Analytic formulas for the topological degree of non-smooth mappings: The odd-dimensional case

Citation for the original published paper (version of record):
Goffeng, M. (2012). Analytic formulas for the topological degree of non-smooth mappings: The odd-dimensional case. Advances in Mathematics, 231(1): 357-377.
http://dx.doi.org/10.1016/j.aim.2012.05.009

Available online at www.sciencedirect.com

SciVerse ScienceDirect

ADVANCES IN Mathematics

Advances in Mathematics 231 (2012) 357–377

www.elsevier.com/locate/aim

Analytic formulas for the topological degree of non-smooth mappings: The odd-dimensional case

Magnus Goffeng *

Department of Mathematical Sciences, Division of Mathematics, Chalmers University of Technology and University of Gothenburg, Sweden

Received 27 April 2010; accepted 10 May 2012
Available online 7 June 2012

Communicated by the Managing Editors of AIM

Abstract

The notion of topological degree is studied for mappings from the boundary of a relatively compact strictly pseudo-convex domain in a Stein manifold into a manifold in terms of index theory of Toeplitz operators on the Hardy space. The index formalism of non-commutative geometry is used to derive analytic integral formulas for the index of a Toeplitz operator with Hölder continuous symbol. The index formula gives an analytic formula for the degree of a Hölder continuous mapping from the boundary of a strictly pseudo-convex domain.

Keywords: Regularized index formulas; Toeplitz operators; Integral representations of holomorphic functions; Index theory; Cyclic cohomology

0. Introduction

This paper is a study of analytic formulas for the degree of a mapping from the boundary of a relatively compact strictly pseudo-convex domain in a Stein manifold. The degree of a continuous mapping between two compact, connected, oriented manifolds of the same dimension is abstractly defined in terms of homology for continuous functions. If the function f is

* Correspondence to: Institut für Analysis, Welfengarten 1, 301 67 Hannover, Germany.
E-mail address: goffeng@math.uni-hannover.de.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.05.009

differentiable, an analytic formula can be derived using Brouwer degree, see [19], or the more global picture of de Rham cohomology. For any form ω of top degree, the form $f^*\omega$ satisfies

$$\int_X f^*\omega = \deg f \int_Y \omega.$$

Without differentiability conditions on f, there are no known analytic formulas beyond the special case of a Hölder continuous mapping $S^1 \to S^1$, which can be found in Chapter 2.α of [9]. The degree of a Hölder continuous function $f : S^1 \to S^1$ of exponent α is expressed by an analytic formula by replacing the de Rham cohomology with the cyclic homology of the algebra of Hölder continuous functions as

$$\deg(f) = \frac{1}{(2\pi i)^{2k}} \int \overline{f(z_0)} \frac{f(z_1) - f(z_0)}{z_1 - z_0} \cdots \frac{f(z_0) - f(z_{2k})}{z_0 - z_{2k}} \mathrm{d}z_0 \ldots \mathrm{d}z_{2k}, \tag{1}$$

whenever $\alpha(2k+1) > 1$. Later, the same technique was used in [22,23] in constructing index formulas for pseudo differential operators with operator-valued symbols. Our aim is to find new formulas for the degree in the multidimensional setting by expressing the degree of a Hölder continuous function as the index of a Toeplitz operator and using the approach of [9].

The motivation to calculate the degree of a non-smooth mapping comes from nonlinear σ-models in physics. For instance, the Skyrme model describing self-interacting mesons in terms of a field $f : X \to Y$, see [1], only has a constant solution if one does not pose a topological restriction, and since the solutions are rarely smooth, but rather in the Sobolev space $W^{1,d}(X, Y)$, one needs a degree defined on non-continuous functions. In the paper [8], the notion of a degree was extended as far as to VMO-mappings in terms of approximation by continuous mappings. See also [7] for a study of the homotopy structure of $W^{1,d}(X, Y)$.

The main idea that will be used in this paper is that the cohomological information of a continuous mapping $f : X \to Y$ between odd-dimensional manifolds can be found in the induced mapping $f^* : K^1(X) \to K^1(Y)$ using the Chern–Simons character. The analytic formula will be obtained by using index theory of Toeplitz operators. The index theory of Toeplitz operators is a well-studied subject for many classes of symbols; see, for instance, [2,6,9,13]. If $X = \partial\Omega$, where Ω is a strictly pseudo-convex domain in a complex manifold, and $f : \partial\Omega \to Y$ is a smooth mapping, the idea can be expressed by the commutative diagram

$$\begin{array}{ccccc} K^1(Y) & \xrightarrow{f^*} & K^1(\partial\Omega) & \xrightarrow{\text{ind}} & \mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{cs}_Y} & & \downarrow{\scriptstyle \mathrm{cs}_{\partial\Omega}} & & \downarrow \\ H^{odd}_{dR}(Y) & \xrightarrow{f^*} & H^{odd}_{dR}(\partial\Omega) & \xrightarrow{\chi_{\partial\Omega}} & \mathbb{C} \end{array} \tag{2}$$

where the mapping ind $: K^1(\partial\Omega) \to \mathbb{Z}$ denotes the index mapping defined in terms of suitable Toeplitz operators on $\partial\Omega$ and

$$\chi_{\partial\Omega}(x) := -\int_{\partial\Omega} x \wedge Td(\Omega).$$

The left part of the diagram (2) is commutative by naturality of the Chern–Simons character and the right part of the diagram is commutative by the Boutet de Monvel index formula.

K-theory is a topological invariant, and the picture of the index map as a pairing in a local homology theory via Chern–Simons characters can be applied to more general classes of functions than the smooth functions. The homology theory present throughout all index theory is cyclic homology. For a Hölder continuous mapping $f : \partial\Omega \to Y$ of exponent α and Ω being a relatively compact strictly pseudo-convex domain in a Stein manifold, the analogy of diagram (2) is

$$\begin{array}{ccccc} K_1(C^\infty(Y)) & \xrightarrow{f^*} & K_1(C^\alpha(\partial\Omega)) & \xrightarrow{\text{ind}} & \mathbb{Z} \\ \downarrow \text{cs}_Y & & \downarrow \text{cs}_{\partial\Omega} & & \downarrow \\ HC_{odd}(C^\infty(Y)) & \xrightarrow{f^*} & HC_{odd}(C^\alpha(\partial\Omega)) & \xrightarrow{\tilde{\chi}_{\partial\Omega}} & \mathbb{C} \end{array} \tag{3}$$

where the mapping $\tilde{\chi}_{\partial\Omega} : HC_{odd}(C^\alpha(\partial\Omega)) \to \mathbb{C}$ is a cyclic cocycle on $C^\alpha(\partial\Omega)$ defined as the Connes–Chern character of the Toeplitz operators on $\partial\Omega$; see more in [9,10]. The condition on Ω to lie in a Stein manifold ensures that the cyclic cocycle $\tilde{\chi}_{\partial\Omega}$ can be defined on Hölder continuous functions; see below in Theorem 4.2. The right-hand side of diagram (3) is commutative by Connes' index formula; see Proposition 4 of Chapter IV.1 of [9]. The dimension in which the Chern–Simons character will take values depends on the Hölder exponent α. More explicitly, the cocycle $\tilde{\chi}_{\partial\Omega}$ can be chosen as a cyclic $2k+1$-cocycle for any $2k+1 > 2n/\alpha$.

The index of a Toeplitz operator T_u on the vector-valued Hardy space $H^2(\partial\Omega) \otimes \mathbb{C}^N$ with smooth symbol $u : \partial\Omega \to \text{GL}_N(\mathbb{C})$ can be calculated using the Boutet de Monvel index formula as $\text{ind}\, T_u = -\int_{\partial\Omega} \text{cs}_{\partial\Omega}[u]$ if the Chern–Simons character $\text{cs}_{\partial\Omega}[u]$ only contains a top degree term. In particular, if $g : Y \to \text{GL}_N(\mathbb{C})$ satisfies that all terms, except for the top-degree term, in $\text{cs}_{\partial\Omega}[g]$ are exact and $f : \partial\Omega \to Y$ is smooth, we can consider the matrix symbol $g \circ f$ on $\partial\Omega$. Naturality of the Chern–Simons character implies the identity

$$\deg f \int_Y \text{cs}_Y[g] = -\text{ind}\, T_{g\circ f},$$

where $T_{g\circ f}$ is a Toeplitz operator on $H^2(\partial\Omega) \otimes \mathbb{C}^N$ with symbol $g \circ f$. This result extends to Hölder continuous functions in the sense that, if we choose g which also satisfies the condition $\int_Y \text{cs}_Y[g] = 1$, we obtain the analytic degree formula:

$$\deg f = \tilde{\chi}_{\partial\Omega}(\text{cs}_{\partial\Omega}[g \circ f]).$$

A drawback of our approach is that it only applies to boundaries of strictly pseudo-convex domains in Stein manifolds. We discuss this drawback at the end of the fourth, and final, section of this paper. The author intends to return to this question in a future paper and to address the problem for even-dimensional manifolds.

The paper is organized as follows. In the first section, we reformulate the degree as an index calculation using the Chern–Simons character from odd K-theory to de Rham cohomology. This result is not remarkable in itself, since the Chern–Simons character is an isomorphism after tensoring with the complex numbers. However, the constructions are explicit and allow us to obtain explicit expressions for a generator of the de Rham cohomology. We will use the complex spin representation of $\mathbb{R}^{2n}$ to construct a smooth function $u : S^{2n-1} \to SU(2^{n-1})$ such that the Chern–Simons character of u is a multiple of the volume element on S^{2n-1}. The function u will then be used to construct a smooth mapping $\tilde{g} : Y \to GL_{2^{n-1}}(\mathbb{C})$ for arbitrary odd-dimensional manifold Y whose Chern–Simons character coincides with $(-1)^n \mathrm{d}V_Y$, where $\mathrm{d}V_Y$ is a normalized volume form on Y; see Theorem 1.6. Thus we obtain for any continuous function $f : \partial\Omega \to Y$ the formula $\deg f = (-1)^{n+1}\text{ind}\, T_{g\circ f}$, as is proved in Theorem 2.1.

In the second section, we will review the theory of Toeplitz operators on the boundary of a strictly pseudo-convex domain. The material in this section is based on [6,9,11,13,16,21]. We will recall the basics from [11,16,21] of integral representations of holomorphic functions on Stein manifolds and the non-orthogonal Henkin–Ramirez projection. We will continue the section by recalling some known results about index formulas and how a certain Schatten class condition can be used to obtain index formulas. The focus will be on the index formula of Connes, see Proposition 4 in Chapter IV.1 of [9], involving cyclic cohomology and how the periodicity operator S in cyclic cohomology can be used to extend cyclic cocycles to larger algebras. In our case, the periodicity operator is used to extend a cyclic cocycle on the algebra $C^\infty(\partial\Omega)$ to a cyclic cocycle on $C^\alpha(\partial\Omega)$. We will also review a theorem of Russo, see [24], which gives a sufficient condition for an integral operator to be of Schatten class.

The third section is devoted to proving that the Szegö projection $P_{\partial\Omega} : L^2(\partial\Omega) \to H^2(\partial\Omega)$ satisfies the property that for any $p > 2n/\alpha$ the commutator $[P_{\partial\Omega}, a]$ is a Schatten class operator of order p for any Hölder continuous functions a on $\partial\Omega$ of exponent α. The statement about the commutator $[P_{\partial\Omega}, a]$ can be reformulated as the corresponding big Hankel operator with symbol a being of Schatten class. We will in fact not look at the Szegö projection, but rather at the non-orthogonal Henkin–Ramirez projection P_{HR} mentioned above. The projection P_{HR} has a particular behavior, making the estimates easier, and an application of Russo's Theorem implies that $P_{HR} - P_{\partial\Omega}$ is Schatten class of order $p > 2n$; see Lemma 3.6.

In the fourth section; we will present the index formula and the degree formula for Hölder continuous functions. Thus, if we let $C_{\partial\Omega}$ denote the Szegö kernel and $\mathrm{d}V$ the volume form on $\partial\Omega$, we obtain the following index formula for u invertible and Hölder continuous on $\partial\Omega$:

$$\operatorname{ind} T_u = -\int_{\partial\Omega^{2k+1}} \operatorname{tr}\left(\prod_{i=0}^{2k}(1 - u(z_i)^{-1}u(z_{i+1}))C_{\partial\Omega}(z_i, z_{i+1})\right)\mathrm{d}V$$

for any $2k+1 > 2n/\alpha$. Here, we identify z_{2k+1} with z_0. Using the index formula for the mapping degree, we finally obtain an analytic formula for the degree of a Hölder continuous mapping from $\partial\Omega$ to a connected, compact, orientable, Riemannian manifold Y. If $f : \partial\Omega \to Y$ is a Hölder continuous function of exponent α, the degree of f can be calculated for $2k + 1 > 2n/\alpha$ from the formula

$$\deg(f) = (-1)^n \int_{\partial\Omega^{2k+1}} \tilde{f}(z_0, z_1, \ldots, z_{2k}) \prod_{j=0}^{2k} C_{\partial\Omega}(z_{j-1}, z_j)\mathrm{d}V,$$

where $\tilde{f} : \partial\Omega^{2k+1} \to \mathbb{C}$ is a function explicitly expressed from f; see more in Eq. (27).

1. The volume form as a Chern–Simons character

In order to represent the mapping degree as an index, we look for a matrix symbol whose Chern–Simons character is cohomologous to the volume form $\mathrm{d}V_Y$ on Y. We will start by considering the case of a $2n - 1$-dimensional sphere, and construct a map into the Lie group $SU(2^{n-1})$ using the complex spinor representation of $Spin(\mathbb{R}^{2n})$. In the complex spin representation, a vector in S^{2n-1} defines a unitary matrix; this construction produces a matrix symbol on odd-dimensional spheres such that its Chern–Simons character spans $H_{dR}^{2n-1}(S^{2n-1})$. The matrix symbol on S^{2n-1} generalizes to an arbitrary connected, compact, oriented manifold Y of dimension $2n - 1$ such that its Chern–Simons character coincides with $(-1)^n\mathrm{d}V_Y$.

Let V denote a real vector space of dimension $2n$ with a non-degenerate inner product g. We take a complex structure J on V which is compatible with the metric, and extend the mapping J to a complex linear mapping on $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. Since $J^2 = -1$, we can decompose $V_{\mathbb{C}} := V^{1,0} \oplus V^{0,1}$ into two eigenspaces of J corresponding to the eigenvalues $\pm i$. If we extend g to a complex bilinear form $g_{\mathbb{C}}$ on $V_{\mathbb{C}}$, and use the isomorphism $Cl_{\mathbb{C}}(V, g) \cong Cl(V_{\mathbb{C}}, g_{\mathbb{C}})$, we can identify the complexified Clifford algebra of V with the complex algebra generated by $2n$ symbols $e_{1,+}, \dots, e_{n,+}, e_{1,-}, \dots, e_{n,-}$ satisfying the relations

$$\{e_{j,+}, e_{k,+}\} = \{e_{j,-}, e_{k,-}\} = 0 \quad \text{and} \quad \{e_{j,+}, e_{k,-}\} = -2\delta_{jk},$$

where $\{\cdot, \cdot\}$ denotes the anti-commutator. The complex algebra $Cl_{\mathbb{C}}(V, g)$ becomes a $*$-algebra in the $*$-operation $e_{j,+}^* := -e_{j,-}$.

The space $S_V := \wedge^* V^{1,0}$ becomes a complex Hilbert space equipped with the sesquilinear form induced from g and J. The vector space S_V will be given the orientation from the lexicographic order on the basis $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ for $i_1 < i_2 < \cdots < i_k$. Define $c : V_{\mathbb{C}} \to End(S_V)$ by

$$c(v).w := \sqrt{2} v \wedge w, \quad \text{for } v \in V^{1,0} \quad \text{and}$$

$$c(v').w := -\sqrt{2} v' \neg w \quad \text{for } v' \in V^{0,1}.$$

The linear mapping c satisfies

$$c(v^*) = c(v)^* \quad \text{and} \quad c(w)c(v) + c(v)c(w) = -2g(w, v),$$

so by the universal property of the Clifford algebra $Cl_{\mathbb{C}}(V, g)$ we can extend c to a $*$-representation $\varphi : Cl_{\mathbb{C}}(V) \to End_{\mathbb{C}}(S_V)$. The space S_V is a 2^n-dimensional Hilbert space which we equip with a $\mathbb{Z}_2$-grading as follows:

$$S_V = S_V^+ \oplus S_V^- := \wedge^{even} V^{1,0} \oplus \wedge^{odd} V^{1,0}.$$

Consider the subalgebra $Cl_{\mathbb{C}}(V)_+$ consisting of an even number of generators. The representation φ restricts to a representation $Cl_{\mathbb{C}}(V)_+ \to End_{\mathbb{C}}(S_V^+)$ and $Cl_{\mathbb{C}}(V)_+ \to End_{\mathbb{C}}(S_V^-)$. We define the 2^{n-1}-dimensional oriented Hilbert space $E_n := S_{\mathbb{C}^n}^+$ when n is even and $E_n := S_{\mathbb{C}^n}^-$ when n is odd. The representation $Cl_{\mathbb{C}}(\mathbb{C}^n)_+ \to End_{\mathbb{C}}(E_n)$ will be denoted by φ_+. For a vector $v \in \mathbb{C}^n$, we can use the fact that $\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n \oplus \mathbb{C}^n$, and define

$$v_+ := \varphi_+(v \oplus 0) \in End_{\mathbb{C}}(E_n) \quad \text{and} \quad v_- := \varphi_+(0 \oplus v) \in End_{\mathbb{C}}(E_n).$$

We will now define a symbol calculus for S^{2n-1}. We choose the standard embedding $S^{2n-1} \subseteq \mathbb{C}^n$ by taking coordinates $z_i : S^{2n-1} \to \mathbb{C}$ satisfying $|z_1|^2 + |z_2|^2 \cdots + |z_n|^2 = 1$. Define the smooth map $u : S^{2n-1} \to Cl_{\mathbb{C}}(\mathbb{R}^{2n})_+$ by

$$u(z) := \frac{1}{2}(e_{1,+} + e_{1,-})(z_+ + \bar{z}_-). \tag{4}$$

Proposition 1.1. *The mapping u satisfies*

$$u(z)^* u(z) = u(z) u(z)^* = 1,$$

so $u : S^{2n-1} \to SU(2^{n-1}) \subseteq End_{\mathbb{C}}(E_n)$ is well defined.

The proof of this proposition is a straightforward calculation using the relations in the Clifford algebra $Cl_{\mathbb{C}}(V,g)$. Observe that, if $n=2$, the mapping u is a diffeomorphism, since we can choose 1 and $e_1\wedge e_2$ as a basis for S_V^+, and in this basis

$$u(z_1,z_2)=\begin{pmatrix}-z_1 & -\bar{z}_2\\ z_2 & -\bar{z}_1\end{pmatrix}.$$

For any N, we can consider the subgroup $SU(N-1)\subseteq SU(N)$ of elements of the form $1\oplus x$. Denoting by e_1 the first basis vector in $\mathbb{C}^N$, we can define a mapping $q: SU(N)\to S^{2N-1}$ by $q(v):=ve_1$. A straightforward calculation shows that q factors over the quotient $SU(N)/SU(N-1)$ and induces a diffeomorphism $SU(N)/SU(N-1)\cong S^{2N-1}$. The function u is in a sense a splitting to q.

Proposition 1.2. *If $\iota: S^{2n-1}\to S^{2^n-1}$ is defined by*

$$\iota(z_1,z_2,\dots z_n):=\begin{cases}(-z_1,z_2,\dots z_n,0,\dots,0) & \text{for } n \text{ even}\\ (-\bar{z}_1,z_2,\dots,z_n,0,\dots,0) & \text{for } n \text{ odd}\end{cases}$$

and $q: SU(2^{n-1})\to S^{2^n-1}$ is the mapping constructed above, the following identity is satisfied:

$$q\circ u=\iota.$$

Proof. We will start with the case when n is even. The first n basis vectors of S_V^+ are $1, e_1\wedge e_2, e_1\wedge e_3,\dots,e_1\wedge e_n$, and

$$q(u(z))=u(z)1=-z_1+z_2e_1\wedge e_2+z_3e_1\wedge e_3+\cdots+z_ne_1\wedge e_n.$$

If n is odd, the first basis vectors of S_V^- are $e_1,e_2,\dots,e_n$. Therefore, we have the equality

$$q(u(z))=u(z)e_1=-\bar{z}_1e_1+z_2e_2+\cdots z_ne_n.\quad\square$$

Consider $\alpha_+:=\varphi_+(\mathrm{d}z\oplus 0)$ and $\alpha_-:=\varphi_+(0\oplus\mathrm{d}\bar{z})$ as elements in $T^*S^{2n-1}\otimes End_{\mathbb{C}}(E_n)$. For an element $\Bbbk=(k_1,\dots,k_{2l-1})\in\{+,-\}^{2l-1}$, we define $\alpha_\Bbbk:=\alpha_{k_1}\alpha_{k_2}\cdots\alpha_{k_{2l-1}}\in\bigwedge^{2l+1}T^*S^{2n-1}\otimes End_{\mathbb{C}}(E_n)$. Define the set Γ_l^+ as the set of $\Bbbk\in\{+,-\}^{2l-1}$ such that the number of $+$ in $\Bbbk$ is l. Similarly, Γ_l^- is defined as the set of $\Bbbk\in\{+,-\}^{2l-1}$ such that the number of $-$ in $\Bbbk$ is l. The number of elements in $\Gamma_l^\pm$ can be calculated as

$$|\Gamma_l^+|=|\Gamma_l^-|=\binom{2l-1}{l-1}=\frac{(2l-1)!}{l!(l-1)!}.$$

Lemma 1.3. *For any $\Bbbk\in\{+,-\}^{2l-1}$, we have the equalities*

$$\mathrm{tr}(z_+\alpha_\Bbbk)=\begin{cases}0 & \text{if}\quad \Bbbk\notin\Gamma_l^-\\ (-1)^n2^{n-1}l!\displaystyle\sum_{m_1,m_2,\dots,m_l}z_{m_1}\mathrm{d}\bar{z}_{m_1}\bigwedge_{j=2}^{l}\mathrm{d}z_{m_j}\wedge\mathrm{d}\bar{z}_{m_j} & \text{if}\quad \Bbbk\in\Gamma_l^-\end{cases}$$

$$\mathrm{tr}(\bar{z}_-\alpha_\Bbbk)=\begin{cases}0 & \text{if}\quad \Bbbk\notin\Gamma_n^+\\ (-1)^{n+1}2^{n-1}l!\displaystyle\sum_{m_1,m_2\dots,m_l}\bar{z}_{m_1}\mathrm{d}z_{m_1}\bigwedge_{j=2}^{l}\mathrm{d}z_{m_j}\wedge\mathrm{d}\bar{z}_{m_j} & \text{if}\quad \Bbbk\in\Gamma_l^+.\end{cases}$$

Here, tr *denotes the matrix trace in $End_{\mathbb{C}}(E_n)$.*

The proof is a straightforward, but rather lengthy, calculation using the relations in the Clifford algebra, so we omit it. We will use the notation $\mathrm{d}V$ for the normalized volume measure on S^{2n-1}:

$$\mathrm{d}V = \frac{(n-1)!}{2\pi^n}\sum_{k=1}^{2n}(-1)^{k-1}x_k\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_{k-1}\wedge\mathrm{d}x_{k+1}\wedge\cdots\wedge\mathrm{d}x_{2n} \tag{5}$$

$$= \frac{(n-1)!}{2(2\pi i)^n}\sum_{k=1}^{n}\bar{z}_k\mathrm{d}z_k\wedge_{j\neq k}(\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j) - z_k\mathrm{d}\bar{z}_k\wedge_{j\neq k}(\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j). \tag{6}$$

That $\mathrm{d}V$ is normalized follows from the fact that the $2n-1$-form ω on S^{2n-1}, defined by

$$\omega = \sum_{k=1}^{2n}(-1)^{k-1}x_k\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_{k-1}\wedge\mathrm{d}x_{k+1}\wedge\cdots\wedge\mathrm{d}x_{2n},$$

satisfies that, if we change to spherical coordinates, the form $r^{2n-1}\mathrm{d}r\wedge\omega$ coincides with the volume form on $\mathbb{C}^n$. Since $\int_{\mathbb{C}}\mathrm{e}^{-|z|^2}\mathrm{d}m = \pi$, where m denotes Lebesgue measure, Fubini's Theorem implies that $\int_{\mathbb{C}^n}\mathrm{e}^{-|z|^2}\mathrm{d}m = \pi^n$ and

$$\pi^n = \int_{\mathbb{C}^n}\mathrm{e}^{-|z|^2}\mathrm{d}m = \int_0^\infty \mathrm{e}^{-r^2}r^{2n-1}\mathrm{d}r\int_{S^{2n-1}}\omega = \frac{(n-1)!}{2}\int_{S^{2n-1}}\omega.$$

Recall that, if $g : Y\to \mathrm{GL}_N(\mathbb{C})$ is a smooth mapping, the Chern–Simons character of g is an element of the odd de Rham cohomology $H^{odd}_{dR}(Y)$, defined as

$$\mathrm{cs}[g] = \sum_{k=0}^{\infty}\frac{(k-1)!}{(2\pi i)^k(2k-1)!}\mathrm{tr}(g^{-1}\mathrm{d}g)^{2k-1}.$$

See more in Chapter 1.8 in [26]. We will denote the $2k-1$-degree term by $\mathrm{cs}_{2k-1}[g]$. The cohomology class of $\mathrm{cs}[g]$ only depends on the homotopy class of g, so the Chern–Simons character induces a group homomorphism $\mathrm{cs} : K_1(C^\infty(Y))\to H^{odd}_{dR}(Y)$.

Lemma 1.4. *The mapping u defined in (4) satisfies*

$$\mathrm{cs}[u] = (-1)^n\mathrm{d}V.$$

Proof. Since the odd de Rham cohomology of S^{2n-1} is spanned by the volume form, it will be sufficient to show that $\mathrm{cs}_{2n-1}[u] = (-1)^n\mathrm{d}V$. First, we observe the identity $u^*\mathrm{d}u = -\mathrm{d}u^*u$, which follows from Proposition 1.1. This fact implies that

$$(u^*\mathrm{d}u)^{2n-1} = (-1)^{n-1}u^*\underbrace{\mathrm{d}u\,\mathrm{d}u^*\ \ldots\ \mathrm{d}u^*\,\mathrm{d}u}_{2n-1\quad\text{factors}}.$$

Our second observation is

$$u^*\mathrm{d}u = -\frac{1}{2}(z+\bar{z})(\mathrm{d}z+\mathrm{d}\bar{z})\quad\text{and}\quad \mathrm{d}u^*\,\mathrm{d}u = -\frac{1}{2}(\mathrm{d}z+\mathrm{d}\bar{z})(\mathrm{d}z+\mathrm{d}\bar{z}).$$

Therefore

$$(u^*\mathrm{d}u)^{2n-1} = -\frac{1}{2^n}(z+\bar{z})(\mathrm{d}z+\mathrm{d}\bar{z})^{2n-1}.$$

Because of Lemma 1.3, we have the equalities

$$\mathrm{tr}((z+\bar{z})(\mathrm{d}z+\mathrm{d}\bar{z})^{2n-1}) = \sum_{\Bbbk\in\Gamma_n^+}\mathrm{tr}(\bar{z}\alpha_{\Bbbk}) + \sum_{\Bbbk\in\Gamma_n^-}\mathrm{tr}(z\alpha_{\Bbbk})$$

$$= \sum_{\Bbbk\in\Gamma_n^+}(-1)^{n+1}2^{n-1}(n-1)!n!\sum_{k=1}^{n}\bar{z}_k\mathrm{d}z_k\wedge_{j\neq k}(\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j)$$

$$+ \sum_{\Bbbk\in\Gamma_n^-}(-1)^{n}2^{n-1}(n-1)!n!\sum_{k=1}^{n}z_k\mathrm{d}\bar{z}_k\wedge_{j\neq k}(\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j)$$

$$= (-1)^{n+1}2^{n-1}(2n-1)!\sum_{k=1}^{n}\big(\bar{z}_k\mathrm{d}z_k\wedge_{j\neq k}(\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j) - z_k\mathrm{d}\bar{z}_k\wedge_{j\neq k}(\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j)\big)$$

$$= \frac{(-1)^{n+1}2^n(2\pi i)^n(2n-1)!}{(n-1)!}\mathrm{d}V.$$

Finally, adding all results together we come to the conclusion of the lemma:

$$\mathrm{tr}(u^*\mathrm{d}u)^{2n-1} = -\frac{1}{2^n}\mathrm{tr}((z+\bar{z})(\mathrm{d}z+\mathrm{d}\bar{z})^{2n-1}) = (-1)^n\frac{(2\pi i)^n(2n-1)!}{(n-1)!}\mathrm{d}V. \quad \square$$

To generalize the construction of u to an arbitrary manifold, we need to cut down u at "infinity". We define the smooth function $\xi_0 : [0,\infty)\to\mathbb{R}$ as

$$\xi_0(x) := \begin{cases} \mathrm{e}^{-\frac{4}{x^2}}, & x>0 \\ 0, & x=0, \end{cases}$$

and the smooth function $\xi : S^{2n-1}\to\mathbb{C}^n$ by

$$\xi(z) := \xi_0(|1-\mathrm{Re}\,(z_1)|)z + (\xi_0(|1-\mathrm{Re}\,(z_1)|)-1,0,0,\ldots,0).$$

By standard methods, it can be proved that, for any natural number k and any vector fields $X_1, X_2, \ldots, X_l$ on S^{2n-1}, the function ξ satisfies

$$|\xi(z)-(-1,0,\ldots,0)| = \mathcal{O}(|1-\mathrm{Re}\,(z_1)|^k) \tag{7}$$

and

$$|X_1X_2\cdots X_l\xi(z)| = \mathcal{O}(|1-\mathrm{Re}\,(z_1)|^k) \quad \text{as} \quad z\to(1,0,\ldots,0). \tag{8}$$

Furthermore, the length of $\xi(z)$ is given by

$$|\xi(z)|^2 = 2(\mathrm{Re}\,(z_1)+1)(\xi_0(|1-\mathrm{Re}\,(z_1)|)^2-\xi_0(|1-\mathrm{Re}\,(z_1)|))+1,$$

so $|\xi(z)|>0$ for all $z\in S^{2n-1}$.

Using the function ξ, we define the smooth function $\tilde{u} : S^{2n-1}\to GL_{2^{n-1}}(\mathbb{C})$ by

$$\tilde{u}(z) := \frac{1}{2}(e_{1,+}+e_{1,-})(\xi(z)_+ + \overline{\xi(z)}_-).$$

The function $\tilde{u}$ is well defined, since

$$\tilde{u}(z)^*\tilde{u}(z) = |\xi(z)|^2 > 0.$$

Observe that we may express $\tilde{u}$ in terms of u as

$$\tilde{u}(z) = \xi_0(|1 - \mathrm{Re}\,(z_1)|)(u(z) - 1) + 1.$$

If we choose a diffeomorphism $\tau : B_{2n-1} \cong S^{2n-1} \setminus \{(1, 0, \dots, 0)\}$, Eqs. (7) and (8) imply that the function $\tau^*\tilde{u}$ can be considered as a smooth function $B_{2n-1} \to GL_{2^{n-1}}(\mathbb{C})$ such that $\tau^*\tilde{u} - 1$ vanishes to infinite order at the boundary of B_{2n-1}. The particular choice of τ as the stereographic projection

$$\tau(y) := \left(2|y|^2 - 1, 2\sqrt{1 - |y|^2}\,y\right)$$

will give a function $\tau^*\tilde{u}$ of the form

$$\begin{aligned} \tau^*\tilde{u}(y) &= \mathrm{e}^{-\frac{1}{(1-|y|^2)^2}}(u(\tau(y)) - 1) + 1 \\ &= \frac{\mathrm{e}^{-\frac{1}{(1-|y|^2)^2}}}{2}(e_{1,+} + e_{1,-})(\tau(y)_+ + \overline{\tau(y)}_-) + 1 - \mathrm{e}^{-\frac{1}{(1-|y|^2)^2}}. \end{aligned}$$

Lemma 1.5. *There is a homotopy of smooth functions $S^{2n-1} \to GL_{2^{n-1}}(\mathbb{C})$ between $\tilde{u}$ and u. Therefore* $\mathrm{cs}[\tilde{u}] - \mathrm{cs}[u]$ *is an exact form.*

Proof. We can take the homotopy $w : S^{2n-1} \times [0, 1] \to GL_{2^{n-1}}(\mathbb{C})$ as

$$w(z, t) = \xi_t(|1 - \mathrm{Re}\,(z_1)|)(u(z) - 1) + 1,$$

where

$$\xi_t(x) := \mathrm{e}^{-\frac{4(1-t)}{x^2}}.$$

Clearly, $w : S^{2n-1} \times [0, 1] \to GL_{2^{n-1}}(\mathbb{C})$ is a smooth function, and $w(z, 0) = \tilde{u}(z)$ and $w(z, 1) = u(z)$. $\square$

In the general case, let Y be a compact, connected, orientable manifold of odd dimension $2n - 1$. If we take an open subset U of Y with coordinates $(x_i)_{i=1}^{2n-1}$ such that

$$U = \left\{x : \sum_{i=1}^{2n-1} |x_i(x)|^2 < 1\right\},$$

the coordinates define a diffeomorphism $\nu : U \cong B_{2n-1}$. We can define the functions $g, \tilde{g} : Y \to GL_{2^{n-1}}(\mathbb{C})$ by

$$g(x) := \begin{cases} u(\tau\nu(x)) & \text{for} \quad x \in U \\ 1 & \text{for} \quad x \notin U \end{cases} \tag{9}$$

$$\tilde{g}(x) := \begin{cases} \tilde{u}(\tau\nu(x)) & \text{for} \quad x \in U \\ 1 & \text{for} \quad x \notin U. \end{cases} \tag{10}$$

If we let $\tilde{\nu} : Y \to S^{2n-1}$ be the Lipschitz continuous function defined by

$$\tilde{\nu}(x) = \begin{cases} \tau(\nu(x)) & \text{for} \quad x \in U \\ (1, 0, \dots, 0) & \text{for} \quad x \notin U, \end{cases} \tag{11}$$

the functions $\tilde{g}$ and g can be expressed as $g = \tilde{\nu}^* u$ and $\tilde{g} = \tilde{\nu}^* \tilde{u}$. The function $\tilde{g}$ is smooth, and the function g is Lipschitz continuous.

Theorem 1.6. *Denoting the normalized volume form on Y by $\mathrm{d}V_Y$, the function $\tilde{g}$ satisfies*

$$\mathrm{cs}[\tilde{g}] = (-1)^n \mathrm{d}V_Y, \tag{12}$$

in $H^{odd}_{dR}(Y)$. Thus, if $f : X \to Y$ is a smooth mapping,

$$\deg(f) = (-1)^n \int_X f^* \mathrm{cs}[\tilde{g}].$$

Proof. By Lemmas 1.4 and 1.5, we have the identities

$$\begin{aligned}\int_Y \mathrm{cs}[\tilde{g}] &= \int_U \mathrm{cs}_{2n-1}[\tilde{g}] = \int_U \tilde{\nu}^* \mathrm{cs}_{2n-1}[\tilde{u}] \\ &= \int_{S^{2n-1}} \mathrm{cs}_{2n-1}[\tilde{u}] = \int_{S^{2n-1}} \mathrm{cs}_{2n-1}[u] = (-1)^n.\end{aligned}$$

Therefore, we have the identity $\mathrm{cs}_{2n-1}[\tilde{g}] = (-1)^n \mathrm{d}V_Y$. Since $\mathrm{cs}[\tilde{g}] - \mathrm{cs}_{2n-1}[\tilde{g}]$ is an exact form on U and vanishes to infinite order at ∂U, the theorem follows. □

2. Toeplitz operators and their index theory

In this section, we will give the basics of integral representations of holomorphic functions and the Henkin–Ramirez integral representation; we will take the facts from [11,16,21] that are relevant for our purposes. After that, we will review the theory of Toeplitz operators on the Hardy space on the boundary of a strictly pseudo-convex domain. We will let M denote a Stein manifold, and we will assume that $\Omega \subseteq M$ is a relatively compact, strictly pseudo-convex domain with smooth boundary.

Consider the Hilbert space $L^2(\partial\Omega)$, in some Riemannian metric on $\partial\Omega$. We will use the notation $H^2(\partial\Omega)$ for the Hardy space, which is defined as the space of functions in $L^2(\partial\Omega)$ with holomorphic extensions to Ω. The subspace $H^2(\partial\Omega) \subseteq L^2(\partial\Omega)$ is a closed subspace so there exists a unique orthogonal projection $P_{\partial\Omega} : L^2(\partial\Omega) \to H^2(\partial\Omega)$ called the Szegö projection. We will consider the Henkin–Ramirez projection, see [15,20] and the generalization in [16] to Stein manifolds, which we will denote by $P_{HR} : L^2(\partial\Omega) \to H^2(\partial\Omega)$ and call the HR projection. The HR projection is not necessarily orthogonal, but is often possible to calculate explicitly, see [21], and easier to estimate. We will briefly review its construction in the case $M = \mathbb{C}^n$ following Chapter VII of [21]. The construction of the HR projection on a general Stein manifold is somewhat more complicated, but the same estimates hold, so we refer the reader to the construction in [16].

The kernel of the HR projection should be thought of as the first terms in a Taylor expansion of the Szegö kernel. This idea is explained in [17]. The HR kernel contains the most singular part of the Szegö kernel, and the HR kernel can be very explicitly estimated at its singularities. This is our reason to use the HR projection instead of the Szegö projection. If Ω is defined by the strictly pluri-subharmonic function ρ, a function $\Phi = \Phi(w, z)$ is defined as the smooth global extension of the Levi polynomial

$$F(w, z) := \sum_{j=1}^n \frac{\partial \rho}{\partial w_j}(w)(w_j - z_j) - \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial w_j \partial w_k}(w)(w_j - z_j)(w_k - z_k)$$

from the diagonal in $\Omega \times \Omega$ to the whole of $\overline{\Omega} \times \overline{\Omega}$; see more in Chapter V.1.1 and Chapter VII.5.1 of [21]. If we take $c > 0$ such that $\partial\bar{\partial}\rho \geq c$, there is an $\varepsilon > 0$ such that

$$2\mathrm{Re}\, \Phi(w, z) \geq \rho(w) - \rho(z) + c|z - w|^2, \quad \text{for} \quad |z - w| < \varepsilon, \tag{13}$$

see more in equation 1.6, Chapter V.1.1 of [21]. By Lemma 1.5 of Chapter VII of [21], the function Φ satisfies the following estimate:

$$\int_{\partial\Omega} \frac{\mathrm{d}V(w)}{|\Phi(w, z)|^{n+\beta}} \lesssim 1, \tag{14}$$

where $\mathrm{d}V$ denotes the volume measure on $\partial\Omega$ if $\beta < 0$, and a similar estimate with the roles of z and w interchanged. Here we used the standard notation $a \lesssim b$ for the statement that there exists a constant $C > 0$ such that $a \leq Cb$.

By Theorem 3.6, Chapter VII of [21] we can associate with Φ a function $H_{\partial\Omega}$ in $\Omega \times \Omega$ holomorphic in its second variable such that, if $g \in L^1(\Omega)$ is holomorphic, it has the integral representation

$$g(z) = \int_{\partial\Omega} H_{\partial\Omega}(w, z) f(w) \mathrm{d}V(w).$$

For the function $H_{\partial\Omega}$, the estimate

$$|H_{\partial\Omega}(z, w)| \lesssim |\Phi(w, z)|^{-n}, \tag{15}$$

holds in $\partial\Omega \times \partial\Omega$; see more in Proposition 3.1, Chapter VII of [21]. Since Φ satisfies the estimate (13), where c is the infimum of $\partial\bar{\partial}\rho$, the construction of an HR projection does give an L^2-bounded projection for strictly pseudo-convex domains. If Ω is weakly pseudo-convex, the situation is more problematic and not that well understood, partly due to problems estimating solutions to the $\bar{\partial}$-equation in weakly pseudo-convex domains. By Proposition 3.8 of Chapter VII.3.1 in [21], the kernel $H_{\partial\Omega}$ satisfies the estimate

$$|H_{\partial\Omega}(z, w) - \overline{H_{\partial\Omega}(w, z)}| \lesssim |\Phi(z, w)|^{-n+1/2}. \tag{16}$$

The estimate (16) will be crucial when proving that $P_{\partial\Omega} - P_{HR}$ is in the Schatten class. The kernel $H_{\partial\Omega}$ determines a bounded operator P_{HR} on $L^2(\partial\Omega)$ by Theorem 3.6 of Chapter VII.3 in [21]. Since the range of P_{HR} is contained in $H^2(\partial\Omega)$ and $g = P_{HR}g$ for any $g \in H^2(\partial\Omega)$, it follows that $P_{HR} : L^2(\partial\Omega) \to H^2(\partial\Omega)$ is a projection.

We will now present some facts about Toeplitz operators on the Hardy space of a relatively compact strictly pseudo-convex domain Ω in a complex manifold M. Our operators are associated with the Szegö projection since the theory becomes somewhat more complicated when a non-orthogonal projection is involved. For any dimension N, we denote by $C(\partial\Omega, M_N)$ the C^*-algebra of continuous functions $\partial\Omega \to M_N$, the algebra of complex $N \times N$-matrices. The algebra $C(\partial\Omega, M_N)$ has a representation $\pi : C(\partial\Omega, M_N) \to \mathcal{B}(L^2(\partial\Omega) \otimes \mathbb{C}^N)$ which is given by pointwise multiplication. We define the linear mapping

$$T : C(\partial\Omega, M_N) \to \mathcal{B}(H^2(\partial\Omega) \otimes \mathbb{C}^N), \quad a \mapsto P_{\partial\Omega}\pi(a)P_{\partial\Omega}.$$

Here, we identify $P_{\partial\Omega}$ with the projection $L^2(\partial\Omega) \otimes \mathbb{C}^N \to H^2(\partial\Omega) \otimes \mathbb{C}^N$. An operator of the form $T(a)$ is called a Toeplitz operator on $\partial\Omega$. Toeplitz operators are well studied; see for instance [6,9,13] and [22]. The representation π satisfies $[P_{\partial\Omega}, \pi(a)] \in \mathcal{K}(L^2(\partial\Omega) \otimes \mathbb{C}^N)$ for any $a \in C(\partial\Omega, M_N)$; see for instance [6] or Theorem 3.1 below. Here, we use the symbol $\mathcal{K}$ to denote the algebra of compact operators. The fact that $P_{\partial\Omega}$ commutes with continuous functions up to a compact operator implies the property

$$T(ab) - T(a)T(b) \in \mathcal{K}(H^2(\partial\Omega) \otimes \mathbb{C}^N). \tag{17}$$

Furthermore, $T(a)$ is compact if and only if $a = 0$. Let us denote the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ by $\mathcal{C}(\mathcal{H})$ and the quotient mapping $\mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ by $\mathfrak{q}$. Eq. (17) implies that the mapping

$$\beta := \mathfrak{q} \circ T : C(\partial\Omega, M_N) \to \mathcal{C}(H^2(\partial\Omega) \otimes \mathbb{C}^N)$$

is an injective $*$-homomorphism.

By the Boutet de Monvel index formula, from [6], if the symbol a is invertible and smooth, the index of the Toeplitz operator $T(a)$ has the analytic expression

$$\mathrm{ind}\,(T(a)) = -\int_{\partial\Omega} \mathrm{cs}[a] \wedge Td(\Omega); \tag{18}$$

see more in Theorem 1 in [6], and the remarks thereafter. The mapping $a \mapsto \mathrm{ind}\,(T(a))$, defined on functions $a : \partial\Omega \to \mathrm{GL}_N(\mathbb{C})$, is homotopy invariant, so it extends to a mapping $\mathrm{ind} : K_1(C^\infty(\partial\Omega)) \to \mathbb{Z}$. Here, $K_1(C^\infty(\partial\Omega))$ denotes the odd K-theory of the Frechet algebra $C^\infty(\partial\Omega)$, which is defined as homotopy classes of invertible matrices with coefficients in $C^\infty(\partial\Omega)$; see more in [5].

Theorem 2.1. *Suppose that $\Omega \subseteq M$ is a relatively compact strictly pseudo-convex bounded domain with smooth boundary, Y is a compact, orientable manifold of dimension $2n-1$, and $g : Y \to GL_{2^{n-1}}(\mathbb{C})$ is the mapping defined in (10). If $f : \partial\Omega \to Y$ is a continuous function, then*

$$\deg(f) = (-1)^{n+1} \mathrm{ind}\,(P_{\partial\Omega}\pi(g \circ f)P_{\partial\Omega}). \tag{19}$$

Proof. If we assume that f is smooth, the index formula of Boutet de Monvel, see above in Eq. (18), implies that the index of $P_{\partial\Omega}\pi(g \circ f)P_{\partial\Omega}$ satisfies

$$\mathrm{ind}\,(P_{\partial\Omega}\pi(g \circ f)P_{\partial\Omega}) = -\int_{\partial\Omega} f^*\mathrm{cs}[\tilde{g}] \wedge Td(\Omega) = -\int_{\partial\Omega} f^*\mathrm{cs}[\tilde{g}] = (-1)^{n+1}\deg(f),$$

where the first equality follows from g and $\tilde{g}$ being homotopic, see Lemma 1.5, and the last two equalities follow from Theorem 1.6. The general case follows from the fact that both hand sides of (19) is homotopy invariant. □

Theorem 2.1 does in some cases hold with even looser regularity conditions on f. Since both sides of Eq. (19) are homotopy invariants, the theorem holds for any class of functions which are homotopic to smooth functions in such sense that both sides in (19) are well defined and depend continuously on the function. For instance, if Ω is a bounded symmetric domain, we may take $f : \partial\Omega \to Y$ to be in the VMO-class. It follows from [3] that, if $w : \partial\Omega \to \mathrm{GL}_N$ has vanishing mean oscillation and Ω is a bounded symmetric domain, the operator $P_{\partial\Omega} w P_{\partial\Omega}$ is Fredholm. By Brezis and Nirenberg [8], the degree of a VMO-function is well defined and depends continuously on f without any restriction on the geometry. To be more precise, there is a one-parameter family $(f_t)_{t\in(0,1)} \subseteq C(\partial\Omega, Y)$ such that $f_t \to f$ in VMO when $t \to 0$ and $\deg(f)$ is defined as $\deg(f_t)$ for t small enough. Since the index of a Fredholm operator is homotopy invariant, the degree of a function $f : \partial\Omega \to Y$ in VMO satisfies

$$\deg f = (-1)^{n+1}\mathrm{ind}\,(P_{\partial\Omega}\pi(g \circ f_t)P_{\partial\Omega}) = (-1)^{n+1}\mathrm{ind}\,(P_{\partial\Omega}\pi(g \circ f)P_{\partial\Omega}).$$

Our next task will be calculating the index of Toeplitz operators with non-smooth symbol. For $p \geq 1$, let $\mathcal{L}^p(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ denote the ideal of Schatten class operators on a separable Hilbert space $\mathcal{H}$, so $T \in \mathcal{L}^p(\mathcal{H})$ if and only if $\mathrm{tr}((T^*T)^{p/2}) < \infty$. An exact description of integral operators belonging to this class exists only for $p = 2$. However, for $p > 2$, there exists a convenient sufficient condition on the kernel, found in [24]. We will return to this subject a little later. Suppose that $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation of a $\mathbb{C}$-algebra $\mathcal{A}$ and P is a projection such that $[P, \pi(a)] \in \mathcal{L}^p(\mathcal{H})$ for all $a \in \mathcal{A}$ and $P - P^* \in \mathcal{L}^p(\mathcal{H})$. Atkinson's Theorem implies that, if a is invertible, $P\pi(a)P$ is Fredholm. The operator $F := 2P - 1$ has the properties

$$F^2 = 1 \quad \text{and} \quad F - F^*, \qquad [F, \pi(a)] \in \mathcal{L}^p(\mathcal{H}). \tag{20}$$

If π and F satisfy the conditions in Eq. (20), the pair (π, F) is called a p-summable odd Fredholm module. If the pair (π, F) satisfies the requirement in Eq. (20) but with $\mathcal{L}^p(\mathcal{H})$ replaced by $\mathcal{K}(\mathcal{H})$, the pair (π, F) is a bounded odd Fredholm module. For a more thorough presentation of Fredholm modules, see, for example, Chapters VII and VIII of [5]. Since our focus is on Toeplitz operators, we will call (π, P) a *Toeplitz pair* if $(\pi, 2P - 1)$ is a bounded odd Fredholm module, and (π, P) is said to be p-summable if $(\pi, 2P - 1)$ is.

The condition that $L := P^* - P \in \mathcal{L}^p(\mathcal{H})$ can be interpreted in terms of the orthogonal projection $\tilde{P}$ to the Hilbert space $P\mathcal{H}$. Using that $\tilde{P}P = P$ and $P\tilde{P} = \tilde{P}$, we obtain the identity

$$\tilde{P}L = \tilde{P}P^* - \tilde{P}P = \tilde{P} - P. \tag{21}$$

Thus the condition $P^* - P \in \mathcal{L}^p(\mathcal{H})$ is equivalent to the property $\tilde{P} - P \in \mathcal{L}^p(\mathcal{H})$.

A Toeplitz pair (π, P) over a topological algebra $\mathcal{A}$ defines a mapping $a \mapsto \mathrm{ind}\,(P\pi(a)P)$ on the invertible elements of $\mathcal{A} \otimes M_N$ for any N. Since the index is homotopy invariant, the association $a \mapsto \mathrm{ind}\,(P\pi(a)P)$ induces the mapping $\mathrm{ind} : K_1(\mathcal{A}) \to \mathbb{Z}$, where $K_1(\mathcal{A})$ denotes the odd K-theory of $\mathcal{A}$; see [5].

Connes placed the index theory for p-summable Toeplitz pairs in a suitable homological picture using cyclic homology in [10]. We will consider Connes' original definition of cyclic cohomology, which simplifies the construction of the Chern–Connes character. The notation $\mathcal{A}^{\otimes k}$ will be used for the kth tensor power of $\mathcal{A}$. The Hochschild differential $b : \mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes k-1}$ is defined as

$$\begin{aligned} b(x_0 \otimes x_1 \otimes \cdots \otimes x_k \otimes x_{k+1}) &:= (-1)^{k+1} x_{k+1}x_0 \otimes x_1 \otimes \cdots \otimes x_k \\ &+ \sum_{j=0}^{k} (-1)^j x_0 \otimes \cdots \otimes x_{j-1} \otimes x_j x_{j+1} \otimes x_{j+2} \otimes \cdots \otimes x_{k+1}. \end{aligned}$$

The cyclic permutation operator $\lambda : \mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes k}$ is defined by

$$\lambda(x_0 \otimes x_1 \otimes \cdot \otimes x_k) = (-1)^k x_k \otimes x_0 \otimes \cdots \otimes x_{k-1}.$$

The complex $C^k_\lambda(\mathcal{A})$ is defined as the space of continuous linear functionals μ on $\mathcal{A}^{\otimes k+1}$ such that $\mu \circ \lambda = \mu$. The Hochschild coboundary operator $\mu \mapsto \mu \circ b$ makes $C^*_\lambda(\mathcal{A})$ into a complex. The cohomology of the complex $C^*_\lambda(\mathcal{A})$ will be denoted by $HC^*(\mathcal{A})$, and is called the cyclic cohomology of $\mathcal{A}$. There is a filtration on cyclic cohomology coming from a linear mapping $S : HC^k(\mathcal{A}) \to HC^{k+2}(\mathcal{A})$, which is called the suspension operator or the periodicity operator. For a definition of the periodicity operator, see [9].

The additive pairing between $HC^{2k+1}(\mathcal{A})$ and the odd K-theory $K_1(\mathcal{A})$ is defined by

$$\langle \mu, u\rangle_k := d_k\ (\mu \otimes \mathrm{tr})\left(\underbrace{(u^{-1}-1)\otimes(u-1)\otimes\cdots\otimes(u^{-1}-1)\otimes(u-1)}_{2k+2\quad \text{factors}}\right),$$

where we choose the same normalization constant d_k as in Proposition 3 of Chapter III.3 of [9]:

$$d_k := \frac{2^{-(2k+1)}}{\sqrt{2i}}\Gamma\left(\frac{2k+3}{2}\right)^{-1}.$$

The choice of normalization implies that, for a cohomology class in $HC^{2k+1}(\mathcal{A})$ represented by the cyclic cocycle μ, the pairing satisfies

$$\langle S\mu, u\rangle_{k+1} = \langle \mu, u\rangle_k;$$

see Proposition 3 in Chapter III.3 of [9]. Following Definition 3 of Chapter IV.1 of [9], we define the Connes–Chern character of a p-summable Toeplitz pair as the cyclic cocycle:

$$\mathrm{ch}_{2k+1}(\pi, P)(a_0, a_1, \ldots, a_{2k+1}) := c_k \mathrm{tr}(\pi(a_0)[P, \pi(a_1)]\cdots[P, \pi(a_{2k+1})]),$$

for $2k+1 \geq p$, where

$$c_k := -\sqrt{2i}\,2^{2k+1}\Gamma\left(\frac{2k+3}{2}\right).$$

This choice of normalization constant implies that

$$S\mathrm{ch}_{2k+1}(\pi, P) = \mathrm{ch}_{2k+3}(\pi, P),$$

by Proposition 2 in Chapter IV.1 of [9].

Theorem 2.2 (*Proposition* 4 *of Chapter* IV.1 *of [9]*)**.** *If (π, P) is a p-summable Toeplitz pair over $\mathcal{A}$, $2k+1 \geq p$, and a is invertible in $\mathcal{A}\otimes M_N$, the index of $P\pi(a)P : P\mathcal{H}\otimes\mathbb{C}^N \to P\mathcal{H}\otimes\mathbb{C}^N$ may be expressed as*

$$\begin{aligned} \mathrm{ind}\,(P\pi(a)P) &= \langle \mathrm{ch}_{2k+1}(\pi, P), a\rangle_k \\ &= -\mathrm{tr}\left(\pi(a^{-1})[P, \pi(a)][P, \pi(a^{-1})]\cdots[P, \pi(a^{-1})][P, \pi(a)]\right) \\ &= -\mathrm{tr}(P - \pi(a^{-1})P\pi(a))^{2k+1}. \end{aligned}$$

The role of the periodicity operator S in the context of index theory is to extend index formulas to larger algebras. Suppose that μ is a cyclic k-cocycle on an algebra $\mathcal{A}$ which is a dense $*$-subalgebra of a C^*-algebra A. As is explained in [9] for functions on S^1 and in [23] for operator-valued symbols, the cyclic $k+2m$-cocycle $S^m\mu$ can be extended to a cyclic cocycle on a larger $*$-subalgebra $\mathcal{A} \subseteq \mathcal{A}' \subseteq A$. When μ is the cyclic cocycle $f_0 \otimes f_1 \mapsto \int f_0 \mathrm{d} f_1$ on $C^\infty(S^1)$, the $2m+1$-cocycle $S^m\mu$ extends to $C^\alpha(S^1)$ whenever $\alpha(2m+1) > 1$ by Proposition 3 in Chapter III2.α of [9] and a formula for $S^m\mu$ is given above in (1). Cyclic cocycles of the form $\mu = \mathrm{ch}(\pi, P)$ appear in index theory, and the periodicity operator can be used to extend index formulas to larger algebras.

The index formula of Theorem 2.2 holds for Toeplitz operators under a Schatten class condition, and to deal with this condition we will need the following theorem of Russo [24] to give a sufficient condition on an integral operator for it to be Schatten class. Let X denote a σ-finite measure space. As in [4], for numbers $1 \leq p, q < \infty$, the mixed (p, q)-norm of a function $k : X \times X \to \mathbb{C}$ is defined by

$$\|k\|_{p,q} := \left(\int_X \left(\int_X |k(x, y)|^p \mathrm{d}x \right)^{\frac{q}{p}} \mathrm{d}y \right)^{\frac{1}{q}}.$$

We denote the space of measurable functions $k : X \times X \to \mathbb{C}$ with finite mixed (p, q)-norm by $L^{(p,q)}(X \times X)$. By Theorem 4.1 of [4], the space $L^{(p,q)}(X \times X)$ becomes a Banach space in the mixed (p, q)-norm which is reflexive if $1 < p, q < \infty$.

The Hermitian conjugate of the function k is defined by $k^*(x, y) := \overline{k(y, x)}$. Clearly, if a bounded operator K has integral kernel k, the Hermitian conjugate K^* has integral kernel k^*.

Theorem 2.3 (*Theorem 1 in [24]*). *Suppose that $K : L^2(X) \to L^2(X)$ is a bounded operator given by an integral kernel k. If $2 < p < \infty$, then*

$$\|K\|_{\mathcal{L}^p(L^2(X))} \leq (\|k\|_{p',p} \|k^*\|_{p',p})^{1/2}, \tag{22}$$

where $p' = p/(p-1)$.

In the statement of the theorem in [24], the assumption that $k \in L^2(X \times X)$ is made. This assumption implies that K is Hilbert–Schmidt and $K \in \mathcal{L}^p(L^2(X))$ for all $p > 2$, so for our purposes it is not interesting. But since L^2-kernels are dense in $L^{(p,q)}$, the non-commutative Fatou Lemma, see Theorem 2.7*d* of [25], implies (22) for any k for which the right-hand side of (22) is finite. Using Theorem 2.3, we obtain the following formula for the trace of the product of integral operators.

Theorem 2.4. *Suppose that $K_j : L^2(X) \to L^2(X)$ are operators with integral kernels k_j for $j = 1, \dots, m$ such that $\|k_j\|_{p',p}, \|k_j^*\|_{p',p} < \infty$ for certain $p > 2$. Then, for $m \geq p$, the operator $K_1 K_2 \cdots K_m$ is a trace class operator, and we have the trace formula*

$$\mathrm{tr}(K_1 K_2 \cdots K_m) = \int_{X^m} \left(\prod_{j=1}^m k_j(x_j, x_{j+1}) \right) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_m,$$

where we identify x_{m+1} with x_1.

Proof. The case $p = m = 2$ follows if, for any $k_1, k_2 \in L^2(X \times X)$, we have the trace formula

$$\mathrm{tr}(K L^*) = \int_{X \times X} k(x, y) \overline{l(x, y)} \mathrm{d}x \mathrm{d}y.$$

Consider the sesquilinear form on $\mathcal{L}^2(L^2(X))$ defined by

$$(K, L) := \mathrm{tr}(K L^*) - \int k(x, y) \overline{l(x, y)} \mathrm{d}x \mathrm{d}y.$$

Since $\mathrm{tr}(K^* K) = \int_{X \times X} |k(x, y)|^2 \mathrm{d}x \mathrm{d}y$, the sesquilinear form satisfies $(K, K) = 0$, and the polarization identity implies that $(K, L) = 0$ for any $K, L \in \mathcal{L}^2(L^2(X))$.

If the operators $K_j : L^2(X) \to L^2(X)$ are Hilbert–Schmidt, or equivalently they satisfy $k_j \in L^2(X \times X)$, we may take $K = K_1$ and $L^* = K_2 K_3 \cdots K_m$, so the case $p = m = 2$ implies that the operators $K_1, K_2, \dots, K_m$ satisfy the statement of the theorem. In the general case, the theorem follows from the non-commutative Fatou Lemma, see Theorem 2.7d of [25], since $\mathcal{L}^2$ is dense in $\mathcal{L}^p$ for $p > 2$. □

3. The Toeplitz pair on the Hardy space

As explained in Section 2, for the representation $\pi : C(\partial\Omega) \to \mathcal{B}(L^2(\partial\Omega))$ and the Szegö projection $P_{\partial\Omega}$, the commutator $[P_{\partial\Omega}, \pi(a)]$ is compact for any continuous a. Thus $(\pi, P_{\partial\Omega})$ is a Toeplitz pair over $C(\partial\Omega)$. To enable the use of the index theory of [9], we will show that the Toeplitz pair $(\pi, P_{\partial\Omega})$ restricted to the subalgebra of Hölder continuous functions $C^\alpha(\partial\Omega) \subseteq C(\partial\Omega)$ becomes p-summable. These results will give us analytic degree formulas for Hölder continuous mappings.

Theorem 3.1. *If Ω is a relatively compact strictly pseudo-convex domain in a Stein manifold of complex dimension n, and P denotes either P_{HR} or $P_{\partial\Omega}$, the operator $[P, \pi(a)]$ belongs to $\mathcal{L}^p(L^2(\partial\Omega))$ for $a \in C^\alpha(\partial\Omega)$ and for all $p > 2n/\alpha$.*

The proof will be based on Theorem 2.3. We will start our proof of Theorem 3.1 by some elementary estimates. We define the measurable function $k_\alpha : \partial\Omega \times \partial\Omega \to \mathbb{C}$ by

$$k_\alpha(z, w) := \frac{|z - w|^\alpha}{|\Phi(w, z)|^n}.$$

Lemma 3.2. *The function k_α satisfies*

$$k_\alpha(z, w) \lesssim |\Phi(w, z)|^{-(n-\frac{\alpha}{2})}$$

for $|z - w| < \varepsilon$.

Proof. By (13), we have the estimate

$$|z - w|^\alpha \lesssim |\Phi(w, z)|^{\alpha/2}.$$

From this estimate, the lemma follows. □

We will use the notation $\mathrm{d}V$ for the volume measure on $\partial\Omega$.

Lemma 3.3. *The function k_α satisfies*

$$\int_{\partial\Omega} |k_\alpha(z, w)|^{p'} \mathrm{d}V(z) \lesssim 1$$

$$\int_{\partial\Omega} |k_\alpha(z, w)|^{p'} \mathrm{d}V(w) \lesssim 1$$

whenever

$$(2n - \alpha)p' < 2n.$$

Proof. We will only prove the first of the estimates in the lemma. The proof of the second estimate goes analogously. Using (13) for Φ, we obtain

$$\int_{\partial\Omega} |k_\alpha(z, w)|^{p'} \mathrm{d}V(z) \lesssim \int_{B_r(w)} |k_\alpha(z, w)|^{p'} \mathrm{d}V(z),$$

since the function Φ satisfies $|\Phi(w,z)| > r^2$ outside $B_r(w)$. By Lemma 3.2, we can estimate the kernel pointwise by Φ, so (14) implies that

$$\int_{B_r(w)} |k_\alpha(z,w)|^{p'} \mathrm{d}V(z) \lesssim \int_{B_r(w)} |\Phi(w,z)|^{-p'(n-\frac{\alpha}{2})} \mathrm{d}V(z) \lesssim 1$$

if $(n - \frac{\alpha}{2})p' < n$. □

Lemma 3.4. *The function k_α satisfies $\|k_\alpha\|_{p',p} < \infty$ and $\|k_\alpha^*\|_{p',p} < \infty$ for $p > 2n/\alpha$.*

Proof. By the first estimate in Lemma 3.3, we can estimate the mixed norms of k_α as

$$\|k_\alpha\|_{p',p}^p \lesssim 1,$$

whenever $(2n - \alpha)p' < 2n$. The statement $(2n - \alpha)p' < 2n$ is equivalent to

$$\frac{1}{p} = 1 - \frac{1}{p'} < \frac{\alpha}{2n},$$

which is equivalent to $p > 2n/\alpha$. Similarly, the second estimate in Lemma 3.3 implies that $\|k_\alpha^*\|_{p',p} < \infty$ under the same condition on p. □

Lemma 3.5. *Suppose that $a \in C^\alpha(\partial\Omega)$, and let κ_a denote the integral kernel of $[P_{HR}, \pi(a)]$. The kernel κ_a satisfies*

$$|\kappa_a(z,w)| \le \|a\|_{C^\alpha(\partial\Omega)} |k_\alpha(z,w)|, \tag{23}$$

where $\|\cdot\|_{C^\alpha(\partial\Omega)}$ denotes the usual norm in $C^\alpha(\partial\Omega)$.

Proof. The integral kernel of $[P_{HR}, \pi(a)]$ is given by

$$\kappa_a(z,w) = (a(z) - a(w)) H_{\partial\Omega}(w,z).$$

Since a is Hölder continuous and $H_{\partial\Omega}$ satisfies Eq. (15), the estimate (23) follows. □

Lemma 3.6. *The HR projection P_{HR} satisfies $P_{HR} - P_{HR}^* \in \mathcal{L}^q(L^2(\partial\Omega))$ for any $q > 2n$. Therefore, $P_{HR} - P_{\partial\Omega} \in \mathcal{L}^q(L^2(\partial\Omega))$ for any $q > 2n$.*

Proof. Let us denote the kernel of the operator $P_{HR} - P_{HR}^*$ by b. By (16), we have the pointwise estimate $|b(z,w)| \lesssim |\Phi(w,z)|^{-n+1/2}$. Applying Lemma 3.4 with $\alpha = 0$ and p' such that $(n - 1/2)q' = np'$, we obtain the inequality $\|b\|_{q',q} < \infty$ for any $q > 2n$. The fact that $P_{HR} - P_{\partial\Omega} \in \mathcal{L}^q(L^2(\partial\Omega))$ follows now from (21). □

Proof of Theorem 3.1. By Lemma 3.5, the integral kernel κ_a of $[P_{HR}, \pi(a)]$ satisfies $|\kappa_a| \le \|a\|_{C^\alpha(\partial\Omega)} k_\alpha$. Theorem 2.3 implies the estimate

$$\|[P_{HR}, \pi(a)]\|_{\mathcal{L}^p(L^2(\partial\Omega))} \le \|a\|_{C^\alpha(\partial\Omega)} (\|k_\alpha\|_{p',p} \|k_\alpha^*\|_{p',p})^{1/2}.$$

By Lemma 3.4, $\|k_\alpha\|_{p',p}, \|k_\alpha^*\|_{p',p} < \infty$ for $p > 2n/\alpha$, so $[P_{HR}, \pi(a)] \in \mathcal{L}^p(L^2(\partial\Omega))$ for $p > 2n/\alpha$. By Lemma 3.6, $P_{HR} - P_{\partial\Omega} \in \mathcal{L}^p(L^2(\partial\Omega))$, so

$$[P_{\partial\Omega}, \pi(a)] = [P_{HR}, \pi(a)] + [P_{\partial\Omega} - P_{HR}, \pi(a)] \in \mathcal{L}^p(L^2(\partial\Omega))$$

for $p > 2n/\alpha$, and the proof of the theorem is complete. □

4. The index and degree formula for boundaries of strictly pseudo-convex domains in Stein manifolds

We may now combine our results on summability of the Toeplitz pairs (P_{HR}, π) and $(P_{\partial\Omega}, \pi)$ into index theorems and degree formulas. The index formula will be proved using the index formula of Connes; see Theorem 2.2.

Theorem 4.1. *Suppose that Ω is a relatively compact strictly pseudo-convex domain with smooth boundary in a Stein manifold of complex dimension n, and denote the corresponding HR kernel by $H_{\partial\Omega}$ and the Szegö kernel by $C_{\partial\Omega}$. If $a : \partial\Omega \to \mathrm{GL}_N$ is Hölder continuous with exponent α, then for $2k+1 > 2n/\alpha$ the index formulas hold:*

$$\operatorname{ind}(P_{\partial\Omega}\pi(a)P_{\partial\Omega}) = \operatorname{ind}(P_{HR}\pi(a)P_{HR}) \tag{24}$$

$$= -\int_{\partial\Omega^{2k+1}} \operatorname{tr}\left(\prod_{j=0}^{2k}(1 - a(z_{j-1})^{-1}a(z_j))H_{\partial\Omega}(z_{j-1}, z_j)\right) \mathrm{d}V^{2k+1} \tag{25}$$

$$= -\int_{\partial\Omega^{2k+1}} \operatorname{tr}\left(\prod_{j=0}^{2k}(1 - a(z_{j-1})^{-1}a(z_j))C_{\partial\Omega}(z_{j-1}, z_j)\right) \mathrm{d}V^{2k+1}, \tag{26}$$

where the integrals in (25) and (26) converge.

Proof. By Theorem 2.2, we have

$$\operatorname{ind}(P_{\partial\Omega}\pi(a)P_{\partial\Omega}) = -\operatorname{tr}(P_{\partial\Omega} - \pi(a^{-1})P_{\partial\Omega}\pi(a))^{2k+1},$$

and by Theorem 2.4 the trace has the form

$$-\operatorname{tr}(P_{\partial\Omega} - \pi(a^{-1})P_{\partial\Omega}\pi(a))^{2k+1} = -\int_{\partial\Omega^{2k+1}} \operatorname{tr}\left(\prod_{j=0}^{2k}(1 - a(z_{j-1})^{-1}a(z_j))C_{\partial\Omega}(z_{j-1}, z_j)\right) \mathrm{d}V^{2k+1}.$$

Similarly, the index for $P_{HR}\pi(a)P_{HR}$ is calculated. The theorem follows from the identity $\operatorname{ind}(P_{\partial\Omega}\pi(a)P_{\partial\Omega}) = \operatorname{ind}(P_{HR}\pi(a)P_{HR})$, since Lemma 3.6 implies that $P_{\partial\Omega}\pi(a)P_{\partial\Omega} - P_{HR}\pi(a)P_{HR}$ is compact. □

Theorem 4.1 has an interpretation in terms of cyclic cohomology. Define the cyclic $2n-1$-cocycle $\chi_{\partial\Omega}$ on $C^\infty(\partial\Omega)$ by

$$\chi_{\partial\Omega} := \sum_{k=0}^{n} S^k \omega_k,$$

where ω_k denotes the cyclic $2n-2k-1$-cocycle given by the Todd class $Td_k(\Omega)$ in degree $2k$ as

$$\omega_k(a_0, a_1, \ldots, a_{2n-2k-1}) := \int_{\partial\Omega} a_0 \mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_{2n-2k-1} \wedge Td_k(\Omega).$$

Similarly to Proposition 13, Chapter III.3 of [9], we have the following theorem.

Theorem 4.2. *The cyclic cocycle $S^m \chi_{\partial\Omega}$ defines the same cyclic cohomology class on $C^\infty(\partial\Omega)$ as*

$$\tilde{\chi}_{\partial\Omega}(a_0, a_1, \ldots, a_{2n+2m-1}) := \int_{\partial\Omega^{2n+2m-1}} \mathrm{tr}\left(a_0(z_0) \prod_{j=1}^{2n+2m-1} (a_j(z_j) - a_j(z_{j-1}))C_{\partial\Omega}(z_{j-1}, z_j)\right) \mathrm{d}V,$$

where we identify $z_{2n+2m-1} = z_0$. Furthermore, the cyclic cocycle $\tilde{\chi}_{\partial\Omega}$ extends to a cyclic $2n+2m-1$-cocycle on $C^\alpha(\partial\Omega)$ if $m > (2n(1-\alpha)+\alpha)/2\alpha$.

Returning to the degree calculations, to express the degree of a Hölder continuous function we will use Theorems 2.1 and 4.1. In order to express the formulas in Theorem 4.1 directly in terms of f, we will need some notation. Let $\langle\cdot,\cdot\rangle$ denote the scalar product on $\mathbb{C}^n$. The symmetric group on m elements will be denoted by S_m. We will consider S_m as the group of bijections on the set $\{1, 2, \ldots, m\}$, and identify the element $m+1$ with 1 in the set $\{1, 2, \ldots, m\}$.

For $2l \leq m$, we will define a function $\varepsilon_l : S_m \to \{0, 1, -1\}$, which we will refer to the order parity. If $\sigma \in S_m$ satisfies that there is an $i \in \{\sigma(1), \sigma(2), \ldots \sigma(2l-1), \sigma(2l)\}$ such that $i+1, i-1 \notin \{\sigma(1), \sigma(2), \ldots \sigma(2l-1), \sigma(2l)\}$, we set $\varepsilon_l(\sigma) = 0$. If σ does not satisfy this condition, the order parity of σ is set as $(-1)^k$, where k is the smallest number of transpositions needed to map the set $\{\sigma(1), \sigma(2), \ldots \sigma(2l-1), \sigma(2l)\}$, with j identified with $j+m$, to a set of the form $\{j_1, j_1+1, j_2, j_2+1, \ldots, j_l, j_l+1\}$, where $1 \leq j_1 < j_2 < \cdots < j_l \leq m$.

Proposition 4.3. *The function u satisfies*

$$\mathrm{tr}\left(\prod_{i=0}^{2k}(1 - u(z_{i-1})^* u(z_i))\right) = \sum_{l=0}^{2k+1} \sum_{\sigma\in S_{2(2k+1)}} (-1)^l 2^{n-l-1} \varepsilon_l(\sigma) \langle z_{\sigma(1)}, z_{\sigma(2)}\rangle \langle z_{\sigma(3)}, z_{\sigma(4)}\rangle \cdots \langle z_{\sigma(2l-1)}, z_{\sigma(2l)}\rangle,$$

where we identify z_m with z_{m+2k+1} for $m = 0, 1, \ldots, 2k$.

Proof. The product in the lemma satisfies the equalities

$$\prod_{i=1}^{2k-1} (1 - u(z_{i-1})^* u(z_i)) = \prod_{i=1}^{2k-1} \left(1 + \frac{1}{2}(z_{i-1,+} + \bar{z}_{i-1,-})(z_{i,+} + \bar{z}_{i,-})\right)$$
$$= \sum_{l=0}^{2k-1} \sum_{i_1<i_2<\cdots<i_l} 2^{-l} \prod_{j=1}^{l} \left((z_{i_j-1,+} + \bar{z}_{i_j-1,-})(z_{i_j,+} + \bar{z}_{i_j,-})\right).$$

The lemma follows from these equalities and degree reasons. □

Let us choose an open subset $U \subseteq Y$ such that there is a diffeomorphism $\nu : U \to B_{2n-1}$. Let $\tilde{\nu}$ be as in Eq. (11), and define the function $\tilde{f} : \partial\Omega^{2k+1} \to \mathbb{C}$ by

$$\tilde{f}(z_0, z_1, \ldots, z_{2k}) := \sum_{\sigma\in S_{2(2k-1)}} \sum_{l=0}^{2k-1} (-1)^l 2^{n-l-1} \varepsilon_l(\sigma) \prod_{i=1}^{l} \langle \tilde{\nu}(f(z_{\sigma(2j-1)})), \tilde{\nu}(f(z_{\sigma(2j)}))\rangle, \tag{27}$$

where we identify z_m with z_{m+2k+1}.

Theorem 4.4. *Suppose that Ω is a relatively compact strictly pseudo-convex domain with smooth boundary in a Stein manifold of complex dimension n, and that Y is a connected, compact, orientable, Riemannian manifold of dimension $2n-1$. If $f:\partial\Omega\to Y$ is a Hölder continuous function of exponent α, the degree of f can be calculated by*

$$\begin{aligned}\deg(f) &= (-1)^n\langle\tilde{\chi}_{\partial\Omega}, g\circ f\rangle_k\\ &= (-1)^n\int_{\partial\Omega^{2k+1}}\tilde{f}(z_0,z_1,\ldots,z_{2k})\prod_{j=0}^{2k}H_{\partial\Omega}(z_{j-1},z_j)\mathrm{d}V\\ &= (-1)^n\int_{\partial\Omega^{2k+1}}\tilde{f}(z_0,z_1,\ldots,z_{2k})\prod_{j=0}^{2k}C_{\partial\Omega}(z_{j-1},z_j)\mathrm{d}V\end{aligned}$$

whenever $2k+1>2n/\alpha$.

Proof. By Theorems 2.1 and 4.1, we have the equality

$$\deg(f) = (-1)^n\int_{\partial\Omega^{2k+1}}\mathrm{tr}\left(\prod_{j=0}^{2k}(1-g(f)(z_j)^*g(f)(z_{j+1}))H_{\partial\Omega}(z_{j-1},z_j)\right)\mathrm{d}V.$$

Proposition 4.3 implies that

$$\mathrm{tr}\left(\prod_{j=0}^{2k}(1-g(f)(z_j)^*g(f)(z_{j+1}))\right) = \tilde{f}(z_0,z_1,\ldots,z_{2k}),$$

from which the theorem follows. □

Let us end this paper by a remark on the restriction in Theorem 4.4 that the domain of f must be the boundary of a strictly pseudo-convex domain in a Stein manifold. The condition on a manifold M to be a Stein manifold of complex dimension n implies that M has the same homotopy type as an n-dimensional CW-complex, since the embedding theorem for Stein manifolds, see for instance [12], implies that a Stein manifold of complex dimension n can be embedded in $\mathbb{C}^{2n+1}$, and by Theorem 7.2 of [18] an n-dimensional complex submanifold of complex Euclidean space has the same homotopy type as a CW-complex of dimension n.

Conversely, if X is a real analytic manifold, then, for any choice of metric on X, the co-sphere bundle S^*X is diffeomorphic to the boundary of a strictly pseudo-convex domain in a Stein manifold; see for instance Proposition 4.3 of [14] or Chapter V.5 of [12]. So the degree of f coincides with the mapping $H^{2n-1}_{dR}(S^*Y)\to H^{2n-1}_{dR}(S^*X)$ that f induces under the Thom isomorphism $H^n_{dR}(X)\cong H^{2n-1}(S^*X)$. Thus the degree of a function $f:X\to Y$ can be expressed using our methods for any real analytic X.

Acknowledgments

The author wishes to thank Grigori Rozenblum and Mats Andersson for valuable discussions.

References

[1] D. Auckly, L. Kapitanski, Holonomy and Skyrme's model, Comm. Math. Phys. 240 (1–2) (2003) 97–122.
[2] P. Baum, R.G Douglas, K-homology and index theory, in: Operator Algebras and Applications, Part I (Kingston, Ont., 1980), in: Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc, Providence, R.I, 1982, pp. 117–173.

[3] D. Bekolle, C.A. Berger, L.A. Coburn, K.H. Zhu, BMO in the Bergman metric on bounded symmetric domains, J. Funct. Anal. 93 (2) (1990) 310–350.
[4] A. Benedek, R. Panzone, The spaces L^P, with mixed norm, Duke Math. J. 28 (3) (1961) 301–324.
[5] B. Blackadar, K-Theory for Operator Algebras, second ed., in: Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998.
[6] L. Boutet de Monvel, On the index of Toeplitz operators of several complex variables, Invent. Math. 50 (3) (1978/79) 249–272.
[7] H. Brezis, L. Li, Topology and Sobolev spaces, J. Funct. Anal. 183 (2) (2001) 321–369.
[8] H. Brezis, L. Nirenberg, Degree theory and BMO. I. Compact manifolds without boundaries, Selecta Math. (N.S.) 1 (2) (1995) 197–263.
[9] A. Connes, Noncommutative Geometry, Academic Press, Inc., San Diego, CA, 1994.
[10] A. Connes, Noncommutative differential geometry, Inst. Hautes Etudes Sci. Publ. Math. 62 (1985) 257–360.
[11] J.E. Fornaess, Embedding strictly pseudoconvex domains in convex domains, Amer. J. Math. 98 (2) (1976) 529–569.
[12] H. Grauert, R. Remmert, Theory of Stein spaces, in: Classics in Mathematics, Springer-Verlag, Berlin, 2004, Translated from the German by Alan Huckleberry. Reprint of the 1979 translation.
[13] E. Guentner, N. Higson, A note on Toeplitz operators, Internat. J. Math. 7 (4) (1996) 501–513.
[14] V. Guillemin, Toeplitz operators in n dimensions, Integral Equations Operator Theory 7 (2) (1984) 145–205.
[15] G.M. Henkin, Integral representation of functions in strongly pseudoconvex regions, and applications to the $\bar{\partial}$-problem, Mat. Sb. (N.S.) 82 (124) (1970) 300–308.
[16] G.M. Henkin, J. Leiterer, Global integral formulas for solving the $\bar{\partial}$-equation on Stein manifolds, Ann. Polon. Math. 39 (1981) 93–116.
[17] N. Kerzman, E.M. Stein, The Szegö kernel in terms of Cauchy-Fantappie kernels, Duke Math. J. 45 (2) (1978) 197–224.
[18] J. Milnor, Morse theory, in: M. Spivak, R. Wells. (Eds.), Based on Lecture Notes, in: Annals of Mathematics Studies, vol. 51, Princeton University Press, Princeton, N.J, 1963, vi+153 pp.
[19] D. O'Regan, Y.J. Cho, Y-Q. Chen, Topological Degree Theory and Applications, in: Series in Mathematical Analysis and Applications, vol. 10, Chapman & Hall/CRC, Boca Raton, FL, 2006.
[20] E. Ramirez de Arellano, Ein divisionsproblem und randintegraldarstellungen in der komplexen analysis, Math. Ann. 184 (1969/1970) 172–187.
[21] R.M. Range, Holomorphic Functions and Integral Representations in Several Complex Variables, in: Graduate Texts in Mathematics, vol. 108, Springer-Verlag, New York, 1986.
[22] G. Rozenblum, On some analytical index formulas related to operator-valued symbols, Electron. J. Differential Equations 17 (2002).
[23] G. Rozenblum, Regularisation of secondary characteristic classes and unusual index formulas for operator-valued symbols, in: Nonlinear Hyperbolic Equations, Spectral Theory, and Wavelet Transformations, in: Oper. Theory Adv. Appl., vol. 145, Birkhäuser, Basel, 2003, pp. 419–437.
[24] B. Russo, On the Hausdorff–Young theorem for integral operators, Pacific J. Math.68 (1) 197.
[25] B. Simon, Trace Ideals and Their Applications, second ed., in: Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, Providence, RI, 2005.
[26] W. Zhang, Lectures on Chern-Weil theory and Witten deformations, in: Nankai Tracts in Mathematics, vol. 4, World Scientific Publishing Co., Inc, River Edge, NJ, 2001, xii+117 pp.