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Coherent multiple Andreev reflections and current resonances in SNS quantum point contacts

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We study coherent multiple Andreev reflections in ballistic superconductor-normal conductor-superconductor junctions with a quantum point contact in the normal region of the junction (superconductor-normal region-quantum point contact-normal region-superconductor) with arbitrary transparency. The presence of superconducting bound states in these junctions gives rise to great enhancement of the subgap current. The effect is most pronounced in low-transparency junctions, \( D \ll 1 \), and in the interval of applied voltage \( \Delta / 2 < eV < 2\Delta \), where the amplitude of the current structures is proportional to the first power of the junction transparency \( D \). The resonant current structures consist of steps and oscillations of the two-particle current and also of multiparticle resonance peaks. The positions of the two-particle current structures have a pronounced temperature dependence, which scales with \( \Delta(T) \), while the positions of the multiparticle resonances have a weak temperature dependence, being mostly determined by the junction geometry. Despite the large, resonant two-particle current, the excess current at large voltage is small and proportional to \( D^2 \).

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I. INTRODUCTION

Transport properties of small conducting structures are strongly influenced by size effects. Oscillation of magnetoresistance in thin metallic films, and quantization of conductance in narrow wires and point contacts are examples of such effects. Size effects in superconducting tunneling have attracted attention since early experiments by Tomasch. In these experiments, oscillations of the tunnel conductance as a function of applied voltage were found for tunneling from a superconductor to a thin superconducting film of a normal conductor-superconductor (NS) proximity bilayer. The geometric resonance nature of the effect was clearly indicated by the dependence of the period of oscillations on the thickness of the superconducting film. Similar conductance oscillations for tunneling into a normal metal film of NS bilayers were reported by Rowell and McMillan. Later on, an even more pronounced effect—steps on the current-voltage characteristics of superconductor-insulator-normal conductor-superconductor (SINS) junctions at applied subgap voltages, \( eV \ll 2\Delta \)—was observed by Rowell for a review see Ref. 4). In addition to the dependence on the thickness of the N film, the period of the current steps also shows temperature dependence, which scales with the temperature dependence of the superconducting gap \( \Delta(T) \). The current steps occur at applied subgap voltages, \( eV \ll 2\Delta \), and they are understood as resonant features due to quasiparticle tunneling through superconducting bound states existing in insulator-normal conductor-superconductor (INS) wells at energies lying within the superconducting gap, \( |E| < \Delta \), de Gennes–Saint-James levels. In the superconducting normal conductor-superconductor junctions, the situation is more complex: in mesoscopic regime when inelastic relaxation plays secondary role, the quasiparticles may undergo multiple Andreev reflections (MAR) before they escape into the reservoirs. Moreover, in the presence of the ac Josephson current, the Andreev reflections are highly coherent. In a number of recent experiments with ballistic SNS devices fabricated with high mobility two-dimensional electron gas (2DEG) the coherent MAR transport regime has been realized. A theory of coherent MAR has been developed earlier for short superconducting junctions, \( L \ll \xi_0 \), where superconducting bound states do not play any significant role. Such a theory is consistent with the physical situation in atomic-size superconducting point contacts. In 2DEG devices the separation of the superconducting electrodes \( L \) is typically larger than 200 nm, which is of the same order of magnitude as the superconducting coherence length, \( \xi_0 = \hbar v_F / \Delta \) (\( v_F \) is the Fermi velocity of the 2D electrons), and superconducting bound states are formed well inside the energy gap. The presence of bound states in the junctions of finite length gives rise to resonances in the MAR transport, which dramatically affects the subgap current.

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transmission coefficient typically exceeding a value 0.75, and the number of conducting modes in the 2DEG channel is controlled by a split gate.

Under these conditions, electrons ballistically move from one electrode to the other while occasionally being scattered by rare impurities or junction interfaces. Under a voltage bias applied to the junction, the transport regime corresponds to fully coherent MAR. To calculate the dc current we will apply the scattering-theory approach generalized for superconducting junctions; see Refs. 26 and 31 and references therein.

The normal electron propagation through the junction is generally described by the N-channel scattering matrix. By assuming the split gate to select only one transport mode, we will characterize the transport through this mode by the energy-dependent transmission amplitude $d(E)$ and reflection amplitudes $r(E)$ and $r'(E)$ (the energy $E$ is counted from the Fermi energy). The scattering amplitudes satisfy the unitarity relations $d^* r + d r^* r' = 0$, $|d|^2 + |r|^2 = 1$. The energy dispersion of the scattering amplitudes will introduce the normal-electron (Breit-Wigner) and superconducting (Andreev) resonances in the scattering problem. The effect of narrow Breit-Wigner resonances on coherent MAR was earlier studied by Johansson et al. and Levi Yeyati et al. Here we will focus on the effect of Andreev resonances and only consider Breit-Wigner resonances, which are wide on the scale of the energy gap. This will allow us to neglect the energy dispersion of the junction transparency, $D = |d|^2 \approx \text{const.}$ However, the scattering phases may depend on the energy, which yields the Andreev resonances. In the simplest case, this dependence is a linear function within the energy interval $|E| \sim \Delta$, and we will write it in the form

\[
d(E) = d_0 e^{iaE}, \quad r(E) = r_0 e^{ibE},
\]

where $a, b$ are constant. In this case, the scattering properties of the normal channel are similar to those of a 1D NIN junction. Indeed, the corresponding 1D transfer matrix,

\[
\hat{T}(E) = \begin{pmatrix}
e^{-iaE/d_0} & r_0^* e^{-i(a-b)E/d_0} \\
r_0 d_0 & e^{iaE/d_0}
\end{pmatrix},
\]

can be decomposed into a product of three transfer matrices,

\[
\hat{T}(E) = \exp(-i \sigma_x L_\tau E/\Delta \xi_0) \hat{T}(0) \exp(-i \sigma_x L_\tau E/\Delta \xi_0) \approx e^{-i \sigma_x k(E) L_\tau} \hat{T}(0) e^{i \sigma_x k(E) L_\tau},
\]

where $\sigma_x$ is a Pauli matrix. The first and the last matrices describe ballistic propagation of an electron, with wave vector $k(E) = \sqrt{2m(E_F + E)/\hbar} \approx k_F + E/\Delta \xi_0$, through the right and left $N$ regions of an effective junction with lengths $L_r = b \Delta \xi_0/2$ and $L_L = (a - b/2) \Delta \xi_0$, respectively (from right to left), and the matrix $\hat{T} = e^{i \sigma_x k_F L_\tau} \hat{T}(0) e^{-i \sigma_x k_F L_\tau}$ describes an effective barrier ($I$).

Quasiparticle propagation through the effective 1D superconductor-normal conductor-insulator-normal conductor-superconductor junction is described by means of the time-dependent Bogoliubov–de Gennes equation.
Since the time dependencies of the wave functions in the two reservoirs are different, the quasiparticle scattering by the junction is inelastic and one has to consider a superposition of plane waves with different energies in order to construct scattering states.

III. CALCULATION OF CURRENT USING SCATTERING STATES

A. Recursion relations for MAR amplitudes

We will now proceed with the construction of recurrences for the scattering amplitudes following the method suggested by Johansson et al.\(^4\) To this end we introduce the wave functions in the left/right normal region (\(ll/r\)) of the junction with respect to the position of the impurity. A particular scattering state, labeled with the energy \(E\) of the incoming quasiparticle, will consist of a superposition of plane waves with energies \(E_n = E + neV\), where \(n\) is an integer, \(-\infty < n < \infty\),

\[
\Psi_j(E) = \sum_{n=-\infty}^{\infty} \exp \left[ -i(E_n + \sigma_z eV/2)t\hbar \right] \begin{pmatrix}
c_n^1r e^{ik_n^1e x} + c_n^1l e^{-ik_n^1e x} \\
c_n^1l e^{-ik_n^1e x} + c_n^1r e^{ik_n^1e x}
\end{pmatrix}.
\]

The normal electron/hole wave vector \(k_n^{e,h}\) is defined here as \(k_n^{e,h} = k(\pm E_n)\), \(k(E) = \sqrt{2m(E_F + E)/\hbar}\). The meaning of the labels for the scattering (MAR) amplitudes \(c_n\) will be explained below.

Continuity of the scattering-state wave function across the left and right NS interfaces determines the relation between the electron and hole amplitudes in the vicinity of each interface,

\[
e^{-i}\begin{pmatrix}
c_n^1r e^{ik_n^1e x} + c_n^1l e^{-ik_n^1e x} \\
c_n^1l e^{-ik_n^1e x} + c_n^1r e^{ik_n^1e x}
\end{pmatrix} = a_n e^{i\sigma_z E}\begin{pmatrix}
c_n^{-1}r e^{ik_n^-e x} + c_n^{-1}l e^{-ik_n^-e x} \\
c_n^{-1}l e^{-ik_n^-e x} + c_n^{-1}r e^{ik_n^-e x}
\end{pmatrix}, \quad \sigma_z = a(E_n), \quad \text{for } n \neq 0,
\]

which describes elastic Andreev reflection (\(l/r\) indices are omitted). It is convenient to consider scattering amplitudes near the impurity (at \(x = \pm 0\)) rather than at the NS interfaces and to rewrite Eq. (8) for such amplitudes, combining the amplitudes of the ballistic propagation through the normal regions with the Andreev reflection amplitude. Then, in vector notation

\[
\begin{pmatrix}
(c_n^1r e^{ik_n^1e x} + c_n^1l e^{-ik_n^1e x})/\Delta \\
(c_n^1l e^{-ik_n^1e x} + c_n^1r e^{ik_n^1e x})/\Delta
\end{pmatrix} = a_n \begin{pmatrix}
c_n^{-1}r e^{ik_n^-e x} + c_n^{-1}l e^{-ik_n^-e x} \\
c_n^{-1}l e^{-ik_n^-e x} + c_n^{-1}r e^{ik_n^-e x}
\end{pmatrix},
\]

\(\Delta = \sqrt{E^2 - E_F^2}\).

In this equation, the \(\pm\) signs in the time-dependent factors refer to the left/right electrode, \(k_{e,h}^{l,r}(E)\) is the wave vector of electronlike/holelike quasiparticles, and \(u(E), v(E)\) are the Bogoliubov amplitudes. The ratio of the Bogoliubov amplitudes equals the amplitude of Andreev reflection for particles incoming from the neighboring normal region,

\[
u/a = \frac{u}{v} = a(E) = \begin{cases}
\frac{(E - sgn(E)\sqrt{E^2 - \Delta^2})}{\Delta}, & |E| > \Delta \\
\frac{(E - i\sqrt{\Delta^2 - E^2})}{\Delta}, & |E| < \Delta.
\end{cases}
\]
The modified relation (8) takes the form

\[ \hat{c}_{n^+} = \hat{U}_n \hat{c}_{n^-}, \quad n \neq 0, \]

where

\[ \hat{U}_n = \exp \left( i \sigma \varepsilon_n L_{1,r}/\Delta \xi_0 \right) \begin{pmatrix} a_n & 0 \\ 0 & a_n^{-1} \end{pmatrix} \exp \left( i \sigma \varepsilon_n L_{1,r}/\Delta \xi_0 \right) = e^{i \sigma \varepsilon_n \phi_n}. \]

The phase \( \phi_n = 2 E_n L_{1,r}/\Delta \xi_0 - \arccos(E_n/\Delta) \), characterizing \( \hat{U}_n \), is real inside the energy gap, \( |E_n| \leq \Delta \), where it describes the total energy-dependent phase shift due to ballistic propagation and Andreev reflection. Outside the gap, \( |E_n| \geq \Delta \), the phase \( \phi_n \) has a imaginary part that describes leakage into the superconducting reservoirs due to incomplete Andreev reflections.

By matching harmonics with the same time dependence in Eq. (7), we derive a relation between scattering amplitudes at the left and the right side of the barrier,

\[ \hat{c}_{l(n+1)} = \hat{T}_l \hat{c}_{l+n}, \quad \hat{c}_{r(n+1)} = \hat{T}_r^{-1} \hat{c}_{r+n}, \]

where the effective barrier transfer matrix \( \hat{T} \) is defined in Eqs. (2) and (3).

The recursion relations in Eqs. (10) and (12) couple the scattering amplitudes \( \hat{c}_{l(n)} \) into an infinitely large equation system. This equation system describing coherent MAR is illustrated by the MAR diagram in Fig. 3.

The electron part of the quasiparticle injected at the left NS interface propagates upwards along the energy axis, the amplitudes for this propagation being labeled with \( c_{l+} \). At the injection energy \( E = E_0 \) (amplitude \( c_{l+0} \)), the quasiparticle is accelerated across the barrier \( (l) \), where the potential drops. Thus, it enters the right normal part of the junction having been accelerated to energy \( E_2 \) (\( c_{l+} \)), and is then again converted into an electron (\( c_{r+} \)). The \( \pm \) indices label the amplitudes after \( (+) \) and before \( (-) \) the Andreev reflection for propagation upwards along the energy axis. There is a similar trajectory of injected holes, which descends in energy, with the MAR amplitudes labeled with \( c_{l-} \).

Due to electron back scattering at the barrier, the upward and downward propagating waves are mixed, e.g., \( c_{l+0} \) being not only forward scattered into \( c_{l+} \), but also back scattered into \( c_{l-} \), which opens up the possibility of interference.

Injection from the left reservoir, shown in Fig. 3, generates a MAR path, which only connects even side bands at the left side of the junction with odd side bands at the right side. Injection from the right reservoir will generate a different MAR path, with even side bands at the right side of the junction, i.e., the diagram in Fig. 3 will effectively be mirrored around the barrier \( (l) \). Thus, there are two independent equation systems for the MAR amplitudes: injection from the left and from right. The \( l, r \) labels in the MAR amplitudes can then be omitted since they are uniquely defined by the source term and the side-band index.

The transport along the energy axis generated by MAR, from energy \( E \) to \( E_n \), is conveniently described by the effective transfer matrix \( \hat{M}_{n0} \),

\[ \hat{c}_{n} = \hat{M}_{n0} \hat{c}_{0}, \quad n > 0; \quad \hat{c}_{n} = [\hat{M}_{0n}]^{-1} \hat{c}_{0}, \quad n < 0, \]

where \( \hat{T}_{l(k)} = \hat{T}^{-1} \) and \( \hat{T}_{2k+1} = \hat{T} \) for the injection from the left (for the injection from the right, the even and odd side-band indices are interchanged). For paths within the superconducting gap, \( |E_n| < \Delta \), the matrix \( \hat{M}_{n0} \) satisfies the standard transfer-matrix equation, \( \hat{M}_{nm} \sigma_z \hat{M}_{nm}^{-1} = \sigma_z \), which provides conservation of probability current along the energy axis,

\[ j_{n=0}^p = \hat{c}_{l+n}^* \sigma_z \hat{c}_{l+n}, \quad j_{n=0}^p = \hat{j}_{r+n}^* \sigma_z \hat{j}_{r+n}, \quad |E_n| < \Delta. \]

An important consequence of the coherence of MAR is the possibility of transmission resonances in energy space. From the form of the \( \hat{M} \) matrix,

\[ \hat{M}_{n0} = \ldots \hat{T}^{-1} e^{i \sigma \varepsilon_i \hat{T}} \ldots, \]

it is evident that when \( \varepsilon_i = m \pi \), the two matrices \( \hat{T}^{-1} \) and \( \hat{T} \) will cancel each other and the probability of transmission through this part will be unity, which leads to resonant enhancement of MAR. The solutions \( E^{(m)} \) of the resonance equation

\[ \varepsilon_i^{(m)} = \varepsilon_i - m \pi = 2 E_0 L_{1,r}/\Delta \xi_0 - \arccos(E_k/\Delta) - m \pi = 0 \]

coincide with the spectrum of the de Gennes–Saint-James levels localized in INS quantum wells. The corresponding bound states are located either on the left or the right side of the junction.
Without loss of generality, the calculations can be performed with a real matrix $\hat{T}$. The transformation to such a real matrix is given by $\hat{T} \rightarrow \hat{V}_1 \hat{T} \hat{V}_2$ with diagonal unitary matrices $\hat{V}_1, \hat{V}_2$ whose elements are constructed with the scattering phases, which are energy-independent. It is clear from Eq. (16) that since these energy-independent matrices commute with the matrices $\hat{U}_n$, they cancel each other and the matrix $\hat{M}_{n0}$ undergoes a similar transformation. This will lead to an overall phase shift of the scattering state, which does not affect the current.

It is interesting to consider the special case of fully transparent junctions, $D = 1$, which has been studied in the literature. In this case, all matrices $\hat{V}_n$ in Eq. (14) are equal to the unity matrix, and the $\hat{M}$ matrix takes the simple form

$$\hat{M}_{n0} = \exp\left(\frac{\imath\pi}{N} - \frac{\imath\pi}{\Delta E}ight).$$

The length of the junction then enters only through the phase of the MAR amplitudes, which drops out of the side-band current. Thus the dc current of fully transparent SNS junctions is independent of length and equal to the current in quantum constrictions. In particular, at zero temperature this current does not show any structure in the subgap current. It is also worth mentioning that in this particular case of fully transparent SNS junctions, the $\hat{M}$ matrix is diagonal and therefore a closed set of recursive relations to a finite set by representing the MAR process above $E_n$ and below $E_0$ by boundary conditions involving reflection amplitudes $r_n^+$ and $r_n^-$, defined as $c_n^+ = c_n^+ r_n^+$ and $c_n^- = c_n^- r_n^-$. This gives the following representation for the vectors in Eq. (9):

$$\hat{c}_n^+ = c_n^+ \begin{pmatrix} 1 \\ r_n^+ \end{pmatrix}, \quad \hat{c}_n^- = c_n^- \begin{pmatrix} r_n^- \\ 1 \end{pmatrix}. \quad (20)$$

The reflection amplitudes $r_n^+$ and $r_n^-$ are independent of the injection, in contrast to the coefficients $c_n^+, c_n^-$. Furthermore, they are determined by the boundary conditions $\hat{c}_\pm = 0$ and can be expressed in terms of the matrix elements of $\hat{M}_n$ and $\hat{M}_{0(-N)}$, where $N \rightarrow \infty$,

$$\lim_{N \rightarrow \infty} \hat{M}_n \begin{pmatrix} 1 \\ r_n^+ \end{pmatrix} = 0, \quad \lim_{N \rightarrow \infty} [\hat{M}_{0(-N)}]^{-1} \begin{pmatrix} r_n^- \\ 1 \end{pmatrix} = 0. \quad (21)$$

In other words, the vectors in Eq. (21) are equal to the asymptotical values of the eigenvectors of $\hat{M}$ matrices corresponding to the eigenvalues, which decrease when $N$ goes to infinity. The advantage of introducing the reflection amplitudes $r_n^+$ and $r_n^-$ is that although they have to be calculated numerically, the recurrences that they obey do not contain resonances, and converge rather quickly. This is in contrast to the matrix $\hat{M}_{n0}$, which does possess resonances, but which can be calculated analytically in a straightforward way for any given $n$.

The solutions of the recursion equations (10) and (12) can now be explicitly written down. For any given energy $E$ we get four different sets of solutions for four scattering states including electron/hole injection from the left and the right. Using the formal expression in Eq. (20) and the matrix elements of $M_{n0} = (m_{11}^1 m_{12}^1 m_{21}^2 m_{22}^2)$, the solutions for injection from the left ($n > 0$) have the form

$$\hat{c}_n^+ = \frac{u(1-a_0^2)e^{-\imath\theta_0}[\delta_{ee} + a_0 r_0^- \delta_{hh}]}{m_{22}^2 + m_{21}^2 r_0^- e^{2\imath\theta_0} - m_{12}^1 r_0^+ e^{2\imath\theta_0} - m_{11}^1 r_0^- r_n^+ e^{2\imath\theta_0} + 2\imath\varepsilon_\theta} \begin{pmatrix} 1 \\ r_n^+ \end{pmatrix}. \quad (22)$$
The solutions for injection from the right can be found by interchanging $e \leftrightarrow h$ and calculating all quantities with respect to injection from the right. The solutions for $n < 0$ are calculated in a similar manner.

C. Calculation of current

Now turning our attention to the current, we calculate it in the normal region next to the barrier, using the wave function in this region, $\Psi$, and assuming quasiparticle equilibrium within the electrodes. The current then takes the form

$$ I(t) = \frac{e}{\hbar k_F} \int_{-\infty}^{\infty} dE (u^2 - v^2)^{-1} \sum_{e,h,l,r} \psi \left( \frac{\partial}{\partial x} \Psi \right) \tanh \left( \frac{|E|}{2k_B T} \right), $$

(23)

where $(u^2 - v^2)^{-1} = |E|/\sqrt{E^2 - \Delta^2} = |E|/\xi$ is the superconducting density of states, and the sum is over the four scattering states at a given energy $E$ associated with the electron-like and hole-like quasiparticles ($e/h$) injected from the left and right ($l/r$). The current can be divided into parts with different time dependence and expressed as a sum over harmonics,

$$ I(t) = \sum_{N} I_N e^{i2\pi Nvt/\hbar}. $$

(24)

Focusing on the dc ($N=0$) component and calculating the contribution of each scattering state at the injection side of the junction, we express the current spectral density $J(E)$ through the probability currents of electrons and holes at energies $E_{2n}$ (Fig. 3),

$$ I_{dc} = \frac{e}{\hbar} \int_{-\infty}^{\Delta} dE J(E), $$

$$ J(E) = \sum_{e,h,l,r} \frac{|E|}{\xi} \sum_{n=-\infty}^{\infty} (\tilde{\psi}_{2n}^+ - \tilde{\psi}_{2n}^- + \tilde{\psi}_{2n}^+ - \tilde{\psi}_{2n}^-). $$

(25)

These currents coincide with the probability currents $j_n^p$, Eq. (15), flowing along the energy axis.

It is convenient to introduce a leakage current $J_n$, defined as the difference of the probability currents before and after Andreev reflection,

$$ J_n = \sum_{e,h,l,r} \frac{|E|}{\xi} (j_n^- - j_n^+). $$

(26)

$J_n$ represents the amount of probability current from all the scattering states injected at energy $E$ and leaking out of the junction at energy $E_n$ (Fig. 3). The leakage current is zero inside the energy gap due to complete Andreev reflection, $J_n = 0$ for $|E_n| < \Delta$ [cf. Eq. (15)].

The explicit expression for the leakage current for $n \neq 0$ follows from Eq. (26) after insertion of Eqs. (22) and (10),

$$ J_n(E) = \sum_{l,r} \frac{1}{m_{22} + e^{2i\varphi_0 r_0 - m_{22}} - e^{2i\varphi_0 r_n + m_{12} - e^{2i\varphi_0 r_0 - m_{12}}} |a_n|^2 |E| \sum_{k=-n}^{\infty} J_k, \quad n > 0, $$

$$ J_{-n}(E) = \sum_{l,r} \frac{1}{m_{22} + e^{2i\varphi_0 r_0 - m_{22}} - e^{2i\varphi_0 r_n + m_{12} - e^{2i\varphi_0 r_0 - m_{12}}} |a_n|^2 |E| \sum_{k=-n}^{\infty} J_k, \quad n < 0, $$

by adding and subtracting consecutive terms in the sum. The spectral density of the dc charge current Eq. (25) can then be written in the form

$$ J(E) = \sum_n nJ_n(E), $$

(30)

since $J_n$ appears in $n$ probability currents. This formula has a clear physical meaning: the contribution to the charge current of the $n$th side band is proportional to the leakage current of the side band times the effective transferred charge $n e$.

The detailed balance of the leakage currents, Eq. (28), allows us explicitly to prove that at zero temperature the scattering processes between (occupied) states with negative
energies, \( E, E_n \leq -\Delta \) do not contribute to the current, in agreement with the Pauli exclusion principle. Indeed, by separating the contributions from side bands with \( n < 0 \) and remembering that the leakage current is zero within the gap, we get for zero temperature,

\[
I_{dc} = \frac{e}{h} \int_{-\Delta}^{-\Delta - eV} dE \sum_{n \neq 0} n J_n(E) = \sum_{n \neq 0} \frac{ne}{h} \left[ \int_{-\Delta}^{-\Delta - eV} dE J_n(E) \right] + \int_{-\Delta - eV}^{-\Delta} dE J_n(E) - \int_{-\Delta}^{-\Delta - eV} dE J_n(E) ,
\]

where the first and the third terms cancel each other by virtue of Eq. (28). At finite temperature, these two terms produce current of thermal excitations while the second term gives the current of real excitations created by the voltage source. Keeping only this term, which dominates at low temperature, we finally get

\[
I_{dc} = \sum_{n > 0} \int_{-\Delta - neV}^{-\Delta} dE J_n(E) \tanh(|E|/2k_B T).
\]

We end this section by noting a technically useful symmetry in the current density, namely, \( J_n(E) = J_n(-E - neV) \), seen from the explicit form of the \( \hat{M}_{n0} \) matrix. This allows us to reduce the integration interval in Eq. (32) to \(-neV/2 < E < -\Delta\).

### IV. CURRENT IN TERMS OF n-PARTICLE PROCESSES

The approach formulated above provides necessary foundations for numerical calculation of the current for arbitrary transparency and length. However, to get a full understanding of the rich subgap structure in the current-voltage characteristics, which may seem quite random, especially for intermediate transparencies and lengths (see Figs. 14–16), we will conduct a detailed analytical study of the limit of low transparency \( D \ll 1 \). The separation of currents into \( n \)-particle currents, Eq. (32), is our basis for analysis and we will study each current \( I_n \) separately.

As explained in the previous section, the de Gennes–Saint-James levels, Eqs. (16) and (17), are important for the current transport through the junction leading to resonant enhancement of the current. Our main attention in this and the next section is on the calculation of the position, height, and width of the main current peaks and oscillations that have the magnitude of order \( D \). To simplify notations, left/right injection indices are omitted in most cases.

#### A. Single-particle current

The single-particle current, which dominates at large applied voltages, has, according to Eq. (32), an onset at \( eV = 2\Delta \). The full numerical solution for the single-particle current is plotted in Fig. 4. The current shows pronounced oscillations and the magnitude of the slope at the current onset strongly depends on the junction length.

![FIG. 4. Single-particle current for symmetric junctions \( L_l = L_r = L/2 \) for different junction lengths; the junction transparency is \( D = 0.1 \). The current onset for the short junction (\( L = 0 \)) disappears for junctions with finite length (bold line); for \( L = n\pi\xi_0 \), the onset appears being roughly \( n + 1 \) times smaller than the onset for \( L = 0 \).](image-url)
From this formula it is clear that for short junctions, $L < \xi_0$ (equal to the DOS in a superconductor), is suppressed for finite-length junctions. The amplitude of the first oscillation increases as the length increases, indicating accumulation of the spectral weight at the energy-gap edge and formation of a bound state for $L = \pi \xi_0$.

Voltage, as well as the integration interval, and therefore the DOS oscillations produce oscillations of the current $I$, as a function of voltage (Rowell-McMillan oscillations). The oscillations become more pronounced when the junction is sufficiently long and the differential conductance may even oscillate behavior. This can be directly related to the smearing of the singularities in the DOS at the gap edge. The length where the crossover between these two behaviors occurs can be taken as a measure of when finite-length effects become important. To estimate this length, we write Eq. (27) for small lengths $L < \xi_0$, near the threshold, $eV = 2\Delta + \Omega$, $\Omega \ll \Delta$, keeping the first-order terms in $D$ in the denominator. For a symmetric junction, $L_i = L_f = L/2$, we get

$$I_1 = \frac{e \Delta \tanh(\Delta/k_B T)}{h} \int_0^\pi \frac{D \sin^2 \theta \, d\theta}{\left(\sin \theta + \frac{D \Delta}{4 \Omega} \right)^2 + \frac{L^2 \Delta}{\xi_0^2 \Omega} \left(1 + \frac{D \Delta}{4 \Omega}\right)}.$$  (36)

From this formula it is clear that for short junctions ($L = 0$), the current onset has the width $\Omega \sim \Delta D/4$. If $L$ is of the order of $\xi_0 \sqrt{\Delta D/2}$, the size of the onset has substantially diminished and there is no visible onset at $eV = 2\Delta$ when $L \gg \xi_0 \sqrt{\Delta D/2}$. This crossover between steep onset and smooth behavior, which happens already for quite short lengths if $D$ is small, can be interpreted in terms of a bound state, which is situated exactly at the gap edge in short junctions ($L = 0$), and which moves down into the gap when $L > 0$, the effect becoming fully pronounced when the distance from the gap edge, $h v_F / L$, exceeds the dispersion of the Andreev state, $\sqrt{\Delta D}$, in symmetric junctions.

When $L_i > \pi \xi_0 / 2$, the lowest quasibound state in the continuum spectrum approaches the gap edge. This leads to an accumulation of the spectral weight at the gap edge and reappearance of the singularity in the DOS, which results in the reappearance of a sharp current onset at $eV = 2\Delta$, but with smaller magnitude; see Fig. 4 ($L_i = \pi \xi_0$).

It is of interest to note that in our calculations, based on the scattering-theory approach, the bound states are not directly involved in the single-particle transport, which therefore is nonresonant and shows no subgap resonance peaks. Within the tunnel-model approach the situation is qualitatively different: the DOS in Eq. (34) usually includes the contribution of the broadened bound states, and therefore the single-particle current exists and has pronounced resonant features at subgap voltages $eV < 2\Delta$. This difference results from the fact that, within the tunnel-model approach, the superconducting bound states are implicitly assumed to be connected to the reservoirs (broadening due to inelastic interaction), which allows a stationary current to flow through the bound states. In contrast, within the scattering approach, the bound states are disconnected from the reservoirs and have zero intrinsic width. In this case the bound states obtain their width only due to higher-order tunneling processes involving Andreev reflections, which are manifested by the resonant multiparticle currents. In practice, the relevance of the multiparticle versus single-particle mechanism of the subgap current transport is determined by physics and depends on the ratio of the corresponding dwelling and relaxation times. In this paper, the inelastic relaxation time $\tau_1$, which determines the width of the single-particle resonances, is assumed to be much larger than the dwelling time of the most important two-particle current, $\tau_1 \gg h v_F / L D$.

### B. Two-particle current

The two-particle current $I_2$ in quantum point contacts ($L \ll \xi_0$) is of order $D^2$ when $eV < 2\Delta$ and of order $D^3 \ln D$ when $eV > 2\Delta$ (Ref. 26). For finite-length junctions, the situation is different. For the MAR paths where the energy of the Andreev reflection coincides with a bound state, the current spectral density $j_2^{\rho\sigma}$ is of the order of unity, due to resonant transmission through this state. For low transparency $D \ll 1$, this gives a sharp concentration of the current density around the resonant energies. In this limit, the two-particle current is well described by the sum of contributions from these resonances, and to evaluate them we examine the energy dependence of $j_2$ close to the resonant energies, $E_1 = \varepsilon^{(m)} + \delta \varepsilon$. Let us consider the contribution to the leakage current $[J_2]^{\rho\sigma}$ from quasiparticles injected from the left. As shown in Appendix B, in this case Eq. (27) reduces to the standard Breit-Wigner resonance form

$$[J_2]^{\rho\sigma} = \Gamma_0^{(m)} \varepsilon_n^{(m)} \frac{\left(\frac{\Delta}{\delta \varepsilon - \delta \varepsilon_{(m)}}\right)^2 + \left(\frac{1}{\Gamma_0^{(m)} + \varepsilon_n^{(m)}}\right)^2}{\frac{2}{\Delta}},$$  (37)

where the tunneling rates $\Gamma_n^{(m)}$ are given by

$$\Gamma_n^{(m)} = N_n (E_n) D/2 \eta_n^{(m)}, \quad n = 0, 2,$$  

and
gies, where the summation is over the positive bound-level energies, and the position of the resonance is shifted by 

$$\eta^{(m)} = \Delta \frac{\partial \varphi}{\partial E_{E_{i}E_{j}}^{(m)}} = \frac{2L_{r}}{\xi_{0}} \frac{\Delta}{\sqrt{\Delta^2 - (E^{(m)})^2}}.$$  

(38)

and the position of the resonance is shifted by 

$$\delta E^{(m)} = \frac{D \Delta}{4 \eta^{(m)}} \text{Im} \left[ \frac{1 + e^{2i\varphi_{0}}}{1 - e^{2i\varphi_{0}}} + \frac{1 + e^{2i\varphi_{2}}}{1 - e^{2i\varphi_{2}}} \right].$$  

(39)

An analogous result is valid for quasiparticles injected from the right.

After integrating over energy, the two-particle current in the resonance approximation may be written in the form 

$$I_{2}(eV) = \sum_{i=r,t} \sum_{m \neq 0} \frac{2e}{h} \theta(eV - E^{(m)} - \Delta) \frac{2\pi D \Delta}{\eta^{(m)}} N^{i}(E^{(m)} - eV)N^{i}(E^{(m)} + eV) f^{(m)}(T, V) \times$$

$$N^{i}(E^{(m)} - eV)N^{i}(E^{(m)} + eV) f^{(m)}(T, V),$$  

(40)

where the summation is over the positive bound-level energies, \(0 < E^{(m)} < \Delta\), and the DOS \(N^{f}\) should be calculated at the injection side of the junction and \(f^{(m)}(T, V) = (1/2) \times [\text{tanh}(eV - E^{(m)})/2k_{B}T] + \text{tanh}(eV + E^{(m)})/2k_{B}T\]). According to Eq. (40), the two-particle current \(I_{2}(eV)\) increases in a steplike manner in the voltage region \(\Delta < eV < 2\Delta\). The steps occur at every voltage where a new resonant channel through a bound state opens up, at \(eV^{(m)} = \Delta + E^{(m)}\). We note that the step positions depend on temperature and approximately scale with \(\Delta(T)\). Each current step has the height of order \(D\).

As seen from Eq. (40), the contribution to the current of a particular bound state \(E^{(m)}\) is modulated, as a function of voltage, by the oscillations of the density of states at the entrance and exit energies, \(N(E^{(m)} \pm eV)\). In other words, the pronounced oscillations of the two-particle current seen in Fig. 6 reflect how close the entrance and exit energies \(E^{(m)} \pm eV\) are to a quasibound state in the continuum. For \(eV > 2\Delta\), the two-particle current \(I_{2}\) oscillates around a constant value with an amplitude of oscillation decreasing as \(\Delta^{2}/eV^{2}\) for large voltages.

It is interesting to compare the resonant structures of the two-particle current with the resonant structures in NINS junctions.\(^{15,16}\) In NINS junctions, the resonant current steps occur at \(eV = E^{(m)}\), and they do not have any modulation because the DOS on the normal side of the junction is constant.

The distance between the resonances and the resonance widths are proportional to the bound-level spacing, and they decrease in long junctions. For sufficiently long junctions, the two-particle current may thus give the appearance of including a series of peaks, as shown on Fig. 7. In symmetric junctions, the bound-state energies at both sides of the barrier will coincide, reducing the number of steps by a factor of 2 and giving current steps of double height.

We will conclude this section by noting that the difference between the full numerical calculation of the two-particle current and the resonant approximation given in Eq. (40) is rather small already when \(D = 0.1\), as can be seen in Fig. 7.

### C. Excess current

Excess current in SNS junctions, i.e., the difference between the current in the superconducting junction and in the normal junction at large voltage,

$$I^{exc} = I - G_{N}V + O(\Delta/eV),$$  

(41)

is commonly considered as a measure of the intensity of Andreev reflection. In tunnel superconductor-insulator-superconductor junctions and low-transmissive point contacts the excess current is small, \(I^{exc} \approx D^{2}e\Delta/\pi\hbar\), \(D \ll 1\), while in fully transparent contacts the excess current is large, \(I^{exc} = 8e\Delta/3\pi\hbar\), \(D = 1\).\(^{26}\) Accordingly, one would expect large excess current in long SNS junctions due to the resonant enhancement of the two-particle current. However, the excess current is small because of a large deficiency, of order \(D\), of the single-particle current caused by the broadening of the current onset at the threshold. As we will show, the single-particle and two-particle currents undergo a fine cancellation, yielding small net excess current of order \(D^{2}\) when \(D \ll 1\).

The excess current has contributions only from the single- and two-particle currents, since all higher-order currents include at least one Andreev reflection outside the gap whose...
probability is of order \((\Delta/eV)^2\). In the limit of large voltage, \(eV \gg \Delta\), the relevant part of the current in Eq. (32) then takes the form

\[
I_1 = \frac{4De}{\hbar} \int_{-\Delta}^{\Delta} dE \frac{(1 - a_0^2)(1 + Ra_0^2)}{1 + R^2a_1^4 - 2R \text{Re}\{e^{2i\phi_0}\}},
\]

\[
I_2 = \frac{8D^2e}{\hbar} \int_{-eV/2}^{eV/2} dE \frac{|a_1|^2}{1 + R^2|a_1|^4 - 2R \text{Re}\{e^{2i\phi_1}\}}.
\]

These equations are written for symmetric junctions, \(L_r = L_l = L/2\), and for zero temperature; small Andreev-reflection amplitudes \(= |a(eV/2)| \ll 1\) have been neglected in Eq. (27). The behavior of the current in Eq. (42) as a function of voltage is presented in Fig. 8 for different lengths. It is clearly seen that the limiting value of the excess current is approached much faster in finite-length SNS junctions compared to point contacts (\(L = 0\)). In Fig. 9 the excess current behavior with respect to the junction length is presented for different transparencies.

To analytically examine the excess current in the limit of small transparency, \(D \ll 1\), it is convenient to start with Eqs. (34) and (40). To first order of \(D\) the excess current assumes the form \(T = 0\),

\[
I_{1}^e = I_{1}^{exc} + I_{2}^{exc},
\]

\[
I_{1}^{exc} = -\frac{4eD\Delta}{\hbar} + \frac{2eD}{\hbar} \int_{\Delta}^{\infty} [N(E) + N'(E) - 2]dE,
\]

\[
I_{2}^{exc} = \frac{\pi De\Delta}{\hbar} \sum_{m > 0} \frac{2\pi De\Delta}{\hbar} \eta^{(m)}.
\]

Let us consider the contributions to the single-particle current from the left electrode,

\[
[I_{1}^{exc}]^l = -\frac{2eD\Delta}{\hbar} + \frac{2eD}{\hbar} \int_{\Delta}^{\infty} [N'(E) - 1]dE.
\]

Inserting \(N'(E)\) from Eq. (35), this equation can be transformed to the form

\[
[I_{1}^{exc}]^l = \frac{2eD}{\hbar} \int_{\Delta}^{\infty} \left( \frac{E\xi}{\xi^2 + 2\Delta \sin^2(2\Delta/\Delta l)} - \frac{E}{\xi} \right) dE
\]

\[
= -\frac{eD}{\hbar} \int_{-\infty}^{\infty} d\xi \frac{\sin^2(2\Delta/\Delta l)}{\xi^2 + 2\Delta \sin^2(2\Delta/\Delta l)},
\]

where \(\xi = \sqrt{E^2 - \Delta^2}\). It is now possible to analytically continue the integral in the upper half plane, which will reduce the integral to a sum over the residues of the poles given by the equation \(\xi^2 + 2\Delta \sin^2(2\Delta/\Delta l)\). Comparing this equation with Eq. (17) we find that the poles coincide with the energies of the bound states in the gap. The excess current contribution from the left-injected single-particle current is thus

\[
[I_{1}^{exc}]^l = -\frac{2D\pi e\Delta}{\hbar} \sum_{m > 0} \frac{1}{\eta^{(m)}} = -[I_{2}^{exc}]^l,
\]

where \([I_{2}^{exc}]^l\) is the contribution to the two-particle current from the bound-state resonances at the left electrode. A similar relation is derived for current from the right electrode. Thus, there is exact cancellation of the excess single-particle and two-particle currents to first order in \(D\).

It is interesting to note that the cancellation effect is related to the conservation of the number of states in a proximity normal metal compared to the conventional normal metal. It follows from Eq. (44) that \(I_{1}\Delta/2eD\Delta\) is equal to the difference between the number of continuum states in the proximity metal and the total number of states in a conventional metal, while, on the other hand, the number of the bound states is equal to

\[
\int_{0}^{\Delta} dE \sum_{m > 0} \delta(\varepsilon(E) - m\pi) = \int_{0}^{\Delta} dE \sum_{m > 0} \delta(\varepsilon - E^{(m)})/\eta^{(m)} = I_{2}\Delta/2eD\Delta
\]

according to Eq. (43).

V. INTERPLAY BETWEEN RESONANCES

For processes with several Andreev reflections \((n \geq 3)\), the possibilities for resonances increase. Every Andreev-
We now expand voltage, two bound-state energies, eV energy of one of the two Andreev reflections coincides with order D

of two overlapping single resonances; see the inset in Fig. 10. The peaks are located in the voltage interval 2Δ/3 < eV < 2Δ; we note that the peak positions are weakly dependent on temperature.

To evaluate the height and the width of these peaks, we study the contribution from overlapping resonances at E1 ≈ E(k) < 0 and at E2 ≈ E(m) > 0. Close to these energies, the phases φ1(k) and φ2(m), defined in Eq. (17), are close to zero, and we find the spectral density for injection of a quasiparticle from the left (see Appendix C),

\[
[J_3]^{(km)} = \frac{D^3 N_l(E) N_r(E)}{|D - 4 \varphi_1(k) \varphi_2(m) + iD(\varphi_1(k) N_l(E_3) + \varphi_2(m) N_r(E))|^2}.
\] (48)

We now expand φ1(k), φ2(m) in the deviation from perfect overlap in energy, δE = (E1 - E(k) + E2 - E(m))/2, and in voltage, δV = V - V(km), and find, using D ≪ 1, from Eq. (48)

\[
[J_3(E)]^{(km)} = \frac{D \Gamma_0^{(km)}}{(\delta E_+ \delta E_- / \Delta^2 - D^2 / 4 \eta(k) \eta(m))^2 + \Lambda^2}.
\] (49)

A. Three-particle current

The three-particle current I3 has a nonresonant value of order D3. However, I3 is enhanced to order D2 when the energy of one of the two Andreev reflections coincides with a bound-state energy.

For the applied voltage equal to the difference between two bound-state energies, eV(km) = E(m) - E(k), two resonances occur simultaneously, i.e., form a resonance consisting of two overlapping single resonances; see the inset in Fig. 10. This will enhance the current to order D close to this voltage, giving a peak in the current-voltage characteristics (CVC). The number of peaks is equal to the number of bound-state pairs. The peaks are located in the voltage interval 2Δ/3 < eV < 2Δ; we note that the peak positions are weakly dependent on temperature.

To evaluate the height and the width of these peaks, we study the contribution from overlapping resonances at E1 ≈ E(k) < 0 and at E2 ≈ E(m) > 0. Close to these energies, the phases φ1(k) and φ2(m), defined in Eq. (17), are close to zero, and we find the spectral density for injection of a quasiparticle from the left (see Appendix C),

\[
[J_3]^{(km)} = \frac{D^3 N_l(E) N_r(E)}{|D - 4 \varphi_1(k) \varphi_2(m) + iD(\varphi_1(k) N_l(E_3) + \varphi_2(m) N_r(E))|^2}.
\] (48)

We now expand φ1(k), φ2(m) in the deviation from perfect overlap in energy, δE = (E1 - E(k) + E2 - E(m))/2, and in voltage, δV = V - V(km), and find, using D ≪ 1, from Eq. (48)

\[
[J_3(E)]^{(km)} = \frac{D \Gamma_0^{(km)}}{(\delta E_+ \delta E_- / \Delta^2 - D^2 / 4 \eta(k) \eta(m))^2 + \Lambda^2}.
\] (49)
The four-particle current has a nonresonant value of order $D^4$, which is enhanced to order $D^5$ when the energy of one of the three Andreev reflections coincides with a bound-state energy. Similar to the three-particle current, overlapping resonances can enhance the magnitude of the current $I_4$ to the order $D$ for those voltages where both the first and the third Andreev reflections coincide with the bound states, as shown in the upper inset in Fig. 12. Indeed, it is clear from the explicit form of $\tilde{M}_{40}=\hat{T}^i_{\xi_1}e^{i\sigma_1\hat{T}}e^{i\xi_2}\hat{\Sigma}e^{i\sigma_2}\hat{T}^i_{\xi_3}e^{-i\xi_3}e^{-i\xi_1}e^{i\sigma_1\hat{T}}$ that when $q_1=k\pi$ and $q_3=m\pi$, then $\tilde{M}_{40}=(-1)^{k+m}\hat{T}^{i\sigma_1\hat{T}}$, i.e., the transparency of the MAR trajectory is enhanced to unity. Other combinations of the resonances, e.g., when the first and the second Andreev reflection occur at bound-state energies, will produce peaks of order $D^2$ or smaller, as described in Appendix D.

Focusing on the double resonances that produce large ($\sim D$) current peaks, we find that in short junctions with just one pair of bound states, $\pm E^{(0)}$, the double resonance will occur at voltage $eV=E^{(0)}$, provided the energy of the bound state is within the interval $\Delta/2\leq E^{(0)}\leq \Delta$. The spectral density of the current has a form similar to that in Eq. (49), the major difference being the small peak splitting, $\delta_{\Delta/\eta} \sim D\Delta/\eta$. The height of the resulting current peak $(k_{\pi}T\ll \Delta)$ is

$$I_{\text{max}}^4 = \frac{\pi D e \Delta}{h} \left[ 1 - a(2E^{(0)})^4 \right]^{1/4} \left[ 1 + a(2E^{(0)})^4 \right]^{1/4},$$

where $a(2E^{(0)})$ is the Andreev-reflection amplitude at energy $2E^{(0)}$.

For longer junctions, there are many possibilities to have overlapping resonances. Two bound states at one side of the junction with energies $E^{(k)}<0$ and $E^{(m)}>0$ can give a peak in $I_4$ if $(E^{(m)}-E^{(k)})/2=eV\geq \Delta/2$. Although the height of all peaks is roughly proportional to $D$, numerically the heights (and widths) of the peaks may vary considerably depending on the position of the second Andreev reflection. If the second Andreev reflection does not occur at the energy of a bound state, the situation is similar to the one described above; see lower inset in Fig. 12. However, if a bound state is close to the energy of the second Andreev reflection, then the current spectral density $I_4^4(E)$ consists of the three full-transmission peaks with widths $\sim D\Delta/\eta$, which are split within the interval $\sim \sqrt{D}\Delta/\eta$ (triple resonance). The triple resonance has larger spectral weight compared to the double resonance, which results in the larger height and width of the current peak.

Rigorously speaking, a triple resonance can only occur in asymmetric junctions because it requires equal distance between neighboring resonances, while the bound-state spectrum in symmetric junctions is not equidistant. However, in long junctions, the deviation from the equidistant spectrum is small, and quasitriple resonances may therefore occur also in long symmetric junctions.

This effect can be observed in Fig. 12, where the four-particle current for a symmetric junction with length $L=7\xi_0>2\pi\xi_0$ consists of three peaks with different heights: the central peak corresponding to the quasitriple resonance, while the two side peaks corresponding to the double resonances with the heights given by Eq. (51).

Finally, it is worth noting that, similar to the situation for the three-particle current, the peaks will form clusters, giving a smaller number of current peaks than the number of pairs of bound states in long junctions.

## C. High-order currents

The studied properties of multiple resonances in three- and four-particle currents allow us to make some general conclusions about resonant behavior of the high-order multiparticle currents that determine the total current at small voltage. The nonresonant magnitude of an $n$-particle current is of order $D^n$ at the threshold voltage, $eV_p=2\Delta/n$, and therefore the total nonresonant current exponentially decreases with the applied voltage (in transparent junctions, $D\sim 1$, the current is exponentially small at $\Delta<\Delta/1-D$). However, multiple resonances may enhance the magnitude of the current by several orders of $D$. The major question of interest here concerns the maximum value of the resonant current, in particular, whether it can be of order $D$ at arbitrary small voltage.

To obtain such large current at small voltage, it is necessary to achieve a transmission probability through a high-order MAR path equal to unity, which implies that the energy of at least every other Andreev reflection must coincide with a bound state (cf. the discussion in the preceding section). For $n>4$, this means that three or more bound states must be approximately equidistant in energy. Since the
bound-state spectrum is nonequidistant, Eq. (17), this is generally not possible if the resonances are narrow; therefore, in junctions with arbitrary geometry and small transmissivity there are no large current peaks below the voltage \( eV = \Delta /2 \).

However, the possibility of a large resonant current exists for junctions with sufficiently large transparency. To find the relevant transparency, let us consider a very long symmetric junction and assume for the moment that the bound-state spectrum is equidistant, \( E^{(m+1)} - E^{(m)} = \text{const} \). Then, from mapping of the \( n \)th order MAR process on a 1D multibarrier structure (see Fig. 13), it is clear that if the applied voltage is commensurate with the level spacing, e.g., \( eV = E^{(m+1)} - E^{(m)} \), the multibarrier structure is periodic, and full transmission is achieved leading to a current peak. This conclusion is valid also for a nonequidistant spectrum if the variation of the interlevel distance does not exceed the width of the full-transmission band. The deviation of the bound-state spectrum from the best linear fit is shown by the thin line.

Adding up the contributions to the current calculated in this paper, we arrive at a rather complex form of CVC at subgap voltages, as shown in Figs. 14–16. Nevertheless, the analysis of the tunnel limit allows us to classify various subgap current structures. Here we will summarize the results of this classification. As a reference system we will take a short (\( L = 0 \)) junction where the form of the CVC is well studied.\(^{28}\) The current structures in short junctions can be interpreted as resonant features due to quasi-bound states situated at the edges of the energy gap,\(^{43}\) the resonant conditions selecting voltages equal to the gap subharmonics, \( eV = 2\Delta /n \). This subharmonic gap structure of the short junction gradually changes with increasing junction length as bound states move down into the gap, giving rise to CVC structures with steps, oscillations, and peaks. The major points are as follows.

(i) The current in the subgap region is considerably enhanced, compared to the short-junction case. This effect is present as soon as the effective length \( L/\xi_0 \) is comparable to, or larger than, the square root of transparency of the junction, \( L/\xi_0 \gg \sqrt{D} \).

(ii) The main onset of the current in short junctions at
$eV = 2\Delta$ shifts downwards in voltage to the value $eV = \Delta + E^{(0)}$ where $E^{(0)}$ is the energy of the bound state. This shift is caused by the resonant two-particle current giving a contribution to the total current of the order of the single-particle current.

(iii) For longer junctions, the current onset transforms into a staircase within the voltage interval $\Delta < eV < 2\Delta$ with the number of steps corresponding to the number of bound states, the step positions being given by $eV = \Delta + E^{(m)}$. This is due to the resonances in the two-particle current transported through bound states. Resonant channels open up, one by one, as the voltage increases and bound states enter the “energy window” available for two-particle processes. The current plateaus are not flat but modulated because of oscillations of the density of continuum states. The period of the modulations is roughly equal to the interlevel distance and it decreases with the junction length. The amplitude of the modulation is roughly equal to the interlevel distance and it decreases with the junction length. Thus, in long junctions, the current structures take the form of a series of peaks.

(iv) There is another series of the current peaks whose positions only weakly depend on temperature and are entirely determined by the bound-state spectrum: $eV = E^{(m)} - E^{(k)}$ and $eV = (E^{(m)} - E^{(k)})/2$. These peaks are caused by the overlap of two resonances in the three- and four-particle currents and the exist in the intervals of applied voltage $2\Delta/3 < eV < 2\Delta$ and $\Delta/2 < eV < \Delta$, respectively. The heights of these peaks are comparable with the heights of the two-particle current structures ($\sim D$).

(v) At voltages smaller than $eV = \Delta/2$ the resonant current structures generally become smaller in magnitude, at least by one order of magnitude, if the junction transparency is sufficiently small ($D \ll 0.1$), and the current decays exponentially when $eV$ approaches zero (although for some particular junction lengths there could be huge ($\sim D$) current peaks caused by multiple resonances). This qualitative difference of the CVC below and above $eV = \Delta/2$ allows one to expect a crossover from power to exponential dependence of CVC in multichannel junctions.

(vi) In transparent junctions, all current structures will persist but become smooth; appreciable current will appear below $eV = \Delta/2$ as soon as $D \gg 1/3$. The current structures completely disappear in fully transparent junctions, $D = 1$, where the CVC does not depend on the junction length; see Fig. 16.

(vii) At voltages larger than $2\Delta$, the current undergoes oscillations, similar to Rowell-McMillan oscillations, and the excess current is approached much faster than in short junctions. In low-transparency junctions the excess current is small, $I^{\text{exc}} \sim D^2$, $D \ll 1$, despite strong Andreev reflection and large pair current $I_2 \sim D$.

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APPENDIX A: APPROXIMATION FOR $r_0^-$ AND $r_{n+}$

In this appendix, the expansion is derived for the reflection amplitudes in Eq. (33) for a quasiparticle injected from the left. From the definition of $r_0^-$ and $r_{(-1)}^-$, Eq. (20), we know

$$\hat{c}_{0^-} = c_{0^-}^0 \left( \frac{r_{0^-}}{1} \right),$$

$$\hat{c}_{(-1)^-} = c_{(-1)^-}^1 \left( \frac{r_{(-1)^-}}{1} \right).$$

They are related as

$$\hat{c}_{0^-} = \hat{M}_{0-1} \hat{U}_{-1} \hat{c}_{(-1)^-},$$

where $\hat{M}_{0-1} = \hat{T}$ and $U_{-1} = e^{i\varphi \sigma_1}$. From this relation, we find $r_{0^-}$ in terms of $r_{(-1)^-}$ as

$$r_{0^-} = \frac{\sqrt{\hat{R}} + r_{(-1)^-} e^{2i\varphi_1}}{1 + \sqrt{\hat{R}} r_{(-1)^-} e^{2i\varphi_1}} = \sqrt{\hat{R} \left( 1 + x/\sqrt{\hat{R}} \right)},$$

where $x = (1 - \sqrt{\hat{R}} r_{(-1)^-} e^{2i\varphi_1 - i/1} (1 + r_{(-1)^-} e^{2i\varphi_1 - i/1})$. When $|x| \ll 1$, we can make an expansion in this parameter to get to the form

$$r_{0^-} = \sqrt{\hat{R} + D} r_{(-1)^-} e^{2i\varphi_1} = \sqrt{\hat{R} + O(a_{-1}^2 D)}.$$  \hspace{1cm} (A5)

Similarly we also get

$$r_{n+} = (-1)^n \sqrt{\hat{R} + D} \frac{r_{(n+1)^+} e^{2i\varphi_{n+1}}}{1 + (-1)^n r_{(n+1)^+} e^{2i\varphi_{n+1}}} = (-1)^n \sqrt{\hat{R} + O(a_{n+1}^2 D)}.$$  \hspace{1cm} (A6)

APPENDIX B: RESONANCE IN TWO-PARTICLE CURRENT

In this appendix, we derive the resonant form of the two-particle current, Eq. (37), for a quasiparticle injected from the left. The definition of $\hat{M}_{20}$ is $\hat{M}_{20} = \hat{T} e^{i\varphi_1 \sigma_1 \hat{T}^{-1}}$, which, using the pseudounitarity of the transfer matrices $\sigma_2 \hat{T} \sigma_2 = \hat{T}^{-1}$, can be written in the form

$$\hat{M}_{20} = \frac{2i}{\sqrt{D}} \sin \varphi_1 \hat{T} \sigma_z + e^{-i\varphi_1 \sigma_1}.$$  \hspace{1cm} (B1)

It simplifies in the limit $D \ll 1$, $|\varphi_1^{(m)}| = |\varphi_1 - m \pi| \ll 1$ to

$$\hat{M}_{20} = \frac{(-1)^k}{D} \left[ 2i \varphi_1^{(m)} (1 + \sigma_z) + D \right].$$  \hspace{1cm} (B2)
Inserting the simplified expansion of \( \hat{M}_{20} \) and the expansion of \( r_{2+} \) and \( r_{0-} \) from Eq. (33) into Eq. (27), as well as putting \( R = 1 \), the leakage current density takes the form

\[
[J_2(E)]^2 = \left( \frac{1 - |a_0|^4}{1 - e^{2|\varphi_0|^2}} \right) \left( \frac{1 - |a_2|^4}{1 - e^{2|\varphi_2|^2}} \right) D^2
\]

\[
\frac{2i\varphi_1^{(m)} + D}{2} \left( \frac{1 + e^{2i\varphi_0} + 1 + e^{2i\varphi_2}}{1 - e^{2i\varphi_2}} \right)
\]

(B3)

We make an expansion of the phase \( \varphi_1^{(m)}(E - E^{(m)})/\Delta = \eta^{(m)} \delta E/\Delta \), where

\[
\eta^{(m)} = \frac{\Delta}{E - E^{(m)}} \frac{\partial \varphi}{\partial E} = \frac{2Lr}{\xi_0} + \frac{\Delta}{\sqrt{\Delta - (E^{(m)})^2}}
\]

(B4)

The two-particle current density now takes a Breit-Wigner form

\[
[J_2(E)]^2 = \left( \frac{\delta E - \delta E^{(m)}}{\Delta} \right)^2 + \left( \frac{\Gamma^{(m)} + \Gamma^{(m)}_0}{2} \right)^2
\]

where the tunneling rates are given by \( \Gamma^{(m)}_0 = N'(E_{0.2})D/2\eta^{(m)} \), where

\[
N'(E_{0.2}) = \text{Re} \left( \frac{1 + e^{2i\varphi_{0.2}}}{1 - e^{2i\varphi_{0.2}}} \right) = \frac{1 - |a_{0.2}|^4}{1 - e^{2|\varphi_{0.2}|^2}}
\]

are equal to the DOS, Eq. (35) at energy \( E_{0.2} \). The resonance is slightly shifted from \( E^{(m)} \) with

\[
\delta E^{(m)} = \frac{D\Delta}{4\eta^{(m)}} \text{Im} \left( \frac{1 + e^{2i\varphi_0} + 1 + e^{2i\varphi_2}}{1 - e^{2i\varphi_2}} \right)
\]

(B7)

**APPENDIX C: RESONANCE IN THREE-PARTICLE CURRENT**

In this appendix, the resonant form of the three-particle current, Eq. (48), is derived. The \( \hat{M}_{30} \) matrix, which by definition is

\[
\hat{M}_{30} = \hat{T}^{-1} e^{i\sigma_z \varphi_2} \hat{T}^{-1} e^{i\sigma_z \varphi_1} \hat{T}^{-1},
\]

(C1)

can be transformed using Eq. (B1) to

\[
\hat{M}_{30} = \hat{T}^{-1} e^{i\sigma_z \varphi_2} \hat{T}^{-1} e^{i\sigma_z \varphi_1} \hat{T}^{-1}
\]

\[
= \left( \frac{2i}{\sqrt{D}} \sin \varphi_2 \hat{T}^{-1} \sigma_z + e^{-i\sigma_z \varphi_2} \right)
\]

\[
\times \hat{T}^{-1} \left( \frac{2i}{\sqrt{D}} \sin \varphi_1 \hat{T} \sigma_z + e^{-i\sigma_z \varphi_1} \right),
\]

which can be written in the form

\[
\hat{M}_{30} = -4 \sin \varphi_1 \sin \varphi_2 \left( \sqrt{D} + \hat{T}^{-1} \hat{T} \right) + 2i\sigma_z \sin(\varphi_1 + \varphi_2)
\]

\[
+ e^{-i\sigma_z \varphi_2} \hat{T}^{-1} e^{-i\sigma_z \varphi_1}.
\]

(C3)

It simplifies in the limit of \( D \ll 1 \), \( |\varphi_1| = |\varphi_1 - k\pi| \ll 1 \), and \( |\varphi_2| = |\varphi_2 - m\pi| \ll 1 \) to

\[
\hat{M}_{30} = \frac{(-1)^{k+m}}{D^{1/2}} \left[ \left( D - 4\varphi_1^{(k)} \varphi_2^{(m)} \right)(1 - \sigma_z) + D i \sigma_z \left\{ \varphi_1^{(k)}(1 - \sigma_z) + \varphi_2^{(m)}(1 + \sigma_z) \right\} \right].
\]

(C4)

Inserting this form of the \( \hat{M}_{30} \) matrix and the expansion (33) for \( r_{0-} \) and \( r_{3+} \) into Eq. (27), as well as putting \( R = 1 \), the probability current density for injection of a quasiparticle from the left takes the form

\[
[J_3(E)]^2 = \left( \frac{(1 - |a_0|^4)(1 - |a_2|^4)D^3}{1 - e^{2|\varphi_0|^2} \sqrt{1 - e^{2|\varphi_2|^2}}} \right)^2
\]

(C5)

\[
Q = (D - 4\varphi_1^{(k)} \varphi_2^{(m)}) + iD \left( \varphi_1^{(k)} \frac{1 + e^{2i\varphi_3}}{1 - e^{2i\varphi_3}} + \varphi_2^{(m)} \frac{1 + e^{2i\varphi_0}}{1 - e^{2i\varphi_0}} \right)
\]

\[
\rightarrow \frac{N'(E_{0.3})D}{4\eta^{(m)}} \text{Im} \left( \frac{1 + e^{2i\varphi_0} + 1 + e^{2i\varphi_2}}{1 - e^{2i\varphi_2}} \right)
\]

where \( D \ll 1 \) is once again used.

Since \( |\varphi_1^{(k)}| \ll 1 \) and \( |\varphi_2^{(m)}| \ll 1 \) and the DOS at energies \( E_{0.3} \), Eq. (35), are equal to

\[
N'(E_{0.3}) = \text{Re} \left( \frac{1 + e^{2i\varphi_3}}{1 - e^{2i\varphi_3}} \right) = \frac{1 - |a_3|^4}{1 - e^{2|\varphi_3|^2}}
\]

(C6)

\[
N'(E_{0.3}) = \text{Re} \left( \frac{1 + e^{2i\varphi_0}}{1 - e^{2i\varphi_0}} \right) = \frac{1 - |a_0|^4}{1 - e^{2|\varphi_0|^2}}
\]

(C7)

we arrive at the form

\[
[J_3(E)]^2 = \frac{N'(E_{0.3})D^3}{|D - 4\varphi_1^{(k)} \varphi_2^{(m)} + iD[\varphi_1^{(k)}N'(E_{0.3}) + \varphi_2^{(m)}N'(E)]|^2}
\]

(C8)

**APPENDIX D: RESONANCE IN FOUR-PARTICLE CURRENT**

In this appendix, we discuss the structure of the resonance in the four-particle current. The matrix

\[
\hat{M}_{40} = \hat{T} e^{i\sigma_z \varphi_1} \hat{T}^{-1} e^{i\sigma_z \varphi_2} \hat{T} e^{i\sigma_z \varphi_3} \hat{T}^{-1}
\]

(D1)

can be written as
From Eq. (2) it is clear that, in general, $\dot{M}_{40} \propto 1/D^2$. When both $\varphi_1$ and $\varphi_3$ are close to a multiple of $\pi$, $\dot{M}_{40} \propto 1/D$. 

$$\dot{M}_{40} = \frac{i\sigma}{D^2}(-8 \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \sqrt{D^2 T^{-1}} + D \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 + D^2 \sin(\varphi_1 + \varphi_3 - \varphi_2)$$

$$+ 2D \sin \varphi_1 \cos(\varphi_3 - \varphi_2) \sqrt{D^2 T^{-1}} + 2D \sin \varphi_2 \cos(\varphi_1 - \varphi_2) \sqrt{D^2 T^{-1}} + \frac{1}{D^2} [-4D \sin \varphi_1 \sin \varphi_2 \cos \varphi_2$$

$$+ 2D \sin \varphi_3 \sin(\varphi_1 - \varphi_2) \sqrt{D^2 T} + 2D \sin \varphi_1 \sin(\varphi_3 - \varphi_2) \sqrt{D^2 T^{-1}} + D^2 \cos(\varphi_1 + \varphi_3 - \varphi_2)].$$

(D2)

Above the subharmonic gap structure at $\epsilon V = 2\Delta/n$ in these junctions can be interpreted as resonances due to superconducting quasibound states situated at the gap edges (Ref. 43).


27. Although the subharmonic gap structure at $\epsilon V = 2\Delta/n$ in these junctions can be interpreted as resonances due to superconducting quasibound states situated at the gap edges (Ref. 43).


41. P. G. de Gennes, Superconductivity of Metals and Alloys (Addison-Wesley, Reading, Massachusetts, 1989).


This results from the fact that the resonances are coupled with the MAR trajectory that crosses the barrier twice (see upper inset in Fig. 12), instead of once as in the three-particle case, and therefore the resonance coupling is weaker. Another difference is that the width of the resonance and thus the height of the current peak is independent of the DOS at the entrance and exit energies, and only depends on the Andreev-reflection probability at the exit and entrance energies.