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## A MAXIMAL FUNCTION CHARACTERIZATION OF THE HARDY SPACE FOR THE GAUSS MEASURE

GIANCARLO MAUCERI, STEFANO MEDA, AND PETER SJÖGREN

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ABSTRACT. An atomic Hardy space  $H^1(\gamma)$  associated to the Gauss measure  $\gamma$  in  $\mathbb{R}^n$  has been introduced by the first two authors. We first prove that it is equivalent to use  $(1, r)$ - or  $(1, \infty)$ -atoms to define this  $H^1(\gamma)$ . For  $n = 1$ , a maximal function characterization of  $H^1(\gamma)$  is found. In arbitrary dimension, we give a description of the nonnegative functions in  $H^1(\gamma)$  and use it to prove that  $L^p(\gamma) \subset H^1(\gamma)$  for  $1 < p \leq \infty$ .

### 1. INTRODUCTION

Denote by  $\gamma$  the Gauss measure on  $\mathbb{R}^n$ , i.e., the probability measure with density  $\gamma_0(x) = \pi^{-n/2} e^{-|x|^2}$  with respect to the Lebesgue measure  $\lambda$ . Harmonic analysis on the measured metric space  $(\mathbb{R}^n, d, \gamma)$ , where  $d$  denotes the Euclidean distance on  $\mathbb{R}^n$ , has been the object of many investigations. In particular, efforts have been made to study operators related to the Ornstein–Uhlenbeck semigroup, with emphasis on maximal operators [33, 16, 27, 13, 21, 22], Riesz transforms [29, 14, 28, 32, 30, 17, 15, 8, 9, 10, 31, 35, 7, 24] and functional calculus [11, 12, 18, 25].

In [24] the first two authors defined an atomic Hardy-type space  $H^1(\gamma)$  and a space  $BMO(\gamma)$  of functions of bounded mean oscillation, associated to  $\gamma$ . We briefly recall their definitions. A closed Euclidean ball  $B$  is called *admissible* at scale  $s > 0$  if

$$r_B \leq s \min(1, 1/|c_B|);$$

here and in the sequel  $r_B$  and  $c_B$  denote the radius and the centre of  $B$ , respectively. We denote by  $\mathcal{B}_s$  the family of all balls admissible at scale  $s$ . For the sake of brevity, we shall refer to balls in  $\mathcal{B}_1$  simply as admissible balls. Further,  $B$  will be called *maximal admissible* if  $r_B = \min(1, 1/|c_B|)$ .

Now let  $r \in (1, \infty]$ . A *Gaussian*  $(1, r)$ -atom is either the constant function 1 or a function  $a$  in  $L^r(\gamma)$  supported in an admissible ball  $B$  and such that

$$(1.1) \quad \int a \, d\gamma = 0 \quad \text{and} \quad \|a\|_r \leq \gamma(B)^{1/r-1};$$

here and in the whole paper,  $\|\cdot\|_r$  denotes the norm in  $L^r(\gamma)$ . In the latter case, we say that the atom  $a$  is associated to the ball  $B$ . The space  $H^{1,r}(\gamma)$  is then the vector

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space of all functions  $f$  in  $L^1(\gamma)$  that admit a decomposition of the form  $\sum_j \lambda_j a_j$ , where the  $a_j$  are Gaussian  $(1, r)$ -atoms and the sequence of complex numbers  $\{\lambda_j\}$  is summable. The norm of  $f$  in  $H^{1,r}(\gamma)$  is defined as the infimum of  $\sum_j |\lambda_j|$  over all representations of  $f$  as above.

In [24] the spaces  $H^{1,r}(\gamma)$  were defined and proved to coincide for all  $1 < r < \infty$ , with equivalent norms. In Section 2 we complement this by proving that they coincide also with the space  $H^{1,\infty}(\gamma)$ . Once this is established, we shall denote the space by  $H^1(\gamma)$  and use the  $H^{1,\infty}(\gamma)$  norm. Further, we shall frequently write atom for the  $(1, \infty)$ -atom.

The space  $BMO(\gamma)$  consists of all functions  $f$  in  $L^1(\gamma)$  such that

$$\sup_{B \in \mathcal{B}_1} \frac{1}{\gamma(B)} \int_B |f - f_B| \, d\gamma < \infty,$$

where  $f_B$  denotes the mean value of  $f$  on  $B$ , taken with respect to the Gauss measure. The norm of a function in  $BMO(\gamma)$  is

$$\|f\|_{BMO(\gamma)} = \|f\|_1 + \sup_{B \in \mathcal{B}_1} \frac{1}{\gamma(B)} \int_B |f - f_B| \, d\gamma.$$

If, in the definitions of  $H^1(\gamma)$  and  $BMO(\gamma)$ , we replace the family  $\mathcal{B}_1$  of admissible balls at scale 1 by  $\mathcal{B}_s$  for any fixed  $s > 0$ , we obtain the same spaces with equivalent norms; see [24]. We remark that a similar  $H^1 - BMO$  theory for more general measured metric spaces has been developed by A. Carbonaro and the first two authors in [1, 2, 3]. See also the papers [19, 20] of L. Liu and D. Yang for related results.

The main motivation for introducing these two spaces was to provide endpoint estimates for singular integrals associated to the Ornstein-Uhlenbeck operator  $\mathcal{L} = -(1/2)\Delta + x \cdot \nabla$ , a natural self-adjoint Laplacian on  $L^2(\gamma)$ . Indeed, in [24] the first two authors proved that the imaginary powers of  $\mathcal{L}$  are bounded from  $H^1(\gamma)$  to  $L^1(\gamma)$  and from  $L^\infty(\gamma)$  to  $BMO(\gamma)$  and that Riesz transforms of the form  $\nabla^\alpha \mathcal{L}^{-|\alpha|}$  and of any order are bounded from  $L^\infty(\gamma)$  to  $BMO(\gamma)$ . In a recent paper [26], the authors proved that boundedness from  $H^1(\gamma)$  to  $L^1(\gamma)$  and from  $L^\infty(\gamma)$  to  $BMO(\gamma)$  holds for any first-order Riesz transform in dimension one, but not always in higher dimensions.

The definition of the space  $H^1(\gamma)$  closely resembles the atomic definition of the classical Hardy space  $H^1(\lambda)$  on  $\mathbb{R}^n$  endowed with the Lebesgue measure  $\lambda$ , but there are two basic differences. First, the measured metric space  $(\mathbb{R}^n, d, \gamma)$  is nondoubling. Further, except for the constant atom, a Gaussian atom must have “small support”, i.e., support contained in an admissible ball. Despite these differences,  $H^1(\gamma)$  shares many of the properties of  $H^1(\lambda)$ . In particular, the topological dual of  $H^1(\gamma)$  is isomorphic to  $BMO(\gamma)$ , an inequality of John-Nirenberg type holds for functions in  $BMO(\gamma)$  and the spaces  $L^p(\gamma)$  are intermediate spaces between  $H^1(\gamma)$  and  $BMO(\gamma)$  for the real and the complex interpolation methods.

It is well known that the classical Hardy space  $H^1(\lambda)$  can be defined in at least three different ways: the *atomic* definition, the *maximal* definition and the definition based on *Riesz transforms* [6, 34].

As shown in [26], in higher dimensions the first-order Ornstein-Uhlenbeck Riesz transforms  $\partial_j \mathcal{L}^{-1/2}$  are unbounded from  $H^1(\gamma)$  to  $L^1(\gamma)$ ; here  $\partial_j = \partial/\partial x_j$ . Thus  $H^1(\gamma)$  does not coincide with the space of all functions in  $L^1(\gamma)$  such that  $\partial_j \mathcal{L}^{-1/2} f \in L^1(\gamma)$  for  $j = 1, \dots, n$ .

This paper arose from the desire to find a maximal characterization of the space  $H^1(\gamma)$ . We recall that the classical space  $H^1(\lambda)$  can be characterized as the space of all functions  $f$  in  $L^1(\lambda)$  whose *grand maximal function*

$$(1.2) \quad \mathcal{M}f(x) = \sup \{ |\phi_t * f(x)| : \phi \in \Phi, t > 0 \}$$

is also in  $L^1(\lambda)$ . Here  $\Phi = \{ \phi \in C_c^1(B(0, 1)) : |D^\alpha \phi| \leq 1 \text{ for } |\alpha| = 0, 1 \}$  and  $\phi_t(x) = t^{-n} \phi(x/t)$ .

To characterize  $H^1(\gamma)$ , we introduce the *local grand maximal function* defined on  $L^1_{\text{loc}}(\mathbb{R}^n, \gamma)$  by

$$\mathcal{M}_{\text{loc}}f(x) = \sup \{ |\phi_t * f(x)| : \phi \in \Phi, 0 < t < \min(1, 1/|x|) \}.$$

In Section 3 we shall prove that, in arbitrary dimensions,  $f \in H^1(\gamma)$  implies  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$ . Moreover, *in dimension one*,  $H^1(\gamma)$  can be characterized as the space of all functions  $f$  in  $L^1(\gamma)$  satisfying  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  and the following additional *global* condition:

$$(1.3) \quad E(f) = \int_0^\infty x \left( \left| \int_x^\infty f \, d\gamma \right| + \left| \int_{-\infty}^{-x} f \, d\gamma \right| \right) d\lambda(x) < \infty.$$

This is Theorem 3.3 below.

Roughly speaking, if we interpret a function  $f$  as a density of electrical charge on the real line, this global condition says that the positive and negative charges nearly balance out, so that the net charges inside the intervals  $(-\infty, -x)$  and  $(x, \infty)$  decay sufficiently fast as  $x$  approaches  $+\infty$ . The condition is violated when the distance between the positive and the negative charges increases too much or the charges do not decay sufficiently fast at infinity. For instance, let  $(a_n)_1^\infty$  and  $(a'_n)_1^\infty$  be increasing sequences in  $(2, \infty)$  such that

$$a_n + 2/a_n < a'_n \quad \text{and} \quad a'_n + 2/a'_n < a_{n+1} < 2a_n$$

for all  $n$ . Then set

$$(1.4) \quad f = \sum_1^\infty c_n \left( \frac{\chi_{(a_n, a_n + 1/a_n)}}{\gamma(a_n, a_n + 1/a_n)} - \frac{\chi_{(a'_n, a'_n + 1/a'_n)}}{\gamma(a'_n, a'_n + 1/a'_n)} \right)$$

for some  $c_n > 0$ . One easily verifies that  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  if and only if  $\sum c_n < \infty$ . But the global condition  $E(f) < \infty$  is equivalent to  $\sum c_n a_n (a'_n - a_n) < \infty$ , which is here a stronger condition.

We have not been able to find a similar characterization of  $H^1(\gamma)$  in higher dimensions. However, in Section 4 we prove in all dimensions that if  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  and the function  $f$  satisfies the stronger global condition

$$E_+(f) = \int |x|^2 |f(x)| \, d\gamma(x) < \infty,$$

then  $f \in H^1(\gamma)$ . Observe that for  $n = 1$  and  $f \geq 0$ , Fubini's theorem implies that the conditions  $E(f) < \infty$  and  $E_+(f) < \infty$  are equivalent. In arbitrary dimensions,  $E_+(f)$  can be used to characterize the nonnegative functions in  $H^1(\gamma)$ ; see Theorem 4.2. This also leads to a simple proof of the inclusions  $L^p(\gamma) \subset H^1(\gamma)$  and  $BMO(\gamma) \subset L^p(\gamma)$  for  $1 < p \leq \infty$ .

We end the introduction with some technical observations and notation. In the following we use repeatedly the fact that on admissible balls at a fixed scale  $s$ ,

the Gauss and the Lebesgue measures are equivalent; i.e., there exists a positive constant  $C(s)$  such that for every measurable subset  $E$  of  $B \in \mathcal{B}_s$ ,

$$(1.5) \quad C(s)^{-1}\gamma(E) \leq \gamma_0(c_B)\lambda(E) \leq C(s)\gamma(E).$$

In particular this implies that the Gauss measure is doubling on balls in  $\mathcal{B}_s$ , with a constant that depends on  $s$  (see [24, Prop. 2.1]). Further, it is straightforward to see that if  $B' \subset B$  are two balls and  $B \in \mathcal{B}_s$ , then  $B'$  is also in  $\mathcal{B}_s$ .

Given a ball  $B$  in  $\mathbb{R}^n$  and a positive number  $\rho$ , we shall denote by  $\rho B$  the ball with the same centre and with radius  $\rho r_B$ .

In the following  $C$  denotes a positive constant whose value may change from occurrence to occurrence and which depends only on the dimension  $n$ , except when otherwise explicitly stated.

## 2. COINCIDENCE OF $H^{1,\infty}(\gamma)$ AND $H^{1,r}(\gamma)$

First we need a lemma which will play a role also in the maximal characterization. It deals with the classical Hardy space  $H^1(\lambda)$  with respect to the Lebesgue measure and the associated standard  $(1, \infty)$ -atoms, called Lebesgue atoms below.

**Lemma 2.1.** *If  $g \in H^1(\lambda)$  and the support of  $g$  is contained in a ball  $B$ , then  $g$  has an atomic decomposition  $g = \sum_k \lambda_k a_k$ , where the  $a_k$  are Lebesgue  $(1, \infty)$ -atoms associated to balls contained in  $2B$  and*

$$(2.1) \quad \sum_k |\lambda_k| \leq C \|g\|_{H^1(\lambda)}.$$

The lemma can be proved by applying [23, Theor. 4.13] to the space of homogeneous type  $B$ , endowed with Euclidean distance and Lebesgue measure. Some related results can be found in [4] and [5].

**Theorem 2.2.** *For every  $r$  in  $(1, \infty)$ , the spaces  $H^{1,r}(\gamma)$  and  $H^{1,\infty}(\gamma)$  coincide, with equivalent norms.*

*Proof.* In this proof, the constants  $C$  may depend on  $r$  and  $n$ . Since any Gaussian  $(1, \infty)$ -atom is also a Gaussian  $(1, r)$ -atom,  $H^{1,\infty}(\gamma)$  is a subspace of  $H^{1,r}(\gamma)$  and  $\|f\|_{H^{1,r}(\gamma)} \leq \|f\|_{H^{1,\infty}(\gamma)}$ . Conversely, suppose that  $a$  is a Gaussian  $(1, r)$ -atom associated to the ball  $B \in \mathcal{B}_1$ . Then the function  $a\gamma_0$  is a multiple of a Lebesgue  $(1, r)$ -atom. Indeed,  $\int a\gamma_0 d\lambda = \int a d\gamma = 0$  and, by the equivalence of the Gauss and Lebesgue measures on admissible balls,

$$\|a\gamma_0\|_{L^r(\lambda)} \leq C \lambda(B)^{1/r-1}.$$

Hence,  $a\gamma_0$  is in  $H^1(\lambda)$  with norm at most  $C$ . By Lemma 2.1, it has a decomposition

$$a\gamma_0 = \sum_j \lambda_j \alpha_j,$$

where each  $\alpha_j$  is a Lebesgue  $(1, \infty)$ -atom associated to a ball  $B_j$  contained in  $2B$ . Moreover

$$\sum_j |\lambda_j| \leq C,$$

and each  $B_j$  is admissible at scale 2. Define  $a_j = \alpha_j \gamma_0^{-1}$ . Then  $\int a_j d\gamma = 0$ , and by the equivalence of the Gauss and Lebesgue measures on  $B_j$ ,

$$\|a_j\|_\infty \leq C\gamma(B_j)^{-1}.$$

Thus the  $a_j$  are multiples of Gaussian  $(1, \infty)$ -atoms. Since  $a = \sum_j \lambda_j a_j$ , we conclude that  $a \in H^{1,\infty}(\gamma)$  and

$$(2.2) \quad \|a\|_{H^{1,\infty}(\gamma)} \leq C \sum_j |\lambda_j| \leq C. \quad \square$$

### 3. THE CHARACTERIZATION OF $H^1(\gamma)$ IN $\mathbb{R}$

In this section, we shall prove that  $f \in H^1(\gamma)$  implies  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  and that, in dimension one, functions in  $H^1(\gamma)$  can be characterized by the two conditions  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  and  $E(f) < \infty$ . We start with a simple but useful lemma dealing with the support of the local grand maximal function.

**Lemma 3.1.** *If  $f \in L^1(\gamma)$  is supported in the admissible ball  $B$ , then  $\text{supp } \mathcal{M}_{\text{loc}}f$  is contained in the ball  $B' = B(c_B, R)$ , where  $R = 4 \min(1, 1/|c_B|)$ .*

*Proof.* Let  $x \in \text{supp } \mathcal{M}_{\text{loc}}f$ . We write  $\rho = |x|$  and  $c = |c_B|$ , so that  $B \subset B(c_B, \min(1, 1/c))$ . The balls  $B$  and  $B(x, \min(1, 1/\rho))$  must intersect, and so

$$(3.1) \quad |x - c_B| \leq \min(1, 1/c) + \min(1, 1/\rho).$$

To prove the lemma, it is enough to show that

$$(3.2) \quad \min(1, 1/\rho) \leq 3 \min(1, 1/c),$$

since it then follows that  $x \in B'$ . Now  $c - \rho \leq |x - c_B|$ , so that (3.1) implies

$$c - \min(1, 1/c) \leq \rho + \min(1, 1/\rho).$$

Considering the cases  $c \leq 1$  and  $c > 1$ , we conclude from this that

$$(3.3) \quad \max(1, c) - \min(1, 1/c) \leq \max(1, \rho) + \min(1, 1/\rho).$$

The function  $t \mapsto t^{-1} - t$ ,  $t > 0$ , and its inverse are clearly decreasing. Considering the values of this function at  $t = \min(1, 1/c)$  and  $\min(1, 1/\rho)/3$ , we see that (3.2) is equivalent to

$$\max(1, c) - \min(1, 1/c) \leq 3 \max(1, \rho) - \frac{1}{3} \min(1, 1/\rho).$$

Because of (3.3), this inequality follows if

$$\max(1, \rho) + \min(1, 1/\rho) \leq 3 \max(1, \rho) - \frac{1}{3} \min(1, 1/\rho)$$

or equivalently  $\frac{4}{3} \min(1, 1/\rho) \leq 2 \max(1, \rho)$ , which is trivially true. We have proved (3.2) and the lemma. □

**Lemma 3.2.** *If  $f$  is in  $H^1(\gamma)$ , then  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  and*

$$\|\mathcal{M}_{\text{loc}}f\|_1 \leq C \|f\|_{H^1(\gamma)}.$$

*Proof.* We shall prove that for any Gaussian atom  $a$ ,

$$(3.4) \quad \|\mathcal{M}_{\text{loc}}a\|_1 \leq C,$$

from which the lemma follows.

Since (3.4) is obvious if  $a$  is the constant function 1, we assume that  $a$  is associated to an admissible ball  $B$ . By the preceding lemma,  $\text{supp } \mathcal{M}_{\text{loc}}f$  is contained in the ball denoted  $B'$ .

The integral of  $\mathcal{M}_{\text{loc}}a$  over  $2B$  with respect to  $\gamma$  is no larger than  $C$ , since  $\mathcal{M}_{\text{loc}}a \leq C \sup |a| \leq C/\gamma(B)$ . To estimate  $\mathcal{M}_{\text{loc}}a$  at a point  $x$  in the remaining

set  $B' \setminus 2B$ , we take  $\phi \in \Phi$  and  $0 < t < \min(1, 1/|x|)$  and estimate  $a * \phi_t(x)$ . We can assume that  $t > d(x, B)$  so that  $t > |x - c_B|/2$ , since otherwise  $\phi_t * a(x)$  will vanish. Write

$$(3.5) \quad \phi_t * a(x) = t^{-n} \int \left( \phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-c_B}{t}\right) \right) a(y) \, dy + t^{-n} \phi\left(\frac{x-c_B}{t}\right) \int a(y) \, dy.$$

Here the first term to the right can be estimated in a standard way by

$$Ct^{-n-1} \int_B |y - c_B| |a(y)| \, dy \leq C |x - c_B|^{-n-1} r_B \gamma_0(c_B)^{-1}.$$

To deal with the second term, we estimate  $\int a(y) \, dy$ , knowing that the integral of  $a$  against  $\gamma$  vanishes. Thus

$$\int a(y) \, dy = \int a(y) \frac{\gamma_0(c_B) - \gamma_0(y)}{\gamma_0(c_B)} \, dy.$$

The fraction appearing here is

$$(3.6) \quad e^{|c_B|^2} \left( e^{-|c_B|^2} - e^{-|y|^2} \right) = 1 - e^{(c_B-y) \cdot (c_B+y)},$$

and the last exponent stays bounded for  $y \in B$ . Thus the modulus of the right-hand side of (3.6) is at most  $C|c_B - y||c_B + y| \leq Cr_B(1 + |c_B|)$ . Since  $\int |a| \, d\gamma \leq 1$ , this implies that

$$\left| \int a(y) \, dy \right| \leq Cr_B(1 + |c_B|)\gamma_0(c_B)^{-1}.$$

For the last term in (3.5), we thus get the bound  $C|x - c_B|^{-n}r_B(1 + |c_B|)\gamma_0(c_B)^{-1}$ .

Putting things together, we conclude that for  $x \in B' \setminus 2B$ ,

$$\mathcal{M}_{\text{loc}}a(x) \leq C|x - c_B|^{-n-1}r_B\gamma_0(c_B)^{-1} + C|x - c_B|^{-n}r_B(1 + |c_B|)\gamma_0(c_B)^{-1}.$$

An integration with respect to  $d\gamma$ , or equivalently  $\gamma_0 \, d\lambda$ , then leads to

$$\int_{B' \setminus 2B} \mathcal{M}_{\text{loc}}a(x) \, d\gamma(x) \leq C + Cr_B(1 + |c_B|) \log \frac{\min(1, 1/|c_B|)}{r_B} \leq C,$$

and (3.4) is proved. □

**Theorem 3.3.** *Let  $n = 1$ , and suppose that  $f$  is a function in  $L^1(\gamma)$ . Then  $f$  is in  $H^1(\gamma)$  if and only if  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  and  $E(f) < \infty$ . The norms  $\|f\|_{H^1(\gamma)}$  and  $\|\mathcal{M}_{\text{loc}}f\|_{L^1(\gamma)} + E(f)$  are equivalent.*

*Proof.* Suppose that  $f \in H^1(\gamma)$ . Then  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  by Lemma 3.2. To prove the necessity of the condition  $E(f) < \infty$ , it suffices to show that  $E(a) < C$  for all Gaussian atoms  $a$ . This is obvious for the exceptional atom 1. If  $a$  is associated to a ball  $B \in \mathcal{B}_1$ , it follows from the inequality

$$\left| \int_{-\infty}^{-x} a \, d\gamma \right| + \left| \int_x^{\infty} a \, d\gamma \right| \leq \mathbf{1}_{(-B) \cup B}(x).$$

Conversely, assume that  $f$  is a function in  $L^1(\gamma)$  such that  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  and  $E(f) < \infty$ . We shall prove that  $f \in H^1(\gamma)$ , by constructing a Gaussian atomic decomposition  $f = \sum_j \lambda_j a_j$  such that  $\sum_j |\lambda_j| \leq C(\|\mathcal{M}_{\text{loc}}f\|_1 + E(f))$ .

Most of the following argument, up to the decomposition (3.15), works also in the  $n$ -dimensional setting. Since we shall need it in the next section, we carry out that part in  $\mathbb{R}^n$ .

By subtracting a multiple of the exceptional atom 1, we may without loss of generality assume that

$$(3.7) \quad \int f \, d\gamma = 0.$$

Let  $\{B_j\}$  be a covering of  $\mathbb{R}^n$  by maximal admissible balls. We can choose this covering in such way that the family  $\{\frac{1}{2}B_j\}$  is disjoint and  $\{4B_j\}$  has bounded overlap [10, Lemma 2.4]. Fix a smooth nonnegative partition of unity  $\{\eta_j\}$  in  $\mathbb{R}^n$  such that  $\text{supp } \eta_j \subset B_j$  and  $\eta_j = 1$  on  $\frac{1}{2}B_j$  and verifying  $|\nabla \eta_j| \leq C/r_{B_j}$ . Thus  $f = \sum_j f \eta_j$ . We now need the following lemma.

**Lemma 3.4.** *For  $g$  in  $L^1_{\text{loc}}(\gamma)$  and  $x \in \mathbb{R}^n$  one has*

$$(3.8) \quad \mathcal{M}_{\text{loc}}(g\eta_j\gamma_0)(x) \leq C \gamma_0(c_{B_j}) \mathcal{M}_{\text{loc}}g(x) \mathbf{1}_{4B_j}(x) \quad \forall j.$$

*Proof.* Since the support of  $\eta_j$  is contained in  $B_j$ , the support of  $\mathcal{M}_{\text{loc}}(g\eta_j\gamma_0)$  is contained in the ball  $4B_j$ , because of Lemma 3.1. Moreover, for  $\phi \in \Phi$  and  $x \in 4B_j$ ,

$$\phi_t * (g\eta_j\gamma_0)(x) = \gamma_0(c_{B_j}) \tilde{\phi}_t * g(x),$$

where  $\tilde{\phi}(z) = \phi(z)\eta_j(x - tz)\gamma_0(x - tz)/\gamma_0(c_{B_j})$ . Thus, to prove (3.8) it suffices to show that there exists a positive constant  $C$  such that  $\tilde{\phi} \in C\Phi$  for  $x \in 4B_j$  and  $0 < t < \min(1, 1/|x|)$ . The support of  $\tilde{\phi}$  is contained in  $B(0, 1)$  and

$$|\tilde{\phi}(z)| \leq \frac{\gamma_0(x - tz)}{\gamma_0(c_{B_j})} \leq C,$$

because for  $|z| \leq 1$ ,

$$|x - tz - c_{B_j}| \leq |x - c_{B_j}| + |tz| \leq C \min(1, 1/|c_{B_j}|).$$

Similarly  $|\nabla \tilde{\phi}(z)| \leq C$ , because the gradients  $\nabla_z \eta_j(x - tz)$  and  $\nabla_z \gamma_0(x - tz)/\gamma_0(c_{B_j})$  give the factors  $t(1 + |c_{B_j}|)$  and  $t|x - tz|\gamma_0(x - tz)/\gamma_0(c_{B_j})$ , respectively, both of which are bounded. This concludes the proof of Lemma 3.4.  $\square$

Continuing the proof of Theorem 3.3, we define  $b_j \in \mathbb{C}$  for each  $j \in \mathbb{N}$  by

$$(3.9) \quad \int_{-\infty}^{\infty} (f - b_j)\eta_j \, d\gamma = 0.$$

Note that since  $\eta_j = 1$  on  $\frac{1}{2}B_j$ ,

$$(3.10) \quad |b_j| = \left| \frac{\int f \eta_j \, d\gamma}{\int \eta_j \, d\gamma} \right| \leq C \frac{1}{\gamma(B_j)} \int_{B_j} |f| \, d\gamma.$$

We now apply Lemma 3.4 with  $g = f - b_j$  and use the subadditivity of  $\mathcal{M}_{\text{loc}}$  combined with (3.10) to get

$$(3.11) \quad \begin{aligned} \int \mathcal{M}_{\text{loc}}((f - b_j)\eta_j\gamma_0) \, d\lambda &\leq C \int_{4B_j} \mathcal{M}_{\text{loc}}f \gamma_0(c_{B_j}) \, d\lambda + C \frac{\gamma(4B_j)}{\gamma(B_j)} \int_{B_j} |f| \, d\gamma \\ &\leq C \int_{4B_j} \mathcal{M}_{\text{loc}}f \, d\gamma. \end{aligned}$$

**Lemma 3.5.** *The function  $(f - b_j)\eta_j\gamma_0$  is in  $H^1(\lambda)$  and*

$$(3.12) \quad \|(f - b_j)\eta_j\gamma_0\|_{H^1(\lambda)} \leq \int_{4B_j} \mathcal{M}_{\text{loc}}f \, d\gamma.$$



*Proof.* By the maximal characterization of the classical space  $H^1(\lambda)$ , it suffices to show that

$$(3.13) \quad \int \mathcal{M}((f - b_j)\eta_j\gamma_0) \, d\lambda(x) \leq C \int_{4B_j} \mathcal{M}_{\text{loc}}f \, d\gamma.$$

Because of (3.11), all that needs to be verified is that

$$(3.14) \quad \int \sup_{\phi \in \Phi} \sup_{t \geq \min(1, 1/|x|)} |((f - b_j)\eta_j\gamma_0) * \phi_t(x)| \, d\lambda(x) \leq C \int_{4B_j} \mathcal{M}_{\text{loc}}f \, d\gamma.$$

To prove (3.14), we split the integral in the left-hand side into the sum

$$\int_{4B_j} \cdots \, d\lambda(x) + \int_{(4B_j)^c} \cdots \, d\lambda(x).$$

If  $x \in 4B_j$ , then for  $\phi \in \Phi$  and  $t \geq \min(1, 1/|x|)$ ,

$$\begin{aligned} |\phi_t * ((f - b_j)\eta_j\gamma_0)(x)| &\leq t^{-n} \int_{B_j} |f(y) - b_j| \, d\gamma(y) \\ &\leq (1 + |x|)^n \int_{B_j} (|f(y)| + |b_j|) \, d\gamma(y) \\ &\leq C(1 + |c_{B_j}|)^n \int_{B_j} |f(y)| \, d\gamma(y), \end{aligned}$$

the last inequality because of (3.10). Hence

$$\int_{4B_j} \cdots \, d\lambda(x) \leq C |4B_j| (1 + |c_{B_j}|)^n \int_{B_j} |f| \, d\gamma \leq C \int_{B_j} \mathcal{M}_{\text{loc}}f \, d\gamma.$$

If  $x \in (4B_j)^c$ , we take  $\phi$  and  $t$  as before and observe that we can assume that  $t > d(x, B_j)$ , since otherwise the convolution in (3.14) will vanish. In view of (3.9) and (3.10), we then get

$$\begin{aligned} |\phi_t * ((f - b_j)\eta_j\gamma_0)(x)| &\leq \int_{B_j} |\phi_t(x - y) - \phi_t(x - c_{B_j})| |f(y) - b_j| \eta_j(y) \, d\gamma(y) \\ &\leq C t^{-n-1} \int_{B_j} |y - c_{B_j}| |f(y) - b_j| \, d\gamma(y) \\ &\leq C \frac{1}{d(x, B_j)^{n+1} r_{B_j}} \int_{B_j} |f(y)| \, d\gamma(y). \end{aligned}$$

This implies that

$$\int_{(4B_j)^c} \cdots \, d\lambda(x) \leq C \int_{B_j} |f| \, d\gamma \leq C \int_{B_j} \mathcal{M}_{\text{loc}}f \, d\gamma.$$

We have proved (3.14) and the lemma. □

We can now finish the proof of Theorem 3.3. By Lemmata 3.5 and 2.1, each function  $(f - b_j)\eta_j\gamma_0$  has an atomic decomposition  $\sum_k \lambda_{jk}\alpha_{jk}$ , where the  $\alpha_{jk}$  are Lebesgue atoms with supports in  $2B_j$  and

$$\sum_k |\lambda_{jk}| \leq C \int_{4B_j} \mathcal{M}_{\text{loc}}f \, d\gamma.$$

As we saw in the proof of Theorem 2.2, each  $a_{jk} = \gamma_0^{-1} \alpha_{jk}$  is a multiple of a Gaussian atom, with a factor which is independent of  $j$  and  $k$ . Thus

$$(3.15) \quad f = \sum_j (f - b_j) \eta_j + \sum_j b_j \eta_j = \sum_j \sum_k \lambda_{jk} a_{jk} + \sum_j b_j \eta_j$$

and

$$\sum_{j,k} |\lambda_{jk}| \leq \sum_j \int_{4B_j} \mathcal{M}_{\text{loc}} f \, d\gamma \leq C \|\mathcal{M}_{\text{loc}} f\|_{L^1(\gamma)}.$$

To complete the proof of Theorem 3.3, we need to find an atomic decomposition of  $\sum_j b_j \eta_j$ . It is here that we must restrict ourselves to the one-dimensional case and that the global condition  $E(f) < \infty$  plays a role.

Choose the intervals  $I_0 = (-1, 1)$ ,  $I_j = (\sqrt{j-1}, \sqrt{j+1})$  for  $j \geq 1$  and  $I_j = -I_{|j|}$  for  $j \leq -1$ . The intervals  $I_j$  have essentially the same properties as the balls  $B_j$  introduced above, and we can use them instead of the  $B_j$  to construct  $\eta_j$  and  $b_j$  as before. To decompose now  $\sum_j b_j \eta_j$ , we first normalise the functions  $\eta_j$ , letting

$$\tilde{\eta}_j = \frac{\eta_j}{\int \eta_j \, d\gamma}.$$

Then  $b_j \eta_j = \int f \eta_j \, d\gamma \tilde{\eta}_j$ , and clearly

$$\sum_{j \geq k} \int f \eta_j \, d\gamma = \int f \mu_k \, d\gamma, \quad k \in \mathbb{Z},$$

where  $\mu_k(x) = \sum_{j \geq k} \eta_j(x)$ . Notice that  $\int f \mu_k \, d\gamma \rightarrow 0$  as  $k \rightarrow \pm\infty$ , in view of (3.7). A summation by parts now yields

$$(3.16) \quad \sum_{j \in \mathbb{Z}} \int f \eta_j \, d\gamma \tilde{\eta}_j = \sum_{k \in \mathbb{Z}} \int f \mu_k \, d\gamma (\tilde{\eta}_k - \tilde{\eta}_{k-1}).$$

But  $\tilde{\eta}_k - \tilde{\eta}_{k-1}$  is  $C$  times a Gaussian atom if we use admissible balls at some scale  $s > 1$  in the definition of atoms. Thus (3.16) is our desired atomic decomposition of  $\sum_j b_j \eta_j$ , provided we can estimate the coefficients by showing that

$$(3.17) \quad \sum_{k \in \mathbb{Z}} \left| \int f \mu_k \, d\gamma \right| \leq C (\|f\|_1 + E(f)).$$

To this end, observe that

$$\int f \mu_k \, d\gamma = \int f(x) \int_{-\infty}^x \mu'_k(y) \, d\lambda(y) \, d\gamma(x) = \int \mu'_k(y) \int_y^\infty f(x) \, d\gamma(x) \, d\lambda(y).$$

Since the support of  $\mu'_k$  is contained in  $I_k$  and

$$|\mu'_k(y)| \leq \frac{C}{|I_k|} \leq C (1 + |c_{I_k}|),$$

we obtain, using also the bounded overlap of the  $I_j$ ,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| \int f \mu_k \, d\gamma \right| &\leq C \sum_k \int_{I_k} (1 + |c_{I_k}|) \left| \int_y^\infty f(x) \, d\gamma(x) \right| \, d\lambda(y) \\ &\leq C \int_{-\infty}^\infty (1 + |y|) \left| \int_y^\infty f \, d\gamma \right| \, d\lambda(y) \\ &= C \int_0^\infty (1 + y) \left( \left| \int_y^\infty f \, d\gamma \right| + \left| \int_{-y}^\infty f \, d\gamma \right| \right) \, d\lambda(y) \\ &\leq C (\|f\|_1 + E(f)); \end{aligned}$$

here we used (3.7). This concludes the proof of Theorem 3.3. □

#### 4. A CHARACTERIZATION OF NONNEGATIVE FUNCTIONS IN $H^1(\gamma)$

The dimension  $n$  is now arbitrary. The following lemma will be needed.

**Lemma 4.1.** *Let  $\phi_0 = \gamma(B(0, 1))^{-1} \mathbf{1}_{B(0,1)}$ . If  $g \in L^\infty$  is supported in a maximal admissible ball  $B$ , then*

$$\left\| g - \int g \, d\gamma \phi_0 \right\|_{H^1(\gamma)} \leq C(1 + |c_B|^2) \gamma(B) \|g\|_{L^\infty}.$$

*Proof.* We shall construct atoms whose supports form a chain connecting  $B(0, 1)$  to  $B$ . First we define a finite sequence of maximal admissible balls

$$\tilde{B}_0 = B(0, 1), \tilde{B}_1, \dots, \tilde{B}_N,$$

all with centres  $c_{\tilde{B}_j}$  on the segment  $[0, c_B]$ . The absolute values  $\rho_j = |c_{\tilde{B}_j}|$  shall be increasing in  $j$ , and the boundary  $\partial\tilde{B}_j$  shall contain  $c_{\tilde{B}_{j-1}}$  for  $j = 1, \dots, N - 1$ , which means that

$$(4.1) \quad \rho_j - \frac{1}{\rho_j} = \rho_{j-1}, \quad j = 1, \dots, N - 1,$$

and  $\rho_0 = 0, \rho_1 = 1$ . Finally,  $N$  is defined so that  $\tilde{B}_{N-1}$  is the first ball of the sequence that contains  $c_B$ , and  $\tilde{B}_N = B$ . Squaring (4.1), we get

$$\rho_j^2 - \rho_{j-1}^2 = 2 - \frac{1}{\rho_j^2} \geq 1,$$

so that  $\rho_{N-1}^2 \geq N - 1$ . It follows that

$$(4.2) \quad N \leq |c_B|^2 + 1.$$

Next, we denote by  $B_j, j = 1, \dots, N$ , the largest ball contained in  $\tilde{B}_j \cap \tilde{B}_{j-1}$ . Notice that the three balls  $\tilde{B}_j, \tilde{B}_{j-1}$  and  $B_j$  have comparable radii and comparable Gaussian measures. Now define functions  $\phi_j$  and  $g_j$  by setting

$$\begin{aligned} \phi_j &= \gamma(B_j)^{-1} \mathbf{1}_{B_j} \quad j = 1, \dots, N, \\ g_j &= \int g \, d\gamma (\phi_j - \phi_{j-1}), \quad j = 1, \dots, N, \\ g_{N+1} &= g - \int g \, d\gamma \phi_N. \end{aligned}$$

Clearly,

$$(4.3) \quad g - \int g \, d\gamma \, \phi_0 = \sum_{j=1}^{N+1} g_j.$$

Each function  $g_j$  is a multiple of an atom. Indeed, its integral against  $\gamma$  vanishes. Moreover, if  $1 \leq j \leq N$ , the support of  $g_j$  is contained in  $\tilde{B}_{j-1}$  and

$$\|g_j\|_\infty \leq (\gamma(B_j)^{-1} + \gamma(B_{j-1})^{-1}) \int |g| \, d\gamma \leq C \gamma(\tilde{B}_{j-1})^{-1} \gamma(B) \|g\|_{L^\infty}.$$

The support of  $\phi_{N+1}$  is contained in  $B$  and

$$\|g_{N+1}\|_\infty \leq \|g\|_\infty + \gamma(B)^{-1} \int |g| \, d\gamma \leq C \|g\|_{L^\infty}.$$

Thus

$$\|g_j\|_{H^1(\gamma)} \leq C \gamma(B) \|g\|_{L^\infty}, \quad j = 1, \dots, N + 1.$$

Summing the coefficients in the atomic decomposition (4.3), we then obtain via (4.2),

$$\left\| g - \int g \, d\gamma \, \phi_0 \right\|_{H^1(\gamma)} \leq C(N + 1) \gamma(B) \|g\|_{L^\infty} \leq C(1 + |c_B|^2) \gamma(B) \|g\|_{L^\infty}.$$

The proof of the lemma is complete. □

**Theorem 4.2.** *Suppose that  $f$  is a function in  $L^1(\gamma)$ . If  $\mathcal{M}_{\text{loc}}f$  is in  $L^1(\gamma)$  and*

$$(4.4) \quad E_+(f) = \int |x|^2 |f(x)| \, d\gamma(x) < \infty,$$

*then  $f$  is in  $H^1(\gamma)$  and*

$$\|f\|_{H^1(\gamma)} \leq C \|\mathcal{M}_{\text{loc}}f\|_1 + CE_+(f).$$

*If  $f$  is nonnegative, the conditions  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  and  $E_+(f) < \infty$  are also necessary for  $f$  to be in  $H^1(\gamma)$ .*

*Proof.* Let  $f$  be a function in  $L^1(\gamma)$  such that  $\mathcal{M}_{\text{loc}}f \in L^1(\gamma)$  and  $E_+(f) < \infty$ . Write  $f = c(f) + f_0$ , where  $c(f) = \int f \, d\gamma$ . Since  $c(f)$  is a multiple of the exceptional atom, it suffices to find an atomic decomposition of  $f_0$ . Note that  $f_0$  satisfies

$$\mathcal{M}_{\text{loc}}f_0 \in L^1(\gamma) \quad \text{and} \quad \int |x|^2 |f_0(x)| \, d\gamma(x) < \infty.$$

Let  $\{B_j\}$  be the covering of  $\mathbb{R}^n$  by maximal admissible balls and  $\{\eta_j\}$  the corresponding partition of unity introduced in the proof of Theorem 3.3. As there, we choose numbers  $b_j \in \mathbb{C}$  such that

$$\int_{-\infty}^{\infty} (f_0 - b_j) \eta_j \, d\gamma = 0 \quad \forall j.$$

Then the argument leading to (3.15) shows that

$$f_0 = \sum_j \sum_k \lambda_{jk} a_{jk} + \sum_j b_j \eta_j,$$

where the  $a_{jk}$  are Gaussian atoms supported in  $4B_j$  and

$$\sum_{j,k} |\lambda_{jk}| \leq C \|\mathcal{M}_{\text{loc}}f_0\|_{L^1(\gamma)}.$$

It remains only to prove that  $\sum_j b_j \eta_j$  is in  $H^1(\gamma)$ . We write  $g_j = b_j \eta_j$  and observe that

$$(4.5) \quad \int \sum_j g_j \, d\gamma = 0$$

because  $f_0$  and the  $a_{ij}$  have integrals zero. Thus

$$\sum_j g_j = \sum_j \left( g_j - \int g_j \, d\gamma \phi_0 \right),$$

where  $\phi_0$  is as in Lemma 4.1. Since (3.10) remains valid for  $f_0$ , we have

$$(4.6) \quad \|g_j\|_\infty \leq C \frac{1}{\gamma(B_j)} \int_{B_j} |f_0| \, d\gamma.$$

Lemma 4.1 thus applies to each  $g_j$ , and using also the bounded overlap of the  $B_j$  we conclude

$$\left\| \sum_j g_j \right\|_{H^1(\gamma)} \leq C \sum_j (1 + |c_{B_j}|^2) \int_{B_j} |f_0| \, d\gamma \leq C \int (1 + |x|^2) |f_0| \, d\gamma.$$

This concludes the proof of the sufficiency and the norm estimate.

The necessity of the condition  $\mathcal{M}_{\text{loc}} f \in L^1(\gamma)$  was obtained in Lemma 3.2.

To prove the necessity of (4.4), let  $0 \leq f \in H^1(\gamma)$ . We first observe that the function  $x \mapsto |x|^2$  is in  $BMO(\gamma)$ . Indeed, its oscillation on any admissible ball is bounded. Since  $BMO(\gamma)$  is a lattice, the functions  $g_k(x) = \min\{|x|^2, k\}$  are in  $BMO(\gamma)$ , uniformly for  $k \geq 1$ . By the monotone convergence theorem and the duality between  $H^1(\gamma)$  and  $BMO(\gamma)$ ,

$$\int |x|^2 f(x) \, d\gamma(x) = \lim_k \int g_k(x) f(x) \, d\gamma(x) \leq C \|f\|_{H^1(\gamma)}.$$

The theorem is proved. □

The following result is a noteworthy consequence of Theorem 4.2.

**Corollary 4.3.** *For  $1 < p \leq \infty$ , one has continuous inclusions  $L^p(\gamma) \subset H^1(\gamma)$  and  $BMO(\gamma) \subset L^{p'}(\gamma)$ , where  $p' = p/(p - 1)$ .*

*Proof.* We claim that the operator  $\mathcal{M}_{\text{loc}}$  is bounded on  $L^p(\gamma)$  for  $1 < p \leq \infty$ . Deferring momentarily the proof of this claim, we complete the proof of the corollary. Suppose that  $f$  is in  $L^p(\gamma)$ . Then  $\mathcal{M}_{\text{loc}} f$  is in  $L^1(\gamma)$ , because

$$\|\mathcal{M}_{\text{loc}} f\|_1 \leq \|\mathcal{M}_{\text{loc}} f\|_p \leq C \|f\|_p < \infty,$$

since  $\gamma(\mathbb{R}^n) = 1$ . Moreover,  $E_+(f) \leq \| |x|^2 \|_{p'} \|f\|_p < \infty$ , by Hölder's inequality. Thus  $f \in H^1(\gamma)$  by Theorem 4.2. It also follows that the inclusion  $L^p(\gamma) \subset H^1(\gamma)$  is continuous, and by duality we get the continuous inclusion  $BMO(\gamma) \subset L^{p'}(\gamma)$ .

It remains to prove the claim. We shall use again the covering  $\{B_j\}$  from the proof of Theorem 3.3. First we observe that the inequality

$$(4.7) \quad \|\mathcal{M}_{\text{loc}} g\|_p \leq C \|g\|_p$$

holds when  $\text{supp } g \subset B_j$ , with a constant  $C$  independent of  $j$ . Indeed,  $\mathcal{M}_{\text{loc}}$  is bounded on  $L^p(\lambda)$ , and  $\mathcal{M}_{\text{loc}} g$  is supported in the ball  $4B_j$ , where the Gaussian measure is essentially proportional to  $d\lambda$ .

Given a function  $f \in L^p(\gamma)$ , we write it as a sum  $f = \sum f_j$  with  $\text{supp } f_j \subset B_j$  and with the sets  $\{f_j \neq 0\}$  pairwise disjoint. We can then apply (4.7) to each  $f_j$  and sum.  $\square$

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