# On the Caffarelli-Kohn-Nirenberg-type inequalities involving critical and supercritical weights



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# On the Caffarelli–Kohn–Nirenberg-type inequalities involving critical and supercritical weights

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**Abstract** The main purpose of this article is to establish the Caffarelli–Kohn–Nirenberg-type (CKN-type) inequalities for all $\alpha \in \mathbf{R}$ and to study the related matters systematically. Roughly speaking, we discuss the characterizations of the CKN-type inequalities for all $\alpha \in \mathbf{R}$ as the variational problems, the existence and nonexistence of the extremal solutions to these variational problems in proper spaces, and the exact values and the asymptotic behaviors of the best constants in both the noncritical case and the critical case.

In the study of the CKN-type inequalities, the presence of weight functions on both sides prevents us from employing effectively the so-called *spherically symmetric rearrangement*. Further the invariance of $\mathbf{R}^n$ by the group of dilatations creates some possible loss of compactness. As a result we see that the existence of extremals, the values of best constants, and their asymptotic behaviors essentially depend upon the relations among parameters in the inequality.

**Contents**



## 1. Introduction and historical remarks

### 1.1. Introduction

We begin by recalling the classical weighted Sobolev inequalities (1.1), which are often called the Caffarelli–Kohn–Nirenberg-type (CKN-type) inequalities.

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There is a positive number $S$ depending only on $p, q, \alpha, \beta$, and $n$ such that we have

$$\int_{\mathbf{R}^n} |\nabla u(x)|^p |x|^{\alpha p}\, dx \ge S\Big(\int_{\mathbf{R}^n} |u(x)|^q |x|^{\beta q}\, dx\Big)^{p/q}, \quad \text{for any } u \in C_{\mathrm{c}}^{\infty}(\mathbf{R}^n), \tag{1.1}$$

where $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_n)$ and $|\nabla u| = \big(\sum_{k=1}^n |\partial u/\partial x_k|^2\big)^{1/2}$.

Here $n \ge 1$, $1 \le p < +\infty$, and $q, \alpha, \beta$ are real numbers satisfying

$$\begin{cases} \alpha > 1 - n/p, \\ (1-\alpha+\beta)p < n, \\ 0 \le 1/p - 1/q = (1-\alpha+\beta)/n, \\ \beta \le \alpha. \end{cases} \tag{1.2}$$

The main purpose of this article is not only to establish the CKN-type inequalities for all $\alpha \in \mathbf{R}$ but also to study the related matters systematically. Roughly speaking, we discuss the characterizations of the imbeddings as the variational problems, the existence and nonexistence of the extremal solutions to these variational problems in proper spaces, and the exact values and the asymptotic behaviors of the best constants.

Now we introduce a crucial parameter $\gamma$ as follows.

DEFINITION 1.1

For $1 \le p < +\infty$, in (1.2) let us set

$$\gamma = \alpha - 1 + \frac{n}{p} = \beta + \frac{n}{q}. \tag{1.3}$$

Under the condition (1.2), we have $0 < \gamma$ as well. By noting that $\alpha p = p(1+\gamma) - n$, $\beta = \gamma q - n$, we can rewrite (1.1) and (1.2) to obtain the following:

$$\int_{\mathbf{R}^n} |\nabla u(x)|^p |x|^{p(1+\gamma)-n}\, dx \ge S\Big(\int_{\mathbf{R}^n} |u(x)|^q |x|^{\gamma q - n}\, dx\Big)^{p/q}, \quad \text{for any } u \in C_{\mathrm{c}}^{\infty}(\mathbf{R}^n), \tag{1.4}$$

where $n \ge 1$, $1 \le p < +\infty$, and $q, \gamma$ are real numbers satisfying

$$\begin{cases} \gamma > 0, \\ q < +\infty, \\ 0 \le 1/p - 1/q \le 1/n. \end{cases} \tag{1.5}$$

Throughout the present article we work with a parameter $\gamma \in \mathbf{R}$ instead of $\alpha$ and $\beta$, so that most of our results become symmetric in $\gamma$ with respect to $\gamma = 0$.

Furthermore we classify the CKN-type inequalities according to the range of the parameter $\gamma$ into three cases.

DEFINITION 1.2
The parameter $\gamma$ is said to be subcritical, critical, and supercritical if $\gamma$ satisfies $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$, respectively.

REMARK 1.1
(1) Here we note that the conditions $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$ are equivalent to $\alpha > 1 - n/p$, $\alpha = 1 - n/p$, and $\alpha < 1 - n/p$, respectively.

(2) In the classical CKN-type inequalities (1.1), it follows from the subcritical condition $\gamma > 0$ that we have $\beta q > -n$; hence the weight functions on both sides are locally integrable on $\mathbf{R}^n$. By this reason these inequalities (1.1) are classified into the subcritical case of the CKN-type inequalities in this article.

## 1.2. Historical remarks

Before we go further into our main results on the CKN-type inequalities involving critical and supercritical cases, we give a brief historical review here. As we have already mentioned, the inequalities (1.1) for $\gamma = \alpha - 1 + n/p > 0$ are often called the CKN-type inequalities. In fact in [CKN] they established general multiplicative inequalities including this type. In [Ho1] we also studied these inequalities among more general imbedding theorems on the weighted Sobolev spaces, where the weights are powers of the distance from a given closed set $F$.

It was also very interesting for us to study further the properties of the embedding operators obtained there. But for a general $F$ it did not seem easy to study these problems in a detailed way. For this reason, in [Ho2] we restricted ourselves to the simplest case in which $F$ consists of a single point, namely, the origin. In this particular case we studied various aspects of related problems and obtained interesting results such as the exact values of the best constant $S = S(p, q, \alpha)$ in certain cases, the existence and nonexistence of the extremals, and so on (see [ACP], [CW1], [CW2], [GPP]).

Recently we have revisited the weighted Hardy–Sobolev inequality in [AH1]. It is easy to see that the classical CKN-type inequality coincides with the weighted Hardy–Sobolev inequality if $\beta = \alpha - 1$, or equivalently $p = q$. To our surprise it was shown that the weighted Hardy–Sobolev inequalities themselves hold for all $\gamma \in \mathbf{R}$ (or equivalently for all $\alpha \in \mathbf{R}$) with some modifications. In fact, even if $\gamma = \alpha - 1 + n/p = 0$ holds, the sharp inequality of Hardy type remains valid as long as the whole space $\mathbf{R}^n$ is replaced by a bounded domain containing the origin and the weight functions on the right-hand side are replaced by the logarithmic ones. Moreover we have successfully improved those weighted Hardy–Sobolev inequalities by finding sharp missing terms, which turned out to be very useful in many ways. For the improved inequalities, see Proposition 1.2 below. (For the complete argument and related applications, see [AH1].)

On the other hand, the counterpart in the CKN-type inequalities to the weighted Hardy–Sobolev inequalities in [AH1] seems to be unknown so far. But it seems reasonable for us to expect that the CKN-type inequalities should remain valid for all $\gamma \in \mathbf{R}$ ($\alpha \in \mathbf{R}$) with a similar modification to that performed in the weighted Hardy–Sobolev inequalities. In this spirit we establish the CKN-type inequalities for all $\gamma \in \mathbf{R}$ ($\alpha \in \mathbf{R}$) and we further study them systematically in the present paper.

To emphasize the meaning of this classification of the CKN-type inequalities and our motivation in this paper, let us recall the results on the weighted Hardy–Sobolev inequalities as the necessary background.

We first review as Proposition 1.1 the classical weighted Hardy–Sobolev inequalities in the noncritical case, and then we also recall as Proposition 1.2 the improved weighted Hardy–Sobolev inequalities with sharp missing terms in [AH1] (see also [AH2], [ANC], [DHA1], [DHA2], [DHA3], [Ho3]). It follows from these results that the weighted Hardy–Sobolev inequalities are valid for all $\gamma \in \mathbf{R}$. Hence it is appropriate to study the CKN-type equalities according to Definition 1.2 on the basis of the (improved) weighted Hardy–Sobolev inequalities.

PROPOSITION 1.1

*Let $n \geq 1$, let $0 \in \Omega$, and let $\Omega$ be a domain of $\mathbf{R}^n$. Assume that $1 < p < +\infty$, and assume that $\gamma \neq 0$. Then we have*

$$\int_\Omega |\nabla u(x)|^p |x|^{(1+\gamma)p-n}\, dx \geq |\gamma|^p \int_\Omega |u(x)|^p |x|^{\gamma p - n}\, dx \tag{1.6}$$

*for any $u \in C_c^\infty(\Omega \setminus \{0\})$.*

In this inequality (1.6), the domain $\Omega$ may be unbounded and the best constant $|\gamma|^p$ is apparently independent of the shape of the domain. In particular we can put $\Omega = \mathbf{R}^n$.

PROPOSITION 1.2

*Let $n \geq 1$, let $0 \in \Omega$, and let $\Omega$ be a bounded domain of $\mathbf{R}^n$.*

(1) Subcritical case ($\gamma > 0$, $1 < p < +\infty$)

*There exist $K = K(n) > 1$ and $C = C(n) > 0$ such that if $R > K \sup_\Omega |x|$ then*

$$\begin{aligned}\int_\Omega |\nabla u(x)|^p |x|^{(1+\gamma)p-n}\, dx &\geq |\gamma|^p \int_\Omega |u(x)|^p |x|^{\gamma p - n}\, dx \\ &\quad + C \int_\Omega |u(x)|^p \Big(\log \frac{R}{|x|}\Big)^{-2} |x|^{\gamma p - n}\, dx\end{aligned} \tag{1.7}$$

*for any $u \in C_c^\infty(\Omega)$.*

(2) Critical case ($\gamma = 0$, $1 < p < +\infty$)

*There exist $K = K(n) > 1$ and $C = C(n) > 0$ such that if $R > K \sup_\Omega |x|$ then*

$$\begin{aligned}\int_\Omega |\nabla u(x)|^p |x|^{p-n}\, dx &\geq \frac{1}{(p')^p} \int_\Omega \frac{|u(x)|^p}{|x|^n} \Big(\log \frac{R}{|x|}\Big)^{-p} dx \\ &\quad + C \int_\Omega \frac{|u(x)|^p}{|x|^n} \Big(\log \frac{R}{|x|}\Big)^{-p} \Big(\log\Big(\log \frac{R}{|x|}\Big)\Big)^{-2} dx\end{aligned} \tag{1.8}$$

*for any $u \in C_c^\infty(\Omega)$. Here $p' = p/(p-1)$.*

(3) Supercritical case ($\gamma < 0,\ 1 < p < +\infty$)
*There exist $K = K(n) > 0$ and $C = C(n) > 0$ such that if $R > K \sup_\Omega |x|$ then*

$$\int_\Omega |\nabla u(x)|^p |x|^{(1+\gamma)p-n}\,dx \ge |\gamma|^p \int_\Omega |u(x)|^p |x|^{\gamma p-n}\,dx + C \int_\Omega |u(x)|^p \Big(\log \frac{R}{|x|}\Big)^{-2} |x|^{\gamma p-n}\,dx \tag{1.9}$$

*for any $u \in C_c^\infty(\Omega \setminus \{0\})$.*

REMARK 1.2

(1) If we replace a bounded domain $\Omega$ by the whole space $\mathbf{R}^n$, then in general we cannot expect any improved weighted Hardy–Sobolev inequalities with a missing term.

(2) If $\gamma = 0$ (the critical case) and $\Omega = \mathbf{R}^n$, then one can show from a capacitary argument that for any compact set $K \subset \mathbf{R}^n$

$$\inf\Big\{\int_{\mathbf{R}^n} |\nabla u(x)|^p |x|^{p-n}\,dx : u \in C_0^\infty(\mathbf{R}^n), u \ge 1 \text{ on } K\Big\} = 0.$$

Therefore we cannot expect the weighted Hardy inequality in the whole space $\mathbf{R}^n$.

## 2. Main results

### 2.1. The CKN-type inequalities

In the subsequent section we employ the following notations:

$$p' = \frac{p}{p-1}, \qquad p^* = \frac{np}{(n-p)_+} \quad \text{for } 1 \le p \le \infty. \tag{2.1}$$

Here we set $t_+ = \max\{0, t\}$ and $1/0 = \infty$.

As we have already mentioned in Section 1, for fixed $p, q$, instead of parameters $\alpha, \beta$ in the CKN-type inequalities we work with a new parameter

$$\gamma = \alpha - 1 + \frac{n}{p} = \beta + \frac{n}{q}. \tag{2.2}$$

Then the range for $p, q, \gamma$ becomes

$$1 \le p \le q < \infty, (0 \le)\tau_{p,q} = \frac{1}{p} - \frac{1}{q} \le \frac{1}{n}, \quad \gamma \in \mathbf{R}. \tag{2.3}$$

From this condition we obtain for a fixed $p$

$$p \le q \le p^* = \frac{np}{n-p} \quad \text{if } 1 \le p < n; \qquad p \le q < p^* = \infty \quad \text{if } n \le p < \infty. \tag{2.4}$$

We recall that the subcritical condition, the critical condition, and the subcritical condition simply correspond to $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$, respectively.

We prepare more notations below.

DEFINITION 2.1

For $\alpha \in \mathbf{R}$ and $R \geq 1$ we set

$$I_\alpha(x) = I_\alpha(|x|) = \frac{1}{|x|^{n-\alpha}} \quad \text{for } x \in \mathbf{R}^n \setminus \{0\}, \tag{2.5}$$

$$A_{1,R}(x) = A_{1,R}(|x|) = \begin{cases} \log \frac{R}{|x|} & \text{for } x \in \overline{B_1} \setminus \{0\}, \\ \log(R|x|) & \text{for } x \in \mathbf{R}^n \setminus B_1. \end{cases} \tag{2.6}$$

When $0 < \alpha < n$ holds, $I_\alpha$ is called a Riesz kernel of order $\alpha$.

Under these notations the CKN-type inequalities have the following forms:
if $\gamma \neq 0$, then

$$\int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x)\, dx \geq S\Big(\int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x)\, dx\Big)^{p/q}. \tag{2.7}$$

If $\gamma = 0$, then for $R > 1$

$$\int_{B_1} |\nabla u(x)|^p I_p(x)\, dx \geq C\Big(\int_{B_1} |u(x)|^q \frac{I_0(x)}{A_{1,R}(x)^{1+q/p'}}\, dx\Big)^{p/q}. \tag{2.8}$$

We introduce function spaces and related norms below.

DEFINITION 2.2

Let $1 \leq p \leq q < \infty$, let $\gamma \in \mathbf{R}$, and let $R \geq 1$. Let $\Omega$ be a domain of $\mathbf{R}^n$, and let $u : \Omega \to \mathbf{R}$.

(1) For $w : \Omega \to \mathbf{R}$ satisfying $w \geq 0$ a.e. on $\Omega$, we set

$$\|u\|_{L^q(\Omega; w)} = \Big(\int_\Omega |u(x)|^q w(x)\, dx\Big)^{1/q}. \tag{2.9}$$

(2) Under the above notation we set

$$\begin{aligned} &\|u\|_{L^q_\gamma(\Omega)} = \|u\|_{L^q(\Omega; I_{q\gamma})}, \qquad \|\nabla u\|_{L^p_{1+\gamma}(\Omega)} = \||\nabla u|\|_{L^p_{1+\gamma}(\Omega)}, \\ &\|u\|_{L^q_{p;R}(\Omega)} = \|u\|_{L^q(\Omega; I_0/A_{1,R}^{1+q/p'})}. \end{aligned} \tag{2.10}$$

(3) We have $L^q_\gamma(\Omega) = \{u : \Omega \to \mathbf{R} \mid \|u\|_{L^q_\gamma(\Omega)} < \infty\}$, $L^q_{p;R}(\Omega) = \{u : \Omega \to \mathbf{R} \mid \|u\|_{L^q_{p;R}(\Omega)} < \infty\}$.

(4) By $W^{1,p}_{\gamma,0}(\Omega)$ we denote the completion of $C^\infty_{\mathrm{c}}(\Omega \setminus \{0\})$ with respect to the norm

$$u \mapsto \|\nabla u\|_{L^p_{1+\gamma}(\Omega)}.$$

(5) Let $\Omega$ be a radially symmetric domain. For any function space $V(\Omega)$ on $\Omega$, we set

$$V(\Omega)_{\mathrm{rad}} = \big\{u \in V(\Omega) \bigm| u \text{ is radial}\big\}. \tag{2.11}$$

We remark on the following fundamental properties concerning the density of smooth functions. (The proof is given in Section 8.)

PROPOSITION 2.1

*Assume that* $1<p<\infty$, *and assume that* $\gamma \in \mathbf{R}$.

(1) *If* $\gamma>0$, *then* $C_c^\infty(\mathbf{R}^n) \subset W_{\gamma,0}^{1,p}(\mathbf{R}^n)$ *and* $C_c^\infty(\mathbf{R}^n)$ *is densely contained in* $W_{\gamma,0}^{1,p}(\mathbf{R}^n)$.

(2) *If* $\gamma<0$, *then* $C_c^\infty(\mathbf{R}^n) \not\subset W_{\gamma,0}^{1,p}(\mathbf{R}^n)$.

(3) *If* $\gamma=0$, *then* $C_c^\infty(B_1) \subset W_{0,0}^{1,p}(B_1)$ *and* $C_c^\infty(B_1)$ *is densely contained in* $W_{0,0}^{1,p}(B_1)$.

Then the CKN-type inequalities are simply represented as follows:
if $\gamma \neq 0$, then

$$\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \geq S\|u\|^p_{L^q_\gamma(\mathbf{R}^n)} \quad \text{for } u \in W_{\gamma,0}^{1,p}(\mathbf{R}^n), \tag{2.12}$$

if $\gamma=0$, then for $R>1$

$$\|\nabla u\|^p_{L^p_1(B_1)} \geq C\|u\|^p_{L^q_{p;R}(B_1)} \quad \text{for } u \in W_{0,0}^{1,p}(B_1). \tag{2.13}$$

REMARK 2.1

(1) When $p=q$ holds, these two inequalities are called the Hardy–Sobolev inequalities. It is known that the best constants $S$ of (2.12) and $C$ of (2.13) coincide with the ones restricted in the radial functional spaces $W_{\gamma,0}^{1,p}(\mathbf{R}^n)_{\mathrm{rad}}$ and $W_{0,0}^{1,p}(B_1)_{\mathrm{rad}}$, respectively, and hence we have

$$S = S^{p,p;\gamma} = \gamma^p, \qquad C = C^{p,p;R} = \frac{1}{(p')^p}. \tag{2.14}$$

(2) It follows from the Hardy–Sobolev inequalities that if $\gamma>0$, then the space $W_{\gamma,0}^{1,p}(\mathbf{R}^n)$ coincides with the completion of $C_c^\infty(\mathbf{R}^n\setminus\{0\})$ with respect to the norm

$$\|u\|_{W_\gamma^{1,p}(\mathbf{R}^n)} = \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|u\|_{L^p_\gamma(\mathbf{R}^n)}, \tag{2.15}$$

and if $\gamma=0$, then the space $W_{0,0}^{1,p}(B_1)$ coincides with the completion of $C_c^\infty(B_1\setminus\{0\})$ with respect to the norm

$$\|u\|_{W_{0;R}^{1,p}(B_1)} = \|\nabla u\|_{L^p_1(B_1)} + \|u\|_{L^p_{p;R}(B_1)} \quad \text{with } R>1. \tag{2.16}$$

Here we note that if $\gamma=0$, then the weight function on the right-hand side of the CKN-type inequality (2.13) is sharp in the following sense. (The proof is given in Section 8.)

PROPOSITION 2.2

*Let* $1<p\leq q<\infty$, *let* $\tau_{p,q}\leq 1/n$, *and let* $R>1$. *Assume that* $w \in C(\overline{B_1}\setminus\{0\})$ *satisfies*

$$w(x)\geq 0 \quad \text{for } x\in \overline{B_1}\setminus\{0\}, \qquad \frac{A_{1,R}(x)^{1+q/p'}}{I_0(x)} w(x) \to \infty \quad \text{as } x\to 0.$$

*Then we have*

$$\inf\Big\{\Big(\frac{\|\nabla u\|_{L^p_1(B_1)}}{\|u\|_{L^q(B_1;w)}}\Big)^p \;\Big|\; u\in W^{1,p}_{0,0}(B_1)\setminus\{0\}\Big\}=0.$$

In what follows we study the validity of the CKN-type inequalities and the behavior of the best constants precisely when the parameters enjoy $1<p\le q<\infty$, $\tau_{p,q}\le 1/n$, and in addition the cases in which $\gamma<0$ and $R=1$ are considered. Moreover when $\gamma=0$, we also establish the CKN-type inequality in the exterior domain $\mathbf{R}^n\setminus\overline{B_1}$ such that

$$\|\nabla u\|^p_{L^p_1(\mathbf{R}^n\setminus\overline{B_1})}\ge \overline{C}\|u\|^p_{L^q_{p;R}(\mathbf{R}^n\setminus\overline{B_1})}\quad \text{for } u\in W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1}). \tag{2.17}$$

## 2.2. Main results in the noncritical case

In this section we describe the results when $\gamma\neq 0$.

DEFINITION 2.3

Let $1\le p\le q<\infty$, and let $\gamma\neq 0$. We have:

Assertion (1):

$$E^{p,q;\gamma}[u]=\Big(\frac{\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}}{\|u\|_{L^q_\gamma(\mathbf{R}^n)}}\Big)^p\quad \text{for } u\in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}. \tag{2.18}$$

Assertion (2):

$$\begin{aligned} S^{p,q;\gamma}&=\inf\big\{E^{p,q;\gamma}[u]\;\big|\; u\in W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}\big\}\\ &=\inf\big\{E^{p,q;\gamma}[u]\;\big|\; u\in C^\infty_c(\mathbf{R}^n\setminus\{0\})\setminus\{0\}\big\},\end{aligned} \tag{2.19}$$

$$\begin{aligned} S^{p,q;\gamma}_{\mathrm{rad}}&=\inf\big\{E^{p,q;\gamma}[u]\;\big|\; u\in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\setminus\{0\}\big\}\\ &=\inf\big\{E^{p,q;\gamma}[u]\;\big|\; u\in C^\infty_c(\mathbf{R}^n\setminus\{0\})_{\mathrm{rad}}\setminus\{0\}\big\}.\end{aligned} \tag{2.20}$$

First of all we state the CKN-type inequalities in the noncritical case.

THEOREM 2.1

*Assume that $1<p\le q<\infty$, assume that $\tau_{p,q}\le 1/n$, and assume that $\gamma\neq 0$. Then, we have $S^{p,q;\gamma}_{\mathrm{rad}}\ge S^{p,q;\gamma}>0$ and the following inequalities:*

$$\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\ge S^{p,q;\gamma}\|u\|^p_{L^q_\gamma(\mathbf{R}^n)}\quad \textit{for } u\in W^{1,p}_{\gamma,0}(\mathbf{R}^n), \tag{2.21}$$

$$\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\ge S^{p,q;\gamma}_{\mathrm{rad}}\|u\|^p_{L^q_\gamma(\mathbf{R}^n)}\quad \textit{for } u\in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}. \tag{2.22}$$

This follows from assertions (1)–(4) of Theorem 2.2. Let us introduce more notation.

DEFINITION 2.4

For $1<p\le q<\infty$, we set

$$
\begin{aligned}
&\gamma_{p,q} = \frac{n-1}{1+q/p'}, \\
&S_{p,q} = \begin{cases} (p')^{p-2+p/q} q^{p/q} \bigl(\frac{\omega_n}{\tau_{p,q}} \mathrm{B}\bigl(\frac{1}{p\tau_{p,q}}, \frac{1}{p'\tau_{p,q}}\bigr)\bigr)^{1-p/q} & \text{if } p<q, \\ 1 & \text{if } p=q. \end{cases}
\end{aligned} \tag{2.23}
$$

Here $\mathrm{B}(\cdot,\cdot)$ is the beta function, and $\omega_n$ is the area of a unit ball.

REMARK 2.2

(1) It holds that

$$
\mathrm{B}\Bigl(\frac{1}{p\tau}, \frac{1}{p'\tau}\Bigr)^{\tau} \to \frac{1}{p^{1/p}(p')^{1/p'}} \quad \text{as } \tau \to 0. \tag{2.24}
$$

In fact for $0<\tau<\min\{1/p, 1/p'\}$, we see that

$$
\begin{aligned}
&t^{1/p-\tau}(1-t)^{1/p'-\tau} \le \frac{1}{(1-2\tau)^{1-2\tau}} \Bigl(\frac{1}{p}-\tau\Bigr)^{1/p-\tau} \Bigl(\frac{1}{p'}-\tau\Bigr)^{1/p'-\tau} \\
&\qquad \text{for } 0 \le t \le 1;
\end{aligned} \tag{2.25}
$$

hence we have

$$
\begin{aligned}
\mathrm{B}\Bigl(\frac{1}{p\tau}, \frac{1}{p'\tau}\Bigr)^{\tau} &= \Bigl(\int_0^1 \bigl(t^{1/p-\tau}(1-t)^{1/p'-\tau}\bigr)^{1/\tau}\, dt\Bigr)^{\tau} \\
&\le \frac{1}{(1-2\tau)^{1-2\tau}} \Bigl(\frac{1}{p}-\tau\Bigr)^{1/p-\tau} \Bigl(\frac{1}{p'}-\tau\Bigr)^{1/p'-\tau} \\
&\to \frac{1}{p^{1/p}(p')^{1/p'}} \quad \text{as } \tau \to 0, \\
\mathrm{B}\Bigl(\frac{1}{p\tau}, \frac{1}{p'\tau}\Bigr)^{\tau} &\ge \Bigl(\int_0^1 \bigl(t^{1/p}(1-t)^{1/p'}\bigr)^{1/\tau}\, dt\Bigr)^{\tau} \\
&\to \max_{0\le t\le 1} t^{1/p}(1-t)^{1/p'} = \frac{1}{p^{1/p}(p')^{1/p'}} \quad \text{as } \tau \to 0.
\end{aligned}
$$

(2) Since $\tau_{p,q} \to 0$ as $q \to p$, it follows from the argument of assertion (1) of this remark that we have

$$
\begin{aligned}
S_{p,q} &= \frac{(p')^{p-1-p\tau_{p,q}}}{(1/p-\tau_{p,q})^{1-p\tau_{p,q}}} \Bigl(\frac{\omega_n}{\tau_{p,q}} \mathrm{B}\Bigl(\frac{1}{p\tau_{p,q}}, \frac{1}{p'\tau_{p,q}}\Bigr)\Bigr)^{p\tau_{p,q}} \to 1 \\
&= S_{p,p} \quad \text{as } q \to p.
\end{aligned} \tag{2.26}
$$

Under these preparations we can compute the best constant $S_{\mathrm{rad}}^{p,q;\gamma}$ of the CKN-type inequality in the radial function space to obtain the exact representation. In the next theorem we describe important relations among the best constants $S_{\mathrm{rad}}^{p,q;\gamma}$ and $S^{p,q;\gamma}$.

THEOREM 2.2

*Assume that $1<p\le q\le \overline{q}<\infty$, and assume that $\tau_{p,q}\le 1/n$. Then it holds that:*

(1) $S^{p,q;\gamma} = S^{p,q;-\gamma}, S^{p,q;\gamma}_{\rm rad} = S^{p,q;-\gamma}_{\rm rad}$ *for* $\gamma \neq 0$.
(2) $S^{p,q;\gamma}_{\rm rad} = S_{p,q}|\gamma|^{p(1-\tau_{p,q})}$ *for* $\gamma \neq 0$.
(3) $S^{p,q;\gamma} = S^{p,q;\gamma}_{\rm rad} = S_{p,q}|\gamma|^{p(1-\tau_{p,q})}$ *for* $0 < |\gamma| \leq \gamma_{p,q}$.
(4) $|\frac{\gamma}{\overline{\gamma}}|^{p(1-\tau_{p,q})} S^{p,q;\overline{\gamma}} \leq S^{p,q;\gamma} \leq |\frac{\overline{\gamma}}{\gamma}|^{p\tau_{p,q}} S^{p,q;\overline{\gamma}}$ *for* $0 < |\gamma| \leq |\overline{\gamma}|$.
(5) $\frac{1}{(2-\gamma_{p,p^*}/\gamma)^p} S^{p,p^*;\gamma_{p,p^*}} \leq S^{p,p^*;\gamma} \leq S^{p,p^*;\gamma_{p,p^*}} = S^{p,p^*;\gamma_{p,p^*}}_{\rm rad}$ *for* $|\gamma| \geq \gamma_{p,p^*} = \frac{n-p}{p}$ *if* $p < n$.
(6) $S^{2,2^*;\gamma} = S^{2,2^*;\gamma_{2,2^*}} = S^{2,2^*;\gamma_{2,2^*}}_{\rm rad}$ *for* $|\gamma| \geq \gamma_{2,2^*} = \frac{n-2}{2}$ *if* $p = 2 < n$.
(7) $S^{p,q;\gamma} \geq (|\gamma|^{p\tau_{q,\overline{q}}} (S^{p,\overline{q};\gamma})^{\tau_{p,q}})^{1/\tau_{p,\overline{q}}}$ *for* $\gamma \neq 0$.
*In particular,* $S^{p,q;\gamma} \geq |\gamma|^{p(1-n\tau_{p,q})} (S^{p,p^*;\gamma})^{n\tau_{p,q}}$ *for* $\gamma \neq 0$ *if* $p < n$.

REMARK 2.3

(1) Assertions (1)–(4) are proved in Sections 3 and 4, and assertions (5)–(7) are established in Section 6, respectively.

(2) It follows from Remark 2.1 and Theorem 2.2(1) that we have

$$S^{p,p;\gamma} = S^{p,p;\gamma}_{\rm rad} = |\gamma|^p \quad \text{for } \gamma \neq 0. \tag{2.27}$$

(3) For $1 < p < n$, the number

$$S^{p,p^*;\gamma_{p,p^*}} = S^{p,p^*;\gamma_{p,p^*}}_{\rm rad} = n\Big(\frac{n-p}{p-1}\Big)^{p-1} \Big(\frac{\omega_n}{p'} \mathrm{B}\Big(\frac{n}{p}, \frac{n}{p'}\Big)\Big)^{p/n} \tag{2.28}$$

coincides with the classical best constant of the Sobolev inequality:

$$\begin{aligned}\|\nabla u\|^p_{L^p(\mathbf{R}^n)} &= \|\nabla u\|^p_{L^p_{1+\gamma_{p,p^*}}(\mathbf{R}^n)} \\ &\geq S\|u\|^p_{L^{p^*}_{\gamma_{p,p^*}}(\mathbf{R}^n)} = S\|u\|^p_{L^{p^*}(\mathbf{R}^n)} \quad \text{for } u \in W^{1,p}_{\gamma_{p,p^*},0}(\mathbf{R}^n).\end{aligned}$$

In particular for $n \geq 3$, $p = 2$, we see that

$$\begin{aligned} S^{2,2^*;\gamma_{2,2^*}} = S^{2,2^*;\gamma_{2,2^*}}_{\rm rad} &= n(n-2)\Big(\frac{\omega_n}{2}\mathrm{B}\Big(\frac{n}{2},\frac{n}{2}\Big)\Big)^{2/n} \\ &= n(n-2)\Big(\frac{\Gamma(n/2)}{\Gamma(n)}\Big)^{2/n}\pi. \end{aligned} \tag{2.29}$$

Here, $\Gamma(\cdot)$ is the gamma function.

Moreover the best constant $S^{p,q;\gamma}$ is a continuous function of the parameters $q$ and $\gamma$. Namely we have the following theorem, which is established in Section 6.

THEOREM 2.3

*For* $1 < p < \infty$, *the maps*

$$([p,p^*] \setminus \{\infty\}) \times (\mathbf{R} \setminus \{0\}) \ni (q;\gamma) \mapsto S^{p,q;\gamma}, \quad S^{p,q;\gamma}_{\rm rad} \in \mathbf{R}, \tag{2.30}$$

*are continuous. In particular, it holds that*

$$S^{p,q;\gamma} \to S^{p,p;\gamma} = |\gamma|^p \quad \textit{as } q \to p. \tag{2.31}$$

In what follows we describe results on the existence and non-existence of extremal functions which attain the best constants of the CKN-type inequalities. In short, the best constant $S^{p,q;\gamma}$ is attained by some element in $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ provided that $p<q<p^*$ is satisfied. On the other hand, if $q=p$, then the corresponding CKN-type inequalities are reduced to the Hardy–Sobolev inequalities and therefore no extremal function exists. When $q=p^*$ holds, then $S^{p,p^*;\gamma}$ is attained provided that $0<|\gamma|\le(n-p)/p=\gamma_{p,p^*}$, but in the case that $|\gamma|>(n-p)/p$, it is unknown in general, except for the case $p=2$, whether $S^{p,p^*;\gamma}$ is achieved by some element or not. If $p=2$ is assumed, then it is shown that no extremal exists provided that $|\gamma|>(n-2)/2$ holds.

THEOREM 2.4

*Assume that* $1<p\le q<\infty$, *assume that* $\tau_{p,q}\le 1/n$, *and assume that* $\gamma\ne 0$. *Then we have the following.*

(1) *If* $p<q$, *then* $S^{p,q;\gamma}_{\mathrm{rad}}$ *is achieved in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\setminus\{0\}$.

(2) *If* $p<q<p^*$, *then* $S^{p,q;\gamma}$ *is achieved in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$.

(3) *If* $p<n$, $q=p^*$, *and* $|\gamma|\le(n-p)/p=\gamma_{p,p^*}$, *then* $S^{p,p^*;\gamma}=S^{p,p^*;\gamma}_{\mathrm{rad}}$ *is achieved in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\setminus\{0\}$.

(4) *If* $p=2<n$, $q=2^*=2n/(n-2)$, *and* $|\gamma|>(n-2)/2=\gamma_{2,2^*}$, *then* $S^{2,2^*;\gamma}=S^{2,2^*;\gamma_{2,2^*}}_{\mathrm{rad}}$ *holds and* $S^{2,2^*;\gamma}$ *is not achieved in* $W^{1,2}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$.

REMARK 2.4

Assertions (1) and (3) are proved in Section 4. On the other hand assertions (2) and (4) are established in Sections 7 and 8, respectively.

PROPOSITION 2.3

*If* $1<p=q<\infty$ *and* $\gamma\ne 0$, *then* $S^{p,p;\gamma}$ *and* $S^{p,p;\gamma}_{\mathrm{rad}}$ *are not achieved in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ *and* $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}\setminus\{0\}$, *respectively.*

This is proved in Section 8.

## 2.3. Main results in the critical case

In this section we state the results in the case of $\gamma=0$. Let us begin by defining various functionals and best constants.

DEFINITION 2.5

Let $1\le p\le q<\infty$, and let $R\ge 1$. We have

Assertion (1)

$$F^{p,q;R}[u]=\Big(\frac{\|\nabla u\|_{L^p_1(B_1)}}{\|u\|_{L^q_{p;R}(B_1)}}\Big)^p\quad\text{for } u\in W^{1,p}_{0,0}(B_1)\setminus\{0\}. \tag{2.32}$$

Assertion (2)

$$
\begin{aligned}
C^{p,q;R} &= \inf\{F^{p,q;R}[u] \mid u \in W_{0,0}^{1,p}(B_1) \setminus \{0\}\} \\
&= \inf\{F^{p,q;R}[u] \mid u \in C_c^{\infty}(B_1 \setminus \{0\}) \setminus \{0\}\},
\end{aligned} \tag{2.33}
$$

$$
\begin{aligned}
C_{\mathrm{rad}}^{p,q;R} &= \inf\{F^{p,q;R}[u] \mid u \in W_{0,0}^{1,p}(B_1)_{\mathrm{rad}} \setminus \{0\}\} \\
&= \inf\{F^{p,q;R}[u] \mid u \in C_c^{\infty}(B_1 \setminus \{0\})_{\mathrm{rad}} \setminus \{0\}\}.
\end{aligned} \tag{2.34}
$$

Assertion (3)

$$
\overline{F}^{p,q;R}[u] = \Big(\frac{\|\nabla u\|_{L_1^p(\mathbf{R}^n \setminus \overline{B_1})}}{\|u\|_{L_{p;R}^q(\mathbf{R}^n \setminus \overline{B_1})}}\Big)^p \quad \text{for } u \in W_{0,0}^{1,p}(\mathbf{R}^n \setminus \overline{B_1}) \setminus \{0\}. \tag{2.35}
$$

Assertion (4)

$$
\begin{aligned}
\overline{C}^{p,q;R} &= \inf\{\overline{F}^{p,q;R}[u] \mid u \in W_{0,0}^{1,p}(\mathbf{R}^n \setminus \overline{B_1}) \setminus \{0\}\} \\
&= \inf\{\overline{F}^{p,q;R}[u] \mid u \in C_c^{\infty}(\mathbf{R}^n \setminus \overline{B_1}) \setminus \{0\}\},
\end{aligned} \tag{2.36}
$$

$$
\begin{aligned}
\overline{C}_{\mathrm{rad}}^{p,q;R} &= \inf\{\overline{F}^{p,q;R}[u] \mid u \in W_{0,0}^{1,p}(\mathbf{R}^n \setminus \overline{B_1})_{\mathrm{rad}} \setminus \{0\}\} \\
&= \inf\{\overline{F}^{p,q;R}[u] \mid u \in C_c^{\infty}(\mathbf{R}^n \setminus \overline{B_1})_{\mathrm{rad}} \setminus \{0\}\}.
\end{aligned} \tag{2.37}
$$

When $R > 1$, we have the following theorem.

THEOREM 2.5

*Assume that $1 < p \le q < \infty$, assume that $\tau_{p,q} \le 1/n$, and assume that $R > 1$. Then, we have $C_{\mathrm{rad}}^{p,q;R} \ge C^{p,q;R} > 0$, $\overline{C}_{\mathrm{rad}}^{p,q;R} \ge \overline{C}^{p,q;R} > 0$, and the following inequalities:*

$$
\|\nabla u\|_{L_1^p(B_1)}^p \ge C^{p,q;R} \|u\|_{L_{p;R}^q(B_1)}^p \quad \textit{for } u \in W_{0,0}^{1,p}(B_1), \tag{2.38}
$$

$$
\|\nabla u\|_{L_1^p(B_1)}^p \ge C_{\mathrm{rad}}^{p,q;R} \|u\|_{L_{p;R}^q(B_1)}^p \quad \textit{for } u \in W_{0,0}^{1,p}(B_1)_{\mathrm{rad}}, \tag{2.39}
$$

$$
\|\nabla u\|_{L_1^p(\mathbf{R}^n \setminus \overline{B_1})}^p \ge \overline{C}^{p,q;R} \|u\|_{L_{p;R}^q(\mathbf{R}^n \setminus \overline{B_1})}^p \quad \textit{for } u \in W_{0,0}^{1,p}(\mathbf{R}^n \setminus \overline{B_1}), \tag{2.40}
$$

$$
\|\nabla u\|_{L_1^p(\mathbf{R}^n \setminus \overline{B_1})}^p \ge \overline{C}_{\mathrm{rad}}^{p,q;R} \|u\|_{L_{p;R}^q(\mathbf{R}^n \setminus \overline{B_1})}^p \quad \textit{for } u \in W_{0,0}^{1,p}(\mathbf{R}^n \setminus \overline{B_1})_{\mathrm{rad}}. \tag{2.41}
$$

REMARK 2.5

If $p \ge n$, these embedding inequalities follow from assertions (3) and (4) of Theorem 2.7. On the other hand if $1 < p < n$, then these are established in Section 5 by using the so-called *nonlinear potential theory*.

When $R = 1$ holds, we have the next result, which is established in Section 4 and partly in Section 8.

THEOREM 2.6

*Assume that* $1 < p \le q < \infty$, *assume that* $\tau_{p,q} \le 1/n$, *and assume that* $R = 1$. *Then we have the following.*

(1) *If* $n = 1$, *then* $C^{p,q;1}_{\mathrm{rad}} \ge C^{p,q;1} > 0$ *and* $\overline{C}^{p,q;1}_{\mathrm{rad}} \ge \overline{C}^{p,q;1} > 0$ *hold. Further the inequalities in Theorem 2.5 are valid with* $R = 1$.

(2) *If* $n \ge 2$, *then* $C^{p,q;1}_{\mathrm{rad}} > 0$ *and* $\overline{C}^{p,q;1}_{\mathrm{rad}} > 0$ *hold. Further the inequalities in Theorem 2.5 are valid with* $R = 1$, *and* $C^{p,q;1} = \overline{C}^{p,q;1} = 0$ *holds.*

Now we introduce more notation.

DEFINITION 2.6

For $1 < p \le q < \infty$ we set

$$R_{p,q} = \exp \frac{1 + q/p'}{(n-1)p'} \quad \text{if } n \ge 2, \qquad C_{p,q} = \frac{S_{p,q}}{(p')^{p(1-\tau_{p,q})}}. \tag{2.42}$$

By virtue of these we can represent in a concrete way $C^{p,q;R}_{\mathrm{rad}}$ and $\overline{C}^{p,q;R}_{\mathrm{rad}}$, which are the best constants in the radial function spaces.

THEOREM 2.7

*Assume that* $1 < p \le q < \infty$, *and assume that* $\tau_{p,q} \le 1/n$. *Then we have the following:*

(1) $C^{p,q;R} = \overline{C}^{p,q;R}, C^{p,q;R}_{\mathrm{rad}} = \overline{C}^{p,q;R}_{\mathrm{rad}}$ *for* $R \ge 1$.

(2) $C^{p,q;R}_{\mathrm{rad}} = \overline{C}^{p,q;R}_{\mathrm{rad}} = C_{p,q}$ *for* $R \ge 1$.

(3) $C^{p,q;R} = C^{p,q;R}_{\mathrm{rad}} = \overline{C}^{p,q;R} = \overline{C}^{p,q;R}_{\mathrm{rad}} = C_{p,q}$ *for* $R \ge \begin{cases} 1 & \text{if } n = 1, \\ R_{p,q} & \text{if } p \ge n \ge 2. \end{cases}$

(4) $C^{p,q;R} = \overline{C}^{p,q;R} \le C^{p,q;\overline{R}} = \overline{C}^{p,q;\overline{R}} \le \big(\frac{\log \overline{R}}{\log R}\big)^p C^{p,q;R} = \big(\frac{\log \overline{R}}{\log R}\big)^p \overline{C}^{p,q;R}$ *for* $1 < R \le \overline{R}$.

REMARK 2.6

(1) Assertions (1) and (4) are established in Section 3 and the rest are done in Section 4.

(2) We have $C_{p,q} \to \frac{1}{(p')^p} = C_{p,p}$ as $q \to p$. Hence we have the following assertion, which is established in Section 6.

(3) From Remark 2.1 and Proposition 3.1 we obtain

$$C^{p,p;R} = C^{p,p;R}_{\mathrm{rad}} = \overline{C}^{p,p;R} = \overline{C}^{p,p;R}_{\mathrm{rad}} = \frac{1}{(p')^p} = C_{p,p} \quad \text{for } R > 1. \tag{2.43}$$

Further the best constant $C^{p,q;R}$ is a continuous function of the parameters $q, R$.

THEOREM 2.8 (1) *For $1<p<\infty$, the maps*

$$([p,p^*]\setminus\{\infty\})\times(1,\infty)\ni(q;R)\mapsto C^{p,q;R}=\overline{C}^{p,q;R},\qquad C^{p,q;R}_{\rm rad}=\overline{C}^{p,q;R}_{\rm rad}\in\mathbf{R}, \tag{2.44}$$

*are continuous.*

(2) *For $n=1$ and $1<p<\infty$, the maps*

$$[p,\infty)\times[1,\infty)\ni(q;R)\mapsto C^{p,q;R}=C^{p,q;R}_{\rm rad}=\overline{C}^{p,q;R}=\overline{C}^{p,q;R}_{\rm rad}\in\mathbf{R} \tag{2.45}$$

*are continuous.*

On the existence of extremal functions we have the next theorem, which is proved in Section 4. When $n\geq 2$, $p<q$, and $R>1$ hold, we do not yet know if $C^{p,q;R}$ and $\overline{C}^{p,q;R}$ are achieved by any extremals or not.

THEOREM 2.9
*Assume that $1<p<q<\infty$, assume that $\tau_{p,q}\leq 1/n$, and assume that $R\geq 1$. Then we have the following.*

(1) *For $R=1$, $C^{p,q;1}_{\rm rad}$ and $\overline{C}^{p,q;1}_{\rm rad}$ are achieved in $W^{1,p}_{0,0}(B_1)_{\rm rad}\setminus\{0\}$ and $W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})_{\rm rad}\setminus\{0\}$, respectively.*

(2) *For $n=1$ and $R=1$, $C^{p,q;1}=C^{p,q;1}_{\rm rad}$ and $\overline{C}^{p,q;1}=\overline{C}^{p,q;1}_{\rm rad}$ are achieved in $W^{1,p}_{0,0}((-1,1))_{\rm rad}\setminus\{0\}$ and $W^{1,p}_{0,0}(\mathbf{R}\setminus[-1,1])_{\rm rad}\setminus\{0\}$, respectively.*

(3) *For $R>1$, $C^{p,q;R}_{\rm rad}$ and $\overline{C}^{p,q;R}_{\rm rad}$ are not achieved in $W^{1,p}_{0,0}(B_1)_{\rm rad}\setminus\{0\}$ and $W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})_{\rm rad}\setminus\{0\}$, respectively.*

We also have the next proposition, which is proved in Section 8.2 together with Theorem 2.4(4).

PROPOSITION 2.4
*Let $1<p=q<\infty$, and let $\tau_{p,q}\leq 1/n$. If $R>1$ is sufficiently large, then $C^{p,p;R}$, $C^{p,p;R}_{\rm rad}$, $\overline{C}^{p,p;R}$, and $\overline{C}^{p,p;R}_{\rm rad}$ are not achieved in $W^{1,p}_{0,0}(B_1)\setminus\{0\}$, $W^{1,p}_{0,0}(B_1)_{\rm rad}\setminus\{0\}$, $W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})\setminus\{0\}$, and $W^{1,p}_{0,0}(\mathbf{R}^n\setminus\overline{B_1})_{\rm rad}\setminus\{0\}$ respectively.*

## 3. Change of variables and the best constants

Here we see the relations among the best constants by the method of change of variables.

DEFINITION 3.1
For $\beta>0$ and $R\geq 1$, we set the following:

(1) $\overline{Y}(y)=\frac{y}{|y|^2}$ for $y\in\mathbf{R}^n\setminus\{0\}$.
(2) $Y_\beta(y)=|y|^{\beta-1}y$ for $y\in\mathbf{R}^n$.
(3) $\tilde{Y}_R(y)=R\exp(-\frac{1}{|y|})\frac{y}{|y|}$ for $y\in\mathbf{R}^n$.

REMARK 3.1

For $\beta > 0$ and $R \geq 1$, we have the inverse maps as follows:

(1) $\overline{Y}^{-1}(x) = \overline{Y}(x) = \frac{x}{|x|^2}$ for $x \in \mathbf{R}^n \setminus \{0\}$.
(2) $Y_\beta^{-1}(x) = Y_{1/\beta}(x) = |x|^{1/\beta - 1}x$ for $x \in \mathbf{R}^n$.
(3) $\tilde{Y}_R^{-1}(x) = \frac{1}{\log(R/|x|)} \frac{x}{|x|}$ for $x \in B_R$.

In what follows we define various operators which are fundamental in the present paper.

DEFINITION 3.2

Let $\beta > 0$, and let $R \geq 1$. Let $\Omega$ be a domain of $\mathbf{R}^n$, and let $u : \Omega \to \mathbf{R}$. We have:

(1) $\overline{T}u(y) = u(\overline{Y}(y)) = u\big(\frac{y}{|y|^2}\big)$ for $y \in \overline{Y}^{-1}(\Omega \setminus \{0\})$.
(2) $T_\beta u(y) = u(Y_\beta(y)) = u(|y|^{\beta-1}y)$ for $y \in Y_{1/\beta}^{-1}(\Omega)$.
(3) For $\Omega \subset B_R$,

$$\tilde{T}_R u(y) = u\big(\tilde{Y}_R(y)\big) = u\bigg(R \exp\Big(-\frac{1}{|y|}\Big)\frac{y}{|y|}\bigg), \quad \text{for } y \in \tilde{Y}_R^{-1}(\Omega).$$

We begin by studying the operator $\overline{T}$. By a direct calculation we have

$$\det(\delta_{ij} + a x_i x_j)_{1 \leq i,j \leq n} = 1 + a|x|^2 \quad \text{for } x \in \mathbf{R}^n, a \in \mathbf{R}. \tag{3.1}$$

Since the Jacobi determinant of the change of variables defined by $x = \overline{Y}(y) = y/|y|^2$ is

$$\det D\overline{Y}(y) = \det\bigg(\frac{1}{|y|^2}\Big(\delta_{ij} - 2\frac{y_i y_j}{|y|^2}\Big)\bigg)_{1 \leq i,j \leq n} = -\frac{1}{|y|^{2n}}, \tag{3.2}$$

we have the following lemma.

LEMMA 3.1

*Assume that $1 \leq p \leq q < \infty$, assume that $\gamma \neq 0$, and assume that $R \geq 1$. Then we have the following:*

(1) $\|u\|_{L^q_\gamma(\mathbf{R}^n)} = \|\overline{T}u\|_{L^q_{-\gamma}(\mathbf{R}^n)}$ *for* $u \in L^q_\gamma(\mathbf{R}^n)$,
$\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \|\nabla[\overline{T}u]\|_{L^p_{1-\gamma}(\mathbf{R}^n)}$ *for* $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$;
(2) $\|u\|_{L^q_{p;R}(B_1)} = \|\overline{T}u\|_{L^q_{p;R}(\mathbf{R}^n \setminus \overline{B_1})}$ *for* $u \in L^q_{p;R}(B_1)$,
$\|\nabla u\|_{L^p_1(B_1)} = \|\nabla[\overline{T}u]\|_{L^p_1(\mathbf{R}^n \setminus \overline{B_1})}$ *for* $u \in W^{1,p}_{0,0}(B_1)$.

For the proof of this, it suffices to note that for $x = y/|y|^2$ we have

$$\Big|(\nabla_x u)\Big(\frac{y}{|y|^2}\Big)\Big|^2 = |y|^4\big|\nabla_y\big(\overline{T}u(y)\big)\big|^2, \quad \text{for } y \in \mathbf{R}^n \setminus \{0\}.$$

As a direct consequence of this we have the next proposition, which proves Theorem 2.2(1) and Theorem 2.7(1). Further we see that in the proofs of Theorems

2.1–2.4, it suffices to assume that $\gamma > 0$, and it suffices to establish the proofs of Theorems 2.5–2.9 in a unit ball $B_1$.

PROPOSITION 3.1

*Assume that $1 \le p \le q < \infty$, assume that $\gamma \ne 0$, and assume that $R \ge 1$. Then we have the following:*

(1) $S^{p,q;\gamma} = S^{p,q;-\gamma}, S_{\text{rad}}^{p,q;\gamma} = S_{\text{rad}}^{p,q;-\gamma}$.
(2) $C^{p,q;R} = \overline{C}^{p,q;R}, C_{\text{rad}}^{p,q;R} = \overline{C}_{\text{rad}}^{p,q;R}$.

*Proof*

From Lemma 3.1 we see that

$$E^{p,q;\gamma}[u] = E^{p,q;-\gamma}[\overline{T}u] \quad \text{for } u \in W_{\gamma,0}^{1,p}(\mathbf{R}^n) \setminus \{0\}, \tag{3.3}$$

$$F^{p,q;R}[u] = \overline{F}^{p,q;R}[\overline{T}u] \quad \text{for } u \in W_{0,0}^{1,p}(B_1) \setminus \{0\}; \tag{3.4}$$

hence the assertions follow. □

In the next lemma we consider the operators $T_\beta$, $\tilde{T}_R$. By $\Delta_{S^{n-1}}$ we denote the Laplace–Beltrami operator on a unit sphere $S^{n-1}$. Then a gradient operator $\Lambda$ on $S^{n-1}$ is defined by

$$\int_{S^{n-1}} (-\Delta_{S^{n-1}} u) v \, dS = \int_{S^{n-1}} \Lambda u \cdot \Lambda v \, dS \quad \text{for } u, v \in C^2(S^{n-1}). \tag{3.5}$$

Here we note that

$$\Delta u = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\Delta_{S^{n-1}} u, \qquad |\nabla u|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}|\Lambda u|^2, \tag{3.6}$$

where

$$r(x) = |x|, \qquad \frac{\partial u}{\partial r}(x) = \frac{x}{|x|} \cdot \nabla u(x). \tag{3.7}$$

The Jacobi determinant of the change of variables $x = Y_\beta(y) = |y|^{\beta-1}y$ is given by

$$\det DY_\beta(y) = \det\left(|y|^{\beta-1}\left(\delta_{ij} + (\beta-1)\frac{y_i y_j}{|y|^2}\right)\right)_{1 \le i,j \le n} = \beta |y|^{n(\beta-1)}. \tag{3.8}$$

Hence by calculations we have the next lemma.

LEMMA 3.2

*Assume that $1 \le p \le q < \infty$, $\gamma > 0$, $R \ge 1$, and $\beta > 0$. Then we have the following:*

(1) $\|u\|_{L_\gamma^q(\mathbf{R}^n)} = \beta^{1/q}\|T_\beta u\|_{L_{\beta\gamma}^q(\mathbf{R}^n)}$ *for* $u \in L_\gamma^q(\mathbf{R}^n)$,
$\|\nabla u\|_{L_{1+\gamma}^p(\mathbf{R}^n)} = \frac{1}{\beta^{1/p'}}\|(|\frac{\partial}{\partial r}[T_\beta u]|^2 + \frac{\beta^2}{r^2}|\Lambda[T_\beta u]|^2)^{1/2}\|_{L_{1+\beta\gamma}^p(\mathbf{R}^n)}$ *for* $u \in W_{\gamma,0}^{1,p}(\mathbf{R}^n)$;

(2) $\|u\|_{L^q_{p;R}(B_1)} = \frac{1}{\beta^{1/p'}}\|T_\beta u\|_{L^q_{p;R^{1/\beta}}(B_1)}$ *for* $u \in L^q_{p;R}(B_1)$,
$\|\nabla u\|_{L^p_1(B_1)} = \frac{1}{\beta^{1/p'}}\|(|\frac{\partial}{\partial r}[T_\beta u]|^2 + \frac{\beta^2}{r^2}|\Lambda[T_\beta u]|^2)^{1/2}\|_{L^p_1(B_1)}$ *for* $u \in W^{1,p}_{0,0}(B_1)$.

As a consequence we have the next proposition, which gives Theorem 2.2(4) and Theorem 2.7(4) as well.

PROPOSITION 3.2

*Assume that* $1 \le p \le q < \infty$. *Then we have the following:*

(1) $\big(\frac{\gamma}{\overline{\gamma}}\big)^{p(1-\tau_{p,q})} S^{p,q;\overline{\gamma}} \le S^{p,q;\gamma} \le \big(\frac{\overline{\gamma}}{\gamma}\big)^{p\tau_{p,q}} S^{p,q;\overline{\gamma}}, S^{p,q;\gamma}_{\mathrm{rad}} = \big(\frac{\gamma}{\overline{\gamma}}\big)^{p(1-\tau_{p,q})} S^{p,q;\overline{\gamma}}_{\mathrm{rad}}$ *for* $0 < \gamma \le \overline{\gamma}$.

*In particular, there is a constant* $\hat{S}_{p,q} \ge 0$ *such that we have*

$$S^{p,q;\gamma}_{\mathrm{rad}} = \hat{S}_{p,q}\gamma^{p(1-\tau_{p,q})} \quad \textit{for } \gamma > 0.$$

(2) $C^{p,q;R} \le C^{p,q;\overline{R}} \le \big(\frac{\log \overline{R}}{\log R}\big)^p C^{p,q;R}, C^{p,q;R}_{\mathrm{rad}} = C^{p,q;\overline{R}}_{\mathrm{rad}}$ *for* $1 < R \le \overline{R}$.

*In particular, there is a constant* $\hat{C}_{p,q} \ge 0$ *such that we have*

$$C^{p,q;R}_{\mathrm{rad}} = \hat{C}_{p,q} \quad \textit{for } R > 1.$$

*Proof*

Let us note that by Remark 3.1, $u = T_{1/\beta}v$ holds for $v = T_\beta u$. Then it follows from Lemma 3.2(1) with $\beta = \gamma/\overline{\gamma}$ that we have

$$\begin{aligned}
&\Big(\frac{\gamma}{\overline{\gamma}}\Big)^{p(1-\tau_{p,q})} E^{p,q;\overline{\gamma}}[T_{\overline{\gamma}/\gamma}u] \\
&\qquad \le E^{p,q;\gamma}[u] \le \Big(\frac{\overline{\gamma}}{\gamma}\Big)^{p\tau_{p,q}} E^{p,q;\overline{\gamma}}[T_{\overline{\gamma}/\gamma}u] \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n) \setminus \{0\}, \\
&E^{p,q;\gamma}[u] = \Big(\frac{\gamma}{\overline{\gamma}}\Big)^{p(1-\tau_{p,q})} E^{p,q;\overline{\gamma}}[T_{\overline{\gamma}/\gamma}u] \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}} \setminus \{0\}.
\end{aligned}$$

From assertion (2) with $\beta = (\log \overline{R})/(\log R)$, we have

$$\begin{aligned}
&F^{p,q;R}[T_{\log \overline{R}/\log R}u] \le F^{p,q;\overline{R}}[u] \\
&\qquad \le \Big(\frac{\log \overline{R}}{\log R}\Big)^p F^{p,q;R}[T_{\log \overline{R}/\log R}u] \quad \text{for } u \in W^{1,p}_{0,0}(B_1) \setminus \{0\}, \\
&F^{p,q;\overline{R}}[u] = F^{p,q;R}[T_{\log \overline{R}/\log R}u] \quad \text{for } u \in W^{1,p}_{0,0}(B_1)_{\mathrm{rad}} \setminus \{0\}.
\end{aligned}$$

Thus the desired assertions follow. □

Further from Proposition 3.2 we have the following proposition.

PROPOSITION 3.3

*Assume that* $1 \le p \le q < \infty$, *assume that* $\overline{\gamma} > 0$, *and assume that* $\underline{R} > 1$. *Then we have the following.*

(1) *If* $S^{p,q;\overline{\gamma}} = S_{\mathrm{rad}}^{p,q;\overline{\gamma}}$ *holds, then*

$$S^{p,q;\gamma} = S_{\mathrm{rad}}^{p,q;\gamma} = \hat{S}_{p,q}\gamma^{p(1-\tau_{p,q})} \quad \text{for } 0 < \gamma \le \overline{\gamma}.$$

(2) *If* $C^{p,q;\underline{R}} = C_{\mathrm{rad}}^{p,q;\underline{R}}$ *holds, then*

$$C^{p,q;R} = C_{\mathrm{rad}}^{p,q;R} = \hat{C}_{p,q} \quad \text{for } R \ge \underline{R}.$$

Lastly we have the next lemma, noting that the Jacobi determinant of the change of variables $x = \tilde{Y}_R(y) = R\exp(-1/|y|)y/|y|$ is given by

$$\tag{3.9} \begin{aligned} \det D\tilde{Y}_R(y) &= \det\Big(\frac{R}{|y|}\exp\Big(-\frac{1}{|y|}\Big)\Big(\delta_{ij} + \Big(\frac{1}{|y|} - 1\Big)\frac{y_i y_j}{|y|^2}\Big)\Big)_{1\le i,j\le n} \\ &= R^n \exp\Big(-\frac{n}{|y|}\Big)\frac{1}{|y|^{n+1}}. \end{aligned}$$

LEMMA 3.3

*Assume that* $1 \le p \le q < \infty$, *and assume that* $R \ge 1$. *Then we have the following:*

$$\|u\|_{L^q_{p;R}(B_1)} = \|\tilde{T}_R u\|_{L^q_{1/p'}(B_{1/\log R})} \quad \text{for } u \in L^q_{p;R}(B_1),$$

$$\|\nabla u\|_{L^p_1(B_1)} = \Big\|\Big(\Big|\frac{\partial}{\partial r}[\tilde{T}_R u]\Big|^2 + \frac{1}{r^4}|\Lambda[\tilde{T}_R u]|^2\Big)^{1/2}\Big\|_{L^p_{1+1/p'}(B_{1/\log R})}$$

*for* $u \in W^{1,p}_{0,0}(B_1)$.

Combining this with Proposition 3.2(2) we obtain the next proposition.

PROPOSITION 3.4

*For* $1 \le p \le q < \infty$ *we have*

$$\tag{3.10} C_{\mathrm{rad}}^{p,q;R} = S_{\mathrm{rad}}^{p,q;1/p'} = \frac{\hat{S}_{p,q}}{(p')^{p(1-\tau_{p,q})}} \quad \text{for } R \ge 1.$$

*Proof*

It follows from Lemma 3.3 that we have

$$\tag{3.11} F^{p,q;R}[u] = E^{p,q;1/p'}[\tilde{T}_R u] \quad \text{for } u \in W^{1,p}_{0,0}(B_1)_{\mathrm{rad}} \setminus \{0\}.$$

Here we note that the operator $\tilde{T}_R u$ is an extension of $T_R u$ to the whole $\mathbf{R}^n$ by setting $\tilde{T}_R u = 0$ on $\mathbf{R}^n \setminus B_{1/\log R}$.

Then we immediately have $C_{\mathrm{rad}}^{p,q;1} = S_{\mathrm{rad}}^{p,q;1/p'}$. From Proposition 3.2(2) we also have

$$\begin{aligned} \hat{C}_{p,q} &= \inf_{R>1} C_{\mathrm{rad}}^{p,q;R} = \inf_{R>1} \inf\big\{F^{p,q;R}[u] \,\big|\, u \in C_{\mathrm{c}}^\infty(B_1 \setminus \{0\})_{\mathrm{rad}} \setminus \{0\}\big\} \\ &= \inf_{R>1} \inf\big\{E^{p,q;1/p'}[\tilde{T}_R u] \,\big|\, u \in C_{\mathrm{c}}^\infty(B_1 \setminus \{0\})_{\mathrm{rad}} \setminus \{0\}\big\} \\ &= \inf\big\{E^{p,q;1/p'}[v] \,\big|\, v \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \{0\})_{\mathrm{rad}} \setminus \{0\}\big\} = S_{\mathrm{rad}}^{p,q;1/p'}. \end{aligned}$$

The assertion follows from this together with assertion 1 and Proposition 3.2(1). □

## 4. Relations among $S_{\mathrm{rad}}^{p,q;\gamma}$, $C_{\mathrm{rad}}^{p,q;R}$, $S^{p,q;\gamma}$, and $C^{p,q;R}$

In this section we exactly determine the best constants $S_{\mathrm{rad}}^{p,q;\gamma}$ and $C_{\mathrm{rad}}^{p,q;R}$ in the radials function spaces, and we study when $S_{\mathrm{rad}}^{p,q;\gamma}$ and $C_{\mathrm{rad}}^{p,q;R}$ should coincide with $S^{p,q;\gamma}$ and $C^{p,q;R}$, respectively.

### 4.1. Variational problems in radially symmetric spaces

In this section we determine the best constants $S_{\mathrm{rad}}^{p,q;\gamma}$ and $C_{\mathrm{rad}}^{p,q;R}$ for $p<q$ by solving corresponding variational problems in radially symmetric spaces employing Talenti's result in an essential way. We begin by introducing variational problems and solutions.

DEFINITION 4.1

Let $1<p<q<\infty$, and let $a,b>0$. We have:

(1) $C_{p,q}^1((0,\infty))=\{u\in C^1((0,\infty))\mid \int_0^\infty |u'(r)|^p r^{1/\tau_{p,q}-1}\,dr<\infty, u(r)\to 0$ as $r\to\infty\}$;

(2) $J^{p,q}[u]=\frac{(\int_0^\infty |u'(r)|^p r^{1/\tau_{p,q}-1}\,dr)^{1/p}}{(\int_0^\infty |u(r)|^q r^{1/\tau_{p,q}-1}\,dr)^{1/q}}$ for $u\in C_{p,q}^1((0,\infty))\setminus\{0\}$;

(3) $\varphi_0(x)=\varphi_0(|x|)=\frac{1}{(a+b|x|^{p'})^{p/(q-p)}}$ for $x\in\mathbf{R}^n\setminus\{0\}$.

(In what follows $\varphi_0$ is also regarded as a function of $r=|x|$ on $(0,\infty)$.)

The next lemma is essentially due to G. Talenti (see [Ta1, Lemma 2]).

LEMMA 4.1

*For $1<p<q<\infty$, we have*

$$J^{p,q}[u]\ge J^{p,q}[\varphi_0]\quad \textit{for } u\in C_{p,q}^1\big((0,\infty)\big)\setminus\{0\}. \tag{4.1}$$

Noting that

$$\int_0^\infty \frac{t^{\alpha-1}}{(1+t)^\beta}\,dt=\mathrm{B}(\alpha,\beta-\alpha)\quad\text{for } 0<\alpha<\beta, \tag{4.2}$$

we have

$$\int_0^\infty |\varphi_0(r)|^q r^{1/\tau_{p,q}-1}\,dr=\frac{1}{(a^{1/p}b^{1/p'})^{1/\tau_{p,q}}}\frac{1}{p'}\mathrm{B}\Big(\frac{1}{p\tau_{p,q}},\frac{1}{p'\tau_{p,q}}\Big), \tag{4.3}$$

$$\begin{aligned}&\int_0^\infty |\varphi_0'(r)|^p r^{1/\tau_{p,q}-1}\,dr\\ &=\frac{1}{(a^{1/p}b^{1/p'})^{p/(q\tau_{p,q})}}\frac{(p')^{p-1}}{(q\tau_{p,q})^p}\mathrm{B}\Big(\frac{1}{p\tau_{p,q}}-1,\frac{1}{p'\tau_{p,q}}+1\Big)\\ &=\frac{1}{(a^{1/p}b^{1/p'})^{p/(q\tau_{p,q})}}\frac{(p')^{p-2}}{q^{p-1}\tau_{p,q}^p}\mathrm{B}\Big(\frac{1}{p\tau_{p,q}},\frac{1}{p'\tau_{p,q}}\Big).\end{aligned} \tag{4.4}$$

Hence we have

$$J^{p,q}[\varphi_0] = \frac{(p')^{1/p'-\tau_{p,q}}}{q^{1/p'}\tau_{p,q}} \mathrm{B}\Big(\frac{1}{p\tau_{p,q}}, \frac{1}{p'\tau_{p,q}}\Big)^{\tau_{p,q}}. \tag{4.5}$$

First of all, for $\gamma > 0$, we have the next proposition and then Theorem 2.4(1) follows. Moreover combining it with Proposition 3.2, Theorem 2.2(2) follows.

PROPOSITION 4.1

*Assume that $1 < p < q < \infty$, and assume that $\gamma > 0$. Then we have the following.*

(1) *The infimum of $S^{p,q;\gamma}_{\mathrm{rad}}$ in $W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}} \setminus \{0\}$ is attained by $u^{p,q;\gamma} = T_{q\tau_{p,q}\gamma}\varphi_0$.*

(2) *In Proposition 3.2(1),*

$$\hat{S}_{p,q} = \big(\omega_n^{\tau_{p,q}}(q\tau_{p,q})^{1-\tau_{p,q}} J^{p,q}[\varphi_0]\big)^p = S_{p,q}.$$

*Proof*

(1) It follows from Lemma 3.2 that we have for $u \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \{0\})_{\mathrm{rad}}$,

$$\|u\|_{L^q_\gamma(\mathbf{R}^n)} = \frac{1}{(q\tau_{p,q}\gamma)^{1/q}} \|T_{1/(q\tau_{p,q}\gamma)}u\|_{L^q_{1/(q\tau_{p,q})}(\mathbf{R}^n)}, \tag{4.6}$$

$$\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = (q\tau_{p,q}\gamma)^{1/p'} \|\nabla[T_{1/(q\tau_{p,q}\gamma)}u]\|_{L^p_{1+1/(q\tau_{p,q})}(\mathbf{R}^n)}. \tag{4.7}$$

Then we have

$$E^{p,q;\gamma}[u] = \big(\omega_n^{\tau_{p,q}}(q\tau_{p,q})^{1-\tau_{p,q}} J^{p,q}[T_{1/(q\tau_{p,q}\gamma)}u]\big)^p \gamma^{p(1-\tau_{p,q})}$$
$$\text{for } u \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \{0\})_{\mathrm{rad}} \setminus \{0\};$$

hence the assertion follows from Lemma 4.1.

(2) This is clear from assertion (1) of this proposition and Proposition 3.2(1).

□

Let us proceed to the case $\gamma = 0$. In this case we have the next proposition, from which Theorem 2.6 and assertions (1) and (3) of Theorem 2.9 follow. Moreover combining it with Proposition 3.2(2), Theorem 2.7(2) follows.

PROPOSITION 4.2

*Assume that $1 < p < q < \infty$, assume that $\gamma = 0$, and assume that $R \geq 1$. Then we have the following.*

(1) *If $R = 1$, the infimum of $C^{p,q;1}_{\mathrm{rad}}$ in $W^{1,p}_{0,0}(B_1)_{\mathrm{rad}} \setminus \{0\}$ is attained by $\tilde{u}^{p,q;1} = \tilde{T}_1^{-1}[T_{q\tau_{p,q}/p'}\varphi_0]$.*

(2) *In Proposition 3.2(2), it holds that*

$$\hat{C}_{p,q} = \left(\omega_n^{\tau_{p,q}}\left(\frac{q\tau_{p,q}}{p'}\right)^{1-\tau_{p,q}} J^{p,q}[\varphi_0]\right)^p = C_{p,q}.$$

(3) *If $R > 1$, then the infimum of $C^{p,q;R}_{\mathrm{rad}}$ is not attained in $W^{1,p}_{0,0}(B_1)_{\mathrm{rad}} \setminus \{0\}$.*

*Proof*

(1) From Lemmas 3.2 and 3.3, we have for $u \in C_c^\infty(B_1 \setminus \{0\})_{\mathrm{rad}}$

$$\begin{aligned}\|u\|_{L^q_{p;R}(B_1)} &= \|\tilde{T}_1 u\|_{L^q_{1/p'}(\mathbf{R}^n)} \\ &= \Big(\frac{p'}{q\tau_{p,q}}\Big)^{1/q} \|T_{p'/(q\tau_{p,q})}[\tilde{T}_1 u]\|_{L^q_{1/(q\tau_{p,q})}(\mathbf{R}^n)},\end{aligned} \tag{4.8}$$

$$\begin{aligned}\|\nabla u\|_{L^p_1(B_1)} &= \|\nabla[\tilde{T}_1 u]\|_{L^p_{1+1/p'}(\mathbf{R}^n)} \\ &= \Big(\frac{q\tau_{p,q}}{p'}\Big)^{1/p'} \big\|\nabla\big[T_{p'/(q\tau_{p,q})}[\tilde{T}_1 u]\big]\big\|_{L^p_{1+1/(q\tau_{p,q})}(\mathbf{R}^n)},\end{aligned} \tag{4.9}$$

and we have

$$F^{p,q;1}[u] = \Big(\omega_n^{\tau_{p,q}} \Big(\frac{q\tau_{p,q}}{p'}\Big)^{1-\tau_{p,q}} J^{p,q}\big[T_{p'/(q\tau_{p,q})}[\tilde{T}_1 u]\big]\Big)^p$$
$$\text{for } u \in C_c^\infty(B_1 \setminus \{0\})_{\mathrm{rad}} \setminus \{0\}.$$

Hence from Lemma 4.1 the desired assertion follows.

(2) This is clear from Propositions 3.2(2) and 3.4.

(3) If $u \in W^{1,p}_{0,0}(B_1)_{\mathrm{rad}} \setminus \{0\}$ for $R > 1$ achieves the infimum of $C^{p,q;R}_{\mathrm{rad}}$, then from the previous result we have

$$F^{p,q;R}[u] = C^{p,q;R}_{\mathrm{rad}} = C_{p,q}.$$

But we have $F^{p,q;R}[u] > F^{p,q;\underline{R}}[u] \geq C_{p,q}$ for any $1 < \underline{R} < R$, and this is a contradiction. □

## 4.2. A generalized rearrangement of functions

We introduce a rearrangement of functions with respect to general weight functions instead of the Lebesgue measure to establish the validity of $S^{p,q;\gamma} = S^{p,q;\gamma}_{\mathrm{rad}}$ and $C^{p,q;R} = C^{p,q;R}_{\mathrm{rad}}$ under additional conditions. In this section we begin by studying a theory of generalized rearrangement of functions (cf. [Ta1], [Ta2]).

DEFINITION 4.2

(1) For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $f \geq 0$ a.e. on $\mathbf{R}^n$, let us set for a (Lebesgue) measurable set $A$

$$\mu_f(A) = \int_A d\mu_f = \int_A f(x)\,dx. \tag{4.10}$$

Then $\mu_f$ is said to be the measure determined by $f$.

(2) We say that $f$ is admissible if and only if $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) \cap C(\mathbf{R}^n \setminus \{0\})_{\mathrm{rad}}$, $f \geq 0$ on $\mathbf{R}^n \setminus \{0\}$, and $f$ is nonincreasing with respect to $r = |x|$.

(3) For an admissible $f$ and a Borel set $A \subset \mathbf{R}^n$ satisfying $0 < \mu_1(A) < +\infty$, let us define $r_f[A] > 0$ by $\mu_f(A) = \mu_f(B_{r_f[A]})$. Then $B_{r_f[A]}$ is said to be the rearrangement set of $A$ by $f$.

(4) For an admissible $f$ and $u : \mathbf{R}^n \to \mathbf{R}$, we set

$$\mu_f[u](t) = \mu_f\big(\{|u| > t\}\big) = \int_{\{|u|>t\}} f(x)\,dx \quad \text{for } t \geq 0, \tag{4.11}$$

$$
\begin{aligned}
\mathcal{R}_f[u](x) &= \mathcal{R}_f[u](|x|) \\
&= \sup\bigl\{t \geq 0 \bigm| \mu_f[u](t) > \mu_f(B_{|x|})\bigr\} \quad \text{for } x \in \mathbf{R}^n \setminus \{0\}.
\end{aligned} \tag{4.12}
$$

Then $\mu_f[u]$ and $\mathcal{R}_f[u]$ are said to be the distribution function of $u$ and the rearrangement function of $u$ with respect to $f$, respectively.

Direct from this definition we see the next proposition.

PROPOSITION 4.3

*Let $1 \leq p < \infty$, and assume that $f$ is admissible. Then, for $u : \mathbf{R}^n \to \mathbf{R}$, we have the following:*

(1) $\mu_f[u](t) = \mu_f[\mathcal{R}_f[u]](t)$ *for* $t \geq 0$;
(2) $\mathcal{R}_f[|u|^p](x) = \mathcal{R}_f[u](x)^p$ *for* $x \in \mathbf{R}^n \setminus \{0\}$;
(3) *if $u$ is radially symmetric and nonincreasing with respect to $r = |x|$, then*

$$\mathcal{R}_f[u](x) = u(x) \quad \textit{for a.e. } x \in \mathbf{R}^n \setminus \{0\}.$$

Further we have the following proposition.

PROPOSITION 4.4

*Let $1 \leq p < \infty$, and assume that $f$ is admissible. Then, for $u, v : \mathbf{R}^n \to \mathbf{R}$, we have the following:*

(1) $\int_{\mathbf{R}^n} |u(x)|^p f(x)\,dx = \int_{\mathbf{R}^n} \mathcal{R}_f[u](x)^p f(x)\,dx$.
(2) $\int_{\mathbf{R}^n} |u(x)v(x)| f(x)\,dx \leq \int_{\mathbf{R}^n} \mathcal{R}_f[u](x)\mathcal{R}_f[v](x) f(x)\,dx$.

*Proof*

(1) Since $|u(x)|^p = p\int_0^\infty \chi_{\{|u|>t\}}(x) t^{p-1}\,dt$ for a.e. $x \in \mathbf{R}^n$, we see that

$$
\begin{aligned}
&\int_{\mathbf{R}^n} |u(x)|^p f(x)\,dx \\
&\qquad = p\int_{\mathbf{R}^n} \Bigl(\int_0^\infty \chi_{\{|u|>t\}}(x) t^{p-1}\,dt\Bigr) f(x)\,dx \\
&\qquad = p\int_0^\infty \Bigl(\int_{\{|u|>t\}} f(x)\,dx\Bigr) t^{p-1}\,dt = p\int_0^\infty \mu_f[u](t) t^{p-1}\,dt,
\end{aligned} \tag{4.13}
$$

and in a similar way

$$\int_{\mathbf{R}^n} \mathcal{R}_f[u](x)^p f(x)\,dx = p\int_0^\infty \mu_f\bigl[\mathcal{R}_f[u]\bigr](t) t^{p-1}\,dt. \tag{4.14}$$

Then the assertion follows from Proposition 4.3(1).

(2.a) First we show that

$$\mu_f\bigl(\bigl\{|u| > t\bigr\} \cap \bigl\{|v| > s\bigr\}\bigr) \leq \mu_f\bigl(\bigl\{\mathcal{R}_f[u] > t\bigr\} \cap \bigl\{\mathcal{R}_f[v] > s\bigr\}\bigr) \quad \text{for } s, t \geq 0. \tag{4.15}$$

If $\mu_f\big(\{|u|>t\}\big)\le\mu_f\big(\{|v|>s\}\big)$, then we have $\{\mathcal{R}_f[u]>t\}\subset\{\mathcal{R}_f[v]>s\}$. So it follows from Proposition 4.3(1) that we have

$$\begin{aligned}\mu_f\big(\{|u|>t\}\cap\{|v|>s\}\big)&\le\mu_f(\{u>t\})\\&=\mu_f\big(\{\mathcal{R}_f[u]>t\}\big)=\mu_f\big(\{\mathcal{R}_f[u]>t\}\cap\{\mathcal{R}_f[v]>s\}\big).\end{aligned}$$

If $\mu_f(\{|v|>s\})\le\mu_f(\{|u|>t\})$, then we see that $\{\mathcal{R}_f[v]>s\}\subset\{\mathcal{R}_f[u]>t\}$; hence in a similar way the desired assertion holds.

(2.b) In a similar way we see that

$$\begin{aligned}\int_{\mathbf{R}^n}|u(x)v(x)|f(x)\,dx&=\int_{\mathbf{R}^n}\Big(\int_0^\infty\chi_{\{|u|>t\}}(x)\,dt\Big)\Big(\int_0^\infty\chi_{\{|v|>s\}}(x)\,ds\Big)f(x)\,dx\\&=\int_0^\infty\int_0^\infty\Big(\int_{\{|u|>t\}\cap\{|v|>s\}}f(x)\,dx\Big)\,ds\,dt\\&=\int_0^\infty\int_0^\infty\mu_f\big(\{|u|>t\}\cap\{|v|>s\}\big)\,ds\,dt\end{aligned}\tag{4.16}$$

and

$$\begin{aligned}&\int_{\mathbf{R}^n}\mathcal{R}_f[u](x)\mathcal{R}_f[v](x)f(x)\,dx\\&\quad=\int_0^\infty\int_0^\infty\mu_f\big(\{\mathcal{R}_f[u]>t\}\cap\{\mathcal{R}_f[v]>s\}\big)\,ds\,dt.\end{aligned}\tag{4.17}$$

The assertion therefore follows from (2.a). □

If $u\in C_c(\mathbf{R}^n)$ is Lipschitz continuous, then $u$ is differentiable for a.e. $x\in\mathbf{R}^n$ and $|\nabla u|\in L^\infty(\mathbf{R}^n)$. For an admissible $f$, we see that $\mathcal{R}_f[u]$ for $u\in C_c^1(\mathbf{R}^n)$ becomes Lipschitz continuous, and hence $\mathcal{R}_f[u]$ is differentiable for a.e. $x\in\mathbf{R}^n$ and $|\nabla[\mathcal{R}_f[u]](x)|\in L^\infty(\mathbf{R}^n)$. Then we have the next proposition, which is established in the Appendix.

PROPOSITION 4.5

*Let $1\le p<\infty$, and assume that $f$ is admissible. Then, for $u\in C_c^1(\mathbf{R}^n)$ we have*

$$\int_{\mathbf{R}^n}\big|\nabla\big[\mathcal{R}_f[u]\big](x)\big|^p\frac{1}{f(x)^{p-1}}\,dx\le\int_{\mathbf{R}^n}|\nabla u(x)|^p\frac{1}{f(x)^{p-1}}\,dx.$$

### 4.3. Application of the theory on rearrangement of functions

In this section we establish $S^{p,q;\gamma}=S^{p,q;\gamma}_{\mathrm{rad}}$ and $C^{p,q;R}=C^{p,q;R}_{\mathrm{rad}}$ under certain assumptions by using the theory on the generalized rearrangement of functions that was developed in the previous section.

First let us consider the case in which $\gamma>0$. Then we have the following proposition, which proves Theorem 2.2(3). Further, making use of Theorem 2.2(4) at the same time, we see that Theorem 2.1(1) follows as well. Here we note that $I_\alpha$ is admissible if $0<\alpha\le n$.

PROPOSITION 4.6

*For $1 \le p \le q < \infty$, $\tau_{p,q} \le 1/n$, and $0 < \gamma \le \gamma_{p,q}$, it holds that $S^{p,q;\gamma} = S^{p,q;\gamma}_{\mathrm{rad}}$.*

*Proof*

By virtue of Proposition 3.3(1), it suffices to consider the case $\gamma = \gamma_{p,q} = (n-1)/(1+q/p')$. Since $0 < q\gamma_{p,q} < n$, by using Propositions 4.3(2), 4.4(1), and 4.5(2), we have for $u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n \setminus \{0\})$

$$\begin{aligned}\|u\|^q_{L^q_{\gamma_{p,q}}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma_{p,q}}(x)\,dx = \int_{\mathbf{R}^n} \mathcal{R}_{I_{q\gamma_{p,q}}}[|u|^q](x) I_{q\gamma_{p,q}}(x)\,dx \\ &= \int_{\mathbf{R}^n} \mathcal{R}_{I_{q\gamma_{p,q}}}[u](x)^q I_{q\gamma_{p,q}}(x)\,dx = \|\mathcal{R}_{I_{q\gamma_{p,q}}}[u]\|^q_{L^q_{\gamma_{p,q}}(\mathbf{R}^n)}, \\ \|\nabla u\|^p_{L^p_{1+\gamma_{p,q}}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |\nabla u(x)|^p \frac{1}{I_{q\gamma_{p,q}}(x)^{p-1}}\,dx \\ &\ge \int_{\mathbf{R}^n} \left|\nabla\left[\mathcal{R}_{I_{q\gamma_{p,q}}}[u]\right](x)\right|^p \frac{1}{I_{q\gamma_{p,q}}(x)^{p-1}}\,dx \\ &= \left\|\nabla\left[\mathcal{R}_{I_{q\gamma_{p,q}}}[u]\right]\right\|^p_{L^p_{1+\gamma_{p,q}}(\mathbf{R}^n)}.\end{aligned}$$

Therefore

$$\begin{aligned}E^{p,q;\gamma_{p,q}}(u) &= \Big(\frac{\|\nabla u\|_{L^p_{1+\gamma_{p,q}}(\mathbf{R}^n)}}{\|u\|_{L^q_{\gamma_{p,q}}(\mathbf{R}^n)}}\Big)^p \ge \Big(\frac{\|\nabla[\mathcal{R}_{I_{q\gamma_{p,q}}}[u]]\|_{L^p_{1+\gamma_{p,q}}(\mathbf{R}^n)}}{\|\mathcal{R}_{I_{q\gamma_{p,q}}}[u]\|_{L^q_{\gamma_{p,q}}(\mathbf{R}^n)}}\Big)^p \\ &\ge S^{p,q;\gamma_{p,q}}_{\mathrm{rad}} \quad \text{for } u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}.\end{aligned} \tag{4.18}$$

This proves the assertion. □

Now we consider the case $\gamma = 0$. Noting that the above argument works only when $p \ge n$, we have the following proposition, which gives Theorem 2.7(3).

PROPOSITION 4.7

*Let $n \ge 2$. If $n \le p \le q < \infty$ and $R \ge R_{p,q}$, then it holds that $C^{p,q;R} = C^{p,q;R}_{\mathrm{rad}}$.*

*Proof*

When $R \ge R_{p,q} = \exp\big((1+q/p')/((n-1)p')\big)$ holds, $I_{n-(n-1)p'}/A^{1+q/p'}_{1,R} : B_1 \setminus \{0\} \to \mathbf{R}$ is positive and decreasing with respect to $r = |x|$. Then, noting that $0 < (n-1)p' < n$, it follows from Propositions 4.4(2), 4.3(2), 4.3(3), and 4.5 that we have for $u \in C^\infty_{\mathrm{c}}(B_1 \setminus \{0\})$

$$\begin{aligned}\|u\|^q_{L^q_{p;R}(B_1)} &= \int_{B_1} |u(x)|^q \Big[\frac{I_{n-(n-1)p'}}{A^{1+q/p'}_{1,R}}\Big](x) I_{(n-1)p'}(x)\,dx \\ &\le \int_{B_1} \mathcal{R}_{I_{(n-1)p'}}[|u|^q](x) \mathcal{R}_{I_{(n-1)p'}}\Big[\frac{I_{n-(n-1)p'}}{A^{1+q/p'}_{1,R}}\Big](x) I_{(n-1)p'}(x)\,dx\end{aligned}$$

$$= \int_{B_1} \mathcal{R}_{I_{(n-1)p'}}[u](x)^q \Big[\frac{I_{n-(n-1)p'}}{A_{1,R}^{1+q/p'}}\Big](x) I_{(n-1)p'}(x)\,dx$$
$$= \|\mathcal{R}_{I_{(n-1)p'}}[u]\|^q_{L^q_{p;R}(B_1)},$$
$$\|\nabla u\|^p_{L^p_1(B_1)} = \int_{B_1} |\nabla u(x)|^p \frac{1}{I_{(n-1)p'}(x)^{p-1}}\,dx$$
$$\geq \int_{B_1} \big|\nabla \big[\mathcal{R}_{I_{(n-1)p'}}[u]\big](x)\big|^p \frac{1}{I_{(n-1)p'}(x)^{p-1}}\,dx$$
$$= \big\|\nabla \big[\mathcal{R}_{I_{(n-1)p'}}[u]\big]\big\|^p_{L^p_1(B_1)}.$$

Therefore we see that

$$F^{p,q;R}(u) = \Big(\frac{\|\nabla u\|_{L^p_1(B_1)}}{\|u\|_{L^q_{p;R}(B_1)}}\Big)^p \geq \Big(\frac{\|\nabla[\mathcal{R}_{I_{(n-1)p'}}[|u|]]\|_{L^p_1(B_1)}}{\|\mathcal{R}_{I_{(n-1)p'}}[|u|]\|_{L^q_{p;R}(B_1)}}\Big)^p \geq C^{p,q;R}_{\mathrm{rad}}$$
$$\text{for } u \in C^\infty_{\mathrm{c}}(B_1 \setminus \{0\}) \setminus \{0\},$$

and this proves the assertion. □

When $n = 1$, $I_{(n-1)p'} = I_0$ is not admissible. Hence we cannot apply the same method directly, but Theorem 2.9(2) follows from the next proposition.

PROPOSITION 4.8

*Let $n = 1$. If $1 < p \leq q < \infty$ and $R \geq 1$, then it holds that $C^{p,q;R} = C^{p,q;R}_{\mathrm{rad}}$.*

*Proof*

(1) Admitting that $(1+t^p)^{1/p} \geq (1+t^q)^{1/q}$ for $t \geq 0$ holds, we have for any $u \in C^\infty_{\mathrm{c}}((-1,1) \setminus \{0\})$

$$(\|u\|^p_{L^q_{p;R}((-1,0))} + \|u\|^p_{L^q_{p;R}((0,1))})^{1/p} \geq (\|u\|^q_{L^q_{p;R}((-1,0))} + \|u\|^q_{L^q_{p;R}((0,1))})^{1/q}$$
$$= \|u\|_{L^q_{p;R}((-1,1))}.$$

Then we also have

$$\frac{\|u'\|_{L^p_1((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}} \geq \min\Big\{\frac{\|u'\|_{L^p_1((-1,0))}}{\|u\|_{L^q_{p;R}((-1,0))}}, \frac{\|u'\|_{L^p_1((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}}\Big\}$$
$$\text{for } u \in C^\infty_{\mathrm{c}}\big((-1,1) \setminus \{0\}\big) \setminus \{0\}.$$

In fact, if $\|u'\|_{L^p_1((-1,0))}/\|u\|_{L^q_{p;R}((-1,0))} \geq \|u'\|_{L^p_1((0,1))}/\|u\|_{L^q_{p;R}((0,1))}$ holds, then we have

$$\frac{\|u'\|_{L^p_1((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}} = \frac{(\|u'\|^p_{L^p_1((-1,0))} + \|u'\|^p_{L^p_1((0,1))})^{1/p}}{\|u\|_{L^q_{p;R}((-1,1))}}$$
$$\geq \frac{1}{\|u\|_{L^q_{p;R}((-1,1))}} \Big(\frac{\|u'\|^p_{L^p_1((0,1))}}{\|u\|^p_{L^q_{p;R}((0,1))}} \|u\|^p_{L^q_{p;R}((-1,0))} + \|u'\|^p_{L^p_1((0,1))}\Big)^{1/p}$$

$$= \frac{\|u'\|_{L_1^p((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}} \frac{\big(\|u\|^p_{L^q_{p;R}((-1,0))} + \|u\|^p_{L^q_{p;R}((0,1))}\big)^{1/p}}{\|u\|_{L^q_{p;R}((-1,1))}}$$

$$\geq \frac{\|u'\|_{L_1^p((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}}.$$

If $\|u'\|_{L_1^p((0,1))}/\|u\|_{L^q_{p;R}((0,1))} \geq \|u'\|_{L_1^p((-1,0))}/\|u\|_{L^q_{p;R}((-1,0))}$, then in a similar way we see that

$$\frac{\|u'\|_{L_1^p((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}} \geq \frac{\|u'\|_{L_1^p((-1,0))}}{\|u\|_{L^q_{p;R}((-1,0))}}.$$

(2) Since we have

$$\begin{aligned}
C^{p,q;R}_{\mathrm{rad}} &= \inf\Big\{\Big(\frac{\|u'\|_{L_1^p((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}}\Big)^p \;\Big|\; u \in C^\infty_{\mathrm{c}}\big((-1,1)\setminus\{0\}\big)_{\mathrm{rad}} \setminus \{0\}\Big\} \\
&= \inf\Big\{\Big(\frac{\|u'\|_{L_1^p((-1,0))}}{\|u\|_{L^q_{p;R}((-1,0))}}\Big)^p \;\Big|\; u \in C^\infty_{\mathrm{c}}\big((-1,1)\setminus\{0\}\big) \setminus \{0\}\Big\} \\
&= \inf\Big\{\Big(\frac{\|u'\|_{L_1^p((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}}\Big)^p \;\Big|\; u \in C^\infty_{\mathrm{c}}\big((-1,1)\setminus\{0\}\big) \setminus \{0\}\Big\},
\end{aligned}$$

it follows from (1) that we have

$$\begin{aligned}
F^{p,q;R}(u) &= \Big(\frac{\|u'\|_{L_1^p((-1,1))}}{\|u\|_{L^q_{p;R}((-1,1))}}\Big)^p \\
&\geq \min\Big\{\Big(\frac{\|u'\|_{L_1^p((-1,0))}}{\|u\|_{L^q_{p;R}((-1,0))}}\Big)^p, \Big(\frac{\|u'\|_{L_1^p((0,1))}}{\|u\|_{L^q_{p;R}((0,1))}}\Big)^p\Big\} \\
&\geq C^{p,q;R}_{\mathrm{rad}} \quad \text{for } u \in C^\infty_{\mathrm{c}}\big((-1,1)\setminus\{0\}\big)\setminus\{0\}.
\end{aligned}$$

Thus the assertion follows. □

## 5. Application of nonlinear potential theory

It follows from Propositions 4.7 and 4.8 that we have Theorem 2.7(3). Then, combining it with Theorem 2.7(4), we find that Theorem 2.5 clearly follows provided that $p \geq n$. Therefore, it suffices to assume that $1 < p < n$ in the rest of the proof of Theorem 2.5. We finish this task by employing the so-called nonlinear potential theory.

DEFINITION 5.1 (MUCKENHOUPT $A_P$-CLASS)

Let $1 < p < \infty$. We say that $w \in C(\mathbf{R}^n \setminus \{0\})$ belongs to $A_p$-class, if and only if $w > 0$ on $\mathbf{R}^n \setminus \{0\}$ and

$$\sup_{x \in \mathbf{R}^n, r>0} \frac{n}{\omega_n r^n} \int_{B_r(x)} w(y)\,dy \Big(\frac{n}{\omega_n r^n} \int_{B_r(x)} \frac{1}{w(y)^{1/(p-1)}}\,dy\Big)^{p-1} < \infty \tag{5.1}$$

are satisfied.

When $w$ belongs to $A_p$-class, simply we describe $w \in A_p(\mathbf{R}^n)$. Let us define

$$(5.2)\quad J_p[w](x,r) = \int_r^\infty \Big( \frac{n}{\omega_n t^n} \int_{B_t(x)} \frac{1}{w(y)^{1/(p-1)}}\, dy \Big) \frac{1}{t^{1+\nu_p}}\, dt \quad \text{for } x \in \mathbf{R}^n, r > 0.$$

Here, $\nu_p = (n-p)/(p-1)$.

Under these notations we have the next lemma, which is due to R. Adams [Ad, Theorem 7.1, Section 7].

LEMMA 5.1

*Let $1 < p < q < \infty$. Assume that $w \in A_p(\mathbf{R}^n)$, assume that $g \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, and assume that $g \geq 0$ a.e. on $\mathbf{R}^n$. Then, the following two assertions are equivalent to each other:*

(1)
$$\sup_{x \in \mathbf{R}^n, r>0} \mu_g\big(B_r(x)\big) J_p[w](x,r)^{q/p'} < \infty,$$

(2) *there is a positive number $C > 0$ such that we have*

$$\|I_1 * f\|_{L^q(\mathbf{R}^n;g)} \leq C \|f\|_{L^p(\mathbf{R}^n;w)} \quad \textit{for any } f \in L^p(\mathbf{R}^n;w).$$

Using this we establish the next proposition. Then, combining it with Theorem 2.7(4), we see that Theorem 2.5 is valid even when $1 < p < n$ holds.

PROPOSITION 5.1

*If $1 < p < q < \infty$, $p < n$, $\tau_{p,q} \leq 1/n$, and $R > 3$, then we have $C^{p,q;R} > 0$.*

Introducing more notation, we verify this using Lemma 5.1.

DEFINITION 5.2

For $1 < p < q < \infty$, $p < n$, $\tau_{p,q} \leq 1/n$, and $R > 1$, we set

$$(5.3)\qquad w_p(x) = w_p(|x|) = \max\big\{I_p(x), 1\big\} \quad \text{for } x \in \mathbf{R}^n \setminus \{0\},$$

$$(5.4)\qquad g_{p,q;R}(x) = g_{p,q;R}(|x|) = \begin{cases} \frac{I_0(x)}{A_{1,R}(x)^{1+q/p'}} & \text{for } x \in \overline{B_1} \setminus \{0\}, \\ 0 & \text{for } x \in \mathbf{R}^n \setminus B_1. \end{cases}$$

To apply Lemma 5.1 to these weight functions, let us prepare more lemmas.

LEMMA 5.2

*For $1 < p < n$, it holds that $w_p \in A_p(\mathbf{R}^n)$.*

*Proof*

Let us set

$$(5.5)\qquad \begin{aligned} \sigma_p[w_p](x,r) &= \frac{n}{\omega_n r^n} \int_{B_r(x)} w_p(y)\, dy \Big( \frac{n}{\omega_n r^n} \int_{B_r(x)} \frac{1}{w_p(y)^{1/(p-1)}}\, dy \Big)^{p-1} \\ &\quad \text{for } x \in \mathbf{R}^n, r > 0, \end{aligned}$$

and show it to be bounded.

(i) First we asssume that $0 \le |x| \le 1$.

(a) If $0 < r \le \min\{|x|/2, 1-|x|\}$, then we see that $B_r(x) \subset B_{|x|+r} \setminus \overline{B_{|x|-r}} \subset B_1$; hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_r(x)} w_p(|x|-r)\,dy\Big(\frac{n}{\omega_n r^n}\int_{B_r(x)} \frac{1}{w_p(|x|+r)^{1/(p-1)}}\,dy\Big)^{p-1}\\ &= \Big(\frac{|x|+r}{|x|-r}\Big)^{n-p} \le \Big(\frac{|x|+|x|/2}{|x|-|x|/2}\Big)^{n-p} = 3^{n-p}.\end{aligned}$$

(b) If $1-|x| \le r \le |x|/2$, we see that $|x| \ge 2/3$; hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_r(x)} w_p(|x|-r)\,dy\Big(\frac{n}{\omega_n r^n}\int_{B_r(x)} dy\Big)^{p-1} = \frac{1}{(|x|-r)^{n-p}}\\ &\le \frac{1}{(|x|-|x|/2)^{n-p}} = \Big(\frac{2}{|x|}\Big)^{n-p} \le 3^{n-p}.\end{aligned}$$

(c) If $|x|/2 \le r \le 1-|x|$, then we see that $B_r(x) \subset B_{|x|+r} \subset B_1$; hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_{|x|+r}} w_p(y)\,dy\Big(\frac{n}{\omega_n r^n}\int_{B_{|x|+r}} \frac{1}{w_p(y)^{1/(p-1)}}\,dy\Big)^{p-1}\\ &= \frac{n}{p}\Big(\frac{n'}{p'}\Big)^{p-1}\Big(\frac{|x|+r}{r}\Big)^{np} \le \frac{n}{p}\Big(\frac{n'}{p'}\Big)^{p-1}\Big(\frac{2r+r}{r}\Big)^{np}\\ &= \frac{n}{p}\Big(\frac{n'}{p'}\Big)^{p-1}3^{np}.\end{aligned}$$

(d) If $r \ge \max\{|x|/2, 1-|x|\}$, then $r \ge 1/3$ and $B_r(x) \subset B_{|x|+r}$. Hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_{|x|+r}} w_p(y)\,dy\Big(\frac{n}{\omega_n r^n}\int_{B_{|x|+r}} \frac{1}{w_p(y)^{1/(p-1)}}\,dy\Big)^{p-1}\\ &= \Big(\Big(\frac{|x|+r}{r}\Big)^n + \Big(\frac{n}{p}-1\Big)\frac{1}{r^n}\Big)\Big(\Big(\frac{|x|+r}{r}\Big)^n - \Big(1-\frac{n'}{p'}\Big)\frac{1}{r^n}\Big)^{p-1}\\ &\le \Big(\Big(\frac{2r+r}{r}\Big)^n + \Big(\frac{n}{p}-1\Big)3^n\Big)\Big(\frac{2r+r}{r}\Big)^{n(p-1)} = \frac{n}{p}3^{np}.\end{aligned}$$

(ii) Second we assume that $|x| \ge 1$.

(a) If $0 < r \le |x|/2$, then $B_r(x) \subset B_{|x|+r} \setminus \overline{B_{|x|-r}}$; hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_r(x)} w_p(|x|-r)\,dy\Big(\frac{n}{\omega_n r^n}\int_{B_r(x)} dy\Big)^{p-1} = \frac{1}{(|x|-r)^{n-p}}\\ &\le \frac{1}{(|x|-|x|/2)^{n-p}} = \Big(\frac{2}{|x|}\Big)^{n-p} \le 2^{n-p}.\end{aligned}$$

(b) If $r \ge |x|/2$, then $r \ge 1/2$ and $B_r(x) \subset B_{|x|+r}$; hence

$$\begin{aligned}\sigma_p[w_p](x,r) &\le \frac{n}{\omega_n r^n}\int_{B_{|x|+r}} w_p(y)\,dy\Big(\frac{n}{\omega_n r^n}\int_{B_{|x|+r}} \frac{1}{w_p(y)^{1/(p-1)}}\,dy\Big)^{p-1}\\ &= \Big(\Big(\frac{|x|+r}{r}\Big)^n + \Big(\frac{n}{p}-1\Big)\frac{1}{r^n}\Big)\Big(\Big(\frac{|x|+r}{r}\Big)^n - \Big(1-\frac{n'}{p'}\Big)\frac{1}{r^n}\Big)^{p-1}\end{aligned}$$

$$\leq \Big(\Big(\frac{2r+r}{r}\Big)^n + \Big(\frac{n}{p}-1\Big)2^n\Big)\Big(\frac{2r+r}{r}\Big)^{n(p-1)}$$
$$= \Big(3^n + \Big(\frac{n}{p}-1\Big)2^n\Big)3^{np}. \qquad \square$$

LEMMA 5.3

*For $1 < p < n$ and $R > 3$, there exist positive numbers $c_p$ and $c_{p;R} > 0$ such that we have the following:*

(1) $J_p[w_p](x,r) \leq \frac{1}{\nu_p}\frac{1}{r^{\nu_p}}$ *for* $x \in \mathbf{R}^n, r > 0$.
(2) $J_p[w_p](x,r) \leq c_p\big(1 + \log\frac{1}{r} + \big(\frac{|x|}{r}\big)^{\nu_p}\big)$ *if* $|x| + r \leq 1$.
(3) $J_p[w_p](x,r) \leq c_{p;R}\big(A_{1,R}(\min\{1,|x|\}) + \big(\frac{\min\{1,|x|\}}{r}\big)^{\nu_p}\big)$ *if* $0 < r \leq \frac{|x|}{2}$.

*Proof*

Let us note that

$$\text{(5.6)}\quad \frac{n}{\omega_n t^n}\int_{B_t(x)} \frac{1}{w_p(y)^{1/(p-1)}}\,dy \leq \min\big\{1,(t+|x|)^{\nu_p}\big\} \leq 1 \quad \text{for } x\in\mathbf{R}^n, t>0.$$

Then

(1)

$$J_p[w_p](x,r) = \int_r^\infty \Big(\frac{n}{\omega_n t^n}\int_{B_t(x)}\frac{1}{w_p(y)^{1/(p-1)}}\,dy\Big)\frac{1}{t^{1+\nu_p}}\,dt \leq \int_r^\infty \frac{1}{t^{1+\nu_p}}\,dt$$
$$= \frac{1}{\nu_p}\frac{1}{r^{\nu_p}} \quad \text{for } x\in\mathbf{R}^n, r>0.$$

(2) If $|x| + r \leq 1$, then we see that $r \leq 1$; hence

$$J_p[w_p](x,r) \leq \int_r^\infty \min\big\{1,(t+|x|)^{\nu_p}\big\}\frac{1}{t^{1+\nu_p}}\,dt$$
$$\leq \int_1^\infty \frac{1}{t^{1+\nu_p}}\,dt + \int_r^1 (t+|x|)^{\nu_p}\frac{1}{t^{1+\nu_p}}\,dt$$
$$= \frac{1}{\nu_p} + \int_r^1 \Big(1+\frac{|x|}{t}\Big)^{\nu_p}\frac{1}{t}\,dt \leq \frac{1}{\nu_p} + 2^{(\nu_p-1)_+}\int_r^1\Big(1+\Big(\frac{|x|}{t}\Big)^{\nu_p}\Big)\frac{1}{t}\,dt$$
$$= \frac{1}{\nu_p} + 2^{(\nu_p-1)_+}\Big(\log\frac{1}{r} + \frac{|x|^{\nu_p}}{\nu_p}\Big(\frac{1}{r^{\nu_p}}-1\Big)\Big)$$
$$\leq \frac{1}{\nu_p} + 2^{(\nu_p-1)_+}\Big(\log\frac{1}{r} + \frac{1}{\nu_p}\Big(\frac{|x|}{r}\Big)^{\nu_p}\Big).$$

(3.a) If $|x| + r \leq 1$ and $0 < r \leq |x|/2$, then $|x| \leq 1$ and $|x|/r \geq 2$. From the argument of assertion (2) of this lemma and

$$\text{(5.7)}\qquad 1 + \log t \leq \tilde{c}_p t^{\nu_p} \quad \text{for } t \geq 1$$

it holds that

$$J_p[w_p](x,r) \le c_p\Big(1+\log\frac{1}{r}+\Big(\frac{|x|}{r}\Big)^{\nu_p}\Big) \le c_p\Big(1+\log\frac{|x|}{r}+\log\frac{R}{|x|}+\Big(\frac{|x|}{r}\Big)^{\nu_p}\Big)$$
$$\le c_p\Big(A_{1,R}(x)+(1+\tilde{c}_p)\Big(\frac{|x|}{r}\Big)^{\nu_p}\Big).$$

(3.b) If $|x|+r\ge 1$, then from assertion (1) of this lemma we see that

$$J_p[w_p](x,r) \le \frac{1}{\nu_p}\frac{1}{r^{\nu_p}} \le \frac{1}{\nu_p}\Big(\frac{|x|+r}{r}\Big)^{\nu_p} \le \frac{2^{(\nu_p-1)_+}}{\nu_p}\Big(1+\Big(\frac{|x|}{r}\Big)^{\nu_p}\Big)$$
$$\le \frac{2^{(\nu_p-1)_+}}{\nu_p}\Big(\frac{A_{1,R}(x)}{\log R}+\Big(\frac{|x|}{r}\Big)^{\nu_p}\Big) \quad \text{if } |x|\le 1,$$
$$J_p[w_p](x,r) \le \frac{1}{\nu_p}\frac{1}{r^{\nu_p}} \le \frac{1}{\nu_p}\Big(A_{1,R}(1)+\Big(\frac{1}{r}\Big)^{\nu_p}\Big) \quad \text{if } |x|\ge 1.$$

□

LEMMA 5.4

*For $1<p<q<\infty$, $p<n$, $\tau_{p,q}\le 1/n$, and $R>3$, there exists a positive number $c_{p,q;R}>0$ such that we have*

$$\mu_{g_{p,q;R}}\big(B_r(x)\big) \le \begin{cases} c_{p,q;R}\, g_{p,q;R}(\min\{1,|x|\})r^n & \textit{if } 0<r\le \frac{1}{2}\min\{1,|x|\}, \\ c_{p,q;R}\frac{1}{A_{1,R}(r)^{q/p'}} & \textit{if } \frac{|x|}{2}\le r\le\frac{1}{2}, \\ c_{p,q;R} & \textit{if } r\ge\frac{1}{2}. \end{cases} \tag{5.8}$$

*Proof*

First we note that for $1<\underline{R}<R$

$$\log\frac{\underline{R}}{r} \ge \frac{\log\underline{R}}{\log R}\log\frac{R}{r} \quad \text{for } 0<r\le 1(<\underline{R}). \tag{5.9}$$

By the definition we have

$$\begin{aligned} \mu_{g_{p,q;R}}\big(B_r(x)\big) &= \int_{B_r(x)} g_{p,q;R}(y)\,dy \\ &= \int_{B_r(x)\cap B_1}\frac{1}{(\log(R/|y|))^{1+q/p'}}\frac{1}{|y|^n}\,dy \quad \text{for } x\in\mathbf{R}^n, r>0. \end{aligned} \tag{5.10}$$

(a) If $0<r\le |x|/2\le 1/2$, then $|x|/2\le |x|-r\le |y|\le |x|+r\le 3|x|/2$ for $y\in B_r(x)$; hence we have, using (5.9) with $\underline{R}=2R/3$,

$$\mu_{g_{p,q;R}}\big(B_r(x)\big) \le \int_{B_r(x)}\frac{1}{(\log(2R/(3|x|)))^{1+q/p'}}\Big(\frac{2}{|x|}\Big)^n dy$$
$$= \frac{\omega_n r^n}{n}\frac{1}{(\log(2R/(3|x|)))^{1+q/p'}}\Big(\frac{2}{|x|}\Big)^n$$
$$\le 2^n\frac{\omega_n}{n}\Big(\frac{\log R}{\log(2R/3)}\Big)^{1+q/p'} g_{p,q;R}(x)r^n.$$

(b) If $0 < r \le 1/2 \le |x|/2$, then $1/2 \le |x|/2 \le |x| - r \le |y|$ for $y \in B_r(x)$; hence we have

$$\mu_{g_{p,q;R}}\big(B_r(x)\big) \le \int_{B_r(x)} \frac{1}{(\log R)^{1+q/p'}} 2^n \, dy = 2^n \frac{\omega_n}{n} g_{p,q;R}(1) r^n.$$

(c) If $1/2 \ge r \ge |x|/2$, then $B_r(x) \subset B_{3r} \subset B_R$; hence we have, using (5.9) with $\underline{R} = R/3$,

$$\begin{aligned}\mu_{g_{p,q;R}}\big(B_r(x)\big) &\le \int_{B_{3r}} \frac{1}{(\log(R/|y|))^{1+q/p'}} \frac{1}{|y|^n} \, dy = \omega_n \frac{p'}{q} \frac{1}{(\log(R/(3r)))^{q/p'}} \\ &\le \omega_n \frac{p'}{q} \Big(\frac{\log R}{\log(R/3)}\Big)^{q/p'} \frac{1}{A_{1,R}(r)^{q/p'}}.\end{aligned}$$

(d) If $r \ge 1/2$, then we have

$$\mu_{g_{p,q;R}}\big(B_r(x)\big) \le \int_{B_1} \frac{1}{(\log(R/|y|))^{1+q/p'}} \frac{1}{|y|^n} \, dy = \omega_n \frac{p'}{q} \frac{1}{(\log R)^{q/p'}}. \qquad \square$$

After all this we have the following lemma.

LEMMA 5.5

*For $1 < p < q < \infty$, $p < n$, $\tau_{p,q} \le 1/n$, and $R > 3$, it holds that*

$$\sup_{x \in \mathbf{R}^n, r > 0} \mu_{g_{p,q;R}}\big(B_r(x)\big) J_p[w_p](x,r)^{q/p'} < \infty. \tag{5.11}$$

*Proof*

(a) If $r \ge 1/2$, it follows from Lemma 5.3(1) and Lemma 5.4 that we have

$$\mu_{g_{p,q;R}}\big(B_r(x)\big) J_p[w_p](x,r)^{q/p'} \le c_{p,q;R} \frac{1}{\nu_p} \frac{1}{r^{\nu_p}} \le c_{p,q;R} \frac{2^{\nu_p}}{\nu_p}.$$

(b) For $0 < r \le \min\{1, |x|\}/2$, it follows from Lemma 5.3(3) and Lemma 5.4 that we have

$$\begin{aligned}&\mu_{g_{p,q;R}}\big(B_r(x)\big) J_p[w_p](x,r)^{q/p'} \\ &\quad \le c_{p,q;R} g_{p,q;R}\big(\min\{1,|x|\}\big) r^n \\ &\qquad \times \bigg(c_{p;R}\Big(\Big(A_{1,R}\big(\min\{1,|x|\}\big) + \frac{\min\{1,|x|\}}{r}\Big)^{\nu_p}\Big)\bigg)^{q/p'} \\ &\quad = \frac{c_{p,q;R} c_{p;R}^{q/p'}}{A_{1,R}(\min\{1,|x|\})} \Big(\frac{r}{\min\{1,|x|\}}\Big)^{nq(1/n - \tau_{p,q})} \\ &\qquad \times \bigg(\Big(\frac{r}{\min\{1,|x|\}}\Big)^{\nu_p} + \frac{1}{A_{1,R}(\min\{1,|x|\})}\bigg)^{q/p'} \\ &\quad \le \frac{c_{p,q;R} c_{p;R}^{q/p'}}{A_{1,R}(1)} \frac{1}{2^{nq(1/n - \tau_{p,q})}} \Big(\frac{1}{2^{\nu_p}} + \frac{1}{A_{1,R}(1)}\Big)^{q/p'}.\end{aligned}$$

(c) Assume that $|x|/2 \le r \le 1/2$. First we deal with the case $|x| + r \le 1$. Then, from Lemma 5.3(2) and Lemma 5.4 we have

$$\begin{aligned}
&\mu_{g_{p,q;R}}\big(B_r(x)\big) J_p[w_p](x,r)^{q/p'} \\
&\quad \le c_{p,q;R} \frac{1}{A_{1,R}(r)^{q/p'}} \left( c_p \left( 1 + \log \frac{1}{r} + \left( \frac{|x|}{r} \right)^{\nu_p} \right) \right)^{q/p'} \\
&\quad \le c_{p,q;R} c_p^{q/p'} \left( \frac{1 + 2^{\nu_p} + \log(1/r)}{\log R + \log(1/r)} \right)^{q/p'} \\
&\quad \le c_{p,q;R} c_p^{q/p'} \left( \max\left\{ 1, \frac{1 + 2^{\nu_p}}{\log R} \right\} \right)^{q/p'}.
\end{aligned}$$

If $|x| + r > 1$, then we have $r > 1/3$. Hence from Lemma 5.3(1) and Lemma 5.4 we have

$$\begin{aligned}
\mu_{g_{p,q;R}}\big(B_r(x)\big) J_p[w_p](x,r)^{q/p'} &\le c_{p,q;R} \frac{1}{A_{1,R}(r)^{q/p'}} \left( \frac{1}{\nu_p} \frac{1}{r^{\nu_p}} \right)^{q/p'} \\
&\le c_{p,q;R} \frac{1}{A_{1,R}(1/2)^{q/p'}} \left( \frac{3^{\nu_p}}{\nu_p} \right)^{q/p'}.
\end{aligned}$$

□

In addition we use the following lemma (cf. [GT, Lemma 7.14]).

LEMMA 5.6

*For $u \in C_c^\infty(\mathbf{R}^n)$, it holds that*

$$u(x) = \frac{1}{\omega_n} \int_{\mathbf{R}^n} \frac{\nabla u(y) \cdot (x - y)}{|x - y|^n} \, dy \quad \text{for } x \in \mathbf{R}^n. \tag{5.12}$$

*In particular*

$$|u(x)| \le \frac{1}{\omega_n} I_1 * [|\nabla u|](x) \quad \text{for } x \in \mathbf{R}^n. \tag{5.13}$$

*Proof*

Noting that

$$u(x) = -\int_0^\infty \nabla u(x + t\omega) \cdot \omega \, dt \quad \text{for } \omega \in S^{n-1}, \tag{5.14}$$

we immediately have

$$\begin{aligned}
\int_{\mathbf{R}^n} \frac{\nabla u(y) \cdot (x - y)}{|x - y|^n} \, dy &= -\int_{\mathbf{R}^n} \frac{\nabla u(x + y) \cdot y}{|y|^n} \, dy \\
&= -\int_{S^{n-1}} \int_0^\infty \frac{\nabla u(x + t\omega) \cdot t\omega}{t^n} t^{n-1} \, dt \, dS(\omega) \\
&= -\int_{S^{n-1}} \int_0^\infty \nabla u(x + t\omega) \cdot \omega \, dt \, dS(\omega) = \int_{S^{n-1}} u(x) \, dS(\omega) \\
&= \omega_n u(x).
\end{aligned}$$

□

Now we are in a position to establish Proposition 5.1.

*Proof of Proposition 5.1*
It follows from Lemmas 5.2, 5.5, and 5.1 that there exists a positive number $C_{p,q;R} > 0$ such that we have

$$\|I_1 * f\|_{L^q(\mathbf{R}^n; g_{p,q;R})} \le C_{p,q;R} \|f\|_{L^p(\mathbf{R}^n; w_p)} \quad \text{for } f \in L^p(\mathbf{R}^n; w_p).$$

Then, from Lemma 5.6 we have

$$\begin{aligned} \|u\|_{L^q_{p;R}(B_1)} = \|u\|_{L^q(\mathbf{R}^n; g_{p,q;R})} &\le \frac{1}{\omega_n} \big\| I_1 * [|\nabla u|] \big\|_{L^q(\mathbf{R}^n; g_{p,q;R})} \\ &\le \frac{C_{p,q;R}}{\omega_n} \|\nabla u\|_{L^p(\mathbf{R}^n; w_p)} \\ &= \frac{C_{p,q;R}}{\omega_n} \|\nabla u\|_{L^p_1(B_1)} \quad \text{for } u \in C_{\mathrm{c}}^\infty(B_1 \setminus \{0\}). \end{aligned}$$ □

## 6. Continuity of the best constants on parameters

In this section we prove that the best constants $S^{p,q;\gamma}$ and $C^{p,q;R}$ are continuous on parameters with $p$ being arbitrarily fixed and we also establish some related estimates. It is clear from Theorem 2.2(2) and Theorem 2.7(2) that the best constants in radial spaces $S^{p,q;\gamma}_{\mathrm{rad}}$ and $C^{p,q;R}_{\mathrm{rad}}$ are continuous functions of $q, \gamma, R$ as well.

### 6.1. The noncritical case ($\gamma \neq 0$)

First in the case $\gamma > 0$, we study the continuity of $S^{p,q;\gamma}$ on $q, \gamma$. Let us introduce the next transformation.

DEFINITION 6.1
Let $1 < p < \infty$, and let $\gamma > 0$. For $u : \mathbf{R}^n \to \mathbf{R}$, we set

$$\hat{T}_\gamma v(x) = \frac{1}{|x|^\gamma} v(x) \quad \text{for } x \in \mathbf{R}^n \setminus \{0\}$$

and set

$$\Phi^{p;\gamma}[v] = \int_{\mathbf{R}^n} \Big| \nabla v(x) - \gamma v(x) \frac{x}{|x|^2} \Big|^p I_p(x)\, dx.$$

Then, it follows from direct calculations and triangle inequalities that we have the following lemma.

LEMMA 6.1
*For $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$, and $\gamma, \overline{\gamma} > 0$, we have the following:*

Assertion (1) $\|\hat{T}_\gamma v\|^q_{L^q_\gamma(\mathbf{R}^n)} = \|v\|^q_{L^q_0(\mathbf{R}^n)}, \|\nabla[\hat{T}_\gamma v]\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} = \Phi^{p;\gamma}[v]$ *for* $v \in \hat{T}_\gamma^{-1}(W^{1,p}_{\gamma,0}(\mathbf{R}^n))$.

Assertion (2)

$$S^{p,q;\gamma} = \inf\Big\{ \frac{\Phi^{p;\gamma}[v]}{\|v\|^p_{L^q_0(\mathbf{R}^n)}} \,\Big|\, v \in \hat{T}^{-1}_\gamma\big(W^{1,p}_{\gamma,0}(\mathbf{R}^n)\big) \setminus \{0\} \Big\}$$
$$= \inf\Big\{ \frac{\Phi^{p;\gamma}[v]}{\|v\|^p_{L^q_0(\mathbf{R}^n)}} \,\Big|\, v \in C^\infty_c\big(\mathbf{R}^n \setminus \{0\}\big) \setminus \{0\} \Big\}.$$

Assertion (3)

$$S^{p,q;\gamma}\|v\|^p_{L^q_0(\mathbf{R}^n)} \le \Phi^{p;\gamma}[v] \quad \textit{for } v \in \hat{T}^{-1}_\gamma\big(W^{1,p}_{\gamma,0}(\mathbf{R}^n)\big).$$

*In particular,*

$$\gamma^p\|v\|^p_{L^p_0(\mathbf{R}^n)} \le \Phi^{p;\gamma}[v] \quad \textit{for } v \in \hat{T}^{-1}_\gamma\big(W^{1,p}_{\gamma,0}(\mathbf{R}^n)\big).$$

Assertion (4) $|\Phi^{p;\gamma}[v]^{1/p} - \Phi^{p;\overline{\gamma}}[v]^{1/p}| \le |\gamma - \overline{\gamma}|\|v\|_{L^p_0(\mathbf{R}^n)}$ *for* $v \in \hat{T}^{-1}_\gamma(W^{1,p}_{\gamma,0}(\mathbf{R}^n)) \cap \hat{T}^{-1}_{\overline{\gamma}}(W^{1,p}_{\overline{\gamma},0}(\mathbf{R}^n))$.

Now let us state a crucial lemma (cf. [CW1, Lemma 3.2, Section 3]).

LEMMA 6.2

*Let* $1 < p \le q < \infty$, *let* $\tau_{p,q} \le 1/n$, *and let* $\gamma > 0$. *Assume that* $\{q_j\}_{j=1}^\infty \subset (p, p^*)$ *satisfies*

$$q_j \to q \quad \textit{as } j \to \infty.$$

*If* $\{v_j\}_{j=1}^\infty \subset C^\infty_c(\mathbf{R}^n \setminus \{0\})$ *and* $\{\Phi^{p;\gamma}[v_j]\}_{j=1}^\infty$ *is bounded, then it holds that*

$$\limsup_{j\to\infty}\big(\|v_j\|^{q_j}_{L^{q_j}_0(\mathbf{R}^n)} - \|v_j\|^q_{L^q_0(\mathbf{R}^n)}\big) \le 0.$$

*Proof*

For $p < \underline{q} < \overline{q} < \tilde{q} < p^*$, let us note that

$$\text{(6.1)} \qquad \begin{aligned} &0 \le t^{\underline{q}} \log\frac{1}{t} \le \frac{1}{e(\underline{q} - p)} t^p \quad \text{for } 0 < t \le 1,\\ &0 \le t^{\overline{q}} \log t \le \frac{1}{e(\tilde{q} - \overline{q})} t^{\tilde{q}} \quad \text{for } t \ge 1. \end{aligned}$$

(a) When $p < q < p^*$ holds, we choose $\underline{q}, \overline{q}$, and $\tilde{q}$ such that $p < \underline{q} \le q_j \le \overline{q} < \tilde{q} < p^*$ for $j \ge 1$. Then it follows from Lemma 6.1 that we have

$$\begin{aligned} &\|v_j\|^{q_j}_{L^{q_j}_0(\mathbf{R}^n)} - \|v_j\|^q_{L^q_0(\mathbf{R}^n)}\\ &\quad = \int_{\mathbf{R}^n} \big(|v_j(x)|^{q_j} - |v_j(x)|^q\big) I_0(x)\, dx\\ &\quad = \int_{\mathbf{R}^n} \Big((q_j - q)\int_0^1 |v_j(x)|^{\theta q_j + (1-\theta)q} \log|v_j(x)|\, d\theta\Big) I_0(x)\, dx\\ &\quad \le |q_j - q|\Big(\int_{\{|v_j|\le 1\}} |v_j(x)|^{\underline{q}}\Big(\log\frac{1}{|v_j(x)|}\Big) I_0(x)\, dx \end{aligned}$$

$$
\begin{aligned}
&+\int_{\{|v_j|\geq 1\}} |v_j(x)|^{\overline{q}}\big(\log|v_j(x)|\big) I_0(x)\,dx\Big)\\
&\leq |q_j-q|\Big(\frac{1}{e(\underline{q}-p)}\int_{\{|v_j|\leq 1\}} |v_j(x)|^p I_0(x)\,dx\\
&\quad+\frac{1}{e(\tilde{q}-\overline{q})}\int_{\{|v_j|\geq 1\}} |v_j(x)|^{\tilde{q}} I_0(x)\,dx\Big)\\
&\leq |q_j-q|\Big(\frac{1}{e(\underline{q}-p)}\|v_j\|^p_{L_0^p(\mathbf{R}^n)}+\frac{1}{e(\tilde{q}-\overline{q})}\|v_j\|^{\tilde{q}}_{L_0^{\tilde{q}}(\mathbf{R}^n)}\Big)\\
&\leq |q_j-q|\left(\frac{1}{e(\underline{q}-p)}\frac{1}{\gamma^p}\Phi^{p;\gamma}[v_j]+\frac{1}{e(\tilde{q}-\overline{q})}\Big(\frac{1}{S^{p,\tilde{q};\gamma}}\Phi^{p;\gamma}[v_j]\Big)^{\tilde{q}/p}\right)\to 0
\end{aligned}
$$

as $j\to\infty$.

(b) When $q=p$ holds, we choose $\overline{q}$ and $\tilde{q}$ such that $p<q_j\leq\overline{q}<\tilde{q}<p^*$ for $j\geq 1$. Then in a similar way as the argument in (a), we have

$$
\begin{aligned}
&\|v_j\|^{q_j}_{L_0^{q_j}(\mathbf{R}^n)}-\|v_j\|^p_{L_0^p(\mathbf{R}^n)}\\
&\quad\leq\int_{\{|v_j|\geq 1\}}\big(|v_j(x)|^{q_j}-|v_j(x)|^p\big) I_0(x)\,dx\\
&\quad=\int_{\{|v_j|\geq 1\}}\Big((q_j-p)\int_0^1 |v_j(x)|^{\theta q_j+(1-\theta)p}\log|v_j(x)|\,d\theta\Big) I_0(x)\,dx\\
&\quad\leq (q_j-p)\frac{1}{e(\tilde{q}-\overline{q})}\Big(\frac{1}{S^{p,\tilde{q};\gamma}}\Phi^{p;\gamma}[v_j]\Big)^{\tilde{q}/p}\to 0\quad\text{as } j\to\infty.
\end{aligned}
$$

(c) When $q=p^*<\infty$ holds, we choose $\underline{q}$ such as $p<\underline{q}\leq q_j<p^*$ for $j\geq 1$. Then in a similar way as the argument in (a), we have

$$
\begin{aligned}
&\|v_j\|^{q_j}_{L_0^{q_j}(\mathbf{R}^n)}-\|v_j\|^{p^*}_{L_0^{p^*}(\mathbf{R}^n)}\\
&\quad\leq\int_{\{|v_j|\leq 1\}}\big(|v_j(x)|^{q_j}-|v_j(x)|^{p^*}\big) I_0(x)\,dx\\
&\quad=\int_{\{|v_j|\geq 1\}}\Big((p^*-q_j)\int_0^1 |v_j(x)|^{\theta q_j+(1-\theta)p^*}\log\frac{1}{|v_j(x)|}\,d\theta\Big) I_0(x)\,dx\\
&\quad\leq (p^*-q_j)\frac{1}{e(\underline{q}-p)}\frac{1}{\gamma^p}\Phi^{p;\gamma}[v_j]\to 0\quad\text{as } j\to\infty.
\end{aligned}
$$

□

Then we have the following proposition, which gives Theorem 2.3.

PROPOSITION 6.1

*Let $1<p\leq q<\infty$, let $\tau_{p,q}\leq 1/n$, and let $\gamma>0$. Assume that $\{(q_j;\gamma_j)\}_{j=1}^{\infty}\subset(p,p^*)\times(0,\infty)$ satisfies*

$$
q_j\to q,\qquad \gamma_j\to\gamma\quad\textit{as } j\to\infty.
$$

*Then, it holds that*

$$S^{p,q_j;\gamma_j} \to S^{p,q;\gamma} \quad \text{as } j \to \infty.$$

*Proof*

(a) We begin by showing that

$$\limsup_{j\to\infty} S^{p,q_j;\gamma_j} \le S^{p,q;\gamma}.$$

For $\varepsilon > 0$, it follows from Lemma 6.1(2) that there exists $v_\varepsilon \in C_c^\infty(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}$ such that

$$\frac{\Phi^{p;\gamma}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^q(\mathbf{R}^n)}} \le S^{p,q;\gamma} + \frac{\varepsilon}{2}.$$

By the Lebesgue convergence theorem we have

$$\|v_\varepsilon\|^{q_j}_{L_0^{q_j}(\mathbf{R}^n)} \to \|v_\varepsilon\|^q_{L_0^q(\mathbf{R}^n)}, \qquad \Phi^{p;\gamma_j}[v_\varepsilon] \to \Phi^{p;\gamma}[v_\varepsilon] \quad \text{as } j \to \infty.$$

Hence for some $j_\varepsilon \in \mathbf{N}$, we have

$$\frac{\Phi^{p;\gamma_j}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^{q_j}(\mathbf{R}^n)}} - \frac{\Phi^{p;\gamma}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^q(\mathbf{R}^n)}} < \frac{\varepsilon}{2} \quad \text{for } j \ge j_\varepsilon.$$

We therefore have

$$S^{p,q_j;\gamma_j} \le \frac{\Phi^{p;\gamma_j}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^{q_j}(\mathbf{R}^n)}} \le \frac{\Phi^{p;\gamma}[v_\varepsilon]}{\|v_\varepsilon\|^p_{L_0^q(\mathbf{R}^n)}} + \frac{\varepsilon}{2} \le S^{p,q;\gamma} + \varepsilon \quad \text{for } j \ge j_\varepsilon.$$

(b) Second, we show that

$$S^{p,q;\gamma} \le \liminf_{j\to\infty} S^{p,q_j;\gamma_j}.$$

By Lemma 6.1(2) there exists $\{v_j\}_{j=1}^\infty \subset C_c^\infty(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}$ such that we have

$$\Phi^{p;\gamma}[v_j] = 1, \qquad \frac{\Phi^{p;\gamma_j}[v_j]}{\|v_j\|^p_{L_0^{q_j}(\mathbf{R}^n)}} \le S^{p,q_j;\gamma_j} + \frac{1}{j} \quad \text{for } j \ge 1.$$

Then from assertions (3) and (4) of Lemma 6.1 we have

$$\Phi^{p;\gamma_j}[v_j]^{1/p} \ge \Phi^{p;\gamma}[v_j]^{1/p} - |\gamma_j - \gamma| \|v_j\|_{L_0^p(\mathbf{R}^n)} \ge \Phi^{p;\gamma}[v_j]^{1/p} - \frac{|\gamma_j - \gamma|}{\gamma} \Phi^{p;\gamma}[v_j]^{1/p}$$

$$= 1 - \frac{|\gamma_j - \gamma|}{\gamma} \quad \text{for } j \ge 1.$$

Combining with (a), we have that there exist $j_1 \in \mathbf{N}$ and $c > 0$ such that

$$\|v_j\|^p_{L_0^{q_j}(\mathbf{R}^n)} \ge \frac{\Phi^{p;\gamma_j}[v_j]}{S^{p,q_j;\gamma_j} + 1/j} \ge c \quad \text{for } j \ge j_1.$$

Letting $\varepsilon$ satisfy $0 < \varepsilon < c$, we find that it follows from Lemma 6.2 that there exists $j_\varepsilon \ge j_1$ such that

$$\|v_j\|^p_{L_0^{q_j}(\mathbf{R}^n)} \le \|v_j\|^p_{L_0^q(\mathbf{R}^n)} + \varepsilon \quad \text{for } j \ge j_\varepsilon.$$

Then from assertions (3) and (4) of Lemma 6.1 we have

$$\begin{aligned} S^{p,q;\gamma} &\le \frac{\Phi^{p;\gamma}[v_j]}{\|v_j\|^p_{L^q_0(\mathbf{R}^n)}} \le \frac{1}{\|v_j\|^p_{L^{q_j}_0(\mathbf{R}^n)} - \varepsilon}(\Phi^{p;\gamma_j}[v_j]^{1/p} + |\gamma_j - \gamma|\|v_j\|_{L^p_0(\mathbf{R}^n)})^p \\ &= \frac{1}{1-\varepsilon/\|v_j\|^p_{L^{q_j}_0(\mathbf{R}^n)}}\left(\left(\frac{\Phi^{p;\gamma_j}[v_j]}{\|v_j\|^p_{L^{q_j}_0(\mathbf{R}^n)}}\right)^{1/p} + |\gamma_j-\gamma|\frac{\|v_j\|_{L^p_0(\mathbf{R}^n)}}{\|v_j\|_{L^{q_j}_0(\mathbf{R}^n)}}\right)^p \\ &\le \frac{1}{1-\varepsilon/c}\left(\left(S^{p,q_j;\gamma_j} + \frac{1}{j}\right)^{1/p} + \frac{|\gamma_j-\gamma|}{c^{1/p}\gamma}\right)^p \quad \text{for } j \ge j_\varepsilon, \end{aligned}$$

and this proves the assertion. □

## 6.2. The critical case ($\gamma = 0$)

In this section we study the continuity of $C^{p,q;R}$ on the parameters $q, R$. Let us introduce the next transformation.

DEFINITION 6.2

Let $1 < p < \infty$, and let $R > 0$. For $u : B_1 \to \mathbf{R}$, we set

$$\hat{T}_{p;R}v(x) = A_{1,R}(x)^{1/p'}v(x) \quad \text{for } x \in B_1 \setminus \{0\}$$

and set

$$\Psi^{p;R}[v] = \int_{B_1}\left|A_{1,R}(x)\nabla v(x) - \frac{1}{p'}v(x)\frac{x}{|x|^2}\right|^p \frac{I_p(x)}{A_{1,R}(x)}\,dx.$$

It follows from direct calculations together with triangle inequalities that we have the following lemma.

LEMMA 6.3

*For $1 < p \le q < \infty$, $\tau_{p,q} \le 1/n$, and $R > 1$ it holds that:*

Assertion (1) $\|\hat{T}_{p;R}v\|^q_{L^q_{p;R}(B_1)} = \|v\|^q_{L^q_{1;R}(B_1)}, \|\nabla[\hat{T}_{p;R}v]\|^p_{L^p_1(B_1)} = \Psi^{p;R}[v]$ *for* $v \in \hat{T}^{-1}_{p;R}(W^{1,p}_{0,0}(B_1))$.

Assertion (2)

$$\begin{aligned} C^{p,q;R} &= \inf\left\{\frac{\Psi^{p;R}[v]}{\|v\|^p_{L^q_{1;R}(B_1)}} \;\middle|\; v \in \hat{T}^{-1}_{p;R}\big(W^{1,p}_{0,0}(B_1)\big) \setminus \{0\}\right\} \\ &= \inf\left\{\frac{\Psi^{p;R}[v]}{\|v\|^p_{L^q_{1;R}(B_1)}} \;\middle|\; v \in C^\infty_{\mathrm{c}}(B_1 \setminus \{0\}) \setminus \{0\}\right\}. \end{aligned}$$

Assertion (3)

$$C^{p,q;R}\|v\|^p_{L^q_{1;R}(B_1)} \le \Psi^{p;R}[v] \quad \textit{for } v \in \hat{T}^{-1}_{p;R}\big(W^{1,p}_{0,0}(B_1)\big).$$

*In particular,*

$$\frac{1}{(p')^p}\|v\|^p_{L^p_{1;R}(B_1)} \le \Psi^{p;R}[v] \quad \textit{for } v \in \hat{T}^{-1}_{p;R}\big(W^{1,p}_{0,0}(B_1)\big).$$

Assertion (4)

$$\int_{B_1} |\nabla v(x)|^p A_{1,R}(x)^{p-1} I_p(x)\,dx \le 2^p \Psi^{p;R}[v] \quad \textit{for } v \in \hat{T}_{p;R}^{-1}\big(W_{0,0}^{1,p}(B_1)\big).$$

Further we show the following lemma.

LEMMA 6.4

*For* $1 < p \le \overline{q} < \infty$, $\tau_{p,\overline{q}} \le 1/n$, *and* $\underline{R} > 1$, *there exist positive numbers* $c_{p;\underline{R}}$, $c_{p,\overline{q};\underline{R}} > 0$ *such that for* $p \le q \le \overline{q}$, $\underline{R} \le R \le \overline{R}$ *we have the following:*

(1) $\Psi^{p;R}[v]^{1/p} \le \big(1 + \big(\frac{\log \overline{R}}{\log R}\big)^{1/p}\big)\Psi^{p;\overline{R}}[v]^{1/p}$ *for* $v \in C_c^\infty(B_1 \setminus \{0\})$.

(2) $|\Psi^{p;\overline{R}}[v]^{1/p} - \Psi^{p;R}[v]^{1/p}| \le c_{p;\underline{R}}(\overline{R} - R)\Psi^{p;\underline{R}}[v]$ *for* $v \in C_c^\infty(B_1 \setminus \{0\})$.

(3) $\big|\|v\|^q_{L^q_{1;\overline{R}}(B_1)} - \|v\|^q_{L^q_{1;R}(B_1)}\big| \le c_{p,\overline{q};\underline{R}}(\overline{R} - R)\Psi^{p;\underline{R}}[v]^{q/p}$ *for* $v \in C_c^\infty(B_1 \setminus \{0\})$.

*Proof*

First we have

$$A_{1,R}(x) \le A_{1,\overline{R}}(x) \le \frac{\log \overline{R}}{\log R} A_{1,R}(x),$$

$$\begin{aligned} A_{1,\overline{R}}(x)^{1/p'} - A_{1,R}(x)^{1/p'} &= \int_0^1 \frac{1}{p'} \frac{\overline{R} - R}{\theta\overline{R} + (1-\theta)R} \frac{1}{A_{1,\theta\overline{R}+(1-\theta)R}(x)^{1/p}}\,d\theta \\ &\le \frac{\overline{R} - R}{p'\underline{R}} \frac{1}{A_{1,\underline{R}}(x)^{1/p}}. \end{aligned}$$

In a similar way,

$$\frac{1}{A_{1,R}(x)^{1/p}} - \frac{1}{A_{1,\overline{R}}(x)^{1/p}} \le \frac{\overline{R} - R}{p\underline{R}} \frac{1}{A_{1,\underline{R}}(x)^{1+1/p}},$$

$$\frac{1}{A_{1,R}(x)} - \frac{1}{A_{1,\overline{R}}(x)} \le \frac{\overline{R} - R}{\underline{R}} \frac{1}{A_{1,\underline{R}}(x)^2} \quad \text{for } x \in B_1 \setminus \{0\}.$$

(1) From Lemma 6.3(3) we have

$$\begin{aligned} \Psi^{p;R}[v]^{1/p} &= \Big\{\int_{B_1} \Big| \frac{A_{1,R}(x)}{A_{1,\overline{R}}(x)}\Big(A_{1,\overline{R}}(x)\nabla v(x) - \frac{1}{p'}v(x)\frac{x}{|x|^2}\Big) \\ &\qquad - \frac{1}{p'}v(x)\Big(1 - \frac{A_{1,R}(x)}{A_{1,\overline{R}}(x)}\Big)\frac{x}{|x|^2}\Big|^p \frac{I_p(x)}{A_{1,R}(x)}\,dx\Big\}^{1/p} \\ &\le \Big(\int_{B_1} \Big|A_{1,\overline{R}}(x)\nabla v(x) - \frac{1}{p'}v(x)\frac{x}{|x|^2}\Big|^p \frac{I_p(x)}{A_{1,\overline{R}}(x)}\,dx\Big)^{1/p} \\ &\qquad + \frac{1}{p'}\Big(\int_{B_1} |v(x)|^p \frac{I_0(x)}{A_{1,R}(x)}\,dx\Big)^{1/p} \end{aligned}$$

$$\le \Psi^{p;\overline{R}}[v]^{1/p} + \frac{1}{p'}\Big(\frac{\log \overline{R}}{\log R}\Big)^{1/p}\|v\|_{L^{p}{}_{1;\overline{R}}(B_1)}$$
$$\le \bigg(1 + \Big(\frac{\log \overline{R}}{\log R}\Big)^{1/p}\bigg)\Psi^{p;\overline{R}}[v]^{1/p}.$$

(2) From assertions (3) and (4) of Lemma 6.3 we have

$$\begin{aligned}
&|\Psi^{p;\overline{R}}[v]^{1/p} - \Psi^{p;R}[v]^{1/p}| \\
&\le \Big\{\int_{B_1}\Big|(A_{1,\overline{R}}(x)^{1/p'} - A_{1,R}(x)^{1/p'})\nabla v(x) \\
&\quad - \frac{1}{p'}\Big(\frac{1}{A_{1,\overline{R}}(x)^{1/p}} - \frac{1}{A_{1,R}(x)^{1/p}}\Big)v(x)\frac{x}{|x|^2}\Big|^p I_p(x)\,dx\Big\}^{1/p} \\
&\le \Big(\int_{B_1}\Big(\frac{\overline{R}-R}{p'\underline{R}}\frac{1}{A_{1,\underline{R}}(x)^{1/p}}|\nabla v(x)|\Big)^p I_p(x)\,dx\Big)^{1/p} \\
&\quad + \frac{1}{p'}\Big(\int_{B_1}\Big(\frac{\overline{R}-R}{p\underline{R}}\frac{1}{A_{1,\underline{R}}(x)^{1+1/p}}|v(x)|\Big)^p I_0(x)\,dx\Big)^{1/p} \\
&\le \frac{\overline{R}-R}{p'\underline{R}\log \underline{R}}\bigg(\Big(\int_{B_1}|\nabla v(x)|^p A_{1,\underline{R}}(x)^{p-1}I_p(x)\,dx\Big)^{1/p} \\
&\quad + \frac{1}{p}\Big(\int_{B_1}|v(x)|^p\frac{I_0(x)}{A_{1,\underline{R}}(x)}\,dx\Big)^{1/p}\bigg) \\
&\le \frac{\overline{R}-R}{p'\underline{R}\log \underline{R}}\Big(2 + \frac{p'}{p}\Big)\Psi^{p;\underline{R}}[v]^{1/p}.
\end{aligned}$$

(3) Using that $t^t \le \max\{1, \bar{t}^{\bar{t}}\}$ for $0 < t \le \bar{t}$ and $\omega_n/(\overline{q}/q)' \le \omega_n/(p/q)'$, we have

$$\Big(\frac{\omega_n}{(\overline{q}/q)'}\Big)^{1/(\overline{q}/q)'} \le \max\Big\{1, \Big(\frac{\omega_n}{(p/q)'}\Big)^{1/(p/q)'}\Big\}.$$

Then by the Hölder inequality and Lemma 6.3(3), it holds that

$$\begin{aligned}
&\big|\|v\|^q_{L^q{}_{1;\overline{R}}(B_1)} - \|v\|^q_{L^q_{1;R}(B_1)}\big| \\
&= \int_{B_1}|v(x)|^q\Big(\frac{1}{A_{1,R}(x)} - \frac{1}{A_{1,\overline{R}}(x)}\Big)I_0(x)\,dx \\
&\le \frac{\overline{R}-R}{\underline{R}}\int_{B_1}|v(x)|^q\frac{1}{A_{1,\underline{R}}(x)}\frac{I_0(x)}{A_{1,\underline{R}}(x)}\,dx \\
&\le \frac{\overline{R}-R}{\underline{R}}\Big(\int_{B_1}(|v(x)|^q)^{\overline{q}/q}\frac{I_0(x)}{A_{1,\underline{R}}(x)}\,dx\Big)^{q/\overline{q}} \\
&\quad \times\Big(\int_{B_1}\Big(\frac{1}{A_{1,\underline{R}}(x)}\Big)^{(\overline{q}/q)'}\frac{I_0(x)}{A_{1,\underline{R}}(x)}\,dx\Big)^{1/(\overline{q}/q)'}
\end{aligned}$$

$$= \frac{\overline{R}-R}{\underline{R}\log\underline{R}}\Big(\frac{\omega_n}{(\overline{q}/q)'}\Big)^{1/(\overline{q}/q)'}\|v\|^q_{L^{\overline{q}}_{1;\underline{R}}(B_1)}$$
$$\le \frac{\overline{R}-R}{\underline{R}\log\underline{R}}\max\Big\{1,\Big(\frac{\omega_n}{(p/q)'}\Big)^{1/(p/q)'}\Big\}\Big(\frac{1}{C^{p,\overline{q};\underline{R}}}\Psi^{p;\underline{R}}[v]\Big)^{q/p}. \qquad \square$$

In a quite similar way to the argument in Lemma 6.2 we can show the following.

LEMMA 6.5

*Let $1<p\le q<\infty$, let $\tau_{p,q}\le 1/n$, and let $R>1$. Assume that $\{q_j\}_{j=1}^\infty\subset(p,p^*)$ satisfies*

$$q_j\to q \quad \text{as } j\to\infty.$$

*If $\{v_j\}_{j=1}^\infty\subset C_c^\infty(B_1\setminus\{0\})$ and $\{\Psi^{p;R}[v_j]\}_{j=1}^\infty$ is bounded, then it holds that*

$$\limsup_{j\to\infty}\big(\|v_j\|^{q_j}_{L^{q_j}_{1;R}(B_1)}-\|v_j\|^q_{L^q_{1;R}(B_1)}\big)\le 0.$$

By using these we have the following proposition, which gives Theorem 2.8.

PROPOSITION 6.2

*Let $1<p\le q<\infty$, let $\tau_{p,q}\le 1/n$, and let $R>1$. Assume that $\{(q_j;R_j)\}_{j=1}^\infty\subset(p,p^*)\times(1,\infty)$ satisfies*

$$q_j\to q,\qquad R_j\to R\quad \text{as } j\to\infty.$$

*Then it holds that*

$$C^{p,q_j;R_j}\to C^{p,q;R}\quad \text{as } j\to\infty.$$

*Proof*

(a) In a similar way to the argument in Proposition 6.1(a), we have

$$\limsup_{j\to\infty}C^{p,q_j;R_j}\le C^{p,q;R}.$$

(b) Next we show that

$$C^{p,q;R}\le\liminf_{j\to\infty}C^{p,q_j;R_j}.$$

To this end, let us take $\overline{q}$ and $\underline{R}$ such that

$$p\le q_j\le\overline{q}\begin{cases}\le p^* & \text{if } p<n,\\ <\infty & \text{if } p\ge n,\end{cases}\qquad 1<\underline{R}\le R_j\quad \text{for } j\ge 1.$$

It follows from Lemma 6.3(2) that there exists $\{v_j\}_{j=1}^\infty\subset C_c^\infty(B_1\setminus\{0\})\setminus\{0\}$ such that

$$\Psi^{p;\underline{R}}[v_j]=1,\qquad \frac{\Psi^{p;R_j}[v_j]}{\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)}}\le C^{p,q_j;R_j}+\frac{1}{j}\quad \text{for } j\ge 1.$$

Since $\underline{R} \le R$ holds, it follows from Lemma 6.4(1) that we have

$$1 = \Psi^{p;\underline{R}}[v_j] \le \left(1 + \left(\frac{\log R}{\log \underline{R}}\right)^{1/p}\right)^p \Psi^{p;R}[v_j] \quad \text{for } j \ge 1.$$

Using assertions (2) and (3) of Lemma 6.4 we also have

$$\begin{aligned} \Psi^{p;R}[v_j]^{1/p} &\le \Psi^{p;R_j}[v_j]^{1/p} + c_{p;\underline{R}}|R_j - R|\Psi^{p;\underline{R}}[v_j]^{1/p} \\ &= \Psi^{p;R_j}[v_j]^{1/p} + c_{p;\underline{R}}|R_j - R|, \\ \|v_j\|^{q_j}_{L^{q_j}_{1;R_j}(B_1)} &\le \|v_j\|^{q_j}_{L^{q_j}_{1;R}(B_1)} + c_{p,\overline{q};\underline{R}}|R_j - R| \quad \text{for } j \ge 1. \end{aligned}$$

Combining this with (a), we have that there exist $j_1 \in \mathbf{N}$ and $c > 0$ such that

$$\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)} \ge \frac{\Psi^{p;R_j}[v_j]}{C^{p,q_j;R_j} + 1/j} \ge c, \qquad |R_j - R| \le \frac{c^{q_j/p}}{c_{p,\overline{q};\underline{R}}} \quad \text{for } j \ge j_1.$$

Now let $\varepsilon$ satisfy $0 < \varepsilon < c$. Then it follows from Lemma 6.5 that there exists $j_\varepsilon \ge j_1$ such that we have

$$\|v_j\|^p_{L^{q_j}_{1;R}(B_1)} \le \|v_j\|^p_{L^q_{1;R}(B_1)} + \varepsilon \quad \text{for } j \ge j_\varepsilon.$$

Then we see

$$\begin{aligned} &C^{p,q;R}\left(\left(1 - \frac{c_{p,\overline{q};\underline{R}}}{c^{q_j/p}}|R_j - R|\right)^{p/q_j} - \frac{\varepsilon}{c}\right) \\ &\le C^{p,q;R}\left(\left(1 - \frac{c_{p,\overline{q};\underline{R}}}{\|v_j\|^{q_j}_{L^{q_j}_{1;R_j}(B_1)}}|R_j - R|\right)^{p/q_j} - \frac{\varepsilon}{\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)}}\right) \\ &= \frac{C^{p,q;R}}{\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)}}\big((\|v_j\|^{q_j}_{L^{q_j}_{1;R_j}(B_1)} - c_{p,\overline{q};\underline{R}}|R_j - R|)^{p/q_j} - \varepsilon\big) \\ &\le \frac{C^{p,q;R}}{\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)}}(\|v_j\|^p_{L^{q_j}_{1;R}(B_1)} - \varepsilon) \\ &\le \frac{C^{p,q;R}\|v_j\|^p_{L^q_{1;R}(B_1)}}{\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)}} \le \frac{\Psi^{p;R}[v_j]}{\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)}} \\ &\le \frac{1}{\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)}}(\Psi^{p;R_j}[v_j]^{1/p} + c_{p;\underline{R}}|R_j - R|)^p \\ &= \frac{\Psi^{p;R_j}[v_j]}{\|v_j\|^p_{L^{q_j}_{1;R_j}(B_1)}}\left(1 + \frac{c_{p;\underline{R}}}{\Psi^{p;R_j}[v_j]^{1/p}}|R_j - R|\right)^p \\ &\le \left(C^{p,q_j;R_j} + \frac{1}{j}\right)\left(1 + c_{p;\underline{R}}\left(1 + \left(\frac{\log R_j}{\log \underline{R}}\right)^{1/p}\right)|R_j - R|\right)^p \quad \text{for } j \ge j_\varepsilon, \end{aligned}$$

and thus, the assertion is established. □

### 6.3. Some estimates for the best constants

In this section we establish assertions (5), (6), and (7) of Theorem 2.2. First Theorem 2.2(7) follows from the next proposition.

PROPOSITION 6.3

*Assume that $1 < p \le q \le \overline{q} < \infty$, and assume that $\tau_{p,\overline{q}} \le 1/n$. Then we have*

$$S^{p,q;\gamma} \ge \left(\gamma^{p\tau_{q,\overline{q}}}(S^{p,\overline{q};\gamma})^{\tau_{p,q}}\right)^{1/\tau_{p,\overline{q}}} \quad \textit{for } \gamma > 0.$$

For the proof we employ the following lemma.

LEMMA 6.6

*Let $1 \le p \le q \le \overline{q} < \infty$, let $\gamma > 0$, and let $\Omega$ be a domain of $\mathbf{R}^n$. Then we have*

$$\|u\|^{\tau_{p,\overline{q}}}_{L^q_\gamma(\Omega)} \le \|u\|^{\tau_{q,\overline{q}}}_{L^p_\gamma(\Omega)}\|u\|^{\tau_{p,q}}_{L^{\overline{q}}_\gamma(\Omega)} \quad \textit{for } u \in L^p_\gamma(\Omega) \cap L^{\overline{q}}_\gamma(\Omega).$$

*Proof*

Noting that $q\tau_{q,\overline{q}}/(p\tau_{p,\overline{q}}) + q\tau_{p,q}/(\overline{q}\tau_{p,\overline{q}}) = 1$, we have

$$\|u\|^q_{L^q_\gamma(\Omega)} = \int_\Omega \left(|u(x)||x|^\gamma\right)^{q\tau_{q,\overline{q}}/\tau_{p,\overline{q}}}\left(|u(x)||x|^\gamma\right)^{q\tau_{p,q}/\tau_{p,\overline{q}}} I_0(x)\,dx.$$

Then the assertion easily follows from this by the aid of the Hölder inequality. □

*Proof of Proposition 6.3*

For $\varepsilon > 0$, there exists a $u_\varepsilon \in C^\infty_c(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}$ such that we have

$$\|u_\varepsilon\|^q_{L^q_\gamma(\mathbf{R}^n)} = 1, \qquad S^{p,q;\gamma} \le \|\nabla u_\varepsilon\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \le S^{p,q;\gamma} + \varepsilon.$$

Then, by Lemma 6.6 and Theorem 2.1 we have

$$\begin{aligned} 1 = \|u_\varepsilon\|^{p\tau_{p,\overline{q}}}_{L^q_\gamma(\mathbf{R}^n)} &\le \|u_\varepsilon\|^{p\tau_{q,\overline{q}}}_{L^p_\gamma(\mathbf{R}^n)}\|u_\varepsilon\|^{p\tau_{p,q}}_{L^{\overline{q}}_\gamma(\mathbf{R}^n)} \\ &\le \left(\frac{1}{\gamma^p}\|\nabla u_\varepsilon\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\right)^{\tau_{q,\overline{q}}}\left(\frac{1}{S^{p,\overline{q};\gamma}}\|\nabla u_\varepsilon\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\right)^{\tau_{p,q}} \\ &\le \frac{1}{\gamma^{p\tau_{p,\overline{q}}}(S^{p,\overline{q};\gamma})^{\tau_{p,q}}}(S^{p,q;\gamma}+\varepsilon)^{\tau_{p,\overline{q}}}, \end{aligned}$$

and this proves the assertion. □

To prove assertions (5) and (6) of Theorem 2.2, we establish the next proposition. Given Theorem 2.2(3), assertions (5) and (6) of Theorem 2.2 follow from assertions (1) and (2) and from assertions (2) and (3) of the next proposition, respectively.

PROPOSITION 6.4

*Let $n \ge 2$, let $1 < p < n$, and let $q = p^*$. Then we have the following.*

Assertion (1)

$$S^{p,p^*;\gamma_{p,p^*}} \le \Big(2 - \frac{\gamma_{p,p^*}}{\gamma}\Big)^p S^{p,p^*;\gamma} \quad \textit{for } \gamma \ge \gamma_{p,p^*}.$$

Assertion (2)

$$S^{p,p^*;\gamma} \le S^{p,p^*;\gamma_{p,p^*}} \quad \textit{for } \gamma \ge \gamma_{p,p^*}.$$

Assertion (3) *When* $p = 2$,

$$S^{2,2^*;\gamma} \le S^{2,2^*;\overline{\gamma}} \quad \textit{for } 0 < \gamma \le \overline{\gamma}.$$

*Proof*

(1) For $\varepsilon > 0$, there exists a $u_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}$ such that we have

$$\|u_\varepsilon\|^{p^*}_{L^{p^*}_\gamma(\mathbf{R}^n)} = 1, \qquad S^{p,p^*;\gamma} \le \|\nabla u_\varepsilon\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \le S^{p,p^*;\gamma} + \varepsilon.$$

Since $n - \gamma_{p,p^*} p^* = 0$, it holds that

$$\|\hat{T}_{\gamma_{p,p^*}-\gamma} u_\varepsilon\|^{p^*}_{L^{p^*}(\mathbf{R}^n)} = \|\hat{T}_{\gamma_{p,p^*}-\gamma} u_\varepsilon\|^{p^*}_{L^{p^*}_{\gamma_{p,p^*}}(\mathbf{R}^n)} = \|u_\varepsilon\|^{p^*}_{L^{p^*}_\gamma(\mathbf{R}^n)} = 1.$$

Noting that $n - (1 + \gamma_{p,p^*})p = 0$ and $n - p(1+\gamma) = (\gamma_{p,p^*} - \gamma)p$, by the Sobolev inequality and the Hardy–Sobolev inequality we have

$$\begin{aligned}
(S^{p,p^*;\gamma_{p,p^*}})^{1/p} &\le \|\nabla[\hat{T}_{\gamma_{p,p^*}-\gamma} u_\varepsilon]\|_{L^p_{1+\gamma_{p,p^*}}(\mathbf{R}^n)} = \|\nabla[\hat{T}_{\gamma_{p,p^*}-\gamma} u_\varepsilon]\|_{L^p(\mathbf{R}^n)} \\
&= \Big(\int_{\mathbf{R}^n} \Big|\nabla u_\varepsilon(x) + (\gamma - \gamma_{p,p^*}) u_\varepsilon(x) \frac{x}{|x|^2}\Big|^p I_{p(1+\gamma)}(x)\,dx\Big)^{1/p} \\
&\le \|\nabla u_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + (\gamma - \gamma_{p,p^*}) \|u_\varepsilon\|_{L^p_\gamma(\mathbf{R}^n)} \\
&\le \|\nabla u_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + (\gamma - \gamma_{p,p^*}) \frac{1}{\gamma} \|\nabla u_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \\
&\le \Big(2 - \frac{\gamma_{p,p^*}}{\gamma}\Big)(S^{p,p^*;\gamma} + \varepsilon)^{1/p}.
\end{aligned}$$

(2) Let $u \in C_{\mathrm{c}}^\infty(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}$, and let $e_1 = (1, 0, \ldots, 0) \in \mathbf{R}^n$. Since $p^*\gamma \ge n$ and $p(1+\gamma) \ge n$ hold, we have

$$\begin{aligned}
\varepsilon^{p^*\gamma - n} \Big\| u\Big(\cdot - \frac{e_1}{\varepsilon}\Big)\Big\|^{p^*}_{L^{p^*}_\gamma(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |u(x)|^{p^*} I_{p^*\gamma}(\varepsilon x + e_1)\,dx \\
&\to \int_{\mathbf{R}^n} |u(x)|^{p^*}\,dx = \|u\|^{p^*}_{L^{p^*}_{\gamma_{p,p^*}}(\mathbf{R}^n)}, \\
\varepsilon^{p(1+\gamma) - n} \Big\| \nabla\Big[u\Big(\cdot - \frac{e_1}{\varepsilon}\Big)\Big]\Big\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(\varepsilon x + e_1)\,dx \\
&\to \int_{\mathbf{R}^n} |\nabla u(x)|^p\,dx \\
&= \|\nabla u\|^p_{L^p_{1+\gamma_{p,p^*}}(\mathbf{R}^n)} \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Therefore

$$S^{p,p^*;\gamma} \le E^{p,p^*;\gamma}\Big[u\Big(\cdot-\frac{e_1}{\varepsilon}\Big)\Big] = \frac{\varepsilon^{p(1+\gamma)-n}\big\|\nabla\big[u\big(\cdot-\frac{e_1}{\varepsilon}\big)\big]\big\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}}{\big(\varepsilon^{p^*\gamma-n}\big\|u\big(\cdot-\frac{e_1}{\varepsilon}\big)\big\|^{p^*}_{L^{p^*}_\gamma(\mathbf{R}^n)}\big)^{p/p^*}}$$
$$\to E^{p,p^*;\gamma_{p,p^*}}[u] \quad \text{as } \varepsilon \to 0,$$

and this proves the assertion.

(3.a) For $u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})\setminus\{0\}$, we set

$$\zeta[u](\gamma) = E^{2,2^*;\gamma}[uI_{n-\gamma}] \quad \text{for } \gamma > 0.$$

If we note that

$$2\int_{\mathbf{R}^n} u(x)\big(x\cdot\nabla u(x)\big)I_0(x)\,dx = \int_{S^{n-1}}\int_0^\infty \frac{\partial}{\partial r}[u^2](r\omega)\,dr\,dS(\omega) = 0,$$

then we obtain

$$\begin{aligned}\zeta[u](\gamma) &= \frac{\|\nabla[uI_{n-\gamma}]\|^2_{L^2_{1+\gamma}(\mathbf{R}^n)}}{\|uI_{n-\gamma}\|^2_{L^{2^*}_\gamma(\mathbf{R}^n)}}\\ &= \frac{1}{\|u\|^2_{L^{2^*}_0(\mathbf{R}^n)}}\int_{\mathbf{R}^n}\big(\gamma^2u(x)^2 - 2\gamma u(x)(x\cdot\nabla u(x)) + |x|^2|\nabla u(x)|^2\big)I_0(x)\,dx\\ &= \frac{1}{\|u\|^2_{L^{2^*}_0(\mathbf{R}^n)}}\big(\gamma^2\|u\|^2_{L^2_0(\mathbf{R}^n)} + \|\nabla u\|^2_{L^2_1(\mathbf{R}^n)}\big) \quad \text{for } \gamma > 0,\end{aligned}$$

and so, we see that $\zeta[u]$ is nondecreasing with respect to $\gamma$.

(3.b) For $0 < \gamma \le \overline{\gamma}$, it follows from (a) that we have

$$S^{2,2^*;\gamma} \le E^{2,2^*;\gamma}\Big[\frac{u}{I_{n-\overline{\gamma}}}I_{n-\gamma}\Big] = \zeta\Big[\frac{u}{I_{n-\overline{\gamma}}}\Big](\gamma) \le \zeta\Big[\frac{u}{I_{n-\overline{\gamma}}}\Big](\overline{\gamma}) = E^{2,2^*;\overline{\gamma}}[u]$$
$$\text{for } u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})\setminus\{0\}.$$

This clearly proves the assertion. □

## 7. Existence of minimizers for the best constants

In this section we prove the existence of minimizers for $S^{p,q;\gamma}$ by the effective use of the so-called concentration compactness principle when $p < q < p^*$ and $\gamma > 0$. We begin by preparing some notations.

DEFINITION 7.1

(1) Let $\psi_1, \rho_1 \in C^\infty_{\mathrm{c}}(\mathbf{R}^n)_{\mathrm{rad}}$ and $\overline{\rho}_1 \in C^\infty(\mathbf{R}^n)_{\mathrm{rad}}$ satisfy

$$0 \le \psi_1 \le 1, \qquad \rho_1 \ge 0, \qquad \overline{\rho}_1 > 0 \quad \text{on } \mathbf{R}^n,$$
$$\psi_1 = 1 \quad \text{on } \overline{B_{1/2}}, \qquad \psi_1 = \rho_1 = 0 \quad \text{on } \mathbf{R}^n\setminus B_1,$$
$$\psi_1' = \frac{\partial\psi_1}{\partial r} \le 0 \quad \text{on } \mathbf{R}^n\setminus\{0\}, \qquad \|\nabla\psi_1\|_{L^\infty(\mathbf{R}^n)} \le 3,$$
$$\|\rho_1\|_{L^1(\mathbf{R}^n)} = \|\overline{\rho}_1\|_{L^1(\mathbf{R}^n)} = 1.$$

(2) For $\varepsilon > 0$

$$\psi_\varepsilon(x) = \psi_\varepsilon(|x|) = \psi_1\Big(\frac{x}{\varepsilon}\Big), \qquad \tilde{\psi}_\varepsilon(x) = \tilde{\psi}_\varepsilon(|x|) = -[\psi_1']\Big(\frac{|x|}{\varepsilon}\Big) = \Big|[\psi_1']\Big(\frac{|x|}{\varepsilon}\Big)\Big|,$$

$$\rho_\varepsilon(x) = \rho_\varepsilon(|x|) = \frac{1}{\varepsilon^n}\rho_1\Big(\frac{x}{\varepsilon}\Big), \qquad \overline{\rho}_\varepsilon(x) = \overline{\rho}_\varepsilon(|x|) = \frac{1}{\varepsilon^n}\overline{\rho}_1\Big(\frac{x}{\varepsilon}\Big) \quad \text{for } x \in \mathbf{R}^n.$$

### 7.1. Preliminaries

In this section we prepare some well-known properties in the theory of concentration compactness due to P. L. Lions, which are useful in the proof of the existence of a minimizer of the best constant $S^{p,q;\gamma}$. We omit the proof of the next fundamental lemma. See [Li1, Lemma 1.1, Section 1.3] for details.

LEMMA 7.1

*Assume that $\{Q_j\}_{j=1}^\infty$ is a sequence of uniformly bounded and nondecreasing functions on $[1,\infty)$. Then, there exist a subsequence $\{Q_{j_k}\}_{k=1}^\infty$ and a nondecreasing function $Q$ on $[1,\infty)$ such that we have*

$$Q_{j_k}(t) \to Q(t) \quad \textit{as } k \to \infty \textit{ for } t > 1.$$

It follows from the Hölder inequality that we have the following lemma.

LEMMA 7.2

*For $1 < p \le q < \infty$, $\gamma > 0$, and $R > 0$, we have*

$$\|u\|_{L^p_\gamma(B_{2R}\setminus\overline{B_R})} \le (\omega_n \log 2)^{\tau_{p,q}} \|u\|_{L^q_\gamma(B_{2R}\setminus\overline{B_R})} \quad \textit{for } u \in L^q_\gamma(B_{2R}\setminus\overline{B_R}).$$

The proof is omitted. It follows from the Rellich lemma that we have the following.

LEMMA 7.3

*For $1 < p \le q < \infty$, $\tau_{p,q} < 1/n$, and $\gamma > 0$, assume that $\Omega$ is a bounded domain of $\mathbf{R}^n$, and assume that $\partial\Omega$ is smooth. Then, the embedding $W^{1,p}_{\gamma,0}(\Omega) \subset L^q_{\gamma+1-n\tau_{p,q}}(\Omega)$ is compact.*

*Proof*

For $u \in C_c^\infty(\Omega\setminus\{0\})$, we have

$$\nabla[uI_{1+\gamma+n/p'}](x) = I_{p(1+\gamma)}(x)^{1/p}\nabla u(x) + \Big(1+\gamma-\frac{n}{p}\Big)I_{p\gamma}(x)^{1/p}u(x)\frac{x}{|x|} \quad \text{for } x \in \Omega.$$

Hence we have

$$\begin{aligned}\|\nabla[uI_{1+\gamma+n/p'}]\|_{L^p(\Omega)} &\le \|I^{1/p}_{p(1+\gamma)}\nabla u\|_{L^p(\Omega)} + \Big|1+\gamma-\frac{n}{p}\Big|\|I^{1/p}_{p\gamma}u\|_{L^p(\Omega)} \\ &= \|\nabla u\|_{L^p_{1+\gamma}(\Omega)} + \Big|1+\gamma-\frac{n}{p}\Big|\|u\|_{L^q_\gamma(\Omega)} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\Omega).\end{aligned}$$

Therefore, if $\{u_j\}_{j=1}^{\infty}$ is bounded in $W_{\gamma,0}^{1,p}(\Omega)$, then $\{u_j I_{1+\gamma+n/p'}\}_{j=1}^{\infty}$ should be bounded in $W_0^{1,p}(\Omega)$ (a classical Sobolev space without a weight), and by the Rellich lemma $\{u_j I_{1+\gamma+n/p'}\}_{j=1}^{\infty}$ has a subsequence $\{u_{j_k} I_{1+\gamma+n/p'}\}_{k=1}^{\infty}$ that converges in $L^q(\Omega)$. Noting that $n-(1+\gamma+n/p')=(n-(1+\gamma)p)/p$ and $(n-(1+\gamma)p)q/p = n-q(1+\gamma-n\tau_{p,q})$, we get that $\{u_{j_k}\}_{k=1}^{\infty}$ converges in $L^q_{\gamma+1-n\tau_{p,q}}(\Omega)$ as well. □

Let us recall a sharp Fatou's lemma, which is essentially due to H. Brézis and E. Lieb [BL] (see also [LL, Section 1.9]).

LEMMA 7.4

*For $1<q<\infty$ and $\gamma>0$, assume that $\{u_j\}_{j=1}^{\infty}$ is bounded in $L^q_\gamma(\mathbf{R}^n)$, and assume that*

$$u_j \to u \quad \text{a.e. on } \mathbf{R}^n \text{ as } j\to\infty.$$

*Then, we have $u\in L^q_\gamma(\mathbf{R}^n)$ and*

$$\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j-u\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \|u\|^q_{L^q_\gamma(\mathbf{R}^n)} \quad \text{as } j\to\infty.$$

*Proof*

For $0<\varepsilon<1$, there exists a positive number $\hat{c}_{q;\varepsilon}>0$ such that we have

$$\big||s+t|^q-|s|^q-|t|^q\big| \le \varepsilon|s|^q+\hat{c}_{q;\varepsilon}|t|^q \quad \text{for } s,t\in\mathbf{R}. \tag{7.1}$$

Since $|u_j|I_{q\gamma}\to|u|I_{q\gamma}$ a.e. on $\mathbf{R}^n$ as $j\to\infty$ by the hypothesis, it follows from Fatou's lemma that

$$\|u\|^q_{L^q_\gamma(\mathbf{R}^n)} \le \liminf_{j\to\infty}\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} \le \sup_{j\ge1}\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)}<\infty;$$

hence we see that $u\in L^q_\gamma(\mathbf{R}^n)$. Then we have $|u|^qI_{q\gamma}\in L^1(\mathbf{R}^n)$ and

$$\big(\big||u_j|^q-|u_j-u|^q-|u|^q\big|-\varepsilon|u_j-u|^q\big)_+I_{q\gamma}\le\hat{c}_{q;\varepsilon}|u|^qI_{q\gamma} \quad \text{a.e. on } \mathbf{R}^n \text{ for } j\ge1.$$

Using Lebesgue's convergence theorem, we have

$$\int_{\mathbf{R}^n}\big[\big(\big||u_j|^q-|u_j-u|^q-|u|^q\big|-\varepsilon|u_j-u|^q\big)_+I_{q\gamma}\big](x)\,dx\to0 \quad \text{as } j\to\infty.$$

After all this we have

$$\begin{aligned}
&\big|\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)}-\|u_j-u\|^q_{L^q_\gamma(\mathbf{R}^n)}-\|u\|^q_{L^q_\gamma(\mathbf{R}^n)}\big|\\
&\quad\le\int_{\mathbf{R}^n}\big[\big||u_j|^q-|u_j-u|^q-|u|^q\big|I_{q\gamma}\big](x)\,dx\\
&\quad=\int_{\mathbf{R}^n}\big[\big(\big||u_j|^q-|u_j-u|^q-|u|^q\big|-\varepsilon|u_j-u|^q\big)I_{q\gamma}\big](x)\,dx\\
&\qquad+\varepsilon\|u_j-u\|^q_{L^q_\gamma(\mathbf{R}^n)}\\
&\quad\le\int_{\mathbf{R}^n}\big[\big(\big||u_j|^q-|u_j-u|^q-|u|^q\big|-\varepsilon|u_j-u|^q\big)_+I_{q\gamma}\big](x)\,dx
\end{aligned}$$

$$+\varepsilon\Big(2\sup_{j\geq 1}\|u_j\|_{L^q_\gamma(\mathbf{R}^n)}\Big)^q$$
$$\to \varepsilon\Big(2\sup_{j\geq 1}\|u_j\|_{L^q_\gamma(\mathbf{R}^n)}\Big)^q \quad \text{as } j\to\infty.$$

Thus the assertion is now established. □

LEMMA 7.5

*For $1<p\leq q<\infty$, $\tau_{p,q}<1/n$, and $\gamma>0$, there exists a positive number $\overline{c}_{p,q;\gamma}>0$ such that we have*

$$\|u\|^p_{L^q_\gamma(B_{|y|/4}(y))}\leq \overline{c}_{p,q;\gamma}(\|\nabla u\|^p_{L^p_{1+\gamma}(B_{|y|/2}(y))}+\|u\|^p_{L^p_\gamma(B_{|y|/2}(y))})$$
$$\textit{for } y\in\mathbf{R}^n\setminus\{0\}, u\in W^{1,p}_{\gamma,0}(\mathbf{R}^n).$$

*Proof*

For $y\in\mathbf{R}^n\setminus\{0\}$ and $u\in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})$ we set

$$K_y u(x)=\psi_{|y|/2}(x-y)u(x) \quad \text{for } x\in\mathbf{R}^n. \tag{7.2}$$

By differentiation we have

$$\nabla[K_yu](x)=\psi_{|y|/2}(x-y)\nabla u(x)-\frac{2}{|y|}\tilde{\psi}_{|y|/2}(x-y)u(x)\frac{x-y}{|x-y|} \quad \text{for } x\in\mathbf{R}^n.$$

Since $\operatorname{supp}(K_yu)\subset\overline{B_{|y|/2}(y)}$ and $|x|\leq 3|y|/2$ for $x\in B_{|y|/2}(y)$, it holds that

$$|\nabla[K_yu](x)|\leq\Big(|\nabla u(x)|+\frac{9}{|x|}|u(x)|\Big)\chi_{B_{|y|/2}(y)}(x) \quad \text{for } x\in\mathbf{R}^n\setminus\{0\}.$$

Noting that

$$|u(x)|\chi_{B_{|y|/4}(y)}(x)\leq|K_yu(x)| \quad \text{for } x\in\mathbf{R}^n,$$

we have

$$\begin{aligned}
&\|u\|^p_{L^q_\gamma(B_{|y|/4}(y))}\\
&\quad=\|u\chi_{B_{|y|/4}(y)}\|^p_{L^q_\gamma(\mathbf{R}^n)}\leq\|K_yu\|^p_{L^q_\gamma(\mathbf{R}^n)}\\
&\quad\leq\frac{1}{S^{p,q;\gamma}}\|\nabla[K_yu]\|^p_{L^q_{1+\gamma}(\mathbf{R}^n)}\leq\frac{1}{S^{p,q;\gamma}}\Big\||\nabla u|+9\frac{|u|}{|\cdot|}\Big\|^p_{L^q_{1+\gamma}(B_{|y|/2}(y))}\\
&\quad\leq\frac{1}{S^{p,q;\gamma}}(\|\nabla u\|_{L^p_{1+\gamma}(B_{|y|/2}(y))}+9\|u\|_{L^q_\gamma(B_{|y|/2}(y))})^p \quad \text{for } u\in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\}),
\end{aligned}$$

and hence the assertion follows. □

LEMMA 7.6

*Let us take $\{z^k\}_{k=1}^\infty\subset\mathbf{R}^n\setminus\{0\}$ and $L\in\mathbf{N}$ such that*

$$\bigcup_{k=1}^\infty B_{|z^k|/4}(z^k)=\mathbf{R}^n\setminus\{0\}, \qquad L=\sup_{x\in\mathbf{R}^n\setminus\{0\}}\sharp\big\{k\in\mathbf{N}\ \big|\ x\in B_{|z^k|/2}(z^k)\big\}<\infty.$$

*Then, for* $1<q<\infty$ *and* $\gamma>0$ *we have*

$$\begin{aligned}\|u\|^q_{L^q_\gamma(\mathbf{R}^n)} &\le \sum_{k=1}^{\infty}\|u\|^q_{L^q_\gamma(B_{|z^k|/4}(z^k))}\\ &\le \sum_{k=1}^{\infty}\|u\|^q_{L^q_\gamma(B_{|z^k|/2}(z^k))}\le L\|u\|^q_{L^q_\gamma(\mathbf{R}^n)}\quad \text{for } u\in L^q_\gamma(\mathbf{R}^n).\end{aligned}$$

*Proof*
By the assumption on $\{z^k\}_{k=1}^{\infty}$ and $L$, it holds that

$$1\le\sum_{k=1}^{\infty}\chi_{B_{|z^k|/4}(z^k)}(x)\le\sum_{k=1}^{\infty}\chi_{B_{|z^k|/2}(z^k)}(x)\le L\quad\text{for } x\in\mathbf{R}^n\setminus\{0\},$$

and this proves the assertion. □

Now we verify the following.

LEMMA 7.7
*Let* $1<p\le q<\infty$*, let* $p\le\tilde{q}<\infty$*, let* $\tau_{p,q}<1/n$*, let* $\tau_{p,\tilde{q}}<1/n$*, and let* $\gamma>0$*. Then, there exist positive numbers* $\theta_{p,q,\tilde{q}}\in(0,1)$ *and* $\overline{c}_{p,q,\tilde{q};\gamma}>0$ *such that we have*

$$\|u\|_{L^q_\gamma(\mathbf{R}^n)}\le\overline{c}_{p,q,\tilde{q};\gamma}\|\nabla u\|^{\theta_{p,q,\tilde{q}}}_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big(\sup_{y\in\mathbf{R}^n\setminus\{0\}}\|u\|_{L^{\tilde{q}}_\gamma(B_{|y|/4}(y))}\Big)^{1-\theta_{p,q,\tilde{q}}}$$

$$\text{for } u\in W^{1,p}_{\gamma,0}(\mathbf{R}^n).$$

*Proof*
(a) Assume that $\tilde{q}<q$. Noting that $1/p-(q/p-1)/\tilde{q}-1/q=(1/p-1/q)(1-q/\tilde{q})<0$ we choose $\overline{q}=\overline{q}_{p,q,\tilde{q}}$ such that

$$\max\Big\{\frac{1}{p^*},\frac{1}{p}-\Big(\frac{q}{p}-1\Big)\frac{1}{\tilde{q}}\Big\}<\frac{1}{\overline{q}}=\frac{1}{\overline{q}_{p,q,\tilde{q}}}<\frac{1}{q},$$

and then we put

$$\theta=\theta_{p,q,\tilde{q}}=\frac{1/\tilde{q}-1/q}{1/\tilde{q}-1/\overline{q}}=\frac{1/\tilde{q}-1/q}{1/\tilde{q}-1/\overline{q}_{p,q,\tilde{q}}}.$$

Then, noting that $\tilde{q}<q<\overline{q}$, $q\theta>p$, and $\tau_{p,\overline{q}}<1/n$, we have that it follows from Lemmas 6.6, 7.5, and 7.6 that

$$\begin{aligned}&\|u\|^q_{L^q_\gamma(\mathbf{R}^n)}\\ &\quad\le\sum_{k=1}^{\infty}\big(\|u\|^{1/\tilde{q}-1/\overline{q}}_{L^q_\gamma(B_{|z^k|/4}(z^k))}\big)^{q/(1/\tilde{q}-1/\overline{q})}\\ &\quad\le\sum_{k=1}^{\infty}\big(\|u\|^{1/q-1/\overline{q}}_{L^{\tilde{q}}_\gamma(B_{|z^k|/4}(z^k))}\|u\|^{1/\tilde{q}-1/q}_{L^{\overline{q}}_\gamma(B_{|z^k|/4}(z^k))}\big)^{q/(1/\tilde{q}-1/\overline{q})}\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=1}^{\infty} \|u\|_{L^{\tilde q}_\gamma(B_{|z^k|/4}(z^k))}^{q(1-\theta)} \|u\|_{L^{\overline{q}}_\gamma(B_{|z^k|/4}(z^k))}^{q\theta} \\
&\le \sum_{k=1}^{\infty} \Big( \sup_{y\in\mathbf{R}^n\setminus\{0\}} \|u\|_{L^{\tilde q}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)} \|u\|_{L^{\overline{q}}_\gamma(\mathbf{R}^n)}^{q\theta-p} \|u\|_{L^{\overline{q}}_\gamma(B_{|z^k|/4}(z^k))}^{p} \\
&\le \Big( \sup_{y\in\mathbf{R}^n\setminus\{0\}} \|u\|_{L^{\tilde q}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)} \|u\|_{L^{\overline{q}}_\gamma(\mathbf{R}^n)}^{q\theta-p} \\
&\quad \cdot \sum_{k=1}^{\infty} \overline{c}_{p,\overline{q};\gamma} (\|\nabla u\|_{L^p_{1+\gamma}(B_{|y|/2}(y))}^p + \|u\|_{L^p_\gamma(B_{|y|/2}(y))}^p) \\
&\le L\overline{c}_{p,\overline{q};\gamma} \Big( \sup_{y\in\mathbf{R}^n\setminus\{0\}} \|u\|_{L^{\tilde q}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)} \|u\|_{L^{\overline{q}}_\gamma(\mathbf{R}^n)}^{q\theta-p} \\
&\quad \cdot (\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}^p + \|u\|_{L^p_\gamma(\mathbf{R}^n)}^p) \\
&\le L\overline{c}_{p,\overline{q};\gamma} \Big( \sup_{y\in\mathbf{R}^n\setminus\{0\}} \|u\|_{L^{\tilde q}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)} \\
&\quad \cdot \Big( \frac{1}{(S^{p,\overline{q};\gamma})^{1/p}} \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big)^{q\theta-p} \Big( \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}^p + \frac{1}{\gamma^p} \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}^p \Big) \\
&= \frac{L\overline{c}_{p,\overline{q};\gamma}}{(S^{p,\overline{q};\gamma})^{q\theta/p-1}} \Big(1+\frac{1}{\gamma^p}\Big) \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}^{q\theta} \Big( \sup_{y\in\mathbf{R}^n\setminus\{0\}} \|u\|_{L^{\tilde q}_\gamma(B_{|y|/4}(y))}\Big)^{q(1-\theta)}
\end{aligned}
$$

for $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$.

(b) Assume that $q \le \tilde q$. Let us take $\overline{q} = \overline{q}_{p,q,\tilde q}$ such that it satisfies $\tilde q \le \overline{q} = \overline{q}_{p,q,\tilde q} < \infty$ and $\tau_{p,\overline{q}} < 1/n$. Then it follows from (a) that there exist positive numbers $\theta_{p,\overline{q},\tilde q} \in (0,1)$ and $\overline{c}_{p,\overline{q},\tilde q;\gamma} > 0$ such that we have

$$
\|u\|_{L^{\overline{q}}_\gamma(\mathbf{R}^n)} \le \overline{c}_{p,\overline{q},\tilde q;\gamma} \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}^{\theta_{p,\overline{q},\tilde q}} \Big( \sup_{y\in\mathbf{R}^n\setminus\{0\}} \|u\|_{L^{\tilde q}_\gamma(B_{|y|/4}(y))}\Big)^{1-\theta_{p,\overline{q},\tilde q}}
$$

for $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$.

Then from Lemma 6.6 we have

$$
\begin{aligned}
\|u\|_{L^q_\gamma(\mathbf{R}^n)}^{1/p-1/\overline{q}} &\le \|u\|_{L^p_\gamma(\mathbf{R}^n)}^{1/q-1/\overline{q}} \|u\|_{L^{\overline{q}}_\gamma(\mathbf{R}^n)}^{1/p-1/q} \\
&\le \frac{1}{\gamma^{1/q-1/\overline{q}}} \|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)}^{1/q-1/\overline{q}} \|u\|_{L^{\overline{q}}_\gamma(\mathbf{R}^n)}^{1/p-1/q} \quad \text{for } u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n).
\end{aligned}
$$

Therefore we have the desired estimate with

$$
\theta_{p,q,\tilde q} = \frac{1/q - 1/\overline{q} + \theta_{p,\overline{q},\tilde q}(1/p-1/q)}{1/p-1/\overline{q}}. \qquad \square
$$

## 7.2. Some properties of minimizing sequences

In this section we study minimizing sequences for the best constants $S^{p,q;\gamma}$ by using the concentration compactness principle on annular domains (cf. [Li2]).

DEFINITION 7.2

Let $1 < p \le q < \infty$, and let $\gamma > 0$. For $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$ we set

$$\rho^{p,q;\gamma}[u] = |u|^q I_{q\gamma} + |\nabla u|^p I_{p(1+\gamma)},$$
$$Q^{p,q;\gamma}[u](t) = \sup_{r>0} \|\rho^{p,q;\gamma}[u]\|_{L^1(B_{tr}\setminus\overline{B_r})} \quad \text{for } t > 1.$$

First of all we show that there exists a minimizing sequence for $S^{p,q;\gamma}$ that does not vanish.

PROPOSITION 7.1

*Assume that $1 < p < q < \infty$, assume that $\tau_{p,q} < 1/n$, and assume that $\gamma > 0$. Then, there exist $\{u_j\}_{j=1}^{\infty} \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n) \setminus \{0\}$, a nondecreasing function $Q : (1, \infty) \to \mathbf{R}$, and positive numbers $\underline{\lambda}, \lambda$ satisfying*

$$0 < \underline{\lambda} \le \lambda \le 1 + S^{p,q;\gamma}$$

*such that:*

(1) $\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} = 1$ *for* $j \ge 1$, $\|\nabla u_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \to S^{p,q;\gamma}$ *as* $j \to \infty$.

(2) $\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{5/4}\setminus\overline{B_{3/4}})} \ge \|u_j\|^q_{L^q_\gamma(B_{5/4}\setminus\overline{B_{3/4}})} \ge \underline{\lambda}$ *for* $j \ge 1$.

(3) $Q^{p,q;\gamma}[u_j](t) \to Q(t)$ *as* $j \to \infty$ *for* $t > 1$; $Q(t) \to \lambda$ *as* $t \to \infty$.

*Proof*

(1)–(2) From Definition 2.3, there exists a sequence $\{v_j\}_{j=1}^{\infty} \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n) \setminus \{0\}$ such that

$$\|v_j\|^q_{L^q_\gamma(\mathbf{R}^n)} = 1 \quad \text{for } j \ge 1, \qquad \|\nabla v_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \to S^{p,q;\gamma} \quad \text{as } j \to \infty. \tag{7.3}$$

Then, from Lemma 7.7 with $\tilde{q} = q$, we have

$$\liminf_{j\to\infty} \sup_{y\in\mathbf{R}^n\setminus\{0\}} \|v_j\|_{L^q_\gamma(B_{|y|/4}(y))} > 0; \tag{7.4}$$

therefore there exist $\underline{\lambda} > 0$ and $\{y^j\}_{j=1}^{\infty} \subset \mathbf{R}^n \setminus \{0\}$ such that

$$\|v_j\|^q_{L^q_\gamma(B_{|y^j|/4}(y^j))} > \underline{\lambda} \quad \text{for } j \ge 1. \tag{7.5}$$

Now putting

$$u_j(x) = |y^j|^\gamma v_j(|y^j|x) \quad \text{for } x \in \mathbf{R}^n, j \ge 1, \tag{7.6}$$

we see that

$$\|u_j\|^q_{L^q_\gamma(B_{5/4}\setminus\overline{B_{3/4}})} \ge \|u_j\|^q_{L^q_\gamma(B_{1/4}(y^j/|y^j|))} = \|v_j\|^q_{L^q_\gamma(B_{|y^j|/4}(y^j))} > \underline{\lambda}, \tag{7.7}$$

$$\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} = \|v_j\|^q_{L^q_\gamma(\mathbf{R}^n)} = 1 \quad \text{for } j \ge 1, \tag{7.8}$$

$$\|\nabla u_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} = \|\nabla v_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \to S^{p,q;\gamma} \quad \text{as } j \to \infty. \tag{7.9}$$

(3) We see that each $Q^{p,q;\gamma}[u_j]$ is nondecreasing on $(1, \infty)$ and that $\{Q^{p,q;\gamma}[u_j]\}_{j=1}^{\infty}$ is uniformly bounded on $(1, \infty)$. Therefore, it follows from

Lemma 7.1 that there exist, by taking a subsequence if necessary, a nondecreasing function $Q:(1,\infty)\to\mathbf{R}$ and a positive number $\lambda\in\mathbf{R}$ such that

$$Q^{p,q;\gamma}[u_j](t)\to Q(t)\quad\text{as } j\to\infty \text{ for } t>1;\qquad Q(t)\to\lambda\quad\text{as } t\to\infty. \tag{7.10}$$

Noting that

$$Q^{p,q;\gamma}[u_j]\Big(\frac{5}{3}\Big)\geq \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{5/4}\setminus\overline{B_{3/4}})}\geq \|u_j\|^q_{L^q_\gamma(B_{5/4}\setminus\overline{B_{3/4}})}>\underline{\lambda}\quad\text{for } j\geq 1, \tag{7.11}$$

we have

$$\underline{\lambda}<Q^{p,q;\gamma}[u_j]\Big(\frac{5}{3}\Big)\leq Q^{p,q;\gamma}[u_j](t)\leq \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)}\quad\text{for } t\geq\frac{5}{3}, j\geq 1. \tag{7.12}$$

Letting $j\to\infty$, we have

$$\underline{\lambda}\leq Q\Big(\frac{5}{3}\Big)\leq Q(t)\leq 1+S^{p,q;\gamma}\quad\text{for } t\geq\frac{5}{3}. \tag{7.13}$$

Then by letting $t\to\infty$, we reach the desired estimate $\underline{\lambda}\leq\lambda\leq 1+S^{p,q;\gamma}$. □

To show that no dichotomy occurs in the minimizing sequence that has been chosen in Proposition 7.1, we prepare the following.

PROPOSITION 7.2

*Assume that $1<p<q<\infty$, assume that $\tau_{p,q}<1/n$, and assume that $\gamma>0$. Let $\{u_j\}_{j=1}^\infty\subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ satisfy properties (1), (2), and (3) in Proposition 7.1. Then for an arbitrary $\varepsilon>0$, there exist $\{v_{\varepsilon,j}\}_{j=1}^\infty\subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)$, $j_\varepsilon\in\mathbf{N}$, and $\tilde{\varepsilon}_{p,q;\varepsilon}>0$ such that we have*

$$\begin{aligned}
&\big|\|\rho^{p,q;\gamma}[v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)}-\lambda\big|\\
&\quad\leq\tilde{\varepsilon}_{p,q;\varepsilon},\big|\|\rho^{p,q;\gamma}[u_j-v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)}-(1+S^{p,q;\gamma}-\lambda)\big|\\
&\quad\leq\tilde{\varepsilon}_{p,q;\varepsilon},\\
&0\leq 1-\|v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}-\|u_j-v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}<2\varepsilon\quad\textit{for } j\geq j_\varepsilon.
\end{aligned} \tag{7.14}$$

*Further it holds that $\tilde{\varepsilon}_{p,q;\varepsilon}\to 0$ as $\varepsilon\to 0$.*

*Proof*

Let $\varepsilon>0$.

(a) From Proposition 7.1(3), there exists $t_\varepsilon>1$ such that we have

$$\lambda-\frac{\varepsilon}{2}<Q(t)\leq\lambda\quad\text{for } t\geq t_\varepsilon. \tag{7.15}$$

Also from Definition 7.2 there exists $\{r_{\varepsilon,j}\}_{j=1}^\infty\cup\{R_{\varepsilon,j}\}_{j=1}^\infty\subset(0,\infty)$ such that we have

$$Q^{p,q;\gamma}[u_j](t_\varepsilon)\leq\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})}+\frac{\varepsilon}{4},\qquad R_{\varepsilon,j}=t_\varepsilon r_{\varepsilon,j}\quad\text{for } j\geq 1. \tag{7.16}$$

Further from assertions (1) and (2) of Proposition 7.1, there exists $j_\varepsilon \in \mathbf{N}$ such that we have

$$0 \le \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} - (1 + S^{p,q;\gamma}) < \varepsilon, \tag{7.17}$$

$$|Q^{p,q;\gamma}[u_j](t_\varepsilon) - Q(t_\varepsilon)| < \frac{\varepsilon}{4}, \qquad |Q^{p,q;\gamma}[u_j](4t_\varepsilon) - Q(4t_\varepsilon)| < \varepsilon \quad \text{for } j \ge j_\varepsilon.$$

(b) Since

$$\lambda - \frac{\varepsilon}{2} < Q(t_\varepsilon) < Q^{p,q;\gamma}[u_j](t_\varepsilon) + \frac{\varepsilon}{4} < \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}}})} + \frac{\varepsilon}{2}, \tag{7.18}$$

$$\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}}})} \le Q^{p,q;\gamma}[u_j](t_\varepsilon) < Q(t_\varepsilon) + \frac{\varepsilon}{4} < \lambda + \frac{\varepsilon}{2} \quad \text{for } j \ge j_\varepsilon,$$

we see that

$$\big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}}})} - \lambda\big| < \varepsilon \quad \text{for } j \ge j_\varepsilon. \tag{7.19}$$

Hence we see that

$$\begin{aligned}
&\big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n \setminus \overline{B_{R_{\varepsilon,j}}})} + \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}})} - (1 + S^{p,q;\gamma} - \lambda)\big| \\
&\quad = \big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} - \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}}})} - (1 + S^{p,q;\gamma}) + \lambda\big| \\
&\quad \le \big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} - (1 + S^{p,q;\gamma})\big| + \big|\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}}})} - \lambda\big| \\
&\quad < 2\varepsilon \quad \text{for } j \ge j_\varepsilon.
\end{aligned} \tag{7.20}$$

Since

$$\begin{aligned}
\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}})} &\le Q^{p,q;\gamma}[u_j](4t_\varepsilon) \\
&\le Q(4t_\varepsilon) + \varepsilon \le \lambda + \varepsilon \quad \text{for } j \ge j_\varepsilon,
\end{aligned} \tag{7.21}$$

we have

$$\begin{aligned}
&\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}} \setminus \overline{B_{R_{\varepsilon,j}}})} + \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}})} \\
&\quad = \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}})} - \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}}})} \\
&\quad < (\lambda + \varepsilon) - (\lambda - \varepsilon) = 2\varepsilon \quad \text{for } j \ge j_\varepsilon.
\end{aligned} \tag{7.22}$$

(c) Let us set $v_{\varepsilon,j}(x) = \psi_{2R_{\varepsilon,j}}(x)(1 - \psi_{r_{\varepsilon,j}}(x))u_j(x)$ for $x \in \mathbf{R}^n, j \ge 1$. Then from Lemma 7.2 and elementary inequalities,

$$(1+t)^p \le 2^{p-1}(1+t^p), \qquad 1 + t^{p/q} \le 2^{1-p/q}(1+t)^{p/q} \quad \text{for } t \ge 0, \tag{7.23}$$

we have

$$\begin{aligned}
&\big|\|\rho^{p,q;\gamma}[v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)} - \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}}})}\big| \\
&\quad = \int_{B_{2R_{\varepsilon,j}} \setminus \overline{B_{R_{\varepsilon,j}}}} \Big\{|\psi_{2R_{\varepsilon,j}}(x)u_j(x)|^q I_{q\gamma}(x) \\
&\qquad + \Big|-\frac{1}{2R_{\varepsilon,j}}\tilde{\psi}_{2R_{\varepsilon,j}}(x)u_j(x)\frac{x}{|x|} + \psi_{2R_{\varepsilon,j}}(x)\nabla u_j(x)\Big|^p I_{p(1+\gamma)}(x)\Big\}\, dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{B_{r_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}}} \Big\{ \big| \big(1-\psi_{r_{\varepsilon,j}}(x)\big) u_j(x) \big|^q I_{q\gamma}(x) \\
&+ \Big| \frac{1}{r_{\varepsilon,j}} \tilde{\psi}_{r_{\varepsilon,j}}(x) u_j(x) \frac{x}{|x|} + \big(1-\psi_{r_{\varepsilon,j}}(x)\big) \nabla u_j(x) \Big|^p I_{p(1+\gamma)}(x) \Big\}\, dx \\
&\le \int_{B_{2R_{\varepsilon,j}} \setminus \overline{B_{R_{\varepsilon,j}}}} \Big( |u_j(x)|^q I_{q\gamma}(x) + 2^{p-1} \Big( \Big( \frac{3|x|}{2R_{\varepsilon,j}} |u_j(x)| \Big)^p I_{p\gamma}(x) \\
&+ |\nabla u_j(x)|^p I_{p(1+\gamma)}(x) \Big) \Big)\, dx \\
&+ \int_{B_{r_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}}} \Big( |u_j(x)|^q I_{q\gamma}(x) + 2^{p-1} \Big( \Big( \frac{3|x|}{r_{\varepsilon,j}} |u_j(x)| \Big)^p I_{p\gamma}(x) \\
&+ |\nabla u_j(x)|^p I_{p(1+\gamma)}(x) \Big) \Big)\, dx \qquad (7.24) \\
&\le 2^{p-1} \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}} \setminus \overline{B_{R_{\varepsilon,j}}})} + \frac{6^p}{2} \|u_j\|^p_{L^p_\gamma(B_{2R_{\varepsilon,j}} \setminus \overline{B_{R_{\varepsilon,j}}})} \\
&+ 2^{p-1} \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}})} + \frac{6^p}{2} \|u_j\|^p_{L^p_\gamma(B_{r_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}})} \\
&\le 2^{p-1} \cdot 2\varepsilon + \frac{6^p}{2} \big( ((\omega_n \log 2)^{\tau_{p,q}} \|u_j\|_{L^q_\gamma(B_{2R_{\varepsilon,j}} \setminus \overline{B_{R_{\varepsilon,j}}})})^p \\
&+ ((\omega_n \log 2)^{\tau_{p,q}} \|u_j\|_{L^q_\gamma(B_{r_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}})})^p \big) \\
&\le 2^p \varepsilon + \frac{1}{2} \big(6(\omega_n \log 2)^{\tau_{p,q}}\big)^p 2^{1-p/q} (\|u_j\|^q_{L^q_\gamma(B_{2R_{\varepsilon,j}} \setminus \overline{B_{R_{\varepsilon,j}}})} \\
&+ \|u_j\|^q_{L^q_\gamma(B_{r_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}})})^{p/q} \\
&\le 2^p \varepsilon + \frac{1}{2^{p/q}} \big(6(\omega_n \log 2)^{\tau_{p,q}}\big)^p (2\varepsilon)^{p/q} = 2^p \varepsilon + \big(6(\omega_n \log 2)^{\tau_{p,q}}\big)^p \varepsilon^{p/q}
\end{aligned}
$$

for $j \ge j_\varepsilon$.

In a similar way we have

$$
\begin{aligned}
&\big| \|\rho^{p,q;\gamma}[u_j - v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)} - \big( \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n \setminus \overline{B_{R_{\varepsilon,j}}})} + \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}})} \big) \big| \\
&= \Big| \int_{B_{2R_{\varepsilon,j}} \setminus \overline{B_{R_{\varepsilon,j}}}} \Big\{ \big( |u_j(x)|^q - |(1-\psi_{2R_{\varepsilon,j}}(x)) u_j(x)|^q \big) I_{q\gamma}(x) \\
&+ \Big( |\nabla u_j(x)|^p - \Big| \frac{1}{2R_{\varepsilon,j}} \tilde{\psi}_{2R_{\varepsilon,j}}(x) u_j(x) \frac{x}{|x|} + (1-\psi_{2R_{\varepsilon,j}}(x)) \nabla u_j(x) \Big|^p \Big) \\
&\times I_{p(1+\gamma)}(x) \Big\}\, dx \\
&+ \int_{B_{r_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}/2}}} \Big\{ (|u_j(x)|^q - |\psi_{r_{\varepsilon,j}}(x) u_j(x)|^q) I_{q\gamma}(x)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(|\nabla u_j(x)|^p-\Big|-\frac{1}{r_{\varepsilon,j}}\tilde{\psi}_{r_{\varepsilon,j}}(x)u_j(x)\frac{x}{|x|}+\psi_{r_{\varepsilon,j}}(x)\nabla u_j(x)\Big|^p\Big)\\
&\times I_{p(1+\gamma)}(x)\Big\}\,dx\Big| \qquad (7.25)\\
&\le \int_{B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}}}\Big\{|u_j(x)|^q I_{q\gamma}(x)+|\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\\
&\quad+2^{p-1}\Big(\Big(\frac{3|x|}{2R_{\varepsilon,j}}|u_j(x)|\Big)^p I_{q\gamma}(x)+|\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\Big)\Big\}\,dx\\
&\quad+\int_{B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}}}\Big\{|u_j(x)|^q I_{q\gamma}(x)+|\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\\
&\quad+2^{p-1}\Big(\Big(\frac{3|x|}{r_{\varepsilon,j}}|u_j(x)|\Big)^p I_{q\gamma}(x)+|\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\Big)\Big\}\,dx\\
&\le 2(2^{p-1}+1)\varepsilon+\big(6(\omega_n\log 2)^{\tau_{p,q}}\big)^p\varepsilon^{p/q}\quad \text{for } j\ge j_\varepsilon.
\end{aligned}
$$

(d) From (7.19), (7.20), (7.24), and (7.25) in (b) and (c), we have

$$
\begin{aligned}
&\big|\|\rho^{p,q;\gamma}[v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)}-\lambda\big|\le 2^p\varepsilon+\big(6(\omega_n\log 2)^{\tau_{p,q}}\big)^p\varepsilon^{p/q}+\varepsilon,\\
&\big|\|\rho^{p,q;\gamma}[u_j-v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)}-(1+S^{p,q;\gamma}-\lambda)\big| \qquad (7.26)\\
&\qquad\le 2(2^{p-1}+1)\varepsilon+\big(6(\omega_n\log 2)^{\tau_{p,q}}\big)^p\varepsilon^{p/q}+2\varepsilon\quad\text{for } j\ge j_\varepsilon.
\end{aligned}
$$

Noting that

$$
\theta^q+(1-\theta)^q\le 1\quad \text{for } 0\le\theta\le 1, \qquad (7.27)
$$

we have

$$
\begin{aligned}
0&\le 1-\big(\psi_{2R_{\varepsilon,j}}(x)(1-\psi_{r_{\varepsilon,j}}(x))\big)^q-\big(1-\psi_{2R_{\varepsilon,j}}(x)(1-\psi_{r_{\varepsilon,j}}(x))\big)^q\\
&\le \chi_{B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}}}(x)+\chi_{B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}}}(x)\quad \text{for } x\in\mathbf{R}^n, j\ge j_\varepsilon.
\end{aligned}
$$

Then from this inequality and (b) we have

$$
\begin{aligned}
0&\le 1-\|v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}-\|u_j-v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}\\
&=\|u_j\|^q_{L^q_\gamma(\mathbf{R}^n)}-\|v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}-\|u_j-v_{\varepsilon,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}\\
&\le \|u_j\|^q_{L^q_\gamma(B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}})}+\|u_j\|^q_{L^q_\gamma(B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})}\\
&\le \|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{2R_{\varepsilon,j}}\setminus\overline{B_{R_{\varepsilon,j}}})}+\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{r_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}/2}})}<2\varepsilon\quad\text{for } j\ge j_\varepsilon.
\end{aligned}
$$

□

PROPOSITION 7.3

*Assume that $1<p<q<\infty$, assume that $\tau_{p,q}<1/n$, and assume that $\gamma>0$. Assume that $\{u_j\}_{j=1}^\infty\subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)\setminus\{0\}$ satisfies property (1) of Proposition 7.1. Then we have $\lambda=1+S^{p,q;\gamma}$.*

*Proof*
On the contrary we assume that $\lambda \neq 1 + S^{p,q;\gamma}$. Then from Proposition 7.1 we should have $0 < \lambda < 1 + S^{p,q;\gamma}$. Let us retain the notations from Proposition 7.2.

(a) Since $\tilde{\varepsilon}_{p,q;\varepsilon} \to 0$ as $\varepsilon \to 0$, there exists some $\varepsilon_0 > 0$ such that

$$0 < \tilde{\varepsilon}_{p,q;\varepsilon} < \frac{1}{2}\min\{\lambda, 1 + S^{p,q;\gamma} - \lambda\} \quad \text{for } 0 < \varepsilon < \varepsilon_0. \tag{7.28}$$

Then from Proposition 7.2 and Theorem 2.1 we have

$$\begin{aligned}
\frac{1}{2}\lambda \le \lambda - \tilde{\varepsilon}_{p,q;\varepsilon} &\le \|\rho^{p,q;\gamma}[v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)} \\
&\le \Big(\frac{1}{S^{p,q;\gamma}}\|\nabla v_{\varepsilon,j}\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big)^{q/p} + \|\nabla v_{\varepsilon,j}\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}, \\
\frac{1}{2}(1 + S^{p,q;\gamma} - \lambda) \le 1 + S^{p,q;\gamma} - \lambda - \tilde{\varepsilon}_{p,q;\varepsilon} &\le \|\rho^{p,q;\gamma}[u_j - v_{\varepsilon,j}]\|_{L^1(\mathbf{R}^n)} \\
&\le \Big(\frac{1}{S^{p,q;\gamma}}\|\nabla [u_j - v_{\varepsilon,j}]\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)}\Big)^{q/p} + \|\nabla [u_j - v_{\varepsilon,j}]\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \\
&\qquad \text{for } j \ge j_\varepsilon, 0 < \varepsilon < \varepsilon_0.
\end{aligned}$$

Hence, for some $\beta > 0$, it holds that

$$\begin{aligned}
\|\nabla v_{\varepsilon,j}\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} &\ge \beta, \\
\|\nabla [u_j - v_{\varepsilon,j}]\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} &\ge \beta \quad \text{for } j \ge j_\varepsilon, 0 < \varepsilon < \varepsilon_0.
\end{aligned} \tag{7.29}$$

(b) Choose a sequence $\{\varepsilon_k\}_{k=1}^\infty \subset (0, \varepsilon_0)$ satisfying $\varepsilon_k \to 0$ as $k \to \infty$. Then from Proposition 7.2 we have

$$0 \le 1 - \|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} \le 2\varepsilon_k \quad \text{for } j \ge j_{\varepsilon_k}, k \ge 1, \tag{7.30}$$

and we see that $\{\|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}\}_{k=1}^\infty$ and $\{\|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}\}_{k=1}^\infty$ are bounded. Hence, by choosing a subsequence with respect to $j$, there exists $\{\overline{\sigma}_k\}_{k=1}^\infty \cup \{\underline{\sigma}_k\}_{k=1}^\infty \subset [0,1]$ such that we have

$$\|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \overline{\sigma}_k, \qquad \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \underline{\sigma}_k \quad \text{as } j \to \infty \text{ for } k \ge 1. \tag{7.31}$$

Since $0 \le 1 - \overline{\sigma}_k - \underline{\sigma}_k \le 2\varepsilon_k$ for $k \ge 1$, by choosing a subsequence with respect to $k$, there exists $\sigma \in [0,1]$ such that we have

$$\overline{\sigma}_k \to \sigma, \qquad \underline{\sigma}_k \to 1 - \sigma \quad \text{as } k \to \infty. \tag{7.32}$$

(c) From (a), Proposition 7.2, and Theorem 2.1, we have

$$\begin{aligned}
&\max\big\{S^{p,q;\gamma}(\|v_{\varepsilon_k,j}\|^p_{L^q_\gamma(\mathbf{R}^n)} + \|u_j - v_{\varepsilon_k,j}\|^p_{L^q_\gamma(\mathbf{R}^n)}), \beta + S^{p,q;\gamma}\|u_j - v_{\varepsilon_k,j}\|^p_{L^q_\gamma(\mathbf{R}^n)}, \\
&\quad \beta + S^{p,q;\gamma}\|v_{\varepsilon_k,j}\|^p_{L^q_\gamma(\mathbf{R}^n)}\big\} \\
&\quad \le \|\nabla v_{\varepsilon_k,j}\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|\nabla [u_j - v_{\varepsilon_k,j}]\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \\
&\quad = \|\rho^{p,q;\gamma}[v_{\varepsilon_k,j}]\|_{L^1(\mathbf{R}^n)} + \|\rho^{p,q;\gamma}[u_j - v_{\varepsilon_k,j}]\|_{L^1(\mathbf{R}^n)} \\
&\qquad - (\|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} + \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)})
\end{aligned}$$

$$\le (\lambda + \tilde{\varepsilon}_{p,q;\varepsilon_k}) + (1 + S^{p,q;\gamma} - \lambda + \tilde{\varepsilon}_{p,q;\varepsilon_k}) - \|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)}$$
$$= S^{p,q;\gamma} + 1 - \|v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|u_j - v_{\varepsilon_k,j}\|^q_{L^q_\gamma(\mathbf{R}^n)} + 2\tilde{\varepsilon}_{p,q;\varepsilon_k}$$
for $j \ge j_{\varepsilon_k}, k \ge 1$.

Therefore letting $j \to \infty$ and $k \to \infty$ and using (b), we have

$$\max\{S^{p,q;\gamma}(\sigma^{p/q} + (1-\sigma)^{p/q}), \beta + S^{p,q;\gamma}(1-\sigma)^{p/q}, S^{p,q;\gamma}\sigma^{p/q} + \beta\} \le S^{p,q;\gamma}, \tag{7.33}$$

and we have $\sigma^{p/q} + (1-\sigma)^{p/q} \le 1$. If we note that

$$\theta^{p/q} + (1-\theta)^{p/q} > 1 \quad \text{for } 0 < \theta < 1, \tag{7.34}$$

we have $\sigma \in \{0,1\}$. Then it holds that $\beta \le 0$, and this is a contradiction. □

Then we have the following.

PROPOSITION 7.4

*Assume that $1 < p < q < \infty$, assume that $\tau_{p,q} < 1/n$, and assume that $\gamma > 0$. Assume that $\{u_j\}_{j=1}^\infty \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n) \setminus \{0\}$ satifies the properties of Proposition 7.1. Then, $\{\rho^{p,q;\gamma}[u_j]\}_{j=1}^\infty$ is tight. Namely, for an arbitrary $\varepsilon > 0$, there exists a constant $R_\varepsilon > 0$ such that we have*

$$\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n \setminus \overline{B_{R_\varepsilon}})} < \varepsilon \quad \textit{for } j \ge 1. \tag{7.35}$$

*In particular, both $\{|u_j|^q I_{q\gamma}\}_{j=1}^\infty$ and $\{|\nabla u_j|^p I_{p(1+\gamma)}\}_{j=1}^\infty$ are tight as well.*

*Proof*

Let $0 < \varepsilon < \underline{\lambda}$.

(a) From Proposition 7.3 we see that $\lambda = 1 + S^{p,q;\gamma}$; hence there exists $t_\varepsilon > 1$ such that we have

$$1 + S^{p,q;\gamma} - \frac{\varepsilon}{4} < Q(t) \le 1 + S^{p,q;\gamma} \quad \text{for } t \ge t_\varepsilon. \tag{7.36}$$

From assertions (1) and (3) of Proposition 7.1 there exists $j_\varepsilon \in \mathbf{N}$ such that we have

$$\begin{aligned} \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)} &< 1 + S^{p,q;\gamma} + \frac{\varepsilon}{4}, \\ |Q^{p,q;\gamma}[u_j](t_\varepsilon) - Q(t_\varepsilon)| &< \frac{\varepsilon}{4} \quad \text{for } j \ge j_\varepsilon. \end{aligned} \tag{7.37}$$

Further, by Definition 7.2 there exists $\{r_{\varepsilon,j}\}_{j=1}^\infty \cup \{R_{\varepsilon,j}\}_{j=1}^\infty \subset (0,\infty)$ such that we have

$$\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}} \setminus \overline{B_{r_{\varepsilon,j}}})} > Q^{p,q;\gamma}[u_j](t_\varepsilon) - \frac{\varepsilon}{4}, \quad R_{\varepsilon,j} = t_\varepsilon r_{\varepsilon,j} \text{ for } j \ge 1. \tag{7.38}$$

Therefore it holds that

$$
\begin{aligned}
\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} &> Q^{p,q;\gamma}[u_j](t_\varepsilon)-\frac{\varepsilon}{4} \\
&> Q(t_\varepsilon)-\frac{\varepsilon}{2} > 1+S^{p,q;\gamma}-\frac{3}{4}\varepsilon \quad \text{for } j\ge j_\varepsilon.
\end{aligned}
\tag{7.39}
$$

Then

$$
r_{\varepsilon,j}\le \frac{5}{4} \quad \text{for } j\ge j_\varepsilon. \tag{7.40}
$$

In fact, if not, we have

$$
(B_{5/4}\setminus\overline{B_{3/4}})\cap(B_{R_{\varepsilon,j_0}}\setminus\overline{B_{r_{\varepsilon,j_0}}})=\varnothing \quad \text{for some } j_0\ge j_\varepsilon; \tag{7.41}
$$

hence we have

$$
\begin{aligned}
1+S^{p,q;\gamma}+\frac{\varepsilon}{4} &> \|\rho^{p,q;\gamma}[u_{j_0}]\|_{L^1(\mathbf{R}^n)} \\
&\ge \|\rho^{p,q;\gamma}[u_{j_0}]\|_{L^1(B_{5/4}\setminus\overline{B_{3/4}})}+\|\rho^{p,q;\gamma}[u_{j_0}]\|_{L^1(B_{R_{\varepsilon,j_0}}\setminus\overline{B_{r_{\varepsilon,j_0}}})} \\
&> \underline{\lambda}+1+S^{p,q;\gamma}-\frac{3}{4}\varepsilon.
\end{aligned}
\tag{7.42}
$$

Then we have $\underline{\lambda}<\varepsilon$, and this is a contradiction.

(b) Let us take a number $R_\varepsilon>0$ such that

$$
R_\varepsilon>\frac{5}{4}t_\varepsilon, \qquad \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n\setminus\overline{B_{R_\varepsilon}})}<\varepsilon \quad \text{for } 1\le j\le j_\varepsilon-1. \tag{7.43}
$$

Since $B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}}\subset B_{R_\varepsilon}$ for $j\ge j_\varepsilon$, we have

$$
\begin{aligned}
&\|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n\setminus\overline{B_{R_\varepsilon}})} \\
&\quad\le \|\rho^{p,q;\gamma}[u_j]\|_{L^1(\mathbf{R}^n)}-\|\rho^{p,q;\gamma}[u_j]\|_{L^1(B_{R_{\varepsilon,j}}\setminus\overline{B_{r_{\varepsilon,j}}})} \\
&\quad< 1+S^{p,q;\gamma}+\frac{\varepsilon}{4}-\Big(1+S^{p,q;\gamma}-\frac{3}{4}\varepsilon\Big)=\varepsilon \quad \text{for } j\ge j_\varepsilon,
\end{aligned}
\tag{7.44}
$$

and this proves the assertion. □

### 7.3. Convergence of minimizing sequence

In this section we investigate the minimizing sequence $\{u_j\}_{j=1}^\infty$ for $S^{p,q;\gamma}$, which is introduced in Proposition 7.1, and we finally prove the existence of a minimizer. To this end we employ the following lemma, which is an easy corollary to [Li2, Lemma 2.1]. Here, by $\mathcal{B}(\mathbf{R}^n)$ we denote a set of all finite Borel measures on $\mathbf{R}^n$, and by $\delta_0$ we denote a Dirac measure with a unit mass at the origin. In a canonical way we see that $L^1(\mathbf{R}^n)\subset\mathcal{B}(\mathbf{R}^n)$. For $\nu\in\mathcal{B}(\mathbf{R}^n)$, by $\nu_{\rm ac}$ and $\nu_{\rm s}$ we denote an absolutely continuous part and a singular part of $\nu$ with respect to the Lebesgue measure, respectively. In this notation we see that $\nu_{\rm ac}\in L^1(\mathbf{R}^n)$ and $\nu=\nu_{\rm ac}+\nu_{\rm s}$.

LEMMA 7.8 ([Li2, LEMMA 1.2])

*Let $1<p\le q<\infty$, let $\mu,\nu\in\mathcal{B}(\mathbf{R}^n)$, let $\mu,\nu\ge 0$, let $\operatorname{supp}\nu_{\rm s}\subset\{0\}$, and let $S>0$.*

*Assume that*

$$S\Big(\int_{\mathbf{R}^n} |\phi(x)|^q \, d\nu(x)\Big)^{p/q} \le \int_{\mathbf{R}^n} |\phi(x)|^p \, d\mu(x) \quad \textit{for } \phi \in C_c^\infty(\mathbf{R}^n \setminus \{0\}). \tag{7.45}$$

*Then there exists a constant* $a_0 \in [0,\infty)$ *such that we have*

$$\nu = a_0\delta_0, \qquad \mu \ge (Sa_0^{p/q})\delta_0. \tag{7.46}$$

For the reader's convenience, let us briefly recall the notion of weak convergence of a sequence of measures. Let us denote by $\mathrm{BC}(\mathbf{R}^n)$ a set of all bounded, continuous functions on $\mathbf{R}^n$, then $\mathcal{B}(\mathbf{R}^n)$ is regarded as a subspace of $\mathrm{BC}(\mathbf{R}^n)'$, which is the dual of $\mathrm{BC}(\mathbf{R}^n)$. A sequence $\{\nu_j\}_{j=1}^\infty \subset \mathcal{B}(\mathbf{R}^n)$ is said to converge weakly to $\nu$ in $\mathrm{BC}(\mathbf{R}^n)'$ if $\{\nu_j\}_{j=1}^\infty$ converges in a weak-$*$ topology to $\nu$ in $\mathrm{BC}(\mathbf{R}^n)'$, that is to say,

$$\int_{\mathbf{R}^n} \phi(x)\, d\nu_j(x) \to \int_{\mathbf{R}^n} \phi(x)\, d\nu(x) \quad \text{ as } j \to \infty \text{ for any } \phi \in \mathrm{BC}(\mathbf{R}^n). \tag{7.47}$$

When $\{\nu_j\}_{j=1}^\infty \subset \mathcal{B}(\mathbf{R}^n)$ converges weakly to $\nu$ in $\mathrm{BC}(\mathbf{R}^n)'$, we simply write

$$\nu_j \rightharpoonup \nu \quad \text{weakly as } j \to \infty. \tag{7.48}$$

We employ the following lemma. (The proof is omitted.)

LEMMA 7.9

*Assume that* $\{\nu_j\}_{j=1}^\infty$ *is bounded in* $\mathcal{B}(\mathbf{R}^n)$. *If* $\{\nu_j\}_{j=1}^\infty$ *is tight, then* $\{\nu_j\}_{j=1}^\infty$ *contains a weakly convergent subsequence.*

If $\{u_j\}_{j=1}^\infty$ satisfies the assertions of Proposition 7.1, then from Proposition 7.4 we see that both $\{|u_j|^q I_{q\gamma}\}_{j=1}^\infty$ and $\{|\nabla u_j|^p I_{p(1+\gamma)}\}_{j=1}^\infty$ are tight. Hence from Lemma 7.9 they both contain weakly convergent subsequences. Further, from Rellich's lemma, Lemma 7.3, and Proposition 7.1 we have the following.

PROPOSITION 7.5

*Assume that* $1 < p < q < \infty$, *assume that* $\tau_{p,q} < 1/n$, *and assume that* $\gamma > 0$. *Then there exist* $\{u_j\}_{j=1}^\infty \subset W_{\gamma,0}^{1,p}(\mathbf{R}^n) \setminus \{0\}$, $u \in W_{\gamma,0}^{1,p}(\mathbf{R}^n) \setminus \{0\}$, *and* $\mu, \nu \in \mathcal{B}(\mathbf{R}^n)$ *such that we have the following:*

(1) $\|u_j\|_{L_\gamma^q(\mathbf{R}^n)}^q = 1$ *for* $j \ge 1$, $\|\nabla u_j\|_{L_{1+\gamma}^p(\mathbf{R}^n)}^p \to S^{p,q;\gamma}$ *as* $j \to \infty$.

(2) $u_j \rightharpoonup u$ *weakly in* $W_{\gamma,0}^{1,p}(\mathbf{R}^n)$, $u_j \to u$ *in* $L_{\mathrm{loc}}^q(\mathbf{R}^n \setminus \{0\}) \cap (L_{\gamma+1-n\tau_{p,q}}^q)_{\mathrm{loc}}(\mathbf{R}^n)$, $u_j \to u$ *a.e. on* $\mathbf{R}^n$ *as* $j \to \infty$.

(3) $|u_j|^q I_{q\gamma} \rightharpoonup \nu, |\nabla u_j|^p I_{p(1+\gamma)} \rightharpoonup \mu$ *weakly as* $j \to \infty$.

(4) $\nu_{\mathrm{ac}} = |u|^q I_{q\gamma}$ *a.e. on* $\mathbf{R}^n, \operatorname{supp} \nu_{\mathrm{s}} \subset \{0\}$.

*Proof*

We prove assertion (4) only. For $\varepsilon > 0$ it follows from assertions (2) and (3) that

$$\int_{\mathbf{R}^n} \phi(x) |\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\, dx \to \int_{\mathbf{R}^n} \phi(x)\, d\mu(x), \tag{7.49}$$

$$\int_{\mathbf{R}^n} \phi(x)|u_j(x)|^q I_{q\gamma}(x)\,dx \to \int_{\mathbf{R}^n} \phi(x)|u(x)|^q I_{q\gamma}(x)\,dx, \quad \text{as } j\to\infty \text{ for } \phi \in C_c^\infty(\mathbf{R}^n \setminus \overline{B_\varepsilon}); \tag{7.50}$$

hence it holds that

$$\int_{\mathbf{R}^n} \phi(x)\big(|u(x)|^q I_{q\gamma}(x) - \nu_{\mathrm{ac}}(x)\big)\,dx = \int_{\mathbf{R}^n} \phi(x)\,d\nu_{\mathrm{s}}(x) \quad \text{for } \phi \in C_c^\infty(\mathbf{R}^n \setminus \overline{B_\varepsilon}). \tag{7.51}$$

Therefore, $|u|^q I_{q\gamma} - \nu_{\mathrm{ac}}$ coincides with $\nu_{\mathrm{s}}$ as measures on $\mathbf{R}^n \setminus \overline{B_\varepsilon}$. Since they are absolutely continuous and singular with respect to the Lebesgue measure, respectively, they should be vanishing as measures on $\mathbf{R}^n \setminus \overline{B_\varepsilon}$. Hence we have

$$|u|^q I_{q\gamma} - \nu_{\mathrm{ac}} = 0 \quad \text{a.e. on } \mathbf{R}^n \setminus \overline{B_\varepsilon}, \operatorname{supp}\nu_{\mathrm{s}} \subset \overline{B_\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$|u|^q I_{q\gamma} - \nu_{\mathrm{ac}} = 0 \quad \text{a.e. on } \mathbf{R}^n, \operatorname{supp}\nu_{\mathrm{s}} \subset \{0\}.$$ □

DEFINITION 7.3

For $\phi \in \mathrm{BC}(\mathbf{R}^n)$ satisfying $\phi > 0$ on $\mathbf{R}^n$, we set

$$\|u\|_{W^{1,p}_\gamma[\phi](\mathbf{R}^n)} = \Big(\int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x)\phi(x)\,dx\Big)^{1/p}. \tag{7.52}$$

By $W^{1,p}_{\gamma,0}[\phi](\mathbf{R}^n)$ we denote the completion of $C_c^\infty(\mathbf{R}^n \setminus \{0\})$ with respect to the norm $\|\cdot\|_{W^{1,p}_\gamma[\phi](\mathbf{R}^n)}$.

In this definition we have

$$\|u\|_{W^{1,p}_\gamma[\phi](\mathbf{R}^n)} \le \|\phi\|_{L^\infty(\mathbf{R}^n)}\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n); \tag{7.53}$$

hence we have a continuous imbedding $W^{1,p}_{\gamma,0}(\mathbf{R}^n) \subset W^{1,p}_{\gamma,0}[\phi](\mathbf{R}^n)$. From this fact we have the next lemma.

LEMMA 7.10

*For $1 < p < \infty$ and $\gamma > 0$, assume that $\{u_j\}_{j=1}^\infty \subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)$ for $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)$ and $\mu \in \mathcal{B}(\mathbf{R}^n)$ satisfy*

$$u_j \rightharpoonup u \quad \textit{weakly in } W^{1,p}_{\gamma,0}(\mathbf{R}^n), \qquad |\nabla u_j|^p I_{p(1+\gamma)} \rightharpoonup \mu \quad \textit{weakly as } j\to\infty. \tag{7.54}$$

*Then, we have*

$$|\nabla u|^p I_{p(1+\gamma)} \le \mu. \tag{7.55}$$

*Proof*

For $\phi \in C_c(\mathbf{R}^n)$ with $\phi \ge 0$ on $\mathbf{R}^n$, it suffices to show that

$$\int_{\mathbf{R}^n} \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \le \int_{\mathbf{R}^n} \phi(x)\,d\mu(x). \tag{7.56}$$

(a) First we show this inequality to be valid assuming that $\phi \in \mathrm{BC}(\mathbf{R}^n)$ satifies $\phi > 0$ on $\mathbf{R}^n$. Since the imbedding $W^{1,p}_{\gamma,0}(\mathbf{R}^n) \subset W^{1,p}_{\gamma,0}[\phi](\mathbf{R}^n)$ is continuous, we see that

$$u_j \rightharpoonup u \quad \text{weakly in } W^{1,p}_{\gamma,0}[\phi](\mathbf{R}^n) \text{ as } j \to \infty.$$

Therefore we have

$$\begin{aligned}\int_{\mathbf{R}^n} \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx &= \|u\|^p_{W^{1,p}_\gamma[\phi](\mathbf{R}^n)} \le \liminf_{j\to\infty} \|u_j\|^p_{W^{1,p}_\gamma[\phi](\mathbf{R}^n)} \\ &= \lim_{j\to\infty} \int_{\mathbf{R}^n} \phi(x)|\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\,dx \\ &= \int_{\mathbf{R}^n} \phi(x)\,d\mu(x).\end{aligned} \tag{7.57}$$

(b) Second we consider the case in which $\phi \in C_{\mathrm{c}}(\mathbf{R}^n)$ and $\phi \ge 0$ on $\mathbf{R}^n$. For $\varepsilon > 0$ it holds that $\overline{\rho}_\varepsilon * \phi \in \mathrm{BC}(\mathbf{R}^n)$ and $\overline{\rho}_\varepsilon * \phi > 0$ on $\mathbf{R}^n$. Then, from (a) we have

$$\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \le \int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)\,d\mu(x) \quad \text{for } \varepsilon > 0.$$

Here noting that $\phi$ is uniformly continuous on $\mathbf{R}^n$, for any $\eta > 0$ we have that there exists a number $r_\eta > 0$ such that

$$|\phi(x-y) - \phi(x)| < \eta \quad \text{for } x \in \mathbf{R}^n, y \in B_{r_\eta}. \tag{7.58}$$

Then

$$\begin{aligned}&\Big|\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx - \int_{\mathbf{R}^n} \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx\Big| \\ &\quad = \Big|\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon(y)\big(\phi(x-y)-\phi(x)\big)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dy\,dx\Big| \\ &\quad \le \int_{\mathbf{R}^n} \Big(\int_{B_{r_\eta}} \eta\overline{\rho}_\varepsilon(y)\,dy + \int_{\mathbf{R}^n\setminus\overline{B_{r_\eta}}} 2\|\phi\|_{L^\infty(\mathbf{R}^n)}\overline{\rho}_\varepsilon(y)\,dy\Big)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \\ &\quad \le (\eta + 2\|\phi\|_{L^\infty(\mathbf{R}^n)}\|\overline{\rho}_1\|_{L^1(\mathbf{R}^n\setminus\overline{B_{r_\eta/\varepsilon}})})\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \to \eta\|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} \\ &\qquad \text{as } \varepsilon \to 0;\end{aligned}$$

hence

$$\begin{aligned}&\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \\ &\qquad \to \int_{\mathbf{R}^n} \phi(x)|\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx \quad \text{as } \varepsilon \to 0.\end{aligned} \tag{7.59}$$

In a similar way we have

$$\int_{\mathbf{R}^n} \overline{\rho}_\varepsilon * \phi(x)\,d\mu(x) \to \int_{\mathbf{R}^n} \phi(x)\,d\mu(x) \quad \text{as } \varepsilon \to 0, \tag{7.60}$$

and the assertion follows. □

Then we have the following proposition.

PROPOSITION 7.6

*Assume that $1 < p < q < \infty$, assume that $\tau_{p,q} < 1/n$, and assume that $\gamma > 0$. Then, in Proposition 7.5, there exists a constant $a_0 \in [0, \infty)$ such that we have*

$$\nu = |u|^q I_{q\gamma} + a_0 \delta_0, \qquad \mu \geq |\nabla u|^p I_{p(1+\gamma)} + (S^{p,q;\gamma} a_0^{p/q}) \delta_0. \tag{7.61}$$

*Proof*

(a) Take an arbitrary $\phi \in C_c^\infty(\mathbf{R}^n)$ satisfying $\operatorname{supp} \phi \subset B_R$. Then it follows from Lemma 7.4 that we have

$$\|\phi u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} - \|\phi(u_j - u)\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \|\phi u\|^q_{L^q_\gamma(\mathbf{R}^n)} \quad \text{as } j \to \infty, \tag{7.62}$$

and from Proposition 7.5(3) we have

$$\begin{aligned} \|\phi u_j\|^q_{L^q_\gamma(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |\phi(x)|^q |u_j(x)|^q I_{q\gamma}(x)\, dx \\ &\to \int_{\mathbf{R}^n} |\phi(x)|^q \, d\nu(x) \quad \text{as } j \to \infty. \end{aligned} \tag{7.63}$$

Hence we have

$$\|\phi(u_j - u)\|^q_{L^q_\gamma(\mathbf{R}^n)} \to \int_{\mathbf{R}^n} |\phi(x)|^q d\nu(x) - \|\phi u\|^q_{L^q_\gamma(\mathbf{R}^n)} \quad \text{as } j \to \infty. \tag{7.64}$$

Since $1/p = \tau_{p,q} + 1/q$, from Hölder's inequality and Proposition 7.5(2) we have

$$\begin{aligned} \|(u_j - u)\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)} &= \big\| |\nabla\phi| (u_j - u) I_{1+\gamma+n/p'} \big\|_{L^p(B_R)} \\ &\leq \|\nabla\phi\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)} \|(u_j - u) I_{1+\gamma+n/p'}\|_{L^q(B_R)} \\ &= \|\nabla\phi\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)} \|u_j - u\|_{L^q_{\gamma+1-n\tau_{p,q}}(B_R)} \\ &\to 0 \quad \text{as } j \to \infty. \end{aligned} \tag{7.65}$$

Here we used the relations $p(n - (1 + \gamma + n/p')) = n - p(1 + \gamma)$ and $q(n - (1 + \gamma + n/p')) = n - q(1 + \gamma - n\tau_{p,q})$. By Proposition 7.5(3) we have

$$\begin{aligned} \|\phi\nabla u_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |\phi(x)|^p |\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\, dx \\ &\to \int_{\mathbf{R}^n} |\phi(x)|^p \, d\mu(x) \quad \text{as } j \to \infty. \end{aligned} \tag{7.66}$$

Then, letting $j \to \infty$ in the inequality

$$\begin{aligned} &(S^{p,q;\gamma})^{1/p} \|\phi(u_j - u)\|_{L^q_\gamma(\mathbf{R}^n)} \\ &\quad \leq \|\nabla[\phi(u_j - u)]\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \\ &\quad \leq \|\phi\nabla[u_j - u]\|_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|(u_j - u)\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \\ &\quad \leq 2^{1/p'} (\|\phi\nabla u_j\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} + \|\phi\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)})^{1/p} \\ &\qquad + \|(u_j - u)\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \quad \text{for } j \geq 1, \end{aligned} \tag{7.67}$$

we get

$$\begin{aligned}&(S^{p,q;\gamma})^{1/p}\Big(\int_{\mathbf{R}^n}|\phi(x)|^q\,d\nu(x)-\int_{\mathbf{R}^n}|\phi(x)|^q|u(x)|^qI_{q\gamma}(x)\,dx\Big)^{1/q}\\ &\le 2^{1/p'}\Big(\int_{\mathbf{R}^n}|\phi(x)|^p\,d\mu(x)+\int_{\mathbf{R}^n}|\phi(x)|^p|\nabla u(x)|^pI_{p(1+\gamma)}(x)\,dx\Big)^{1/p}\\ &\text{for } \phi\in C_{\mathrm{c}}^\infty(\mathbf{R}^n).\end{aligned}\tag{7.68}$$

Since $\operatorname{supp}(\nu-|u|^qI_{q\gamma})_{\mathrm{s}}\subset\{0\}$ by Proposition 7.5(4), it follows from Lemma 7.8 that we have for some $a_0\in[0,\infty)$

$$\nu-|u|^qI_{q\gamma}=a_0\delta_0.\tag{7.69}$$

Further by letting $j\to\infty$ in the inequality

$$\begin{aligned}(S^{p,q;\gamma})^{1/p}\|\phi u_j\|_{L^q_\gamma(\mathbf{R}^n)}&\le\|\nabla[\phi u_j]\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\\ &\le\|\phi\nabla u_j\|_{L^p_{1+\gamma}(\mathbf{R}^n)}+\|u_j\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\quad\text{for } j\ge1,\end{aligned}$$

we have

$$\begin{aligned}&(S^{p,q;\gamma})^{1/p}\big(\|\phi u\|^q_{L^q_\gamma(\mathbf{R}^n)}+a_0|\phi(0)|^q\big)^{1/q}\\ &\le\Big(\int_{\mathbf{R}^n}|\phi(x)|^p\,d\mu(x)\Big)^{1/p}+\|u\nabla\phi\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\quad\text{for } \phi\in C_{\mathrm{c}}^\infty(\mathbf{R}^n).\end{aligned}\tag{7.70}$$

(b) Let $\varepsilon>0$, and let $\psi_\varepsilon$ be given as in Definition 7.1. Noting that $1/p=\tau_{p,q}+1/q$, by Hölder's inequality we have

$$\begin{aligned}&\|u\nabla\psi_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\\ &=\frac{1}{\varepsilon}\Big(\int_{B_\varepsilon}\big(|\tilde\psi_\varepsilon(x)||x|\cdot|u(x)||x|^\gamma\big)^pI_0(x)\,dx\Big)^{1/p}\\ &\le\frac{1}{\varepsilon}\Big(\int_{\mathbf{R}^n}\big(|\tilde\psi_\varepsilon(x)||x|\big)^{1/\tau_{p,q}}I_0(x)\,dx\Big)^{\tau_{p,q}}\Big(\int_{B_\varepsilon}\big(|u(x)||x|^\gamma\big)^qI_0(x)\,dx\Big)^{1/q}\\ &=\|\tilde\psi_1\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)}\|u\|_{L^q_\gamma(B_\varepsilon)}.\end{aligned}\tag{7.71}$$

Hence, by virtue of (a) we have

$$\begin{aligned}(S^{p,q;\gamma})^{1/p}a_0^{1/q}&\le(S^{p,q;\gamma})^{1/p}(\|\psi_\varepsilon u\|^q_{L^q_\gamma(\mathbf{R}^n)}+a_0)^{1/q}\\ &\le\Big(\int_{\mathbf{R}^n}|\psi_\varepsilon(x)|^p\,d\mu(x)\Big)^{1/p}+\|u\nabla\psi_\varepsilon\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\\ &\le\Big(\int_{B_\varepsilon}d\mu(x)\Big)^{1/p}+\|\tilde\psi_1\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)}\|u\|_{L^q_\gamma(B_\varepsilon)}\\ &=\mu(B_\varepsilon)^{1/p}+\|\tilde\psi_1\|_{L^{1/\tau_{p,q}}(\mathbf{R}^n)}\|u\|_{L^q_\gamma(B_\varepsilon)}\to\mu(\{0\})^{1/p}\quad\text{as }\varepsilon\to0;\end{aligned}\tag{7.72}$$

hence

$$\mu(\{0\})\ge S^{p,q;\gamma}a_0^{p/q},\qquad\mu\ge(S^{p,q;\gamma}a_0^{p/q})\delta_0.\tag{7.73}$$

On the other hand, by Lemma 7.10 $|\nabla u|^p I_{p(1+\gamma)} \le \mu$ holds, and we have

$$\mu \ge |\nabla u|^p I_{p(1+\gamma)} + (S^{p,q;\gamma} a_0^{p/q})\delta_0. \tag{7.74}$$ □

After all this we have the following proposition, which proves Theorem 2.4(2).

PROPOSITION 7.7
*Assume that $1 < p < q < \infty$, assume that $\tau_{p,q} < 1/n$, and assume that $\gamma > 0$. Then, in Proposition 7.6, it holds that $a_0 = 0$ and*

$$\|u\|^q_{L^q_\gamma(\mathbf{R}^n)} = 1, \qquad \|\nabla u\|^p_{L^p_{1+\gamma}(\mathbf{R}^n)} = S^{p,q;\gamma}. \tag{7.75}$$

*Proof*
By Proposition 7.5(3) we have

$$\int_{\mathbf{R}^n} |u_j(x)|^q I_{q\gamma}(x)\,dx \to \int_{\mathbf{R}^n} d\nu(x),$$
$$\int_{\mathbf{R}^n} |\nabla u_j(x)|^p I_{p(1+\gamma)}(x)\,dx \to \int_{\mathbf{R}^n} d\mu(x) \quad \text{as } j \to \infty.$$

Combining Proposition 7.5(1) with Proposition 7.6 we have

$$1 = \int_{\mathbf{R}^n} d\nu(x) = \int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x)\,dx + a_0 > a_0, \qquad S^{p,q;\gamma} = \int_{\mathbf{R}^n} d\mu(x). \tag{7.76}$$

Moreover by Proposition 7.6 and Theorem 2.1 we have

$$\begin{aligned} S^{p,q;\gamma} &= \int_{\mathbf{R}^n} d\mu(x) \ge \int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx + S^{p,q;\gamma} a_0^{p/q} \\ &\ge S^{p,q;\gamma}\Big(\Big(\int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x)\,dx\Big)^{p/q} + a_0^{p/q}\Big) \\ &= S^{p,q;\gamma}\big((1-a_0)^{p/q} + a_0^{p/q}\big), \end{aligned} \tag{7.77}$$

and then $(1-a_0)^{p/q} + a_0^{p/q} \le 1$ and $a_0 = 0$ follow. In particular, we have

$$\begin{aligned} 1 &= \int_{\mathbf{R}^n} |u(x)|^q I_{q\gamma}(x)\,dx, \\ S^{p,q;\gamma} &= \int_{\mathbf{R}^n} d\mu(x) \ge \int_{\mathbf{R}^n} |\nabla u(x)|^p I_{p(1+\gamma)}(x)\,dx, \end{aligned} \tag{7.78}$$

and this proves the assertion. □

## 8. Proofs of Propositions 2.1 and 2.2 and some assertions

In this section we establish Propositions 2.1 and 2.2 and the propositions on the nonexistence of minimizers and the failure of some embedding inequalities whose proofs have been postponed.

### 8.1. Proofs of Propositions 2.1 and 2.2

To prove Propositions 2.1 and 2.2, let us prepare a cutoff function.

DEFINITION 8.1

For $0 < \varepsilon < 1$ and $0 < \eta < 1/4$ we set

$$\phi_{\varepsilon,\eta}(x) = \phi_{\varepsilon,\eta}(|x|) = \begin{cases} 1 & \text{for } x \in \overline{B_{3\varepsilon\eta}}, \\ \frac{\log(\varepsilon(1-\eta)/|x|)}{\log((1-\eta)/(3\eta))} & \text{for } x \in \overline{B_{\varepsilon(1-\eta)}} \setminus B_{3\varepsilon\eta}, \\ 0 & \text{for } x \in \mathbf{R}^n \setminus B_{\varepsilon(1-\eta)}, \end{cases} \tag{8.1}$$

and we set

$$\psi_{\varepsilon,\eta}(x) = \psi_{\varepsilon,\eta}(|x|) = \phi_{\varepsilon,\eta} * \rho_{\varepsilon\eta}(x) \quad \text{for } x \in \mathbf{R}^n. \tag{8.2}$$

LEMMA 8.1

*Let $1 < p \le q < \infty$, let $\gamma \ge 0$, let $R \ge 1$, and let $0 < \alpha < 1/p'$. Then there exist positive numbers $c_{p;\gamma}, \overline{c}_{p;\alpha}, \underline{c}_{p,q;\alpha} > 0$ such that we have for $0 < \varepsilon < 1$ and $0 < \eta < 1/8$ the following:*

(1) $\psi_{\varepsilon,\eta} \in C_c^\infty(\mathbf{R}^n)_{\mathrm{rad}}$, $0 \le \psi_{\varepsilon,\eta} \le 1$ *on* $\mathbf{R}^n, \psi_{\varepsilon,\eta} = 1$ *on* $\overline{B_{2\varepsilon\eta}}, \psi_{\varepsilon,\eta} = 0$ *on* $\mathbf{R}^n \setminus B_\varepsilon$.

(2) $\|\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \le c_{p;\gamma}\varepsilon^{1+\gamma}, \|\nabla\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \le \begin{cases} \frac{c_{p;\gamma}\varepsilon^\gamma}{\log(1/\eta)} & \text{if } \gamma > 0, \\ \frac{c_{p;0}}{(\log(1/\eta))^{1/p'}} & \text{if } \gamma = 0. \end{cases}$

(3) $\|\nabla[A^\alpha_{1,R}\psi_{\varepsilon,\eta}]\|_{L^p_1(B_1)} \le \overline{c}_{p;\alpha} A_{1,R}(\varepsilon)^\alpha \big(\frac{1}{(\log(1/\eta))^{1/p'}} + \frac{1}{A_{1,R}(\varepsilon)^{1/p'}}\big)$, $\|A^\alpha_{1,R}\psi_{\varepsilon,\eta}\|_{L^q_{p;R}(B_1)} \ge \frac{\underline{c}_{p,q;\alpha}}{A_{1,R}(2\varepsilon\eta)^{1/p'-\alpha}}$.

*Proof*

We see that $\phi_{\varepsilon,\eta} \in W^{1,\infty}(\mathbf{R}^n)$, and the first derivatives of $\phi_{\varepsilon,\eta}$ in a distribution sense are given by

$$\nabla\phi_{\varepsilon,\eta}(x) = -\frac{1}{\log((1-\eta)/(3\eta))}\chi_{B_{\varepsilon(1-\eta)}\setminus\overline{B_{3\varepsilon\eta}}}(x)\frac{x}{|x|^2} \quad \text{for a.e. } x \in \mathbf{R}^n. \tag{8.3}$$

Particularly we have

$$\begin{aligned} |\nabla\phi_{\varepsilon,\eta}(x)| &\le \frac{1}{\log(1/(4\eta))}\frac{1}{|x|}\chi_{B_{\varepsilon(1-\eta)}\setminus\overline{B_{3\varepsilon\eta}}}(x) \quad \text{for a.e. } x \in \mathbf{R}^n, \\ |\nabla\phi_{\varepsilon,\eta}(x-y)| &\le \frac{1}{\log(1/(4\eta))}\frac{1}{|x|-\varepsilon\eta}\chi_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}(x) \quad \text{for a.e. } x \in \mathbf{R}^n, y \in B_{\varepsilon\eta}. \end{aligned} \tag{8.4}$$

Here we note that

$$\begin{aligned} &0 \le \phi_{\varepsilon,\eta}(x) \le \chi_{B_{\varepsilon(1-\eta)}}(x), \\ &0 \le \phi_{\varepsilon,\eta}(x-y) \le \chi_{B_\varepsilon}(x) \quad \text{for a.e. } x \in \mathbf{R}^n, y \in B_{\varepsilon\eta}. \end{aligned} \tag{8.5}$$

Assertion (1) is now clear, hence we prove assertions (2) and (3) below.

Assertion (2)

$$\|\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)} = \Big(\int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n}\phi_{\varepsilon,\eta}(x-y)\rho_{\varepsilon\eta}(y)\,dy\Big)^p I_{p(1+\gamma)}(x)\,dx\Big)^{1/p}$$

$$\le \Big(\int_{B_\varepsilon}\Big(\int_{B_{\varepsilon\eta}}\rho_{\varepsilon\eta}(y)\,dy\Big)^p I_{p(1+\gamma)}(x)\,dx\Big)^{1/p}$$
$$= \Big(\int_{B_\varepsilon} I_{p(1+\gamma)}(x)\,dx\Big)^{1/p} = \Big(\frac{\omega_n}{p(1+\gamma)}\Big)^{1/p}\varepsilon^{1+\gamma},$$

$\|\nabla\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)}$

$$\le \Big(\int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n}|\nabla\phi_{\varepsilon,\eta}(x-y)|\rho_{\varepsilon\eta}(y)\,dy\Big)^p I_{p(1+\gamma)}(x)\,dx\Big)^{1/p}$$
$$\le \Big(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}\Big(\int_{B_{\varepsilon\eta}}\frac{1}{\log(1/(4\eta))}\frac{1}{|x|-\varepsilon\eta}\rho_{\varepsilon\eta}(y)\,dy\Big)^p I_{p(1+\gamma)}(x)\,dx\Big)^{1/p}$$
$$= \frac{1}{\log(1/(4\eta))}\Big(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}\frac{1}{(1-\varepsilon\eta/|x|)^p} I_{p\gamma}(x)\,dx\Big)^{1/p}$$
$$\le \frac{2}{\log(1/(4\eta))}\Big(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}} I_{p\gamma}(x)\,dx\Big)^{1/p}$$
$$= \begin{cases} 2\big(\frac{\omega_n}{p\gamma}\big)^{1/p}\frac{\varepsilon^\gamma(1-(2\eta)^\gamma)}{\log(1/(4\eta))} \le 2\big(\frac{\omega_n}{p\gamma}\big)^{1/p}\frac{\varepsilon^\gamma}{\log(1/(4\eta))} & \text{if } \gamma>0,\\ 2\omega_n^{1/p}\frac{\left(\log(1/(2\eta))\right)^{1/p}}{\log(1/(4\eta))} & \text{if } \gamma=0.\end{cases}$$

Assertion (3)

$\|A_{1,R}^\alpha\nabla\psi_{\varepsilon,\eta}\|_{L^p_1(B_1)}$

$$= \Big(\int_{B_1}\Big(\int_{\mathbf{R}^n}|\nabla\phi_{\varepsilon,\eta}(x-y)|\rho_{\varepsilon\eta}(y)\,dy\Big)^p A_{1,R}(x)^{p\alpha} I_p(x)\,dx\Big)^{1/p}$$
$$\le \Big(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}\Big(\int_{B_{\varepsilon\eta}}\frac{1}{\log(1/(4\eta))}\frac{1}{|x|-\varepsilon\eta}\rho_{\varepsilon\eta}(y)\,dy\Big)^p A_{1,R}(x)^{p\alpha} I_p(x)\,dx\Big)^{1/p}$$
$$= \frac{1}{\log(1/(4\eta))}\Big(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}}\Big(\frac{A_{1,R}(x)^\alpha}{1-\varepsilon\eta/|x|}\Big)^p I_0(x)\,dx\Big)^{1/p}$$
$$\le \frac{2A_{1,R}(\varepsilon)^\alpha}{\log(1/(4\eta))}\Big(\int_{B_\varepsilon\setminus\overline{B_{2\varepsilon\eta}}} I_0(x)\,dx\Big)^{1/p}$$
$$= 2\omega_n^{1/p}A_{1,R}(\varepsilon)^\alpha\frac{(\log(1/(2\eta)))^{1/p}}{\log(1/(4\eta))},$$

$\|\psi_{\varepsilon,\eta}\nabla[A_{1,R}^\alpha]\|_{L^p_1(B_1)}$

$$= \alpha\Big(\int_{B_1}\psi_{\varepsilon,\eta}(x)^p\frac{I_0(x)}{A_{1,R}(x)^{p(1-\alpha)}}\,dx\Big)^{1/p}$$
$$\le \alpha\Big(\int_{B_\varepsilon}\frac{I_0(x)}{A_{1,R}(x)^{p(1-\alpha)}}\,dx\Big)^{1/p}$$
$$= \alpha\Big(\frac{\omega_n}{p(1/p'-\alpha)}\Big)^{1/p}\frac{1}{A_{1,R}(\varepsilon)^{1/p'-\alpha}},$$

$$\begin{aligned}
&\|\psi_{\varepsilon,\eta}A^{\alpha}_{1,R}\|_{L^q_{p;R}(B_1)}\\
&\quad=\Big(\int_{B_1}\psi_{\varepsilon,\eta}(x)^q\,\frac{I_0(x)}{A_{1,R}(x)^{1+q(1/p'-\alpha)}}\,dx\Big)^{1/q}\\
&\quad\geq\Big(\int_{B_{2\varepsilon\eta}}\frac{I_0(x)}{A_{1,R}(x)^{1+q(1/p'-\alpha)}}\,dx\Big)^{1/q}=\Big(\frac{\omega_n}{q(1/p'-\alpha)}\Big)^{1/q}\frac{1}{A_{1,R}(2\varepsilon\eta)^{1/p'-\alpha}}.
\end{aligned}$$

□

By virtue of these we are able to verify Propositions 2.1 and 2.2.

*Proof of Proposition 2.1*
(1) For $\gamma>0$ it suffices to show that $C^\infty_{\mathrm{c}}(\mathbf{R}^n)\subset W^{1,p}_{\gamma,0}(\mathbf{R}^n)$. Take and fix a $u\in C^\infty_{\mathrm{c}}(\mathbf{R}^n)$. Then, for $0<\varepsilon<1$ and $0<\eta<1/8$ we see that $u(1-\psi_{\varepsilon,\eta})\in C^\infty_{\mathrm{c}}(\mathbf{R}^n\setminus\{0\})$ holds; hence by Lemma 8.1(2), we obtain

$$\begin{aligned}
&\big\|\nabla[u(1-\psi_{\varepsilon,\eta})-u]\big\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\\
&\quad=\|\nabla[u\psi_{\varepsilon,\eta}]\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\\
&\quad\leq\|\nabla u\|_{L^\infty(\mathbf{R}^n)}\|\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)}+\|u\|_{L^\infty(\mathbf{R}^n)}\|\nabla\psi_{\varepsilon,\eta}\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\\
&\quad\leq c_{p;\gamma}\Big(\|\nabla u\|_{L^\infty(\mathbf{R}^n)}\varepsilon^{1+\gamma}+\|u\|_{L^\infty(\mathbf{R}^n)}\frac{\varepsilon^\gamma}{\log(1/\eta)}\Big)\to 0\quad\text{as }\varepsilon\to0,\eta\to0.
\end{aligned}\tag{8.6}$$

Assertion (2) is now clear; hence we proceed to assertion (3).

(3). It suffices to prove $C^\infty_{\mathrm{c}}(B_1)\subset W^{1,p}_{0,0}(B_1)$. Let $u\in C^\infty_{\mathrm{c}}(B_1)$. Then, for $0<\varepsilon<1$ and $0<\eta<1/8$, we see that $u(1-\psi_{\varepsilon,\eta})\in C^\infty_{\mathrm{c}}(B_1\setminus\{0\})$, and hence by Lemma 8.1(2) we have

$$\begin{aligned}
&\|\nabla[u(1-\psi_{\varepsilon,\eta})-u]\|_{L^p_1(B_1)}\\
&\quad=\|\nabla[u\psi_{\varepsilon,\eta}]\|_{L^p_1(B_1)}\\
&\quad\leq\|\nabla u\|_{L^\infty(B_1)}\|\psi_{\varepsilon,\eta}\|_{L^p_1(B_1)}+\|u\|_{L^\infty(B_1)}\|\nabla\psi_{\varepsilon,\eta}\|_{L^p_1(B_1)}\\
&\quad\leq c_{p;0}\Big(\|\nabla u\|_{L^\infty(B_1)}\varepsilon+\|u\|_{L^\infty(B_1)}\frac{1}{(\log(1/\eta))^{1/p'}}\Big)\to 0\quad\text{as }\varepsilon\to0,\eta\to0.
\end{aligned}\tag{8.7}$$

□

*Proof of Proposition 2.2*
(a) First we show that if $0<\alpha<1/p'$, it holds that

$$A^{\alpha}_{1,R}\psi_{\varepsilon,\eta}\in W^{1,p}_{0,0}(B_1)\quad\text{for }0<\varepsilon<1,0<\eta<\frac{1}{8}.\tag{8.8}$$

For $0<\delta<\min\{2\varepsilon\eta,1/8\}$, noting that $A^{\alpha}_{1,R}\psi_{\varepsilon,\eta}(1-\psi_{\delta,\delta})\in C^\infty_{\mathrm{c}}(B_1\setminus\{0\})$ and $\psi_{\varepsilon,\eta}\psi_{\delta,\delta}=\psi_{\delta,\delta}$, we have that it follows from Lemma 8.1(3) that

$$\begin{aligned}
&\big\|\nabla[A^{\alpha}_{1,R}\psi_{\varepsilon,\eta}(1-\psi_{\delta,\delta})-A^{\alpha}_{1,R}\psi_{\varepsilon,\eta}]\big\|_{L^p_1(B_1)}\\
&\quad=\|\nabla[A^{\alpha}_{1,R}\psi_{\varepsilon,\eta}\psi_{\delta,\delta}]\|_{L^p_1(B_1)}
\end{aligned}$$

$$= \|\nabla[A_{1,R}^{\alpha}\psi_{\delta,\delta}]\|_{L_1^p(B_1)} \\ \le \overline{c}_{p;\alpha} A_{1,R}(\delta)^{\alpha}\Big(\frac{1}{(\log(1/\delta))^{1/p'}} + \frac{1}{A_{1,R}(\delta)^{1/p'}}\Big) \to 0 \quad \text{as } \delta \to 0.$$

(b) By the assumption, for an arbitrary $\varepsilon > 0$ there exists $0 < \eta_\varepsilon < 1/8$ such that we have

$$\frac{I_0(x)}{A_{1,R}(x)^{1+q/p'}} \le \varepsilon w(x) \quad \text{for all } x \in \overline{B_{\eta_\varepsilon}} \setminus \{0\}. \tag{8.9}$$

Then, if $0 < \eta < \eta_\varepsilon$, we see that

$$\|A_{1,R}^{\alpha}\psi_{\eta,\eta}\|_{L^q_{p;R}(B_1)} = \|A_{1,R}^{\alpha}\psi_{\eta,\eta}\|_{L^q_{p;R}(B_{\eta_\varepsilon})} \\ \le \varepsilon\|A_{1,R}^{\alpha}\psi_{\eta,\eta}\|_{L^q(B_{\eta_\varepsilon};w)} = \varepsilon\|A_{1,R}^{\alpha}\psi_{\eta,\eta}\|_{L^q(B_1;w)}.$$

Hence using Lemma 8.1(3), we have

$$\begin{aligned} &\frac{\|\nabla[A_{1,R}^{\alpha}\psi_{\eta,\eta}]\|_{L_1^p(B_1)}}{\|A_{1,R}^{\alpha}\psi_{\eta,\eta}\|_{L^q(B_1;w)}} \\ &\quad\le \frac{\|\nabla[A_{1,R}^{\alpha}\psi_{\eta,\eta}]\|_{L_1^p(B_1)}}{\|A_{1,R}^{\alpha}\psi_{\eta,\eta}\|_{L^q_{p;R}(B_1)}}\varepsilon \\ &\quad\le \frac{\overline{c}_{p;\alpha}}{\underline{c}_{p,q;\alpha}} A_{1,R}(\eta)^{\alpha}\Big(\frac{1}{(\log(1/\eta))^{1/p'}} + \frac{1}{A_{1,R}(\eta)^{1/p'}}\Big) A_{1,R}(2\eta^2)^{1/p'-\alpha}\varepsilon \\ &\quad\to 2^{1+1/p'-\alpha}\frac{\overline{c}_{p;\alpha}}{\underline{c}_{p,q;\alpha}}\varepsilon \qquad \text{as } \eta \to 0. \end{aligned} \tag{8.10}$$

Thus the assertion follows. □

## 8.2. Nonexistence of minimizers

In this section we verify Theorem 2.4(4), Proposition 2.3, and Proposition 2.4. We remark that both Theorem 2.4(4) and Proposition 2.4 follow from improved Hardy–Sobolev inequalities with sharp missing terms.

First Theorem 2.4(4) follows from the next lemma whose proof can be found in [Ho2, Lemma 4.2, Section 4].

LEMMA 8.2 (HORIUCHI)

*If $n \ge 3$, $p = 2 < q = 2^* = 2n/(n-2)$, and $\gamma > \gamma_{2,2^*} = (n-2)/2$, then $S^{2,2^*;\gamma} = S^{2,2^*;\gamma_{2,2^*}} = S_{\mathrm{rad}}^{2,2^*;\gamma_{2,2^*}}$ and*

$$\|\nabla u\|^2_{L^2_{1+\gamma}(\mathbf{R}^n)} \ge S_{\mathrm{rad}}^{2,2^*;\gamma_{2,2^*}}\|u\|^2_{L^{2^*}_{\gamma}(\mathbf{R}^n)} + (\gamma^2 - \gamma^2_{2,2^*})\|u\|^2_{L^2_{\gamma}(\mathbf{R}^n)} \\ \text{for } u \in W^{1,2}_{\gamma,0}(\mathbf{R}^n). \tag{8.11}$$

Proposition 2.4 follows from Lemma 8.3 below, which can be found in [AH1, Theorem 2.1(2), Section 2]. Here we put for $R > e$

$$A_{2,R}(x) = A_{2,R}(|x|) = \log A_{1,R}(x) = \log\Big(\log\frac{R}{|x|}\Big) \quad \text{for } x \in \overline{B_1} \setminus \{0\}. \tag{8.12}$$

LEMMA 8.3 (ANDO–HORIUCHI)
*For $1 < p = q < \infty$ there exist positive numbers $R_p > 0$ and $C > 0$ such that we have for $R \geq R_p$*

$$\|\nabla u\|^p_{L^p_1(B_1)} \geq \frac{1}{(p')^p}\|u\|^p_{L^p_{p;R}(B_1)} + C\int_{B_1}|u(x)|^p \frac{I_0(x)}{A_{1,R}(x)^p A_{2,R}(x)^2}\,dx \tag{8.13}$$
$$\text{for } u \in W^{1,p}_{0,0}(B_1).$$

Now we proceed to the proof of Proposition 2.3. To this end we employ the next proposition.

PROPOSITION 8.1
*Let $1 < p = q < \infty$, and let $\gamma > 0$. If $w \in C(\mathbf{R}^n \setminus \{0\})$ satisfies*

$$w(x) \geq 0 \quad \text{for } x \in \mathbf{R}^n \setminus \{0\}, \qquad \frac{(\log(1/|x|))^p}{I_{p\gamma}(x)} w(x) \to \infty \quad \text{as } x \to 0, \tag{8.14}$$

*then it holds that*

$$\inf\Big\{\frac{\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} - (S^{p,p;\gamma})^{1/p}\|u\|_{L^p_\gamma(\mathbf{R}^n)}}{\|u\|_{L^p(\mathbf{R}^n;w)}} \,\Big|\, u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n \setminus \{0\}) \setminus \{0\}\Big\} \tag{8.15}$$
$$= 0.$$

*Proof*
(a) If $R > 1$, then it follows from the assumption that we have

$$\frac{A_{1,R}(x)^{1+p/p'}}{I_0(x)}\frac{w(x)}{I_{p\gamma+n}(x)} = \frac{A_{1,R}(x)^p}{I_{p\gamma}(x)} w(x) \to \infty \quad \text{as } x \to 0. \tag{8.16}$$

Hence by Proposition 2.2

$$\inf\Big\{\Big(\frac{\|\nabla v\|_{L^p_1(B_1)}}{\|v\|_{L^p(B_1;w/I_{p\gamma+n})}}\Big)^p \,\Big|\, v \in C^\infty_{\mathrm{c}}(B_1 \setminus \{0\}) \setminus \{0\}\Big\} = 0. \tag{8.17}$$

(b) On the contrary we assume that the assertion is false. Since $S^{p,p;\gamma} = \gamma^p$ holds, there exists a number $C > 0$ such that we have

$$\|\nabla u\|_{L^p_{1+\gamma}(\mathbf{R}^n)} \geq \gamma\|u\|_{L^p_\gamma(\mathbf{R}^n)} + C\|u\|_{L^p(\mathbf{R}^n;w)} \quad \text{for } u \in C^\infty_{\mathrm{c}}(\mathbf{R}^n \setminus \{0\}). \tag{8.18}$$

Using Lemma 6.1 we have

$$\begin{aligned}
&\gamma\|v\|_{L^p_0(\mathbf{R}^n)} + C\|v\|_{L^q(\mathbf{R}^n;w/I_{p\gamma+n})}\\
&\quad = \gamma\|\hat{T}_\gamma v\|_{L^p_\gamma(\mathbf{R}^n)} + C\|\hat{T}_\gamma v\|_{L^p(\mathbf{R}^n;w)} \leq \|\nabla[\hat{T}_\gamma v]\|_{L^p_{1+\gamma}(\mathbf{R}^n)}\\
&\quad = \Big(\int_{\mathbf{R}^n}\Big|\nabla v(x) - \gamma v(x)\frac{x}{|x|^2}\Big|^p I_p(x)\,dx\Big)^{1/p}\\
&\quad \leq \|\nabla v\|_{L^p_1(\mathbf{R}^n)} + \gamma\|v\|_{L^p_0(\mathbf{R}^n)} \quad \text{for } v \in C^\infty_{\mathrm{c}}(\mathbf{R}^n \setminus \{0\}),\\
&\|\nabla v\|_{L^p_1(B_1)} \geq C\|v\|_{L^p(B_1;w/I_{p\gamma+n})} \quad \text{for } v \in C^\infty_{\mathrm{c}}(B_1 \setminus \{0\}).
\end{aligned}$$

This contradicts (a). □

Let us recall the result due to [AH1, proof of Theorem 2.1, Section 4.1].

LEMMA 8.4 (ANDO–HORIUCHI)

*Assume that* $1 < p = q < \infty$, *and assume that* $\gamma > 0$. *If* $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n) \setminus \{0\}$ *is a minimizer for* $S^{p,p;\gamma}$, *then* $u$ *is radially symmetric with respect to the origin and has a constant sign. Moreover if* $u \geq 0$ *on* $\mathbf{R}^n$, *then* $u$ *is a monotonically decreasing function of* $r = |x|$ *and satisfies*

$$u(x)|x|^{\gamma} \to 0 \quad \text{as } |x| \to 0, \qquad u(x)|x|^{\gamma} \to 0 \quad \text{as } |x| \to \infty. \tag{8.19}$$

From this we have the next proposition from which Proposition 2.3 follows.

PROPOSITION 8.2

*Assume that* $1 < p = q < \infty$, *and assume that* $\gamma > 0$. *Then, there exists no minimizer for* $S^{p,p;\gamma}$ *in* $W^{1,p}_{\gamma,0}(\mathbf{R}^n) \setminus \{0\}$.

*Proof*

(a) Assume that there exists a minimizer $u \in W^{1,p}_{\gamma,0}(\mathbf{R}^n) \setminus \{0\}$ for $S^{p,p;\gamma}$. Then it follows from the variational principle that we have

$$\begin{aligned} &\int_{\mathbf{R}^n} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \phi(x) I_{p(1+\gamma)}(x)\, dx \\ &\quad = \gamma^p \int_{\mathbf{R}^n} |u(x)|^{p-2} u(x) \phi(x) I_{p\gamma}(x)\, dx \quad \text{for } \phi \in W^{1,p}_{\gamma,0}(\mathbf{R}^n). \end{aligned} \tag{8.20}$$

By Lemma 8.4 $u$ should be radially symmetric and satisfies

$$u > 0 \quad \text{on } \mathbf{R}^n, \qquad \frac{\partial u}{\partial r} < 0 \quad \text{on } \mathbf{R}^n \setminus \{0\}. \tag{8.21}$$

Hence we have for $\phi \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}$

$$-\int_0^\infty \Big(-\frac{\partial u}{\partial r}(r)\Big)^{p-1} \frac{\partial \phi}{\partial r}(r) r^{p(1+\gamma)-1}\, dr = \gamma^p \int_0^\infty u(r)^{p-1} \phi(r) r^{p\gamma-1}\, dr. \tag{8.22}$$

Since $u : (0,\infty) \to (0, u(0))$ is surjective, we have the inverse $R : (0, u(0)) \to (0,\infty)$, and by Lemma 8.4 it holds that

$$R(\varepsilon)^{\gamma} \varepsilon = u\big(R(\varepsilon)\big) R(\varepsilon)^{\gamma} \to 0 \quad \text{as } \varepsilon \to 0. \tag{8.23}$$

(b) For $0 < \varepsilon < u(0)$, we set

$$u_\varepsilon(x) = \big(u(x) - \varepsilon\big)_+ = \begin{cases} u(x) - \varepsilon & \text{for } x \in B_{R(\varepsilon)}, \\ 0 & \text{for } x \in \mathbf{R}^n \setminus B_{R(\varepsilon)}. \end{cases} \tag{8.24}$$

Then $u_\varepsilon \in W^{1,p}_{\gamma,0}(\mathbf{R}^n)_{\mathrm{rad}}$ and its derivative in a sense of distribution is given by

$$\frac{\partial u_\varepsilon}{\partial r}(x) = \begin{cases} \frac{\partial u}{\partial r}(x) & \text{for } x \in B_{R(\varepsilon)}, \\ 0 & \text{for } x \in \mathbf{R}^n \setminus \overline{B_{R(\varepsilon)}}. \end{cases} \tag{8.25}$$

Therefore from (a) we have

$$
(8.26)\quad -\int_0^\infty \Big(-\frac{\partial u}{\partial r}(r)\Big)^{p-1}\frac{\partial u_\varepsilon}{\partial r}(r)r^{p(1+\gamma)-1}\,dr = \gamma^p\int_0^\infty u(r)^{p-1}u_\varepsilon(r)r^{p\gamma-1}\,dr,
$$

$$
\int_0^{R(\varepsilon)} \Big(-\frac{\partial u}{\partial r}(r)\Big)^{p} r^{p(1+\gamma)-1}\,dr
= \gamma^p\int_0^{R(\varepsilon)} u(r)^p r^{p\gamma-1}dr - \varepsilon\gamma^p\int_0^{R(\varepsilon)} u(r)^{p-1}r^{p\gamma-1}\,dr.
$$

Setting

$$
(8.27)\qquad v(r) = u(r)r^\gamma \quad \text{for } r>0,
$$

we have

$$
\int_0^{R(\varepsilon)} \Big(\gamma v(r) - \frac{\partial v}{\partial r}(r)\Big)^p \frac{1}{r}\,dr
$$

$$
(8.28)\qquad = \gamma^p\int_0^{R(\varepsilon)} \big(v(r)-\varepsilon r^\gamma\big)v(r)^{p-1}\frac{1}{r}\,dr \quad \text{for } 0<\varepsilon<u(0),
$$

$$
v\big(R(\varepsilon)\big) = u\big(R(\varepsilon)\big)R(\varepsilon)^\gamma \to 0 \quad \text{as } \varepsilon\to 0.
$$

(c) Since there exists a number $c_p>0$ such that

$$
(8.29)\qquad |1-t|^p - 1 + pt \ge c_p\frac{t^2}{1+t^2} \quad \text{for } t\in\mathbf{R},
$$

we have for $r>0$

$$
\Big(\gamma v(r) - \frac{\partial v}{\partial r}(r)\Big)^p
\ge \gamma^p v(r)^p - p\gamma^{p-1}v(r)^{p-1}\frac{\partial v}{\partial r}(r)r + c_p\gamma^p\frac{v(r)^p\frac{\partial v}{\partial r}(r)^2r^2}{\gamma^2v(r)^2+v(r)^p\frac{\partial v}{\partial r}(r)^2r^2}.
$$

By using Lemma 8.4, we have

$$
\gamma^p\int_0^{R(\varepsilon)} (v(r)-\varepsilon r^\gamma)v(r)^{p-1}\frac{1}{r}\,dr
$$

$$
(8.30)\qquad = \int_0^{R(\varepsilon)} \Big(\gamma v(r)-\frac{\partial v}{\partial r}(r)\Big)^p\frac{1}{r}\,dr
\ge \int_0^{R(\varepsilon)} \Big(\gamma^p v(r)^p\frac{1}{r} - p\gamma^{p-1}v(r)^{p-1}\frac{\partial v}{\partial r}(r)
+ c_p\gamma^p\frac{v(r)^p\frac{\partial v}{\partial r}(r)^2 r}{\gamma^2v(r)^2+v(r)^p\frac{\partial v}{\partial r}(r)^2r^2}\Big)\,dr
$$

$$
(8.31)\qquad = \gamma^p\int_0^{R(\varepsilon)} \Big(v(r)^p\frac{1}{r} + c_p\frac{v(r)^p\frac{\partial v}{\partial r}(r)^2 r}{\gamma^2v(r)^2+v(r)^p\frac{\partial v}{\partial r}(r)^2r^2}\Big)\,dr - \gamma^{p-1}v\big(R(\varepsilon)\big)^p
$$

for $0<\varepsilon<u(0)$.

Therefore we have

$$(8.32)\qquad \begin{aligned} 0 &\le c_p \int_0^{R(\varepsilon)} \frac{v(r)^p \frac{\partial v}{\partial r}(r)^2 r}{\gamma^2 v(r)^2 + v(r)^p \frac{\partial v}{\partial r}(r)^2 r^2}\, dr \\ &\le \frac{1}{\gamma} v\big(R(\varepsilon)\big)^p - \varepsilon \int_0^{R(\varepsilon)} v(r)^{p-1} r^{\gamma-1}\, dr \\ &\le \frac{1}{\gamma} v\big(R(\varepsilon)\big)^p \quad \text{for } 0 < \varepsilon < u(0), \end{aligned}$$

and then, if we let $\varepsilon \to 0$, it follows from (b) that

$$(8.33)\qquad \int_0^\infty \frac{v(r)^p \frac{\partial v}{\partial r}(r)^2 r}{\gamma^2 v(r)^2 + v(r)^p \frac{\partial v}{\partial r}(r)^2 r^2}\, dr = 0.$$

Thus we have a constant $c$ such that

$$0 = v(r)^{p/2} \frac{\partial v}{\partial r}(r) = \frac{2}{p+2} \frac{\partial}{\partial r}[v^{(p+2)/2}](r), \qquad c = v(r) = u(r) r^\gamma \quad \text{for } r > 0,$$

and this contradicts Lemma 8.4. □

### 8.3. Failure of embedding inequalities

In this section we prove $C^{p,q;1} = 0$ provided that $n \ge 2$ and $p < q$. Combining this fact with assertions (1) and (2) of Proposition 4.2, we have Theorem 2.6(2).

PROPOSITION 8.3

*Assume that $n \ge 2$, assume that $1 < p < q < \infty$, assume that $\tau_{p,q} \le 1/n$, and assume that $R = 1$. Then it holds that $C^{p,q;1} = 0$.*

Let us set

$$B'_r = \{x' \in \mathbf{R}^{n-1} \mid |x'| < r\} \quad \text{for } r > 0, \qquad (B_1)_+ = \{x = (x', x_n) \in B_1 \mid x_n > 0\},$$

and let us prepare the following.

LEMMA 8.5

*For $n \ge 2$, we set*

$$(8.34)\qquad \begin{aligned} \varphi(x) &= \varphi(x', x_n) = \big(x', \varphi_n(x)\big), \\ \varphi_n(x) &= \varphi_n(x', x_n) = (1 - |x'|^2)^{1/2} - x_n \quad \text{for } x = (x', x_n) \in (B_1)_+. \end{aligned}$$

*Then we have the following:*

(1) *$\varphi : (B_1)_+ \to (B_1)_+$ is a diffeomorphism and $\varphi^{-1} = \varphi$ is valid. In particular, we have*

$$\varphi_n\big(\varphi(x)\big) = x_n \quad \text{for } x \in (B_1)_+.$$

(2) $\det D\varphi(x) = -1$ *for $x \in (B_1)_+$.*

(3) $1 - x_n \le |\varphi(x)| = (|x'|^2 + \varphi_n(x)^2)^{1/2} \le 1 + x_n$ *for $x \in (B_1)_+$.*

*Proof of Proposition 8.3*

(a) Let us fix an $\alpha > 0$. For $0 < \varepsilon < 1/2$ we set

$$u_\varepsilon(x) = \begin{cases} \psi_\varepsilon(\varphi(x))\varphi_n(x)^{1+\alpha} & \text{for } x \in (B_1)_+, \\ 0 & \text{for } x \in B_1 \setminus (B_1)_+. \end{cases} \tag{8.35}$$

Here $\psi_\varepsilon$ is given as in Definition 7.1. Then, we see that $u_\varepsilon \in W^{1,p}_{0,0}(B_1)$ and

$$\partial_{x_j}[|\varphi|](x) = \frac{x_j x_n}{|\varphi(x)|(1-|x'|^2)^{1/2}} \quad (1 \le j \le n-1),$$

$$\partial_{x_n}[|\varphi|](x) = -\frac{\varphi_n(x)}{|\varphi(x)|} \quad \text{for } x \in (B_1)_+.$$

Then we have

$$\begin{aligned}
|\nabla u_\varepsilon(x)|^2 &= \Big(\frac{1}{\varepsilon}\frac{\tilde\psi_\varepsilon(\varphi(x))}{|\varphi(x)|}\varphi_n(x)x_n + (1+\alpha)\psi_\varepsilon\big(\varphi(x)\big)\Big)^2\varphi_n(x)^{2\alpha}\frac{|x'|^2}{1-|x'|^2} \\
&\quad + \Big(\frac{1}{\varepsilon}\frac{\tilde\psi_\varepsilon(\varphi(x))}{|\varphi(x)|}\varphi_n(x)^2 - (1+\alpha)\psi_\varepsilon\big(\varphi(x)\big)\Big)^2\varphi_n(x)^{2\alpha} \\
&\le 2\Big(\frac{1}{\varepsilon^2}\frac{\tilde\psi_\varepsilon(\varphi(x))^2}{|\varphi(x)|^2}\varphi_n(x)^2\big(|x'|^2x_n^2 + (1-|x'|^2)\varphi_n(x)^2\big) \\
&\quad + (1+\alpha)^2\psi_\varepsilon\big(\varphi(x)\big)^2\Big)\frac{\varphi_n(x)^{2\alpha}}{1-|x'|^2} \\
&\le 2\Big(\frac{1}{\varepsilon^2}\tilde\psi_\varepsilon\big(\varphi(x)\big)^2\varphi_n(x)^2 + (1+\alpha)^2\psi_\varepsilon\big(\varphi(x)\big)^2\Big)\frac{\varphi_n(x)^{2\alpha}}{1-|x'|^2} \\
&\quad \text{for } x \in (B_1)_+.
\end{aligned} \tag{8.36}$$

(b) By using (a) and Lemma 8.5 we have

$$\big|u_\varepsilon\big(\varphi(y)\big)\big| = \psi_\varepsilon(y)y_n^{1+\alpha}, \tag{8.37}$$

$$\begin{aligned}
\big|[\nabla u_\varepsilon]\big(\varphi(y)\big)\big|^2 &\le 2\Big(\frac{1}{\varepsilon^2}\tilde\psi_\varepsilon(y)^2y_n^2 + (1+\alpha)^2\psi_\varepsilon(y)^2\Big)\frac{y_n^{2\alpha}}{1-|y'|^2} \\
&\le \frac{8}{3}\Big(\frac{1}{\varepsilon^2}\tilde\psi_\varepsilon(y)^2y_n^2 + (1+\alpha)^2\psi_\varepsilon(y)^2\Big)y_n^{2\alpha} \\
&\quad \text{for } y \in (B_1)_+ \cap B_{1/2}.
\end{aligned}$$

Noting Lemma 8.5(3) and

$$\frac{1}{t}\log\frac{1}{1-t} \le 2\log 2 \quad \text{for } 0 < t \le \frac{1}{2}, \tag{8.38}$$

we see that

$$I_0\big(\varphi(y)\big) \ge \frac{1}{(1+1/2)^n} = \Big(\frac{2}{3}\Big)^n, \tag{8.39}$$

$$I_p\big(\varphi(y)\big) \le \max\Big\{\Big(1+\frac{1}{2}\Big)^{p-n}, \Big(\frac{1}{1-1/2}\Big)^{n-p}\Big\} \le 2^{|n-p|},$$

$$A_{1,1}\big(\varphi(y)\big) = \log\frac{1}{|\varphi(y)|} \le \log\frac{1}{1-y_n} \le (2\log 2)y_n \quad \text{for } y \in (B_1)_+ \cap B_{1/2}.$$

Then, we also have

$$\begin{aligned}
\|u_\varepsilon\|^q_{L^q_{p;1}(B_1)} &= \int_{(B_1)_+} |u_\varepsilon(x)|^q \frac{I_0(x)}{A_{1,1}(x)^{1+q/p'}}\,dx \\
&= \int_{(B_1)_+} \big|u_\varepsilon\big(\varphi(y)\big)\big|^q \frac{I_0(\varphi(y))}{A_{1,1}(\varphi(y))^{1+q/p'}}\,dy \\
&\ge \int_{(B_1)_+\cap B_{\varepsilon/2}} \big(\psi_\varepsilon(y)y_n^{1+\alpha}\big)^q \frac{1}{((2\log 2)y_n)^{1+q/p'}}\Big(\frac{2}{3}\Big)^n\,dy \qquad (8.40)\\
&\ge \frac{1}{(2\log 2)^{1+q/p'}}\Big(\frac{2}{3}\Big)^n \int_{B'_{\varepsilon/4}\times(0,\varepsilon/4)} y_n^{q(\alpha+1/p)-1}\,dy \\
&= \frac{1}{(2\log 2)^{1+q/p'}}\Big(\frac{2}{3}\Big)^n \frac{p}{q}\frac{\omega_{n-1}}{(n-1)(1+p\alpha)}\Big(\frac{\varepsilon}{4}\Big)^{n-1+q(\alpha+1/p)},
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla u_\varepsilon\|^p_{L^p_1(B_1)} &= \int_{(B_1)_+} |\nabla u_\varepsilon(x)|^p I_p(x)\,dx \\
&= \int_{(B_1)_+} \big|\nabla u_\varepsilon\big(\varphi(y)\big)\big|^p I_p\big(\varphi(y)\big)\,dy \\
&\le \int_{(B_1)_+\cap B_\varepsilon} \Big(\frac{8}{3}\Big(\frac{1}{\varepsilon^2}\tilde{\psi}_\varepsilon(y)^2 y_n^2 + (1+\alpha)^2\psi_\varepsilon(y)^2\Big)y_n^{2\alpha}\Big)^{p/2} 2^{|n-p|}\,dy \qquad (8.41)\\
&\le 2^{|n-p|}\Big(\frac{8}{3}\big(9+(1+\alpha)^2\big)\Big)^{p/2} \int_{B'_\varepsilon\times(0,\varepsilon)} y_n^{p\alpha}\,dy \\
&= 2^{|n-p|}\Big(\frac{8}{3}\big(9+(1+\alpha)^2\big)\Big)^{p/2} \frac{\omega_{n-1}}{(n-1)(1+p\alpha)}\varepsilon^{n+p\alpha}.
\end{aligned}$$

Since $n+p\alpha-(p/q)(n-1+q(\alpha+1/p)) = (n-1)(1-p/q) > 0$ holds, we have

$$F^{p,q;1}(u_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0. \qquad \square$$

## Appendix: Proof of Proposition 4.5

*Proof of Proposition 4.5*

In this Appendix we give a proof of Proposition 4.5, which had been postponed in Section 4.2. According to Definition 4.2, by $\mu_1$ we denote the ($n$-dimensional) Lebesgue measure, and hence by $\mu_1[u]$ and $\mathcal{R}_1[u]$ we denote the distribution function and the rearrangement function of $f$ with respect to the constant function 1, respectively.

REMARK A.1

If $u \in C_c(\mathbf{R}^n)$ is Lipschitz continuous, then $u$ is differentiable almost everywhere on $\mathbf{R}^n$ and the first-order derivatives of $u$ in the distribution sense $\partial u/\partial x_i : \mathbf{R}^n \to$

$\mathbf{R}$ $(1 \le i \le n)$ coincide with those of $u$ in the classical sense almost everywhere on $\mathbf{R}^n$, that is,

$$\frac{\partial u}{\partial x_i}(x) = D_i u(x)\Big(= \lim_{t\to 0}\frac{1}{t}\big(u(x+te_i) - u(x)\big)\Big) \quad \text{for a.e. } x \in \mathbf{R}^n \ (1 \le i \le n).$$

Here $e_i = (\delta_{ij})_{1\le j\le n} \in \mathbf{R}^n$ is a unit vector on the $x_i$-axis.

DEFINITION A.1

For an admissible $f$ and a Lipschitz continuous $u \in C_{\mathrm{c}}(\mathbf{R}^n)$, we define the following:

(1) $Z[u] = \{x \in \mathbf{R}^n \mid u \text{ is differentiable at } x \text{ and } \nabla u(x) \ne 0\}$, $Z_0[u] = \{x \in \mathbf{R}^n \mid u \text{ is differentiable at } x \text{ and } \nabla u(x) = 0\}$.

(2) $h_f[u](t) = \mu_f(Z_0[u] \cap \{|u| > t\})$ for $t > 0$.

REMARK A.2

When $u \in C_{\mathrm{c}}(\mathbf{R}^n)$ is Lipschitz continuous, by Remark A.1 we see that

$$\mu_1\big(\mathbf{R}^n \setminus (Z[u] \cup Z_0[u])\big) = 0.$$

Further it follows from Definitions 4.2 and A.1 that $\mu_f[u], h_f[u] : (0,\infty) \to [0,\infty)$ are nonincreasing and right-continuous.

To study $\mathcal{R}_f[u]$, we employ the next lemma concerning single-variable functions. When $n = 1$, we set $\mu_f = \tilde{\mu}_f$ in Definition 4.2(1) temporarily.

LEMMA A.1

*Assume that $I \subset \mathbf{R}$ is an open interval, and assume that $v : I \to \mathbf{R}$ is a bounded variation and right-continuous. Then we have the following:*

(1) *The distributional derivative $v'$ of $v$ is a Borel measure on $I$ and we have*

$$v(s_1) - v(s_0) = \int_{(s_0,s_1]} dv' \quad \text{for } (s_0, s_1] \subset I.$$

*Moreover for an arbitrary Borel set $A \subset I$, we can set*

$$|v'|(A) = \int_A d|v'|.$$

(2) *$v$ is differentiable almost everywhere on $I$ and we have*

$$Dv(s)\Big(= \lim_{t\to 0}\frac{1}{t}\big(v(s+t) - v(s)\big)\Big) = [v']_{\mathrm{ac}}(s) \quad \text{for a.e. } s \in I.$$

*Here, $[v']_{\mathrm{ac}}$ denotes the absolutely continuous part of $v'$ with respect to (the 1-dimensional Lebesgue measure) $\tilde{\mu}_1$.*

By using Lemma A.1 we can show the next lemma.

LEMMA A.2

*Assume that $g \in L^1_{\mathrm{loc}}([0,\infty))$, assume that $g \ge 0$ a.e. on $[0,\infty)$, assume that*

$v \in C_{\mathrm{c}}([0,\infty))$ *is nonincreasing, and Lipschitz continuous, and assume that* $v \geq 0$ *on* $[0,\infty)$*. Let us set*

$$\tilde{Z}[v] = \big\{ s \in (0,\infty) \;\big|\; v \text{ is differentiable at } s \text{ and } Dv(s) \neq 0 \big\},$$
$$\tilde{Z}_0[v] = \big\{ s \in (0,\infty) \;\big|\; v \text{ is differentiable at } s \text{ and } Dv(s) = 0 \big\},$$
$$\tilde{h}_g[v](t) = \tilde{\mu}_g(\tilde{Z}_0[v] \cap \{v > t\}) \quad \text{for } t > 0.$$

*Then we have the following:*

(1) $\tilde{\mu}_1((0,\infty) \setminus (\tilde{Z}[v] \cup \tilde{Z}_0[v])) = 0, \tilde{\mu}_1(v(\tilde{Z}_0[v])) = 0.$

(2) $\tilde{h}_g[v] : (0,\infty) \to [0,\infty)$ *is nonincreasing and right-continuous. In particular,* $\tilde{h}_g[v]$ *is a bounded variation, and* $\tilde{h}_g[v]'$ *is a Borel measure on* $(0,\infty)$*. Further, for an arbitrary Borel set* $A \subset (0,\infty)$*, we have*

$$\tilde{\mu}_g(\tilde{Z}_0[v] \cap \{v \in A\}) = |\tilde{h}_g[v]'|(A).$$

*Here,* $\{v \in A\} = \{s \in [0,\infty) \mid v(s) \in A\}$*.*

(3) $[\tilde{h}_g[v]']_{\mathrm{ac}} = 0$ *a.e. on* $(0,\infty)$*.*

*Proof*

(1) Since $v$ is absolutely continuous on $[0,\infty)$, $v$ is differentiable almost everywhere in $(0,\infty)$ and

$$Dv = v' \quad \text{a.e. on } (0,\infty), \qquad v' = [v']_{\mathrm{ac}} \quad \text{in } \mathcal{D}'\big((0,\infty)\big).$$

In particular we have $\tilde{\mu}_1\big((0,\infty) \setminus (\tilde{Z}[v] \cup \tilde{Z}_0[v])\big) = 0$. Moreover it follows from Lemma A.1(1) that

$$\tilde{\mu}_1\big(v((s_0,s_1])\big) \leq \int_{(s_0,s_1]} |v'(s)|\, ds \quad \text{for } (s_0,s_1] \subset I,$$

and we have for an arbitrary Borel set $A \subset (0,\infty)$

$$\tilde{\mu}_1\big(v(A)\big) \leq \int_A |v'(s)|\, ds.$$

Here, $v(A) = \{v(s) \mid s \in A\}$. Particularly we have

$$\tilde{\mu}_1\big(v(\tilde{Z}_0[v])\big) \leq \int_{\tilde{Z}_0[v]} |v'(s)|\, ds = \int_{\tilde{Z}_0[v]} |Dv(s)|\, ds = 0.$$

(2) We easily see that $\tilde{h}_g[v]$ is nonincreasing and right-continuous. Then, it follows from Lemma A.1(1) that

$$\begin{aligned}
\tilde{\mu}_g\big(\tilde{Z}_0[v] \cap \big\{v \in (t_0,t_1]\big\}\big) &= \tilde{\mu}_g(\tilde{Z}_0[v] \cap \{v > t_0\}) - \tilde{\mu}_g(\tilde{Z}_0[v] \cap \{v > t_1\}) \\
&= \tilde{h}_g[v](t_0) - \tilde{h}_g[v](t_1) \\
&= -\int_{(t_0,t_1]} d[\tilde{h}_g[v]'] = \int_{(t_0,t_1]} d|\tilde{h}_g[v]'| \\
&= |\tilde{h}_g[v]'|\big((t_0,t_1]\big) \quad \text{for } (t_0,t_1] \subset (0,\infty),
\end{aligned}$$

and this proves the assertion.

(3) From assertion (1) of this lemma we have $\tilde{\mu}_1(v(\tilde{Z}_0[v]))=0$; hence there exists a Borel set $F[v]\subset(0,\infty)$ such that $v(\tilde{Z}_0[v])\subset F[v]$, $\tilde{\mu}_1(F[v])=0$. Since $\tilde{Z}_0[v]\subset\{v\in F[v]\}$ and $\tilde{Z}_0[v]\cap\{v\in(0,\infty)\setminus F[v]\}=\varnothing$, it follows from assertion (2) of this lemma that we have

$$\begin{aligned}\int_{(0,\infty)}\big|[\tilde{h}_g[v]']_{\mathrm{ac}}(t)\big|\,dt&=\int_{(0,\infty)\setminus F[v]}\big|\big[\tilde{h}_g[v]'\big]_{\mathrm{ac}}(t)\big|\,dt\le\int_{(0,\infty)\setminus F[v]}d|\tilde{h}_g[v]'|\\&=|\tilde{h}_g[v]'|\big((0,\infty)\setminus F[v]\big)\\&=\tilde{\mu}_g\big(\tilde{Z}_0[v]\cap\big\{v\in(0,\infty)\setminus F[v]\big\}\big)=0,\end{aligned}$$

and this proves the assertion. □

By this lemma and Sard's lemma we have the next proposition.

PROPOSITION A.1

*Let $f$ be admissible, and let $u\in C_{\mathrm{c}}^1(\mathbf{R}^n)$. Then we have the following:*

(1) $\big[h_f[\mathcal{R}_f[u]]'\big]_{\mathrm{ac}}=0$ *a.e. on* $(0,\infty)$.
(2) $[h_f[u]']_{\mathrm{ac}}=0$ *a.e. on* $(0,\infty)$.

*Proof*

(1) Let us set

$$g_f(s)=f\left(\left(\frac{s}{\omega_n}\right)^{1/n}\right),\qquad v_f[u](s)=\mathcal{R}_f[u]\left(\left(\frac{s}{\omega_n}\right)^{1/n}\right)\quad\text{for } s\ge 0.$$

Then we see that $g_f\in L^1_{\mathrm{loc}}([0,\infty))$, $g_f\ge 0$ almost everywhere on $[0,\infty)$, $v_f[u]\in C_{\mathrm{c}}([0,\infty))$ is nonincreasing, and $v_f[u]\ge 0$ on $[0,\infty)$. Further we have

$$\tilde{Z}_0\big[v_f[u]\big]=\left\{s\in(0,\infty)\;\Big|\;\left(\frac{s}{\omega_n}\right)^{1/n}e_1\in Z_0\big[\mathcal{R}_f[u]\big]\right\}.$$

Noting that $f,\mathcal{R}_f[u]$ are radially symmetric, we also have

$$\begin{aligned}\tilde{h}_{g_f}\big[v_f[u]\big](t)&=\tilde{\mu}_{g_f}\big(\tilde{Z}_0\big[v_f[u]\big]\cap\big\{v_f[u]>t\big\}\big)\\&=\int_{\tilde{Z}_0[v_f[u]]\cap\{v_f[u]>t\}}f\left(\left(\frac{s}{\omega_n}\right)^{1/n}\right)ds\\&=n\omega_n\int_{\{r\in(0,\infty)|re_1\in Z_0[\mathcal{R}_f[u]],\mathcal{R}_f[u](re_1)>t\}}f(r)r^{n-1}\,dr\\&=n\int_{Z_0[\mathcal{R}_f[u]]\cap\{\mathcal{R}_f[u]>t\}}f(x)\,dx=n\mu_f\big(Z_0\big[\mathcal{R}_f[u]\big]\cap\big\{\mathcal{R}_f[u]>t\big\}\big)\\&=nh_f\big[\mathcal{R}_f[u]\big](t)\quad\text{for } t>0.\end{aligned}$$

Thus the assertion follows from Lemma A.2(3).

(2) First, from Sard's lemma we see $\tilde{\mu}_1(|u|(Z_0[u]))=0$, where $|u|(Z_0[u])=\{|u(x)|\mid x\in Z_0[u]\}$. Since $\operatorname{supp}u$ is compact, for $t\in(0,\infty)\setminus|u|(Z_0[u])$ there exists

$\varepsilon_t > 0$ satisfying $\{t - \varepsilon_t < |u| < t + \varepsilon_t\} \cap Z_0[u] = \varnothing$ such that we have

$$Z_0[u] \cap \bigl\{|u| > s\bigr\} = Z_0[u] \cap \bigl\{|u| > t\bigr\},$$
$$h_f[u](s) = h_f[u](t) \quad \text{for } s \in (t - \varepsilon_t, t + \varepsilon_t).$$

Then

$$D\bigl[h_f[u]\bigr](t) = 0 \quad \text{for } t \in (0, \infty) \setminus |u|(Z_0[u]),$$

and hence from Lemma A.1(2) we have

$$\bigl[h_f[u]'\bigr]_{\mathrm{ac}}(t) = D\bigl[h_f[u]\bigr](t) = 0 \quad \text{for a.e. } t > 0. \qquad \square$$

By $H^{n-1}$ we denote the $(n-1)$-dimensional Hausdorff measure.

LEMMA A.3
*Let $1 \le p < \infty$, and let $f$ be admissible. Then we have the following:*

(1) *For an arbitrary Borel set $A \subset \mathbf{R}^n$ satisfying $0 < \mu_1(A) < \infty$, we have*

$$\int_{\partial B_{r_f[A]}} dH^{n-1} \le \int_{\partial A} dH^{n-1}.$$

(2) *For an arbitrary $u \in C_{\mathrm{c}}(\mathbf{R}^n)$, we have*

$$\int_{\{\mathcal{R}_f[u]=t\}} dH^{n-1} \le \int_{\{|u|=t\}} dH^{n-1} \quad \textit{for a.e. } t > 0.$$

*Proof*
(1) Since $\mathcal{R}_1[\chi_A] = \chi_{B_{r_1[A]}}$, from Proposition 4.4(2) we have

$$\mu_f(B_{r_f[A]}) = \mu_f(A) = \int_{\mathbf{R}^n} \chi_A(x) f(x)\, dx \le \int_{\mathbf{R}^n} \mathcal{R}_1[\chi_A](x) \mathcal{R}_1[f](x)\, dx$$
$$= \int_{\mathbf{R}^n} \chi_{B_{r_1[A]}}(x) f(x)\, dx = \mu_f(B_{r_1[A]}).$$

Then we see that $r_f[A] \le r_1[A]$, $B_{r_f[A]} \subset B_{r_1[A]}$. Noting that $\mu_1(A) = \mu_1(B_{r_1[A]})$, we have that it follows from the isometric inequality that

$$\int_{\partial B_{r_f[A]}} dH^{n-1} \le \int_{\partial B_{r_1[A]}} dH^{n-1} \le \int_{\partial A} dH^{n-1}.$$

(2) If $t \in (\|u\|_{L^\infty(\mathbf{R}^n)}, \infty)$, then the assertion clearly holds. Now we set

$$H_f[u] = \bigl\{t \in (0, \|u\|_{L^\infty(\mathbf{R}^n)}] \bigm| \mu_1\bigl(\{\mathcal{R}_f[u] = t\}\bigr) > 0\bigr\}.$$

Then we see that $\tilde{\mu}_1(H_f[u]) = 0$. Then for $t \in (0, \|u\|_{L^\infty(\mathbf{R}^n)}) \setminus H_f[u]$, we have

$$\bigl\{\mathcal{R}_f[u] = t\bigr\} = \partial\bigl\{\mathcal{R}_f[u] > t\bigr\} = \partial B_{r_f[\{|u|>t\}]}, \qquad \partial\bigl\{|u| > t\bigr\} = \bigl\{|u| = t\bigr\}.$$

Therefore the assertion follows from assertion (1). $\square$

In the subsequent argument we employ the following co-area formula (see, e.g., [Ma, Theorem 1.2.4]).

LEMMA A.4 (THE CO-AREA FORMULA)

*Let $1 \le p < \infty$. Assume that $u \in C_c(\mathbf{R}^n)$ is Lipschitz continuous, and assume that $g \ge 0$ a.e. on $\mathbf{R}^n$. Then we have*

$$\int_{\mathbf{R}^n} |\nabla u(x)|^p g(x)\,dx = \int_{-\infty}^{\infty} \int_{\{u=s\}} |\nabla u(x)|^{p-1} g(x)\,dH^{n-1}(x)\,ds.$$

From this we have the following proposition (cf. [CF, Lemma 3.1]).

PROPOSITION A.2

*Assume that $f$ is admissible, and assume that $u \in C_c(\mathbf{R}^n)$ is Lipschitz continuous. Then we have*

$$\mu_f[u](t) = h_f[u](t) + \int_t^{\infty} \int_{\{|u|=s\}\cap Z[u]} \frac{f(x)}{|\nabla u(x)|}\,dH^{n-1}(x)\,ds \quad \textit{for } t > 0.$$

*Proof*

For $t > 0$, setting $p = 1$ and

$$g_t(x) = \begin{cases} \frac{\chi_{\{|u|>t\}}(x)}{|\nabla u(x)|} f(x) & \text{for } x \in Z[u], \\ 0 & \text{for } x \in \mathbf{R}^n \setminus Z[u], \end{cases}$$

we apply Lemma A.4 to obtain

$$\begin{aligned} \mu_f\big(Z[u] \cap \{|u| > t\}\big) &= \int_{Z[u]} \chi_{\{|u|>t\}}(x) f(x)\,dx = \int_{\mathbf{R}^n} \big|\nabla[|u|](x)\big| g_t(x)\,dx \\ &= \int_{-\infty}^{\infty} \int_{\{|u|=s\}} g_t(x)\,dH^{n-1}(x)\,ds \\ &= \int_{-\infty}^{\infty} \int_{\{|u|=s\}\cap Z[u]} \frac{\chi_{\{|u|>t\}}(x)}{|\nabla u(x)|} f(x)\,dH^{n-1}(x)\,ds \\ &= \int_t^{\infty} \int_{\{|u|=s\}\cap Z[u]} \frac{f(x)}{|\nabla u(x)|}\,dH^{n-1}(x)\,ds. \end{aligned}$$

Therefore we see that

$$\begin{aligned} \mu_f[u](t) &= \mu_f\big((Z_0[u] \cup Z[u]) \cap \{|u| > t\}\big) \\ &= \mu_f\big(Z_0[u] \cap \{|u| > t\}\big) + \mu_f\big(Z[u] \cap \{|u| > t\}\big) \\ &= h_f[u](t) + \int_t^{\infty} \int_{\{|u|=s\}\cap Z[u]} \frac{f(x)}{|\nabla u(x)|}\,dH^{n-1}(x)\,ds. \end{aligned}$$ □

Using this we show the next proposition.

PROPOSITION A.3

*Assume that $f$ is admissible, and assume that $u \in C_c^1(\mathbf{R}^n)$. Then we have the following:*

Assertion (1)

$$\int_{\{\mathcal{R}_f[u]=t\}\cap Z[\mathcal{R}_f[u]]} \frac{f(x)}{\big|\nabla[\mathcal{R}_f[u]](x)\big|}\, dH^{n-1}(x) = \int_{\{|u|=t\}\cap Z[u]} \frac{f(x)}{|\nabla u(x)|}\, dH^{n-1}(x)$$

for a.e. $t > 0$.

Assertion (2)

$$\int_{\{\mathcal{R}_f[u]=t\}\cap Z[\mathcal{R}_f[u]]} \Big(\frac{\big|\nabla[\mathcal{R}_f[u]](x)\big|}{f(x)}\Big)^{p-1} dH^{n-1}(x)$$
$$\le \int_{\{|u|=t\}} \Big(\frac{|\nabla u(x)|}{f(x)}\Big)^{p-1} dH^{n-1}(x) \quad \textit{for a.e. } t > 0.$$

*Proof*
(1) By Proposition A.2, we have

$$\mu_f\big[\mathcal{R}_f[u]\big](t) = h_f\big[\mathcal{R}_f[u]\big](t) + \int_t^\infty \int_{\{\mathcal{R}_f[u]=s\}\cap Z[\mathcal{R}_f[u]]} \frac{f(x)}{\big|\nabla[\mathcal{R}_f[u]](x)\big|}\, dH^{n-1}(x)\, ds,$$
$$\mu_f[u](t) = h_f[u](t) + \int_t^\infty \int_{\{|u|=s\}\cap Z[u]} \frac{f(x)}{|\nabla u(x)|}\, dH^{n-1}(x)\, ds \quad \text{for } t > 0.$$

Then, by Proposition A.1, we have

$$\big[\mu_f[\mathcal{R}_f[u]]'\big]_{\mathrm{ac}}(t) = -\int_{\{\mathcal{R}_f[u]=t\}\cap Z[\mathcal{R}_f[u]]} \frac{f(x)}{\big|\nabla[\mathcal{R}_f[u]](x)\big|}\, dH^{n-1}(x),$$
$$\big[\mu_f[u]'\big]_{\mathrm{ac}}(t) = -\int_{\{|u|=t\}\cap Z[u]} \frac{f(x)}{|\nabla u(x)|}\, dH^{n-1}(x) \quad \text{for a.e. } t > 0.$$

Noting that $\mu_f[\mathcal{R}_f[u]] = \mu_f[u]$ holds by Proposition 4.3(1), we have that the assertion follows.
(2) By virtue of Sard's lemma, first we see that

$$\int_{\{|u|=t\}\cap Z_0[u]} dH^{n-1}(x) = 0 \quad \text{for a.e. } t > 0,$$

and then by Hölder's inequality we have

$$\Big(\int_{\{|u|=t\}\cap Z[u]} dH^{n-1}(x)\Big)^p$$
$$= \Big(\int_{\{|u|=t\}\cap Z[u]} \Big(\frac{f(x)}{|\nabla u(x)|}\Big)^{1/p'} \Big(\frac{|\nabla u(x)|}{f(x)}\Big)^{1/p'} dH^{n-1}(x)\Big)^p$$
$$\le \Big(\int_{\{|u|=t\}\cap Z[u]} \frac{f(x)}{|\nabla u(x)|} dH^{n-1}(x)\Big)^{p-1} \int_{\{|u|=t\}} \Big(\frac{|\nabla u(x)|}{f(x)}\Big)^{p-1} dH^{n-1}(x)$$

for $t > 0$.

Since $f, \mathcal{R}_f[u], \big|\nabla[\mathcal{R}_f[u]]\big|$ are radially symmetric, we have

$$\Big(\int_{\{\mathcal{R}_f[u]=t\}\cap Z[\mathcal{R}_f[u]]} dH^{n-1}(x)\Big)^p$$
$$= \Big(\int_{\{\mathcal{R}_f[u]=t\}\cap Z[\mathcal{R}_f[u]]} \frac{f(x)}{|\nabla[\mathcal{R}_f[u]](x)|}\, dH^{n-1}(x)\Big)^{p-1}$$
$$\times \int_{\{\mathcal{R}_f[u]=t\}\cap Z[\mathcal{R}_f[u]]} \Big(\frac{|\nabla[\mathcal{R}_f[u]](x)|}{f(x)}\Big)^{p-1} dH^{n-1}(x) \quad \text{for } t > 0.$$

Therefore, using assertions (1) and (2) of Lemma A.3, we have

$$\int_{\{\mathcal{R}_f[u]=t\}\cap Z[\mathcal{R}_f[u]]} \Big(\frac{|\nabla[\mathcal{R}_f[u]](x)|}{f(x)}\Big)^{p-1} dH^{n-1}(x)$$
$$\leq \frac{\big(\int_{\{\mathcal{R}_f[u]=t\}} dH^{n-1}(x)\big)^p}{\big(\int_{\{\mathcal{R}_f[u]=t\}\cap Z[\mathcal{R}_f[u]]} \frac{f(x)}{|\nabla[\mathcal{R}_f[u]](x)|}\, dH^{n-1}(x)\big)^{p-1}}$$
$$\leq \frac{\big(\int_{\{|u|=t\}\cap(Z[u]\cup Z_0[u])} dH^{n-1}(x)\big)^p}{\big(\int_{\{|u|=t\}\cap Z[u]} \frac{f(x)}{|\nabla u(x)|}\, dH^{n-1}(x)\big)^{p-1}}$$
$$= \frac{\big(\int_{\{|u|=t\}\cap Z[u]} dH^{n-1}(x)\big)^p}{\big(\int_{\{|u|=t\}\cap Z[u]} \frac{f(x)}{|\nabla u(x)|}\, dH^{n-1}(x)\big)^{p-1}}$$
$$\leq \int_{\{|u|=t\}} \Big(\frac{|\nabla u(x)|}{f(x)}\Big)^{p-1} dH^{n-1}(x) \quad \text{for a.e. } t \in (0, \|u\|_{L^\infty(\mathbf{R}^n)}).$$

Therefore the assertion follows. □

After all this, Proposition 4.5 follows from Lemma A.4 and Proposition A.3(2).

*Proof of Proposition 4.5*
By Lemma A.4 and Proposition A.3(2), we have

$$\int_{\mathbf{R}^n} \big|\nabla\big[\mathcal{R}_f[u]\big](x)\big|^p \frac{1}{f(x)^{p-1}}\, dx$$
$$= \int_{-\infty}^{\infty}\int_{\{\mathcal{R}_f[u]=t\}} \big|\nabla\big[\mathcal{R}_f[u]\big](x)\big|^{p-1} \frac{1}{f(x)^{p-1}}\, dH^{n-1}(x)\, dt$$
$$= \int_0^{\infty}\int_{\{\mathcal{R}_f[u]=t\}\cap Z[\mathcal{R}_f[u]]} \Big(\frac{|\nabla[\mathcal{R}_f[u]](x)|}{f(x)}\Big)^{p-1} dH^{n-1}(x)\, dt$$
$$\leq \int_0^{\infty}\int_{\{|u|=t\}} \Big(\frac{|\nabla u(x)|}{f(x)}\Big)^{p-1} dH^{n-1}(x)\, dt$$
$$= \int_{-\infty}^{\infty}\int_{\{|u|=t\}} |\nabla u(x)|^{p-1} \frac{1}{f(x)^{p-1}}\, dH^{n-1}(x)\, dt$$
$$= \int_{\mathbf{R}^n} |\nabla u(x)|^p \frac{1}{f(x)^{p-1}}\, dx.$$
□

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## References

[ACP] B. Abdellaoui, E. Colorado, and I. Peral, *Some improved Caffarelli–Kohn–Nirenberg inequalities*, Calc. Var. Partial Differential Equations **23** (2005), 327–345.

[Ad] R. Adams, *Weighted nonlinear potential theory*, Trans. Amer. Math. Soc. **297** (1986), 73–94.

[ANC] N. Adimurthi, M. Chaudhuri, and M. Ramaswamy, *An improved Hardy-Sobolev inequality and its application*, Proc. Amer. Math. Soc. **130** (2001), 489–505.

[AH1] H. Ando and T. Horiuchi, *Missing terms in the weighted Hardy–Sobolev inequalities and its application*, Kyoto J. Math. **52** (2012), 759–796.

[AH2] ———, *Weighted Hardy inequalities with infinitely many sharp missing terms*, to appear.

[BL] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[CKN] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compos. Math. **53** (1984), 259–275.

[CW1] F. Catrina and Z.-Q. Wang, *On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math. **54** (2001), 229–258.

[CW2] ———, *Positive bound states having prescribed symmetry for a class of nonlinear elliptic equations in* $\mathbf{R}^n$, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), 157–178.

[CF] A. Cianchi and N. Fusco, *Functions of bounded variation and rearrangements*, Arch. Ration. Mech. Anal. **165** (2002), 1–40.

[DHA1] A. Detalla, T. Horiuchi, and H. Ando, *Missing terms in Hardy–Sobolev inequalities*, Proc. Japan Acad. Ser. A Math. Sci. **80** (2004), 160–165.

[DHA2] ———, *Missing terms in Hardy–Sobolev inequalities and its application*, Far East J. Math. Sci. **14** (2004), 333–359.

[DHA3] ———, *Sharp remainder terms of Hardy–Sobolev inequalities*, Math. J. Ibaraki Univ. **37** (2005), 39–52.

[GPP] J. Garcia-Azoreto, I. Peral, and A. Primo, *A borderline case in elliptic problems involving weights of Caffarelli–Kohn–Nirenberg type*, Nonlinear Anal. **67** (2007), 1878–1894.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.

[Ho1] T. Horiuchi, *The imbedding theorems for weighted Sobolev spaces*, J. Math. Kyoto **29** (1989), 365–403.

[Ho2] ______, *Best constant in weighted Sobolev inequality with weights being powers of distance from the origin*, J. Inequal. Appl. **1** (1997), 275–292.

[Ho3] ______, *Missing terms in generalized Hardy's inequalities and its application*, J. Math. Kyoto **43** (2003), 235–260.

[LL] E. H. Lieb and M. Loss, *Analysis*, Amer. Math. Soc., Providence, 2001.

[Li1] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1*, Ann. Inst. Henri Poincaré **1** (1984), 109–145; *part 2*, 223–284.

[Li2] ______, *The concentration-compactness principle in the calculus of variations. The limit case, part 1*, Rev. Mat. Iberoam. **1** (1985), 145–201; *part 2*, 45–121.

[Ma] V. G. Maz'ja, *Sobolev Spaces*, Springer, Berlin, 1985.

[Ta1] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.

[Ta2] ______, *Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl. (4) **120** (1979), 160–184.

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