DOKTORSAVHANDLINGAR
VID
CHALMERS TEKNISKA HÖGSKOLA
Nr 55

ON THE ASYMPTOTIC PROPERTIES OF THE SCATTERING MATRIX IN COMPLEX ANGULAR MOMENTUM

A study in potential scattering theory

BY

OLLE BRANDER

GÖTEBORG 1966

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AV

OLLE BRANDER

Tekn. lic.

AKADEMISK AVHANDLING

SOM MED TILLSTÅND AV SEKTIONEN FÖR TEKNISK FYSIK VID CHALMERS TEKNISKA HÖGSKOLA FRAMLÄGGES TILL OFFENTLIG GRANSKNING FÖR TEKNOLOGIE DOKTORSGRADS VINNANDE TISDAGEN DEN 24 MAJ 1966 KL. 10.15 Å HÖRSALEN I ADMINISTRATIONSBYGGNADEN VID CHALMERS TEKNISKA HÖGSKOLA, SVEN HULTINS GATA, GÖTEBORG. AVHANDLINGEN FÖRSVARAS PÅ ENGELSKA.

GÖTEBORG

ELANDERS BOKTRYCKERI AKTIEBOLAG

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This thesis consists of three papers:

A. O. Brander: An asymptotic expansion for the Green's function of nonrelativistic potential scattering theory and the asymptotic character of the Born series for the Jost function for large complex angular momenta; *Nuovo Cimento 42 A*, 39 (1966).

B. O. Brander: On angular momentum analyticity in hard cor potential scattering; *Physics Letters 4*, 218 (1963).

C. O. Brander: Asymptotic behaviour of the S-matrix in complex angular momentum for singular potentials; *Arkiv för Fysik 32*, 131 (1966).

The interest in potential scattering theory has in recent years been concerned mainly with the problem of analyticity. The reason for this can be traced to the importance of dispersion theory for high energy physics, and to the need for a frame within which explicit calculations can be made to test the assumptions usually made in dispersion theory.

The theory of scattering on nonsingular potentials is by now fairly complete. By nonsingular potentials we mean here potentials less singular at the origin than the centrifugal barrier. One of the most difficult problems of this theory concerns the asymptotic behaviour of the scattering amplitude for large complex angular momenta. This behaviour is of importance for the properties of the scattering amplitude in the momentum transfer variable, and for the validity of the Mandelstam representation, a double dispersion integral representation. Paper *A* contains a detailed study of this asymptotic behaviour. It is emphasized there that the asymptotic behaviour for large complex angular momenta depends in a complicated way on the interrelation between the parts of the potential for small and large distances and on its smoothness.

The papers *B* and *C* are concerned with the theory of scattering on potentials more singular at the origin than the centrifugal barrier, a field which has attracted much less attention so far. Paper *B* is a note about the influence on angular momentum analyticity of a hard core of the type often used in nuclear physics, that is, corresponding to an infinite repulsive potential inside the hard core radius.

In paper *C* the same problem is treated for singular potentials, as in paper *A* for nonsingular potentials. A new formalism is developed, leading to an asymptotic series for the Jost function, valid for large complex angular momenta. The first term of this asymptotic series is the result obtained in the JWKB-approximation. Asymptotic formulae are given for the positions and residues of the Regge poles. Furthermore, it is shown how the asymptotic positions of the poles depend critically on the behaviour of the potential at the origin, and that they are very insensitive to changes in the potential elsewhere.

I wish to thank Professor N. Svartholm for his continuous encouragement and advise in the course of my graduate study and research work. Thanks are also due to Professors S. Lundqvist and J. Nilsson for many stimulating discussions and valuable suggestions. For competent help with programming and numerical calculations in connection with this thesis I thank Mr. G. Björkman. To all other members of the staff at the Institute of Theoretical Physics in Göteborg I also wish to express my sincere thanks.

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GÖTEBORG 1966
ELANDERS BOKTRYCKERI AKTIEBOLAG

Bilaga till Doktorsavhandlingar, CTH

O. Brander
1º Marzo 1966
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O. BRANDER

An Asymptotic Expansion for the Green's Function of Nonrelativistic Potential-Scattering Theory and the Asymptotic Character of the Born Series for the Jost Function for Large Complex Angular Momenta

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TIPOGRAFIA COMPOSITORI
1966

An Asymptotic Expansion for the Green's Function of Nonrelativistic Potential-Scattering Theory and the Asymptotic Character of the Born Series for the Jost Function for Large Complex Angular Momenta.

O. BRANDER

Institute of Theoretical Physics - Göteborg

(ricevuto il 13 Agosto 1965)

Summary. — From a generalization of the Langer uniform asymptotic formula for Bessel functions of large order, an asymptotic expansion is constructed for the scattering Green's function. From this expansion a simple uniform bound for the Green's function follows, which contains the well-known nonuniform $1/k$ bound. This uniform bound is then used to study the asymptotic properties for large complex angular momenta of the Born series for the Jost function in nonrelativistic potential scattering theory for a great number of potentials. Among other things the following result is obtained. If the potential has a meromorphic continuation, with a finite number of complex poles, into the half-plane $\mathrm{Re}\, r > 0$ satisfying the condition $\int\limits_0^a |rV(r)\,\mathrm{d}r| < \infty$, $\int\limits_a^\infty |V(r)\,\mathrm{d}r| < \infty$, $0 < a < \infty$, on all rays in this half-plane, then the Born series for the Jost function is an asymptotic series in the Erdélyi sense and the Jost function tends to 1 when $|\lambda| \to \infty$ in any direction in the region $|\arg \lambda| \leqslant \pi/2$. Thus the Sommerfeld-Watson transformation is allowed for such potentials. For parallels to the imaginary r-axis the above condition may be slightly relaxed to include, for instance, the Yukawa potential. Our method also gives some information about the necessary conditions to be imposed on the potential in order to make the Sommerfeld-Watson transformation permissible. Although strict mathematical proof is lacking, the following conclusions have been reached. The potential must have a smooth connection between small and large values of r, and it must not decrease faster than exponentially when $r \to \infty$, in order to make the Sommerfeld-Watson transformation permissible.

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1. – Introduction.

The asymptotic properties for large complex angular momenta of the Jost function in nonrelativistic potential-scattering theory have been studied by a large number of authors ([1-12]) by several different methods.

Among these methods we shall use the most straightforward one ([3,13,14]), which involves direct estimates of the terms in the Born series with the help of uniform approximations to the Green's function. This method has previously met with moderate success only, because the nonuniform behaviour of Bessel functions of high order with respect to their argument makes it difficult to obtain good uniform bounds for the Green's function.

However, recent work on the old Langer uniform asymptotic formula for the Bessel functions of high order ([15-17]), including error bounds ([18,19]), has made it possible to handle those difficulties. In this way we have managed to obtain a very good uniform bound for the Green's function, which makes it possible, for instance, to reproduce and to some extent generalize the strong results of Martin ([10]) and Bessis ([11]) concerning the asymptotic properties of the Jost function.

Section **2** is a short review of the relevant formulae for the Bessel functions, and in Section **3** the corresponding expressions for the scattering Green's func-

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([1]) T. Regge: *Nuovo Cimento*, **14**, 951 (1959).

([2]) A. Bottino, A. M. Longoni and T. Regge: *Nuovo Cimento*, **23**, 954 (1962).

([3]) B. P. Desai and R. G. Newton: *Phys. Rev.*, **129**, 1445 (1963).

([4]) A. M. Jaffe and Y. S. Kim: *Phys. Rev.*, **129**, 2818 (1963).

([5]) F. Calogero: *Nuovo Cimento*, **28**, 66, 761 (1963).

([6]) V. de Alfaro, E. Predazzi and C. Rossetti: *Nuovo Cimento*, **30**, 522 (1963).

([7]) L. Brown, D. I. Fivel, B. W. Lee and R. F. Sawyer: *Ann. of Phys.*, **23**, 187 (1963).

([8]) A. O. Barut and J. Dilley: *Journ. Math. Phys.*, **4**, 1401 (1963).

([9]) R. Jakšić: *Acta Phys. Austriaca*, **17**, 29 (1963).

([10]) A. Martin: *Nuovo Cimento*, **31**, 1229 (1964).

([11]) D. Bessis: *Nuovo Cimento*, **33**, 797 (1964).

([12]) K. Chadan and J. Y. Guennegues: *Nuovo Cimento*, **34**, 665 (1964).

([13]) R. G. Newton: *The Complex j-Plane* (New York, 1964).

([14]) E. P. Wigner, Ed.: *Dispersion Relations and Their Connection with Causality* (New York, London, 1964), p. 97.

([15]) R. E. Langer: *Trans. Amer. Math. Soc.*, **33**, 23 (1931); **34**, 447 (1932); **67**, 461 (1949).

([16]) T. M. Cherry: *Trans. Amer. Math. Soc.*, **68**, 224 (1950).

([17]) F. W. J. Olver: *Phil. Trans.*, A **247**, 307, 328 (1954).

([16]) F. W. J. Olver: *Journ. Soc. Indust. Appl. Math.*, **11**, 748 (1963); **12**, 200 (1964); *Natl. Phys. Lab. Math. Tables*, vol. **6** (London, 1962).

([19]) C. H. Wilcox, Ed.: *Asymptotic Solutions of Differential Equations and their Applications* (New York, London, Sidney, 1964), p. 163.

tion are constructed. In particular a simple uniform bound is obtained for the Green's function, which contains the usual $1/k$ bound.

This bound is then used in Sections **4** and **5** to study the asymptotic character of the Born series for the Jost function when the complex angular momentum is large. In Section **4** we examine the Jost function for general potentials, and also for potentials, which very rapidly tend to zero at infinity. Then we study in Section **5** the Jost function for potentials that have analytic continuations outside the real r-axis. Finally, in Section **6** we conclude with a short discussion of the results. An Appendix contains estimates of an important integral.

2. – Asymptotic formulae for the Bessel functions.

For our work we shall require approximations to and bounds for integrals containing Bessel functions of high order and of argument varying from 0 to ∞. To obtain such approximations, we start from the uniform asymptotic expansion [15-17].

$$H_\lambda^{(1,2)}(\lambda x) \underset{\lambda\to\infty}{\sim} \left(\frac{u}{w}\right)^{\frac{1}{2}}\left[\exp[\pm i\pi/6]\, H_{\frac{1}{3}}^{(1,2)}(\lambda u)\sum_{s=0}^{\infty}\frac{a_s(u)}{\lambda^{2s}} + \right.$$
$$\left. + \lambda^{-1}\exp[\mp i\pi/6]\left(\tfrac{3}{2}u\right)^{\frac{1}{3}} H_{\frac{2}{3}}^{(1,2)}(\lambda u)\sum_{s=0}^{\infty}\frac{b_s(u)}{\lambda^{2s}}\right], \tag{2.1}$$

which is valid uniformly in x for $|\arg\lambda|\leqslant\frac{1}{2}\pi$, $|\arg x|\leqslant\frac{1}{2}\pi$.

The complex variables w and u in eq. (2.1) are defined by

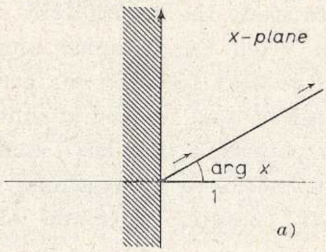
$$w = (x^2-1)^{\frac{1}{2}}, \qquad u = w - \operatorname{arctg} w, \tag{2.2}$$

and u has the asymptotic form

$$u = \begin{cases} x - \frac{1}{2}\pi + O(x^{-1}) & \text{for} \quad |x|\gg 1, \\ \frac{1}{3}w^3 + O(w^5) & \text{for} \quad |x-1|\ll 1, \\ [-\ln(\frac{1}{2}ex) + O(x^2)]\exp[\frac{3}{2}\pi i\,\operatorname{sign}(\arg x)] & \text{for} \quad |x|\ll 1. \end{cases} \tag{2.3}$$

The properties of the λ-independent functions a_s and b_s ($a_0\equiv 1$) are discussed below.

Figure 1 shows the mapping from the right-hand half of the complex x-plane to the complex u-plane. The origin in the u-plane corresponds to the point $x=1$, and the origin in the x-plane corresponds to the point at infinity in the direction $\arg u = \pm\frac{3}{2}\pi$ in the u-plane, where the plus sign is to be taken for x above, and the minus sign for x below a cut from 0 to 1 on the real x-axis. In this way the first quadrant of the x-plane is mapped on an L-shaped region in a first Riemann sheet $0 \leqslant \arg u \leqslant \frac{3}{2}\pi$ of the u-plane, and the fourth quadrant on a second Riemann sheet $-\frac{3}{2}\pi \leqslant \arg u \leqslant 0$.



These two Riemann sheets can be mapped on a single sheet of a z-plane,

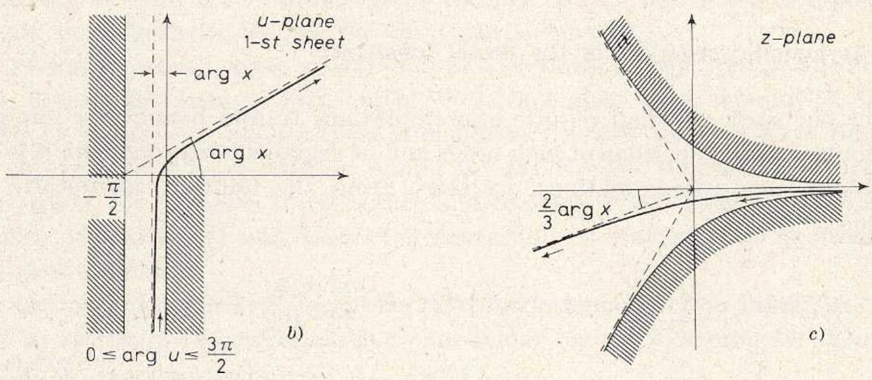


Fig. 1. – *a*) The complex x-plane; *b*) The complex u-plane; *c*) The complex z-plane.

which is also given in Fig. 1, by the transformation

$$z = (\tfrac{3}{2}u)^{\frac{2}{3}} \exp[-i\pi \operatorname{sign}(\arg u)] \,. \tag{2.4}$$

In the z-plane, the two lips of the cut in the x-plane have both been mapped on the real z-axis, and the cut is no longer needed.

In the variable z, eq. (2.1) becomes

$$\begin{aligned} \boldsymbol{H}_\lambda^{(1,2)}(\lambda x) \underset{\lambda\to\infty}{\simeq} 2\,\lambda^{-\frac{1}{3}}\left(\frac{4z}{1-x^2}\right)^{\frac{1}{4}}\Big[&\exp[\mp i\pi/3]\operatorname{Ai}(\exp[\pm\tfrac{2}{3}i\pi]\lambda^{\frac{2}{3}}z)\sum_{s=0}^{\infty}\frac{A_s(z)}{\lambda^{2s}} + \\ &+\exp[\pm i\pi/3]\,\lambda^{-\frac{1}{3}}\operatorname{Ai}'(\exp[\pm\tfrac{2}{3}i\pi]\lambda^{\frac{2}{3}}z)\sum_{s=0}^{\infty}\frac{B_s(z)}{\lambda^{2s}}\Big] \,. \end{aligned} \tag{2.5}$$

Here Ai and Ai′ are the Airy function and its derivative, defined in the standard way (see LUKE [20]). The functions A_s and B_s ($A_0 \equiv 1$) are λ-independent

[20] Y. L. LUKE: *Integrals of Bessel Functions* (New York, Toronto, London, 1962).

analytic functions of z in the region shown in Fig. 1. For $|z| \to \infty$ within this region, $A_s(z)$ and $z^{\frac{1}{2}} B_s(z)$ are finite. Explicit expressions for the functions A_s and B_s have been given by OLVER ([17]).

The Airy function, as well as the factor in front of it in eq. (2.5), is analytic in z, also at $z=0$, and from a mathematical point of view the expression (2.5) is thus preferable to eq. (2.1). However, after having observed that the point $z=0$ is not a singular point, we can very well return to eq. (2.1) in order to work with the Hankel functions.

The properties of the functions a_s and b_s of. eq. (2.1) now follow from the just above mentioned properties of A_s and B_s. Thus it is found that a_s and b_s are analytic functions of u in the L-shaped regions of Fig. 1, finite at $u=0$, and that when $|u| \to \infty$ in the L-shaped region, $a_s(u)$ and $u^{\frac{1}{3}} b_s(u)$ are finite.

The Bessel functions to be studied usually have the argument kr, where k is the wave number, and r is the radial variable. Then

$$x = \frac{kr}{\lambda} \tag{2.6}$$

and for fixed complex k and λ, and real r varying from 0 to ∞, x varies along a ray in the complex x-plane, as shown in Fig. 1. By the transformations (2.2) and (2.4) this ray is mapped on certain curves in the u- and z-planes, which curves are also shown in Fig. 1.

In the integrals to be studied, the integrations go over unbounded intervals of r, and it is thus important for the asymptotic series (2.1) to be uniformly valid in x for unbounded regions of u. That this is actually the case was proved by OLVER ([17]). The region of validity given by him contains the region we have given for eq. (2.1), except that he does not include the case $\arg \lambda = \pm \frac{1}{2}\pi$. However, use of the standard continuation formulae for Bessel functions shows that there is no Stokes line at $\arg \lambda = \pm \frac{1}{2}\pi$, which could form a natural boundary for the validity of the asymptotic expansion.

Bounds for the rest terms of expansions like eq. (2.1) have recently been given by OLVER ([18,19]), and therefore, this expansion can, for large λ (which means $|\lambda| \geqslant 3$ according to OLVER), be used to get good numerical estimates with controllable accuracy.

However, a simple estimate, reproducing the behaviour of the function qualitatively will also be most useful. Such an estimate can be obtained by taking the first term of eq. (2.1) and using what is known as the Levinson bound for Bessel functions of fixed real order. The resulting bound is

$$|H_\lambda^{(1,2)}(\lambda x)| \leqslant c[1 + O(\lambda^{-1})]\,|\lambda|^{-\frac{1}{3}} \sigma(\lambda, u) \exp[\mp \operatorname{Im}(\lambda u)]\,, \tag{2.7}$$

where c is a constant and

$$\sigma(\lambda, u) = \frac{|\lambda u|^{\frac{1}{6}}}{|w|^{\frac{1}{4}}(1+|\lambda u|)^{\frac{1}{6}}}. \tag{2.8}$$

The properties of the function σ are studied in the Appendix.

Equation (2.7) is only valid if

$$\begin{cases} -\pi \leqslant \arg(\lambda u) \leqslant 2\pi & \text{for } H^{(1)}, \\ -2\pi \leqslant \arg(\lambda u) \leqslant \pi & \text{for } H^{(2)}, \end{cases} \tag{2.9}$$

because only then is the Levinson bound valid. The limits of the region (2.9) correspond to Stokes lines for the asymptotic expansions of $H_{\frac{1}{3}}^{(1,2)}(\lambda u)$, and outside this region only a weaker statement is possible, namely

$$|H_{\lambda}^{(1,2)}(\lambda x)| \leqslant c[1 + O(\lambda^{-1})]|\lambda|^{-\frac{1}{3}}\sigma(\lambda, u)\exp[|\operatorname{Im}(\lambda u)|]. \tag{2.7'}$$

A lower limit for the constant c can be obtained by comparison with the asymptotic form when $x \to \infty$, and an upper limit has been given by Bessis [11] in the special case $\operatorname{Im}(\lambda u) = 0$. However, instead of pressing the value of the constant, we recommend using eq. (2.1) with the Olver bound for the rest term.

We end this Section with a general remark on asymptotic expansions.

Consider a sequence of functions $\varphi_n(x)$, $n = 0, 1, 2, \ldots$, which satisfies

$$\varphi_{n+1}(x) = o(\varphi_n(x)) \quad \text{as } x \to x_0. \tag{2.10}$$

Such a sequence is called an asymptotic sequence [21] or scale. Consider further a function $f(x)$ and suppose that there exist constants a_n such that for each $N = 0, 1, 2, \ldots$

$$f(x) - \sum_{n=0}^{N-1} a_n \varphi_n(x) = o(\varphi_N(x)) \quad \text{as } x \to x_0. \tag{2.11}$$

Then the formal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

is called an asymptotic expansion of the Poincaré type for $f(x)$ when $x \to x_0$.

Suppose further that there exist functions $f_n(x)$, $n = 0, 1, 2, \ldots$, such that

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[21] A. Erdélyi: *Asymptotic Expansions* (New York, 1956).

for each $N = 0, 1, 2, \ldots$

$$f(x) - \sum_{n=0}^{N-1} f_n(x) = o(\varphi_N(x)) \qquad \text{as } x \to x_0 . \tag{2.12}$$

Then the formal series

$$\sum_{n=0}^{\infty} f_n(x)$$

is called an asymptotic expansion of the Erdélyi [22] type for $f(x)$ with respect to the scale $\varphi_n(x)$ when $x \to x_0$.

The Erdélyi type of expansion is more general than the Poincaré type, because the limit

$$\lim_{x \to x_0} \frac{f_{n+1}(x)}{f_n(x)}$$

is not required to exist. On the other hand, the nice property of uniqueness of the Poincaré type expansion is not shared by the Erdélyi type expansion.

It should be noted that the expansion (2.5) is not of the Poincaré type, but can be regarded as the sum of two Poincaré type expansions. It is also an Erdélyi type expansion with respect to the scale.

$$\varphi_n(\lambda) = \lambda^{-\frac{1}{2}-n+\varepsilon}, \qquad n = 0, 1, 2, \ldots; \; \varepsilon > 0; \; \lambda \to \infty . \tag{2.13}$$

3. – Asymptotic formulae for the Green's function.

We shall now study the Green's function

$$G(r, r') = \tfrac{1}{4} i\pi (rr')^{\frac{1}{2}} [H_\lambda^{(1)}(kr) H_\lambda^{(2)}(kr') - H_\lambda^{(1)}(kr') H_\lambda^{(2)}(kr)] \tag{3.1}$$

for large complex λ. Substitution of the first term of eq. (2.1) into this equation gives

$$G(r, r') = \frac{1}{4}\, i\pi [1 + O(\lambda^{-1})] \left(\frac{rr'uu'}{ww'}\right)^{\frac{1}{2}} [H_{\frac{1}{3}}^{(1)}(\lambda u) H_{\frac{1}{3}}^{(2)}(\lambda u') - H_{\frac{1}{3}}^{(1)}(\lambda u') H_{\frac{1}{3}}^{(2)}(\lambda u)] , \tag{3.2}$$

which, because of the symmetry of the Green's function is valid for all values of $\arg \lambda$ and $\arg k$ in the interval $[-\pi, \pi]$, uniformly in r and r'. The only exceptions are the neighbourhoods of the zeros of G, where the error term must be modified.

([22]) A. Erdélyi: *Arch. Rat. Mech. Anal.*, **7**, 1 (1961); A. Erdélyi and M. Wyman: *Arch. Rat. Mech. Anal.*, **14**, 217 (1963).

Equation (3.2) is the first term of an asymptotic expansion. This expansion can be obtained by substitution of the whole expansion (2.1) into eq. (3.1). This operation is allowed because Poincaré type powers series asymptotic expansions may be multiplied together term by term ([21]), and eq. (2.1) contains two such expansions. The result is the following combination of four Poincaré type asymptotic expansions:

$$
\begin{aligned}
(3.3)\qquad G(r, r') \underset{\lambda\to\infty}{\sim} g_0(r, r') \sum_{s=0}^{\infty} \frac{c_s(u, u')}{\lambda^{2s}} + g_1(r, r') \sum_{s=0}^{\infty} \frac{d_s(u, u')}{\lambda^{2s+1}} - \\
- g_1(r', r) \sum_{s=0}^{\infty} \frac{d_s(u', u)}{\lambda^{2s+1}} + g_2(r, r') \sum_{s=0}^{\infty} \frac{e_s(u, u')}{\lambda^{2s+2}} ,
\end{aligned}
$$

where

$$
(3.4)\qquad \left\{
\begin{aligned}
g_0(r, r') &= \frac{1}{4} i\pi \left(\frac{rr'uu'}{ww'}\right)^{\frac{1}{2}} [H^{(1)}_{\frac{1}{3}}(\lambda u)\, H^{(2)}_{\frac{1}{3}}(\lambda u') - H^{(1)}_{\frac{1}{3}}(\lambda u')\, H^{(2)}_{\frac{1}{3}}(\lambda u)] ,\\
g_1(r, r') &= \frac{1}{4} i\pi \left(\frac{rr'uu'}{ww'}\right)^{\frac{1}{2}} \left(\frac{3}{2} u'\right)^{\frac{1}{3}} [\exp[i\pi/3] H^{(1)}_{\frac{1}{3}}(\lambda u)\, H^{(2)}_{\frac{2}{3}}(\lambda u') - \\
&\qquad\qquad - \exp[-i\pi/3]\, H^{(1)}_{\frac{2}{3}}(\lambda u')\, H^{(2)}_{\frac{1}{3}}(\lambda u)] ,\\
g_2(r, r') &= \frac{1}{4} i\pi \left(\frac{rr'uu'}{ww'}\right)^{\frac{1}{2}} \left(\frac{9}{4} uu'\right)^{\frac{1}{3}} [H^{(1)}_{\frac{2}{3}}(\lambda u)\, H^{(2)}_{\frac{2}{3}}(\lambda u') - H^{(1)}_{\frac{2}{3}}(\lambda u')\, H^{(2)}_{\frac{2}{3}}(\lambda u)] ,
\end{aligned}
\right.
$$

and

$$
(3.5)\qquad \left\{
\begin{aligned}
c_s(u, u') &= \sum_{t=0}^{s} a_t(u)\, a_{s-t}(u') ,\\
d_s(u, u') &= \sum_{t=0}^{s} a_t(u)\, b_{s-t}(u') ,\\
e_s(u, u') &= \sum_{t=0}^{s} b_t(u)\, b_{s-t}(u') .
\end{aligned}
\right.
$$

The functions c_s, d_s and e_s are uniformly bounded. When u and u' approach infinity even $u'^{\frac{1}{3}} d_s$ and $u^{\frac{1}{3}} u'^{\frac{1}{3}} e_s$ are bounded. This implies that eq. (3.3) is an asymptotic expansion of the Erdélyi type as $\lambda \to \infty$, with respect to the scale

$$
(3.6)\qquad \varphi_n(\lambda) = \lambda^{-\frac{1}{3}-n+\varepsilon} , \qquad \varepsilon > 0 .
$$

For $|\lambda|$ larger than a given limit, depending on the accuracy required, the expansion eq. (3.3) can now be used to get good numerical estimates for the Green's function. The error can again be controlled with the Olver bound for the rest term of eq. (2.1).

However, the qualitative behaviour of the Green's function can be reproduced with the bound (2.7) for the Hankel function inserted into eq. (3.1):

$$(3.7)\qquad |G(r,r')| \leqslant \tfrac{1}{2}\pi c^2[1+O(\lambda^{-1})]\,|\lambda|^{-1}(rr')^{\frac{1}{2}}\sigma(\lambda,u)\sigma(\lambda,u')\exp\big[|\mathrm{Im}\,(\lambda u-\lambda u')|\big]\,.$$

This bound is valid uniformly in r and r' when $|\lambda|$ is not too small and $\arg\lambda$, $\arg k$ are in the interval $[-\pi,\pi]$. The symmetric form of the Green's function makes unnecessary the restriction eq. (2.9), which was necessary for eq. (2.7). This can be shown by using the standard continuation formulae for the Hankel functions of eq. (3.2).

Equation (3.7) reproduces the functional form of the absolute value of the first term of the asymptotic expansion (3.3), except when u and u' are both very small, that is when r and r' are both near λ/k. For such values of r and r' the two terms of g_0 cancel each other out so that we have in fact

$$(3.8)\qquad g_0(r,r') = \mathrm{const}\,(rr')^{\frac{1}{2}}\big[x-x'+O\big((x-1)^2\big)+O\big((x'-1)^2\big)\big]\,,$$

with $x = kr/\lambda$, whereas the bound (3.7) gives only

$$(3.9)\qquad |g_0(r,r')| \leqslant \mathrm{const}\,(rr')^{\frac{1}{2}}|\lambda|^{-\frac{4}{3}} \qquad \text{for } x\approx x'\approx 1\,.$$

However, for all other values of r and r', eq. (3.7) gives a very good bound for the Green's function. Thus when r and r' are both large we obtain

$$(3.10)\qquad |G(r,r')| \leqslant \mathrm{const}\,|k|^{-1}\exp\big[(r-r')\,|\mathrm{Im}\,k|\big] \qquad \text{for } r\geqslant r'\gg\left|\frac{\lambda}{k}\right|,$$

which is the well known $1/k$ bound. When r and r' are both small we get

$$(3.11)\qquad |G(r,r')| \leqslant \mathrm{const}\,|\lambda|^{-1}(rr')^{\frac{1}{2}}\left(\frac{r}{r'}\right)^{|\mathrm{Re}\,\lambda|} \qquad \text{for } r'\leqslant r\ll\left|\frac{\lambda}{k}\right|,$$

which is the bound used in ref. ([3]).

Our uniform bound is to be compared to that of Newton (Appendix A of ref. ([13])), which is unable to reproduce the $1/k$ bound, but contains an exponential factor also for real k (*).

Finally, since G is finite when $\lambda\to 0$, and the right-hand side of eq. (3.7) is different from zero when $\lambda\to 0$, the bound (3.7) must for some finite value

(*) *Note added in proof.* - Compare also V. DE ALFARO and T. REGGE: *Potential Scattering* (Amsterdam, 1965), where a bound resembling ours is given.

of the constant be valid for all λ. Therefore, we have

$$(3.7') \qquad |G(r, r')| \leqslant C\,|\lambda|^{-1}(rr')^{\frac{1}{2}}\sigma(\lambda, u)\,\sigma(\lambda, u')\exp[\,|\mathrm{Im}(\lambda u - \lambda u')|\,]\,,$$

for all r, r', k and λ.

4. – The Jost function for general potentials.

The Jost solutions $f^{\pm}(r)$ of the radial Schrödinger equation satisfy the integral equation

$$(4.1) \qquad f^{\pm}(r) = f_0^{\pm}(r) + \int\limits_r^{\infty} G(r, r')V(r')f^{\pm}(r')\,\mathrm{d}r'\,,$$

where

$$(4.2) \qquad f_0^{\pm}(r) = i^{\mp(\lambda+\frac{1}{2})}(\tfrac{1}{2}\pi kr)^{\frac{1}{2}}H_{\lambda}^{(2,1)}(kr) \underset{r\to\infty}{\simeq} \exp[\mp ikr]\,.$$

For simplicity, we shall often suppress the variables k and λ in the notation.

The resolvent kernel of eq. (4.1) has the Neumann series

$$(4.3) \qquad K(r, r') = G(r, r')V(r') + \sum_{i=1}^{\infty}\int\limits_r^{r_2}\mathrm{d}r_1\int\limits_{r_1}^{r_3}\mathrm{d}r_2 \ldots \int\limits_{r_{i-1}}^{r'}\mathrm{d}r_i \prod_{\substack{j=1\\(r_0=r,\,r_{i+1}=r')}}^{i+1} G(r_{j-1}, r_j)V(r_j)\,.$$

This series is convergent as soon as the integrals converge, since it belongs to a Volterra integral equation. We shall derive a bound for its rest term

$$(4.4) \qquad R_N(r, r') = \sum_{i=N}^{\infty}\int\limits_r^{r_2}\mathrm{d}r_1 \ldots \int\limits_{r_{i-1}}^{r'}\mathrm{d}r_i \prod_{\substack{j=1\\(r_0=r,\,r_{i+1}=r')}}^{i+1} G(r_{j-1}, r_j)V(r_j) =$$

$$= \int\limits_r^{r_2}\mathrm{d}r_1 \ldots \int\limits_{r_{N-1}}^{r'}\mathrm{d}r_N \prod_{\substack{j=1\\(r_0=r)}}^{N}\{G(r_{j-1}, r_j)V(r_j)\}\,K(r_N, r')\,,$$

by using the bound (3.7′) for the Green's function. For simplicity we then assume that k is real and positive. This assumption is not necessary, but it simplifies the reasoning a great deal, and also corresponds to that part of the Regge trajectories that we want to study.

The simplification of the reasoning comes from the fact that for such k, $\mathrm{Im}\,(\lambda u)$ is a monotonous function of r. This fact is most easily seen in the u-plane of Fig. 1, by noting that $|\lambda|^{-1}\,\mathrm{Im}\,(\lambda u)$ is the distance from the point u on the curve to a straight line through the origin making the angle $-\arg\lambda$ with the positive real axis. If k is real, this line is parallel to the asymptote of the curve, and $\operatorname{Im}(\lambda u)$ is a monotonous function of r.

With the notation

$$|G(r, r')|_{as} = C\,|\lambda|^{-1}(rr')^{\frac{1}{2}}\sigma(\lambda, u')\sigma(\lambda, u) \exp\left[|\operatorname{Im}(\lambda u - \lambda u')|\right] \tag{4.5}$$

for the right-hand side of eq. (3.7′), we now find that

$$|G(r, r_1)|_{as}\,|G(r_1, r')|_{as} = C\,|\lambda|^{-1} r_1\, \sigma^2(\lambda, u_1)\,|G(r, r')|_{as} \,. \tag{4.6}$$

Therefore, the integral equation satisfied by the quantity

$$k(r, r') = K(r, r')\,|G(r, r')|_{as}^{-1}\,,$$

that is

$$k(r, r') = \frac{G(r, r')}{|G(r, r')|_{as}}\,V(r') + \int_r^{r'} G(r, r_1)V(r_1)\,\frac{|G(r_1, r')|_{as}}{|G(r, r')|_{as}}\,k(r_1, r')\,\mathrm{d}r_1 \tag{4.7}$$

implies that

$$|k(r, r')| \leqslant |V(r')| + C\,|\lambda|^{-1}\int_r^{r'} r_1\,\sigma^2(\lambda, u_1)\,|V(r_1)|\,|k(r_1, r')|\,\mathrm{d}r_1 \tag{4.8}$$

which, in turn, by a lemma of Titchmarsch [23], implies the following bound for the resolvent kernel

$$|K(r, r')| \leqslant |G(r, r')|_{as}\,|V(r')|\exp\left[C\,|\lambda|^{-1}\int_r^{r'} r_1\sigma^2(\lambda, u_1)\,|V(r_1)|\,\mathrm{d}r_1\right]. \tag{4.9}$$

If the integral

$$v(\lambda) = C\,|\lambda|^{-1}\int_0^{\infty} r_1\,\sigma^2(\lambda, u_1)\,|V(r_1)|\,\mathrm{d}r_1 \tag{4.10}$$

is convergent, the exponential is uniformly bounded, and we arrive at the bound

$$|K(r, r')| \leqslant \exp\left[v(\lambda)\right]|G(r, r')|_{as}\,|V(r')|\,. \tag{4.11}$$

For the rest term eq. (4.4) this implies

$$|R_N(r, r')| \leqslant \left[v(\lambda)\right]^N \exp\left[v(\lambda)\right]|G(r, r')|_{as}\,|V(r')|\,. \tag{4.12}$$

([23]) E. C. TICHMARSCH: *Eigenfunction Expansions* (Oxford, 1946).

From eq. (2.3) we get the following asymptotic form of σ^2

$$\sigma^2(\lambda, u) \approx \begin{cases} \left|\dfrac{\lambda}{kr}\right| & \text{for } |x| \gg 1\,, \\ |\tfrac{1}{3}\lambda|^{\frac{1}{3}} & \text{for } x \approx 1\,, \\ 1 & \text{for } |x| \ll 1\,. \end{cases} \tag{4.13}$$

This, in addition to the observation that σ is everywhere finite, gives the following condition for the convergence of the integral of eq. (4.10) (a is a finite constant $\neq 0$)

$$\int_0^a r\,|V(r)|\,\mathrm{d}r < \infty \quad \text{and} \quad \int_a^\infty |V(r)|\,\mathrm{d}r < \infty\,. \tag{4.14}$$

Let us call the potentials satisfying this condition class A potentials. For this large class of potentials the Neumann series (4.3) is thus absolutely convergent, and the rest term has the bound (4.12) (*).

Substitution of this Neumann series into

$$f^\pm(r) = f_0^\pm(r) + \int_r^\infty K(r, r')\, f_0^\pm(r')\,\mathrm{d}r'\,, \tag{4.15}$$

now gives the Born series for the Jost solutions

$$f^\pm(r) = f_0^\pm(r) + \sum_{i=0}^{\infty} \int_r^{r_2} \mathrm{d}r_1 \ldots \int_{r_i}^{\infty} \mathrm{d}r_{i+1} \prod_{\substack{j=1 \\ (r_0=r)}}^{i+1} \{G(r_{j-1}, r_j)V(r_j)\}\, f_0^\pm(r_{i+1})\,. \tag{4.16}$$

For $r > 0$ all the integrals in this expression converge when the potential is of class A, and the Born series is absolutely convergent. This follows from writing the Green's function in the form

$$G(r, r') = \frac{i}{2k}\,[f_0^+(r) f_0^-(r') - f_0^+(r')\, f_0^-(r)]\,, \tag{4.17}$$

and observing that the crucial point is the convergence at infinity of integrals

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(*) *Note added in proof.* - For potentials not satisfying the first of the conditions (4.14) see O. BRANDER: *Ark. f. Fys.* (to be published).

like

$$\int^{\infty} f_0^{\pm}(r)V(r)f_0^{\pm}(r)\,\mathrm{d}r \underset{r\to\infty}{\sim} \int^{\infty} \exp[\mp ikr]V(r)\exp[\mp ikr]\,\mathrm{d}r\,, \tag{4.18}$$

which are absolutely convergent for real k and potentials of the class A.

For $r=0$ the integrals of eq. (4.16) diverge. However, this is not serious, because those integrals are to be compared to the first term of eq. (4.16), which behaves like

$$f_0^{\pm}(r) = O(r^{\frac{1}{2}-\lambda}) \tag{4.19}$$

when $r\to 0$ for $\operatorname{Re}\lambda>0$. What we need to show is thus only that the following limit exists:

$$\lim_{r\to 0}\frac{1}{f_0^{\pm}(r)}\int_r^{\infty} K(r,r')f_0^{\pm}(r')\,\mathrm{d}r'\;. \tag{4.20}$$

To this end we observe that

$$f_0^{\pm}(r) = i^{\mp(\lambda-\frac{1}{2})}(\tfrac{1}{2}\pi kr)^{\frac{1}{2}}(\sin\pi\lambda)^{-1}[J_{-\lambda}(kr) - i^{\pm 2\lambda}J_{\lambda}(kr)] \tag{4.21}$$

and

$$G(r,r') = \tfrac{1}{2}i\,\pi(rr')^{\frac{1}{2}}(\sin\pi\lambda)^{-1}[J_{-\lambda}(kr)J_{\lambda}(kr') - J_{-\lambda}(kr')J_{\lambda}(kr)]\;, \tag{4.22}$$

which together with eq. (4.16) gives the following expression for the limit (4.20):

$$\lim_{r\to 0}\frac{f^{\pm}(r)-f_0^{\pm}(r)}{f_0^{\pm}(r)} = i^{\pm(\lambda-\frac{1}{2})}\left(\frac{\pi}{2k}\right)^{\frac{1}{2}}\left[\int_0^{\infty} r_1^{\frac{1}{2}}\,\mathrm{d}r_1 J_{\lambda}(kr_1)V(r_1)f_0^{\pm}(r_1) + \right.$$
$$\left. + \sum_{i=1}^{\infty}\int_0^{r_2}\mathrm{d}r_1\ldots\int_{r_i}^{\infty}\mathrm{d}r_{i+1}r_1^{\frac{1}{2}}J_{\lambda}(kr_1)V(r_1)\prod_{j=2}^{i+1}\{G(r_{j-1},r_j)V(r_j)\}f_0^{\pm}(r_{i+1})\right]. \tag{4.23}$$

In this equation all the integrals converge for class A potentials because near $r=0$

$$r^{\frac{1}{2}}J_{\lambda}(kr)f_0^{\pm}(r) = O(r)\;. \tag{4.24}$$

Then this series is absolutely convergent like the series of eq. (4.16).

However, eq. (4.23) is nothing but the Born series for the Jost function. because the Jost function can be defined as the limit

$$f^{\pm}(\lambda,k) = \lim_{r\to 0}\frac{f^{\pm}(r)}{f_0^{\pm}(r)}\;. \tag{4.25}$$

We shall now study the rest term of the Born series for the Jost function, This rest term,

$$(4.26)\qquad r_N^{\pm}(\lambda) \equiv f^{\pm}(\lambda, k) - 1 - i^{\pm(\lambda-\frac{1}{2})}\left(\frac{\pi}{2k}\right)^{\frac{1}{2}}\Bigg[\int_0^{\infty} r_1^{\frac{1}{2}}\,\mathrm{d}r_1 J_\lambda(kr_1)V(r_1)f_0^{\pm}(r_1) + \sum_{i=1}^{N-2}\int_0^{r_2}\mathrm{d}r_1 \dots \int_{r_i}^{\infty}\mathrm{d}r_{i+1}\, r_1^{\frac{1}{2}} J_\lambda(kr_1)V(r_1)\prod_{j=2}^{i+1}\{G(r_{j-1}, r_j)V(r_j)\}f_0^{\pm}(r_{i+1})\Bigg],$$

has the following relation to the rest term (4.4),

$$(4.27)\qquad r_N^{\pm}(\lambda) = i^{\pm(\lambda-\frac{1}{2})}\left(\frac{\pi}{2k}\right)^{\frac{1}{2}}\int_0^{r_2}\mathrm{d}r_1\int_{r_1}^{\infty}\mathrm{d}r_2\, r_1^{\frac{1}{2}} J_\lambda(kr_1)V(r_1)R_{N-2}(r_1, r_2)f_0^{\pm}(r_2),$$

and we are interested in its behaviour when $|\lambda| \to \infty$ in the right-hand half-plane.

In the rest of this paper we assume that λ is in the first quadrant. This is no restriction, because the fourth quadrant can be reached by complex conjugation,

$$(4.28)\qquad f^{\pm}(\lambda^*, k) = [f^{\mp}(\lambda, k)]^* .$$

To get a bound for the rest term (4.27) we now use eq. (4.12) and the bounds of Sect. 2 for the Bessel functions involved. These are for λ in the first quadrant

$$(4.29)\qquad \begin{cases} |J_\lambda(kr)| \leqslant \frac{1}{2}c|\lambda|^{-\frac{1}{2}}[1 + O(\lambda^{-1})]\sigma(\lambda, u)\exp[-\operatorname{Im}(\lambda u)], \\ |f_0^{+}(r)| \leqslant \sqrt{\dfrac{\pi}{2}}\,c\left|\dfrac{kr}{\lambda}\right|^{\frac{1}{2}}[1 + O(\lambda^{-1})]\sigma(\lambda, u)\exp\left[\operatorname{Im}(\lambda u) + \dfrac{\pi}{2}\operatorname{Im}\lambda\right], \\ |f_0^{-}(r)| \leqslant \sqrt{\dfrac{\pi}{2}}\,c\left|\dfrac{kr}{\lambda}\right|^{\frac{1}{2}}[1 + O(\lambda^{-1})]\sigma(\lambda, u)\exp\left[|\operatorname{Im}(\lambda u)| - \dfrac{\pi}{2}\operatorname{Im}\lambda\right], \end{cases}$$

where the absolute value sign in the exponent of the last formula comes from $\arg(\lambda u)$ leaving the region (2.9) so that eq. (2.7′) has to be used. In this way we obtain the bound

$$(4.30)\qquad |r_0^{\pm}(\lambda)| \leqslant [1 + O(\lambda^{-1})]v^{\pm}(\lambda)[v(\lambda)]^{N-1}\exp[v(\lambda)],$$

where

$$(4.31)\qquad \begin{cases} v^{+}(\lambda) = v(\lambda), \\ v^{-}(\lambda) = C|\lambda|^{-1}\displaystyle\int_0^{\infty} r\,\mathrm{d}r\,|V(r)|\sigma^2(\lambda, u)\exp[|\operatorname{Im}(\lambda u)| - \operatorname{Im}(\lambda u)]. \end{cases}$$

In the Appendix the function $v(\lambda)$ is studied for large λ, and it is shown that it approaches zero,

$$v(\lambda) = o(1) \qquad \text{when } |\lambda| \to \infty, \tag{4.32}$$

for all potentials of the class A. If the potential also satisfies

$$V(r) = O(r^{-1-\alpha}) \qquad \text{when } r \to \infty, \tag{4.33}$$

for some positive α, it is further shown that

$$v(\lambda) = \begin{cases} O(\lambda^{-\alpha}) & \text{if } \alpha < 1, \\ O(\lambda^{-1}) & \text{if } \alpha > 1, \end{cases} \tag{4.34}$$

when $\lambda \to \infty$.

The eqs. (4.30)–(4.32) now imply that the Born series for the Jost function $f^+(\lambda, k)$ is an asymptotic series of the Erdélyi type when $\lambda \to \infty$ in the first quadrant with respect to the scale

$$\varphi_N(\lambda) = [v(\lambda)]^{N-\varepsilon}, \qquad \varepsilon > 0 . \tag{4.35}$$

In particular, we find that

$$f^+(\lambda, k) = 1 + O\big(v(\lambda)\big) \tag{4.36}$$

when $\lambda \to \infty$ in the first quadrant.

However, for $f^-(\lambda, k)$ we cannot say so much in general, because without knowing more about the potential than that it belongs to class A, we can only draw the conclusion from eq. (4.31) that

$$v^-(\lambda) \leqslant v(\lambda) \exp[\pi \operatorname{Im} \lambda], \tag{4.37}$$

because $\operatorname{Im}(\lambda u)$ in the exponent of eq. (4.31) becomes negative for large r and approaches $-(\pi/2) \operatorname{Im} \lambda$. This result is not sufficient to prove anything about the asymptotic character of the Born series for $f^-(\lambda,k)$ when $\lambda \to \infty$ in a complex direction in the first quadrant.

However, the change of sign of $\operatorname{Im}(\lambda u)$ for $\operatorname{Im} \lambda < \pi/2 - \varepsilon$ occurs as can be seen in Fig. 1, at a value of r slightly less than $|\lambda/k|$, say at $\gamma\lambda$. Then

$$v^-(\lambda) \leqslant v(\lambda) + C|\lambda|^{-1} \exp[\pi \operatorname{Im} \lambda] \int_{\gamma\lambda}^{\infty} r\,\mathrm{d}r\,|V(r)|\,\sigma^2(\lambda, u) , \tag{4.38}$$

and we see that if only the potential goes sufficiently fast to zero at infinity, we will have

$$v^{-}(\lambda) = O(v(\lambda))$$

when $\lambda \to \infty$ in $\arg\lambda \in [0, (\pi/2) - \varepsilon]$, so that eq. (4.36) will be valid also for $f^{-}(\lambda, k)$, except for parallels to the imaginary axis.

« Sufficiently fast » is here taken to be faster than any exponential $\exp[-\mu r]$, because then the above argument is valid for all values of $\gamma > 0$, that is all k and ε. We thus define: a potential is said to belong to class B if

B.1) it belongs to class A, and

B.2) it approaches zero at infinity faster than any exponential $\exp[-\mu r]$.

When $\arg\lambda = \pi/2$, we find that $\operatorname{Im}(\lambda u) = -(\pi/2)\operatorname{Im}\lambda$ for all r, and the preceding arguments are not applicable. In fact, for fast-decreasing potentials the Jost function will grow indefinitely when $\lambda \to \infty$ parallel to the imaginary axis. This is a well-known fact for square-well potentials and potentials with an abrupt cut-off, but as we shall see it is also valid for more regular class B potentials, like the Gauss potential $\exp[-p^2r^2]$.

For the Born approximation to the Jost function when $\operatorname{Im}\lambda \to \infty$ we find

$$f_B^{-}(\lambda, k) \underset{\operatorname{Im}\lambda\to\infty}{\sim} 1 - \frac{\pi\exp[-i\pi\lambda]}{2\sin\pi\lambda}\int_0^\infty r\,\mathrm{d}r\,V(r)J_\lambda^2(kr)\,. \tag{4.40}$$

For the Gauss potential the integral in this expression can be calculated ([24]),

$$\begin{aligned}\int_0^\infty \exp[-p^2r^2]J_\lambda^2(kr)\,r\,\mathrm{d}r &= \frac{1}{2p^2}\exp\left[-\frac{k^2}{2p^2}\right]I_\lambda\left(\frac{k^2}{2p^2}\right) = \\ &= O\left[\frac{1}{\lambda}\left(\frac{ek^2}{4p^2\lambda}\right)^{2\lambda}\right] \qquad \text{as } |\lambda|\to\infty,\end{aligned} \tag{4.41}$$

which means that for this potential

$$|f_B^{-}(\lambda, k)| \underset{\lambda\to i\infty}{\sim} \text{const}\cdot\frac{1}{|\lambda|}\exp[\pi\operatorname{Im}\lambda]\,, \tag{4.42}$$

where the constant can be shown to be different from zero.

We believe the exponential growth of f^{-} along the positive imaginary axis

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([24]) G. N. WATSON: *A Treatise on the Theory of Bessel Functions* (Cambridge, 1922). p. 395.

978

to be common to all class B potentials. However, this is difficult to prove, because a proof has to use lower bounds to integrals with oscillating integrands, and such bounds are much more difficult to find than upper bounds.

Therefore, let us content ourselves with the rigorous result (4.42), the well-known result for cut-off potentials, and the following qualitative explanation:

If the potential decreases fast at infinity, contributions from small and medium values of r will dominate in the integral for $v^-(\lambda)$, and also in the integral of eq. (4.40). However, for small and medium values of r and $\lambda \to \infty$, the asymptoticform of J_λ contains a factor $\Gamma^{-1}(1+\lambda)$. This factor has the property of going fast to zero at infinity in all directions of the right-hand half-plane, except in directions parallel to the imaginary axis, where it grows exponentially.

To conclude, we find that when $\lambda \to \infty$ in the first quadrant,

$$(4.43) \qquad \begin{cases} f^+(\lambda, k) = 1 + O(v(\lambda)) & \text{for } \arg \lambda \in \left[0, \dfrac{\pi}{2}\right], \\ f^-(\lambda, k) = 1 + O(v(\lambda)) & \text{for } \arg \lambda \in \left[0, \dfrac{\pi}{2} - \varepsilon\right], \\ f^-(\lambda, k) = o\left(\exp[\pi \operatorname{Im} \lambda]\right) & \text{for } \arg \lambda = \dfrac{\pi}{2}, \end{cases}$$

for all class B potentials. Further we know that the exponential growth in the last equation cannot be eliminated for square-well potentials, cut-off potentials and the Gauss potential. Neither can the exponential growth be eliminated, in our opinion, for any other class B potential or for potentials with discontinuities.

However, it is a well-known fact that there are potentials, like the Yukawa potential and superpositions of Yukawa potentials, for which the exponential growth in the last of eqs. (4.43) can be eliminated.

In the next Section we shall show how this comes about. We do this not only to show the power of our uniform bounds, but also to emphasize how the behaviour of the Jost function in the complex λ-plane, and especially on the imaginary axis, depends in a complicated way on the interrelation between the parts of the potential for small and large r.

5. – The Jost function for meromorphic potentials.

As we pointed out in the preceding Section, a change of sign of $\operatorname{Im}(\lambda u)$ in the integration interval of eq. (4.31) made it difficult to obtain a sharp enough bound for the rest term of the Born series for the Jost function.

However, if the potential has a meromorphic continuation into the right-hand half of the complex r-plane, it may be possible to change the integration contour and overcome this difficulty.

In the complex u-plane the integration along the real r-axis corresponds to integration along the curve C, shown in Fig. 2. This figure shows the com-

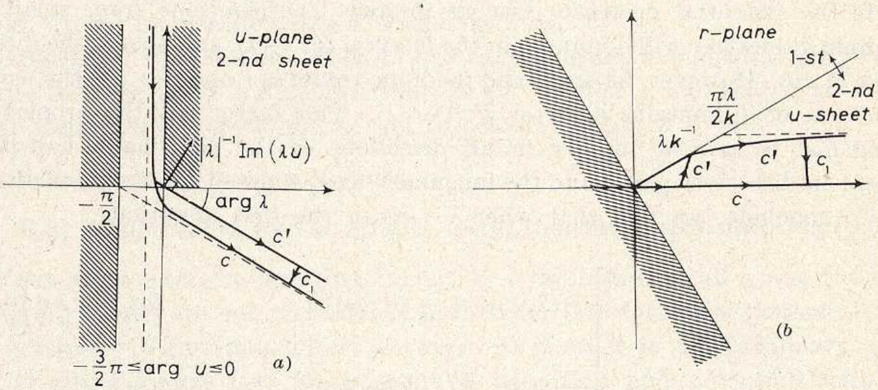

Fig. 2*a*) and *b*). – Integration contours in the complex u- and r-planes for $0 < \arg \lambda < \pi/2$.

plex u- and r-planes for $0 < \arg \lambda < \pi/2$, and it is clear from it that $\operatorname{Im}(\lambda u)$ is positive for small r, goes through zero for a value of r slightly less than $|\lambda/k|$, and is negative for all larger r. This suggests that the integration contour should be changed to C', on which $\operatorname{Im}(\lambda u)$ is nonnegative.

In the figure, C' is shown also in the r-plane, and it lies there in the sector $0 \leqslant \arg r \leqslant \arg \lambda < \pi/2$. Therefore, for this change of contour to be allowed, first of all the potential has to have an analytic continuation into this sector at least for large r. Second, the integral on the curve C_1 in Fig. 2 has to go to zero when the distance of C_1 from the origin goes to infinity.

To meet this need we construct the following potential class: A potential $V(r)$ is said to belong to class C if

C.1) it belongs to class A,

C.2) it has for large r an analytic continuation into the sector $|\arg r| \leqslant (\pi/2) - \varepsilon$, for any $\varepsilon > 0$, and

C.3) in any direction in this sector the continuation satisfies

$$\int^{\infty} |V(r)\,\mathrm{d}r| < \infty .$$

For a class C potential, we now prove that the integral on C_1 approaches zero. The crucial integral is

$$I_1^{\pm} = \int_{C_1} r^{\frac{1}{2}} \mathrm{d}r \, J_\lambda(kr) V(r) f^{\pm}(r) \,. \tag{5.1}$$

When $r \to \infty$ with fixed λ we find that

$$\begin{cases} J_\lambda(kr) \underset{r\to\infty}{\simeq} \left(\dfrac{\pi}{2}\, kr\right)^{-\frac{1}{2}} \cos\left(kr - \frac{1}{2}\pi\lambda - \frac{1}{4}\pi\right), \\ f^{\pm}(r) \underset{r\to\infty}{\simeq} f_0^{\pm}(r) \underset{r\to\infty}{\simeq} \exp\left[\mp ikr\right], \end{cases} \tag{5.2}$$

and thus the integrability condition C.3 for the potential is sufficient to make $I_1^{\pm}$ approach zero when the distance of C_1 from the origin approaches infinity.

Now we can make the whole analysis of Sect. **4** once again with all the integrations on the curve C' instead of on the real r-axis. The result will be formally identical, but with $v(\lambda)$ and $v^{\pm}(\lambda)$ replaced by the quantity

$$\tilde{v}(\lambda) = C\,|\lambda|^{-1} \int_{C'} |r\sigma^2(\lambda, u) V(r)\, \mathrm{d}r| \,. \tag{5.3}$$

In the same way as in the Appendix for $v(\lambda)$, it can be proved that for all class C potentials

$$\tilde{v}(\lambda) = o(1) \tag{5.4}$$

when $\lambda \to \infty$ in $|\arg\lambda| \leqslant (\pi/2) - \varepsilon$. Then for all those potentials, the Born series for $f^-(\lambda, k)$ is an asymptotic series of the Erdélyi type with respect to the scale

$$\varphi_N(\lambda) = [\tilde{v}(\lambda)]^{N-\varepsilon'} \,, \qquad \varepsilon' > 0 \,, \tag{5.5}$$

when $\lambda \to \infty$ in $|\arg\lambda| \leqslant (\pi/2) - \varepsilon$.

If further

$$V(r) = O(r^{-1-\alpha}) \,, \qquad \alpha > 0 \,, \tag{5.6}$$

when $r \to \infty$ in $|\arg r| \leqslant (\pi/2) - \varepsilon$, we find as before for $v(\lambda)$ that

$$\tilde{v}(\lambda) = \begin{cases} O(\lambda^{-\alpha}) & \text{if } \alpha < 1 \,, \\ O(\lambda^{-1}) & \text{fi } \alpha > 1 \,, \end{cases} \tag{5.7}$$

when $\lambda \to \infty$ in $\arg\lambda| \leqslant (\pi/2) - \varepsilon$.

We still have to discuss the case of λ approaching infinity in directions parallel to the imaginary axis. For $\arg \lambda = (\pi/2)$, the complex u- and r-planes and the integration contours C and C' are shown in Fig. 3. In order for $\operatorname{Im}(\lambda u)$ to be nonnegative on C', C and C' now have to be different also for small r. Therefore, we now have to require the potential to be continuable into the whole right-hand half of the r-plane, with the possible exception of a finite number of complex poles. We also have to require the potential not to be too singular for small r, so that the integral on the curve C_2 in Fig. 3 is zero in the limit when C_2 shrinks to the origin in the r-plane.

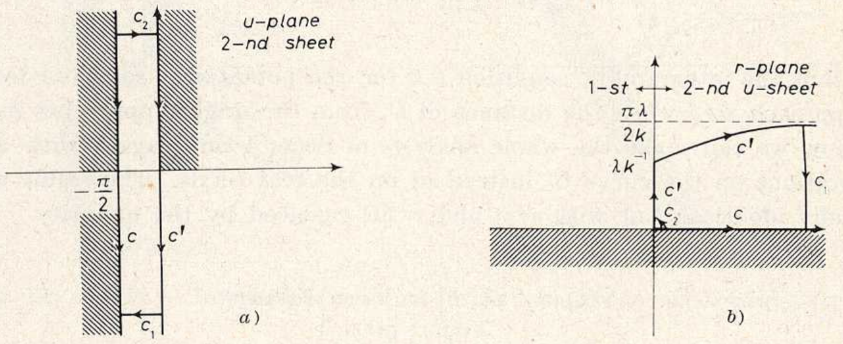

Fig. 3*a*) and *b*). – Integration contours in the complex u- and r-planes for $\arg \lambda = \pi/2$.

We now construct the following class of potentials: A potential $V(r)$ is said to belong to class D if

D.1) it belongs to class A,

D.2) it has a meromorphic continuation into the half-plane $\operatorname{Re} r > 0$, with a finite number of complex poles, and

D.3) in any direction in this half-plane, including parallels to the imaginary axis, the continuation satisfies

$$\int_0 |rV(r)\,\mathrm{d}r| < \infty\,, \qquad \int^\infty |V(r)\,\mathrm{d}r| < \infty\,.$$

Class D is contained in class C, because the poles must, as they are finite in number, all be within a finite distance from the origin. Thus the meromorphic continuation is analytic for large $|r|$.

The change of integration contour from C to C' is permitted for all potentials of the class C, if due care is taken of the contribution from the complex poles of the potential. This follows exactly as for $\arg \lambda < \pi/2$, except that we now also have to discuss the integral

$$I_2^{\pm} = \int_{C_2} r^{\frac{1}{2}} \mathrm{d}r J_\lambda(kr) V(r) f^{\pm}(r) . \tag{5.8}$$

When $r \to 0$ we find that

$$\begin{cases} J_\lambda(kr) = O(r^\lambda) , \\ f^{\pm}(r) = O(r^{\frac{1}{2}-\lambda}), \end{cases} \tag{5.9}$$

and thus the integrability condition D.3 for the potential is sufficient to make $I_2^{\pm}$ approach zero when C_2 shrinks to the origin in the r-plane.

Repeating now the analysis of Sect. **4** once more, with the modified integration contour C', the result will again be formally identical, and the quantity that replaces $v(\lambda)$ and $v^{\pm}(\lambda)$ this time is

$$\begin{aligned} \tilde{v}(\lambda) = C|\lambda|^{-1} \int_{C'} |r\sigma^2(\lambda, u) V(r)\, \mathrm{d}r| + \\ + 2\pi C|\lambda|^{-1} \sum_i |r_i \sigma^2(\lambda, u_i) \operatorname{Res} V(r_i)| \exp\left[|\operatorname{Im}(\lambda u_i)| - \operatorname{Im}(\lambda u_i)\right] . \end{aligned} \tag{5.10}$$

In this expression the last term is the contribution from the complex poles of the potential. These poles are λ-independent, and therefore, in the limit $\lambda \to \infty$ both σ^2 and the exponential factor approach 1, so that the contribution from the poles is $O(\lambda^{-1})$.

Exactly as for the class C potentials, it now follows that

$$\tilde{v}(\lambda) = o(1) \tag{5.11}$$

as $\lambda \to +i\infty$ for all class D potentials. Thus for those potentials, the Born series for the Jost function is an asymptotic series of the Erdélyi type with respect to the scale

$$\varphi_N(\lambda) = [\tilde{v}(\lambda)]^{N-\varepsilon}, \qquad \varepsilon > 0 , \tag{5.12}$$

when $\lambda \to \infty$ in any direction in $\operatorname{Re}\lambda \geqslant 0$.

If further

$$V(r) = O(r^{-1-\alpha}), \qquad \alpha > 0 , \tag{5.13}$$

when $r \to \infty$ in $|\arg r| \leqslant \pi/2$, we obtain in the same way as before

$$\tilde{v}(\lambda) = \begin{cases} O(\lambda^{-\alpha}) & \text{if } \alpha < 1 , \\ O(\lambda^{-1}) & \text{if } \alpha > 1 , \end{cases} \tag{5.14}$$

when $\lambda \to \infty$ in $|\arg \lambda| \leqslant \pi/2$.

We now turn our attention to the Yukawa potential, which belongs to class C, but not to class D, because it does not go fast enough to zero at infinity in directions parallel to the imaginary r-axis.

However, the treatment used above for class D potentials also gives some information for the Yukawa potential. In fact, the estimates of the Appendix, applied to $\tilde{v}(\lambda)$ with the integration contour of Fig. 3, give for the Yukawa potential

$$\tilde{v}(\lambda) = O(1) \qquad \text{when } \lambda \to +i\infty, \tag{5.15}$$

or

$$f^{-}(\lambda, k) = O(1) \qquad \text{when } \lambda \to +i\infty. \tag{5.16}$$

This result is not sufficient to show that the Born series is an asymptotic series, it only shows that the higher-order Born terms do not dominate when $\lambda \to i\infty$. It is also insufficient for proving that the Sommerfeld-Watson transformation is allowed, because eq. (5.16) does not exclude the possibility of $f^{-} \to 0$ when $\lambda \to i\infty$.

These difficulties are exactly the same as those encountered by MARTIN [10]. This is not surprising, because when we changed to the integration contour C' of Fig. 3, our method became very similar to that of MARTIN.

The method of improving eq. (5.16) given by MARTIN can be used in our formalism also. It consists in iterating the integral equation once, and obtaining a better bound for the new inhomogeneous term.

However, the asymptotic properties of the Born series for the Jost function with the Yukawa potential when $\lambda \to i\infty$ are known from the work of CALOGERO [5]. His result is that the Born series is an asymptotic series in the Erdélyi sense with respect to the scale

$$\varphi_N(\lambda) = \lambda^{-\frac{1}{2}N+\varepsilon}, \qquad \varepsilon > 0,$$

when $\lambda \to i\infty$. This result cannot be improved by using our method.

6. – Discussion.

We have shown above, that using uniform asymptotic estimates obtained from the formula (2.1) for Bessel functions, we get a very straightforward and powerful formalism for the study of asymptotic properties for large complex angular momenta. Within this formalism we can study the asymptotic character in the right-hand half of the complex λ-plane of the Born series for the Jost function for many different potentials, both fast decreasing and slowly decreasing.

For slowly decreasing potentials with analytic continuations, our method becomes very similar to that of MARTIN [10] and BESSIS [11]. However, we make slightly weaker assumptions about the potential (our class D), and we also study the whole Born series.

The bounds of Bessis for the location of Regge poles could possibly be improved by a modification of our method. Taking explicitly one or more terms of the asymptotic expansions, and using the Olver ([18]) bound for the rest term, one should even be able to calculate the location of the outer Regge poles (say with $|\lambda|\geqslant 3$) with controllable accuracy.

The Yukawa potential is just outside our potential class D, and is in a way the limiting case between the slowly decreasing and fast decreasing potentials.

For potentials decreasing faster than exponentially, that is of class B, we have shown above that the Born series for the Jost function is an asymptotic series when $\operatorname{Re}\lambda\to\infty$. This is the same result as that obtained for class C potentials, and therefore, we expect it to be common to a very large class of potentials, almost as large as class A.

However, when $\lambda\to i\infty$ it seems to be impossible for the Jost function to approach 1, except for smooth potentials which do not go faster to zero at infinity than exponentially, that is potentials like the Yukawa potential or the class D potentials. This is because, as we have seen, the inner parts of the potential tend to give contributions to the Jost function that grow exponentially on the imaginary axis, and these contributions must be counterbalanced by contributions from the outer parts of the potential. For this compensation to be effective, the outer parts of the potential must not be too small, and they must also have a smooth connection to the inner parts.

It thus seems likely that the Sommerfeld-Watson transformation is not allowed for any fast-decreasing potentials. We shall now give some further independent support for this supposition.

The partial-wave expansion for the scattering amplitude,

$$A(k,\theta)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\,\frac{f^{+}(l+\frac12,k)-f^{-}(l+\frac12,k)}{f^{-}(l+\frac12,k)}\,P_l(\cos\theta)\,, \tag{6.1}$$

can be tested for convergence with our uniform bounds. Since f^- approaches 1 on the real λ-axis, what we need for this test is an estimate of the difference $\Delta f=f^{+}-f^{-}$ on the real axis.

For this difference eqs. (4.21) and (4.23) give the expression

$$\Delta f(\lambda,k)=-\,i\pi\int_0^{\infty}r\,\mathrm{d}r\,V(r)J_\lambda^2(kr)-i\pi\int_0^{r'}r^{\frac12}\mathrm{d}r\int_r^{\infty}r'^{\frac12}\mathrm{d}r'J_\lambda(kr)V(r)K(r,r')J_\lambda(kr')\,, \tag{6.2}$$

and eqs. (4.11) and (4.29) give when applied to this equation the bound

$$|\Delta f(\lambda,k)|\leqslant[1+O(\lambda^{-1})]\,v_0(\lambda)\,[1+v(\lambda)\exp[v(\lambda)]]\,, \tag{6.3}$$

where

$$v_0(\lambda) = \frac{1}{4}\pi c^2 |\lambda|^{-1} \int_0^\infty r\,\mathrm{d}r\,|V(r)|\,\sigma^2(\lambda, u)\exp[-2\,\mathrm{Im}\,(\lambda u)]\,. \tag{6.4}$$

This integral can now be estimated for different potentials. As an illustrating example we take a potential decreasing so fast that the contribution from small r dominates, which probably means faster than $\exp[-2kr\ln r]$. Then we get

$$\Delta f(\lambda, k) = O(v_0(\lambda)) = O(\lambda^{-2\lambda}) \qquad \text{when } \lambda \to +\infty. \tag{6.5}$$

The partial-wave expansion is thus convergent in the whole complex $\cos\theta$-plane for such potentials. This means that the Lehmann ellipse covers the whole plane, and the scattering amplitude is an entire function of the momentum transfer squared.

We now test this entire function for its order ([25]). For large z, $P_l(z)$ behaves like z^l, and the partial-wave expansion becomes almost a power series. Therefore, the order of the entire function can be obtained by the formula ([25])

$$\varrho = \limsup_{n\to\infty} \frac{n\ln n}{-\ln|a_n|}\,, \tag{6.6}$$

where a_n is the coefficient of z^n in the power series. Thus the order is $\varrho = \frac{1}{2}$ for the scattering amplitude as an entire function of the momentum transfer squared, if the potential decreases faster than $\exp[-2kr\ln r]$. This result agrees with the result of Nussenzveig ([26]) for cut-off potentials.

For a potential decreasing slower than $\exp[-2kr\ln r]$, but still belonging to class B, eq. (6.5) will not be satisfied, and the scattering amplitude will be an entire function of the momentum transfer squared of order larger than $\frac{1}{2}$.

* * *

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([25]) R. P. Boas: *Entire Functions* (New York, 1954).

([26]) H. M. Nussenzveig: *Ann. of Phys.*, **21**, 344 (1963).

APPENDIX

In this Appendix we shall study bounds for large complex λ of the integral

$$v(\lambda) = C\,|\lambda|^{-1}\int_0^\infty \mathrm{d}r\, r\,|V(r)|\,\sigma^2(\lambda, u)\,, \tag{A.1}$$

where

$$\sigma(\lambda, u) = \frac{|\lambda u|^{\frac{1}{6}}}{|w|^{\frac{1}{2}}(1+|\lambda u|)^{\frac{1}{6}}}\,. \tag{A.2}$$

According to eq. (2.3) the function σ has the following asymptotic form ($x = kr/\lambda$)

$$\sigma(\lambda, u) = \begin{cases} |x|^{-\frac{1}{2}}\left[1 + O\left(\dfrac{1}{kr}\right) + O(x^{-2})\right] & \text{for} \quad |x| \gg 1\,, \\ |\frac{1}{3}\lambda|^{\frac{1}{6}}[1 + O(w^2) + O(\lambda w^3)] & \text{for } |x-1| \ll 1\,, \\ 1 + O\left(\dfrac{1}{\ln x}\right) & \text{for} \quad |x| \ll 1\,. \end{cases} \tag{A.3}$$

When r varies from 0 to ∞, σ thus varies from 1 through a maximum for finite r to zero in the limit $r \to \infty$. If the quotient λ/k is real, this maximum is $|\frac{1}{3}\lambda|^{\frac{1}{6}}$ and occurs at $r = \lambda/k$. For nonreal λ/k the maximum is smaller, because, as can be seen from eq. (A.2), σ cannot be large unless w is small.

Therefore, the integral (A.1) is largest for real λ/k, and to get a bound valid for all $\arg\lambda$ we need only consider such λ/k.

From the asymptotic form of σ we saw in Sect. **4** that if the integral (A.1) is to be convergent, the potential has to be of the class A, namely it must satisfy eq. (4.14).

We shall now show that for any potential of class A, $v(\lambda)$ approaches zero when $\lambda \to \infty$.

To do this, we cut the integration interval into four,

$$[0, a], \quad \left[a, \left(\frac{\lambda}{k}\right)^\beta\right], \quad \left[\left(\frac{\lambda}{k}\right)^\beta, \frac{\lambda}{k}(1+\varepsilon)\right], \quad \left[\frac{\lambda}{k}(1+\varepsilon), \infty\right], \tag{A.4}$$

where a, β and ε are positive constants, $\beta < 1$, and number the corresponding parts of $v(\lambda)$ with an index $i = 1, 2, 3, 4$. Further we observe that

$$\sigma^2(\lambda, u) < \frac{1}{|w|}\,. \tag{A.5}$$

Despite its singularity at $x=1$, this bound is useful because it is integrable.

From the first interval we get the contribution

$$\text{(A.6)} \qquad v_1(\lambda) = C\,|\lambda|^{-1}\left[1 + O\left(\frac{1}{\ln(ka/\lambda)}\right)\right]\int_0^a \mathrm{d}r\, r\,|V(r)|\,,$$

which approaches zero as $|\lambda|^{-1}$ when $\lambda\to\infty$.

For the contribution from the second interval we get the following bound:

$$\text{(A.7)} \qquad v_2(\lambda) < C\,|\lambda|^{-1}\int_a^{(\lambda/k)^\beta}\left|\frac{rV(r)}{w}\right|\mathrm{d}r \leqslant$$

$$\leqslant C\,|k|^{-\beta}|\lambda|^{-1+\beta}\left[1 + O\left(\frac{1}{(1-\beta)\ln(\lambda/k)}\right)\right]\int_a^\infty |V(r)|\,\mathrm{d}r\,,$$

which approaches zero when $\lambda\to\infty$, provided $\beta<1$.

For the contribution from the third interval we get the bound

$$\text{(A.8)} \qquad v_3(\lambda) < C\,|k|^{-1}\int_{(\lambda/k)^{\beta-1}}^{1+\varepsilon}\left|\frac{rV(r)}{\sqrt{x^2-1}}\right|\mathrm{d}x \leqslant C\,|k|^{-1}\,|rV(r)|_{r=(\lambda/k)^\beta}\int_0^{1+\varepsilon}\left|\frac{\mathrm{d}x}{\sqrt{x^2-1}}\right|,$$

which approaches zero when $\lambda\to\infty$, provided $\beta>0$, because of the integrability condition for $|V(r)|$.

Finally, in the fourth interval we observe that

$$\text{(A.9)} \qquad \frac{1}{|w|} = \frac{1}{(x^2-1)^{\frac12}} \leqslant \frac{1}{(2\varepsilon+\varepsilon^2)^{\frac12}} = \gamma\,,$$

where γ is a finite constant when $\varepsilon>0$. Thus

$$\text{(A.10)} \qquad v_4(\lambda) < \gamma C\,|k|^{-1}\int_{(\lambda/k)(1+\varepsilon)}^{\infty}|V(r)|\,\mathrm{d}r\,,$$

and the contribution from this part of the integration interval also approaches zero when $\lambda\to\infty$.

This concludes the proof that

$$\text{(A.11)} \qquad v(\lambda) = o(1) \qquad \text{when } |\lambda|\to\infty$$

for all class A potentials.

We now also want to show how fast $v(\lambda)$ approaches zero when the potential is not only integrable, but also satisfies

$$V(r) = O(r^{-1-\alpha})\,, \qquad \alpha > 0, \text{ when } r \to \infty, \tag{A.12}$$

that is, that there exist constants b, B such that

$$|V(r)| \leqslant B r^{-1-\alpha} \qquad \text{for } r \geqslant b\,. \tag{A.13}$$

For this potential, we take the common endpoint of the second and third intervals (A.4) as $(1-\varepsilon')\lambda/k$ instead of $(\lambda/k)^\beta$. In the second interval we then find that

$$\frac{1}{|w|} \leqslant \frac{1}{(2\varepsilon' - \varepsilon'^2)^{\frac{1}{2}}} = \gamma' < \infty \qquad \text{for } 0 < \varepsilon' < 1, \tag{A.14}$$

in analogy with eq. (A.9). For the contribution to $v(\lambda)$ from this interval we thus get the bound

$$v_2'(\lambda) < \gamma' C |\lambda|^{-1} \int_a^{(\lambda/k)(1-\varepsilon')} r^{-\alpha} \mathrm{d}r = \frac{\gamma' C}{|1-\alpha|} |\lambda|^{-1} \left| \left[\frac{\lambda}{k}(1-\varepsilon')\right]^{1-\alpha} - a^{1-\alpha} \right|, \tag{A.15}$$

provided a is chosen larger than the b of eq. (A.13). Equation (A.15) implies

$$v_2'(\lambda) = \begin{cases} O(\lambda^{-\alpha}) & \text{if } \alpha < 1, \\ O(\lambda^{-1}) & \text{if } \alpha > 1, \end{cases} \tag{A.16}$$

when $\lambda \to \infty$.

The contribution to $v(\lambda)$ from the new third interval can be estimated with eq. (A.8) with $\beta = 1$ to be

$$v_3'(\lambda) = O(\lambda^{-\alpha}) \qquad \text{when } \lambda \to \infty. \tag{A.17}$$

The last two equations, together with eqs. (A.6) and (A.10), now imply that for a potential satisfying eq. (A.12)

$$v(\lambda) = \begin{cases} O(\lambda^{-\alpha}) & \text{if } \alpha < 1, \\ O(\lambda^{-1}) & \text{if } \alpha > 1, \end{cases} \tag{A.18}$$

when $\lambda \to \infty$.

RIASSUNTO (*)

Da una generalizzazione della formula asintotica uniforme di Langer per funzioni di Bessel di ordine superiore si costruisce uno sviluppo asintotico della funzione di Green per lo scattering. Da questo sviluppo segue un semplice limite uniforme per la funzione di Green, che contiene il ben noto limite non uniforme $1/k$. Si usa poi questo limite uniforme per studiare le proprietà asintotiche per grandi momenti angolari complessi della serie di Born per la funzione di Jost nella teoria dello scattering non relativistico del potenziale per un gran numero di potenziali. Tra l'altro si ottengono i seguenti risultati. Se il potenziale ha una continuazione meromorfica, con un numero finito di poli complessi, nel semipiano $\mathrm{Re}\, r > 0$, che soddisfa la condizione $\int_0^a |rV(r)\,\mathrm{d}r| < \infty$, $\int_a^\infty |V(r)\,\mathrm{d}r| < \infty$, $0 < a < \infty$, su tutti i raggi di questo semipiano, allora la serie di Born per la funzione di Jost è una serie asintotica nel senso di Erdélyi e la funzione di Jost tende ad 1 quando $|\lambda| \to \infty$ in ogni direzione nella regione $|\arg \lambda| \leqslant \pi/2$. Così è permessa per tali potenziali la trasformazione di Sommerfeld-Watson. Per parallele all'asse r immaginario la suddetta condizione può essere un pò mitigata in modo da includere, per esempio, il potenziale di Yukawa. Il nostro metodo dà anche alcune informazioni sulle condizioni necessarie da imporre al potenziale per rendere lecita la trasformazione di Sommerfeld-Watson. Sebbene manchi una dimostrazione matematica rigorosa, si è giunti alle seguenti conclusioni. Il potenziale deve avere una connessione uniforme fra piccoli e grandi valori di r e non deve decrescere più rapidamente che in modo esponenziale quando $r \to \infty$ perchè sia permessa la trasformazione di Sommerfeld-Watson.

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ON ANGULAR MOMENTUM ANALYTICITY IN HARD CORE POTENTIAL SCATTERING

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ON ANGULAR MOMENTUM ANALYTICITY IN HARD CORE POTENTIAL SCATTERING

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Recently Predazzi and Regge [1)] and, independently, Limić [2)] have studied potential scattering in the presence of analytic hard cores, which for small r behave like $r^{-\alpha}$, $\alpha > 2$. Predazzi and Regge showed that in the presence of an analytic hard core the S-matrix can be continued in a trivial way to the half plane Re $\lambda < 0$ by means of the formula

$$S(-\lambda) = S(\lambda) \exp(-2i\pi\lambda) \,. \tag{1}$$

In this note it will be shown that this result holds also in the case of the usual non-analytic hard core, that is the one that corresponds to a potential $= +\infty$ for $r < r_0$. As is well known (e.g., ref. [3)]), a hard core of this form is equivalent to the boundary condition

$$\varphi(\lambda,k,r_0) = 0 \,, \quad \frac{\partial\varphi}{\partial r}(\lambda,k,r_0) = 1 \tag{2}$$

for the wave function φ. Because this boundary condition is independent of the angular momentum $l = \lambda - \frac{1}{2}$, and the Schrödinger equation contains λ only through λ^2, it follows at once that φ is an even function of λ.

With the notation of Froissart [4)] we can write the integral equation corresponding to the Schrödinger equation with the boundary condition (2) in the following way

$$\varphi(\lambda,k,r) = k^{-1} G_\lambda(kr,kr_0) + k^{-1}\int_{r_0}^{r} G_\lambda(kr,kr')\, V(r')\, \varphi(\lambda,k,r')\, dr' \,. \tag{3}$$

As usual, the S-matrix is defined through the asymptotic behaviour of φ

$$\varphi(\lambda,k,r) \underset{r\to\infty}{\sim} (2ik)^{-1}\,[W_2(\lambda,k)\, h_\lambda^{(1)}(kr) + W_1(\lambda,k)\, h_\lambda^{(2)}(kr)] \,, \tag{4}$$

to be

$$S(\lambda,k) = \frac{W_2(\lambda,k)}{W_1(\lambda,k)} \,. \tag{5}$$

Now [5)]

$$h_{-\lambda}^{(1)}(kr) = e^{i\pi\lambda}\, h_\lambda^{(1)}(kr) \,, \qquad h_{-\lambda}^{(2)}(kr) = e^{-i\pi\lambda}\, h_\lambda^{(2)}(kr) \,, \tag{6}$$

and therefore, because φ is an even function of λ, eq. (1) holds also with our hard core.

To find the analytic properties of the S-matrix we can use the method of Froissart [4)] on the integral equation (3). Iteration of (3) gives for W_1 the following formal expression [4)]

$$W_1 = k^{-1} h_\lambda^{(1)}(kr_0) + \sum_{n=1}^{\infty} k^{-n} \int_{r_0<r_1<\ldots<r_n<\infty} h_\lambda^{(1)}(kr_n) \times \prod_{m=1}^{n} G_\lambda(kr_m, kr_{m-1})\, V(r_m)\, dr_m \,. \tag{7}$$

In this equation there are no divergence problems at the lower limit of integration ($r_0 > 0$). Therefore, W_1 is an entire function of λ for any fixed $k \neq 0$ which does not cause the integral to diverge at infinity. For the k dependence the considerations of Froissart go through unchanged, to give the following final result:

Provided $e^{\mu r}\, V(r)$ is finite from the edge of the hard core to infinity for some positive μ, the S-matrix for potential scattering with a hard core is meromorphic in the topological product of the finite λ and k planes, except for the following branch points in the k plane

$$k = 0 \quad \text{and} \quad k = \pm\,[(m+1)/2i]\,\mathcal{P} \,, \tag{8}$$

where $\mathcal{P}$ is the set of singularities of

$$\tilde{u}(q) = \int_{r_0}^{\infty} V(r)\, e^{-qr}\, dr \,. \tag{9}$$

At infinity, the hard core will cause essential singularities for the S-matrix both in λ and k. Those singularities will prevent one both from making a Sommerfeld-Watson transformation, and from writing down a Mandelstam representation. However, by subtracting the contribution from a pure hard core one can go around this difficulty. The details about this procedure will be published elsewhere.

The author wishes to thank Dr. J. S. Bell for calling his attention to the Predazzi and Regge article, and for an illuminating discussion. I also wish to express my gratitude to Professor N. Svartholm for his constant kind interest in my work, and to Dr. J. Nilsson for stimulating discussions.

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References

1) E. Predazzi and T. Regge, Nuovo Cimento 24 (1962) 518 (note the error in eq. 3.3).
2) N. Limić, Nuovo Cimento 26 (1962) 581.
3) R. G. Newton, J. Math. Phys. 1 (1960) 319.
4) M. Froissart, J. Math. Phys. 3 (1962) 922.
5) G. N. Watson, A treatise on the theory of Bessel functions (Cambridge, 1922) p. 74.

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Asymptotic behaviour of the *S*-matrix in complex angular momentum for singular potentials

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Asymptotic behaviour of the S-matrix in complex angular momentum for singular potentials

By O. Brander

ABSTRACT

Using the theory for the asymptotic solution of differential equations, initiated by Langer and recently developed in great detail by Olver, the asymptotic properties of singular potential scattering theory are studied. More detailed results are obtained than those arrived at earlier (Jakšić and Limić; Tiktopoulos) with the JWKB-method. In particular, explicit formulae for for the positions and residues of the Regge poles are obtained. For potentials behaving at the origin like r^{-2-m}, $m>0$, or like $-r^{-2}\ln r$, explicit calculations are made. It is shown that all potentials having an analytic continuation into the right hand half r-plane satisfying

$$\int^{\infty} |V(r)\,dr| < \infty \quad \text{and} \quad \int_0 |[V(r) - r^{-2-m}]\, r^{1+(m/2)}\, dr| < \infty$$

in all directions of this half plane, except possibly for parallels to the imaginary axis, have the same asymptotic distribution of Regge poles as r^{-2-m}. For potentials with a logarithmic singularity at the origin the corresponding conditions on the analytic continuation are

$$\int^{\infty} |V(r)\,dr| < \infty \quad \text{and} \quad \int_0 |[V(r) + r^{-2}\ln r]\,(-\ln r)^{\frac{1}{2}}\, r\, \mathrm{d}r| < \infty$$

in all directions of the half-plane, including parallels to the imaginary axis, in order that $V(r)$ shall have the same asymptotic distribution of Regge poles as $-r^{-2}\ln r$.

1. Introduction

The theory of scattering for potentials with a singular repulsive core has been the subject of a large number of investigations in the last few years. The main reason for this seems to be a hope to learn more about unrenormalizable field theories from their connection with singular potentials [1]. Other reasons are the possible use of those potentials to explain the high-energy large-angle behaviour of the proton–proton scattering amplitude [2, 3], and a desire to investigate the somewhat unusual analytic properties of the scattering amplitude from such potentials, first stressed by Predazzi and Regge [4].

The simplest singular potentials to study are those with a nonanalytic hard core, like those which have been used since many years as phenomenological nucleon–nucleon potentials. The analytic properties of the scattering amplitude for such potentials were discussed in ref. [5].

The analytic properties for general singular potentials have been investigated by a great number of authors [6]. The $1/r^4$-potential, for which the Schrödinger equation reduces to the Mathieu equation [7], has been studied most extensively.

A powerful method for the explicit construction of the S-matrix for singular potentials has been developed by Cornille [8]. An interesting approximation method has been studied by Calogero [9].

Some mathematical aspects of the theory of scattering from singular potentials were recently discussed in ref. [10].

It is the purpose of this paper to examine in detail the asymptotic properties with regard to the complex angular momentum of wave functions, Jost functions and S-matrix in nonrelativistic potential scattering theory with singular potentials. We shall also discuss the limit of large, positive energies.

The method to be used can be described as a generalisation of the JWKB-method, giving an asymptotic series, of which the usual JWKB-approximation is the first term, and also giving at each order of approximation bounds for the rest term. The method is due to Langer, Cherry and Olver [11], and it was originally used to obtain an asymptotic series for Bessel functions of large order, valid uniformly in the argument of the function. This latter series for the Bessel functions was used in ref. [12] by the present author to discuss the asymptotic properties in the complex angular momentum plane for non-singular potentials. References to earlier work can be found there and in Olver [11].

In the present investigation we first concentrate on the power potential r^{-2-m}, $m>0$. For this potential the Schrödinger equation is, in section 2, solved asymptotically by the Olver method, and asymptotic series are obtained for the wave functions. These asymptotic series are valid uniformly in the radial variable, and they have well defined asymptotic properties when the angular momentum, the energy or the strength of the potential goes to infinity.

In section 3 we extract information concerning the asymptotic form of the Jost functions and the S-matrix in the complex angular momentum plane from the series which were previously obtained for the wave functions. In particular, we obtain explicit formulae for the positions and residues of the Regge poles.

Finally, in section 4 we discuss to what extent the power potential can be modified without changing the asymptotic distribution of Regge poles. Results pertaining to potentials with logarithmic singularities at the origin are also given there.

Five appendices containing mathematical details end the paper.

2. Asymptotic solution of the Schrödinger equation

In this section we shall use the method of Olver [11] to solve asymptotically in a large parameter the Schrödinger equation for a singular potential.

Consider a potential $V(r)$ more singular at the origin than the centrifugal barrier and satisfying the conditions

(1) $V(r) > 0$ when $0 < r < \delta$ for some positive δ.

(2) $$\int_0^\delta \frac{dr}{\sqrt{V(r)}}\left[\frac{1}{r^2}+\left(\frac{V'(r)}{V(r)}\right)^2+\frac{|V''(r)|}{V(r)}\right]<\infty.$$

(3) $r^2 V(r)$ monotonic for $0 < r < \delta$.

(4) $V(r)$ has an analytic continuation into the right-hand half r-plane outside the origin, satisfying

$$\int^\infty |V(r)\,dr| < \infty$$

in all directions of this half-plane, except possibly for parallels to the imaginary r-axis.

Conditions 1 to 3 are the Limić conditions [6], which ensure that the potential is repulsive and nonoscillating at short distances. Condition 4 is weak enough to include most of the physically interesting potentials, but strong enough to avoid complications with our method.

The most interesting singular potentials are those which at the origin behave like r^{-2-m}, $m>0$ or $r^{-2}(-\ln r)^m, m>0$. Those potentials can be represented in the following way

$$V(r) = r^{-2-m}\xi(r) \tag{2.1}$$

and $$V(r) = r^{-2}(-\ln r)^m \xi(r). \tag{2.2}$$

The function $\xi(r)$ is an analytic function in the right-hand half-plane and it approaches the value 1 at the origin. At infinity $\xi(r)$ must not grow too fast in order that condition 4 shall be satisfied.

In the rest of this section, and in the next, we shall consider a "power potential"

$$V(r) = r^{-2-m}, \quad m > 0, \tag{2.1$'$}$$

as explicit calculations are simple for this potential. However, the theory applies equally well to the potentials (2.1) and (2.2), as will be discussed in section 4.

For the power potential (2.1′) the Schrödinger equation reads,

$$\left[\frac{d^2}{dr^2}+\lambda^2 P(r;g,k,\lambda)+\frac{1}{4r^2}\right]\varphi(r)=0, \tag{2.3}$$

where $$P(r;g,k,\lambda)=\frac{k^2}{\lambda^2}-\frac{1}{r^2}-\frac{g^2}{\lambda^2}r^{-2-m}, \tag{2.4}$$

and the 'coupling constant' g^2 is a measure of the strength of the potential. It is expedient to introduce the following new variable (Bertocchi *et al.* [6]):

$$s=\frac{r}{r_0},\quad r_0=\left(\frac{g}{k}\right)^{2/(2+m)}, \tag{2.5}$$

because then
$$\left[\frac{d^2}{ds^2}+\lambda^2 p(s;f)+\frac{1}{4s^2}\right]\varphi(r_0 s)=0 \tag{2.6}$$

with
$$p(s;f)=f^2-s^{-2}-f^2 s^{-2-m} \tag{2.7}$$

and
$$f=\lambda^{-1} g^{2/(2+m)} k^{m/(2+m)}, \tag{2.8}$$

so that, formally, we have just two independent parameters, λ and f. However, to interpret the result we shall return to the physical parameters.

We observe [3] that for g and k constant, and λ approaching infinity, we have $f\to 0$, which according to eq. (2.8) corresponds to the weak coupling limit. On the other hand, for λ and g constant and k approaching infinity we have $f\to\infty$, which is the strong coupling limit. We shall obtain formulae valid both in the weak and the strong coupling limits.

It is well known from the theory of the JWKB-approximation that the zeros of $p(s;f)$, the turning points, play a fundamental role. Therefore, we begin by examining the location of those points.

It is easy to see that for large f and integer m p has $2+m$ simple zeros distributed around the unit circle. When m is not an integer, then we make a cut along the negative real s-axis and a number of zeros appear also on other Riemann sheets.

It is also easy to find the zeros when f is small. Two of them can be found near the points $\pm f^{-1}$, whereas the other m zeros are found near the origin.

Let s_0 be that zero, which is near f^{-1} for f small, and let s_1 be the zero which is near the origin for f small and which has the smallest positive argument for real positive g, k and λ. Then

$$\begin{cases} s_0=f^{-1}\left[1+\frac{1}{2}f^{2+m}+O(f^{4+2m})\right], \\ s_1=e^{i\pi/m}f^{2/m}\left[1+\dfrac{1}{m}e^{-2i\pi/m}f^{2+4/m}+O(f^{4+8/m})\right], \end{cases} \tag{2.9}$$

in the limit $f\to 0$.

In Fig. 1 we illustrate for $m=2$ how the two zeros s_0 and s_1 move when f is varied for some fixed values of $\arg f$. We see that when $|f|\to\infty$, s_0 approaches unity for $|\arg f|<\pi/(2+m)$, but for $-\arg f>\pi/(2+m)$ it is the zero s_1 that approaches unity. When $|f|\to\infty$ we have the asymptotic formulae

$$s_0=\begin{cases} 1+\dfrac{1}{2+m}f^{-2}+O(f^{-4}) \quad \text{for} \quad |\arg f|<\dfrac{\pi}{2+m} \\ e^{2i\pi/(2+m)}\left[1+\dfrac{1}{2+m}e^{-4i\pi/(2+m)}f^{-2}+O(f^{-4})\right] \text{ for } \dfrac{\pi}{2+m}<-\arg f<\dfrac{3\pi}{2+m} \\ e^{4i\pi/(2+m)}\left[1+\dfrac{1}{2+m}e^{-8i\pi/(2+m)}f^{-2}+O(f^{-4})\right] \text{ for } \dfrac{3\pi}{2+m}<-\arg f<\dfrac{5\pi}{2+m} \quad \text{etc.} \end{cases} \tag{2.10 a}$$

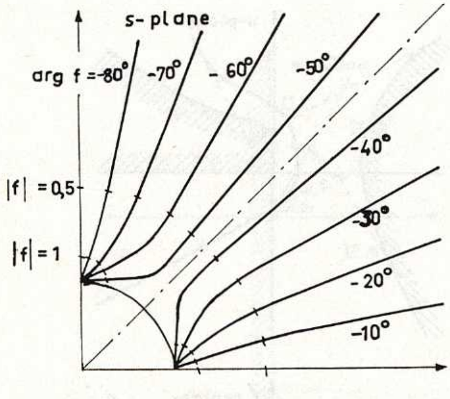

Fig. 1 A

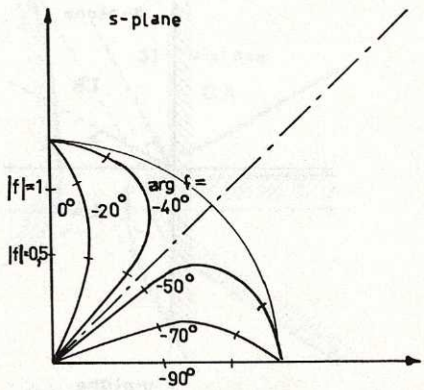

Fig. 1 B

Fig. 1 A. For arg f fixed and $|f|$ varying from 0 to ∞, the zero s_0 moves in from infinity to a point on the unit circle. Figure drawn for $m=2$.

Fig. 1 B. For arg f fixed and $|f|$ varying from 0 to ∞, the zero s_1 moves out from the origin to a point on the unit circle. Figure drawn for $m=2$.

We see that if we vary $|f|$ and keep arg f constant, s_0 stays within that sector of width $2\pi/(2+m)$ from which it originated for small f.

For s_1 we have the asymptotic formula

$$s_1 = \begin{cases} e^{2i\pi/(2+m)}\left[1+\dfrac{1}{2+m}e^{-4i\pi/(2+m)}f^{-2}+O(f^{-4})\right] & \text{for} \quad 0 \leqslant -\arg f < \dfrac{\pi}{2+m} \\ 1+\dfrac{1}{2+m}f^{-2}+O(f^{-4}) \quad \text{for} \quad \dfrac{\pi}{2+m} < -\arg f \leqslant \dfrac{\pi}{2} \end{cases} \tag{2.10 b}$$

when $|f|\to\infty$.

The situation is now the following. We have a second order differential equation with two irregular singular points at 0 and ∞ and a number of turning points, which move when the parameters are varied. Except for the Mathieu equation, corresponding to $m=2$, there is as yet no general theory of such equations (Fubini *et al.* [6]).

However, we shall show that, when one parameter is large, e.g. λ, then it is possible to use the method of Olver [11–13] to obtain asymptotic series for the solutions of such equations, and these series are valid uniformly in the variable in a region containing the two singular points and one of the turning points. This is accomplished by transforming the differential equation in such a way, that the irregular singular points fall at $\pm\infty$, and that the conditions at infinity for the application of theorem B of Olver [11] are satisfied. The rest is then a straight-forward application of this theorem. The difficulties which are encountered are mainly related to the complicated nature of the transformation.

We shall use different transformations of the differential equation depending on at which of the two turning points s_0 and s_1 we want the expansion to be

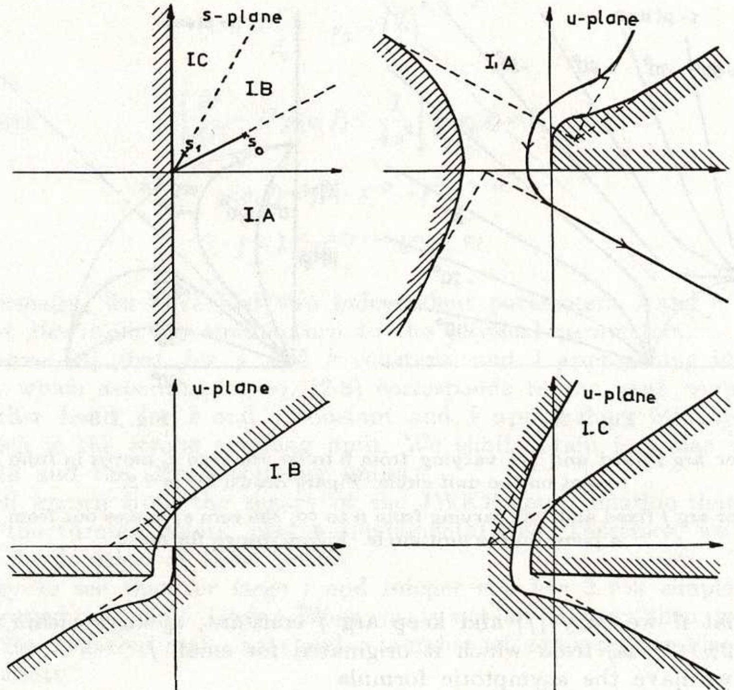

Fig. 2 A, B, C and D. The complex s- and u-planes for $\arg s_1 > \arg s_0$. Figure drawn for $m = 2$, $f = 0.5 \exp(-i\pi/6)$.

valid. The two transformations are obtained by introducing the new independent variables z and y, respectively, where

$$z = -(\tfrac{3}{2}u)^{\frac{2}{3}}; \quad u = \int_{s_0}^{s} p^{\frac{1}{2}}(s; f)\, ds, \tag{2.11}$$

and

$$y = -(\tfrac{3}{2}v)^{\frac{2}{3}}; \quad v = \int_{s_1}^{s} p^{\frac{1}{2}}(s; f)\, ds. \tag{2.12}$$

To make $p^{\frac{1}{2}}$ single-valued, we introduce cuts from each of the zeros of p to the origin. We consider only the right-hand half r-plane, that is, assuming r_0 to be real, the right-hand half s-plane.

The transformations (2.11) and (2.12) are conformal mappings of the complex s-plane onto the complex z- and y-planes, respectively, except for the zeros of p. At those points the mapping is not conformal. However, the mapping (2.11) is conformal as s_0 and the mapping (2.12) is conformal at s_1 due to the exponent $\frac{2}{3}$. This is shown in appendix A.

In appendix B the transformations (2.11) and (2.12) are studied, and formulae valid in the asymptotic regions are derived. For the special case of small $|f|$

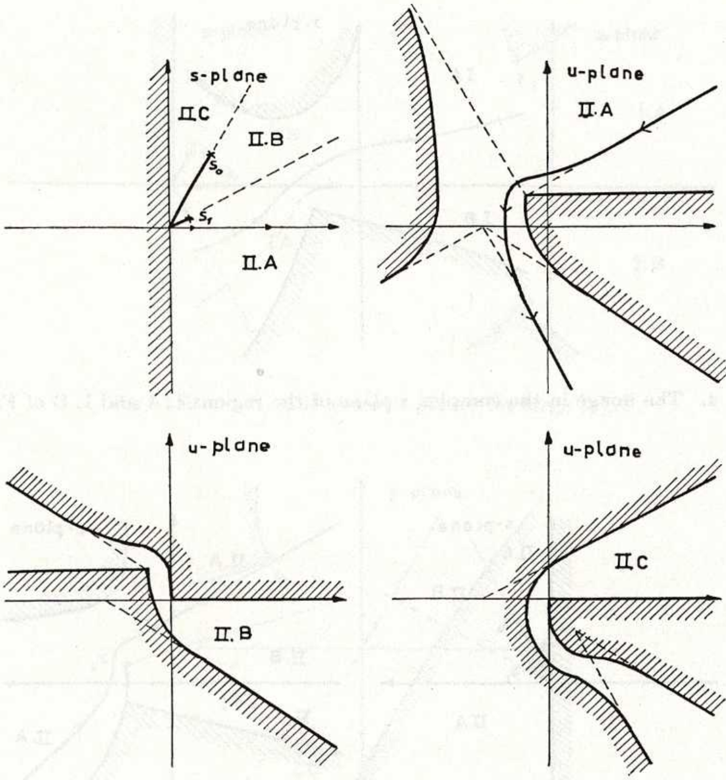

Fig. 3 A, B, C and D. The complex s- and u-planes for $\arg s_1 < \arg s_0$. Figure drawn for $m = 2$, $f = 0.5 \exp(-2i\pi/6)$.

the form of the mapping (2.11) is illustrated in the Figs. 2–5, and that of the mapping (2.12) in the Fig. 6. The v-plane is not shown separately, since it differs from the u-plane only by a shift of the origin.

Also, the y-plane is not shown for the case I, when $\arg s_1 > \arg s_0$. In this case the mapping (2.12) is not $1-1$ on any region containing the whole positive real s-axis and the turning point s_1, and, therefore, it can not be used for our purpose. On the other hand, when $\arg s_1 > \arg s_0$, that is for $|\arg f| < \pi/(2+m)$, then the transformation (2.11) will give all the results we need. It is only for $\arg s_1 < \arg s_0$ that we will need both transformations to get the complete asymptotic behaviour of the S-matrix.

For large $|f|$ and for $|\arg f| < \pi/(2+m)$ the mapping (2.11) is illustrated in Fig. 7 and (2.12) in Fig. 8. For $-\frac{1}{2}\pi \leqslant \arg f \leqslant -\pi/(2+m)$ and $|f|$ large s_1 is the zero close to unity according to eq. (2.10 b) and the y-plane will look like the z-plane of Fig. 7.

Introducing the new dependent variables

$$\psi(z) = \left(\frac{ds}{dz}\right)^{-\frac{1}{2}} \varphi(r_0 s) \quad \text{and} \quad \chi(y) = \left(\frac{ds}{dy}\right)^{-\frac{1}{2}} \varphi(r_0 s), \tag{2.13}$$

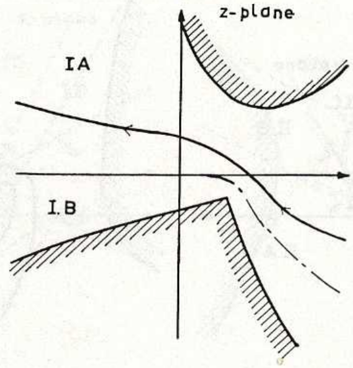

Fig. 4. The image in the complex z-plane of the regions I. A and I. B of Fig. 2.

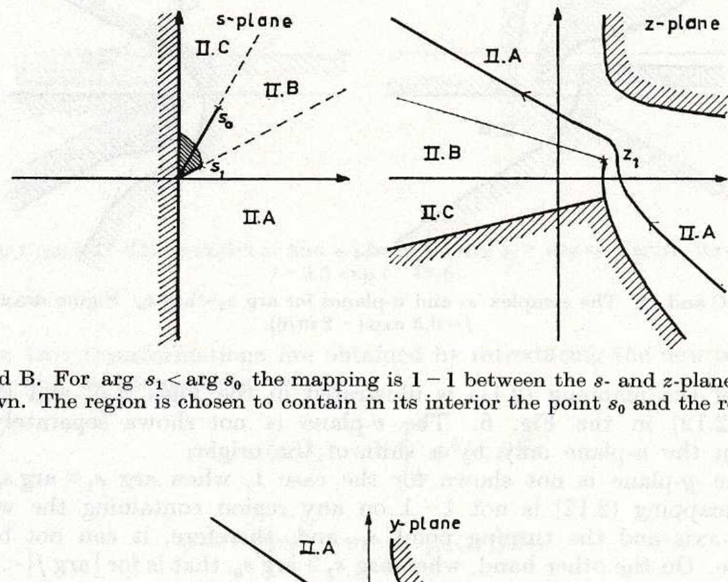

Fig. 5 A and B. For $\arg s_1 < \arg s_0$ the mapping is $1-1$ between the s- and z-planes for the region shown. The region is chosen to contain in its interior the point s_0 and the real s-axis.

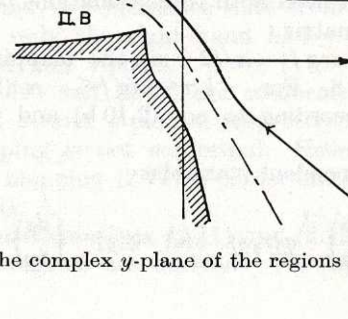

Fig. 6. The image in the complex y-plane of the regions II. A and II. B of Fig. 3.

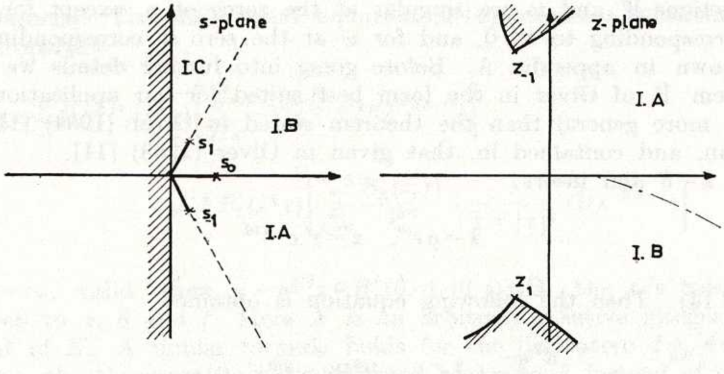

Fig. 7 A and B. The s- and z-planes for large $|f|$.

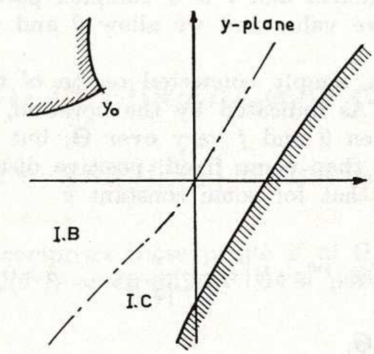

Fig. 8. The y-plane for large $|f|$.

the differential equation (2.6) is given by

$$\frac{d^2\psi}{dz^2}=[\lambda^2 z+W(z;f)]\,\psi(z), \tag{2.14}$$

where
$$W(z;f)=\frac{5}{16\,z^2}+\frac{z}{4\,s^2 p(s;f)}-\frac{zp''(s;f)}{4[p(s;f)]^2}+\frac{5z[p'(s;f)]^2}{16[p(s;f)]^3}. \tag{2.15}$$

Primes denote differentiation with respect to s. Analogously, the transformation (2.12) gives

$$\frac{d^2\chi}{dy^2}=[\lambda^2 y+w(y;f)]\,\chi(y), \tag{2.16}$$

where
$$w(y;f)=\frac{5}{16\,y^2}+\frac{y}{4s^2 p(s;f)}-\frac{yp''(s;f)}{4\,[p(s;f)]^2}+\frac{5\,y[p'(s;f)]^2}{16\,[p(s;f)]^3}. \tag{2.17}$$

The functions W and w are singular at the zeros of p, except for W at the zero s_0, corresponding to $z=0$, and for w at the zero s_1, corresponding to $y=0$. This is shown in appendix A. Before going into further details we shall now give theorem B of Olver in the form best suited for our application. In this form it is more general than the theorem stated in Olver (1954) [11] but less general than, and contained in, that given in Olver (1958) [11].

Put $\arg \lambda=\theta$ and insert

$$\lambda=\varrho e^{i\theta}, \quad z=z' e^{-\frac{2}{3} i\theta} \tag{2.18}$$

into eq. (2.14). Then the following equation is obtained

$$\frac{d^2\psi}{dz'^2}=[\varrho^2 z'+e^{-\frac{4}{3}i\theta} W(e^{-\frac{2}{3}i\theta} z'; f)]\psi, \tag{2.14'}$$

where ϱ, θ are real parameters and f is a complex parameter. We assume that ϱ takes on a large positive value and we allow θ and f to vary over a certain region $\mathbf{\Theta}$.

Let $\mathbf{D}_z(\theta, f)$ be an open, simply connected region of the z'-plane, in which W of eq. (2.14′) is regular. As indicated by the notation, we allow the boundaries of the region to vary when θ and f vary over $\mathbf{\Theta}$, but they are not allowed to come closer to the origin than some fixed, positive distance b.

Let us further assume that for some constant c

$$|W(e^{-\frac{2}{3}i\theta} z'; f)|<\frac{c}{1+|z'|^{\frac{1}{2}+\sigma}}, \quad \sigma>0, \tag{2.19}$$

for all $z' \in \mathbf{D}_z(\theta, f)$, $(\theta, f) \in \mathbf{\Theta}$.

Next, we define $\mathbf{G}_z(\theta, f)$ to be a closed subdomain of $\mathbf{D}_z(\theta, f)$, having the properties

(i) $\mathbf{G}_z(\theta, f)$ contains the circle $|z'| \leqslant b$.

(ii) The distance between each boundary point z_0' of $\mathbf{G}_z(\theta, f)$ and each boundary point of $\mathbf{D}_z(\theta, f)$ is not less than $d/|z_0'|^{\frac{1}{2}}$, where d is a positive constant, assignable independently of θ and f.

(iii) For some path lying wholly in $\mathbf{G}_z(\theta, f)$

$$\int_0^{z'} \frac{|dt|}{1+|t|^{1+\sigma_1}}<\text{const}, \quad \sigma_1=\min(\sigma, \tfrac{3}{2}),,$$

uniformly in z', θ and f.

Using the notation

$$P_j(x)=\mathrm{Ai}\,(\varrho_j x); \quad \varrho_1=1, \quad \varrho_{2,3}=e^{\pm\frac{2}{3}\pi i}, \tag{2.20}$$

where Ai stands for the Airy function, we can now state the following theorem [11]:

Olver's theorem. The differential equation (2.14) possesses solutions $\psi_j(z;\lambda,f)$ with the properties

$$\psi_j(z;\lambda,f)=P_j(\lambda^{\frac{2}{3}}z)\left[\sum_{n=0}^{N}\frac{A_n(z;f)}{\lambda^{2n}}+O(\lambda^{-2N-1})\right]$$

$$+\lambda^{-\frac{4}{3}}P_j'(\lambda^{\frac{2}{3}}z)\left[\sum_{n=0}^{N-1}\frac{B_n(z;f)}{\lambda^{2n}}+\frac{1}{1+|z|^{\frac{1}{2}}}O(\lambda^{-2N+1})\right] \tag{2.21}$$

as $\varrho=|\lambda|\to\infty$, valid when $z'=e^{\frac{2}{3}i\theta}z\in\mathbf{H}_z^j(\theta,f),(\theta,f)\in\mathbf{\Theta}$, the O's being uniform with respect to z, θ and f. Here N is an arbitrary positive integer and ψ_j is independent of N. A similar formula holds for the derivative $d\psi_j/dz$.

In stating the theorem, we have returned to z and λ instead of using z', ϱ and θ, because the theorem is slightly simpler in that formulation.

The coefficients A_n and B_n are defined as follows,

$$\begin{cases} A_0\equiv 1,\\ B_n(z;f)=\frac{1}{2}z^{-\frac{1}{2}}\displaystyle\int_0^z z^{-\frac{1}{2}}[W(z;f)A_n(z;f)-A_n''(z;f)]\,dz,\\ A_{n+1}(z;f)=-\frac{1}{2}B_n'(z;f)+\frac{1}{2}\displaystyle\int W(z;f)B_n(z;f)\,dz,\end{cases} \tag{2.22}$$

and the region $\mathbf{H}_z^j(\theta,f)$ comprises those points z' of $\mathbf{G}_z(\theta,f)$ which can be joined to a point $a_j(\theta,f)$, in $\mathbf{G}_z(\theta,f)$ or at infinity, by a path $\mathcal{P}_j$ having the following properties

(i) $\mathcal{P}_j$ lies in $\mathbf{G}_z(\theta,f)$.

(ii) $\mathcal{P}_j$ comprises a finite number of Jordan arcs, each with a parametric equation of the form $t=t(\tau)$, where τ is the real parameter of the arc; $t''(\tau)$ is continuous and $t'(\tau)$ does not vanish. If $a_j(\theta,f)$ is at infinity it is on a straight line $\mathcal{L}_j$ in $\mathbf{G}_z(\theta,f)$, and $\mathcal{P}_j$ coincides with $\mathcal{L}_j$ for sufficiently large $|t|$.

(iii) $$\int_{\mathcal{P}_j}\frac{|dt|}{1+|t|^{1+\sigma_1}}<\text{const}$$

uniformly in z', θ and f.

(iv) As t traverses $\mathcal{P}_j$ from $a_j(\theta,f)$ to z', $|\exp\{\frac{2}{3}(\varrho_j t)^{\frac{3}{2}}\}|$ is monotonic decreasing, where ϱ_j is defined by eq. (2.20).

As the point $a_j(\theta,f)$, $j=1,2,3$, we take a point at infinity in the corresponding sector $\mathbf{S}_j$: $|\arg(\varrho_j\lambda^{\frac{2}{3}}z)|<\frac{1}{3}\pi$. $\mathbf{S}_j$ is the sector in which the solution ψ_j is asymptotically small.

Next we must investigate the transformations (2.11) and (2.12) to see whether the conditions for the application of Olver's theorem are satisfied.

Asymptotically, eqs. (2.7) and (2.11) imply that

$$u \sim \begin{cases} fs & \text{as} \quad |s| \to \infty \\ \pm i \dfrac{2f}{m} s^{-m/2} & \text{as} \quad |s| \to 0 \end{cases} \tag{2.23}$$

for the "power potential". Together with eqs. (2.11) and (2.15) this gives that

$$W(z; f) = \begin{cases} O(zf^{-2}s^{-2}) = O(z^{-2}) & \text{as} \quad z \to +\infty \\ O(zf^{-2}s^{m}) = O(z^{-2}) & \text{as} \quad z \to -\infty, \end{cases} \tag{2.24}$$

which shows that the condition (2.19) is satisfied, provided $\mathbf{D}_z(\theta, f)$ does not contain the turning points z_i, $i = 1, 2, \ldots$, corresponding to the zeros s_i of $p(s; f)$ or the immediate neighbourhood of those points. As we have already discussed, there is no singularity of W at $z = 0$ corresponding to the zero s_0.

The remaining point to prove now is that there exists a simply connected region of the z-plane containing

(i) the circle $|z| \leqslant b$ and

(ii) the whole image of the positive real r-axis.

After a rotation through the angle $\frac{2}{3}\theta$ this region is to be used as $\mathbf{D}_z(\theta, f)$. The condition (i) is necessary for the Olver theorem to be applicable, the condition (ii) is necessary in order for the resulting series to be valid uniformly in $r \in (0, \infty)$.

For small f the two conditions above are satisfied by the regions given in the Figs. 4 and 5 for arg $s_1 >$ or $<$ arg s_0, respectively, except for the fact that for large m one might have to restrict the range of arg s when s is small so that the regions will be simply connected.

The allowed values of arg f for small f are

$$|\arg f| \leqslant \frac{\pi}{2} - \varepsilon, \quad \varepsilon > 0, \quad \text{small } |f|, \tag{2.25 a}$$

where the ε is introduced to exclude the neighbourhood of the point z_1, where W is singular.

For large f the restriction of arg f is

$$|\arg f| < \frac{\pi}{2+m}, \quad \text{large } |f|, \tag{2.25 b}$$

because in that domain of $|f|$ the construction of Fig. 5 is not possible, since the image of the positive real s-axis will not be contained in the same Riemann sheet as the image of the point s_0. In the two equations above we still assume that arg $r_0 = 0$.

When $\mathbf{D}_z(\theta, f)$ has been chosen as discussed above, $\mathbf{G}_z(\theta, f)$ can be obtained from $\mathbf{D}_z(\theta, f)$ just by slicing off a thin strip along the boundary according to the prescription (ii) for $\mathbf{G}_z(\theta, f)$.

Finally, $\mathbf{H}_z^j(\theta, f)$ is to be constructed according to the prescriptions (i–iv) above. To ensure that the origin and the whole image of the real positive r-axis will remain inside $\mathbf{H}_z^j(\theta, f)$ we must restrict the angle θ such that the point z_1 and the image of small, real, positive r fall inside the sector $\mathbf{S}_1$. The restrictions (2.25) together with this restriction for θ yields that Olver's theorem is applicable in the z-plane provided (θ, f) is inside one the following two regions,

$$\mathbf{\Theta}_0': \begin{cases} |\arg f| < \dfrac{\pi}{2+m} \\ \text{any } |f| \\ -\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2} \end{cases} \tag{2.26a}$$

and

$$\mathbf{\Theta}_0'': \begin{cases} \dfrac{\pi}{2+m} < -\arg f \leqslant \dfrac{\pi}{2} - \varepsilon \\ |f| \leqslant a_1 \\ -\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2} - b_1, \end{cases} \tag{2.26b}$$

where a_1 and b_1 are positive numbers and $b_1 \to 0$ if $a_1 \to 0$.

Let us now consider the y-plane. For small f the y-plane is shown in Fig. 6. For large f it is shown in Fig. 8 for $0 \leqslant -\arg f < \pi/(2+m)$ and in Fig. 7 for $\pi/(2+m) < -\arg f \leqslant \frac{1}{2}\pi$. Constructing now regions $\mathbf{D}_y(\theta, f)$, $\mathbf{G}_y(\theta, f)$ and $\mathbf{H}_y^j(\theta, f)$ in the y-plane in complete analogy to the construction above with regard to the z-plane, we find that Olver's theorem is applicable in the y-plane if (θ, f) is inside the following region,

$$\mathbf{\Theta}_1: \begin{cases} \dfrac{\pi}{2+m} < -\arg f \leqslant \dfrac{\pi}{2} \\ \text{any } |f| \\ -\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}. \end{cases} \tag{2.27}$$

Let us now define the physical wave functions $\varphi^{\pm}$ and φ_R as those solutions of eq. (2.3), which satisfy the boundary conditions

$$\varphi^{\pm}(r) \underset{r\to\infty}{\sim} e^{\mp ikr} = e^{\mp i\lambda f s} \tag{2.28}$$

and for the "power potential",

$$\varphi_R(r) \underset{r\to 0}{\sim} r^{(2+m)/4} \exp\left\{-\frac{2g}{m} r^{-\frac{1}{2}m}\right\} = (r_0 s)^{(2+m)/4} \exp\left\{-\frac{2\lambda f}{m} s^{-\frac{1}{2}m}\right\}, \tag{2.29}$$

respectively.

From the asymptotic form of the mapping from r to z, eq. (2.23), it then follows that φ_R is asymptotically small in the same sector as ψ_1, φ^+ in the same sector as ψ_3 and φ^- in the same sector as ψ_2. Therefore, according to eq. (2.13) we must have the following relations between φ and ψ,

$$\varphi_R(r) = C_R(\lambda, f)\left(-\frac{z}{P}\right)^{\frac{1}{4}} \psi_1(z) \tag{2.30}$$

and

$$\varphi^{\pm}(r) = C_{\pm}(\lambda, f)\left(-\frac{z}{P}\right)^{\frac{1}{4}} \psi_{3,2}(z). \tag{2.31}$$

The functions C_R and $C_{\pm}$ are independent of r and z, and P is given by eq. (2.4).

Similarly, application of Olver's theorem to eq. (2.16) gives

$$\chi_j(y) \underset{\lambda\to\infty}{\sim} P_j(\lambda^{\frac{2}{3}} y) \sum_{n=0}^{\infty} \frac{a_n(y;f)}{\lambda^{2n}} + \lambda^{-\frac{4}{3}} P_j'(\lambda^{\frac{2}{3}} z) \sum_{n=0}^{\infty} \frac{b_n(y;f)}{\lambda^{2n}}, \tag{2.32}$$

with the same meaning of the index j as above, with $a_0 \equiv 1$ and

$$\left\{\begin{aligned} b_n(y;f) &= \tfrac{1}{2} y^{-\frac{1}{2}} \int_0^y y^{-\frac{1}{2}} [w(y;f)\, a_n(y;f) - a_n''(y;f)]\, dy, \\ a_{n+1}(y;f) &= -\tfrac{1}{2} b_n'(y;f) + \tfrac{1}{2} \int w(y;f)\, b_n(y;f)\, dy. \end{aligned}\right. \tag{2.33}$$

The analogues of the eqs. (2.30) and (2.31) are

$$\varphi_R(r) = c_R(\lambda, f)\left(-\frac{y}{P}\right)^{\frac{1}{4}} \chi_1(y) \tag{2.34}$$

and

$$\varphi^{\pm}(r) = c_{\pm}(\lambda, f)\left(-\frac{y}{P}\right)^{\frac{1}{4}} \chi_{3,2}(y). \tag{2.35}$$

It now remains to determine the factors $C_{R,\pm}$ and $c_{R,\pm}$. This can be done by comparing the asymptotic forms of the two sides of the eqs. (2.30), (2.31), (2.34) and (2.35).

To do this we need the following formulae from appendix B

$$\lambda u \sim \begin{cases} kr - \frac{1}{2}\pi\lambda + \delta(\lambda, f) & \text{as} \quad r \to \infty \\ i\,\dfrac{2g}{m}\, r^{-\frac{1}{2}m} + \delta_0(\lambda, f) + \Delta(\lambda, f) & \text{as} \quad r \to 0 \end{cases} \tag{2.36}$$

and

$$\lambda v \sim \begin{cases} kr - \frac{1}{2}\pi\lambda + \delta(\lambda, f) - \Delta(\lambda, f) & \text{as} \quad r \to \infty \\ i\,\dfrac{2g}{m}\, r^{-\frac{1}{2}m} + \delta_0(\lambda, f) & \text{as} \quad r \to 0. \end{cases} \tag{2.37}$$

Here,

$$\delta(\lambda, f) = \lim_{r\to\infty} (\lambda u - kr + \tfrac{1}{2}\pi\lambda) \tag{2.38}$$

is the usual JWKB phase-shift,

$$\delta_0(y,f)=\lim_{r\to 0}\left(\lambda v - i\,\frac{2g}{m}\,r^{-\frac{1}{4}m}\right), \tag{2.39}$$

and

$$\Delta(\lambda,f)=\lambda u-\lambda v=\lambda\int_{s_0}^{s_1} p^{\frac{1}{2}}(s;f)\,ds. \tag{2.40}$$

Further, the following limits are involved

$$\begin{cases}\alpha_{1n}(f)=\lim\limits_{z\to+\infty} A_n(z;f); & \alpha_{2n}(f)=\lim\limits_{y\to+\infty} a_n(y;f);\\ \alpha_{3n}(f)=\lim\limits_{z\to-\infty} A_n(z;f); & \alpha_{4n}(f)=\lim\limits_{y\to-\infty} a_n(y;f)\end{cases} \tag{2.41}$$

and

$$\begin{cases}\beta_{1n}(f)=\lim\limits_{z\to+\infty} z^{\frac{1}{2}}B_n(z;f); & \beta_{2n}(f)=\lim\limits_{y\to+\infty} y^{\frac{1}{2}}b_n(y;f);\\ \beta_{3n}(f)=\lim\limits_{z\to-\infty} z^{\frac{1}{2}}B_n(z;f); & \beta_{4n}(f)=\lim\limits_{y\to-\infty} y^{\frac{1}{2}}b_n(y;f).\end{cases} \tag{2.42}$$

That these limits exist is shown explicitly in Lemma 1 of Olver (1954) [11]. For each $n>0$, two of the four α's can be chosen equal to zero by ajusting the integration constants of eqs. (2.22) and (2.33), say $\alpha_{3n}=\alpha_{4n}=0$.

The following formulae are obtained for the C's

$$\begin{cases}C_R^{-1}(\lambda,f)\underset{\lambda\to\infty}{\sim}\dfrac{1}{2\sqrt{\pi}}\,r_0^{-\frac{1}{4}m}f^{-\frac{1}{2}}\lambda^{-1/6}\,e^{i\Delta(\lambda,f)+i\delta_0(\lambda,f)}\left[1+\sum\limits_{n=1}^{\infty}\dfrac{\alpha_{1n}(f)}{\lambda^{2n}}-\sum\limits_{n=0}^{\infty}\dfrac{\beta_{1n}(f)}{\lambda^{2n+1}}\right]\\ C_\pm^{-1}(\lambda,f)\underset{\lambda\to\infty}{\sim}\dfrac{1}{2\sqrt{\pi}}\,r_0^{\frac{1}{2}}f^{-\frac{1}{2}}\lambda^{-1/6}\,i^{\pm(\lambda-1/6)}\,e^{\mp i\delta(\lambda,f)}\left[1+\sum\limits_{n=0}^{\infty}\dfrac{\beta_{3n}(f)}{\lambda^{2n+1}}\right]\end{cases} \tag{2.43}$$

and

$$\begin{cases}c_R^{-1}(\lambda,f)\underset{\lambda\to\infty}{\sim}\dfrac{1}{2\sqrt{\pi}}\,r_0^{-\frac{1}{4}m}f^{-\frac{1}{2}}\lambda^{-1/6}\,e^{i\delta_0(\lambda,f)}\left[1+\sum\limits_{n=1}^{\infty}\dfrac{\alpha_{2n}(f)}{\lambda^{2n}}-\sum\limits_{n=0}^{\infty}\dfrac{\beta_{2n}(f)}{\lambda^{2n+1}}\right].\\ c_\pm^{-1}(\lambda,f)\underset{\lambda\to\infty}{\sim}\dfrac{1}{2\sqrt{\pi}}\,r_0^{\frac{1}{2}}f^{-\frac{1}{2}}\lambda^{-1/6}\,i^{\pm(\lambda-1/6)}\,e^{\mp i\delta(\lambda,f)\pm i\Delta(\lambda,f)}\left[1+\sum\limits_{n=0}^{\infty}\dfrac{\beta_{4n}(f)}{\lambda^{2n+1}}\right].\end{cases} \tag{2.44}$$

The derivation of these formulae is given in appendix C.

The eqs. (2.21), (2.22), (2.30), (2.31) and (2.43) now give the complete asymptotic series of the physical wave functions as $|\lambda|\to\infty$, considering f as a free parameter. The series is valid uniformly in $r\in(0,\infty)$ and $(\theta,f)\in\mathbf{\Theta}_0$, and is an asymptotic series in the sense of Erdélyi [12] with respect to the scale

$$\lambda^{-n-\eta},\quad \eta>0, \tag{2.45}$$

as $|\lambda|\to\infty$.

Now let us express these results in terms of the physical parameters g, k and λ. Then f varies with λ in such a way that the coefficients of the asymptotic series become λ-dependent, and the series is very far from being a Poincaré type asymptotic series. However, due to the uniform validity in f, the series is still an Erdélyi type asymptotic series with respect to the scale (2.45).

Eqs. (2.32–35) and (2.44) also give an asymptotic series for the physical wave functions as $|\lambda| \to \infty$, valid uniformly in $r \in (0, \infty)$ and $(\theta, f) \in \mathbf{\Theta}_1$. This series is an Erdélyi type asymptotic series with respect to the same scale (2.45) as the preceding series.

Just as the asymptotic series for the Bessel functions the asymptotic series for the wave functions have asymptotic properties also for $|f| \to \infty$ [3, 13] with finite λ. According to eq. (2.8) this limit corresponds to the strong coupling or high energy limit.

In order to study the asymptotic properties when $|f| \to \infty$ we consider the bounds for the rest term of the expansion (2.21), given by Olver in ref. [13]. We do not need to reproduce their detailed form here, because our considerations only use the fact that the error bound is proportional to the total variation of the coefficient of the first neglected term over the path $\mathcal{P}_j$ of Olver's theorem, and is thus proportional to

$$\mathcal{V}_j(A_n) = \int_{\mathcal{P}_j} \left| \frac{d}{dt} \{A_n(t; f)\}\, dt \right| \tag{2.46}$$

or

$$\mathcal{V}_j(z^{\frac{1}{2}} B_n) = \int_{\mathcal{P}_j} \left| \frac{d}{dt} \{z^{\frac{1}{2}} B_n(t; f)\}\, dt \right|. \tag{2.47}$$

Now, when $|f| \to \infty$,

$$p(s; f) \underset{f \to \infty}{\sim} f^2 (1 - s^{-2-m}), \tag{2.48}$$

so that

$$u \underset{f \to \infty}{\sim} f \int_{s_0}^{s} (1 - s^{-2-m})^{\frac{1}{2}}\, ds \tag{2.49}$$

and thus we have that

$$W(z; f) = O(f^{-\frac{4}{3}}) \quad \text{as} \quad |f| \to \infty. \tag{2.50}$$

These expressions are to be inserted into the recursion formulae (2.22).

To see the effect of this, we make the following scale transformations in eq. (2.22)

$$\begin{cases} p \to f^2\, p; & u \to fu; \\ z \to f^{\frac{2}{3}}\, z; & W \to f^{-\frac{4}{3}} W. \end{cases} \tag{2.51}$$

Then the coefficients transform in the following way

$$\begin{cases} A_n \to f^{-2n} A_n, \\ z^{\frac{1}{2}} B_n \to f^{-2n-1} z^{\frac{1}{2}} B_n. \end{cases} \tag{2.52}$$

Thus, the eqs. (2.48–50) imply that when $|f| \to \infty$ then

$$\begin{cases} A_n(z;f) = O(f^{-2n}), \\ z^{\frac{1}{2}} B_n(z;f) = O(f^{-2n-1}), \end{cases} \tag{2.53}$$

uniformly in z. The same behaviour for large $|f|$ is true for the total variations (2.46) and (2.47), and the asymptotic series is an asymptotic series in the Erdélyi sense with respect to the scale

$$(\lambda f)^{-n+\eta}, \quad \eta > 0, \tag{2.54}$$

as $|f| \to \infty$.

Combined with eqs. (2.8) and (2.45) this gives the scale

$$\{|\lambda| + |g^2 k^m|^{1/(2+m)}\}^{-n+\eta}, \quad \eta > 0, \tag{2.55}$$

for the Erdélyi type asymptotic series for the wave functions. This was proved by Tiktopoulos [3] for the error in the JWKB approximation, that is, for $n = 1$.

3. Jost function, *S*-matrix and Regge poles

The asymptotic formulae for the wave functions, which were derived in the preceding section, can be used to obtain asymptotic formulae for the Jost functions.

Let us define the Jost functions as the coefficients $f_0^{\pm}$ of the following connection formula,

$$\varphi_R(r) = \frac{1}{2ik}\left[i^{\lambda+\frac{1}{2}} f_0^-(\lambda, f)\, \varphi_+(r) - i^{-\lambda-\frac{1}{2}} f_0^+(\lambda, f)\, \varphi_-(r)\right], \tag{3.1}$$

between the regular and the asymptotically in- and outgoing wave functions.

Between the Airy functions used in the preceding section there exists a linear relation,

$$\mathrm{Ai}\,(x) = e^{\frac{1}{3}\pi i}\,\mathrm{Ai}\,(e^{-\frac{2}{3}\pi i} x) + e^{-\frac{1}{3}\pi i}\,\mathrm{Ai}\,(e^{\frac{2}{3}\pi i} x), \tag{3.2}$$

which implies that the asymptotic formulae of section 2 satisfy eq. (3.1) with the coefficients given by

$$f_0^{\pm}(\lambda, f) \underset{\lambda\to\infty}{\sim} 2ik i^{\pm(\lambda - 1/6)}\, C_R(\lambda, f)\, C_{\pm}^{-1}(\lambda, f) \tag{3.3}$$

when eqs. (2.28) and (2.29) are used, or

$$f_0^{\pm}(\lambda, f) \underset{\lambda\to\infty}{\sim} 2ik i^{\pm(\lambda - 1/6)}\, c_R(\lambda, f)\, c_{\pm}^{-1}(\lambda, f) \tag{3.4}$$

when eqs. (2.34) and (2.35) are used for the wave functions. Using eqs. (2.43) and (2.44) we get the following asymptotic formulae for the Jost functions

$$f_0^{\pm}(\lambda, f) \underset{\lambda\to\infty}{\sim} 2ikr_0^{\frac{1}{2}+\frac{1}{4}m}\left[1 + \sum_{n=1}^{\infty} \frac{\gamma_{1n}(f)}{\lambda^n}\right] e^{\pm i\delta(\lambda, f) - i\Delta(\lambda, f) - i\delta_0(\lambda, f)} \tag{3.5}$$

with $$1+\sum_{n=1}^{\infty}\frac{\gamma_{1n}(f)}{\lambda^n}=\left[1+\sum_{n=1}^{\infty}\frac{\alpha_{1n}(f)}{\lambda^{2n}}-\sum_{n=0}^{\infty}\frac{\beta_{1n}(f)}{\lambda^{2n+1}}\right]^{-1}\left[1+\sum_{n=0}^{\infty}\frac{\beta_{3n}(f)}{\lambda^{2n+1}}\right], \tag{3.6}$$

or $$f_0^{\pm}(\lambda,f)\underset{\lambda\to\infty}{\sim}2ikr_0^{\frac{1}{2}+\frac{1}{4}m}\left[1+\sum_{n=1}^{\infty}\frac{\gamma_{2n}(f)}{\lambda^n}\right]e^{\pm i\delta(\lambda,f)\mp i\Delta(\lambda,f)-i\delta_0(\lambda,f)} \tag{3.7}$$

with $$1+\sum_{n=1}^{\infty}\frac{\gamma_{2n}(f)}{\lambda^n}=\left[1+\sum_{n=1}^{\infty}\frac{\alpha_{2n}(f)}{\lambda^{2n}}-\sum_{n=0}^{\infty}\frac{\beta_{2n}(f)}{\lambda^{2n+1}}\right]^{-1}\left[1+\sum_{n=0}^{\infty}\frac{\beta_{4n}(f)}{\lambda^{2n+1}}\right]. \tag{3.8}$$

Here the eq. (3.5) is to be used when $(\theta,f)\in\boldsymbol{\Theta}_0$ and the eq. (3.7) when $(\theta,f)\in\boldsymbol{\Theta}_1$. We note that the expressions for $f_0^+(\lambda,f)$ only differ by the coefficients of the series inside the square brackets. However, those series are Poincaré type asymptotic series when $\lambda\to\infty$ for any fixed f, and since such asymptotic series are unique, we must have

$$\gamma_{1n}(f)=\gamma_{2n}(f)=\gamma_n(f) \tag{3.9}$$

in order for eqs. (3.5) and (3.7) to represent the same function $f_0^+(\lambda,f)$. Thus the asymptotic formula

$$f_0^+(\lambda,f)\underset{\lambda\to\infty}{\sim}2ikr_0^{\frac{1}{2}+\frac{1}{4}m}\,e^{-i\delta_0(\lambda,f)}\left[1+\sum_{n=1}^{\infty}\frac{\gamma_n(f)}{\lambda^n}\right]e^{i\delta(\lambda,f)-i\Delta(\lambda,f)} \tag{3.10}$$

is valid both in $\boldsymbol{\Theta}_0$ and $\boldsymbol{\Theta}_1$.

For $f_0^-(\lambda,f)$ we get different exponential factors in the eqs. (3.5) and (3.7). This implies that the correct asymptotic expression for $f_0^-(\lambda,f)$ is a sum of two contributions, each asymptotically dominating in its region of θ and f.

$$f_0^-(\lambda,f)\underset{\lambda\to\infty}{\sim}2ikr_0^{\frac{1}{2}+\frac{1}{4}m}e^{-i\delta_0(\lambda,f)}\left[1+\sum_{n=1}^{\infty}\frac{\gamma_n(f)}{\lambda^n}\right]e^{-i\delta(\lambda,f)}\left[e^{i\Delta(\lambda,f)}+e^{-i\Delta(\lambda,f)}\right]. \tag{3.11}$$

On the boundary between these two regions, that is, on the curve $\operatorname{Im}\Delta=0$ there appear zeros of $f_0^-(\lambda,f)$ at those points of $\boldsymbol{\Theta}_0\cup\boldsymbol{\Theta}_1$ for which

$$\Delta(\lambda,f)=(n+\tfrac{1}{2})\pi,\quad n=0,\pm1,\pm2,\dots. \tag{3.12}$$

For the S-matrix we get

$$S(\lambda,f)=\frac{f_0^+(\lambda,f)}{f_0^-(\lambda,f)}\underset{\lambda\to\infty}{\sim}\frac{e^{2i\delta(\lambda,f)}}{1+e^{2i\Delta(\lambda,f)}},\quad(\theta,f)\in\boldsymbol{\Theta}_0\cup\boldsymbol{\Theta}_1. \tag{3.13}$$

Let us now return to the physical variables g,k and λ and assume that g and k are real and positive and such that $\arg f=-\theta$. Then the range of θ becomes $0\leqslant\theta\leqslant\pi/2$ and from appendix B, eq. (B. 17), we have that

$$\Delta(\lambda,f)\underset{\lambda\to\infty}{\sim}i\left[\left(1+\frac{2}{m}\right)\lambda\ln\lambda-\lambda\ln\frac{ek}{2}-\frac{2\lambda}{m}\ln\frac{eg}{2}-\frac{i\pi\lambda}{m}\right]+O(\lambda^{-1}). \tag{3.14}$$

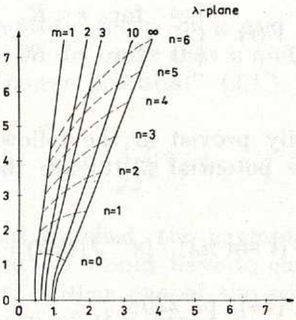

Fig. 9. The asymptotic distribution of Regge poles for the power potential.

Together with eq. (3.12) this gives an asymptotic formula for the distribution of the Regge poles of the S-matrix in the first quadrant of the λ-plane. The asymptotic form of the S-matrix in the other quadrants follows from the following two relations,

$$[f_0^{\pm}(\lambda, f)]^* = f_0^{\mp}(\lambda^*, f^*) \tag{3.15}$$

and

$$f_0^{\pm}(-\lambda, -f) = e^{\mp i\pi\lambda} f_0^{\pm}(\lambda, f), \tag{3.16}$$

which are both consequences of eq. (3.1), and the form of the boundary conditions for the wave functions. One finds that to each pole λ_n in the first quadrant there corresponds a pole in the third quadrant at $-\lambda_n$ and two zeros at $\mp\lambda_n^*$ in the second and fourth quadrant, respectively.

In Fig. 9 we illustrate the asymptotic distribution of Regge poles for fixed and real g and k and a few different m-values.

There are several important facts that one should note about those Regge poles for the "power potential" (2.1′):

(i) The poles are infinite in number.

(ii) Seen from the origin the angle between the imaginary λ-axis and a pole λ_n approaches zero when $n \to \infty$.

(iii) The poles are not bounded to the right by any vertical line, that is, when $n \to \infty$ we have that $\operatorname{Re} \lambda_n \to \infty$.

(iv) The residue of the S-matrix at the pole λ_n is

$$\operatorname{Res}_n \mathrm{S} \sim \tfrac{1}{2} i [\Delta'(\lambda_n, f_n)]^{-1} e^{2i\delta(\lambda_n, f_n)}, \tag{3.17}$$

where the derivative of Δ is taken with respect to λ, with g and k held fixed, and f varying with λ.

(v) When m increases from a value near zero to a large value all the Regge poles move out from the neighbourhood of the imaginary λ-axis to certain positions, which they reach asymptotically when $m \to \infty$. These asymptotic positions coincide with the positions of the Regge poles (Berendt [5]) for the nonanalytic hard core potential

$$V(r)=\begin{cases}\infty & \text{for} \quad r<R\\ 0 & \text{for} \quad r>R.\end{cases} \tag{3.18}$$

for $R=1$.

This last fact is most easily proved in the following way. It is well known that the Regge poles for the potential (3.18) are the zeros of the Hankel function $H_\lambda^{(2)}(kR)$. Writing

$$H_\lambda^{(2)}(kR)=(i\sin\pi\lambda)^{-1}[e^{i\pi\lambda}J_\lambda(kR)-J_{-\lambda}(kR)] \tag{3.19}$$

and observing that (Watson [14], p. 225)

$$J_\lambda(kR)\underset{\lambda\to\infty}{\sim}\frac{1}{\sqrt{2\pi\lambda}}\left[1+\sum_{i=1}^{\infty}\frac{c_i}{\lambda^i}\right]e^{-\lambda\ln\lambda+\lambda\ln(ekR/2)}, \tag{3.20}$$

we find that the zeros of the Hankel function are asymptotically given by the eqs. (3.12) and (3.14) for $m=\infty$ and $R=1$. One should also note that the "power potential" (2.1′) approaches the hard core potential (3.18) as $m\to\infty$ for $R=1$.

The first three of the points above, (i–iii), have been proved before by a number of authors [6], the fourth point was proved in the special case of $m=2$ by Vogt and Wannier [7], but the points (iv) for general m and (v) seem to be new results.

It was pointed out by Tiktopoulos in ref. [3] that eq. (3.12) is equivalent to a Bohr–Sommerfeld phase-integral condition.

4. Other potentials

In this section we shall discuss to what extent the theory of the preceding two sections is applicable when the potential is not a simple "power potential" as in (2.1′).

Let us first discuss the conditions at $r=0$. Consider a potential of the type (2.1),

$$V(r)=r^{-2-m}\xi(r),$$

where $\xi(r)$ approaches 1 when $r\to 0$. It follows from the discussion of section 2 that the Limić conditions imply the condition (2.19) for the applicability of the Olver theorem.

For a potential of the type (2.2),

$$V(r)=r^{-2}(-\ln r)^m\,\xi(r)$$

the second of the Limić conditions is satisfied only for $m>2$. However, the condition (2.19) is valid for all $m>0$, as is shown in appendix E. Therefore, we can get a slight generalization for this potential and a closer adaptation to the present method by replacing the second of the Limić conditions by the less explicit condition

$$W(z;f)=O(z^{-\frac{1}{2}-\sigma}),\quad \sigma>0, \tag{4.1}$$

when $z\to-\infty$.

In appendix D the asymptotic forms of u and v are calculated for the potential (2.1) in the limit $f \to 0$. In order that u and v shall not deviate too much from their values for the "power potential" (2.1′), we assume that

$$\int^{\infty} |V(r)\,dr| < \infty \quad \text{and} \quad \int_0 |[V(r) - r^{-2-m}]\, r^{1+\frac{1}{2}m}\,dr| < \infty. \tag{4.2}$$

If these conditions were not satisfied, the asymptotic form of the wave functions would be different, and we would have to change the boundary conditions used in section 2. However, neither one of the conditions (4.2) is necessary for the applicability of the theory of this paper.

In fact, for eq. (2.19) to be satisfied at $r=\infty$ it would be sufficient to have only $V(r)\to 0$ as $r\to\infty$. This follows from the same argument as that leading to eq. (2.24). Thus, the theory could be applied to potentials with a Coulomb tail, provided that the boundary condition at infinity is modified accordingly.

For potentials satisfying eq. (4.2) we prove, in appendix D, that

$$\Delta(\lambda, f) = i\left[\left(1+\frac{2}{m}\right)\lambda \ln \lambda - \lambda \ln \frac{ek}{2} - \frac{2\lambda}{m} \ln \frac{eg}{2} - \frac{i\pi\lambda}{m}\right] + o(1) \tag{4.3}$$

as $\lambda\to\infty$. This is a very important result, since it implies that the whole class of potentials satisfying eq. (4.2) have the same asymptotic distribution of Regge poles as the "power potential" (2.1′). Let us call this class of potentials the power potential class:

A potential is said to belong to the power potential class if

(1) as a function of r it has an analytic continuation into the whole right half-plane outside the origin, and

(2) in all directions in the half-plane, except possibly for parallels to the imaginary axis, this continuation satisfies eq. (4.2).

Now let us turn to the potentials (2.2) which have a logarithmic singularity at the origin. We begin with $\xi(r)\equiv 1$ and transform the Schrödinger equation by introducing the new variable s,

$$s = \frac{r}{r_0}, \quad r_0 = \frac{g}{k}. \tag{4.4}$$

The new equation reads

$$\left[\frac{d^2}{ds^2} + g^2 - \frac{\lambda^2 - \frac{1}{4}}{s^2} - g^2 \frac{(-\ln s - \ln r_0)^m}{s^2}\right] \varphi(s) = 0, \tag{4.5}$$

and in this form the energy dependence enters only through the potential term.

Let us for simplicity consider only the case $m=1$. Then we can put

$$\nu^2 = \lambda^2 - g^2 \ln r_0, \quad f = \nu^{-1} g, \tag{4.6}$$

and write the Schrödinger equation in the form

$$\left[\frac{d^2}{ds^2}+\nu^2 p(s;f)+\frac{1}{4s^2}\right]\varphi(s)=0 \tag{4.7}$$

with
$$p(s;f)=f^2-s^{-2}+f^2 s^{-2}\ln s. \tag{4.8}$$

Eq. (4.7) is in the correct form for the application of the theory of section 2 with ν corresponding to λ and with f as a free parameter in the same sense as there.

The asymptotic series, which is obtained in this theory, will be asymptotic as $\nu\to\infty$ or as $\nu f\to\infty$ or, to be more precise, the series obtained for the wave functions will be an asymptotic series in the Erdélyi sense with respect to the scale

$$\left\{|g|+\left|\lambda^2+g^2\ln\frac{k}{g}\right|^{\frac{1}{2}}\right\}^{-n+\eta},\quad \eta>0,\quad n=1,2,\ldots, \tag{4.9}$$

as $|\lambda|\to\infty$, $|k|\to\infty$ or $|g|\to\infty$.

A similar but nonrigorous consideration for $m\neq 1$ would give the scale

$$\left\{|g|+\left|\lambda^2+g^2\left(\ln\frac{k}{g}\right)^m\right|^{\frac{1}{2}}\right\}^{-n+\eta},\quad \eta>0,\quad n=1,2,\ldots \tag{4.9'}$$

For the zeros of the $p(s;f)$ of eq. (4.8) we obtain when f is small

$$\begin{cases} s_0=f^{-1}[1+O(f^2\ln f)], \\ s_1=\exp(f^{-2})\{1+O[\exp(2f^{-2})]\},\ \ \mathrm{Re}\, f^{-2}<0. \end{cases} \tag{4.10}$$

When $\mathrm{Re}\, f^{-2}\geqslant 0$, that is when $|\arg f|\leqslant\frac{1}{4}\pi$ then s_0 is the only zero of $p(s;f)$. In fact, s_1 is important only for f very near the imaginary axis because as can be seen from the equation

$$\arg s_1=\mathrm{Im}\, f^{-2}=|f|^{-2}\sin(-2\arg f), \tag{4.11}$$

s_1 is in the first quadrant only for

$$-\frac{\pi}{2}\leqslant\arg f\leqslant-\frac{\pi}{2}+\frac{\pi}{4}|f|^2+O(f^6). \tag{4.12}$$

Thus, for most values of $\arg f$ s_1 is far away in other Riemann sheets.

In order to make $p^{\frac{1}{2}}(s;f)$ single-valued we make cuts from s_0 and s_1 to the origin. According to eq. (4.11) these cuts coincide for

$$\arg f=-\frac{\pi}{2}+\frac{\pi}{4}|f|^2-\frac{\pi}{8}|f|^4+O(f^6), \tag{4.13}$$

and for all values of $\arg f$ between this value and $-\frac{1}{2}\pi$ we have that $\arg s_1<\arg\ s_0$.

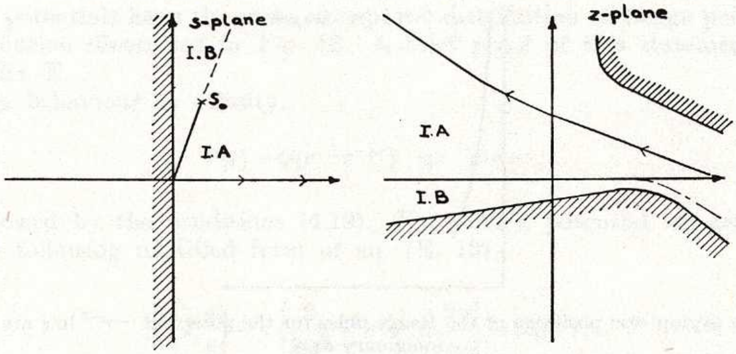

Fig. 10. The complex z-plane for the potential $-r^{-2}\ln r$.

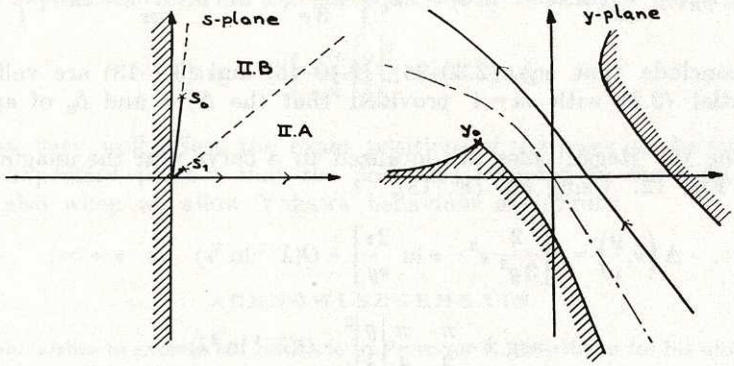

Fig. 11. The complex y-plane for the potential $-r^{-2}\ln r$.

In appendix E the asymptotic form of the transformations (2.11) and (2.12) for small f is calculated for the logarithmically singular potential (2.2) with $m=1$. The result is illustrated in the Figs. 10 and 11. It is clear that appropriate regions $\mathbf{D}_z(\theta, f)$, $\mathbf{G}_z(\theta, f)$ and $\mathbf{H}'_z(\theta, f)$ can be constructed as in section 2, provided $\arg s_1 \geqslant \varepsilon > 0$ which for real g means

$$0 \leqslant \arg \nu \leqslant \frac{\pi}{2} - \tfrac{1}{2}\varepsilon |f|^2 + O(f^6). \tag{4.14}$$

On the other hand, the corresponding y-regions can be constructed when $\arg s_1 < \arg s_0$, that is, for

$$\frac{\pi}{2} - \frac{\pi}{4}|f|^2 + \frac{\pi}{8}|f|^4 + O(f^6) \leqslant \arg \nu \leqslant \frac{\pi}{2}. \tag{4.15}$$

These two regions overlap. This is necessary for the theory of section 3 to be applicable also in this case. Finally, fixing an appropriate boundary condition for the regular function at the origin,

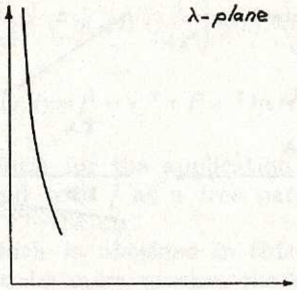

Fig. 12. The asymptotic positions of the Regge poles for the potential $-r^{-2}\ln r$ are close to the imaginary axis.

$$\varphi_R(r) \underset{r\to 0}{\sim} \text{const}\,(-\ln r)^{-\frac{1}{4}} \exp\left\{-\frac{2g}{3\nu}(-\ln r)^{\frac{3}{2}} - \frac{\lambda^2}{g\nu}(-\ln r)^{\frac{1}{2}}\right\} \tag{4.16}$$

we can conclude that eqs. (2.30–35), (2.40–45) and (3.1–13) are valid also for the potential (2.2) with $m=1$ provided that the Δ, δ and δ_0 of appendix E are used.

This time the Regge poles are localized to a curve near the imaginary λ-axis, shown in Fig. 12. Using eq. (E. 13),

$$\Delta\left(\nu, \frac{g}{\nu}\right) = i\left[\frac{2}{3g^2}\nu^3 - \nu \ln\frac{2\nu}{eg}\right] + O(\lambda^{-1}\ln^2\nu) \quad \text{as} \quad \nu\to\infty,$$

we get
$$\arg\lambda = \frac{\pi}{2} - \frac{\pi}{4}\left|\frac{g}{\lambda}\right|^2 + O(\lambda^{-4}\ln^2\lambda) \tag{4.17}$$

as $|\lambda|\to\infty$. Since $|\nu|\to\infty$ also when $k\to\infty$ for finite λ we can get also the high energy behaviour from eq. (E. 13). The result is that the Regge poles move asymptotically towards the imaginary axis also in this limit.

The main difference from the "power potential" case is that this time the poles are bounded to the right by some vertical line. Moreover, this is valid uniformly for all energies.

This result is very interesting since it indicates that a Mandelstam representation with a finite number of subtractions might be valid for those potentials. We intend to discuss this more fully in a future publication.

Let us define the following class of potentials: A potential $V(r)$ is said to belong to the logarithmic potential class if

(1) as a function of r it has an analytic continuation into the whole right half-plane outside the origin, and

(2) in all directions in the half-plane, including parallels to the imaginary axis, this continuation satisfies

$$\int^{\infty} |V(r)\,dr| < \infty, \quad \int_0 |[V(r) + r^{-2}\ln r]\,(-\ln r)^{\frac{1}{2}}\,r\,dr| < \infty. \tag{4.18}$$

All those potentials have the same asymptotic distribution of Regge poles, namely the distribution illustrated in Fig. 12. A brief proof of this statement is given in appendix E.

Yukawa behaviour at infinity,

$$V(r)=O(r^{-1}e^{-\mu r}) \quad \text{as} \quad r\to\infty, \tag{4.19}$$

is not allowed by the conditions (4.18). For such a potential we get from eq. (E. 7) the following modified form of eq. (E. 13)

$$\Delta\left(\nu,\frac{g}{\nu}\right)=i\left\{\frac{2}{3g^2}\nu^3-\nu\ln\frac{2\nu}{eg}\right\}+O(\ln\nu). \tag{4.20}$$

The larger error term of this equation as compared to eq. (E. 13) does not affect the asymptotic form of the curve on which the Regge poles lie,

$$\arg\lambda=\frac{\pi}{2}-\frac{\pi}{4}\left|\frac{g}{\lambda}\right|^2+O(\lambda^{-3}\ln\lambda), \tag{4.17'}$$

but it may very well affect the exact positions of the poles on the curve. However, the important point is that the poles are bounded to the right by a vertical line also when we allow Yukawa behaviour at infinity.

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APPENDIX A

In this appendix we shall show that the transformation (2.11) is conformal at $s=s_0$, corresponding to $z=0$, and that $W(z;f)$ is analytic at $z=0$.

Consider the Taylor expansion of $p(s;f)$ around $s=s_0$,

$$p(s;f)=a_1x+\tfrac{1}{2}a_2x^2+O(x^3), \quad x=s-s_0. \tag{A.1}$$

This expansion is uniformly and absolutely convergent inside a circle of positive radius, say $b>0$. When inserted into eq. (2.11), eq. (A.1) gives

$$u=a_1^{\frac{1}{2}}\int_0^x x^{\frac{1}{2}}\left[1+\frac{a_2}{2a_1}x+O(x^2)\right]^{\frac{1}{2}}dx=\tfrac{2}{3}a_1^{\frac{1}{2}}x^{\frac{3}{2}}+\frac{a_2}{10a_1^{\frac{1}{2}}}x^{5/2}+O(x^{7/2}) \tag{A.2}$$

and $$z = -a_1^{\frac{3}{2}}\, x \left[1 + \frac{a_2}{10\, a_1}\, x + O(x^2) \right]. \tag{A.3}$$

Thus z as a function of s has no branch point at $s = s_0$, it is analytic, and the mapping is conformal.

Next consider W, defined in eq. (2.15). Put

$$W(z; f) = \frac{5}{16\, z^2} + z\Omega(s; f), \tag{A.4}$$

and derive from eq. (A.1) the power series expansion of Ω around $s = s_0$. A straightforward calculation gives

$$\Omega(s; f) = \frac{5}{16\, a_1}\, x^{-3} \left[1 - \frac{3\, a_2}{10\, a_1}\, x + O(x^2) \right], \tag{A.5}$$

which together with eq. (A.3) implies that

$$\frac{1}{z}\, W(z; f) = \frac{5}{16\, z^3} + \frac{5}{16\, a_1\, x^3} \left[1 - \frac{3\, a_2}{10\, a_1}\, x + O(x^2) \right] = O\left(\frac{1}{x}\right), \tag{A.6}$$

that is $$W(z; f) = O(1) \quad \text{as} \quad z \to 0. \tag{A.7}$$

Thus W has no pole at $z = 0$ and W, being a rational function of functions analytic in a neighbourhood of $z = 0$, must be analytic at $s = 0$.

It is clear that the above considerations apply equally well to the transformation (2.12) at $s = s_1$ corresponding to $y = 0$, and to w defined in eq. (2.17) at $y = 0$.

APPENDIX B

In this appendix we shall study the asymptotic forms of the transformations (2.11) and (2.12) for the "power potential".

Let us begin with the asymptotic forms of the transformations when $f \to 0$. By writing

$$p(s; f) = s^{-2}(f^2\, s^2 - 1 - f^2\, s_0^{-m}) - f^2\, s^{-2}(s^{-m} - s_0^{-m}) \tag{B.1}$$

and $$p^{\frac{1}{2}}(s; f) = s^{-1}(f^2\, s^2 - 1 - f^2\, s_0^{-m})^{\frac{1}{2}} + O\left(\frac{f^2}{s}\, \frac{s^{-m} - s_0^{-m}}{(f^2\, s^2 - 1 - f^2\, s_0^{-m})^{\frac{1}{2}}} \right), \tag{B.2}$$

we obtain for small f a good approximation to $p^{\frac{1}{2}}$ in a large environment of the point $s = s_0$.

As we are interested in u, which is the integral of $p^{\frac{1}{2}}$, we need an estimate of the integral of the error term in eq. (B.2),

$$f\int_{s_0}^{s}\frac{ds}{s}\frac{s^{-m}-s_0^{-m}}{(s^2-s_0^2)^{\frac{1}{2}}}=\begin{cases}O\left(f s_0^{-m}\int_{s_0}^{s}\frac{ds}{s^2}\right)=O(f^{2+m}) & \text{for } |s|\gtrsim|s_0| \\ O\left(f s_0^{-1}\int_{s_0}^{s}\frac{ds}{s^{1+m}}\right)=O(f^2) & \text{for } |s|\gtrsim 1.\end{cases} \tag{B.3}$$

We note that the singularity of the integrand at $s=-s_0$ is integrable, and does not affect the result above.

It follows that

$$u=\int_{s_0}^{s}\frac{ds}{s}(f^2s^2-1-f^2s_0^{-m})^{\frac{1}{2}}+O(f^2) \tag{B.4}$$

for all s outside a circle of radius 0(1) around the origin.

The integral in eq. (B.4) is easy to calculate and we obtain

$$u=f(s^2-s_0^2)^{\frac{1}{2}}-fs_0\operatorname{arctg}\left(\frac{s^2}{s_0^2}-1\right)^{\frac{1}{2}}+O(f^2)$$

$$=\pm i\left\{fs_0\left(1-\frac{s^2}{s_0^2}\right)^{\frac{1}{2}}+\tfrac{1}{2}\ln\frac{1-(1-s^2/s_0^2)^{\frac{1}{2}}}{1+(1-s^2/s_0^2)^{\frac{1}{2}}}\right\}+O(f^2). \tag{B.5}$$

The phase factor of this last equation is to be taken as

$$\exp\{\tfrac{3}{2}i\pi\operatorname{sign}(\arg s-\arg s_0)\}.$$

This implies that for large s

$$u=fs-\tfrac{1}{2}\pi+O(f^{2+m})+O(s^{-1}), \tag{B.6}$$

where we used the stronger form of the error term according to eq. (B.3). This in turn implies that the JWKB phase-shift

$$\delta(\lambda,f)=\lim_{s\to\infty}(\lambda u-\lambda fs+\tfrac{1}{2}\pi\lambda) \tag{B.7}$$

satisfies

$$\delta(\lambda,f)=O(\lambda f^{2+m}) \tag{B.8}$$

as $f\to 0$.

For small s we try another approximation

$$p^{\frac{1}{2}}(s;f)=s^{-1}(f^2s_1^2-1-f^2s^{-m})^{\frac{1}{2}}+O\left(\frac{f}{s}\frac{s_1^2-s^2}{(s_1^{-m}-s^{-m})^{\frac{1}{2}}}\right), \tag{B.9}$$

analogous to eq. (B.2) but this time constructed to be well behaved near the point $s=s_1$. We note that the error term is large near the points $s=s_1\exp(2\pi in/m)$, $n=\pm1,\pm2,\ldots$. However, it is integrable also at those points and we get for the error in v

$$f\int_{s_1}^{s}\frac{ds}{s}\frac{s_1^2-s^2}{(s_1^{-m}-s^{-m})^{\frac{1}{2}}}=\begin{cases} O\left(fs_1^2\int_{s_1}^{s}s^{-1+\frac{1}{2}m}\,ds\right)=O(f^{2+4/m}) & \text{for } |s|\lesssim|s_1| \\ O\left(fs_1^{\frac{1}{2}m}\int_{s_1}^{s}s\,ds\right)=O(f^2) & \text{for } |s|\lesssim 1. \end{cases} \tag{B.10}$$

Thus
$$v=f\int_{s_1}^{s}\frac{ds}{s}(s_1^{-m}-s^{-m})^{\frac{1}{2}}+O(f^2) \tag{B.11}$$

for s inside a circle of radius $O(1)$ around the origin. Calculation of the integral in eq. (B.11) gives

$$v=\frac{2f}{m}\left\{-s^{-\frac{1}{2}m}\left[\left(\frac{s}{s_1}\right)^m-1\right]^{\frac{1}{2}}+s_1^{-\frac{1}{2}m}\ln\left(\left[\left(\frac{s}{s_1}\right)^m-1\right]^{\frac{1}{2}}+\left(\frac{s}{s_1}\right)^{\frac{1}{2}m}\right)\right\}+O(f^2). \tag{B.12}$$

In particular, with the stronger form of the error term according to eq. (B.10), we obtain for small s that

$$v=\pm i\frac{2f}{m}s^{-\frac{1}{2}m}+O(f^{2+4/m})+O(s^{\frac{1}{2}m}), \tag{B.13}$$

the sign being sign $(-\arg s+\arg s_1)$.

This implies that for the function

$$\delta_0(\lambda,f)=\lim_{s\to 0}\left(\lambda v\mp i\frac{2\lambda f}{m}s^{-\frac{1}{2}m}\right), \tag{B.14}$$

we have that
$$\delta_0(\lambda,f)=O(\lambda f^{2+4/m}) \tag{B.15}$$

as $f\to 0$.

Eqs. (B.5) and (B.12) have a common region of validity around $s=1$. Therefore, these two equations can be used to calculate the function Δ, defined by

$$\Delta(\lambda,f)=\lambda u-\lambda v=\lambda\int_{s_0}^{s_1}p^{\frac{1}{2}}(s;f)\,ds. \tag{B.16}$$

A straightforward calculation gives

$$\Delta(\lambda,f)=\pm i\lambda\left\{\left(1+\frac{2}{m}\right)\ln\frac{2}{ef}-\frac{i\pi}{m}\right\}+O(\lambda f^2) \tag{B.17}$$

as $f\to 0$, where the sign is to be taken as sign $(\arg s_0-\arg s_1)$.

Let us define the functions δ, δ_0 and Δ by eqs. (B.7), (B.14) and (B.16), respectively, also when f is not small. Then the following asymptotic formulae are valid for all f,

$$u = \begin{cases} fs - \tfrac{1}{2}\pi + \lambda^{-1}\delta(\lambda, f) + O(s^{-1}) & \text{as} \quad s \to \infty \\ i\dfrac{2f}{m}\, s^{-\frac{1}{2}m} + \lambda^{-1}\Delta(\lambda, f) + \lambda^{-1}\delta_0(\lambda, f) + O(s^{\frac{1}{2}m}) & \text{as} \quad s \to 0 \end{cases} \tag{B.18}$$

and

$$v = \begin{cases} fs - \tfrac{1}{2}\pi + \lambda^{-1}\delta(\lambda, f) - \lambda^{-1}\Delta(\lambda, f) + O(s^{-1}) & \text{as} \quad s \to \infty \\ i\dfrac{2f}{m}\, s^{-\frac{1}{2}m} + \lambda^{-1}\delta_0(\lambda, f) + O(s^{\frac{1}{2}m}) & \text{as} \quad s \to 0. \end{cases} \tag{B.19}$$

The asymptotic forms of δ, δ_0 and Δ as $f \to 0$ are given by the eqs. (B.8), (B.15) and (B.17). As $f \to \infty$, u and v become proportional to f and

$$\left.\begin{aligned} \delta\,(\lambda, f) &= O(\lambda f) \\ \delta_0(\lambda, f) &= O(\lambda f) \\ \Delta\,(\lambda, f) &= O(\lambda f) \end{aligned}\right\} \quad \text{as} \quad f \to \infty. \tag{B.20}$$

APPENDIX C

In this appendix we shall give the derivation of the formulae (2.43) and (2.44) of the text.

We begin by studying the asymptotic form of the expression

$$\left(-\frac{z}{P}\right)^{\frac{1}{4}} \psi_{3,2}(z) \underset{\lambda\to\infty}{\sim} r_0^{\frac{1}{2}} \left(-\frac{z}{p(s; f)}\right)^{\frac{1}{4}} \left[\mathrm{Ai}\,(e^{\mp\frac{2}{3}\pi i}\lambda^{\frac{2}{3}} z) \sum_{n=0}^{\infty} \frac{A_n(z; f)}{\lambda^{2n}} + e^{\mp\frac{2}{3}\pi i}\lambda^{-\frac{4}{3}}\,\mathrm{Ai}'\,(e^{\mp\frac{2}{3}\pi i}\lambda^{\frac{2}{3}} z) \sum_{n=0}^{\infty} \frac{B_n(z; f)}{\lambda^{2n}}\right] \tag{C.1}$$

when $s \to \infty$.

From appendix B we have that

$$u \underset{s\to\infty}{\sim} fs - \tfrac{1}{2}\pi + \lambda^{-1}\delta(\lambda, f)$$

and using the fact that

$$\begin{cases} \mathrm{Ai}\,(z) \underset{|z|\to\infty}{\sim} \dfrac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp\,(-\tfrac{2}{3}z^{\frac{3}{2}}) \\ \mathrm{Ai}'(z) \underset{|z|\to\infty}{\sim} -\dfrac{1}{2\sqrt{\pi}} z^{\frac{1}{4}} \exp\,(-\tfrac{2}{3}z^{\frac{3}{2}}) \end{cases} \tag{C.2}$$

we obtain after a straightforward calculation the result

$$\left(-\frac{z}{P}\right)^{\frac{1}{4}} \psi_{3,2}(z) \underset{\substack{\lambda\to\infty \\ s\to\infty}}{\sim} \frac{1}{2\sqrt{\pi}} r_0^{\frac{1}{2}} f^{-\frac{1}{2}} \lambda^{-1/6} i^{\pm(\lambda - 1/6)} e^{\mp i\delta(\lambda, f)} \left[1 + \sum_{n=0}^{\infty} \frac{\beta_{3n}(f)}{\lambda^{2n+1}}\right] \exp\,\{\mp i\lambda f s\}. \tag{C.3}$$

The functions β of this equation were defined in eq. (2.42).

Using eqs. (2.28) and (C.3) on the left and right hand sides of eq. (2.31) we now get that

$$C_{\pm}^{-1}(\lambda, f) \underset{\lambda\to\infty}{\sim} \frac{1}{2\sqrt{\pi}} r_0^{\frac{1}{2}} f^{-\frac{1}{2}} \lambda^{-1/6} \left[1 + \sum_{n=0}^{\infty} \frac{\beta_{3n}(f)}{\lambda^{2n+1}}\right] i^{\pm(\lambda - 1/6)} e^{\mp i\delta(\lambda, f)}. \tag{C.4}$$

The expression in the square brackets is an asymptotic series of the Poincaré type when $\lambda \to \infty$ (or when $f \to \infty$, as follows from the discussion at the end of section 2 above). This is in contrast to e.g. eq. (C.1), where the series is only asymptotic in the Erdélyi sense. We will use this observation in section 3.

The same technique can now be used to obtain the functions $c_{\pm}$, C_R and c_R of the eqs. (2.35), (2.30) and (2.34).

Using the formula

$$v \underset{s\to\infty}{\sim} fs - \tfrac{1}{2}\pi + \lambda^{-1}\delta(\lambda, f) - \lambda^{-1}\Delta(\lambda, f)$$

from appendix B, we get

$$c_{\pm}^{-1}(\lambda, f) \underset{\lambda\to\infty}{\sim} \frac{1}{2\sqrt{\pi}} r_0^{\frac{1}{2}} f^{-\frac{1}{2}} \lambda^{-1/6} \left[1 + \sum_{n=0}^{\infty} \frac{\beta_{4n}(f)}{\lambda^{2n+1}}\right] i^{\pm(\lambda - 1/6)} e^{\mp i\delta(\lambda, f) \pm i\Delta(\lambda, f)}. \tag{C.5}$$

Similarly, from

$$u \underset{s\to 0}{\sim} i\frac{2f}{m} s^{-\frac{1}{2}m} + \lambda^{-1}\Delta(\lambda, f) + \lambda^{-1}\delta_0(\lambda, f),$$

we get that

$$\left(-\frac{z}{P}\right)^{\frac{1}{4}} \psi_1(z) \underset{\substack{\lambda\to\infty\\ s\to 0}}{\sim} \frac{1}{2\sqrt{\pi}} r_0^{\frac{1}{2}} f^{-\frac{1}{2}} \lambda^{-1/6} e^{i\Delta(\lambda, f) + i\delta_0(\lambda, f)}$$

$$\times \left[\sum_{n=0}^{\infty} \frac{\alpha_{1n}(f)}{\lambda^{2n}} - \sum_{n=0}^{\infty} \frac{\beta_{1n}(f)}{\lambda^{2n+1}}\right] s^{\frac{1}{4}(2+m)} \exp\left\{-\frac{2\lambda f}{m} s^{-\frac{1}{2}m}\right\}, \tag{C.6}$$

which implies that

$$C_R^{-1}(\lambda, f) \underset{\lambda\to\infty}{\sim} \frac{1}{2\sqrt{\pi}} r_0^{-\frac{1}{4}m} f^{-\frac{1}{2}} \lambda^{-1/6} \left[\sum_{n=0}^{\infty} \frac{\alpha_{1n}(f)}{\lambda^{2n}} - \sum_{n=0}^{\infty} \frac{\beta_{1n}(f)}{\lambda^{2n+1}}\right] e^{i\Delta(\lambda, f) + i\delta_0(\lambda, f)}. \tag{C.7}$$

Finally, from

$$v \underset{s\to 0}{\sim} i\frac{2f}{m} s^{-\frac{1}{2}m} + \lambda^{-1}\delta_0(\lambda, f)$$

we obtain in the same way

$$c_R^{-1}(\lambda, f) \underset{\lambda\to\infty}{\sim} \frac{1}{2\sqrt{\pi}} r_0^{-\frac{1}{4}m} f^{-\frac{1}{2}} \lambda^{-1/6} \left[\sum_{n=0}^{\infty} \frac{\alpha_{2n}(f)}{\lambda^{2n}} - \sum_{n=0}^{\infty} \frac{\beta_{2n}(f)}{\lambda^{2n+1}}\right] e^{i\delta_0(\lambda, f)}. \tag{C.8}$$

The expressions within the square brackets in eqs. (C.5–8) are Poincaré type asymptotic series, just as the corresponding expression in eq. (C.4).

APPENDIX D

In this appendix we shall study the influence of the factor $\xi(r)$ of eq. (2.1) on the calculations of appendix B for the "power potential".

To this end we put

$$p(s;f)=s^{-2}[f^2 s^2-1-f^2 s_0^{-m}\,\xi(s_0)]-f^2 s^{-2}[s^{-m}\,\xi(s)-s_0^{-m}\,\xi(s_0)] \tag{D.1}$$

and observe that $$\xi(s)=o(s^{1+m}) \quad \text{as} \quad s\to\infty \tag{D.2}$$

and that $$s_0=f^{-1}[1+o(f)] \quad \text{as} \quad f\to 0. \tag{D.3}$$

Using the method of appendix B we get also here eq. (B.5) for u, but this time with the error term

$$f\int_{s_0}^{s}\frac{ds}{s}\frac{s^{-m}\,\xi(s)-s_0^{-m}\,\xi(s_0)}{(s^2-s_0^2)^{\frac{1}{2}}}=O\left(f\int_{s_0}^{s}|V(s)\,ds|\right)=o(f) \tag{D.4}$$

for $|s|\gtrsim|s_0|$ and $f\to 0$. Here we used condition 4 of section 2 on the potential at infinity. For $|s|<|s_0|$ we instead split $p(s;f)$ in the following way,

$$p(s;f)=s^{-2}[f^2 s^2-1-f^2 s^{-m}]-f^2 s^{-2-m}[\xi(s)-1]. \tag{D.5}$$

By neglecting the last term we make the following error in the calculation of u,

$$f\int_{s_0}^{s}ds\,\frac{s^{-1-m}[\xi(s)-1]}{(s^2-f^{-2}-s^{-m})^{\frac{1}{2}}}=O\left(f\int_{s_0}^{s}|V(s)\,ds|\right)=O(f) \quad \text{if} \quad |s|\gtrsim 1, \tag{D.6}$$

as $f\to 0$. From the first term of eq. (D.5) we then get the same u as in appendix B, eq. (B.5).

Also for small s the splitting (D.5) of $p(s;f)$ is effective, but in order for the error, made by neglecting the second term, to be small, $\xi(s)$ has to approach 1 sufficiently fast, such that the integral (D.6) is convergent at $s=0$. The condition for this is

$$\int_0|ds\,s^{-1-\frac{1}{2}m}[\xi(s)-1]|=\int_0|ds\,s^{1+\frac{1}{2}m}[V(s)-s^{-2-m}]|<\infty. \tag{D.7}$$

This is a condition on how much the potential may deviate from the power potential near the origin, and still have the same asymptotic properties.

When eq. (D.7) is satisfied, we have

$$f\int_{s_1}^{s}ds\,\frac{s^{-1-m}[\xi(s)-1]}{(s^2-f^{-2}-s^{-m})^{\frac{1}{2}}}=\begin{cases}O\left(f\displaystyle\int_{s_1}^{s}ds\,s^{-1-\frac{1}{2}m}[\xi(s)-1]\right)=o(f) & \text{if} \quad |s|\lesssim|s_1|\\ O\left(f^2\displaystyle\int_{s_1}^{s}ds\,s[V(s)-s^{-2-m}]\right)=O(f^2 s) & \text{if} \quad |s_1|\lesssim|s|\lesssim|s_0|\end{cases} \tag{D.8}$$

as $f \to 0$. From the first term of eq. (D.5) we then get the same v as in appendix B, eq. (B.12).

This last assertion as well as the corresponding one after eq. (D.6) is true only if we can neglect the differences between this appendix and appendix B with regard to the lower limits of integration for u and v. To justify this we first observe that eq. (D.7) implies that

$$s_1 = e^{i\pi/m} f^{2/m} [1 + o(f)]. \tag{D.9}$$

It then follows from eqs. (A.1) and (A.2) together with eqs. (D.3) and (D.9) that the differences of lower limits of integration only correspond to an error $o(f^{\frac{3}{2}})$ in u and v.

We have thus proven that the asymptotic forms of appendix B for u and v in the limit $f \to 0$ are changed at most by $o(f)$ if a $\xi(r)$ such that

$$\int^{\infty} |V(r)\, dr| < \infty \quad \text{and} \quad \int_0 |[V(r) - r^{-2-m}]\, r^{1+\frac{1}{2}m}\, dr| < \infty. \tag{D.10}$$

is inserted.

Joining the expressions for u and v at some point s, $1 < |s| < |s_0|$, we now get the following expression for $\Delta(\lambda, f)$,

$$\Delta(\lambda, f) = i\lambda \left\{ \left(1 + \frac{2}{m}\right) \ln \frac{2}{ef} - \frac{i\pi}{m} \right\} + o(\lambda f) \tag{D.11}$$

as $f \to 0$. This is the same result as in appendix B, except for a change of the error term. For the JWKB phase-shift there is also a change in the error term,

$$\delta(\lambda, f) = o(\lambda f) \quad \text{as} \quad f \to 0, \tag{D.12}$$

but for δ_0 eq. (B.15) is still valid.

If we happen to know more about the behaviour of the potential at infinity than what is contained in the condition

$$\int^{\infty} |V(r)\, dr| < \infty,$$

then the error bounds may be sharpened. Say, for example, that we have the physically plausible situation that a Yukawa potential is dominating at infinity, that is,

$$V(r) = O(r^{-1} e^{-\mu r}) \tag{D.13}$$

as $r \to \infty$. Then eq. (D.4) instead reads

$$f \int_{s_0}^{s} ds \, \frac{s^{-m}\xi(s) - s_0^{-m}\xi(s_0)}{(s^2 - s_0^2)^{\frac{1}{2}}} = O\left(f s_0^{-m} \xi(s_0) \int_{s_0}^{s} \frac{ds}{s^2}\right) = O(f e^{-\mu r_0 f^{-1}}) \tag{D.14}$$

and thus
$$\delta(\lambda, f) = O\left(\exp\left\{-\frac{\mu\lambda}{k}\right\}\right) \tag{D.15}$$

when $\lambda \to \infty$, and eq. (B.17) is valid for Δ.

APPENDIX E

In this appendix we shall consider potentials with a logarithmic singularity at the origin.

For a potential of the form (2.2),
$$V(r) = r^{-2}(-\ln r)^m \xi(r),$$

with $\xi(r) \to 1$ as $r \to 0$ we have that
$$u \underset{r\to 0}{\sim} if \int^r \frac{dr}{r}(-\ln r)^{\frac{1}{2}m} \underset{r\to 0}{\sim} -\frac{2if}{2+m}(-\ln r)^{1+\frac{1}{2}m} \tag{E.1}$$

and thus
$$z \underset{r\to 0}{\sim} \text{const}\,(-\ln r)^{\frac{1}{2}(2+m)}. \tag{E.2}$$

In order to test the condition (2.19) at $r = 0$ we calculate the asymptotic form of $W(z; f)$. Observing that cancellations take place between the largest terms, we obtain
$$W(z; f) = O[z(-\ln r)^{-2-m}] + O[zr(-\ln r)^{-m}\xi'(r)] + O[zr^2(-\ln r)^{-m}\xi''(r)] \tag{E.3}$$

as $r \to 0$, $z \to \infty$. Only assuming the very reasonable condition
$$r\xi'(r) = O[(-\ln r)^{-1}], \quad r^2\xi''(r) = O[(-\ln r)^{-1}], \tag{E.4}$$

it follows that
$$W(z; f) = O[z(-\ln r)^{-1-m}] = O(z^{-\frac{1}{2}-3m/(4+2m)}) \tag{E.5}$$

as $z \to \infty$ and eq. (2.19) is satisfied in this limit for any $m > 0$.

Let us now specialize to $m = 1$, $\xi(r) \equiv 1$ and calculate the asymptotic forms when $f \to 0$ of the variables u and v explicitly.

We begin by writing
$$p(s; f) = s^{-2}(f^2 s^2 - 1 + f^2 \ln s_0) + f^2 s^{-2} \ln \frac{s}{s_0}, \tag{E.6}$$

and estimate the error in u from neglecting the last term of eq. (E.6) to
$$f\int_{s_0}^{s} \frac{ds}{s} \frac{\ln(s/s_0)}{(s^2 - f^{-2} + \ln s_0)^{\frac{1}{2}}} = \begin{cases} O\left(f^2 \ln^2 \dfrac{s}{s_0}\right) \text{ for any } s \\ O\left(f \displaystyle\int^s |V(s)\,ds|\right) = o(f) \text{ as } s \to \infty, \end{cases} \tag{E.7}$$

when $f \to 0$.

Thus, we have that outside a circle of radius not less than $|s_0| \exp(-|f|^{-\frac{1}{2}})$ and centered at the origin u is given by

$$u = f(s^2 - s_0^2)^{\frac{1}{2}} - fs_0 \operatorname{arctg}\left(\frac{s^2}{s_0^2} - 1\right)^{\frac{1}{2}} + o(f). \tag{E.8}$$

Except for the error term this is the same expression as in appendix B.

To get an approximation for very small s we write

$$p(s; f) = -s^{-2}(1 - f^2 \ln s) + f^2 \tag{E.9}$$

and estimate the error in v from neglecting the last term of eq. (E.9) to

$$f \int_{s_1}^{s} \frac{s\, ds}{(f^{-2} - \ln s)^{\frac{1}{2}}} = \begin{cases} O(f^2 s_1^2) & \text{if } |s| \lesssim |s_1| \\ O(f^2) & \text{if } |s| \lesssim 1 \end{cases} \tag{E.10}$$

as $f \to 0$.

Thus inside a circle of radius $O(1)$ and centered at the origin v is given by

$$v = -\frac{2i}{3f^2}(1 - f^2 \ln s)^{\frac{3}{2}} + O(f^2). \tag{E.11}$$

In particular, we obtain that for small f

$$v = -\tfrac{2}{3} if(-\ln s)^{\frac{3}{2}} - if^{-1}(-\ln s)^{\frac{1}{2}} + O[f^{-3}(-\ln s)^{-\frac{1}{2}})] + O(f^2 s_1^2) \tag{E.12}$$

as $s \to 0$.

For $\Delta(\nu, f)$ we obtain from the eqs. (E.7), (E.8) and (E.11)

$$\Delta(\nu, f) = \pm i\nu \left\{\frac{2}{3f^2} - \ln \frac{2}{ef}\right\} + O(\nu f^2 \ln^2 f), \tag{E.13}$$

as $f \to 0$, where the sign is to be taken as sign $(\arg s_1 - \arg s_0)$.

Allowing now a factor $\xi(s)$ in the potential (2.2) with $m = 1$ we still get eq. (E.8), provided $|V(s)|$ is integrable to infinity. For the behaviour at small s, we get in analogy with appendix D that eq. (E.11) is valid with the error term $o(f)$, provided the potential satisfies

$$\int_0 |[V(r) + r^{-2} \ln r](-\ln r)^{\frac{1}{2}} r\, dr| < \infty. \tag{E.14}$$

REFERENCES

1. Khuri, N. N., and Pais, A., Rev. Mod. Phys. *36*, 590 (1964);
Pais, A., and Wu, T. T., J. math. Phys. *5*, 799 (1964) and Phys. Rev. *134*, B1303 (1964);
Wu, T. T., Phys. Rev. *136*, B1176 (1964);
Tiktopoulos, G., and Treiman, S. B., Phys. Rev. *134*, B844 (1964);
Bastai, A., Bertocchi, L., Fubini, S., Furlan, G., and Tonin, M., Nuovo Cimento *30*, 1512 (1963);

Arbuzov, B. A., Filippov, A. T., and Khrustalev, O. A., Phys. Letters *8*, 205 (1964);
Calogero, F., and Cassandro, M., Nuovo Cimento *34*, 1712 (1964) and *37*, 760 (1965);
Aly, H. H., Riazuddin, and Zimerman, A. H., Phys. Rev. *136*, B1174 (1964) and Nuovo Cimento *35*, 324 (1965);
Burdet, G., and Giffon, M., University of Lyon preprint LYCEN/6543.
2. Orear, J., Phys. Rev. Letters *12*, 112 (1964).
3. Tiktopoulos, G., Phys. Rev. *138*, B1550 (1965).
4. Predazzi, E., and Regge, T., Nuovo Cimento *24*, 518 (1962).
5. Nussenzveig, H. M., Ann. Phys. (N.Y.) *21*, 344 (1963);
Berendt, G., Nuovo Cimento *27*, 550 (1963);
Durso, J. W., and Signell, P., J. math. Phys. *5*, 350 (1964);
Brander, O., Phys. Letters *4*, 218 (1963);
Ciulli, S., Ghika, Gr., and Stihi, M., Nuovo Cimento *32*, 955 (1964).
6. Limić, N., Nuovo Cimento *26*, 581 (1962);
Challifour, J., and Eden, R. J., J. math. Phys. *4*, 359 (1963);
Giffon, M., and Predazzi, E., Nuovo Cimento *33*, 1374 (1964);
Bertocchi, L., Fubini, S., and Furlan, G., Nuovo Cimento *35*, 633 (1965);
Dombey, N., Nuovo Cimento *37*, 1743 (1965);
Del Giudice, E., and Galzenati, E., Nuovo Cimento *38*, 443 and *40 A*, 739 (1965);
Fubini, S., and Stroffolini, R., Nuovo Cimento *37*, 1812 (1965);
Jakšić, B., and Limić, N., J. math. Phys. *7*, 88 (1966) and Commun. math. Phys. (to be published);
de Alfaro, V., and Predazzi, E., Nuovo Cimento *39*, 235 (1965);
Burdet, G., Giffon, M., and Goldberg, J., University of Lyon preprint LYCEN/6540.
7. Vogt, E., and Wannier, G. H., Phys. Rev. *95*, 1190 (1954);
Wannier, G. H., Quart. Appl. Math. *11*, 33 (1953);
Spector, R. M., J. math. Phys. *5*, 1185 (1964);
Aly, H. H., and Müller, H. J. W., J. math. Phys. *7*, 1 (1966).
8. Cornille, H., Nuovo Cimento *33*, 434 (1964), *36*, 1316, *38*, 1243 and *39*, 557 (1965);
Cornille, H., and Predazzi, E., Nuovo Cimento *35*, 879 (1965);
Cornille, H., Burdet, G., and Giffon, M., Nuovo Cimento *38*, 647 (1965).
9. Calogero, F., Phys. Rev. *135*, B693 (1964); Nuovo Cimento *37*, 756 (1965) and University of Roma preprints.
10. Meetz, K., Nuovo Cimento *34*, 690 (1964);
Kupsch, J., and Sandhas, W., University of Bonn preprint;
Limić, N., Nuovo Cimento *36*, 100 (1965) and Commun. math. Phys. *1*, 321 (1966).
11. Olver, F. W. J., Phil. Trans. *A247*, 307, 328 (1954); *A249*, 65 (1956) and *A250*, 479 (1958).
12. Brander, O., Nuovo Cimento *42 A*, 39 (1966).
13. Olver, F. W. J., J. Soc. Indust. Appl. Math. *12*, 200 (1964);
Wilcox, C. H., ed: Asymptotic Solutions of Differential Equations and their Applications (New York, London, Sidney, 1964) p. 163.
14. Watson, G. N., A Treatise on the Theory of Bessel Functions (Cambridge, 1922).

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