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A new generalization of the Lelong number

Aron Lagerberg

Abstract. We will introduce a quantity which measures the singularity of a plurisubharmonic function φ relative to another plurisubharmonic function ψ, at a point a. We denote this quantity by $\nu_{a,\psi}(\varphi)$. It can be seen as a generalization of the classical Lelong number in a natural way: if $\psi=(n-1)\log|\cdot-a|$, where n is the dimension of the set where φ is defined, then $\nu_{a,\psi}(\varphi)$ coincides with the classical Lelong number of φ at the point a. The main theorem of this article says that the upper level sets of our generalized Lelong number, i.e. the sets of the form $\{z:\nu_{z,\psi}(\varphi)\geq c\}$ where $c>0$, are in fact analytic sets, provided that the *weight* ψ satisfies some additional conditions.

1. Introduction

In what follows, we let Ω denote an open subset of $\mathbb{C}^n$, φ a plurisubharmonic function in Ω, and ψ a plurisubharmonic function in $\mathbb{C}^n$. When we are dealing with constants, we often let the same symbol denote different values when the explicit value does not concern us. The object of this paper is to introduce a generalization of the classical Lelong number: The quantity we will consider depends on two plurisubharmonic functions φ and ψ and it will be a measurement of the singularity of φ relative to ψ. Moreover, if we let $\psi(z)=(n-1)\log|z-a|$ we get back the classical Lelong number of φ at the point a. The main theorem of this paper (Theorem 3.9) tells us that this generalized Lelong number satisfies a semicontinuity property of the same type as the classical Lelong number does, namely, its super level sets define analytic varieties. Also, we investigate some further properties that this quantity satisfies. The paper is organized as follows: in this introduction we define the generalized Lelong number and discuss the motivation behind it. In Section 2 we explore some basic properties and examples of the generalized Lelong number, obtaining as corollaries classical results concerning the classical Lelong number. Section 3 concerns the theorem stating that the upper level sets of the generalized Lelong number defines an analytic set. In Section 4 we prove a theorem due to Demailly, which states that one can approximate plurisubharmonic functions well with Bergman functions with respect to a certain weight.

Let us begin by recalling some relevant definitions. For $r>0$ define

$$\nu_a(\varphi,r):=\frac{\sup_{|z-a|=r}\varphi(z)}{\log r}. \tag{1.1}$$

The function in the numerator can actually be seen to be a convex function of $\log r$ (cf. [7]). Furthermore, the fraction is increasing in r, and so the limit as r tends to 0 exists.

Definition 1.1. The (classical) *Lelong number* of φ at $a\in\Omega$ is defined as

$$\nu_a(\varphi)=\lim_{r\to 0}\nu_a(\varphi,r). \tag{1.2}$$

As can be seen from the definition, the Lelong number compares the behaviour of φ to that of $\log|z-a|$, as $z\to a$. In fact (cf. [6]), the following is true: if $\nu_a(\varphi)=\tau$ then, near the point a,

$$\varphi(z)\le\tau\log|z-a|+O(1), \tag{1.3}$$

and τ is the best constant possible.

Two other ways to represent the classical Lelong number are given by the equalities

$$\nu_a(\varphi)=\liminf_{z\to a}\frac{\varphi(z)}{\log|z-a|}$$

and

$$\nu_a(\varphi)=\lim_{r\to 0}\int_{B(a,r)}(dd^c\varphi(z))\wedge(dd^c\log|z-a|^2)^{n-1},$$

where $B(a,r)=\{z:|z-a|<r\}$. The first of these equalities is a simple consequence of (1.1) (cf. [8]), while the other follows from Stokes' theorem (cf. [3]). Two generalizations of the Lelong number, due to Rashkovskii and Demailly respectively, come from exchanging $\log|z-a|$ for a different plurisubharmonic function ψ in the characterizations of the Lelong number above (cf. [8] and [3], respectively). To that effect, the relative type of φ with respect to a function ψ is given by

$$\sigma_a(\varphi,\psi)=\liminf_{z\to a}\frac{\varphi(z)}{\psi(z-a)}, \tag{1.4}$$

and *Demailly's generalized Lelong number* of φ with respect to ψ is given by

$$\nu_{\text{Demailly}}(\varphi,\psi)=\lim_{r\to 0}\int_{\psi<\log r}(dd^c\varphi)\wedge(dd^c\psi)^{n-1}. \tag{1.5}$$

Of course, in each of the definitions above, ψ needs to satisfy some regularity conditions; for our discussion it suffices to assume that e^{ψ} is continuous and $\{z:\psi(z)=-\infty\}=\{0\}$.

The inspiration for this article comes from an observation made by Berndtsson, who in the article [2] related the classical Lelong number to the convergence of a certain integral.

Theorem 1.2. *For* $a\in\Omega$,

$$\nu_a(\varphi)\geq 1 \quad\Longleftrightarrow\quad \int_a e^{-2\varphi(\zeta)-2(n-1)\log|\zeta-a|}\,d\lambda(\zeta)=\infty,$$

where λ *is Lebesgue measure on* $\mathbb{C}^n$.

We will give a simplified proof of this theorem in Section 2. Using this result, assuming $\nu_a(\varphi)=\tau$ so that $\nu_a(\varphi/\tau)=1$, we see that

$$\int_a e^{-2\varphi(\zeta)/s-2(n-1)\log|\zeta-a|}\,d\lambda(\zeta)=\int_a e^{-2\tau\varphi(\zeta)/s\tau-2(n-1)\log|\zeta-a|}\,d\lambda(\zeta)$$

is finite if and only if $s>\tau$. Thus Theorem 1.2 implies that the classical Lelong number coincides with the number given by

$$\inf\{s>0:\zeta\longmapsto e^{-2\varphi(\zeta)/s-2(n-1)\log|\zeta-a|}\in L^1_{\mathrm{loc}}(a)\}.$$

In [2] the following generalization of the classical Lelong number is indicated.

Definition 1.3. The *generalized Lelong number of* φ at $a\in\Omega$ with respect to a plurisubharmonic function ψ, is defined as

$$\nu_{a,\psi}(\varphi)=\inf\{s>0:\zeta\longmapsto e^{-2\varphi(\zeta)/s-2\psi(\zeta-a)}\in L^1_{\mathrm{loc}}(a)\}.$$

Obviously, some condition regarding the integrability of $e^{-2\psi}$ is needed for the definition to provide us with something of interest; for our purpose, it is sufficient to assume that

$$e^{-2(1+\tau)\psi}\in L^1_{\mathrm{loc}}(0) \tag{1.6}$$

for some $\tau>0$. We single out the following special case of the generalized Lelong number.

Definition 1.4. For $t\in[0,n)$ we define

$$\nu_{a,t}(\varphi)=\inf\{s>0:\zeta\longmapsto e^{-2\varphi(\zeta)/s-2t\log|\zeta-a|}\in L^1_{\mathrm{loc}}(a)\},$$

that is, $\nu_{a,t}(\varphi):=\nu_{a,\psi}(\varphi)$, with $\psi=t\log|\cdot|$.

Theorem 1.2 shows that $\nu_{a,n-1}$ equals the classical Lelong number, which we will denote by just ν_a. For $t=0$, $\nu_{a,t}=\nu_{a,0}$ equals another well-known quantity, the so called *integrability index of* φ *at* a. Thus $\nu_{a,t}$ can be regarded as a family of numbers which interpolate between the classical Lelong number and the integrability index of φ, as t ranges between 0 and $n-1$. One should put this in context with the following important inequality, due to Skoda (cf. [10]).

Theorem 1.5. (Skoda's inequality) *For* $\varphi\in\mathrm{PSH}(\Omega)$,

$$\nu_{z,0}(\varphi)\leq\nu_{z,n-1}(\varphi)\leq n\nu_{z,0}(\varphi). \tag{1.7}$$

Later, we will prove the following generalization of Skoda's inequality:

$$\nu_{z,0}(\varphi)\leq\nu_{z,n-1}(\varphi)\leq(n-t)\nu_{z,t}(\varphi)\leq n\nu_{z,0}(\varphi)\quad\text{for } 0\leq t\leq n-1.$$

Remark 1.6. Observe that when $n=1$ the Lelong number and integrability index of a function coincide. This follows from, for instance, Theorem 1.2.

A well-known important result concerning classical Lelong numbers, due to Siu (cf. [9]), is the following.

Theorem 1.7. *The sets* $\{z\in\Omega:\nu_z(\varphi)\geq\tau\}$ *are analytic subsets of* Ω *for* $\tau>0$.

In fact both the relative type and Demailly's generalized Lelong number defined above, enjoy similar analyticity properties, provided that $e^{2\psi}$ is Hölder continuous. A natural question arises: does an equivalent statement to Siu's analyticity theorem hold for Berndtsson's generalized Lelong number? More precisely, are the sets

$$\{z\in\Omega:\nu_{z,\psi}(\varphi)\geq\tau\}$$

analytic for $\tau>0$? In the case where the weight $e^{2\psi}$ is Hölder continuous the affirmative answer is the content of Theorem 3.9. The main idea of the proof is due to Kiselman (cf. [6]) and consists of his technique of "attenuating the singularities of φ". However, this is here done in a different manner than in [6], following results from [2]. Attenuating the singularities means that we construct a plurisubharmonic function Ψ satisfying the following properties: if the generalized Lelong number of φ is large then its classical Lelong number is positive, and if the generalized Lelong number of φ is small then its classical Lelong number vanishes. Using this function we can then realize the set $\{z\in\Omega:\nu_{z,\psi}(\varphi)\geq\tau\}$ as an intersection of analytic sets, which by basic properties of analytic sets is analytic.

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2. Properties and examples

We begin with listing some properties which the generalized Lelong number satisfies.

Lemma 2.1. *Let $\varphi, \varphi' \in \mathrm{PSH}(\Omega)$, and assume that ψ satisfies* (1.6). *Then the following are true*:

(1) *For $c>0$, $\nu_{a,\psi}(c\varphi)=c\nu_{a,\psi}(\varphi)$;*

(2) *If $\varphi\le\varphi'$ on some neighbourhood of $a\in\Omega$, then $\nu_{a,\psi}(\varphi)\ge\nu_{a,\psi}(\varphi')$;*

(3) $\nu_{a,\psi}(\max\{\varphi,\varphi'\})\ge\min\{\nu_{a,\psi}(\varphi),\nu_{a,\psi}(\varphi')\}$;

(4) $\nu_{a,\psi}(\varphi+\varphi')\le\nu_{a,\psi}(\varphi)+\nu_{a,\psi}(\varphi')$;

(5) *Assume that φ satisfies $(dd^c\varphi)^n=0$ on a punctured neighbourhood of a and that $\nu_{a,0}(\varphi)\le\sigma_a(\varphi,\psi):=\sigma$, where σ_a denotes the relative type as defined by* (1.4). *Then,*

$$\nu_{a,\psi}(\varphi)\le\frac{\nu_{a,0}(\varphi)}{1-\nu_{a,0}(\varphi)/\sigma}.$$

If $\nu_{a,0}(\varphi)>\sigma_a(\varphi,\psi)$ then $\nu_{a,\psi}(\varphi)=0$.

Proof. The first properties, (1), (2) and (3), are immediate consequences of the definition.

(4) This is a simple application of Hölder's inequality.

(5) In [8] it is deduced that, under the assumptions on φ,

$$\varphi(z)\le\sigma_a(\varphi,\psi)\psi(z)+O(1),$$

as $z\to a$ (cf. (1.3)). Thus, if we choose $r>0$ small enough, and $\sigma\le\nu_{a,0}(\varphi)$,

$$\int_{B(a,r)} e^{-2\varphi(\zeta)/s-2\psi(\zeta-a)}\,d\lambda(\zeta)\le C\int_{B(a,r)} e^{-2\varphi(\zeta)/s-2\varphi(\zeta)/\sigma}\,d\lambda(\zeta)$$

which is finite if (remember that $\nu_{a,0}$ denotes the integrability index)

$$\frac{1}{s}+\frac{1}{\sigma}<\frac{1}{\nu_{a,0}(\varphi)},$$

i.e. if

$$s>\frac{\nu_{a,0}(\varphi)}{1-\nu_{a,0}(\varphi)/\sigma}.$$

Thus we obtain

$$\nu(\varphi,\psi) \le \frac{\nu_{a,0}(\varphi)}{1-\nu_{a,0}(\varphi)/\sigma}.$$

On the other hand, it is evident that if $\sigma>\nu_{a,0}(\varphi)$ then the integral above will always be infinite, whence $\nu(\varphi,\psi)=0$. $\square$

We proceed by listing properties which the special case $\nu_{z,t\psi}$ satisfies.

Lemma 2.2. *For φ,ψ plurisubharmonic we have*:

(1) *The function*

$$t \longmapsto \frac{1}{\nu_{z,t\psi}(\varphi)}$$

is concave while

$$t \longmapsto \nu_{z,t\psi}(\varphi)$$

is convex, both for $t\in[0,\nu_{z,0}(\psi)]$.

(2) *The function*

$$t \longmapsto \Big(\frac{1}{\nu_{z,0}(\psi)} - t\Big)\nu_{z,t\psi}(\varphi)$$

is decreasing for $t\in[0,1/\nu_{z,0}(\psi)]$.

(3) *The following inequalities hold*:

$$\nu_{z,0}(\varphi) \le \nu_{z,n-1}(\varphi) \le (n-t)\nu_{z,t}(\varphi) \le n\nu_{z,0}(\varphi) \quad \textit{for } 0\le t\le n-1. \tag{2.1}$$

Proof. (1) Assume that $z=0$ and put

$$f(t) = \frac{1}{\nu_{0,t}(\varphi)} = \sup\{s>0 : e^{-s2\varphi-2t\psi} \in L^1_{\mathrm{loc}}(0)\}.$$

By the definition of concavity, we need to show that for every $a,b\in[0,1/\nu_{0,0}(\psi)]$ and $\tau\in(0,1)$ the inequality

$$f(\tau a+(1-\tau)b) \ge \tau f(a)+(1-\tau)f(b)$$

holds. Applying Hölder's inequality once again, with $p=1/\tau$ and $q=1/(1-\tau)$, we see that

$$\int_0 e^{-2(\tau f(a)+(1-\tau)f(b))\varphi-2(\tau a+(1-\tau)b)\psi}\,d\lambda \\ \le \Big(\int_0 e^{-2f(a)\varphi-2a\psi}\,d\lambda\Big)^{\tau}\Big(\int_0 e^{-2f(b)\varphi-2b\psi}\,d\lambda\Big)^{1-\tau},$$

which implies that $f(\tau a+(1-\tau)b)\geq\tau f(a)+(1-\tau)f(b)$. Thus f is a concave function of t for $t\in[0,1/\nu_{0,0}(\psi)]$. The exact same calculations with $f(t)=\nu_{0,t\psi}(\varphi)$ give convexity of $t\mapsto\nu_{z,t\psi}(\varphi)$. Note however that this statement is weaker than saying that $t\mapsto 1/\nu_{z,t\psi}(\varphi)$ is concave.

(2) One can show that if $f(t)\geq 0$ is a concave function with $f(0)=0$, then $t\mapsto f/t$ is decreasing. Since $t\mapsto 1/\nu_{0,(1/\nu_{0,0}(\psi)-t)\psi}(\varphi)$ is concave for $t\in[0,1/\nu_{0,0}(\psi)]$ by property (1) and is equal to 0 for $t=0$ by the condition on ψ, we see that

$$\frac{1}{t\nu_{0,(1/\nu_{0,0}(\psi)-t)\psi}(\varphi)}$$

is decreasing in t for $t\in[0,1/\nu_{0,0}(\psi)]$. This implies that $t\mapsto(1/\nu_{0,0}(\psi)-t)\nu_{0,t\psi}(\varphi)$ is a decreasing function on $[0,1/\nu_{0,0}(\psi)]$.

(3) If we accept Skoda's inequality (1.7), the only new information is the inequality

$$\nu_{0,n-1}(\varphi)\leq(n-t)\nu_{0,t}(\varphi)\leq n\nu_{0,0}(\varphi),\quad 0\leq t\leq n-1,$$

which follows immediately from property (2) with $\psi=\log|\cdot|$, that is, the fact that $t\mapsto(n-t)\nu_{0,t}(\varphi)$ is decreasing in t. $\square$

Remark 2.3. The proof of property (1) in Lemma 2.2 can easily be adapted to show that something stronger holds: the function

$$\psi\longmapsto\frac{1}{\nu_{z,\psi}(\varphi)}$$

is concave on the set of plurisubharmonic functions ψ.

We proceed by calculating two special cases of the generalized Lelong number, which will give us some insight into what it measures.

Example 2.4. We calculate $\nu_{0,t}(\varphi)$, where $\varphi(z)=\frac{1}{2}\log|z_1\bar z_1+...+z_k\bar z_k|$, $z=(z_1,...,z_n)$, and k lies between 1 and n. Thus we want to decide for which $s>0$ the following integral goes from being finite into being infinite:

$$\int_\Delta\frac{d\lambda}{|z_1\bar z_1+...+z_k\bar z_k|^{1/s}|z|^{2t}},$$

where Δ is some arbitrarily small polydisc containing the origin. In this integral we put $z''=(z_{k+1},...,z_n)$ and introduce polar coordinates with respect to $z'=(z_1,...,z_k)$ to obtain that it is equal to

$$C\int_{\Delta''}\int_0^1\frac{R^{2k-1-2/s}}{\left|R^2+|z''|^2\right|^t}\,dR\,d\lambda_{n-k}(z''),\tag{2.2}$$

where $\Delta''=\{(z_{k+1},...,z_n):z\in\Delta\}$, λ_k is k-dimensional Lebesgue measure and C is some constant depending only on the dimension. (By dimension we always mean complex dimension, unless otherwise stated.) This integral is easily seen to be finite if and only if $2k-2/s>0$ and $2k-2t-2/s>2k-2n$. In other words

$$\nu_{0,t}\left(\tfrac{1}{2}\log|z_1\bar{z}_1+...+z_k\bar{z}_k|\right)=\max\left\{\frac{1}{k},\frac{1}{n-t}\right\}.$$

This example shows that when we look at sets of the type $\{z:z_1=...=z_k=0\}$ in $\mathbb{C}^n$, where $z=(z_1,...,z_n)$, the generalized Lelong number, as a function of t, thus senses the (co)dimension of the set: it is constant, and equal to the integrability index of $\frac{1}{2}\log|z_1\bar{z}_1+...+z_k\bar{z}_k|$, when t is so small so that $n-t$ is larger than k – the codimension of the set – and then grows linearly to 1, which is the Lelong number of $\frac{1}{2}\log|z_1\bar{z}_1+...+z_k\bar{z}_k|$.

Example 2.5. A similar calculation to that of Example 2.4 reveals that

$$\nu_{0,t}(\log|z_1^{\alpha_1}...z_k^{\alpha_k}|)=\max\left\{\alpha_1,...,\alpha_k,\frac{\sum_{i=1}^k\alpha_i}{n-t}\right\}\quad\text{for } t\in[0,n).$$

Thus, when considering sets of the form $\{z:z_1...z_k=0\}$, which is the union of the k coordinate planes $\{z:z_k=0\}$ (corresponding to the function $\varphi=\log|z_1...z_k|$), the generalized Lelong number senses how many coordinate planes the union is taken over.

Remark 2.6. These two examples show us that the two rightmost inequalities in (2.1) are sharp. More precisely: using φ from Example 2.4, we see that if $t=n-k$,

$$\nu_{0,n-1}(\varphi)=1=(n-t)\nu_{0,t}(\varphi).$$

However, if $\varphi(z)=\log|z_1...z_n|$, we see by Example 2.5 that

$$(n-t)\nu_{0,t}(\varphi)=n=n\nu_{0,0}(\varphi)\quad\text{for every } t\in[0,n).$$

Using our generalized Lelong number it is now easy to obtain a classical result due to Siu (cf. [9]), namely that $\nu_{a,n-1}$ is invariant under biholomorphic coordinate changes. This will be a corollary of the following proposition.

Proposition 2.7. *If* $f\colon\Omega\to\Omega$ *is biholomorphic,* $f(0)=0$ *and* $\det f'(0)\neq0$, *then*

$$\nu_{0,t}(\varphi\circ f)=\nu_{0,t}(\varphi).$$

Proof. By a change of coordinates we obtain

$$\int_0 e^{-2\varphi\circ f(\zeta)/s-2t\log|\zeta|/s}\,d\lambda(\zeta)=\int_0 e^{-2\varphi(z)/s-2t\log|f^{-1}(z)|}|\det f'(z)|^{-1}\,d\lambda(z),$$

which by the assumptions on f is finite if and only if

$$\int_0 e^{-2\varphi(\zeta)/s-2t\log|\zeta|}\,d\lambda(\zeta)<\infty. \qquad \square$$

Since for $t=n-1$ we get the classical Lelong number, we obtain the following corollary.

Corollary 2.8. *The classical Lelong number is invariant under biholomorphic changes of coordinates.*

Example 2.9. Let $V\subset\Omega$ be a variety and pick a point $x\in V$ where V is smooth. We can then find a neighbourhood U of x and $f_1,...,f_k\in\mathcal{O}(U)$ such that

$$V\cap U=\{z:f_1(z)=...=f_k(z)=0\}.$$

Since V was smooth at x, we can change coordinates via a function $g\colon U\to U$ such that in these new coordinates

$$V\cap U=\{z:z_1=...=z_l=0\}$$

for some $l\le k$. This means that $f_i\circ g=z_i$ for $1\le i\le l$ and $f_i\circ g=0$ for $i\ge l$. By Proposition 2.7 we have that

$$\nu_{x,t}\left(\sum_{i=1}^{k}|f_i|^2\right)=\nu_{x,t}\left(\sum_{i=1}^{l}|z_i|^2\right),$$

and thus, by Example 2.4 we see that $\nu_{x,t}\left(\sum_{i=1}^{k}|f_i|^2\right)$ senses the codimension of V at x.

When considering the generalized Lelong number, our next technical lemma shows that we can "move" parts of the singularity from the plurisubharmonic function to the weight, provided the singularity is sufficiently large.

Lemma 2.10. *Let $\delta>0$, and let ψ be a plurisubharmonic function such that $e^{-2(1+\tau)\psi}\in L^1_{\mathrm{loc}}(0)$ for some $\tau>0$. If $\nu_{a,\psi}(\varphi)=1+\delta$, then with $0<\varepsilon<\tau\delta$ we have that*

$$\int_a e^{-2\varphi(\zeta)-2(1-\varepsilon)\psi(\zeta-a)}\,d\lambda(\zeta)=\infty.$$

Proof. The hypothesis implies that for every neighbourhood U of a,

$$\int_U e^{-2\varphi(\zeta)/(1+\delta')-2\psi(\zeta-a)}\,d\lambda(\zeta)=\infty,$$

when $\delta'<\delta$. The function $\zeta\mapsto e^{-2(1-\varepsilon)\psi(\zeta-a)}$ is locally integrable around a, and we apply Hölder's inequality with respect to the measure $e^{-2(1-\varepsilon)\psi(\zeta-a)}\,d\lambda(\zeta)$ on U, with $p=1+\delta'$ and $q=(1+\delta')/\delta'$, to obtain

$$\begin{aligned}\int_U & e^{-2\varphi(\zeta)/(1+\delta')-2\psi(\zeta-a)}\,d\lambda(\zeta)\\ &=\int_U e^{-2\varphi(\zeta)/(1+\delta')}e^{-2(1-\varepsilon)\psi(\zeta-a)}e^{-2\varepsilon\psi(\zeta-a)}\,d\lambda(\zeta)\\ &\leq\left(\int_U e^{-2\varphi(\zeta)}e^{-2(1-\varepsilon)\psi(\zeta-a)}\,d\lambda(\zeta)\right)^{1/p}\left(\int_U e^{-2(\varepsilon q+1-\varepsilon)\psi(\zeta-a)}\,d\lambda(\zeta)\right)^{1/q}.\end{aligned}$$

Since the left-hand side is infinite by hypothesis, and the second integral on the right-hand side converges (after possibly shrinking U, since $\varepsilon q+1-\varepsilon\leq 1+\tau$, if $\varepsilon<\delta'\tau$), we see that

$$\left(\int_U e^{-2\varphi(\zeta)-2(1-\varepsilon)\psi(\zeta-a)}\,d\lambda(\zeta)\right)^{1/p}=\infty.$$

This implies the desired conclusion, as δ' can be choose arbitrarily close to δ. □

We will now give a proof of the Skoda inequality (1.7), based on the Ohsawa–Takegoshi extension theorem. We will also use the same technique to give a simple proof of Theorem 1.2. We begin by recalling the statement of the Ohsawa–Takegoshi theorem (see e.g. [1]):

Assume that V is a smooth hypersurface in $\mathbb{C}^n$ which in local coordinates can be written as $V=\{z:z_n=0\}$, and let U be a neighbourhood in $\mathbb{C}^n$ whose intersection with V is non-empty. We also assume that φ is such that $\int_V e^{-2\varphi}\,d\lambda<\infty$. Then, if $h_0\in\mathcal{O}(V\cap U)$, there exists an $h\in\mathcal{O}(U)$ with $h=h_0$ on V which satisfies the estimate

$$\int_U \frac{|h|^2e^{-2\varphi}}{|z_n|^{2-2\delta}}\,d\lambda\leq C_\delta\int_{U\cap V}|h_0|^2e^{-2\varphi}\,d\lambda, \tag{2.3}$$

for $0<\delta<1$ and some constant C_δ depending only on U, δ and φ.

The hard part of Skoda's inequality, and the part we will show, is the implication $\nu_{z,n-1}(\varphi)<1\Rightarrow\nu_{z,0}(\varphi)<1$. We record the core of the argument as a lemma (cf. [4], Proposition 2.2).

Lemma 2.11. *Let $\varphi\in\mathrm{PSH}(\Omega)$ and let $x\in\Omega$. Assume there exists a complex line L through x for which*

$$\int_{L\cap\Omega} e^{-2\varphi}\,d\lambda_1<\infty,$$

then there exists a neighbourhood $\omega\subset\Omega$ of x for which

$$\int_{\omega} e^{-2\varphi}\,d\lambda<\infty.$$

That is, in order to prove that $e^{-2\varphi}$ is locally integrable at a point, we need only to find a complex line where the statement holds.

Proof. It suffices, of course, to prove this for $x=0$. Assume that L is a complex line through the origin for which $\int_{L\cap\Omega} e^{-2\varphi}\,d\lambda_1<\infty$. Applying the Ohsawa–Takegoshi extension theorem inductively, we can extend the function

$$1\in L^2(L\cap\Omega, e^{-\varphi})\cap\mathcal{O}(L\cap\Omega)$$

to a function $h\in L^2(\Omega, e^{-\varphi})\cap\mathcal{O}(\Omega)$, where Ω is a neighbourhood in $\mathbb{C}^n$, with a bound on the L^2 norm:

$$\int_{\Omega} |h|^2 e^{-2\varphi}\,d\lambda\le C\int_{\Omega\cap L} e^{-2\varphi}\,d\lambda_1<\infty$$

for some constant C. This inequality is obtained from (2.3) by just discarding the denominator figuring in the left-hand-side integral. Since h is equal to 1 on L, the quantity $|h|^2$ is comparable to 1 in a neighbourhood ω of the origin. Thus we obtain

$$\int_{\omega} e^{-2\varphi}\,d\lambda<\infty$$

which is what we aimed for. $\square$

Proof of Skoda's inequality. Remember, we want to prove the implication $\nu_{z,n-1}(\varphi)<1\Rightarrow\nu_{z,0}(\varphi)<1$. To that effect, assume that $\nu_{z,n-1}(\varphi)<1$. It is well known that the Lelong number (at the origin) of a function φ is equal to the Lelong number of the same function restricted to a generic complex line passing through the origin (we will prove this later, see Lemma 2.16), which coincides with the integrability index on that line. Thus we can find a complex line L for which $\int_{\Omega\cap L} e^{-2\varphi}\,d\lambda_1<\infty$ and so by Lemma 2.11 we see that $\nu_{z,0}(\varphi)<1$. $\square$

One might hope that knowledge of the dimension of the set where $\nu_{z,n-1}(\varphi)\ge c$ would enable us to sharpen the estimate of Skoda's inequality. The following example shows that unfortunately this information is not sufficient to succeed.

Example 2.12. Let $n=2$ and $\varphi(z_1,z_2)=\log(|z_1|^2+|z_2|^{2a})$. Then one calculates:
– $\nu_{0,n-1}(\varphi)=2$,
– $\nu_{0,0}(\varphi)=2/(1+1/a)$,
– $\{z:\nu_{z,n-1}(\varphi)\geq 1\}=\{0\}$.
This is the best scenario possible: the dimension of the upper-level set of the Lelong number is 0 and *still* the lower bound of the Skoda inequality is sharp, which one realizes by letting $a\to\infty$.

Let us see how we can apply the full strength of the estimate (2.3) of the Ohsawa–Takegoshi theorem to obtain a proof of Theorem 1.2. We recall the statement of Theorem 1.2:

$$\nu_a(\varphi)\geq 1 \quad\Longleftrightarrow\quad \int_a e^{-2\varphi(\zeta)-2(n-1)\log|\zeta-a|}\,d\lambda(\zeta)=\infty.$$

Proof of Theorem 1.2. Assume that $a=0$, let Ω be a small neighbourhood of the origin in $\mathbb{C}^n$, and let $\varphi\in\mathrm{PSH}(\Omega)$ satisfy $\nu_0(\varphi)<1$. Then we know that the restriction of $e^{-2\varphi}$ to a generic complex line is integrable. However, since a rotation of φ will not effect φ's integrability properties in $\mathbb{C}^n$, we may assume that the line is given by $\{z:z_2=...=z_n=0\}$. In fact we can assume that φ is integrable along every coordinate axis. Thus φ satisfies

$$\int_{\{z:z_2=...=z_n=0\}\cap\Omega} e^{-2\varphi}\,d\lambda_1<\infty. \tag{2.4}$$

We want to prove that

$$\int_0 e^{-2\varphi(\zeta)-2(n-1)\log|\zeta|}\,d\lambda(\zeta)<\infty.$$

If we consider the constant function 1 as an element of $\mathcal{O}(\mathbb{C}\cap\Omega)$ then, by the argument above, we obtain a function $h\in\mathcal{O}(\mathbb{C}^2\cap\Omega)$, comparable to 1 in Ω, and thus acquiring the inequality

$$\int_{\mathbb{C}^2\cap\Omega'}\frac{e^{-2\varphi}}{|z_1|^{2-2\delta}}\,d\lambda_2(z)\leq C_\delta\int_{\mathbb{C}\cap\Omega}e^{-2\varphi}d\lambda_1<\infty,$$

with $\Omega'\subset\Omega$. Since $0<\delta<1$ the function $\varphi+(1-\delta)\log|z_1|$ is plurisubharmonic in Ω. Thus we can repeat the argument with φ exchanged for $\varphi+(1-\delta)\log|z_1|$ to obtain

$$\int_{\mathbb{C}^3\cap\Omega''}\frac{e^{-2\varphi}}{|z_1|^{2-2\delta}|z_2|^{2-2\delta}}\,d\lambda_3(z)\leq C_\delta\int_{\mathbb{C}^2\cap\Omega}\frac{e^{-2\varphi}}{|z_1|^{2-2\delta}}\,d\lambda_2(z)<\infty,$$

with $\Omega''\subset\Omega'$. Iterating this procedure it is easy to realize that, after possibly shrinking Ω, we obtain the inequality
(2.5)

$$\int_\Omega \frac{e^{-2\varphi}}{|z_1|^{2-2\delta}...|z_{n-1}|^{2-2\delta}}\,d\lambda(z)\le C_\delta \int_{\mathbb{C}^{n-1}\cap\Omega} \frac{e^{-2\varphi}}{|z_1|^{2-2\delta}...|z_{n-2}|^{2-2\delta}}\,d\lambda_{n-1}(z)<\infty.$$

Using the trivial estimate

$$\int_\Omega \frac{e^{-2\varphi}}{|z|^{2(n-1)(1-\delta)}}\,d\lambda(z)\le \int_\Omega \frac{e^{-2\varphi}}{|z_1|^{2-2\delta}...|z_{n-1}|^{2-2\delta}}\,d\lambda(z)$$

we see that

$$\int_\Omega e^{-2\varphi-2(n-1)(1-\delta)\log|z|}\,d\lambda(z)\le C_\delta<\infty,\quad \delta>0,$$

which, by Lemma 2.10 with $\psi(\zeta)=(n-1)\log|\zeta|$, implies that

$$\int_\Omega e^{-2\varphi/(1+\delta(n-1))-2(n-1)\log|z|}\,d\lambda(z)\le C_\delta<\infty,\quad \delta>0.$$

In this argument, since $\nu_0(\varphi)<1$, we can exchange φ for $\varphi/(1-r)$, where $r>0$, and still have $\nu_0(\varphi/(1-r))<1$. Thus, the hypothesis implies that $\nu_{0,n-1}(\varphi)<1$.

The other direction is simpler: by introducing polar coordinates we see that

$$\int_a e^{-2\varphi(\zeta)-2(n-1)\log|\zeta-a|}\,d\lambda(\zeta)=C\int_{\omega\in S^{2n-2}}\int_{t\in\mathbb{C},|t|<1} e^{-2\varphi(a+t\omega)}\,d\lambda_1\,d\omega.$$

If $\nu_a(\varphi)\ge1$ then the integral of φ over almost every complex line through a is infinite, and thus the above integral is infinite which is the same as saying that $\nu_{0,n-1}(\varphi)\ge1$. We have proved Theorem 1.2. $\square$

We will now describe the relation between the generalized Lelong number $\nu_{a,k}$ and restrictions to linear subspaces of dimension k. In order to do this, we will have to recall the natural measure on the Grassmannian induced by the Haar measure on $U(n)$, where $U(n)$ denotes the unitary group of $\mathbb{C}^n$. Also, let ϑ denote the unique unit Haar measure on $U(n)$. Then we can define a measure $d\mu$ on the Grassmannian $G(k,n)$ – the set of k-dimensional subspaces of $\mathbb{C}^n$ – by setting for some fixed $V\in G(k,n)$ and $A\subset G(k,n)$, the mass of A to be $\mu(A)=\vartheta\{M\in U(n):MV\in A\}$. This means that if $P:U(n)\to G(k,n)$ is the function $P(M)=MV$, then

$$\mu=P_*(\vartheta).$$

The measure μ is easily seen to be invariant under actions of $U(n)$, that is, $\mu(MA)=\mu(A)$ if $M\in U(n)$, and also to be independent of our choice of V. For a function f defined on $G(k,n)$, we have

$$\int_{T\in G(k,n)} f(T)\,d\mu=\int_{M\in U(n)} f(MV)\,d\vartheta.$$

We deduce that, for $g\in C_c^\infty(\mathbb{C}^n)$,

$$\int_{T\in G(k,n)}\int_T g(z)\,d\lambda_k(z)\,d\mu=\int_{M\in U(n)}\int_{MV} g(z)\,d\lambda_k(z)\,d\vartheta.$$

After changing to polar coordinates the above integral becomes

$$\int_0^\infty\int_{M\in U(n)}\int_{S_{MV}^{2k-1}} \rho^{2k-1} g(\rho\omega)\,dS(\omega)\,d\vartheta\,d\rho,$$

where S_{MV}^{2k-1} denotes the sphere of real dimension $2k-1$ in the k-dimensional subspace defined by MV. Consider the linear functional on $C(\rho S^{2n-1})$, defined by

$$I_\rho(g):=\rho^{2n-1}\int_{M\in U(n)}\int_{S_{MV}^{2k-1}} g(\rho\omega)\,dS(\omega)\,d\vartheta. \tag{2.6}$$

Notice that, even though this functional is defined for functions on S^{2n-1} while the integration takes place on the sphere S^{2k-1}, it is invariant under rotations on the sphere S^{2n-1}. By the Riesz representation theorem, this functional is given by integration against a measure $d\gamma$ on ρS^{2n-1}, i.e.,

$$I_\rho(g)=\int_{\rho S^{2n-1}} g(\omega)\,d\gamma(\omega), \tag{2.7}$$

where, since I_ρ is rotationally invariant, the measure $d\gamma$ is rotationally invariant as well. Thus $d\gamma$ is equal to the surface measure on ρS^{2n-1} multiplied by a constant $c(\rho)$. This constant is easily determined by evaluating $I_\rho(1)$ using the two expressions (2.6) and (2.7) above (remember that ϑ was normalized so that $\vartheta(U(n))=1$):

$$\rho^{2n-1}c(\rho)=I_\rho(1)=\rho^{2n-1}\int_{S_{MV}^{2k-1}} dS(\omega).$$

So $c(\rho)=c_k=\int_{S^{2k-1}} dS(\omega)$ and is therefore independent of ρ. Thus we see that the integral

$$\int_{T\in G(k,n)}\int_T g(z)\,d\lambda_k\,d\mu=\int_0^\infty \rho^{2(k-n)} I_\rho(g)\,d\rho$$

is equal to

$$c_k \int_0^\infty \rho^{2(k-n)} \int_{\rho S^{2n-1}} g(\omega)\, dS(\omega)\, d\rho = c_k \int_{\mathbb{C}^n} |z|^{2(k-n)} g(z)\, d\lambda.$$

Exchanging $g(z)$ for $g(z)|z|^{2(n-k)}$ we have proven the following formula, which generalizes the formula for changing to polar coordinates in an integral.

Lemma 2.13. *For an integrable function* g,

$$\int_{\mathbb{C}^n} g\, d\lambda = \frac{1}{c_k} \int_{T\in G(k,n)} \int_T |z|^{2(n-k)} g(z)\, d\lambda_k\, d\mu.$$

Proof. We have proven the formula under the condition that $g\in C_c^\infty(\mathbb{C}^n)$. The general case follows by approximating an arbitrary integrable function g by functions in $C_c^\infty(\mathbb{C}^n)$. □

Of course, a similar formula holds if we instead consider k-planes through some arbitrary point $a\in\mathbb{C}^n$, and in the above discussion assume the spheres to be centred around the point a. This remark applies to the following result as well.

Proposition 2.14. *Let k be an integer satisfying $0\le k\le n-1$. Then the following are equivalent*:

(a) $\nu_{0,n-k}(\varphi)<1$;
(b) $\nu_{0,0}(\varphi|_T)<1$ *for almost every* $T\in G(k,n)$;
(c) $\nu_{0,0}(\varphi|_T)<1$ *for some* $T\in G(k,n)$.

Proof. The assumption $\nu_{0,n-k}(\varphi)<1$ means that

$$\int_{B(0,r)} e^{-2\varphi(\zeta)/(1-\delta)-2(n-k)\log|\zeta|}\, d\lambda < \infty$$

for some $r>0$, and $\delta>0$ small. Using Lemma 2.13 this integral equals

$$\int_{T\in G(k,n)} \int_{B(0,r)\cap T} e^{-2\varphi(\zeta)/(1-\delta)}\, d\lambda_k\, d\mu.$$

Thus $\int_{B(0,r)\cap T} e^{-2\varphi(\zeta)/(1-\delta)}\, d\lambda_k$ must be finite for almost every T (since by the lemma, $d\mu$ is a multiple of the Lebesgue measure), which implies that $\nu_{0,0}(\varphi|_T)<1$ for almost every $T\in G(k,n)$. This, of course, implies that $\nu_{0,0}(\varphi|_T)<1$ for some $T\in G(k,n)$. On the other hand, if $\int_{B(0,r')\cap T} e^{-2\varphi(\zeta)/(1-\delta)}\, d\lambda_k<\infty$ for some T and

$\delta>0$, then the exact same argument involved in proving the Skoda inequality (using the Ohsawa–Takegoshi theorem), shows that in fact

$$\int_{B(0,r)} e^{-2\varphi(\zeta)/(1-\delta')-2(n-k)\log|z|}\,d\lambda<\infty,$$

for some small $\delta'>0$, which implies that $\nu_{0,n-k}(\varphi)<1$. $\square$

A similar argument gives us the following classical statement (cf. [9]).

Theorem 2.15. *For a generic $V\in G(k,n)$ we have that*

$$\nu_{0,n-1}(\varphi)=\nu_{0,k-1}(\varphi|_V).$$

Proof. Assume that $\nu_{0,n-1}(\varphi)<1$. Then by Lemma 2.13, with the function $g(\zeta)=\exp(-2\varphi(\zeta)-2(n-1)\log|\zeta|)$ we get that for some small $\delta>0$,

$$\begin{aligned}\infty &> \int_{B(0,r)} e^{-2\varphi(\zeta)/(1-\delta)-2(n-1)\log|\zeta|}\,d\lambda\\ &=\int_{V\in G(k,n)}\int_{B(0,r)\cap V} e^{-2\varphi(\zeta)/(1-\delta)-2(k-1)\log|\zeta|}\,d\lambda_k\,d\mu.\end{aligned}$$

Thus $\nu_0(\varphi|_V)<1$ for a generic $V\in G(k,n)$. The other direction is proved by using the same Ohsawa–Takegoshi argument as before. $\square$

Taking $k=1$ we obtain again the following classical result.

Theorem 2.16. *For almost every complex line L through a point a,*

$$\nu_a(\varphi)=\nu_a(\varphi|_L).$$

That is, the Lelong number coincides with what it generically is on complex lines. Moreover, ν_{n-k} coincides with the integrability index of φ restricted a generic k-plane.

3. The analyticity property of the upper level sets of the generalized Lelong number

In this section we prove that the upper level sets of our generalized Lelong number are analytic, provided that the weight is "good enough". This we accomplish by considering the Bergman function, whose definition we will soon recall. First, however, we begin with discussing which properties the weight need to satisfy in order to be "good enough".

• We assume that ψ satisfies the following condition: there exists an $M>0$ such that

$$\psi(z) \geq M \log |z|$$

for z close to 0. It might be worth mentioning that in the case of analytic singularities, i.e., if $\psi=\log |f|$, where $f=(f_1, ..., f_n)$ is a tuple of holomorphic functions with common intersection locus at the a single point, the least of all M for which

$$\log |f(z)| \geq M \log |z|$$

is called the *Łojasiewicz exponent* of f.

• We assume as before that

$$e^{-2(1+\tau)\psi} \in L^1_{\text{loc}}(0)$$

for some $\tau>0$.

• We also assume that $e^{2\psi}$ is Hölder continuous with exponent α in a neighbourhood of the origin.

• Finally we assume that $\nu_0(\psi)=l>0$, so that ψ carries some singularity at the origin.

Definition 3.1. We say that a plurisubharmonic function ψ is an *admissible weight*, and write $\psi \in W(\tau, l, M, \alpha)$, if it satisfies the four properties above.

Admissible weights satisfy the following property, which we will make use of in the proof of the analyticity.

Lemma 3.2. *Assume that $e^{2\psi}$ is Hölder continuous at the origin, with exponent α and satisfies $\psi \geq M \log |z|$ near the origin. Then there exists an $R>0$ and a constant $C>0$, such that for every $k \in \mathbb{N}$,*

$$e^{-2\psi(\zeta-a')} \geq Ce^{-2\psi(\zeta)},$$

where $2^{-(k-1)} \geq |\zeta| \geq 2^{-k}$ and $|a'|=2^{-Rk}$.

Proof. Fix $k \in \mathbb{N}$. We want to show that

$$e^{-2\psi(\zeta-a')} \geq Ce^{-2\psi(\zeta)}$$

for $2^{-(k-1)} \geq |\zeta| \geq 2^{-k}$ and $|a'|=2^{-Rk}$. Since $|\zeta|^R \geq |a'|$ the assumption gives us that

$$e^{2\psi(\zeta)} \geq |\zeta|^{2M} \geq |a'|^{\alpha}$$

if $R\alpha \geq 2M$ (which is a condition independent of k). Now, using the Hölder continuity, we obtain

$$e^{2\psi(\zeta-a')} \leq e^{2\psi(\zeta)} + c|a'|^{\alpha} \leq ce^{2\psi(\zeta)},$$

which implies that

$$e^{-2\psi(\zeta-a')} \geq Ce^{-2\psi(\zeta)}$$

for some constant $C>0$. $\square$

Example 3.3. Examples of plurisubharmonic functions which are admissible weights are given by

$$\psi = \log\left(\sum_{i=1}^{n} |f_i|^{\alpha_i}\right),$$

where $f=(f_1, ..., f_n)$ is an n-tuple of holomorphic functions with common zero locus at the origin. Here we have to assume that $\alpha_i>0$ are as small as needed in order for a $\tau>0$ to exist for which

$$e^{-2(1+\tau)\psi} \in L^1_{\mathrm{loc}}(0).$$

Then e^{ψ} is Hölder continuous with Hölder exponent $\min\{1, \alpha_i\}$, and ψ has Lelong number equal to $\min_i \alpha_i \nu_0(\log|f_i|)$. Also, the Łojasiewicz exponent, which is the smallest M for which $\sum_{i=1}^{n} |f_i|^{\alpha_i} \geq |z|^M$ is easily seen to be finite.

We now define the Bergman kernel with respect to a weight.

Definition 3.4. Let $a\in\Omega$, $\varphi\in\mathrm{PSH}(\Omega)$ and $\psi\in W(\tau, l, M, \alpha)$. We define $\mathcal{H}_a = \mathcal{O}(\Omega)\cap L^2(\Omega, e^{-2\varphi(\,\cdot\,)-2\psi(\,\cdot\,-a)})$, which is a separable Hilbert space. The *Bergman kernel* for a point $z\in\Omega$ is defined as the unique function $B_a^{\psi}(\zeta, z)$, holomorphic in ζ, satisfying

$$h(z) = \int_{\Omega} h(\zeta)\overline{B_a^{\psi}(\zeta, z)}e^{-2\varphi(\zeta)-2\psi(\zeta-a)}\, d\lambda(\zeta)$$

for every $h\in\mathcal{H}_a$.

The existence of the Bergman kernel is a (rather easy) consequence of the Riesz representation theorem for Hilbert spaces. Closely related to the Bergman kernel is the Bergman function.

Definition 3.5. For $a\in\Omega$ the *Bergman function* at a point $\zeta\in\Omega$ is defined as

$$B_a^{\psi}(\zeta) := B_a^{\psi}(\zeta, \zeta).$$

We define

$$\Lambda(a)=\left\{h\in\mathcal{O}(\Omega):\int_\Omega |h(\zeta)|^2\, e^{-2\varphi(\zeta)-2\psi(\zeta-a)}d\lambda(\zeta)\le 1\right\},$$

that is, the set of functions in $\mathcal{H}_a$ of norm less than or equal to 1. Let us calculate the norm of $\zeta\mapsto B_a^\psi(\zeta,z)$:

$$\|B_a^\psi(\,\cdot\,,z)\|^2=\int_\Omega B_a^\psi(\zeta,z)\overline{B_a^\psi(\zeta,z)}e^{-2\varphi(\zeta)-2\psi(\zeta-a)}\,d\lambda(\zeta)=B_a^\psi(z,z),$$

which in particular implies that $B_a^\psi(z,z)$ is given by a non-negative real number. Consequently

$$s(\zeta)=\frac{B_a^\psi(\zeta,z)}{\sqrt{B_a^\psi(z,z)}}\in\Lambda(z)$$

and so

$$|s(z)|^2=B_a^\psi(z). \tag{3.1}$$

Also, we have that

$$\sup_{h\in\Lambda(a)}|h(z)|^2=\sup_{h\in\Lambda(a)}|(h,B_a^\psi(\,\cdot\,,z))|^2=\|B_a^\psi(\,\cdot\,,z)\|^2=B_a^\psi(z), \tag{3.2}$$

where $(\,\cdot\,,\cdot\,)$ denotes the inner product in $\mathcal{H}_a$, which gives us the following extremely useful characterization of the Bergman function,

$$B_a^\psi(z):=\sup\left\{|h(z)|^2:h\in\mathcal{O}(\Omega)\text{ and }\int_\Omega |h(\zeta)|^2\, e^{-2\varphi(\zeta)-2\psi(\zeta-a)}\,d\lambda(\zeta)\le 1\right\}, \tag{3.3}$$

and (3.1) means that this supremum is actually realized by the function s.

Bergman functions enjoy several nice properties, and one, critical for our purposes, is provided by the following theorem of Berndtsson (cf. [2]).

Theorem 3.6. *If Ω is pseudoconvex, then the function $(a,z)\mapsto\log B_a^\psi(z)$ is plurisubharmonic in (a,z).*

Thus we can talk about the Lelong number of the function $z\mapsto\log B_z^\psi(z)$ in Ω, and the following proposition relates it to the generalized Lelong number of φ. More specifically, it says that if the generalized Lelong number of φ with respect to ψ is larger than 1, then the classical Lelong number of $z\mapsto\log B_z^\psi(z)$ is larger than 0, and if the generalized Lelong number is smaller than 1 the classical number is 0. In the terminology of Kiselman, we say that $\log B_z^\psi(z)$ attenuates the singularities of φ.

Recall that by Lemma 3.2 we can find an $R>0$ for which

$$e^{-2\psi(\zeta-a')} \geq Ce^{-2\psi(\zeta)}$$

for $2^{-(k-1)} \leq |\zeta| \leq 2^{-k}$ and $|a'|=2^{-Rk}$, for every $k\in\mathbb{N}$.

Also, by Lemma 2.10 we can choose an $\varepsilon<\delta\tau$ (arbitrarily close to $\delta\tau$) for which

$$\int_{B(a,1/2^N)} e^{-2\varphi(\zeta)-2(1-\varepsilon)\psi(\zeta-a)}\, d\lambda(\zeta) = \infty, \tag{3.4}$$

if we fix an $N>0$ large enough.

Proposition 3.7. *Let $\delta>0$ be small, let Ω be an open and pseudoconvex set containing the point a and let $\psi\in W(\tau,l,M,\alpha)$. Assume that*

$$\nu_{a,\psi}(\varphi) = 1+\delta.$$

Then, with $C_\delta=\delta\tau l$, the classical Lelong number of $\log B^{\psi}_{\cdot}(\,\cdot\,)$ at a is larger than or equal to C_δ/R, that is

$$\nu_a(\log B^{\psi}_{\cdot}(\,\cdot\,)) \geq \frac{C_\delta}{R}.$$

On the other hand, if we assume that

$$\nu_{a,\psi}(\varphi) < 1,$$

then

$$\nu_a(\log B^{\psi}_{\cdot}(\,\cdot\,)) = 0.$$

Without loss of generality, we can assume that $a=0$. Recalling the definition of the classical Lelong number, we see that we want to show that

$$\lim_{r\to 0} \frac{\sup_{|z|=r} \log B^{\psi}_{z}(z)}{\log r} \geq \frac{C_\delta}{R}. \tag{3.5}$$

The idea of the proof is the following. The assumption $\nu_{0,\psi}(\varphi)=1+\delta$ essentially means that

$$\int_0 e^{-2\varphi/(1+\delta')-2\psi}\, d\lambda = \infty$$

for every $\delta'<\delta$. Thus, the weight $\varphi+\psi$ has a rather large singularity at the origin. If we move the singularity of ψ by translating it to an arbitrary point a', then if a' is small enough, the weight $\varphi(\zeta)+\psi(\zeta-a')$ will have a rather large singularity at the point a'. As we will show, the singularity will actually be so large that if $h \in \Lambda(a')$, that is if h is holomorphic and satisfies

$$\int_\Omega |h(\zeta)|^2 \, e^{-2\varphi(\zeta)-\psi(\zeta-a')} \, d\lambda(\zeta) \leq 1,$$

then h is forced to be small at the point a'. In fact,

$$|h(a')|^2 \leq |a'|^{C_\delta/R}. \tag{3.6}$$

This would be enough to prove the proposition, but the following observations show that in fact it will be enough to show something weaker. By Cauchy estimates we will see that it suffices to find just some point z_0 near the origin for which $|h(z_0)|^2 \leq |z_0|^{C_\delta}$, if $h \in \Lambda(a')$. This simplifies things considerably. Also, since we, in (3.5), are dealing with a limit, it suffices to find a sequence r_k tending to 0, for which the inequality (3.5) holds. This means that instead of applying the above idea to arbitrary points a' near the origin, we merely need to apply it to points of a sequence a_k tending to 0. We now turn to the details.

Lemma 3.8. *Assume that $\nu_{0,\psi}(\varphi) = 1+\delta$. Fix any sequence $a_k \to 0$ with $|a_k| = 2^{-Rk}$ and for every k choose a corresponding $h^k \in \Lambda(a_k)$. Then $\{a_k\}_{k=1}^\infty$ contains a subsequence $\{a_{k_j}\}_{j=1}^\infty$, for which there exists $b_{k_j} \in B(0, 2^{-k_j})$ with*

$$|h^{k_j}(b_{k_j})| \leq |b_{k_j}|^{C_\delta}.$$

Proof. The lemma will be proved with $C_\delta = \varepsilon l$. The general case with $C_\delta = \delta\tau l$ then follows, since ε can be chosen arbitrarily close to $\delta\tau$. We will prove the lemma by arguing via contradiction. The negation of the statement is the following: For every k larger than some finite number, which we can assume to be the N figuring in (3.4),

$$|h^k(\zeta)| > |\zeta|^{C_\delta}$$

for every $\zeta \in B(0, 2^{-k})$.

Let us assume this negation. Then, for every a_k we have, since $h^k \in \Lambda(a_k)$,

$$1 \geq \int_{B(0,2^{-k})} |\zeta|^{C_\delta} \, e^{-2\varphi(\zeta)-2\psi(\zeta-a_k)} \, d\lambda(\zeta) \geq \int_{A(k)} |\zeta|^{C_\delta} \, e^{-2\varphi(\zeta)-2\psi(\zeta-a_k)} \, d\lambda(\zeta),$$

where $A(k)$ denotes the annulus $B(0, 2^{-k}) \setminus B(0, 2^{-k-1})$. Since for $\zeta \in A(k)$,

$$e^{-2\psi(\zeta-a_k)} \geq C e^{-2\psi(\zeta)}$$

by Lemma 3.2, we deduce that

$$C \geq \int_{A(k)} |\zeta|^{C_\delta} e^{-2\varphi(\zeta)-2\psi(\zeta)} \, d\lambda(\zeta). \tag{3.7}$$

Now, remember that ε was chosen so that

$$\infty = \int_{B(0,1/2^N)} e^{-2\varphi(\zeta)-2(1-\varepsilon)\psi(\zeta)} \, d\lambda(\zeta).$$

Thus, by covering the ball $B(0,1/2^N)$ by annuli $B(0,2^{-k})\backslash B(0,2^{-k-1})$, we get

$$\begin{aligned}\infty &= \int_{B(0,1/2^N)} e^{-2\varphi(\zeta)-2(1-\varepsilon)\psi(\zeta)} \, d\lambda(\zeta) \leq C \sum_{k=1}^{\infty} \int_{A(k)} |\zeta|^{2\varepsilon l} e^{-2\varphi(\zeta)-2\psi(\zeta)} \, d\lambda(\zeta) \\ &\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon l} \int_{A(k)} |\zeta|^{\varepsilon l} e^{-2\varphi(\zeta)-2\psi(\zeta)} \, d\lambda(\zeta) \leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon l} < \infty,\end{aligned}$$

where we in the first inequality use the fact that $\nu_0(\psi)<l$ implies that $e^{2\varepsilon\psi(\zeta)} \leq C|\zeta|^{2\varepsilon l}$ for $|\zeta| \leq 2^{-N}$, if N is large enough, and in the last but one inequality use (3.7). This establishes the desired contradiction. □

Proof of Proposition 3.7. Fix a point $a \in \Omega$ with $|a|=2^{-Rk}$ for some $k \in \mathbb{N}$, and choose an $h \in \Lambda(a)$ for which $|h(b)| \leq |b|^{C_\delta}$ for some $b \in B(0,2^{-k})$. We claim that in fact such an h satisfies the estimate

$$|h(a)| \leq D|a|^{C_\delta/R} \tag{3.8}$$

for some constant $D \geq 0$ which does not depend on h nor k: Since φ and ψ are locally bounded from above, every $h \in \Lambda(a)$ satisfies:

$$\int_{\Omega'} |h(\zeta)|^2 \, d\lambda(\zeta) \leq C$$

for every $\Omega' \Subset \Omega$. Fix such an Ω' with $B(0,2^{-k}) \subset \Omega'$. Then, by applying Cauchy estimates on h, we see that

$$|h'(\zeta)|^2 \leq C,$$

in Ω', where C is some constant independent of k. By using a first order Taylor expansion of h, we conclude that

$$|h(a)| \leq |h(b)| + C|a-b| \leq |b|^{C_\delta} + C|b| \leq D\,|a|^{C_\delta/R}$$

for some constant $D \geq 0$ independent of h and k, as promised.

Let us complete the proof. Assume that $\nu_{0,\psi}(\varphi)=1+\delta$. Fix a sequence $\{a_k\}_{k=1}^{\infty}\subset\Omega$ with $|a_k|=2^{-Rk}$ satisfying

$$\sup_{|z|=2^{-Rk}} \log B_z^{\psi}(z) = \log B_{a_k}^{\psi}(a_k)$$

for every k. This we can accomplish since a plurisubharmonic function attains its supremum on any compact set. In view of (3.1), which says that we can actually find a holomorphic function in $\Lambda(z)$ realizing the Bergman function at z, we can find for each k, a function $h^k\in\Lambda(a_k)$ for which

$$h^k(a_k) = B_{a_k}^{\psi}(a_k). \tag{3.9}$$

By Lemma 3.8 we can extract a subsequence $\{a_{k_j}\}_{j=1}^{\infty}$ with a corresponding sequence $\{b_{k_j}\}_{j=1}^{\infty}$, where $|b_{k_j}|=2^{-k_j}$, for which

$$|h^{k_j}(b_{k_j})| \le |b_{k_j}|^{C_\delta}.$$

The estimate (3.8) implies that

$$\log B_{a_{k_j}}^{\psi}(a_{k_j}) \le \log |a_{k_j}|^{C_\delta/R} + D.$$

Thus we obtain (observe that the denominators are *negative*),

$$\lim_{r\to 0} \frac{\sup_{|z|=r} \log B_z^{\psi}(z)}{\log r} = \lim_{j\to\infty} \frac{\sup_{|z|=2^{-Rk_j}} \log B_z^{\psi}(z)}{\log 2^{-Rk_j}} = \lim_{j\to\infty} \frac{\log B_{a_{k_j}}^{\psi}(a_{k_j})}{\log 2^{-Rk_j}} \ge \frac{C_\delta}{R},$$

and we are done in this case.

If $\nu_a(\varphi,\psi)<1$, then an application of Hörmander's L^2-methods (cf. [5]) provides us with a holomorphic function h, satisfying $|h(a)|^2>0$, and the integral over Ω of h with respect to the weight $e^{-2\varphi(\,\cdot\,)-2\psi(\,\cdot\,-a)}$ is less than 1. In view of (3.3) this implies that $B_a^{\psi}(a)>0$. Hence $\nu_a(\varphi,\psi)=0$. □

We can now prove the analogue of Siu's theorem for our generalized Lelong number, using an argument due to Kiselman.

Theorem 3.9. *Let $\Omega\in\mathbb{C}^n$ be open and pseudoconvex and φ be a plurisubharmonic function in Ω. Then if $\rho>0$,*

$$\{z\in\Omega : \nu_{z,\psi}(\varphi)\ge\rho\}$$

is an analytic subset of Ω.

Proof. We first note that if $\psi=0$ then $\nu_{a,0}(\varphi)$ is the same as the integrability index of φ for which the conclusion of the theorem holds (see e.g. [6]). Using the notation of Proposition 3.7 we define

$$\Psi(z)=3n\frac{\log B_z^{\psi}(z)}{C_\delta/R}, \quad z\in\Omega.$$

The core of the proof is to show that

$$\{z\in\Omega:\nu_{z,\psi}(\varphi)\geq 1+\delta\}\subset\{z\in\Omega:e^{-2\Psi}\notin L^1_{\mathrm{loc}}(z)\}\subset\{z\in\Omega:\nu_{z,\psi}(\varphi)\geq 1\}.$$

This we can accomplish as follows.

If for $a\in\Omega$ we have that $\nu_{a,\psi}(\varphi)\geq 1+\delta$ then due to Proposition 3.7, the classical Lelong number of Ψ at a is greater than $3n$ since

$$\nu_a(\Psi)\geq\frac{3nC_\delta/R}{C_\delta/R}=3n.$$

By Skoda's inequality (1.7) we have that $\nu_a(\Psi)\leq n\nu_{a,0}(\Psi)$ which shows that the integrability index of Ψ at a is larger than or equal to 3. In particular, this implies that $e^{-2\Psi(\,\cdot\,)}$ is not locally integrable at a and proves the first of the inclusions.

For the second one, assume that

$$\nu_{a,\psi}(\varphi)<1.$$

This implies that $e^{-2\varphi(\zeta)-2\psi(\zeta-a)}$ is locally integrable at a. As noted above, an application of Hörmander's L^2-methods gives us a holomorphic function h in Ω such that $|h(a)|^2>0$, and the integral of h with respect to the weight $e^{-2\varphi(\,\cdot\,)-2\psi(\,\cdot\,-a)}$ is less than 1. Thus the function $z\mapsto B_z^{\psi}(z)$, being defined as that supremum of the modulus square of all holomorphic functions whose integral with respect to our weight is less than or equal to 1, is strictly positive at a, which implies that

$$\Psi(a)>-\infty.$$

But (see e.g. [5]) e^{-2u} is locally integrable around the points where u is finite for every plurisubharmonic function u, and thus we see that

$$e^{-2\Psi}\in L^1_{\mathrm{loc}}(a),$$

which proves the second inclusion.

As noted above, we know that set $\{z\in\Omega:e^{-2\Psi}\notin L^1_{\mathrm{loc}}(z)\}$ is analytic in Ω. Thus, by rescaling we obtain analytic sets $Z_{\delta,\rho}$ such that

$$\{z\in\Omega:\nu_{z,\psi}(\varphi)\geq\rho\}\subset Z_{\delta,\rho}\subset\left\{z\in\Omega:\nu_{z,\psi}(\varphi)\geq\frac{\rho}{1+\delta}\right\},$$

which implies that

$$\{z\in\Omega:\nu_{z,\psi}(\varphi)\geq\rho\}=\bigcap_{\delta>0}Z_{\delta,\rho}.$$

Since the intersection of any number of analytic sets is analytic, we are done. □

As a consequence of this theorem we can define the following concept, introduced in the classical case by Siu [9].

Definition 3.10. For an analytic set Z in Ω, we define the *generic generalized Lelong number* of φ by

$$m_Z^\psi(\varphi)=\inf\{\nu_{z,\psi}(\varphi):z\in Z\}.$$

We have the following lemma by precisely the same argument as in the classical case.

Lemma 3.11. *$\nu_{z,\psi}(\varphi)=m_Z^\psi(\varphi)$ for $z\in Z\setminus Z'$, where Z' is a union of countably many proper analytic subsets of Z.*

Proof. Put $Z'=\bigcup_{c>m_Z^\psi,c\in\mathbb{Q}}Z\cap E_c^\psi$, where $E_c^\psi=\{z\in Z:\nu_{z,\psi}(\varphi)\geq c\}$. Then each $Z\cap E_c^\psi$ is an analytic proper subset of Z and $\nu_{z,\psi}(\varphi)=m_Z^\psi(\varphi)$ on $Z\setminus Z'$ by construction. □

4. Approximation of plurisubharmonic functions by Bergman kernels

A well-known result due to Demailly (see for instance [4]) makes it possible to approximate a plurisubharmonic function φ by the logarithm of the Bergman function Ψ^m with respect to the weight $e^{-2m\varphi}$, as m tends to infinity. Furthermore, the approximation is continuous with respect to the (classical) Lelong number, i.e.

$$\nu_{z,n-1}(\Psi^m)\to\nu_{z,n-1}(\varphi),\quad \text{as } m\to\infty.$$

We will now show that the same holds true using the Bergman function with respect to the weight $e^{-2m\varphi-2\psi(\,\cdot\,-x)}$, where x is the point at which we evaluate the Bergman function. The argument mimics closely that of Demailly (cf. [4]), with some minor changes to fit our case. To begin with, we modify the construction of $\mathcal{H}_a$ slightly.

Definition 4.1. For each $m\in\mathbb{N}$ and $a\in\Omega$ we let

$$\mathcal{H}_a^m=\mathcal{O}(\Omega)\cap L^2(\Omega,e^{-2m\varphi(\,\cdot\,)-2\psi(\,\cdot\,-a)}).$$

Denote by B_a^m (for notational convenience we suppress the dependence on ψ) the Bergman function for $\mathcal{H}_a^m$, and put

$$\Psi_a^m(z) = \frac{1}{2m}\log B_a^m(z)$$

for $z\in\Omega$. Fix $a\in\Omega$. If $h\in\mathcal{H}_a^m$ has norm bounded by 1, the mean value property for holomorphic functions shows that for $r=r(a)>0$ small enough,

$$\begin{aligned}|h(a)|^2 &\leq \frac{n!}{\pi^n r^{2n}}\int_{|a-\zeta|<r}|h(\zeta)|^2\,d\lambda(\zeta)\\ &\leq \frac{n!}{\pi^n r^{2n}}e^{\sup_{|a-\zeta|<r}(2m\varphi(\zeta)+2\psi(\zeta-a))}\int_{|a-\zeta|<r}|h(\zeta)|^2e^{-2m\varphi(\zeta)-2\psi(\zeta-a)}\,d\lambda(\zeta).\end{aligned}$$

Thus, if we assume that

$$\psi(\zeta-a)\leq l\log|\zeta-a|, \tag{4.1}$$

we have that

$$\begin{aligned}\Psi_a^m(a) &\leq \sup_{|a-\zeta|<r}\biggl(\varphi(\zeta)+\frac{1}{2m}2\psi(\zeta-a)\biggr)-\frac{1}{2m}\log r^{2n}+C\\ &\leq \sup_{|a-\zeta|<r}\varphi(\zeta)+\frac{1}{m}(l-n)\log r+\frac{C}{m}.\end{aligned}$$

Now, assume that

$$\psi(\zeta)\geq(n-\delta)\log|\zeta| \tag{4.2}$$

for some small, fixed $\delta>0$. Fix a point a for which $\varphi(a)>-\infty$. By considering the 0-dimensional variety $\{a\}$, we obtain, by the Ohsawa–Takegoshi theorem (see Section 2), that for every $\xi\in\mathbb{C}$, there exists an $h\in\mathcal{O}(\Omega)$, satisfying $h(a)=\xi$, and a constant $C_\delta>0$, depending only on the dimension and δ, such that

$$\int_\Omega |h(\zeta)|^2e^{-2m\varphi(\zeta)-2(n-\delta)\log|\zeta-a|}\,d\lambda(\zeta)\leq C_\delta e^{-2m\varphi(a)}|\xi|^2.$$

By the assumption (4.2) this implies that

$$\int_\Omega |h(\zeta)|^2e^{-2m\varphi(\zeta)-2\psi(\zeta-a)}\,d\lambda(\zeta)\leq C_\delta e^{-2m\varphi(a)}|\xi|^2.$$

Since this holds for every ξ we can choose a ξ such that the right-hand-side is equal to 1, i.e. $C_\delta e^{-2m\varphi(a)}|\xi|^2=1$. Then h satisfies

$$\int_\Omega |h(\zeta)|^2e^{-2m\varphi(\zeta)-2\psi(\zeta-a)}\,d\lambda(\zeta)\leq 1$$

and

$$\log|h(a)|^2 = \log|\xi|^2 = -\log C_\delta + 2m\varphi(a).$$

Thus,

$$\Psi_a^m(a) \geq \varphi(a) - \frac{\log C_\delta}{2m}.$$

If a is such that $\varphi(a)=-\infty$ this inequality is trivial. Thus, for every m and $z\in\Omega$ we have that

$$\varphi(z) - \frac{1}{2m}\log C_\delta \leq \Psi_z^m(z) \leq \sup_{|z-\zeta|<r} \varphi(\zeta) + \frac{1}{m}(l-n)\log r + C. \tag{4.3}$$

We now want to show that this approximation behaves well with respect to the generalized Lelong number with weight ψ.

To this end, fix $a\in\Omega$, let $\rho>\nu_{a,\psi}(\Psi_a^m(\,\cdot\,))$, and put

$$p = 1 + m\rho \quad \text{and} \quad q = 1 + \frac{1}{m\rho}.$$

Then $1/p+1/q=1$ and we apply Hölder's inequality to the following integral, with $r(a)$ so small that $\{\zeta:|a-\zeta|<r(a)\}\Subset\Omega$,

$$\int_{|\zeta-a|<r} e^{-2m/p\varphi - 2\psi(\zeta-a)}\,d\lambda(\zeta) = \int_{|\zeta-a|<r} e^{-2m/p\varphi - 2\psi(\zeta-a)} B_a^m(\zeta)^{1/p} B_a^m(\zeta)^{-1/p}\,d\lambda(\zeta),$$

to obtain, since $-q/p=-1/m\rho$, that it is dominated by

$$\left(\int_{|\zeta-a|<r} B_a^m(\zeta) e^{-2m\varphi} e^{-2\psi(\zeta-a)}\,d\lambda(\zeta)\right)^{1/p} \times \left(\int_{|\zeta-a|<r} B_a^m(\zeta)^{-1/m\rho} e^{-2\psi(\zeta-a)}\,d\lambda(\zeta)\right)^{1/q}.$$

The first integral can be seen to be finite (cf. [4], p. 546), as well as the second integral, as it equals

$$\int_{|\zeta-a|<r} e^{-2\log B_a^m(\zeta)/2m\rho - 2\psi(\zeta-a)}\,d\lambda(\zeta),$$

which is finite due to the assumption on ρ. Since $p/m=1/m+\rho$ this implies the inequality

$$\nu_{a,\psi}(\varphi(\,\cdot\,)) \leq \nu_{a,\psi}(\Psi_a^m(\,\cdot\,)) + \frac{1}{m}. \tag{4.4}$$

We need the following lemma.

Lemma 4.2. *For every $m\in\mathbb{N}$ and $a\in\Omega$ we have that*

$$B_a^m(\zeta)\geq C|\zeta-a|^{(n+2)(n-\delta)/l}B_\zeta^m(\zeta)$$

for every ζ in a small neighbourhood of a, where $C>0$ is a constant not depending on ζ or a.

Proof. For a fixed point a in Ω, choose $\zeta\neq a$ with distance less than $\min\{1,\operatorname{dist}(a,\partial\Omega)\}$ to each other, and let

$$M=\frac{n-\delta}{l}.$$

First, we claim that for

$$z\in B(\zeta,2^{-M}|\zeta-a|^M),$$

the inequality

$$e^{-2\psi(a-z)}\leq e^{-2\psi(\zeta-z)} \tag{4.5}$$

holds. Indeed, due to the assumptions (4.1) and (4.2), for such z,

$$e^{\psi(\zeta-z)}\leq|\zeta-z|^l\leq|a-z|^{n-\delta}\leq e^{\psi(a-z)},$$

since

$$|\zeta-z|\leq 2^{-M}|\zeta-a|^M\leq|z-a|^M,$$

where in the second inequality we used that $|\zeta-a|\leq 2|z-a|$. Now, let

$$h\in\Lambda^m(\zeta):=\left\{h\in\mathcal{O}(\Omega):\int_\Omega|h(z)|^2\,e^{-2m\varphi(z)-2\psi(z-\zeta)}\,d\lambda(z)\leq 1\right\}$$

be such that $h(\zeta)=B_\zeta^m(\zeta)$. To simplify notation we assume that $m=1$, but the following calculations remains valid for any m. Take a smooth function θ with support in $B(\zeta,2^{-M}|\zeta-a|^M)$ satisfying $\theta=1$ in a neighbourhood of $B(\zeta,2^{-(1+M)}|\zeta-a|^M)$, and

$$|\bar\partial\theta(z)|\leq\frac{1}{|a-\zeta|^{2M}}. \tag{4.6}$$

Thus for every point $z\in\operatorname{supp}\theta$ the inequality (4.5) holds. Moreover, we have that

$$e^{-2(n+1)\log|z-\zeta|}=\frac{1}{|z-\zeta|^{2(n+1)}}\leq\frac{2^{(n+1)(M+1)}}{|a-\zeta|^{2M(n+1)}}$$

for $z\in\operatorname{supp}\bar\partial\theta$. Putting this information together we obtain the estimate

$$\int_\Omega |\bar\partial\theta h|^2 e^{-2\varphi(z)-2\psi(a-z)-2(n+1)\log|z-\zeta|}\,d\lambda(z)$$

$$\le \frac{2^{(n+1)(M+1)}}{|a-\zeta|^{2M(n+1)}}\int_\Omega |\bar\partial\theta h|^2 e^{-2\varphi(z)-2\psi(\zeta-z)}\,d\lambda(z) \le \frac{2^{(n+1)(M+1)}}{|a-\zeta|^{2M(n+2)}}, \tag{4.7}$$

since $h\in\Lambda(\zeta)$. Thus, by standard L^2-estimates, we can solve the equation

$$\bar\partial u = \bar\partial(\theta h) = \bar\partial\theta h \tag{4.8}$$

with respect to the weight $e^{-2\varphi(z)-2\psi(a-z)-2(n+1)\log|z-\zeta|}$. The singularity of the weight forces u to vanish at ζ, so if we define $F=\theta h-u$, then $F(\zeta)=h(\zeta)$, and F is holomorphic in Ω. Moreover, by the triangle inequality,

$$\left(\int_\Omega |F|^2 e^{-2\varphi(z)-2\psi(a-z)}\,d\lambda(z)\right)^{1/2} \le \left(\int_\Omega |\theta h|^2 e^{-2\varphi(z)-2\psi(a-z)}\,d\lambda(z)\right)^{1/2} + \left(\int_\Omega |u|^2 e^{-2\varphi(z)-2\psi(a-z)}\,d\lambda(z)\right)^{1/2}.$$

Using (4.5) we have that

$$\int_\Omega |\theta h|^2 e^{-2\varphi(z)-2\psi(a-z)}\,d\lambda(z) \le \int_\Omega |\theta h|^2 e^{-2\varphi(z)-2\psi(\zeta-z)}\,d\lambda(z) \le 1,$$

and we also see that

$$\int_\Omega |u|^2 e^{-2\varphi(z)-2\psi(a-z)}\,d\lambda(z) \le \int_\Omega |u|^2 e^{-2\varphi(z)-2\psi(a-z)-2(n+1)\log|z-\zeta|}\,d\lambda(z)$$
$$\le C'\int_\Omega |\bar\partial\theta h|^2 e^{-2\varphi(z)-2\psi(a-z)-2(n+1)\log|z-\zeta|}\,d\lambda(z),$$

where the first inequality comes from the assumption that $|z-\zeta|\le|a-\zeta|\le 1$, and the last inequality, as well as the constant C' (which only depends on Ω), comes from the L^2-estimate obtained from solving (4.8). Using (4.7) we arrive at

$$\int_\Omega |F|^2 e^{-2\varphi(z)-2\psi(a-z)}\,d\lambda(z) \le 1 + C'\frac{2^{(n+1)(M+1)}}{|a-\zeta|^{2M(n+2)}} \le \frac{C_1}{|a-\zeta|^{2M(n+2)}},$$

where C_1 is a constant independent of ζ and a. Thus, if we define the function

$$\widetilde{F}(z) = \frac{|a-\zeta|^{M(n+2)}}{\sqrt{C_1}}F(z)$$

then $\widetilde{F}$ belongs to $\Lambda(a)$ and satisfies

$$|\widetilde{F}(\zeta)| = C_1^{-1/2} |B_\zeta^m(\zeta)|\, |a-\zeta|^{M(n+2)}.$$

This shows that for every a and ζ (the inequality is trivial if $\zeta = a$),

$$|B_a^m(\zeta)| \geq C |B_\zeta^m(\zeta)|\, |a-\zeta|^{M(n+2)},$$

where $C = C_1^{-1/2}$ does not depend on ζ or a. $\square$

Using the lemma we see that for each $a \in \Omega$,

$$\frac{1}{2m} \log |B_a^m(\zeta)| \geq \frac{1}{2m} \log |B_\zeta^m(\zeta)| + \frac{M(n+2)}{2m} \log |a-\zeta| + \frac{C}{m}$$

for ζ close to a, which implies that

$$\nu_{a,\psi}(\Psi_a^m(\,\cdot\,)) \leq \nu_{a,\psi}(\Psi_\cdot^m(\,\cdot\,)) + \frac{C}{2m}.$$

Combining with (4.4) we obtain that

$$\nu_{a,\psi}(\varphi(\,\cdot\,)) \leq \nu_{a,\psi}(\Psi_\cdot^m(\,\cdot\,)) + \frac{C}{m}$$

for every $a \in \Omega$.

On the other hand, the left-hand estimate of (4.3) implies that

$$\nu_{a,\psi}(\varphi(\,\cdot\,)) \geq \nu_{a,\psi}(\Psi_\cdot^m(\,\cdot\,)).$$

Thus we have proved the following theorem.

Theorem 4.3. *Assume that ψ satisfies*

$$l \log |z| \geq \psi(z) \geq (n-\delta) \log |z| \tag{4.9}$$

for some small, fixed $\delta > 0$. Then for $\varphi \in \mathrm{PSH}(\Omega)$, $m \in \mathbb{N}$, $z, a \in \Omega$ and every $r < d(z, \partial\Omega)$ we have that

$$\varphi(z) - \frac{1}{2m} \log C_\delta \leq \Psi_z^m(z) \leq \sup_{|z-\zeta|<r} \varphi(\zeta) + \frac{1}{m}(l-n) \log r + \frac{C}{m} \tag{4.10}$$

and

$$\nu_{a,\psi}(\varphi(\,\cdot\,)) - \frac{C}{m} \leq \nu_{a,\psi}(\Psi_\cdot^m(\,\cdot\,)) \leq \nu_{a,\psi}(\varphi(\,\cdot\,)), \tag{4.11}$$

where C is a constant depending on Ω, l and δ. In particular, $\Psi_z^m(z)$ converges to $\varphi(z)$, as $m \to \infty$, both pointwise and in L^1_{loc}.

Remark 4.4. If $\psi=0$, as in the original theorem of Demailly, then Ψ_a^m are plurisubharmonic functions with analytic singularities, and thus we approximate φ with plurisubharmonic functions with analytic singularities. For instance, this gives a very simple proof of Siu's analyticity theorem for the classical Lelong number. In our setting however, it is unclear, and an interesting question, if the presence of ψ allows for Ψ_a^m to have analytic singularities.

Remark 4.5. One can show that when comparing the approximations Ψ_z^m to the classical Lelong number we can obtain, instead of (4.11), the inequalities

$$\nu_{a,n-1}(\varphi(\,\cdot\,))-\frac{n-l}{m}\leq\nu_{a,n-1}(\Psi_{\cdot}^m(\,\cdot\,))\leq\nu_{a,n-1}(\varphi(\,\cdot\,)).$$

The approximation of Demailly, that is with $\psi=0$, satisfied these inequalities with $l=0$.

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