. We write $g_j = b_j \eta_j$ and observe that

$$\int \sum_j g_j \,\mathrm{d}\gamma = 0 \tag{4.5}$$

because f_0 and the a_{ij} have integrals zero. Thus

$$\sum_j g_j = \sum_j \Big(g_j - \int g_j \,\mathrm{d}\gamma \, \phi_0 \Big),$$

where ϕ_0 is as in Lemma 4.1. Since (3.10) remains valid for f_0, we have

$$\|g_j\|_\infty \leq C \, \frac{1}{\gamma(B_j)} \int_{B_j} |f_0| \,\mathrm{d}\gamma. \tag{4.6}$$

Lemma 4.1 thus applies to each g_j, and using also the bounded overlap of the B_j we conclude

$$\|\sum_j g_j\|_{H^1(\gamma)} \leq C \sum_j \big(1 + |c_{B_j}|^2\big) \int_{B_j} |f_0| \,\mathrm{d}\gamma \leq C \int \big(1 + |x|^2\big) \, |f_0| \,\mathrm{d}\gamma.$$

This concludes the proof of the sufficiency and the norm estimate.

The necessity of the condition $\mathcal{M}_{\mathrm{loc}} f \in L^1(\gamma)$ was obtained in Lemma 3.2.

To prove the necessity of (4.4), let $0 \leq f \in H^1(\gamma)$. We first observe that the function $x \mapsto |x|^2$ is in $BMO(\gamma)$. Indeed, its oscillation on any admissible ball is bounded. Since $BMO(\gamma)$ is a lattice, the functions $g_k(x) = \min \left\{ |x|^2, k \right\}$ are in $BMO(\gamma)$, uniformly for $k \geq 1$. By the monotone convergence theorem and the duality between $H^1(\gamma)$ and $BMO(\gamma)$,

$$\int |x|^2 \, f(x) \,\mathrm{d}\gamma(x) = \lim_k \int g_k(x) f(x) \,\mathrm{d}\gamma(x) \leq C \|f\|_{H^1(\gamma)}.$$

The theorem is proved. □

The following result is a noteworthy consequence of Theorem 4.2.

Corollary 4.3. *For $1 < p \leq \infty$, one has continuous inclusions $L^p(\gamma) \subset H^1(\gamma)$ and $BMO(\gamma) \subset L^{p'}(\gamma)$, where $p' = p/(p-1)$.*

Proof. We claim that the operator $\mathcal{M}_{\mathrm{loc}}$ is bounded on $L^p(\gamma)$ for $1 < p \leq \infty$. Deferring momentarily the proof of this claim, we complete the proof of the corollary. Suppose that f is in $L^p(\gamma)$. Then $\mathcal{M}_{\mathrm{loc}} f$ is in $L^1(\gamma)$, because

$$\|\mathcal{M}_{\mathrm{loc}} f\|_1 \leq \|\mathcal{M}_{\mathrm{loc}} f\|_p \leq C \|f\|_p < \infty,$$

since $\gamma(\mathbb{R}^n) = 1$. Moreover, $E_+(f) \leq \||x|^2\|_{p'} \, \|f\|_p < \infty$, by Hölder's inequality. Thus $f \in H^1(\gamma)$ by Theorem 4.2. It also follows that the inclusion $L^p(\gamma) \subset H^1(\gamma)$ is continuous, and by duality we get the continuous inclusion $BMO(\gamma) \subset L^{p'}(\gamma)$.

It remains to prove the claim. We shall use again the covering $\{B_j\}$ from the proof of Theorem 3.3. First we observe that the inequality

$$\|\mathcal{M}_{\mathrm{loc}} g\|_p \leq C \|g\|_p \tag{4.7}$$

holds when $\operatorname{supp} g \subset B_j$, with a constant C independent of j. Indeed, $\mathcal{M}_{\mathrm{loc}}$ is bounded on $L^p(\lambda)$, and $\mathcal{M}_{\mathrm{loc}} g$ is supported in the ball $4B_j$, where the Gaussian measure is essentially proportional to $d\lambda$.

Given a function $f \in L^p(\gamma)$, we write it as a sum $f = \sum f_j$ with $\operatorname{supp} f_j \subset B_j$ and with the sets $\{f_j \neq 0\}$ pairwise disjoint. We can then apply (4.7) to each f_j and sum. ☐

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