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CONTRIBUTIONS TO THE DYNAMICS OF FLEXIBLE STRINGS

With particular Attention to the Steady Rotatory Motion of the Elastic String

BY

ERLAND YHLAND

GÖTEBORG 1961

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AV

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Introduction

The present thesis is devoted to a theoretical study of some problems concerning the dynamics of strings.* Preponderant interest is centred on a particular class of problems related to the steady rotatory motion of an elastic string. The study of such motions is of importance in textile mechanics.

The treatment is theoretical in the sense that no experiments are made in order to verify the results, and no particular attention has been paid to problems which may have greater interest from the practical point of view. Although admittedly highly desirable, a systematic investigation of such problems would be very voluminous as well as laborious. It should also be taken into consideration that theoretical works in this field are rare, indeed, some themes discussed here have not been treated previously.

No serious endeavours have been made to attain a high degree of concentration, neither has the author's efforts primarily been directed towards making the exposition mathematically rigorous at every phase. In some places a more penetrating analysis might be expected and there the author has duly commented on this fact.

The thesis is divided into six chapters and two appendices.

In the first chapter the author derives the basic equations related to arbitrary motions of a *homogeneous* and *linearly elastic* string. These equations are valid in an arbitrary curvilinear coordinate system in space. Tensor notation is used to some extent.

The second chapter is divided into two main parts. The first part contains a brief discussion of the mathematical character of the basic equations. These equations constitute a system of quasi-linear partial differential equations which are shown to be hyperbolic for the elastic string, while they are non-hyperbolic, if the string is inextensible. The second part of the chapter is devoted to a short discussion of discontinuities propagating along the string.

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* By a *string* we understand here a *one-dimensional, continuous medium of perfect flexibility*. The terms '*thread*' and '*filament*' are also common in the literature.

In the third chapter the basic equations of string motion are specialized to be valid in a system of cylindrical coordinates. Dimensionless variables are introduced, and a linearization process is outlined for a small time-dependent motion superimposed on a steady rotatory string motion; it is assumed that no external forces do act on the string.

The fourth and fifth chapters contain the principal results obtained by the author. These chapters are devoted to the study of inextensible and elastic strings in steady rotatory motion, in the absence of external forces. A more precise description of the problems treated is given *infra*.

In Chapter six we consider a time-dependent motion in the form of small vibrations superimposed on a simple kind of steady rotatory motion of an inextensible string. This chapter is intended as an introduction to the investigation of more general, time-dependent motions of a rotating string. Such investigations may assume an interest with reference to the stability of steady motions.

Two appendices on mathematical minutiae conclude the thesis.

As mentioned above the main subject of the work is the study of a particular class of steady rotatory string motions.

A string motion is called *steady rotatory* (or *steady rotational*), *if the string moves with a velocity, the tangential velocity which does not depend on time, along a curve which is fixed in a frame of reference, the latter rotating with constant angular velocity round an axis fixed in space.*

As regards the properties of the string we confine ourselves to strings that are *linearly elastic* and *homogeneous*. A string is *linearly elastic* in our sense of the term, *if the logarithmic strain depends linearly on the string tension*. In the case of small strain the results obtained are valid for most kinds of elastic strings. *A homogeneous string in unstressed state has the same mass per unit length in every point*. An inextensible string is obtained by putting the elastic constant equal to zero.

As to the kinds of motion treated we restrict ourselves to cases where *no external forces act on the string*. This implies that we disregard gravity and air resistance. Such limitations need no particular justification in a theoretical work like this. We also ignore the special motions which take place in a plane perpendicular to the axis of rotation.

By a *string property* we understand *any dependent variable of the basic equations* formulated in Chapter 1, or equations that can be deduced from the aforesaid basic equations. The properties of the steady string motions specified here are governed by a system of ordinary nonlinear differential equations. The latter system is derived in Chapter 3.

From our theoretical point of view it is of interest to investigate and solve this differential system under as general conditions as possible. In this matter we are confronted with certain difficulties as there are no theorems available on the existence and uniqueness of general boundary value problems in connection with nonlinear differential systems. If we exclude string motions involving finite strain in connection with non-zero tangential velocity, however, the system will become amenable to elementary methods of integration.

In order to facilitate the determination of explicit solutions we divide the class of problems, consisting of all possible boundary value problems connected with our differential system, into a number of subclasses.

Within each subclass treated in this thesis we establish a basic problem which is representative of the class in the following sense: If a problem belonging to a certain subclass has a solution (*i. e.*, if the corresponding motion really exists), then that solution is contained in the solution of the basic problem. The basic problems are suitably chosen as one-point (initial value) problems or closely related problems. Their solutions are unique, but this circumstance does not imply uniqueness of the solution of a particular boundary value problem; in fact, several problems which possess an interest in this connection actually are eigenvalue problems.

The separation into subclasses is made according to three different bases of division.

In the first place we must distinguish between string motions that take place with *positive reduced string tension* and motions with *negative reduced string tension*. In the author's terminology the *reduced string tension* is the quantity $\overline{T}=T-mv^2$, where $T\geq 0$ is the string tension, m the mass of the unit length, and v the tangential velocity of the string. Steady rotatory motions with $\overline{T}>0$ are treated in Chapter 4, and those with $\overline{T}<0$ are briefly discussed in Chapter 5.

The second basis of division is related to the magnitude of the strain of the string and we consequently establish the following division: (i) *inextensible strings*; (ii) *elastic strings subject to small strain*; (iii) *linearly elastic strings subject to finite strain*. As concerns the last kind of strings we confine the treatment to motions which take place without tangential velocity.

The third basis of division is related to the curve assumed by the string as observed in a co-rotating frame of reference. If this curve *intersects* the *axis of rotation*, we have a special (although important) case which must be treated separately from the *general non-intersection* case.

We must make a further division for motions with negative reduced string tension, $\overline{T}<0$, as shown in Chapter 5.

A schematic diagram, arranged according to the subclasses enumerated above, is placed at the end of this introduction; it shows the designations of those basic problems which are solved in the thesis.

We must now briefly survey the literature in the field and in neighbouring fields. The survey follows the contents of the chapters of the thesis and is not intended to be complete.

The basic equations of inextensible and elastic strings with a fixed, rectangular Cartesian coordinate system as frame of reference are stated in several textbooks; in some of these works particular kinds of problems are discussed, see for instance Beghin, [B1], and Routh, [R2]. Hamel, [H3], discusses the basic string equations as special cases of rod equations, and valuable information on the string theory may be extracted from the rigorous treatment of stress and strain in connection with finite deformations of rods (and shells) by Ericksen and Truesdell, [E1].

No discussion on the transformation of the basic string equations into curvilinear frames of reference seems to exist in the literature. Neither has the author been able to find any general investigations of the mathematical nature of the basic equations or the propagation of discontinuities along elastic strings. Strong discontinuities on inextensible strings are analysed by Beghin, [B1], and Pailloux, [P1], [P3], and in connection with a particular problem by Hamel, [H4].

A large number of papers are devoted to the study of spinning problems; a survey and bibliography containing 37 papers up to 1955 may be found in three papers by Detry, [D1]. The factor of gravity is disregarded in all these papers on spinning theory and practice, and the string is regarded throughout as inextensible; solutions which also take air resistance into consideration, however, are of particular interest. The tangential velocity is neglected since it is of minor importance in most spinning problems, but MACK, [M1], has shown that it is possible to solve the problems here called (I1) and (I2) by means of quadrature.

Steady rotatory motions of inextensible strings taking place in a plane perpendicular to the axis of rotation (the gravitational effects being neglected) are treated by HAMEL, [H5], and MACK, [M2].

The problem of an inextensible string intersecting the axis of rotation and being in steady rotatory motion without tangential velocity round a vertical axis is rigorously analysed by KOLODNER, [K2], whose treatment also takes gravity into account. The stability of particular motions is studied by CAUGHEY, [C5], HALL and HUNTER, [H6], KOLODNER, [K2], and NERONOFF, [N2].

A special type of vibrational motion superimposed on a steadily rotating string is discussed by STEVENSON, [S4], *cf.* also WRIGHT, [W1].

We shall end this introduction with a few comments on the disposition of the thesis and on particular concepts and notations that are used throughout the work.

The decimal paragraphing system is employed and the equations are numbered conformably. The footnotes are numbered consecutively throughout the thesis.

The bibliography is arranged in approximate alphabetic order and each separate work is given a code number, *e. g.* [B1] refers to H. BEGHIN: Cours de mécanique théorique et appliquée. These code numbers are used in the references in the text of the thesis.

Each chapter contains a list of the main symbols and conventions used in that chapter.

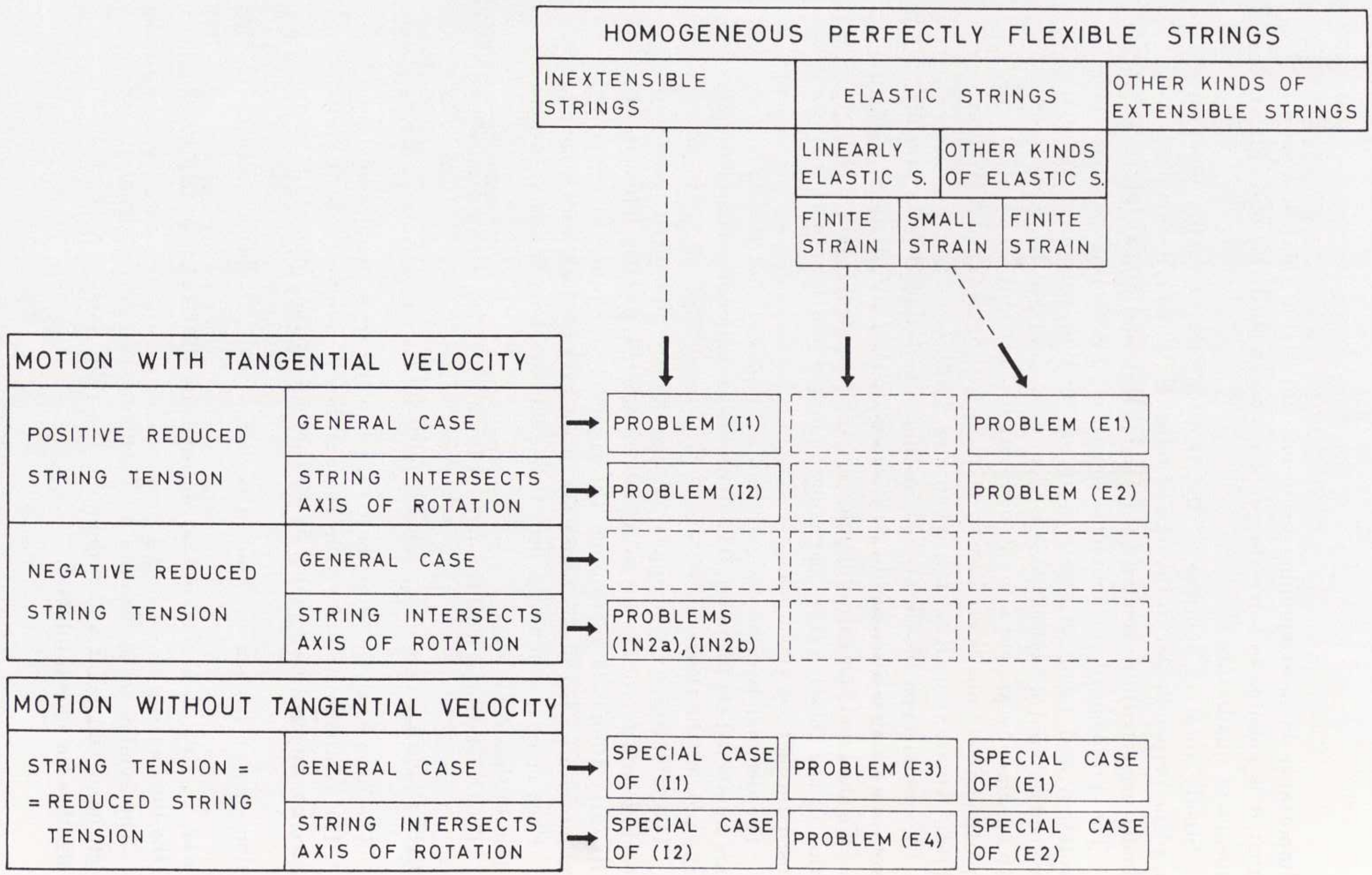
The O-notation is used throughout the work. A function $f(x)$ satisfies

$f(x)=O(\Phi(x))$, $x\to 0$, if $\dfrac{f(x)}{\Phi(x)}$ is bounded for $x\to 0$.

The term analytic (function) is used in the sense regular analytic.

The symbol $\equiv$ is employed in three different connotations: (i) $f(x)\equiv$ $\equiv$const. means that equality holds throughout the interval of definition of $f(x)$; (ii) $u\equiv\dfrac{\partial h}{\partial \tau}$ that u is defined as $\dfrac{\partial h}{\partial \tau}$; (iii) $x'\equiv x'(u,t)\equiv$ $\equiv(x_1,\ldots,x_9,T,u)$ merely that we use three different ways of writing (the row matrix) x'.

The symbol (a, b) signifies an open interval with a denoting the left-hand end point and b denoting the right-hand end point. $[a, b]$ signifies the closed interval.



Schematic diagram showing those problems concerning steady rotatory motion of various kinds of flexible strings that are treated in the thesis. Broken rectangles indicate problems that have not been the objects of treatment.

Contents

1. The Basic Equations of String Motions

Symbols and conventions:

y^i, $i=1,2,3$	Rectangular, right-handed, Cartesian coordinates, fixed in space. Coordinates of the string points.
x^i, $i=1,2,3$	Arbitrary, curvilinear coordinates, fixed in space. Coordinates of the string points.
σ	Coordinate along the curve of the string. $\sigma=$const. denotes a particular string point.
τ, t	Time coordinates.
s	Arc length of the string curve.
S	Surface generated by the string during its motion.
v^i	Velocity vector of the string point.
f^i	Acceleration vector of the string point.
z	Independent variable, defined by Eq. (1.2—2).
u, q	String properties, defined by Eq. (1.2—4).
m	Mass of the unit length of the string.
m_0	Mass of the unit length of the string for $T=0$.
T	String tension (force).
k	Elastic constant, see Eq. (1.2—18).
A	Arbitrary string property, scalar or vector.
A^i	Arbitrary, contravariant string property.
g_{ik}	Metric tensor of the curvilinear coordinate system x^i.
$\left\{ {i \atop km} \right\}$	Christoffel symbol of the second kind.
δ^p_k	Kronecker symbol.
$F^i=F_i$	Rectangular Cartesian components of the external force acting on the unit length of the string.

K^i, K_i Contravariant and covariant components of the external force in a curvilinear coordinate system.

H Quantity of string mass passing per unit of time through a particular surface $F(y^i, t) = 0$ which is fixed in space or moving through space.

Symbols not included in the above list will occur now and then in the text. All indices have the range of values $1, 2, 3$. The summation convention is used. Concerning the tensor notation see for instance [M3].

1.1. Description of the String and its Motion

In this section we shall give a description of the fundamental properties of the string and the motions that we are going to study in this work.

1.11. The String

The string is defined and various assumptions concerning it are made as enunciated in the following points.

1) A point of the string, or *string point*, is a point which is fixed with respect to the material of the string. (A string point consequently follows a certain 'molecule' of the string.)

2) The string is assumed to have zero thickness and its mass is therefore distributed along a curve, *the string curve*, consisting of the string points.

3) An *element* of the string consists of the string material carried by an arc element of the string curve. The definitions of the length and the mass of the string element are self-evident.

4) The string is *perfectly flexible* and the state of stress in a string point is completely determined by *the string tension* which is directed along the tangent of the string curve. It is postulated that *the string tension cannot be negative, i. e.* the string cannot sustain compressive forces.

5) The string is *elastic, i. e.* the length of the string element is solely determined by the string tension. We shall in general confine the treatment to a *linearly elastic string, i. e.* to the case of a linear relationship

between strain (defined in a certain manner, see section 1.25) and the string tension.

6) The string is *homogeneous*, *i. e.* the mass of the unit length is the same in every point, when the string is in unstressed state.

1.12. **The Motion of the String**

1) When a string (of finite or infinite length) moves through space, it generates a surface S. Let the equation of S (in a *rectangular, Cartesian* coordinate system) be

$$y^i = y^i(\sigma, \tau). \tag{1.1—1}$$

Let the independent variable τ be *the time*. It is then clear that a curve on the surface S with the equation $y^i = y^i(\sigma, \tau_0)$, $\tau_0 = \text{const.}$, is the string curve at the time τ_0.

Let the curve $y^i = y^i(\sigma_0, \tau)$, $\sigma_0 = \text{arbitrary constant}$, be the path of a certain string point; the velocity vector of that string point is then directed along the tangent to the curve $y^i = y^i(\sigma_0, \tau)$. Let us specify σ as *the unstretched length* of the string measured from some arbitrary string point. It is obvious that the independent variables σ and τ constitute a coordinate system on the surface S.

2) In general we shall assume that there exists no point on S in which the tangent vectors to the coordinate lines $\sigma = \text{const.}$ and

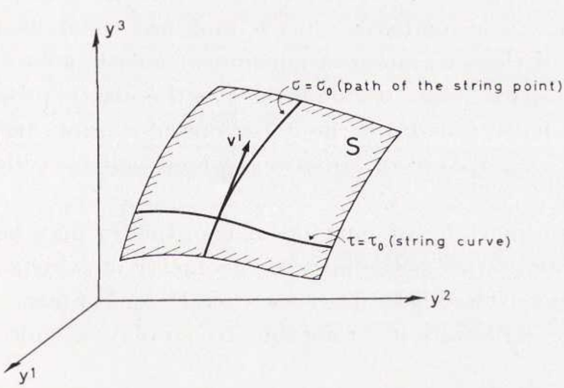


Fig. I. Schematic sketch showing part of the surface S generated by a string, and the coordinate lines $\sigma = \sigma_0$ and $\tau = \tau_0$ on S. The velocity vector v^i of the string point $\sigma = \sigma_0$ at the time $\tau = \tau_0$ is also shown.

τ=const. are collinear. This assumption implies that we exclude cases where adjacent points of the string have the same path. When every point of the string describes the same path we have the kind of motion that is discussed in section 1.4.

3) We shall assume that the functions $y^i(\sigma, \tau)$ possess sufficient properties of continuity. They must in general be twice continuously differentiable with respect to σ and τ. Along discrete lines on S, however, there may appear *discontinuities* of different kinds, *cf*. Ch. 2.

4) Later on we shall use other *surface coordinates* than σ, τ and also introduce *curvilinear space coordinates* instead of the rectangular, Cartesian coordinates y^i used so far.

1.2. Kinematics and Kinetics of the String

It is the purpose of this section to establish a system of equations for the linearly elastic, homogeneous string. The system will be complete in the sense that we can deduce from it *any* equation which, in terms of mechanics, has a meaning for such a string.[1] Naturally, the string may also have to satisfy initial and boundary conditions. The mentioned system can be formulated in different ways. We shall start from the expressions for the velocity and acceleration vectors with σ, τ as independent variables and a rectangular, Cartesian system as the frame of reference. The corresponding equations of motion and kinematical conditions are well known and may be found in the standard textbooks, *e. g.* [B1], [R2], [S3]. We next introduce a new set of surface coordinates z, t on S and use them as independent variables, and then a system of curvilinear coordinates as a frame of reference in space. Then we shall discuss the elastic properties of the string and finally put down the equations of motion and gather our expressions in a system of equations which will be called *the string equations*.

The transformation into curvilinear coordinates may be carried out in conventional terms of the intrinsic geometry of a surface embedded in a three-dimensional Euclidean space, *cf*. [M3]. For later applications, however, we will find it preferable to employ a more specialized technique.

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[1] In certain points of discontinuity conditions obtain which cannot be deduced from our complete system, *cf*. Ch. 2.

1.21. Velocity and Acceleration Vectors in Lagrangian Formulation

The velocity vector v^i and the acceleration vector f^i of the string point are determined by the expressions

$$
\begin{aligned}
v^i &= v^i(\sigma, \tau) = \frac{\partial y^i}{\partial \tau}, \\
f^i &= f^i(\sigma, \tau) = \frac{\partial^2 y^i}{\partial \tau^2}.
\end{aligned}
\tag{1.2—1}
$$

where the variables are the same as in section 1.1.

1.22. A New Set of Independent Variables

Let z and t be coordinates on S defined by

$$
\begin{aligned}
z &= h(\sigma, \tau), \\
t &= \tau.
\end{aligned}
\tag{1.2—2}
$$

We shall assume that this transformation is one to one, and that h is at least twice continuously differentiable with respect to σ and τ. We also assume that

$$
\frac{\partial(z, t)}{\partial(\sigma, \tau)} = \frac{\partial h}{\partial \sigma} \neq 0. \tag{1.2—3}
$$

From now on we shall regard every *property of the string*, *i. e.* every dependent variable, as a function of z and t.

Let $q = q(z, t)$, $u = u(z, t)$ be string properties defined by

$$
\begin{aligned}
\frac{\partial h}{\partial \sigma} &\equiv \frac{m_0}{mq}, \\
\frac{\partial h}{\partial \tau} &\equiv u,
\end{aligned}
\tag{1.2—4}
$$

where $m = m(T) = m(T(z, t))$ is the mass of the unit length of the string. For an elastic and homogeneous string m depends explicitly of the string tension T only, and m_0 is the value of m when $T = 0$. The physical interpretations of q and u depend on the choice of h and will be discussed later.

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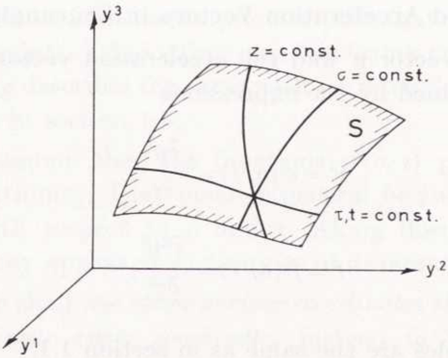

Fig. II. Sketch illustrating the significance of the transformation Eq. (1.2—2) from the surface coordinates σ,τ into the surface coordinates z,t.

We now get the following expressions for the transformation of the derivatives of an arbitrary string property A (scalar or vector)

$$\frac{\partial A}{\partial \sigma} = \frac{m_0}{mq}\,\frac{\partial A}{\partial z}\,,$$
$$\frac{\partial A}{\partial \tau} = \frac{\partial A}{\partial t} + u\,\frac{\partial A}{\partial z} \equiv \frac{dA}{dt}\,, \tag{1.2—5}$$

where, from the mechanics of continuous media, we adopt the term *substantial time derivative*, symbolized by $\frac{d}{dt}$. From Eq. (1.2—1) we obtain the following expressions for the velocity and acceleration vectors of the string point

$$v^i = \frac{\partial y^i}{\partial t} + u\,\frac{\partial y^i}{\partial z} = \frac{dy^i}{dt}\,,$$
$$f^i = \left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial z}\right)\left(\frac{\partial y^i}{\partial t} + u\,\frac{\partial y^i}{\partial z}\right) = \frac{d^2 y^i}{dt^2}\,. \tag{1.2—6}$$

It may be observed that the substantial time derivative generally does not commute with the partial derivatives.

We may remark that if, for some special string problem, we know the string properties m, q, u as functions of z and t, we naturally can also determine the paths of the string points, *i. e.* the curves σ=const. on the surface S. Let the inverse transformation to Eq. (1.2—2) be

$$\sigma = g(z, t),$$

$$\tau = t.$$

Then, from the identities in Eq. (1.2—4), we have

$$\frac{\partial g}{\partial t} = -\frac{mqu}{m_0}.$$

The right member of this differential equation is a known function of z and t, and together with an initial condition, for instance,

$$g(z, t_0) = g_0(z)$$

the differential equation determines the function g. The path of an arbitrary string point $\sigma = \sigma_1$ is then the curve $g(z, t) = \sigma_1$.

We shall now discuss the physical interpretations of the string properties q and u.

Let s be the arc length of the string curve measured from an arbitrary point. By the definition of σ we then obtain the following equation for the mass of a string element of the length ds

$$mds = m_0 d\sigma. \tag{1.2—7}$$

Studying the change of an arbitrary string property A due to a small displacement along the string curve we obtain the equality

$$\frac{\partial A}{\partial \sigma} d\sigma = \frac{\partial A}{\partial z} dz.$$

Together with Eq. (1.2—5) this yields $d\sigma = \dfrac{mq}{m_0} dz$; and from Eq. (1.2—7) we then get

$$dz = \frac{1}{q} ds. \tag{1.2—8}$$

Observing that z is a coordinate along the string curve (t=const.), we may interpret q as a measure of *the longitudinal scale* along that curve.

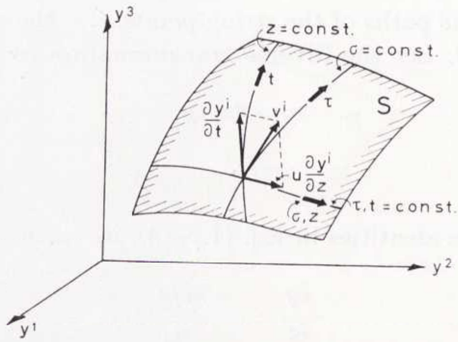

Fig. III. The decomposition of the velocity v^i into components along the coordinate lines z=const. and t=const. The arrows along the coordinate lines indicate positive directions of the coordinates, and in the case reproduced here q and u are both assumed to be positive quantities.

By the definition of s the unit tangent vector along the string curve is $\dfrac{\partial y^i}{\partial s} = \dfrac{1}{q} \dfrac{\partial y^i}{\partial z}$. From the expressions for the velocity vector v^i in Eq. (1.2—6) we can then conclude that uq may be interpreted as *the component of the velocity vector along the tangent to the string curve* (*cf*. Fig. III).

Finally we consider two special choices of the coordinate z, *i. e.* the function $h(\sigma, \tau)$.

1) Let z be *the arc length* of the string curve, *i. e.* let $z \equiv s$. Choose an arbitrary curve on the surface S which does not touch any string curve and let this curve have the equation $y^i = y^i(s_0, t)$, s_0=const. Provided the motion is known, every curve s=const. on S is then determined. From Eq. (1.2—8) we have

$$q = q(z, t) \equiv 1 \qquad (1.2-9)$$

and $u = u(z, t)$ is then the component of the velocity vector along the string tangent.

2) Let z be *one of the coordinates* y^i, *say* y^3, and let the string curves nowhere touch a plane y^3=const. Then the curves $y^i = y^i(z, t)$, z=const., are lines of intersection between the planes y^3=const. and the surface S. We get

$$\frac{\partial y^3}{\partial z} \equiv 1, \; \frac{\partial y^3}{\partial t} \equiv 0. \qquad (1.2-10)$$

According to Eq. (1.2—8) $\frac{1}{q}$ becomes the direction cosine of the tangent to the string curve with respect to the y^3 axis. By perpendicular projection on the y^i axes we obtain u as the y^3 component of the velocity vector.

1.23. Kinematical Conditions

Let z, t be the same set of coordinates on the surface S as previously. Let s, t represent this set when $z=s$ is chosen as the arc length of the string curve. Since $y^i=y^i(s,t)$ is the equation of the surface S, $\frac{\partial y^i}{\partial s}$ is the unit tangent vector to the string curve and the equations

$$\frac{\partial y^i}{\partial s}\frac{\partial y^i}{\partial s}=1, \tag{1.2—11}$$

$$\left.\begin{aligned} &\frac{\partial y^i}{\partial s}\frac{\partial^2 y^i}{\partial s^2}=\frac{\partial y^i}{\partial s}\frac{\partial^2 y^i}{\partial t\partial s}=0,\\ &\frac{\partial y^i}{\partial s}\frac{d}{dt}\left(\frac{\partial y^i}{\partial s}\right)=0 \end{aligned}\right\} \tag{1.2—12}$$

hold everywhere on the surface S, provided that the derivatives exist. Eq. (1.2—12) imply that $\frac{\partial y^i}{\partial s}$ is perpendicular to its own derivative in an arbitrary direction on S. Instead of Eq. (1.2—11) we get in the general case

$$\frac{\partial y^i}{\partial z}\frac{\partial y^i}{\partial z}=q^2. \tag{1.2—13}$$

There is another kinematical condition. The string properties q and u are both derived from the function $h=h(\sigma,\tau)$ which defines the curves $z=$const. on S, *cf*. Eq. (1.2—4). By hypothesis, the partial derivatives of h with respect to σ and τ commute, and by means of the expressions (1.2—5) we get

$$\frac{\partial(mq)}{\partial t}+\frac{\partial(mqu)}{\partial z}=0. \tag{1.2—14}$$

In the special case of $z=s$ we have $q\equiv 1$ and Eq. (1.2—14) will then be slightly simplified. Eq. (1.2—14) corresponds to the equation of continuity for a general continuous medium. It can be derived directly from a kinematical study of the motion of a string element during a small interval of time.

1.24. Curvilinear Coordinates

In the preceding sections we have studied the motion of the string with a rectangular Cartesian system (coordinates designated y^i, $i=1,2,3$) as a frame of reference in space. We now introduce a *general, curvilinear system* with the coordinates x^i, $i=1,2,3$, and the *metric tensor* g_{ik}, *cf.* [M3]. We transform the expressions (1.2—6) for the velocity and acceleration vectors and the kinematical conditions (1.2—13) and (1.2—14) to be valid in a general curvilinear coordinate system. For the contravariant tangent vectors $\dfrac{\partial y^i}{\partial z}$ and $\dfrac{\partial y^i}{\partial t}$ we have

$$\frac{\partial y^i}{\partial z} \to \frac{\partial x^i}{\partial z}, \qquad \frac{\partial y^i}{\partial t} \to \frac{\partial x^i}{\partial t}, \tag{1.2—15}$$

where $\to$ means: corresponds to.

For the partial derivatives of an arbitrary contravariant vector with space components A^i, defined on the surface S, we obtain

$$\begin{aligned} \frac{\partial A^i}{\partial z} &\to \frac{\delta A^i}{\delta z} \equiv \frac{\partial A^i}{\partial z} + \begin{Bmatrix} i \\ km \end{Bmatrix} A^k \frac{\partial x^m}{\partial z}, \\ \frac{\partial A^i}{\partial t} &\to \frac{\delta A^i}{\delta t} \equiv \frac{\partial A^i}{\partial t} + \begin{Bmatrix} i \\ km \end{Bmatrix} A^k \frac{\partial x^m}{\partial t}, \end{aligned} \tag{1.2—16}$$

where $\dfrac{\delta A^i}{\delta z}$ is called *the intrinsic derivative* of A^i with respect to z.

In Eq. (1.2—16) $\begin{Bmatrix} i \\ km \end{Bmatrix}$ is the CHRISTOFFEL symbol of the second kind defined by

$$\begin{Bmatrix} i \\ km \end{Bmatrix} = \frac{1}{2} g^{ip} \left(\frac{\partial g_{mp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^m} - \frac{\partial g_{km}}{\partial x^p} \right),$$

where *the conjugate metric tensor* g^{ip} is defined by

$$g_{pi}\, g^{ik} = \delta_p^k = \begin{cases} 1 & \text{if } k = p \\ 0 & \text{if } k \neq p \end{cases}.$$

and δ_p^k is the KRONECKER symbol.

Scalar functions defined on S are not affected by the change of reference system in space. We therefore get the following expressions corresponding to Eqs. (1.2—6), (1.2—13) and (1.2—14)

$$\begin{aligned} v^i &= \frac{dx^i}{dt} = \frac{\partial x^i}{\partial t} + u \frac{\partial x^i}{\partial z}, \\ f^i &= \frac{\delta v^i}{\delta t} + u \frac{\delta v^i}{\delta z} = \frac{\partial v^i}{\partial t} + u \frac{\partial v^i}{\partial z} + \begin{Bmatrix} i \\ km \end{Bmatrix} v^k v^m, \\ & g_{ik} \frac{\partial x^i}{\partial z} \frac{\partial x^k}{\partial z} = q^2, \\ & \frac{\partial (mq)}{\partial t} + \frac{\partial (mqu)}{\partial z} = 0. \end{aligned} \tag{1.2—17}$$

1.25. Conditions Governing the Properties of an Elastic String

So far we have not made any assumptions concerning the elastic properties of the material in the string.

We shall assume that the string is in a state of simple tension and confine ourselves to the case of a linearly elastic string, which we shall define as a string in which the strain is proportional to the string tension. It may be remarked that owing to the lateral contraction of the string the stress-strain relation in general will not be linear.

By the strain we shall understand the logarithmic or natural strain, *cf.* [H1], p. 9. The strain is generally assumed to be finite, and some of the results obtained later may be extended to apply to a wider range of elastic strings. For an element of the length l of a linearly elastic string subjected to the tension T we have by definition

$$l = l_0 \exp(kT), \quad m = m_0 \exp(-kT), \tag{1.2—18}$$

where l_0 and m_0 are the length and the mass per unit length, respectively, of the unstretched element; $k \geq 0$ is the elastic constant of the string and $k=0$ means that the string is inextensible.[2]

From Eq. (1.2—18) we have

$$l=l_0(1+kT)+O((kT)^2), \quad m=m_0(1-kT)+O((kT)^2), \; kT \to 0. \quad (1.2-19)$$

Neglecting terms $O((kT)^2)$ we get the classical expressions valid for small strain.

1.26. Equations of Motion. The Complete Equations

Taking s to be the arc length of the string curve, as we have done previously, we get the following wellknown *equation of motion* for the string element

$$\frac{\partial}{\partial s}\left(T\frac{\partial y^i}{\partial s}\right)-mf^i+F^i=0, \qquad (1.2-20)$$

where T is the string tension, f^i the acceleration vector, F^i the external force on the unit length of the string,[3] and m the mass of the unit length. y^i are rectangular, Cartesian coordinates in space. From sections 1.22, 1.23 and 1.25 we have expressions for f^i, the kinematical conditions and the conditions valid for a linearly elastic string.

Gathering all these equations into a single system we get the following equations for a linearly elastic string

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(T\frac{\partial y^i}{\partial s}\right)-m\,\frac{d^2y^i}{dt^2}+F^i=0,\\
&\frac{\partial y^i}{\partial s}\,\frac{\partial y^i}{\partial s}=1,\\
&\frac{\partial m}{\partial t}+\frac{\partial (mu)}{\partial s}=0,\\
&m=m_0\exp(-kT).
\end{aligned} \qquad (1.2-21)$$

This system is complete in the sense discussed in the beginning of section 1.2. System (1.2—21) consists of five (scalar) partial differential

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[2] The elastic properties of the material of the string cannot, of course, be completely described by a single constant.

[3] If the external force on the unit mass of the string is P^i, we must replace F^i by mP^i.

equations and one transcendental equation for the six dependent variables y^i, T, u, m, and we have s, t as independent variables. By means of inner multiplication by $\dfrac{\partial y^i}{\partial s}$ in the first equation of the system and taking into consideration the conditions of Eq. (1.2—12) we get the following equations

$$\frac{\partial T}{\partial s} - m\left(\frac{du}{dt} + \frac{\partial y^i}{\partial s}\frac{\partial^2 y^i}{\partial t^2}\right) + F^i\frac{\partial y^i}{\partial s} = 0,$$
$$\frac{\partial}{\partial s}(T - mu^2) - \frac{\partial}{\partial t}(mu) - m\frac{\partial y^i}{\partial s}\frac{\partial^2 y^i}{\partial t^2} + F^i\frac{\partial y^i}{\partial s} = 0. \qquad (1.2-22)$$

These two equations are equivalent and are equations of motion in the direction of the string tangent.

We can, of course, formulate the equations (1.2—21) with z and t as independent variables (s simply means a particular choice of z). We then obtain

$$\frac{\partial}{\partial z}\left(\frac{T}{q}\frac{\partial y^i}{\partial z}\right) - mq\frac{d^2 y^i}{dt^2} + qF^i = 0,$$
$$\frac{\partial y^i}{\partial z}\frac{\partial y^i}{\partial z} = q^2, \qquad (1.2-23)$$
$$\frac{\partial(mq)}{\partial t} + \frac{\partial(mqu)}{\partial z} = 0,$$
$$m = m_0 \exp(-kT).$$

Finally we transform the string equations (1.2—23) to be valid in a general, curvilinear coordinate system x^i.

By means of the expressions (1.2—16) and (1.2—17) we get

$$\frac{\delta}{\delta z}\left(\frac{T}{q}\frac{\partial x^i}{\partial z}\right) - mq\left(\frac{\delta}{\delta t} + u\frac{\delta}{\delta z}\right)\left(\frac{\partial x^i}{\partial t} + u\frac{\partial x^i}{\partial z}\right) + qK^i = 0,$$
$$g_{ik}\frac{\partial x^i}{\partial z}\frac{\partial x^k}{\partial z} = q^2, \qquad (1.2-24)$$
$$\frac{\partial(mq)}{\partial t} + \frac{\partial(mqu)}{\partial z} = 0,$$
$$m = m_0 \exp(-kT).$$

In this system the contravariant force components, K^i, are determined from the Cartesian components $F^i = F_i$ by means of the expression

$$K^i = F^p \frac{\partial x^i}{\partial y^p} = g^{ki} F_p \frac{\partial y^p}{\partial x^k}. \qquad (1.2-25)$$

Essentially, the systems (1.2—23) and (1.2—24) possess the same mathematical properties. The following comments on system (1.2—24), with obvious modifications, are consequently valid also for system (1.2—23). System (1.2—24) consists of five (scalar) partial differential equations and one transcendental equation for seven (scalar) dependent variables: x^i, T, m, q, u. There are two independent variables z and t. We have one more dependent variable than equations, but we are free to choose the curves $z=$const. in an arbitrary manner on the surface S, generated by the string in its motion.

In the following considerations the dependent variables will generally be called *string properties*, and the systems (1.2—24) or (1.2—23) will be called *the string equations*. These equations naturally do not yield unique solutions of problems concerning the motions of a string. In addition to the string equations we must possess a suitable system of initial values and boundary conditions.

The questions of the existence and uniqueness of the solutions to the system (1.2—24) present difficult problems which sometimes become even more complicated by the presence of strong discontinuities in the string properties. A brief discussion of the string equations, starting from system (1.2—21), is given in Chapter 2.

1.3. The Flow of String Mass Through a Surface

Sometimes it is of interest to determine the quantity of string mass passing through a particular surface in space during the motion of a string. The surface may be fixed or it may be moving through space.

1) Let us consider a surface D in a rectangular, Cartesian system in space, and let t be the time. Let the equation of D be

$$F(y^i, t) = 0, \qquad (1.3-1)$$

where F is continuously differentiable with respect to y^i and t. Let h^i be the velocity vector of a point P on D at the time t, and n_i a normal unit vector of D at P.[4]

[4] It should be observed that Eq. (1.3—1) does not completely determine h^i but only the component along n_i. There is here no necessity of a more precise statement.

2) Let the positive side of D, at the time t, be defined as the set consisting of all the points y^i in space that satisfy the inequality $F > 0$. Let the negative side of D be defined accordingly.

3) Let v^i be the velocity vector of a string point coincident with P at the time t.

From Eq. (1.3—1) we then obtain

$$\frac{\partial F}{\partial y^i} h^i + \frac{\partial F}{\partial t} = 0, \qquad (1.3-2)$$

$$n_i = \frac{\partial F}{\partial y^i} \left(\frac{\partial F}{\partial y^k} \frac{\partial F}{\partial y^k} \right)^{-1/2}. \qquad (1.3-3)$$

As usual we take s to be the arc length of the string curve, and $y^i = y^i(s,t)$ to be the equation of the surface S, generated by the motion of the string. We shall assume that the string does not touch the surface D in the point P, and we then obtain

$$n_i \frac{\partial y^i}{\partial s} \neq 0. \qquad (1.3-4)$$

The quantity of string mass H passing through the surface D in the vicinity of P per unit of time is determined by

$$H = m \left(n_i \frac{\partial y^i}{\partial s} \right)^{-1} n_k (v^k - h^k). \qquad (1.3-5)$$

Combining the last equation with Eqs. (1.3—2) and (1.3—3) we get

$$H = m \left(\frac{\partial F}{\partial y^i} \frac{\partial y^i}{\partial s} \right)^{-1} \left(v^k \frac{\partial F}{\partial y^k} + \frac{\partial F}{\partial t} \right).$$

We next write the equation of the surface S in the form $y^i = y^i(z,t)$ with z as in section 1.22. Finally, adopting a general, curvilinear coordinate system x^i, and writing the equation of the surface D as $F(x^i,t) = 0$, we get

$$H = mq \left[u + \left(\frac{\partial F}{\partial x^i} \frac{\partial x^i}{\partial z} \right)^{-1} \left(\frac{\partial F}{\partial x^k} \frac{\partial x^k}{\partial t} + \frac{\partial F}{\partial t} \right) \right], \qquad (1.3-6)$$

where by Eq. (1.3—4) $\frac{\partial F}{\partial x^i} \frac{\partial x^i}{\partial z} \neq 0$. It should be observed that H may have positive or negative sign. If we choose the arc length s in such a manner that it increases on passing from the negative to the positive side of D, then $n_i \dfrac{\partial y^i}{\partial s} > 0$ and $H > 0$ means that string mass is flowing from the negative into the positive side of D.

1.4. The String Motion in a Fixed Path

We remarked in section 1.12, point (2) that the case in which the string moves in a fixed path must be discussed separately. We now consider such a motion in a rectangular, Cartesian system y^i, fixed in space, and take s to be the arc length of the string curve measured from a fixed point. Using the same notations as previously we have the following string properties:

$y^i = y^i(s)$ equation of the string curve;

$u = u(s,t)$ velocity of the string point, directed along the tangent vector $\dfrac{\partial y^i}{\partial s}$;

$T = T(s,t)$ string tension;

$m = m(s,t)$ mass of the unit length of the string.

Taking $\dfrac{\partial y^i}{\partial t} \equiv 0$ into consideration, simple reflection will indicate that we may make use of the string equations (1.2—21). Rewriting the first of the latter equations we get

$$\begin{aligned}
&\frac{\partial}{\partial s}\left[(T - mu^2)\frac{\partial y^i}{\partial s}\right] + \frac{\partial (mu)}{\partial t}\frac{\partial y^i}{\partial s} + F^i = 0,\\
&\frac{\partial y^i}{\partial s}\frac{\partial y^i}{\partial s} = 1,\\
&\frac{\partial m}{\partial t} + \frac{\partial (mu)}{\partial s} = 0,\\
&m = m_0 \exp(-kT).
\end{aligned} \tag{1.4—1}$$

We shall not attempt a general discussion of this system but merely make a few comments.

1) If the external force F^i on the unit length of the string does not depend on time, and if the string is inextensible, *i. e.* $k=0$, then the motion may take place with a constant velocity u. We then obtain

$$\frac{\partial}{\partial s}\left[(T-mu^2)\frac{\partial y^i}{\partial s}\right]+F^i=0. \tag{1.4—2}$$

This equation implies that the string curve will be the same as the curve of a string which is in statical equilibrium subjected to the external force F^i, provided that m is the same and similar boundary conditions hold. If the string tension is $\overline{T}=\overline{T}(s)$ in the equilibrium case, the tension of the moving string is $T=\overline{T}+mu^2$. This result is well known, *cf.* [R2], p. 400.

2) If the external force is $F^i\equiv 0$, the system (1.4—1) is satisfied by $T=mu^2=$const., m, $u=$const., for arbitrary functions $y^i(s)$. The string curve may thus have an arbitrary shape and may also contain points of strong discontinuities.

The latter fact is a consequence of results which will be presented in section 2.22. (A strong discontinuity propagates with the velocity $c=\left(\frac{T}{m}\right)^{1/2}$ relative to the string points, and from the preceding formulae we get $c=u$. The discontinuity is then fixed in space, on condition that it moves backwards along the string; in the opposite case the string cannot possibly move in a fixed path.)

When the string is inextensible the motion now discussed is a special case of the motion discussed in (1).

The result obtained for a linearly elastic string may be used for all other kinds of elastic strings, since $T=$const. implies $m=$const. for every homogeneous elastic string.

2. Discussion of the String Equations

Symbols and conventions:

y^i	Fixed, rectangular, right-handed, Cartesian coordinates. Coordinates of the string points.
s	Arc length of the string curve.
t	Time coordinate.
T	String tension.
m	Mass of the unit length of the string.
m_0	— — — — — — — — — — — — — — in unstrained state.
u	Tangential velocity of the string point.
F^i	Rectangular, Cartesian components of the external force acting on the unit length of the string.
$x=[x_k]$	Column matrix with elements x_k, see Eq. (2.1—2).
$x'=(x_k)$	Row matrix, transpose of x.
A, B	Square matrices, see Eq. (2.1—4), Eq. (2.1—5).
A^{-1}	Inverse of A.
I	Unit (three-row) matrix.
$h=[h_k]$	Column matrix with elements h_k, see Eq. (2.1—3).
$\tau^{(i)}$	Row matrix, eigenvector with ordinal number i of the matrix pair A, B, see Eq. (2.1—8).
$\varrho^{(i)}$	Eigenvalue with ordinal number i of the matrix pair A, B.
P, Q	Variables defined by Eq. (2.1—14).
E	Domain of the complex x, s, t space.
ξ, η	Coordinates in the s, t plane.
D	Jacobian of ξ, η with respect to s, t.
$\{A\}$	Weak discontinuity in the string property A, see Eq. (2.2—3).

$l^{(i)}$ Path of propagation in the s, t plane of a weak discontinuity; i is the ordinal number.

$w^{(i)}$ Propagation velocity in relation to the string points of a weak discontinuity propagating along the path $l^{(i)}$.

$\tau^i = \dfrac{\partial y^i}{\partial s}$ Unit vector along the string tangent.

U Elastic potential of the unit mass of the string.

c Propagation velocity of a strong discontinuity in relation to the string points.

C The quantity of string mass passed by a strong discontinuity per unit of time.

Symbols not included in the above list will occur now and then in the text. Indices run from one to three, if not otherwise specified. The summation convention is used. Partial derivatives are generally written by the use of indices, *e. g.* f_s means $\dfrac{\partial f}{\partial s}$.

2.1. Classification of the String Equations

2.11. Introduction

A definite problem concerning the motion of a string naturally cannot be regarded as properly formulated unless the string equations have a unique solution. This proposition immediately leads to an inquiry into *existence* and *uniqueness theorems* for *initial* and *boundary value problems* connected with the string equations. Particular interest will be centred on initial value problems of the infinite string and mixed initial — boundary value problems of the semi-infinite and finite string. General problems of these kinds are complicated, especially because of the presence of strong discontinuities,[5] and we shall consider it to be outside the scope of this work to give a thorough discussion of such problems. In order to get some information concerning these questions, however, we shall classify the string equations in the usual manner according to the theory of partial differential systems. We shall also briefly discuss the propagation of discontinuities along the string.

[5] Such cases are discussed by Hamel, [H4], and Pailloux, [P1], [P3].

Anticipating the results, we assert that the string equations form a *quasi-linear*, differential system which in the case of a *linearly elastic string* (and all other kinds of elastic strings, too) is *hyperbolic* in the sense used by COURANT and others, *cf.* [C2], and [S1], p. 127. For the *inextensible string* the system is *not hyperbolic*. Further, the string curve is a *characteristic curve* for both kinds of string.[6] The latter fact means that a problem with initial values prescribed along the string (which may be finite or infinite) generally will not be a proper problem.

Special kinds of mixed initial — boundary value problems have been investigated in connection with quasi-linear hyperbolic systems, *cf.* [B3], [C2], [C6]. Some of these problems may be of interest with reference to the elastic string. The results obtained will not be valid for the inextensible string.

To sum up, most dynamical problems related to a string, extensible or inextensible, finite or infinite, are not easily attacked by rigorous methods.

2.12. A First Order System of String Equations

We now turn to the study of the string in a rectangular, Cartesian coordinate system y^i, and use the arc length s of the string curve and the time t as independent variables. From the string equations (1.2—21) we easily obtain the following, first order differential system[7]

$$Ax_s + Bx_t + h = 0. \qquad (2.1-1)$$

In this equation we have adopted the index notation for the partial derivatives of the dependent variable x, *e. g.* $x_s = \dfrac{\partial x}{\partial s}$,&c. Further, x and h are column vectors. The coefficients A and B are square matrices. Writing x' and h' for the transposes of x and h we have (with y^i as in Chapter 1)

$$\left.\begin{aligned} & x' \equiv x'(u, t) \equiv (x_1, \ldots, x_9, T, u), \text{ where} \\ & x_i = y^i,\ i = 1,2,3, \\ & x_{i+3} = y^i_s, \\ & x_{i+6} = y^i_t; \end{aligned}\right\} \qquad (2.1-2)$$

---

[6] Anticipating later treatment we may mention here that strong discontinuities as well as weak ones propagate along characteristics, *cf.* section 2.22.

[7] The symbols x, x_i in this chapter should not be confused with the curvilinear coordinates x^i appearing in Chapter 1.

$$\left.\begin{aligned} &h' \equiv (h_1, \ldots, h_{11}), \text{ where} \\ &h_k = -x_{k+3},\ k=1,2,3, \\ &h_k = F^{k-3},\ k=4,5,6, \\ &h_k \equiv 0,\ k=7, \ldots, 11. \end{aligned}\right\} \tag{2.1—3}$$

The coefficient matrices A and B are expressed (in partitioned form) by

$$A = \left[\begin{array}{c|c|c|c} I & & & \\ \hline & (T-mu^2)I & -2muI & \begin{matrix} x_4 & -mux_4 \\ x_5 & -mux_5 \\ x_6 & -mux_6 \end{matrix} \\ \hline & & -I & \\ \hline & \begin{matrix} ux_4 & ux_5 & ux_6 \\ 0 & 0 & 0 \end{matrix} & & \begin{matrix} 0 & 0 \\ -ku & 1 \end{matrix} \end{array}\right], \tag{2.1—4}$$

$$B = \left[\begin{array}{c|c|c|c} & & & \\ \hline & & -mI & \begin{matrix} 0 & -mx_4 \\ 0 & -mx_5 \\ 0 & -mx_6 \end{matrix} \\ \hline & I & & \\ \hline & \begin{matrix} x_4 & x_5 & x_6 \\ 0 & 0 & 0 \end{matrix} & & \begin{matrix} 0 & 0 \\ -k & 0 \end{matrix} \end{array}\right]. \tag{2.1—5}$$

In these expressions I stands for the unit three-row matrix and every excluded element is identically zero, *i. e.* zero for every x, s, t. F^i, $i=1,2,3$, appearing in Eq. (2.1—3) are the rectangular, Cartesian components of the external force. In equation (2.1—1) we have preferred to take the last of the equations (1.2—12) as a kinematical condition instead of Eq. (1.2—11) which was used in Eq. (1.2—21). In order to make the system Eq. (2.1—1) equivalent to Eq. (1.2—21) the condition

$$x_i\, x_i = 1,\ i=4,5,6 \tag{2.1—6}$$

must hold along some non-characteristic curve, *cf.* [K1], p. 11.

Since the sum Eq. (2.1—6) appears frequently in subsequent deductions, we shall, for the sake of brevity, imply $x_i\, x_i = 1$ throughout. In Eq. (2.1—1) m and T are connected by the elastic condition

$$m = m_0 \exp(-kT),$$

see section 1.25. This equation is also implied throughout.

The equation (2.1—1) forms a quasi-linear system of partial differential equations in two independent variables s, t and eleven dependent variables $x \equiv (x_1, \ldots, x_9, T, u)$.[8] The coefficient matrices A and B are functions of the dependent variable x, but not of s and t. Further, A and B are analytic everywhere in the (complex, eleven-dimensional) x space. This is evident, since the elements of A and B are polynomials in m and the elements of the vector x and, by the elastic condition above, m is analytic in T for every T (which is an element of x). In the most general case the vector $h = h(s, t, x)$, *i. e.* $F^i = F^i(s, t, x)$, but as we do not discuss the inhomogeneous string h is unlikely to depend on s explicitely. If F^i emanates from a scleronomic force field h does not depend on t explicitely, and in the common, gravitational field F^i depends on the string tension T only. If the external force $F^i(s, t, x)$ in the general case is analytic in its variables in a domain of the (thirteen-dimensional, complex) s, t, x space, then, by definition, the differential system Eq. (2.1—1) is analytic in that domain.

If the coefficient matrix A is non-singular, *i. e.* $\det A \neq 0$, the inverse A^{-1} of A exists, and it is then possible to solve Eq. (2.1—1) with respect to x_s. $\det A \neq 0$ holds in those points of the x space where

$$\begin{aligned} &u \neq 0, \\ &T - mu^2 \neq 0, \\ &1 - kmu^2 \neq 0, \\ &x_4,\, x_5,\, x_6 \text{ not all} = 0 \end{aligned} \qquad (2.1-7)$$

hold true simultaneously. The last condition is satisfied by the implication of Eq. (2.1—6). In several cases, however, the first two conditions are not satisfied everywhere on the string, *cf.* the discussion at the end of section 2.14.

2.13. Characteristics of the String Equations Hyperbolic Systems

Proceeding as in the works [S1] or [C2], we seek scalars ϱ and vectors τ and p which satisfy the equation

$$\tau'(Ax_s + Bx_t + h) = p'(x_s + \varrho x_t) + \tau' h = 0 \qquad (2.1-8)$$

---

[8] Concerning the term quasi-linear see for instance [S1], p. 36.

The vectors τ are called *eigenvectors* and the scalars ϱ *eigenvalues* of the matrix pair A, B. This procedure means that we seek linear combinations of the equations in the system (2.1—1) which will contain the derivatives of each of the components of x in one and the same direction in the s, t plane. The latter direction is called a characteristic direction; it is determined by the ordinary differential equation

$$\frac{dt}{ds}=\varrho. \tag{2.1—9}$$

The integral curves of this differential equation are called *characteristics*. The differential system Eq. (2.1—1) is a special one, in so far as A, B and h depend explicitely on x but not on the independent variables s and t. The same is consequently true of ϱ, τ and p also. It is, however, of no particular importance as concerns the theoretical treatment of the system (2.1—1). From Eq. (2.1—8) we get

$$\tau'(B-\varrho A)=0, \tag{2.1—10}$$

$$\det(B-\varrho A)=0. \tag{2.1—11}$$

The latter equations determine ϱ and τ, and if A and B are of the order n, with A nonsingular, there are n roots $\varrho^{(k)}$ of the equation (2.1—11). Further, $\varrho^{(k)}$, $k=1, \ldots, n$, is an eigenvalue of the matrix $A^{-1}B$, provided A is nonsingular. If A and B are both real, $\varrho^{(k)}$ may be real or complex; distinct or multiple.

The differential system Eq. (2.1—1) is called hyperbolic in a point (x_0, s_0, t_0) of the $n+2$-dimensional x, s, t space, if every eigenvalue $\varrho^{(k)}$ is real and the eigenvectors $\tau^{(i)}$ span the n-dimensional space at that point.[9] If a system is hyperbolic in every point of a domain E in the $n+2$-dimensional space it is called hyperbolic in E. The fact that a system is hyperbolic implies the existence of matrices K and K^{-1} such that $K^{-1}A^{-1}BK$ is a diagonal matrix. If A, B, K and the vector h are analytic in E, the system is called analytic hyperbolic in E, see [L1], p. 235.

It is well known that weak discontinuities in the functions $x_i(s, t)$ propagate along the characteristics, *cf.* section 2.21.

The equations (2.1—8) are frequently called the characteristic system (corresponding to Eq. (2.1—1)). Finally, we may point out

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[9] The eigenvectors span the n-dimensional space, if, and only if, the rank of B-ϱA is n-q_k for a q_k-fold eigenvalue $\varrho^{(k)}$, and the same holds true for every eigenvalue.

that the significance of the characteristics and the characteristic system is not affected by transformations of the independent variables, *cf.* [C3], p. 143.

2.14. The Elastic String

In the case of a linearly elastic string we have the elastic constant $k>0$. We easily obtain the following expressions:

$$\det A = u P^2 Q, \tag{2.1-12}$$

$$\left.\begin{aligned} &\varrho^{(1)}, \ldots, \varrho^{(4)} \equiv 0, \\ &\varrho^{(5)} = \frac{1}{u}, \\ &\varrho^{(6)},\ \varrho^{(7)} = -\frac{1}{Q}\left[mku \pm (mk)^{1/2}\right], \\ &\varrho^{(8)},\ \varrho^{(9)} = -\frac{1}{P}\left[mu + (mT)^{1/2}\right], \\ &\varrho^{(10)}, \varrho^{(11)} = -\frac{1}{P}\left[mu - (mT)^{1/2}\right], \end{aligned}\right\} \tag{2.1-13}$$

where Eq. (2.1—6) is implied to hold and

$$\begin{aligned} P &= T - mu^2, \\ Q &= 1 - mku^2. \end{aligned} \tag{2.1-14}$$

We may remark that

1) If $T>0$, *i. e.* if the string tension is positive, every eigenvalue $\varrho^{(i)}$, $i=1, \ldots, 11$, is real, and by an investigation of the rank of $B-\varrho^{(i)} A$ we can conclude that the eigenvectors $\tau^{(i)}$ span the eleven-dimensional space.

2) We let the domain E of the s, t, x space consist of all points satisfying[10]

$$\begin{aligned} &T>0, \\ &u, P, Q \neq 0, \\ &x_k x_k = 1,\ k=4, 5, 6. \end{aligned} \tag{2.1-15}$$

[10] The last condition in Eq. (2.1—15) may be replaced by a less restrictive one, for instance, x_4, x_5, x_6 not all zero, *cf.* remarks in connection with Eq. (2.1—6).

The differential system Eq. (2.1—1), or Eq. (2.1—8), is then analytic hyperbolic in E. It must be pointed out that while $T<0$ is excluded by hypothesis, *vide* section 1.11, T, u, $P=0$ may occur in string points that are interesting from the physical aspect. At a free end of a string $T=0$, in a string point fixed in space $u=0$, and motions where $P=0$ are discussed in section 1.4. All non-hyperbolic points of the s, t, x space are therefore not without interest.

3) The curves $x_i = x_i(s, t)$, $\dfrac{dt}{ds} = \varrho^{(k)}$, $i = 1, 2, 3$; $k = 1, \ldots, 11$, *i. e.* the curves on the surface S (generated by the string in its motion through space) corresponding to the characteristics, have simple physical interpretations. The eigenvalue $\varrho = 0$ corresponds to the string curve, $\varrho^{(5)}$ to the paths of the string points and the others to the paths of propagation of certain discontinuities which will be discussed in section 2.2. The fact that the string curve itself corresponds to a characteristic complicates the treatment of an initial value problem (in the strict sense of the term), as was pointed out in the introduction to section 2.1.

2.15. **The Inextensible String**

The inextensible string is characterized by $k=0$, implying $m = m_0 = \text{const}$. We then get the following expressions

$$\det A = uP^2, \tag{2.1—16}$$

$$\left.\begin{aligned}
&\varrho^{(1)}, \varrho^{(2)}, \varrho^{(3)}, \varrho^{(4)}, \varrho^{(6)}, \varrho^{(7)} \equiv 0, \\
&\varrho^{(5)} = \frac{1}{u}, \\
&\varrho^{(8)}, \varrho^{(9)} = -\frac{1}{P}\left[m_0 u + (m_0 T)^{1/2}\right], \\
&\varrho^{(10)}, \varrho^{(11)} = -\frac{1}{P}\left[m_0 u - (m_0 T)^{1/2}\right].
\end{aligned}\right\} \tag{2.1—17}$$

By investigating the rank of B we find that only five linearly independent eigenvectors correspond to the sixfold eigenvalue $\varrho \equiv 0$. The eigenvectors therefore do not span the eleven-dimensional space, and the system Eq. (2.1—1), or Eq. (2.1—8), is consequently not hyperbolic in any point of the s, t, x space in the case of an inextensible string.

2.2. The Propagation of Discontinuities Along the String

It is well known from the theory of the inextensible string that discontinuities of the string properties may propagate along a string, see for instance [B1], p. 412, [H4] and [P3].

Since there does not appear to exist any general survey of the subject, the author considers it suitable to discuss the matter here.

We shall consider the string motion with rectangular Cartesians as space coordinates and the s, t coordinates of Chapter 1 as independent variables. We shall use the terms *weak* and *strong discontinuities*.

1) Weak discontinuities are discontinuities in $y^i_{ss}, y^i_{st}, y^i_{tt}$, $(i=1,2,3)$, $T_s, T_t, m_s, m_t, u_s, u_t$, or discontinuities of derivatives of higher order.

2) Strong discontinuities are discontinuities in y^i_s, v^i, $(i=1,2,3)$, T, m, u.

It should be pointed out that weak discontinuities may be analyzed by using the string equations in their characteristical form as a starting point, while the presence of strong discontinuities necessitates a particular treatment.

2.21. Weak Discontinuities

In essential the discussion in this section follows that in [S1], p. 135 *et seq*. We start from the string equations (1.2—21) and assume the external force F^i to be continuous in s, t and y^i. This assumption means that we disregard discontinuities caused by an external force.[11] If the latter is a field force, for instance the weight of the string, our assumption is justified; but it is not valid in several cases of physical interest.

We now introduce a new set of coordinates ξ, η in the s, t plane by the one to one transformation

$$\left.\begin{aligned} \xi&=\xi(s, t), \\ \eta&=\eta(s, t), \end{aligned}\right\} \qquad (2.2-1)$$

$$D=\frac{\partial(\xi, \eta)}{\partial(s, t)} \neq 0. \qquad (2.2-2)$$

[11] To make the investigation valid for discontinuities in derivatives of higher order than those appearing in Eq. (1.2—21) we must, of course, make further assumptions regarding the differentiability of F^i.

Assuming ξ and η to be sufficiently smooth we can use them as independent variables in the string equations (1.2—21). The curve $\xi(s, t)=0$ divides the s, t plane into two domains, (1) and (2). An arbitrary property $A=A(s, t)=A(s(\xi, \eta), t(\xi, \eta))$ (scalar or vector) of the string is said to possess a discontinuity $\{A\}$ along $\xi(s, t)=0$ if

$$\{A\}=\lim_{P_1, P_2 \to P} [A(P_1)-A(P_2)] \neq 0, \qquad (2.2-3)$$

where $A(Q)$ is the value of $A(s, t)$ at a point Q; P is a point on $\xi=0$ and P_1, P_2 are points in the domains (1) and (2) approaching P. The property A then shows a jump $\{A\}$ when crossing the curve $\xi=0$. Let the following conditions hold true:

1) The string properties satisfy Eq. (1.2—21) in the open domains (1) and (2).

2) y^i_s, y^i_t, T, m, u are continuous when crossing the curve $\xi=0$.

3) The intrinsic derivatives[12] $(y^i_s)_\eta$, $(y^i_t)_\eta$, T_η, m_η, u_η are continuous on $\xi=0$.

4) The external derivatives $(y^i_s)_\xi$, $(y^i_t)_\xi$, T_ξ, m_ξ, u_ξ may be discontinuous on $\xi=0$.

For the intrinsic derivative A_η of the string property A we obtain (by equations (2.2—1), (2.2—2))

$$A_\eta=A_s\, s_\eta+A_t\, t_\eta=-\frac{1}{D}(A_s\, \xi_t-A_t\, \xi_s).$$

Taking A to be any one of the properties listed in condition (2), we obtain by condition (3)

$$\{A_s\}+\varrho\{A_t\}=0, \qquad (2.2-4)$$

where

$$\varrho=\frac{dt}{ds}=-\frac{\xi_s}{\xi_t} \qquad (2.2-5)$$

is the derivative of t with respect to s along the curve $\xi(s, t)=0$. From the string equations (1.2—21) and the kinematical conditions Eq. (1.2—12) we obtain the following system for the weak discontinuities of the string properties,

---

[12] *i. e.* the derivatives along the curve $\xi=0$.

$$\{T_s - muu_s - mu_t\}\, y_s^i + (T - mu^2)\, \{y_{ss}^i\} - m\,\{y_{tt}^i\} - 2mu\,\{y_{st}^i\} = 0,$$

$$y_s^i\{y_{ss}^i\} = y_s^i\{y_{st}^i\} = 0, \tag{2.2-6}$$

$$\{u_s\} = k\{T_t + uT_s\},$$

$$\{m_s\} = -km\{T_s\},\ \{m_t\} = -km\{T_t\},$$

where we have summation on $i = 1,2,3.$[13] The Eqs. (2.2−6) together with Eq. (2.2−4) and Eq. (2.2−5) give the essential information about the weak discontinuities and their paths of propagation. There are some differences between the elastic string, $k > 0$, and the inextensible string, $k = 0$. It is easy to verify that the possible paths of propagation of discontinuities coincide with the characteristics of the string equations discussed in sections 2.13, 2.14 and 2.15. No discontinuity can propagate, however, along the characteristics determined by $\varrho = \dfrac{1}{u}$.[14]

We shall now discuss the elastic and the inextensible string separately.

a) For the *elastic string* we have $k > 0$. Let the path of propagation (in the s, t plane), corresponding to the directional derivative $\dfrac{dt}{ds} = \varrho^{(p)}$, be symbolized by $l^{(p)}$, $p = 1, \ldots, 5$.

We conclude from Eq. (2.2−6) that along $l^{(1)}$

$$\varrho^{(1)} = 0,$$

$$\{y_{tt}^i\} = -\{u_t\}y_s^i \neq 0,\ i = 1,2,3, \tag{2.2-7}$$

while all other properties remain continuous on $l^{(1)}$. For the vector y_{tt}^i a discontinuity parallel to the tangent vector y_s^i (of the string) consequently propagates instantaneously along the string. It should be observed that the acceleration vector remains continuous, since the discontinuities $\{y_{tt}^i\}$ and $\{u_t\}$ cancel each other.

[13] The application of this method to problems containing discontinuities of vectors in common space requires no particular investigation.

[14] It may be pointed out that this discrepancy is due to the fact that on discussing the characteristics we only make use of the last of the three conditions in Eq. (1.2−12), while we here apply the first two.

We next obtain, from Eq. (2.2—6), the following relations, valid along $l^{(2)}$ and $l^{(3)}$.[15]

$$\varrho^{(2)},\ \varrho^{(3)} = -\frac{1}{P}\left[mu \pm (mT)^{1/2}\right],$$

$$\{y^i_{ss}\} = -\varrho^{(p)}\ \{y^i_{st}\} = (\varrho^{(p)})^2 \{y^i_{tt}\} \neq 0, \qquad (2.2-8)$$

$$p=2,\ 3;\ i=1,2,3,$$

where $P=T-mu^2$. The other properties of the string remain continuous along $l^{(2)}$ and $l^{(3)}$. The physical interpretation of the above discontinuities is a jump in the acceleration vector perpendicular to the tangent vector y^i_s of the string curve.

Along $l^{(4)}$ and $l^{(5)}$, finally,

$$\varrho^{(4)},\ \varrho^{(5)} = -\frac{1}{Q}\left[mku \pm (mk)^{1/2}\right],$$

$$\{u_s\} = \pm \{T_s\}\left(\frac{k}{m}\right)^{1/2}, \qquad (2.2-9)$$

$$\{T_t\} = (\varrho^{(p)})^{-1}\{T_s\},\quad p=4,\ 5,$$

$$\{m_s\} = -km\{T_s\},\ \{m_t\} = -km\{T_t\},$$

while the second order derivatives of y^i remain continuous. Physically, these discontinuities are interpreted by a jump in the acceleration vector parallel to y^i_s, a jump in the derivatives of the string tension T and corresponding jumps in the derivatives of the mass m of the unit length of the string.

Concerning the determination of the path $l^{(i)}$, $i=1, \ldots, 5$, we may remark, that owing to the nonlinear character of the string equations the differential equations (2.2—5) of the $l^{(i)}$'s contain the dependent variables of the string equations. The paths $l^{(i)}$ consequently cannot be determined until the string equations are solved. For the propagation velocities $w^{(p)}$ in relation to the string points we obtain

$$w^{(p)} = (\varrho^{(p)})^{-1} - u, \qquad (2.2-10)$$

since we know from section 1.22 that u (in connection with the s, t coordinates) means the velocity of the string point along the string. Therefore

---

[15] The notation here is different from that used in sections 2.14 and 2.15.

$$w^{(1)} = \infty,$$

$$w^{(2)}, w^{(3)} = \mp \left(\frac{T}{m} \right)^{1/2}, \tag{2.2—11}$$

$$w^{(4)}, w^{(5)} = \mp (mk)^{-1/2}.$$

As mentioned previously the first expression means that the corresponding discontinuity propagates instantaneously. The expressions for $w^{(2)}$ and $w^{(4)}$, ($w^{(3)}$ and $w^{(5)}$) imply that the corresponding discontinuities, see Eq. (2.2—8), propagate with the same relative velocity in either direction. Comparison with the results relating to small strain in the theory of elasticity indicates that $w^{(2)}$, $w^{(3)}$ correspond to the propagation velocity of transversal waves along a tight string, while $w^{(4)}$, $w^{(5)}$ correspond to the propagation velocity of longitudinal waves along a thin rod, *cf.* [R1], pag. 271 and 245.[16] These facts become clear, if we let the string possess the cross-section area A, YOUNG's modulus of the material E, the stress σ, and the density q. We then get

$$\left.\begin{aligned} \sigma &= \frac{T}{A} = EkT, \\ m &= Aq, \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} w^{(2)}, w^{(3)} &= \mp \left(\frac{\sigma}{q} \right)^{1/2}, \\ w^{(4)}, w^{(5)} &= \mp \left(\frac{E}{q} \right)^{1/2}. \end{aligned}\right\} \tag{2.2—12}$$

This agreement is hardly surprising, remembering the physical interpretation of the discontinuities.

Finally, we may remark that the equations (2.2—6) only give information about the relative magnitudes of the discontinuities. It is well known that it is possible to deduce conditions determining the jumps $\{A\}$ of the string properties A from the string equations in their characteristic form, Eq. (2.1—8). These conditions appear as ordinary differential equations for the $\{A\}$'s along the characteristics (*i. e.* the paths of propagation of the discontinuities). In the absence of solutions to the string equations the above equations will yield

[16] It may be remarked that our results are valid for finite strain, provided the material of the string is linearly elastic in the sense stated in section 1.25.

only general, well-known results, and we therefore confine ourselves to pointing out their most important consequences. The latter may be stated as follows. There are only two possibilities for the jump $\{A\}$, *viz.*

$$\begin{aligned} &\{A\}\equiv 0 \text{ on the characteristic, or} \\ &\{A\}\neq 0. \end{aligned} \qquad (2.2-13)$$

This means that a weak discontinuity cannot (in the absence of discontinuous external forces) be formed in or vanish from an interior point of a string. It must be forced on the end points or be present in the initial state.

b) For the *inextensible string*, $k=0$, we obtain from Eq. (2.2—6)

$$\begin{aligned} &\varrho^{(1)}=0, \\ &\{y^i_{tt}\}=-\{u_t\}y^i_s\neq 0,\ i=1,2,3, \\ &\{T_t\}\neq 0. \end{aligned} \qquad (2.2-14)$$

In addition to the discontinuities occurring on an elastic string, a jump in the time derivate of the string tension may then propagate instantaneously along the string. The expressions of Eq. (2.2—8) hold (with the obvious change from m to m_0=const.) for the paths of propagation $l^{(2)}$ and $l^{(3)}$. The propagation velocities given in Eq. (2.2—11) and the comments on the elastic string also hold with appropriate modifications (the paths $l^{(4)}$ and $l^{(5)}$ may be regarded as contained in $l^{(1)}$).

2.22. **Strong Discontinuities**

We now consider the case when our linearly elastic string has a discontinuity in one or more of the properties y^i_s, v^i, T, m, u, *i. e.* the string tangent, the velocity of the string point, the tension, the mass of the unit length, and the velocity component along the string tangent. We shall assume that the discontinuities are not caused by external point forces. It is well known that only one kind of strong discontinuity can occur on the inextensible string, see for instance [B1], [P1]. We shall find that this is the case also of the linearly elastic string, and probably of all other kinds of elastic strings, too.

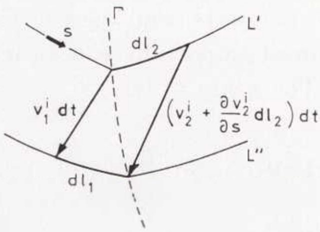

Fig. IV. Schematic sketch of the propagation of a (strong) discontinuity in the tangent vector $\frac{\partial y^i}{\partial s}$. The path of the point of discontinuity through space is designated Γ.

We adopt the following notations (see Fig. IV):

L' — The string curve at the time t.

L'' — — — — — — — — — — — — — $t+dt$.

Γ — The path of the discontinuity point in space.

s — The arc length of the string curve.

$s_\Gamma = s_\Gamma(t)$ — The value of s at the point of discontinuity (at the time t). Indices 1 and 2 denote string properties immediately to the left, $s=s_\Gamma-0$, of the point s_Γ and to the right, $s=s_\Gamma+0$, respectively.

$dl_1(dl_2)$ — The length of an element of the string (passed by the discontinuity during the time dt) in the left (right) position; dM is the mass of the element.

$\tau^i = y_s^i$, $i=1,2,3$ — The unit tangent vector of the string curve.

$U=U(T)$ — The elastic potential of the unit mass of the string.[17]

$C=\frac{dM}{dt}$ — The quantity of string mass passed by the discontinuity per unit of time.

$c=\frac{dl}{dt}$ — The propagation velocity of the discontinuity in relation to the string.

The figure is drawn with reference to a discontinuity propagating in the direction of positive s.

[17] The existence $U(T)$ is an immediate consequence of our assumption that the length of an element depends explicitely on the tension only; *cf.* section 1.25.

Neglecting low order terms, we obtain

$$
\begin{aligned}
& m_1 \exp(kT_1) = m_2 \exp(kT_2) = m_0, \\
& m_1\, dl_1 = m_2\, dl_2 = dM, \\
& \tau_1^i\, dl_1 + v_1^i\, dt = \tau_2^i\, dl_2 + v_2^i\, dt, \qquad (2.2-15) \\
& (v_1^i - v_2^i)\, dM = (T_2 \tau_2^i - T_1 \tau_1^i)\, dt, \\
& (U_1 - U_2 + \tfrac{1}{2} v_1^i v_1^i - \tfrac{1}{2} v_2^i v_2^i)\, dM = (T_2 \tau_2^i v_2^i - T_1 \tau_1^i v_1^i)\, dt,
\end{aligned}
$$

where we have summation on $i=1,2,3$. In this system the first equation is the elastic condition, the second implies conservation of mass, the third is a kinematical condition obtained from Figure IV, the fourth implies conservation of momentum, and the fifth is an energy equation. It may be remarked that a discontinuous external force will not affect the discontinuity, provided it is not a point force. After some further calculations we get from Eq. (2.2—15)

$$
\begin{aligned}
& m_1 c_1 = m_2 c_2 = C, \\
& [T_1 - m_1 (c_1)^2]\, \tau_1^i = [T_2 - m_2 (c_2)^2]\, \tau_2^i, \qquad (2.2-16) \\
& 2\, (C)^2\, (U_2 - U_1) = (T_2)^2 - (T_1)^2.
\end{aligned}
$$

As the string is linearly elastic we get the following expression for U

$$
\begin{aligned}
U &= \frac{k}{m_0} \int T \exp(kT)\, dT \\
&= U_0 + (m_0 k)^{-1} \exp(kT)\, (kT - 1), \qquad (2.2-17)
\end{aligned}
$$

where U_0 is a constant.

We conclude from Eq. (2.2—16) and Eq. (2.2—17) that only one kind of discontinuity can exist. We get

$$
\begin{aligned}
\tau_1^i \neq \tau_2^i \;\Rightarrow\; & T_1 = T_2 = T, \\
& m_1 = m_2 = m, \qquad (2.2-18) \\
& c_1 = c_2 = c = \pm \left(\frac{T}{m}\right)^{1/2}.
\end{aligned}
$$

Only the tangent vector is consequently discontinuous.

This kind of discontinuity is possible, of course, also in the case of an inextensible string. On comparing the above expression with Eq. (2.2—11) we can observe that the strong discontinuity moves along the string at the same relative velocity as a weak discontinuity of that component of acceleration which is perpendicular to the string tangent. The strong discontinuity therefore propagates along a characteristic of the string equations, as HAMEL, [H4], points out (in a case of plane motion).

3. The String Equations in Cylindrical Coordinates

Symbols and conventions:

x^1	Cylindrical coordinate, radial distance.
x^2	— — — — — — — — — — — —, polar angle.
x^3	— — — — — — — — — — — —, axial distance.
s	Arc length of the string curve.
z	Independent variable, defined by Eq. (3.1—6).
t	The time.
T	String tension.
$\overline{T}$	Reduced string tension, defined by Eq. (3.1—10).
q, u	String properties, defined as in Chapter 1 and physically interpreted on pag. 50.
m	Mass of the unit length of the string.
K^i	Contravariant, cylindrical components of the external force acting on the unit length of the string.
a	Characteristical length.
ω	Angular velocity of the string at steady rotatory motion. $\dfrac{1}{\omega}$ is used as a characteristical time.
T_0	Characteristical force.
τ	Independent variable, dimensionless time, defined by Eq. (3.1—6).
ξ	Dependent variable, dimensionless radial distance, defined by Eq. (3.1—8).
$\varphi = x^2$	Dependent variable, see above.
σ	Dependent variable, dimensionless string tension, defined by Eq. (3.1—8).
$\bar{\sigma}$	Dependent variable, dimensionless reduced string tension, defined by Eq. (3.1—10).

γ Dependent dimensionless variable, defined by Eq. (3.1—8) and physically interpreted on pag. 50.

ϱ Dependent variable, dimensionless string mass per unit length.

μ Dependent variable, dimensionless axial velocity of the string point, see Eq. (3.1—8).

p_1, α Dimensionless string parameters, defined by Eq. (3.1—9).

y_i, $i=1, \ldots, 6$ Dimensionless, dependent variables for a small, time-dependent motion superimposed on a steady rotatory motion of a string, see Eq. (3.2—2).

h Dimensionless constant, see Eq. (3.2—5).

In this chapter derivatives are generally written by means of indices, *e.g.* $y_z = \dfrac{\partial y}{\partial z}$.

3.1. Dimensionless Variables

In this section we shall establish a complete system of equations for the linearly elastic string which hold in a cylindrical coordinate system fixed in space. In subsequent applications it will be convenient to use dimensionless variables, and we shall introduce such variables from the first. As we shall not deal with plane motion, the time and the axial distance of the point (on the string curve) will serve as independent variables in the string equations. Let us start from the system (1.2—24) and let x^1 be the distance from the axis, x^2 the polar angle, and x^3 the axial coordinate of a point on the string curve. For the metric tensor we have

$$g_{11} \equiv 1, \quad g_{22} = (x^1)^2, \quad g_{33} \equiv 1, \qquad g_{ik} \equiv 0, \quad i \neq k. \tag{3.1—1}$$

The Christoffel symbols are determined by

$$\left\{ {2 \atop 1\,2} \right\} = \left\{ {2 \atop 2\,1} \right\} = \frac{1}{x^1}, \quad \left\{ {1 \atop 2\,2} \right\} = -x^1; \qquad \textit{all other Christoffel symbols} \equiv 0. \tag{3.1—2}$$

After some calculations we get from system (1.2—24)

$$\left(\frac{T}{q}\,x^1_z\right)_z - x^1\,\frac{T}{q}\,x^2_z - mq\left[\frac{d^2x^1}{dt^2} - x^1\left(\frac{dx^2}{dt}\right)^2\right] + qK^1 = 0,$$

$$\left(\frac{T}{q}\,x^2_z\right)_z + \frac{2}{x^1}\,\frac{T}{q}\;x^1_z\,x^2_z - mq\left[\frac{d^2x^2}{dt^2} + \frac{2}{x^1}\,\frac{dx^1}{dt}\,\frac{dx^2}{dt}\right] + qK^2 = 0,$$

$$\left(\frac{T}{q}\,x^3_z\right)_z - mq\,\frac{d^2x^3}{dt^2} + qK^3 = 0, \tag{3.1—3}$$

$$(x^1_z)^2 + (x^1x^2_z)^2 + (x^3_z)^2 = (q)^2,$$

$$(mq)_t + (mqu)_z = 0,$$

$$m = m_0\,\exp\,(-kT),$$

where

$$x^1_z = \frac{\partial x^1}{\partial z},\quad x^1_t = \frac{\partial x^1}{\partial t},$$

$$\frac{dx^1}{dt} = x^1_t + ux^1_z,\ \text{etc.} \tag{3.1—4}$$

K^i, $i=1,2,3$, are the contravariant, cylindrical components of the external force (exerted on the unit length of the string), determined from the Cartesian components by Eq. (1.2—25). In order to facilitate the transition to dimensionless dependent and independent variables we now introduce *characteristical quantities of length, time, and force:*

$$a = \textit{characteristical length},$$

$$\frac{1}{\omega} = \textit{characteristical time}, \tag{3.1—5}$$

$$T_0 = \textit{characteristical force}.$$

On studying particular problems we shall later identify these parameters with well-defined physical quantities. Let now z and τ be *dimensionless independent variables* defined by

$$x^3 = az,$$

$$t = \frac{\tau}{\omega}. \tag{3.1—6}$$

By this definition we have

$$x^3_z \equiv a, \quad x^3_{zz} = x^3_t \equiv 0,$$
$$\frac{\partial}{\partial t} = \omega \frac{\partial}{\partial \tau}. \tag{3.1—7}$$

From the discussion at the end of section 1.22 we can conclude (putting $y^3 = x^3 = az$) that $\frac{a}{q}$ is the direction cosine of the tangent vector of the string with respect to the x^3 axis. Further, au is the component of velocity in the direction of the x^3 axis. The string variables $q(z, \tau)$ and $u(z, \tau)$ are then given physical interpretations. The transformation Eq. (1.2—2), by which the independent variable z was introduced, has a sense only if $\frac{1}{q} \neq 0$, *i. e.* if the string curve does not touch any plane $x^3 = \text{const}$. Assuming this condition to hold we choose $q(z, \tau) \geq a > 0$ (it should be noted that the equations (3.1—3) are invariant under a change from q to $-q$). The velocity component au may, of course, be positive, zero or negative.

In order to complete the transition to dimensionless variables we introduce the variables ξ, φ, σ, γ, μ, ϱ, functions of z, τ, by means of the following expressions:[18]

$$x^1 = a\xi, \quad x^2 = \varphi, \quad \frac{T}{q} = \frac{T_0}{a}\sigma,$$
$$q = a\gamma, \quad u = \omega\mu, \quad m = m_0\varrho. \tag{3.1—8}$$

The variables defined in this manner have *simple physical interpretations.* ξ is the dimensionless form of the distance of the point (ξ, φ, z) (on the string curve) from the z axis, φ is the polar angle measured from a line fixed in space. Concerning σ we observe that $\frac{a}{q}T$ is the axial component of the string tension vector and that σ therefore is the dimensionless form of this force component. $\frac{1}{\gamma}$ is the axial component of the unit tangent vector of the string curve, hence $\gamma \geq 1$. μ is the dimensionless axial component of the velocity vector. ϱ is the dimensionless form of the mass of the unit length of the string, restricted by $1 \geq \varrho > 0$.

[18] The variables σ, τ which occur in this chapter should not be confused with the σ,τ in Chapter 1.

We shall find it convenient to use two dimensionless parameters p_1 and α, defined by

$$p_1 = \frac{m_0 a^2 \omega^2}{2T_0}, \qquad \alpha = kT_0. \tag{3.1—9}$$

The constants p_1 and α, $p_1 > 0$, $\alpha \geq 0$, will be called *string parameters.* It can easily be verified that α is the logarithmic strain of a string element subjected to the tension T_0. The inextensible string is characterized by $\alpha = 0$.

New variables may also suitably be introduced instead of T and σ. Let $\overline{T}$ and $\bar{\sigma}$ be defined by

$$\overline{T} = T - mu^2q^2, \qquad \bar{\sigma} = \overline{T}\frac{a}{qT_0} = \sigma - 2p_1\,\varrho\gamma\mu^2. \tag{3.1—10}$$

$\overline{T}$ will be called the *reduced string tension* and $\bar{\sigma}$ the *dimensionless reduced string tension,* although the latter is not the dimensionless form of $\overline{T}$.

After straight-forward calculations the system (3.1—3) takes the form[19]

$$\begin{aligned}
&(\bar{\sigma}\xi_z)_z - \bar{\sigma}\xi\varphi_z^2 - 2p_1[(\varrho\gamma\mu\xi_z)_\tau + \varrho\gamma(\xi_{\tau\tau} + \mu\xi_{z\tau}) - \varrho\gamma\xi(\varphi_\tau^2 + 2\mu\varphi_z\varphi_\tau)] + \\
&\qquad\qquad + \gamma\frac{a}{T_0}K^1 = 0, \\
&(\bar{\sigma}\xi^2\varphi_z)_z - 2p_1[(\varrho\gamma\mu\xi^2\varphi_z)_\tau + \varrho\gamma(\xi^2\varphi_\tau)_\tau + \varrho\gamma\mu(\xi^2\varphi_\tau)_z] + \\
&\qquad\qquad + \gamma\xi^2\frac{a^2}{T_0}K^2 = 0, \\
&\bar{\sigma}_z - 2p_1(\varrho\gamma\mu)_\tau + \gamma\frac{a}{T_0}K^3 = 0, \\
&\xi_z^2 + (\xi\varphi_z)^2 + 1 = \gamma^2, \\
&(\varrho\gamma)_\tau + (\varrho\gamma\mu)_z = 0, \\
&\varrho = \exp[-\alpha\gamma(\bar{\sigma} + 2p_1\varrho\gamma\mu^2)].
\end{aligned} \tag{3.1—11}$$

[19] The superscripts in K^1, K^2, K^3 indicate the components of the force vector as in Eq. (3.1—3), but everywhere else a superscript means raising to a power. This inconsistency should not cause any confusion.

Combining the first three equations in this system we get

$$(\bar{\sigma}\gamma)_z - 2p_1[(\varrho\gamma^2\mu)_\tau + \varrho\gamma\mu\gamma_\tau + \varrho\xi_z(\xi_{\tau\tau} - \xi\varphi_\tau^2) + \varrho\varphi_z(\xi^2\varphi_\tau)_\tau] +$$

$$+ \frac{a}{T_0}(\xi_z K^1 + a\xi^2\varphi_z K^2 + K^3) = 0. \qquad (3.1-12)$$

This equation corresponds to the second one in Eq. (1.2—22); it is the equation of motion (for the string element) along the string tangent.

The system (3.1—11) consists of five nonlinear partial differential equations and one transcendental equation, the elastic condition. There are two independent variables, z and τ, and six unknown functions, ξ, φ, $\bar{\sigma}$, γ, μ, ϱ. We can add the equation (3.1—12) to the system without changing its main properties.

3.2. String Equations for a Small Motion Dependent on Time and Superimposed on a Steady Rotatory Motion; External Forces Being Absent

In this section we shall specialize the string equations (3.1—11) and (3.1—12) to apply to a small time-dependent motion superimposed on a steady rotatory motion. These motions will be described more exactly below. It is assumed that no external forces affect the motions. Linearizing the string equations in the usual manner, they will be separated into 'steady' terms and 'time-dependent' terms. In Chapter 6 we shall make an investigation into a simple special case of such a motion.

3.21. Description of the Motion. Linearization

The following conditions are postulated to hold for the motion:

1) No external forces act on the string. Therefore

$$K^1 = K^2 = K^3 = 0 \qquad (3.2-1)$$

hold everywhere in the strip $z \in (z_1, z_2)$, $\tau \geq 0$ which we shall take to be the domain of definition of the motion. A point force equal to the positive (negative) of the string tension vector acts on the string at the boundary $z = z_2$, $(z = z_1)$.

2) Within the strip the dependent variables can be expanded in power series of a small, real parameter ε according to the following expressions[20]

$$\begin{aligned}
\xi(z,\tau)&=\xi_0(z)+\varepsilon y_1(z,\tau)+O(\varepsilon^2),\quad \varepsilon\to 0,\\
\varphi(z,\tau)&=\tau+\varphi_0(z)+\varepsilon y_2(z,\tau)+O(\varepsilon^2),\\
\bar{\sigma}(z,\tau)&=\bar{\sigma}_0(z)+\varepsilon y_3(z,\tau)+O(\varepsilon^2),\\
\gamma(z,\tau)&=\gamma_0(z)+\varepsilon y_4(z,\tau)+O(\varepsilon^2),\\
\mu(z,\tau)&=\mu_0(z)+\varepsilon y_5(z,\tau)+O(\varepsilon^2),\\
\varrho(z,\tau)&=\varrho_0(z)+\varepsilon y_6(z,\tau)+O(\varepsilon^2).
\end{aligned}\tag{3.2—2}$$

Condition (1) implies that we neglect the effect of gravity on the string and the air-drag.[21] Terms not containing ε in Eq. (3.2—2) will be called steady terms and the others time-dependent terms. Thus $\varepsilon=0$ corresponds to a *steady motion*. The time-dependent terms constitute the so-called *time-dependent motion*. The functions $y_i(z,\tau)$, $i=1,\ldots,6$, approximate the time-dependent motion. The expressions (3.2—2) imply that, in the steady motion every string point, viewed from a coordinate system rotating about the z axis with the constant angular velocity ω, will describe the same path $\xi=\xi_0(z)$, $\varphi=\varphi_0(z)$ with a velocity uq which we shall prefer to call the tangential velocity. It is easily concluded from Eq. (3.1—7) that the angular velocity is ω. We have consequently now given a *physical interpretation of the characteristical time* $\dfrac{1}{\omega}$.

Adding the equation (3.1—12) to the system (3.1—11) and substituting the expressions (3.2—2) for the unknown functions, we get a system each equation of which may be arranged according to powers of ε.

We now assume this system to be satisfied in such a way that the coefficient of each power of ε is zero in each of the equations.

We then get

a) A non-linear system of ordinary differential equations (including a transcendental equation, the elastic condition) for the steady motion.

---

[20] Naturally, we must assume that the expansion coefficients (functions of z and τ) are sufficiently regular to guarantee the validity of the subsequent deductions.

[21] A special case of steady rotatory motion in a gravitational field has been treated by Kolodner, [K2], and steady rotatory motion influenced by air-drag by Mack, [M1]; in those papers the string is assumed to be inextensible.

b) A homogeneous, linear system of partial differential equations for the unknown functions $y_i(z, \tau)$, $i=1, \ldots, 6$, together with an algebraic equation linear in y_i (and homogeneous). The coefficients of this system are functions of z formed by the steady terms.

c) A rest-system consisting of homogeneous, linear partial differential equations for higher order terms.

It is of great interest to study a time-dependent motion caused by a small external disturbance acting on a string which initially has a steady rotatory motion. The functions $y_i(z, \tau)$, solutions of the system (b), form a first order approximation of the time-dependent motion. In such a case the parameter ε is an appropriate measure of the disturbance. The system (b) will be non-homogeneous, if the disturbance has the form of an external, distributed force acting on the string in the interval in question. In order to be able to discuss the time-dependent system (b) one must first, of course, determine the essentials of a steady motion which is governed by (a).

The procedure outlined for the time-dependent motion may be called a *linearization,* and (in general) the following two problems will then result:

a′) A two-point boundary value problem for the system (a).

b′) A mixed initial-boundary value problem for the system (b).

Questions of convergence of the method (*viz.* the expansion of the time-dependent motion in powers of ε) will also arise.

We shall thoroughly discuss the equations of steady motion in Chapter 4. A strange type of steady motion is briefly discussed in Chapter 5, and in Chapter 6 we shall study a simple vibration problem for an inextensible string.

3.22. The Equations of Steady Motion

From Eqs. (3.1—11) and (3.2—2) we obtain the following system valid for the steady terms

$$
\begin{aligned}
&(\bar{\sigma}\xi_z)_z-\bar{\sigma}\xi\varphi_z^2+2p_1\varrho\gamma\xi+4p_1\varrho\gamma\mu\xi\varphi_z=0,\\
&(\bar{\sigma}\xi^2\varphi_z)_z-2p_1\varrho\gamma\mu(\xi^2)_z=0,\\
&\bar{\sigma}_z=0,\\
&(\bar{\sigma}\gamma)_z+2p_1\varrho\xi\xi_z=0,\\
&\xi_z^2+(\xi\varphi_z)^2+1=\gamma^2,\\
&(\varrho\gamma\mu)_z=0,\\
&\varrho=\exp\left[-\alpha\gamma(\bar{\sigma}+2p_1\varrho\gamma\mu^2)\right],
\end{aligned}
\qquad (3.2-3)
$$

where index zero has been omitted for the sake of brevity. The fourth equation of the system corresponds to Eq. (3.1—12) and the first four equations are consequently not independent. From the equation $(\varrho\gamma\mu)_z=0$ we get $\varrho\gamma\mu=\text{const.}$ along the string. This fact has a simple physical interpretation as is shown later, see remark (1). Since $\varrho(z)$, $\gamma(z)<\infty$ according to assumptions made previously, we have two possibilities for $\mu(z)$, *viz.*

$$\mu(z)\neq 0, \text{ or } \mu(z)\equiv 0.$$

The tangential velocity of a string point is $uq=a\omega\mu\gamma$; in a steady rotatory motion this velocity consequently cannot be zero in one point unless it is zero everywhere, a result which is almost self-evident. For $\mu(z)\neq 0$ the system Eq. (3.2—3) gives

$$\begin{aligned} &\bar{\sigma}(\xi_{zz}-\xi\varphi_z^2)+2p_1h\frac{1}{\mu}\xi+4p_1h\xi\varphi_z=0,\\ &\bar{\sigma}(\xi^2\varphi_z)_z-4p_1h\xi\xi_z=0, \\ &\bar{\sigma}\gamma_z+2p_1\varrho\xi\xi_z=0,\\ &\xi_z^2+(\xi\varphi_z)^2+1=\gamma^2, \end{aligned} \qquad (3.2-4)$$

where the following equations hold

$$\begin{aligned} &\varrho=\exp\left[-\alpha\gamma(\bar{\sigma}+2p_1h\mu)\right],\\ &\varrho\gamma\mu=h=\text{const.} \end{aligned} \qquad (3.2-5)$$

and

$$\bar{\sigma}=\text{const.} \qquad (3.2-6)$$

For a motion where $\mu(z)\equiv 0$ the first equation of the system (3.2—4) becomes

$$\bar{\sigma}(\xi_{zz}-\xi\varphi_z^2)+2p_1\varrho\gamma\xi=0, \qquad (3.2-7)$$

while the others are valid if we put $h=0$. In the case of an inextensible string $\alpha=0\Rightarrow\varrho\equiv 1$ and the tangential velocity of the string point is therefore the same everywhere.

The system Eq. (3.2—4) is a nonlinear system of differential equations for the unknown functions $\xi(z)$, $\varphi(z)$, $\gamma(z)$, $\mu(z)$ and $\varrho(z)$; these functions are further connected by Eq. (3.2—5). α, p_1 are constants, *cf.* Eq. (3.1—9). The system is determinate since there are three independent differential equations and two other equations. These equations, or equivalent equations, may be derived directly from the conservation laws of classical mechanics, as is generally done for the case $\alpha=0$ (inextensible string) in works on textile mechanics, see for instance [M1].

In the following considerations we shall take it as understood that the equations (3.2—4) to (3.2—7) are defined on some interval (z_1, z_2) of the z axis. We may remark that

1) The same quantity of string mass flows per unit of time through every plane $z=$const. This is readily apparent from section 1.3, Eq. (1.3—6). In the latter equation we put $F(x^i, t)=x^3-$ const. $=0$, and consequently $\dfrac{\partial F}{\partial x^1}=\dfrac{\partial F}{\partial x^2}\equiv 0$ and by Eq. (3.1—7) $\dfrac{\partial x^3}{\partial t}\equiv 0$. Hence

$$\frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial t}\equiv 0 \Rightarrow H=mqu=m_0 a\omega h=\text{const.}$$

2) From Eq. (3.1—10) and Eq. (3.2—6) we obtain $\overline{T}\dfrac{a}{q}=\bar{\sigma}T_0=$const. Now, $q>0$ which implies that $\overline{T}$ cannot change sign along the string. Since $\overline{T}$ is the difference between two functions which are positive definite, three different possibilities exist (we shall take no interest in the pathological case $T<0$), *viz.*

a) $\overline{T}>0 \qquad \Rightarrow \bar{\sigma}>0,$

b) $-mu^2q^2\leq\overline{T}<0 \qquad \Rightarrow -2p_1\varrho\gamma\mu^2\leq\bar{\sigma}<0$

c) $\overline{T}=0 \qquad \Rightarrow \bar{\sigma}=0$

Case (c) does not possess any interest, since by Eq. (3.2—4) it implies either $\xi\equiv 0$, or $p_1=0\Rightarrow\omega=0$ (though a strange case of motion with $\xi\equiv$const. $\neq 0$, $\varphi_z=-1/2\mu=$const. has, unfortunately, been passed over). The other two cases correspond to entirely different types of steady motion. Case (a) is exhaustively discussed in Chaper 4. Several special types of case (a) have been treated by various authors. Case (b), which obviously is possible only if $\mu\neq 0$ (*i. e.* non-zero tangential velocity), has apparently not been discussed before. We shall devote Chapter 5 to a brief investigation of this case.

3) In case (a) we find it suitable to choose $\bar{\sigma}=1$, implying $T_0=\overline{T}a/q$. The denomination of T_0 as the z component of the reduced string

tension $\overline{T}$ is justified, since $\dfrac{a}{q}$ is the direction cosine of the string tangent with respect to the z axis.[23] *In this way the characteristical force T_0 is given a physical interpretation.* In case (b) we analogously choose $\bar{\sigma}=-1$.

3.23. A Special Case of the Linearized Time-Dependent Motion

The mixed initial-boundary value problem (b′) in section 3.21, which results from the linearization process, will in general turn out to be quite complicated. In the case of an elastic string the differential system is a linear hyperbolic system,[24] but the coefficient functions determined by the steady motion will become complicated. For the inextensible string the differential system is not hyperbolic, but on the other hand the coefficient functions are comparatively simple.

We now deduce the equations of a linearized time-dependent rotary motion for an inextensible string in the case of a steady motion characterized by the following condition

$$\varphi_0(z)=\mu_0(z)\equiv 0. \tag{3.2—8}$$

This means that in steady motion the string is contained in a plane which rotates with constant angular velocity about the z axis. The string itself has no tangential velocity. The steady motion is therefore a simple and well-known motion, satisfying the following equations

$$\begin{aligned} &\xi_{zz}+2p_1\gamma\xi=0,\\ &\gamma_z+2p_1\xi\xi_z=0,\\ &\xi_z^2+1=\gamma^2. \end{aligned} \tag{3.2—9}$$

Putting $\bar{\sigma}=1$ (*cf.* remark (3) in section 3.22) these equations are immediately obtained from Eqs. (3.2—4) and (3.2—7), and it can easily be seen that only two of them are independent. We next apply

---

[23] Actually, we may define the reduced string tension as the contravariant vector $\overline{T}\,\dfrac{\partial x^i}{\partial s}$.

[24] In a recent paper Ždanovič, [Z1], has investigated initial value problems for hyperbolic differential systems on the plane with homogeneous two-point boundary conditions (by means of separation of the variables).

the linearization process outlined in section 3.21 to the system (3.1—11). For $\alpha=0$, $K^i\equiv0$ $(i=1,2,3)$, $\varphi_0(z)=\mu_0(z)\equiv0$ we then get

$$
\begin{aligned}
& y_{1zz}-2p_1\gamma(y_{1\tau\tau}-y_1-y_{1z}\xi\xi_z\gamma^{-2}-2y_{2\tau}\xi)+y_3\xi_{zz}=0,\\
& y_{2zz}-2p_1\gamma(y_{2\tau\tau}+2y_{1\tau}\xi^{-1}+2y_5\xi^{-1}\xi_z)+2y_{2z}\,\xi^{-1}\,\xi_z=0,\\
& y_{3z}-2p_1\gamma y_{5\tau}=0, \qquad\qquad (3.2-10)\\
& y_{1z\tau}\,\xi_z+\gamma(y_5\gamma)_z=0,\\
& y_4-y_{1z}\xi_z\gamma^{-1}=0,
\end{aligned}
$$

where we have omitted index zero for the steady terms just as in equations (3.2—9). This is a homogeneous system of linear partial differential equations for the unknown functions $y_i(z,\tau)$, $i=1,\ldots,5$. The coefficient functions satisfy Eq. (3.2—9) and they consequently depend on z but not on τ; p_1 is a string parameter, *cf.* Eq. (3.1—9). The system is not hyperbolic, and this was not to be expected since we are dealing with an inextensible string, *cf.* section 2.15.

The unknown function $y_4(z,\tau)$ appears only in the last equation. This equation may therefore be regarded as defining y_4, and we can exclude it without changing the properties of the system. We shall return to this kind of motion in Chapter 6.

4. The Steady Rotatory Motion of the Linearly Elastic String in the Case of Positive Reduced String Tension and in the Absence of External Forces

Symbols and conventions:

Most of the symbols used in this chapter are the same as those in Chapter 3; the following list is therefore incomplete.

p_2 Dimensionless string parameter, defined by Eq. (4.2—4).

β — — — — — — — — — — — — — — , — — — — — Eq. (4.3—3).

b — — — — — — constant, initial value, see Eq. (4.1—2). a and b also occur as end points of an interval of the z axis.

k Modulus of various elliptic functions and integrals. k also denotes an elastic constant.

k_i, $i=1, \ldots, 4$ Constants, determined by Eq. (4.2—13) or Eq. (4.3—14).

m, n_1, n_2, n_2' Constants, determined by Eqs. (4.3—13), (4.3—14) and (4.3—16).

λ, δ Constants, determined by Eq. (4.3—19).

$\varkappa_0$, $\varkappa_1$, $\varkappa_2$ Constants, determined by Eq. (4.3—20).

The symbols denoting elliptic functions and integrals agree with those used by Byrd and Friedman, [B2].

4.1. Introduction

Before we start the systematic treatment of the steady rotatory motion of the elastic string we may suitably give a survey of those steady-motion problems which will be treated later.

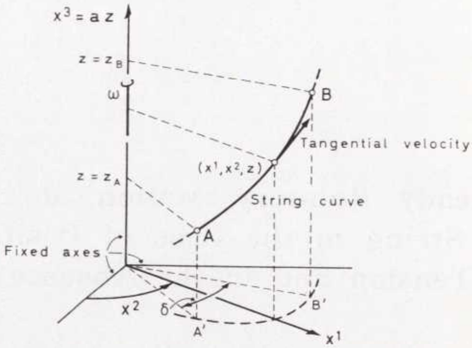

Fig. V. Sketch of the steady motion described in section 4.11. The broken curve $A'B'$ is the trace of the string curve in the plane $x^3 = z = 0$. In the case reproduced here the string is assumed to flow from A to B.

4.11. A Two-Point Steady-Motion Problem

The following problem may be considered typical for a certain class of problems in textile mechanics. *The following premisses are given*:

1) A flexible, linearly elastic string with the mass m_0 per unit length in unstrained state, and the constant of elasticity k (k defined as in section 1.25).

2) A cylindrical coordinate system x^1, x^2, x^3 fixed in space, *cf.* Fig. V. We define the dimensionless coordinate z by $x^3 = az$, where a is the characteristical length in the problem.

3) Two moving points A and B. Their motions are defined by

$$x^1_A,\ z_A = \text{const.},\ x^2_A = \omega t,$$

$$x^1_B,\ z_B = \text{const.},\ x^2_B = \omega t + \delta,$$

where ω, $\delta = \text{const.}$, and $z_B > z_A$. The points A and B are consequently fixed in a coordinate system rotating with the constant angular velocity ω about the z axis.

4) A quantity of string mass $H = \text{const.}$ which flows through A towards B (or in the opposite direction) per unit of time along a string curve containing A and B. It is assumed that the motion is steady in our sense of the term, *cf.* section 3.21; the string curve therefore does not change its shape.

5) No external forces act on the string in the interval (z_A, z_B), *i. e.* the effects of gravity and air-drag are neglected.

6) One more condition in addition to those given in (3). This condition may be

a) Information concerning the total mass of the string elements in the interval (z_A, z_B). This is a probable condition for motions without tangential velocity, *i. e.* for $H=0$.

b) Information concerning the magnitude of the string tension in some string point on the interval $[z_A, z_B]$.

c) A more general condition than (a) or (b), *e. g.* a connection between the total string mass in the interval (z_A, z_B) and the string tension at the point A. Such a condition and still more general ones may occur in problems of textile mechanics.

Necessary requirements include

7) All data for the string in the interval (z_A, z_B).[25] We consequently require:

a) The equation of the string curve

$$x^1 = x^1(z),$$

$$x^2 = \omega t + \varphi(z).$$

b) The tangential velocity or the z component of the velocity vector.

c) The string tension $T = T(z)$ which by the elastic condition determines $m = m(z)$.

It may be remarked that

8) The string equations of the steady rotatory motion, Eq. (3.2—4), must hold on (z_A, z_B).

9) It is not known from the beginning whether the reduced string tension $\overline{T}$ is positive or negative, *cf.* section 3.22, remark (2). Apparently a solution of the problem always exists for $\overline{T} > 0$. Solutions for $\overline{T} < 0$ exist, at least in some special cases, *cf.* Chapter 5.

10) From a physical point of view it might be expected that the informations in points (1) to (6) should uniquely determine the solution

[25] In the case of positive reduced string tension (*cf.* section 3.22, remark (2)) the interval of definition of the occurring functions may always be extended to be the whole z axis.

of the problem, but this will not be true in general.[26] Apparently no existence or uniqueness theorems directly applicable to two-point problems of the present type are to be found in the mathematical literature. Questions concerning existence and uniqueness of two-point problems are left open in this thesis.

11) It is not easy to obtain explicit solutions of a two-point problem by direct attack. We shall therefore set one-point problems (initial value problems) with conditions for the string tension of the kind (6) (b). It will then be demonstrated that if a two-point problem does have a solution it is contained in the solution of a one-point problem. The uniqueness of the solutions of the one-point problems occurring here follows from elementary theorems on ordinary differential systems.

4.12. Two Steady-Motion Problems Relating to an Inextensible String

We now set two basic problems for an inextensible string. They are equivalent to the two-point problems discussed in the preceding section. The first one, called (I1), refers to a case, where the string curve does not intersect the z axis. In the second one, (I2), the string curve has at least one point of intersection with the z axis. Both problems, of course, start from the system Eq. (3.2—4).

(I1) Problem: *To find functions $\xi(z)$, $\varphi(z)$, $\gamma(z)$, defined on some interval (a, b), $a<0<b$, such that they satisfy*

a) *The differential system*

$$(\xi^2\varphi_z)_z - 4p_1h\xi\xi_z = 0,$$

$$\gamma_z + 2p_1\xi\xi_z = 0, \qquad (4.1-1)$$

$$\xi_z^2 + (\xi\varphi_z)^2 + 1 = \gamma^2.$$

b) *The initial values*[27]

$$\xi(0)=1,\ \varphi(0)=0,\ \varphi_z(0)=b,\ \gamma(0)=(1+b^2)^{1/2}, \qquad (4.1-2)$$

[26] In an (unpublished) paper on the theory of ringspinning the author obtained an infinitely enumerable set of solutions to such a problem. Instable motions, indicating non-uniqueness, are well-known in textile technology, *cf.* also [N1].

[27] The initial value b is not to be confused with the right end point of the interval (a,b).

where the following inequality is valid

$$b^2-2p_1(1+b^2)^{1/2}-4p_1hb\geq 0. \tag{4.1—3}$$

(I2) Problem: *To find functions* $\xi(z)$, $\varphi(z)$, $\gamma(z)$, *defined on some interval* (a, b), $a<0<b$, *such that they satisfy*

a) *The differential system Eq.* (4.1—1)

b) *The initial values*

$$\xi(0)=\varphi(0)=0. \tag{4.1—4}$$

c) *The condition*

$$\xi_z=0 \text{ for } \xi=1. \tag{4.1—5}$$

In problems (I1) and (I2) p_1 and h are real constants and $p_1>0$, while h is not restricted. Omitting the first of the equations and putting $\alpha=0$, $\bar{\sigma}=1$ in Eq. (3.2—4) we immediately obtain the system Eq. (4.1—1). Problems (I1) and (I2) will be completely solved in section 4.2. Anticipating the results in that section we may remark that

1) The existence and uniqueness of the solutions of (I1) and (I2) are guaranted by basic theorems in the theory of ordinary differential systems. The solutions will be obtained in terms of elliptic functions and integrals by straight-forward calculations. The latter property of the steady rotatory motion of an inextensible string has also been discovered by Mack, *cf.* [M1]. Mack used a method of representation entirely different from ours and does not make a penetrating study of the matter as the air-drag problem constitutes his major interest. We shall examine the subject thoroughly, especially since we need the results when studying the elastic string.

2) Problem (I1) is a one-point problem in the strict sense of the term, whereas (I2) is not.

3) The functions $\xi(z)$ in (I1) and (I2) turn out to be periodic functions, we may take their intervals of definition to be the whole z axis. The condition Eq. (4.1—2) means that $\xi(0)=1$ is an extremum of $\xi(z)$, and condition Eq. (4.1—3) that we choose it to be a minimum. These two conditions therefore do not imply any specialization, but merely a normalization of the function $\xi(z)$. In fact, $\xi(0)=1$ means that *the characteristical length* a, *cf.* (3.1—5), *is the smallest distance between the string curve and the* z *axis*. Similarly, the condition Eq. (4.1—5) for problem (I2) means that a *is the greatest distance between the string curve and the* z *axis.*

4) Problems (I1) and (I2) will prove to be equivalent to the two-point problems of section 4.11, *cf.* section 4.23.

4.13. Two Steady-Motion Problems for an Elastic String in the Case of Small Strain

In section 4.3 we shall treat those two basic problems for an elastic string which are analogous to problems (I1) and (I2). We shall assume that the strain $\alpha = kT_0$ is small and so obtain solutions valid in the vicinity of $\alpha = 0$ and correct up to and including terms $O(\alpha)$, $\alpha \to 0$. A common linearization process will lead to equations that are singular at the turning points $\xi_z = 0$, and in order to avoid such difficulties we shall adopt a special method. The solutions so obtained will be valid on any finite z interval. We shall call these problems (E1) and (E2), corresponding to (I1) and (I2), respectively.

4.14. Two Steady-Motion Problems for a Linearly Elastic String and Finite Strain

In section 4.4 we shall treat two basic problems related to the steady motion of a linearly elastic string, when the latter has no tangential velocity. These problems are analogous to problems (I1) and (I2) and will be called (E3) and (E4), respectively. No restrictions are imposed on the magnitude of the strain.

4.2. The Steady Rotatory Motion of the Inextensible String in the Case of Positive Reduced String Tension

Two problems, (I1) and (I2), related to the inextensible string were set in section 4.12. We now proceed to the solution of these problems, and at the end of the section we shall briefly discuss the possibilities of finding solutions of two-point problems, starting from the solutions of (I1) or (I2).

4.21. The Existence and Uniqueness of the Solution of Problem (I1)

The differential system Eq. (4.1—1) is easily transformed into the vector differential equation

$$x_z = A(x), \tag{4.2—1}$$

where x and $A(x)$ are column vectors in the five-dimensional space. Writing x' for the transpose of x we have

$$x' \equiv (x_i) = (\xi, \xi_z, \varphi, \xi\varphi_z, \gamma), \qquad i = 1, \ldots, 5, \tag{4.2-2}$$

and

$$A(x) = \begin{bmatrix} x_2 \\ x_4^2 x_1^{-1} - 2p_1 x_1 x_5 - 2p_2 x_4 \\ x_4 x_1^{-1} \\ 2p_2 x_2 - x_2 x_4 x_1^{-1} \\ -2p_1 x_1 x_2 \end{bmatrix}, \tag{4.2-3}$$

where p_2 is a constant

$$p_2 = 2p_1 h. \tag{4.2-4}$$

The initial values defined by Eq. (4.1−2) obviously correspond to

$$x'(0) \equiv k' = (1, 0, 0, b, (1+b^2)^{1/2}). \tag{4.2-5}$$

From the above equations we can conclude that the elements A_i of A are rational functions of the elements x_n of x; $i, n = 1, \ldots, 5$. We write G for the domain $Re(x_1) > 0$ of the complex five-dimensional space, and it is evident that $A(x)$ is analytic and regular in G and also that $x(0) = k \in G$. From theorems on ordinary differential systems it therefore follows that problem (I1) has a unique solution, $x(z)$, which is analytic in the neighbourhood of the point $z = 0$, *cf.* for instance [H2], [P2]. This neighbourhood may be extended to consist of all points z such that $x(z)$ is in the interior of G. Moreover, the solution is analytic in the parameters p_1, p_2 and the initial value b.

4.22. The Existence and Uniqueness of the Solution of problem (I2)

The transformation of the string equations from rectangular Cartesians to cylindrical coordinates is singular for $x_1 = 0$. Strictly speaking, this fact means that the equations (4.1−1) become meaningless for

$\xi=0$ ($x_1=a\xi$, $a=$const.) and that consequently problem (I2) is not a proper one. From the first of the equations (4.1—1), however, we conclude that

$$\xi^2\varphi_z-p_2\xi^2=\text{const.}$$

Now, this constant may be zero for some kind of motion and then

$$\varphi_z=p_2=\text{const.} \tag{4.2—6}$$

Further, a string curve intersecting the axis of rotation ($\xi=0$) is permitted by this kind of motion, a motion which can easily be shown to correspond to the solution of problem (I2).

We now set a one-point problem equivalent to the problem (I2).

(I2′) Problem: *To find functions $x_i(z)$, $i=1,2,3,5$, defined on some interval (a, b), $a<0<b$, such that they satisfy*

a) *A differential system* Eq. (4.2—1) *where, using the same notations as in the preceding section*

$$x'\equiv(x_i)=(x_1, x_2, x_3, x_5),$$
$$A(x)=\begin{bmatrix} x_2 \\ -p_2^2x_1-2p_1x_1x_5 \\ p_2 \\ -2p_1x_1x_2 \end{bmatrix}. \tag{4.2—7}$$

b) *The initial value*

$$x'(0)=(0, b_2, 0, (1+b_2^2)^{1/2}). \tag{4.2—8}$$

As previously, p_1, p_2 are real constants and b_2 is a positive constant. The initial value $x_2(0)=b_2$ is equivalent to the condition $\xi_z=0$ for $\xi=1$, *cf.* Eq. (4.1—5), since the solution of problem (I2) yields $\xi_z(0)>0$, say $\xi_z(0)=b_2$, see Eq. (4.2—20).

Now, the elements of $A(x)$ are integral functions of x_i, $i=1,2,3,5$, so that the solution of (I2′), and in consequence the solution of (I2), is analytic in z for every z such that x_i is finite. Moreover, the solution is analytic in the parameters p_1, p_2 and the initial value b_2.

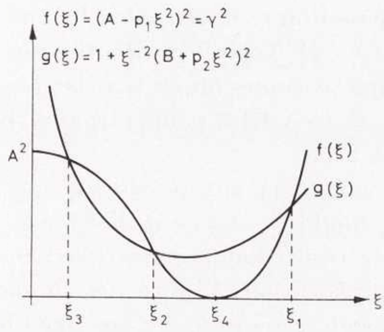

Fig. VI. The graph explains the situation of case (1). For $\xi \in (\xi_3, \xi_2)$ the conditions $\xi_z{}^2 > 0$ and $\gamma > 1$ are satisfied. For $\xi > \xi_1$ one has $\xi_z{}^2 > 0$ but $\gamma < 0$, *cf.* note 28. In case (2) the two points of intersection, $\xi = \xi_2$ and $\xi = \xi_3$, disappear.

4.23. The Solutions of Problems (I1) and (I2)

Writing $p_2 = 2p_1h$ we get from Eq. (4.1—1) for problem (I1)

$$\begin{aligned} &\gamma + p_1\xi^2 = A, \\ &\xi^2\varphi_z - p_2\xi^2 = B, \\ &\xi_z^2 = (A - p_1\xi^2)^2 - 1 - \xi^{-2}(B + p_2\xi^2)^2, \end{aligned} \tag{4.2—9}$$

where A and B are real constants (arbitrary, so far). We take $B \neq 0$, since $B = 0$ leads to problem (I2).

There are now two possible cases:

1) A and B are such that ξ_z regarded as a function of ξ has three real zeros, $\xi_1 > \xi_2 > \xi_3 > 0$.[28] It is not difficult to show that the expressions Eq. (4.2—9) are physically significant in the interval $[\xi_3, \xi_2]$ only (see Fig. VI).

2) A and B are such that ξ_z has only one zero. It is easily demonstrated that this case has no physical significance.

It is now clear that for *any steady rotatory motion with positive reduced string tension and such that the string curve does not intersect the axis of rotation, the function* $\xi(z)$ *can assume the values* $\xi_3 \leq \xi \leq \xi_2$ *only*. Further, $\xi_z = 0$ for $\xi = \xi_2$ and $\xi = \xi_3$. It is evident that no physical

[28] It should be noted that (by definition) $\gamma > 1$ and that we may take ξ to be positive. The constant p_1 is also positive.

specialization is imposed on such a motion by the introduction of the initial values Eq. (4.1—2). Together with the restrictions formulated in Eq. (4.1—3) the initial values imply that letting $\xi_3=1$ we normalize the string curve (*cf.* section 4.12, point (3)) and that we fix it so that $\xi=1$ occurs at $\varphi=0$, $z=0$.

By varying b we can cover any possible string motion of the type in question here. It should be observed that for given $p_1>0$ and p_2 we cannot choose b arbitrarily but are restricted by Eq. (4.1—3). The latter restriction implies that b must be chosen outside the open interval (b_2, b_1) where $b_2<0$ and $b_1>0$ are roots of the equation

$$f(b)\equiv b^2-2p_1(1+b^2)^{1/2}-2p_2b=0. \tag{4.2—10}$$

From Eq. (4.2—9) we now obtain

$$\begin{aligned}(\xi\xi_z)^2&=p_1^2(\xi^2-1)(k_2-\xi^2)(k_1-\xi^2),\\ \xi^2\varphi_z&=b+p_2(\xi^2-1),\\ \gamma&=(1+b^2)^{1/2}-p_1(\xi^2-1),\end{aligned} \tag{4.2—11}$$

where we have to regard

$$\xi^2\in[1, k_2]. \tag{4.2—12}$$

We have introduced k_1 and k_2 instead of ξ_1^2 and ξ_2^2. The following relations are valid:

$$\begin{aligned}k_1&=Q+\frac{1}{p_1}[p_1^2Q^2-(b-p_2)^2]^{1/2},\\ k_2&=Q-\frac{1}{p_1}[p_1^2Q^2-(b-p_2)^2]^{1/2},\\ Q&=\frac{1}{2}\left[1+\left(\frac{p_2}{p_1}\right)^2+\frac{2}{p_1}(1+b^2)^{1/2}\right],\\ f(b)&\equiv b^2-2p_1(1+b^2)^{1/2}-2p_2b\geq 0.\end{aligned} \tag{4.2—13}$$

Excluding the special cases $b=b_1$ or b_2, $f(b_1)=f(b_2)=0$, we can integrate the first of the equations (4.2—11) directly, we then obtain $z=z(\xi)$ as an elliptic integral of the first kind, *cf.* [B2], pag. 72. Inverting, we get $\xi=\xi(z)$ as a periodic function; its domain of definition may be taken to be the whole z axis. Then, $\gamma(z)$ is obtained directly from the last of the equations (4.2—11) and $\varphi(z)$ from the second one. After some calculation we get *the following solution of problem* (I1):

$$\begin{aligned}
&\xi(z)=[1+(k_2-1)\mathrm{sn}^2(\zeta(z))]^{1/2},\\
&\varphi(z)=p_2 z+\frac{b-p_2}{p_1}(k_1-1)^{-1/2}\Pi(\zeta(z),k_2-1),\\
&\gamma(z)=(1+b^2)^{1/2}-p_1(k_2-1)\mathrm{sn}^2(\zeta(z)),\\
&\zeta(z)=zp_1(k_1-1)^{1/2},\ z\in(-\infty,\infty).
\end{aligned} \tag{4.2—14}$$

In these expressions the modulus k of $\mathrm{sn}(\zeta)=\mathrm{sn}(\zeta,k)$ and $\Pi(\zeta,k_2-1)=$ $=\Pi(\zeta,k_2-1,k)$ is determined by

$$k^2=\frac{k_2-1}{k_1-1},\ 0<k<1, \tag{4.2—15}$$

where k_1 and k_2 are the same as in Eq. (4.2—13) and $f(b)>0$ holds. Further, the constants p_1 and p_2 are real and $p_1>0$, while p_2 is not restricted. The solution of problem (I1) given by Eq. (4.2—14) will be discussed in the next section.

In the special cases $f(b)=0$ *we have*

$$\begin{aligned}
&\xi(z)\equiv 1,\\
&\varphi(z)=bz,\\
&\gamma(z)\equiv(1+b^2)^{1/2},
\end{aligned} \tag{4.2—16}$$

where b is a root of the equation (4.2—10). In these two cases the string curve is a *circular helix.*

We now turn to problem (I2). From system Eq. (4.1—1) and the discussion in section 4.22 we obtain

$$\begin{aligned}
&\gamma+p_1\xi^2=A,\\
&\varphi_z=p_2,\\
&\xi_z^2=(A-p_1\xi^2)^2-1-p_2^2\xi^2.
\end{aligned} \tag{4.2—17}$$

It is clear that $A>1$ (*cf.* note 28) and it is easily deduced that the preceding expressions will be physically significant for $\xi\in[-\xi_2,\xi_2]$ only, where $\xi=\xi_2$ is the smallest positive zero of ξ_z regarded as a

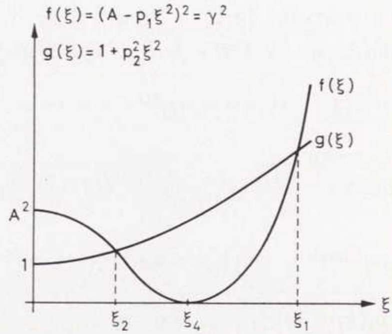

Fig. VII. The graph shows that the conditions $\xi_z^2 > 0$ and $\gamma > 1$ are satisfied for $|\xi| < \xi_2$. For $|\xi| > \xi_1$ one has $\gamma < 0$, *cf.* note 28.

function of ξ (see Fig. VII). Normalizing the string curve, we write $\xi_2 = 1$ and take $\xi = \varphi = 0$ for $z = 0$, *i. e.* we introduce the conditions Eqs. (4.1—4) and (4.1—5). Consequently *any steady rotatory motion with positive reduced string tension such that the string curve intersects the axis of rotation is determined by the solution of problem* (I2). From Eq. (4.2—17) we obtain

$$\begin{aligned} \xi_z^2 &= \left(\frac{p_1}{k}\right)^2 (1-\xi^2)(1-k^2\xi^2), \\ \varphi_z &= p_2, \\ \gamma &= (1+p_2^2)^{1/2} + p_1(1-\xi^2), \end{aligned} \tag{4.2—18}$$

where the range of ξ is $[-1, 1]$ and k is determined by

$$k = \left[1 + \left(\frac{p_2}{p_1}\right)^2 + \frac{2}{p_1}(1+p_2^2)^{1/2}\right]^{-1/2}. \tag{4.2—19}$$

After some elementary calculations we get *the following solution of problem* (I2):

$$\begin{aligned} \xi(z) &= \operatorname{sn}\left(z\,\frac{p_1}{k}\right), \\ \varphi(z) &= p_2 z, \\ \gamma(z) &= (1+p_2^2)^{1/2} + p_1 \operatorname{cn}^2\left(z, \frac{p_1}{k}\right), \\ z &\in (-\infty, \infty). \end{aligned} \tag{4.2—20}$$

The modulus k of sn $\left(z\frac{p_1}{k}\right)$ and cn $\left(z\frac{p_1}{k}\right)$ is determined by Eq. (4.2—19). p_1 and p_2 are real constants and p_1 is positive, while p_2 is not restricted.

4.24. Discussion of the Solutions of Problems (I1) and (I2)

In this section we shall briefly discuss the solutions of problems (I1) and (I2) with special regard to the possibilities of obtaining a complete determination of all the string properties, starting from the physical conditions in one or two points (*cf.* section 4.11). It is only to be expected that one-point problems are more easily approached than two-point problems. Three physical quantities will be considered known *a priori*, *viz.* m, ω, and H (the string mass per unit length, the angular velocity of the string curve, and the quantity of string mass flowing through every surface z=const. per unit of time). These quantities are constants in the case of an inextensible string in steady motion. We first discuss the solution of problem (I1) and start by transforming Eq. (4.2—14) into physical variables. We also add expressions for the arc length s of the string curve and the string tension T. From equations (1.2—8) and (3.1—8) we get $ds=a\gamma dz$ and from Eqs. (3.1—8) and (3.1—10) $T=T_0(\gamma\bar{\sigma}+2p_1\varrho\gamma^2\mu^2)$. Now, $\bar{\sigma}\equiv 1$, $\varrho\equiv 1$ and $\gamma\mu\equiv h=\frac{p_2}{2p_1}$, so we obtain $T=T_0\left(\gamma+\frac{p_2^2}{2p_1}\right)$. We recall that $x^1=a\xi$, φ, $x^3=az$ are the cylindrical coordinates of the string point observed from a frame of reference rotating with the string curve. With respect to such a frame, every string point has a constant velocity $uq=a\omega h=\frac{H}{m}$ (*cf.* section 3.22, point (1)). After some calculation we obtain the following arrangement of *the physical string properties as functions of the axial coordinate z* (*cf.* Eq. (4.2—14)):

$$
\begin{aligned}
&x^1(z)=a[1+(k_2-1)\mathrm{sn}^2(\zeta(z))]^{1/2},\\
&\varphi(z)=p_2z+\frac{b-p_2}{p_1}(k_1-1)^{-1/2}\Pi(\zeta(z),k_2-1),\\
&s(z)=a\{z(1+b^2)^{1/2}-(k_1-1)^{1/2}[\zeta(z)-E(\zeta(z))]\}, \qquad (4.2-21)\\
&T(z)=T_0\left[\frac{p_2^2}{2p_1}+(1+b^2)^{1/2}-p_1(k_2-1)\,\mathrm{sn}^2(\zeta(z))\right],\\
&\zeta(z)=zp_1(k_1-1)^{1/2},\ z\in(-\infty,\infty).
\end{aligned}
$$

According to Eqs. (4.2—15) and (4.2—13) the constants k, k_1 and k_2 may be expressed in terms of p_1, p_2 and b. T_0 and a may also be expressed in terms of p_1, p_2 (and, of course, in the *a priori* constants m, ω, H). We then get from Eqs. (3.1—9), (4.2—4) and the above equality $h = \dfrac{H}{a\omega m}$

$$a = \frac{p_1}{p_2}\frac{2H}{m\omega},$$
$$T_0 = \frac{p_1}{p_2^2}\frac{2H^2}{m}. \qquad (4.2-22)$$

The three constants p_1, p_2 and b are the dimensionless parameters of the string,[29] and if they are known, it then follows from the expressions Eq. (4.2—21) that all the physical string properties are known in every point of the string curve. Instead of these three parameters we may use a, T_0 and b which have the advantage of possessing simple physical interpretations (a is the smallest distance from any point of the string curve to the axis of rotation, T_0 is the axial component of the reduced string tension vector, and b is the cotangent of the angle of elevation of the string curve at a point of distance a from the axis of rotation). We prefer to use p_1, p_2 and b, since they are dimensionless and yield simple expressions. In order to obtain the expressions Eq. (4.2—21) we have introduced a simple set of *one-point conditions, viz.*

$$x^1(0)=a,\ x^1_z(0)=0,\ \varphi(0)=0,$$
$$\varphi_z(0)=b,\ s(0)=0, \qquad (4.2-23)$$

and we have also introduced the parameter T_0.[30] It is most improbable, however, that T_0 and such simple conditions as Eq. (4.2—23) should be available in a physical problem, *cf.* section 4.11. *The main difficulty in solving a particular problem of steady rotatory motion of an inextensible string will consist in determining the string parameters* p_1, p_2, b (*or*

[29] A more precise statement would be: p_1, p_2 and b are the three parameters of the (inextensible) string in case of a steady rotatory motion such that the string curve does not intersect the axis of rotation.

[30] We could also introduce T_0 by means of an initial value of the string tension T, *viz.* $T(0) = \dfrac{H^2}{m} + T_0(1+b^2)^{1/2}$. This fact can be verified by Eqs. (4.2—21) and (4.2—22).

a, T_0, *b*) *and adjusting the string curve in such a way with respect to the cylindrical coordinate system that the available set of boundary conditions and other conditions are satisfied.* For instance, let the motion satisfy conditions (3) of section 4.11. We then have to find string parameters p_1, p_2, b and z_A, z_B that satisfy

$$\begin{aligned} &x^1(z_A)=x^1_A, \quad x^1(z_B)=x^1_B, \\ &\varphi(z_B)-\varphi(z_A)=\delta, \\ &z_B-z_A=\frac{1}{a}C=\frac{p_2}{p_1}\frac{m\omega}{2H}C, \end{aligned} \tag{4.2—24}$$

where x^1_A, x^1_B, δ and C are known quantities and $x^1(z)$ and $\varphi(z)$ are the same as in Eq. (4.2—21). (It should be noted that C is the physical distance between the planes $z=z_A$ and $z=z_B$.) In order to be able to determine the unknown quantities we need one more condition, as predicted in point (6) of section 4.11. The determination of p_1, p_2, b, z_A, z_B will generally involve intricate computations, especially in such cases where several different string motions have to be joined.

It must be pointed out that we cannot expect the determination of the set p_1, p_2, b, z_A, z_B to be unique. It is known that infinitely many solutions are theoretically possible for a simple motion corresponding to a special case of problem (I2), *cf.* the paper by NERONOFF, [N1], containing a discussion on stability; see also [H6], [K2], and note 26. This state of things has nothing to do with the uniqueness of the solutions of the problems (I1) and (I2) as stated here.

So far, we have discussed the case of a steady rotatory motion, when the string curve does not intersect the axis of rotation (corresponding to the problem (I1)). We now turn to the case of an intersection between the string curve and the axis of rotation, corresponding to problem (I2). This is a most special case, in so far as there are small chances that any set of one-point or two-point conditions would produce such an intersection. In several textile devices, however, for example spinning machines, the string is forced to pass through a small guide eye placed centrally with reference to the axis of rotation. It is not difficult to show that the solution of problem (I1) turns into the solution of (I2) when we pass to the limits $a\to 0$, $b\to p_2$.[31] The solu-

[31] It should be observed that a has different meanings in problems (I1) and (I2), respectively.

tion of (I2) consequently provides an approximate solution of (I1) for the kind of problem outlined above.

Transforming the solution Eq. (4.2—20) of the problem (I2) into *physical variables* and adding expressions for the arc length of the string curve s and the string tension T we get

$$
\begin{aligned}
x^1(z) &= a \operatorname{sn}\left(z \frac{p_1}{k}\right), \\
\varphi(z) &= p_2 z, \\
s\,(z) &= a\left[-z\left(\frac{p_2^2}{p_1}+(1+p_2)^{1/2}\right)+\frac{1}{k}E\left(z\frac{p_1}{k}\right)\right], \\
T(z) &= T_0\left[\frac{p_2^2}{2p_1}+(1+p_2)^{1/2}+p_1 \operatorname{cn}^2\left(z\frac{p_1}{k}\right)\right],
\end{aligned}
\tag{4.2—25}
$$

where k is determined by Eq. (4.2—19) and we choose $s(0)=0$. The equalities Eq. (4.2—22) hold true in this case too. There are two independent parameters p_1, p_2 in Eq. (4.2—25). Alternatively, we may use a, T_0 instead of p_1 and p_2. The comments on Eq. (4.2—21) are in the main valid for Eq. (4.2—25) also, but the computational difficulties are considerably smaller for Eq. (4.2—25), since there are only two parameters to be determined and the elliptic integral of the third kind in the expression for $\varphi(z)$ has disappeared.

4.3. The Steady Rotatory Motion of an Elastic String in the Case of Positive Reduced String Tension and Small Strain

In this section we treat the basic problems (E1) and (E2), *i. e.* the analogues for an elastic string to the problems (I1) and (I2). We shall assume that the reduced string tension is positive, *i. e.* $\overline{T}>0$, and that the strain of the string is small everywhere, *i. e.* $\alpha=kT_0\ll 1$, *cf.* section 3.22. According to point (3) of section 3.22 we put $\bar{\sigma}=1$ in the system Eqs. (3.2—4), (3.2—5). Excluding the first equation of Eq. (3.2—4) we have

$$
\begin{aligned}
&(\xi^2\varphi_z)_z-2p_2\xi\xi_z=0, \\
&\gamma_z+2p_1\varrho\xi\xi_z=0, \\
&\xi_z^2+(\xi\varphi_z)^2+1=\gamma^2,
\end{aligned}
\tag{4.3—1}
$$

and from Eq. (3.2—5)

$$\varrho = \exp\left[-\frac{\beta}{p_1}\left(\gamma + \frac{p_2^2}{2p_1}\frac{1}{\varrho}\right)\right]. \tag{4.3—2}$$

The *parameters* p_1, p_2 and β of these equations are connected with the *a priori constants* m_0, ω, H, k and the *characteristical quantities* a, T_0 according to

$$p_1 = \frac{1}{2}\, m_0\, \omega^2 \frac{a^2}{T_0},$$

$$p_2 = 2p_1\, h = \omega\, H \frac{a}{T_0}, \tag{4.3—3}$$

$$\beta = \alpha\, p_1 = \frac{1}{2}\, m_0 \omega^2 k a^2.$$

The system Eq. (4.3—1) consists of three ordinary, nonlinear differential equations for the functions $\xi(z)$, $\varphi(z)$, $\gamma(z)$. Eq. (4.3—2) is a transcendental equation which determines ϱ as a function of γ. The parameters are real and $p_1 > 0$, $p_2 \neq 0$, $0 \leq \beta \ll 1$. We exclude the special case $p_2 = 0$ (which implies zero tangential velocity), since this case will be treated in section 4.4. The function $\varrho(\gamma) = \varrho(\gamma, \beta)$ turns out to be multiple-valued mathematically, but if we take $\varrho(\gamma) \to 1$ as $\beta \to 0$, then $\varrho(\gamma)$ is analytic in γ and β for every γ and $p_1 > 0$, if β is sufficiently small.[32] This is a consequence of a theorem on implicit analytic functions, see for instance [S2], p. 170.

Compared to the corresponding system Eq. (4.1—1) of the inextensible string Eq. (4.3—1) is further complicated by the presence of the variable $\varrho = \varrho(\gamma)$. This complication is not serious from the mathematical point of view, however, since Eq. (4.3—1) is analytic in its variables and parameters. The solution of a one-point problem in connection with Eq. (4.3—1) will consequently be analytic in the independent variable z, the initial values and the parameters (in the neighbourhood of $\beta = 0$). The solution therefore admits of series expansions after powers of β. This property is most valuable, since the exact solution of the problem (for arbitrary values of β), though it probably might be found, is too complicated to be of practical interest. We will therefore seek approximations, linear in β, to the

---

[32] It should be observed that $\beta = 0$ means an inextensible string for which $\varrho \equiv 1$.

solution of our problem. The search for linear approximations does not need any special justification; it corresponds to the common approximations in the classical theory of elasticity.

It would be natural to apply a linearization on approaching the problem of approximation, *i. e.* to add terms linear in β to the solutions of the corresponding problems for the inextensible string. Neglecting terms $O(\beta^2), \beta \to 0$, a system of linear differential equations would then result from Eq. (4.3—1). This system, however, is singular at the turning points $\xi_z = 0$, as may be seen from Eq. (4.3—1). Such a difficulty might be surmounted by an application of the PLK method, *cf.*, for instance, [T2], but the attempts in this direction by the author have failed so far. Moreover, the coefficients of the differential system will consist of elliptic functions and such a system is not easily treated, however simple the boundary conditions may be.[33]

In order to avoid the difficulties just pointed out we shall apply a method which is a most special one, since its application is confined to this very problem, or at least to a small class of problems.[34] The details of the method will be elucidated later on.

In subsequent deductions we shall frequently meet statements in the form, the function $f = f(x, p_1, p_2, \beta)$ is analytic in its variable x and the parameter β. It should be understood that the statement is true for every x of current interest, if β is in a sufficiently small neighbourhood of zero and the parameters p_1, p_2 are real, and $p_1 > 0$.

We now return to the system Eqs. (4.3—1), (4.3—2). Expanding after powers of β and making some elementary calculations, we get

$$\begin{aligned} \varrho &= 1 - \frac{\beta}{p_1}\left(\gamma + \frac{p_2^2}{2p_1}\right) + \beta^2 G_1, \\ \gamma &= A - p_1(\xi^2 - 1)\left[1 - \frac{1}{2}\beta(2Q - \xi^2)\right] + \beta^2 G_2, \end{aligned} \qquad (4.3-4)$$

where $G_1 = G_1(\gamma, \beta)$ is analytic in the variable γ and the parameter β, and further $G_1 = O(1), \beta \to 0$. $G_2 = G_2(\xi, \beta)$ is analytic in ξ and β and $G_2 = O(1), \beta \to 0$. A is an arbitrary constant and Q is a constant determined by

$$2Q = 1 + \left(\frac{p_2}{p_1}\right)^2 + \frac{2}{p_1} A. \qquad (4.3-5)$$

---

[33] *Cf.* Chapter 6 where a related problem is discussed.

[34] A mathematician might well ask if it deserves to be termed a method.

From Eq. (4.3—1) we also obtain

$$\xi_z^2 = \gamma^2 - 1 - \xi^{-2}(B + p_2\xi^2)^2,$$
$$\varphi_z = \xi^{-2}(B + p_2\xi^2), \tag{4.3—6}$$

where B is an arbitrary constant.

From now on we must consider the two problems (E1) and (E2) separately.

4.31. **Problems (E1) and (E2)**

We now formulate the basic problem (E1) for the steady rotatory motion of an elastic string, when the strain is small and the string curve does not intersect the axis of rotation.

Let the functions $\xi'(z, \beta)$, $\varphi'(z, \beta)$, $\gamma'(z, \beta)$ *of the variable* z *and the parameter* β *be defined, for* z *on some interval* $[a', b']$, $a' < 0 < b'$, *and for* β *in a neighbourhood of* $\beta = 0$. *Let these functions satisfy*

a) *The differential system Eq.* (4.3—1), *where* $\varrho = \varrho(\gamma, \beta)$ *is defined by Eq.* (4.3—2).

b) *The initial values*

$$\xi'(0, \beta) = 1, \ \varphi'(0, \beta) = 0,$$
$$\varphi_z'(0, \beta) = b, \ \gamma'(0, \beta) = (1 + b^2)^{1/2}, \tag{4.3—7}$$

where b *is a real constant such that for some positive number* M

$$f(b) = b^2 - 2p_1(1 + b^2)^{1/2} - 2p_2 b > M. \tag{4.3—8}$$

We call the functions $\xi(z, \beta)$, $\varphi(z, \beta)$, $\gamma(z, \beta)$ *a solution of the problem* (E1) *on the interval* $(a, b) \subset [a', b']$, $a < 0 < b$ (*cf.* note 27), *if, for* $z \in (a, b)$ *and in a neighbourhood of* $\beta = 0$, *they satisfy*

c) *The initial values Eq.* (4.3—7).

d) *The conditions*

$$\xi - \xi', \ \varphi - \varphi', \ \gamma - \gamma' = O(\beta^2), \ \beta \to 0. \tag{4.3—9}$$

e) *Some further conditions of continuity.*

The following comments may be made concerning problem (E1):

1) The condition Eq. (4.3—8) means that $\xi'(0, \beta) = 1$ is a local minimum of $\xi'(z, \beta)$ provided that β is small enough. This condition implies no specialization, *cf.* point (3) of section 4.12.

2) The conditions of continuity imposed on ξ, φ, γ will depend on the kind of string property we want to approximate. If, for instance, we want the approximation of $\frac{d^n \xi'}{dz^n}$, (linear in β), then $\xi(z, \beta)$ must be of the class C^n and the condition $\frac{d^n}{dz^n}(\xi - \xi') = O(\beta^2), \beta \to 0$, must hold. The method employed in this thesis will provide functions $\xi(z, \beta)$, $\varphi(z, \beta)$, $\gamma(z, \beta)$ which are analytic in z and β, and further supply approximations of derivatives of any order.

3) It will become clear in the next section that $\xi'(z, \beta)$ must be a periodic function of z; the domain of definition of ξ', φ', γ' may accordingly be taken to be the whole z axis. The functions ξ, φ, γ (solution of (E1)) may also be defined on the whole z axis and provide the required approximations valid for any finite z, if β is small enough.

We now proceed to the problem (E2) which is the counterpart of (E1) when the string curve intersects the axis of rotation. *We use a formulation of the problem* (E2) *similar to that given for* (E1), *with the exception that we replace the conditions Eq.* (4.3—7) *by the following expressions*:

$$\xi'(0, \beta) = \varphi'(0, \beta) = 0, \tag{4.3—10}$$

$$\xi'_z = 0 \quad \text{for } \xi' = 1. \tag{4.3—11}$$

Comments (2) and (3) on problem (E1) are valid for (E2), too, and Eq. (4.3—10) means that $z=0$ is a point of intersection between the string curve and the axis of rotation. The condition Eq. (4.3—11) means that the amplitude of $\xi'(z, \beta)$ is normalized to be 1 and does not imply any specialization, *cf.* point (3) in section 4.12.

4.32. The Solution of Problem (E1)

Proceeding now to the solution of problem (E1) it is evident that the functions $\xi'(z, \beta)$, $\varphi'(z, \beta)$, $\gamma'(z, \beta)$ must satisfy equations (4.3—4) and (4.3—6). We may here make the same investigations concerning the constants A and B as we did in section 4.23. Since the influence of the parameter β on γ' and ξ'_z may be made as small as we like, the result will be essentially the same as in the case of the inextensible string. We can therefore conclude that A and B must be such that ξ'_z, regarded as a function of ξ' and β, has three zeros $\xi'_1(\beta) > \xi'_2(\beta) >$ $> \xi_3(\beta) > 0$. These zeros are analytic in β and are real for any real β.

The expressions Eqs. (4.3—4) and (4.3—6) are physically significant for $\xi' \in [\xi_3', \xi_2']$ only. We may put $\xi_3'(\beta)=1$ which means that we let the string parameter a be the smallest distance between the string curve and the axis of rotation. Just as in problem (I1) we can introduce the initial values Eq. (4.3—7) without loss of general validity. The special case $\xi_2'=\xi_3'$ is excluded by the inequality Eq. (4.3—8). We consider this helix case separately in section 4.34. We shall find it convenient to write $(\xi_2')^2=k_2 n_2'$ where $n_2'=1+O(\beta)$, $\beta\to 0$. (It should be observed that k_2 here is the same parameter as in Eq. (4.2—11).

It is our aim to find an approximation $\xi(z, \beta)$ of $\xi'(z, \beta)$, and the key to the problem is a quartic in $(\xi_z')^2$ which provides an approximation, linear in β, of the expression for $(\xi_z')^2$, *i. e.* the first of Eqs. (4.3—6). The next step in the derivation of this quartic consists in expressing ξ_z' as a function of ξ' and β in a suitable form. The expression so obtained must be valid for $(\xi')^2 \in [1, k_2 n_2']$. After some elementary calculations we get the following expressions which (*in the case of small strain*) *hold true for any steady rotatory motion of an elastic string which does not intersect the axis of rotation,*

$$
\begin{aligned}
&(\xi'\xi_z')^2 = p_1^2(1+m\beta)((\xi')^2-1)(k_2 n_2'-(\xi')^2)\times\\
&\qquad\times(k_1 n_1-(\xi')^2)(1+\beta(\xi')^2)(1+\beta^2 G_3),\\
&(\xi')^2\varphi_z' = b+p_2((\xi')^2-1),\\
&\gamma' = (1+b^2)^{1/2} - p_1((\xi')^2-1)\left[1-\tfrac{1}{2}\beta(2Q_1-(\xi')^2)\right]+\beta^2 G_2,
\end{aligned}
\tag{4.3—12}
$$

where $(\xi')^2 \in [1, k_2 n_2']$, $G_2 = G_2(\xi', \beta)$ and $G_3 = G_3(\xi', \beta)$ are analytic in ξ' and β and are real for real values of ξ' and β. They possess the properties $G_2 = O(1)$, $G_3 = O(1)$, $\beta\to 0$. Moreover, $n_1 = n_1(\beta)$ and $n_2' = n_2'(\beta)$ are analytic in β. The following expressions determine the parameters in Eq. (4.3—12):

$$
\left.
\begin{aligned}
&m = -(k_1+k_4),\\
&n_1 = n_1(\beta) = 1-\beta\,\frac{k_2 k_4}{k_1-k_2},\\
&n_2' = n_2'(\beta) = 1+\beta\,\frac{k_1 k_3}{k_1-k_2}+O(\beta^2),\ \ \beta\to 0,
\end{aligned}
\right\}
\tag{4.3—13}
$$

$$
\left.
\begin{aligned}
&k_1=Q_1+R,\quad k_2=Q_1-R,\\
&k_3=Q_2+R,\quad k_4=Q_2-R,\\
&2Q_1=1+\left(\frac{p_2}{p_1}\right)^2+\frac{2}{p_1}\,(1+b^2)^{1/2},\\
&2Q_2=1-\left(\frac{p_2}{p_1}\right)^2,\\
&R=\frac{1}{p_1}\,[p_1^2Q_1^2-(b-p_2)^2]^{1/2}.
\end{aligned}
\right\}
\qquad (4.3-14)
$$

The constants m, k_1, k_2, k_3, k_4 are all real for $p_1>0$ and we have $k_1>k_2>1$. These facts can be verified directly, but they can also be concluded from section 4.23 and inequality (4.3—8). (In section 4.23 k_1 and k_2 are real and have the same meaning as here, and if $f(b)>0$ in Eq. (4.2—13), then $k_1>k_2>1$.) Therefore, n_1 and n_2' are real if β is real, and obviously $n_1=1+O(\beta)$, $n_2'=1+O(\beta)$, $\beta\to 0$. The inequalities $k_1n_1>k_2n_2'>1$ then hold, if β is sufficiently small.

Regarding z as a function of $t\equiv(\xi')^2$, the fact that z_t is singular of the order $O(x^{-1/2})$, $x\to 0$, at $x=1$ and $x=k_2n_2'$ is readily obtained from the first of the equations (4.3—12). Within this interval z_t is finite and positive (by a proper choice of sign), if β is sufficiently small. If we then put $z(1,\beta)=0$, the function $z=z(t,\beta)$ is defined for $t\in[1,k_2n_2']$ and can be obtained by means of quadrature. Moreover, $z(t,\beta)$ is strictly increasing and the inverse function $(\xi')^2\equiv\equiv t\equiv t(z,\beta)$ consequently exists and is defined on $z\in[0, z(k_2n_2',\beta)]$. Further, $t(z,\beta)$ may be re-defined as a periodic function of z with the period $2z(k_2n_2',\beta)$. Once we have determined $(\xi')^2$ as a function of z and the parameter β, $\gamma'=\gamma'(z,\beta)$ will be obtained directly from the last of Eqs. (4.3—12). Integrating once with respect to z we get $\varphi'=\varphi'(z,\beta)$ from the second of the equations (4.3—12). In order to be able to determine the functions $\xi'(z,\beta)$, $\varphi'(z,\beta)$, $\gamma'(z,\beta)$ we must calculate the functions G_2 and G_3 that appear in Eq. (4.3—12), *e. g.* by computing their series expansions after powers of β. It would be very laborious to do so and we therefore restrict ourselves to the search for functions $\xi(z,\beta)$, $\varphi(z,\beta)$, $\gamma(z,\beta)$ satisfying conditions (c), (d) and (e) of section 4.31. In other words, we proceed to find a solution of problem (E1).

As a solution of the problem (E1) *we choose functions* $\xi(z, \beta)$, $\varphi(z, \beta)$, $\gamma(z, \beta)$ *such that they satisfy*

a) *The system*

$$\begin{aligned} &\xi^2=t,\ \xi>0,\\ &t_z^2=4p_1^2(1+m\beta)(t-1)(k_2n_2-t)(k_1n_1-t)(1+\beta t),\\ &t\varphi_z=b+p_2(t-1),\\ &\gamma=(1+b^2)^{1/2}-p_1(t-1)[1-\tfrac{1}{2}\beta(2Q_1-t)], \end{aligned} \tag{4.3—15}$$

where $t\in[1, k_2n_2]$; *and* n_2 *is determined by*

$$n_2=1+\beta\frac{k_1k_3}{k_1-k_2}. \tag{4.3—16}$$

Eqs. (4.3—13) *and* (4.3—14) *are valid for the other parameters. b is assumed to satisfy Eq.* (4.3—8).

b) *The initial values* Eq. (4.3—7), *i. e.*

$$t(0, \beta)=1,\ \varphi(0, \beta)=0. \tag{4.3—17}$$

In order to be sure that ξ, φ, γ do constitute a solution of (E1) we must show that the conditions Eq. (4.3—9) are satisfied everywhere in the domain of definition of the functions. At first glance it might appear evident that these conditions should be satisfied, since the system Eq. (4.3—15) is obtained from Eq. (4.3—12) by omitting terms of the order $O(\beta^2)$, $\beta\to 0$, but owing to the singularities of the integrand in the integral which determines $z=z(t, \beta)$ things become somewhat complicated. We therefore discuss this question in greater detail in Appendix A1.

After some calculations we get *the following solution of problem* (E1):

$$\begin{aligned} &\xi^2(z, \beta)=1+(\varkappa_2-1)(1+\beta)\mathrm{sn}^2(\lambda z)\,[1+\beta\varkappa_2-\beta(\varkappa_2-1)\mathrm{sn}^2(\lambda z)]^{-1},\\ &\varphi(z, \beta)=z[p_2(1+\beta)-\beta b]+(b-p_2)(1+\beta)\,\frac{1}{\lambda}\,\Pi(\lambda z, \delta^2),\\ &\gamma(z, \beta)=(1+b^2)^{1/2}-p_1(\xi^2-1)\left[1-\tfrac{1}{2}\beta(2Q_1-\xi^2)\right], \end{aligned} \tag{4.3—18}$$

where $\xi>0$ *and* $z\in[a, b]\subset(-\infty, \infty)$.

The parameters λ, k, δ^2 appearing in Eq. (4.3—18) are determined by

$$\lambda=\lambda(\beta)=\varkappa_0[(\varkappa_1-1)(1+\beta\varkappa_2)]^{1/2},$$

$$k^2=k^2(\beta)=(\varkappa_2-1)(1+\beta\varkappa_1)(\varkappa_1-1)^{-1}(1+\beta\varkappa_2)^{-1},\ k>0, \qquad (4.3-19)$$

$$\delta^2=\delta^2(\beta)=(\varkappa_2-1)(1+\beta\varkappa_2)^{-1},$$

where k is the modulus of $\mathrm{sn}(\lambda z)=\mathrm{sn}(\lambda z, k)$ and $\Pi(\lambda z, \delta^2)=\Pi(\lambda z, \delta^2, k)$. For the parameters $\varkappa_0$, $\varkappa_1$, $\varkappa_2$ we have

$$\varkappa_0^2=\varkappa_0^2(\beta)=p_1^2(1+m\beta)=p_1^2[1-\beta(k_1+k_4)],\ \varkappa_0>0,$$

$$\varkappa_1=\varkappa_1(\beta)=k_1n_1=k_1[1-\beta k_2k_4(k_1-k_2)^{-1}], \qquad (4.3-20)$$

$$\varkappa_2=\varkappa_2(\beta)=k_2n_2=k_2[1+\beta k_1k_3(k_1-k_2)^{-1}],$$

where k_1, k_2, k_3, k_4 are given by Eq. (4.3—14) which also determines Q_1. If the four parameters β, p_1, p_2, b are given, then the solution Eq. (4.3—18) of problem (E1) is completely determined. As was shown in a previous discussion $\varkappa_1>\varkappa_2>1$ and consequently $0<k<1$ for sufficiently small values of β.

For $\beta=0$ the solution of (E1) turns into the solution of (I1). This trivial result is readily perceived from Eqs. (4.3—18) and (4.2—14).

By the method applied here we have obtained an approximation $\xi(z, \beta)$ of the function $\xi'(z, \beta)$, where ξ' is determined by Eq. (4.3—12) and the initial value $\xi'(0, \beta)=1$. The approximation is true up to terms linear in β. It is also possible to obtain approximations of higher order, since we can write the first equation of Eq. (4.3—12)

$$(\xi'\xi_z')^2=Q_n(\xi', \beta)(1+\beta^nH),$$

where Q_n is a polynomial in $(\xi')^2$ the degree of which is increasing with n, and $H=H(\xi', \beta)$ is analytic in ξ' and β. Let $\xi=\xi(z, \beta)$ be the nth order approximation of $\xi'(z, \beta)$ with respect to β. As previously for $n=1$, we can determine ξ from the differential equation

$$(\xi\xi_z)^2=Q_n(\xi, \beta)$$

and the initial value $\xi(0, \beta)=1$. We obtain $\xi(z, \beta)$ as the inverse function of a hyperelliptic integral the class of which depends on n, *cf.* [B2], p. 252.

4.33. The Solution of Problem (E2)

We start by considering the functions $\xi'(z, \beta)$, $\varphi'(z, \beta)$, $\gamma'(z, \beta)$ which satisfy Eqs. (4.3—4) and (4.3—6), just as we did for problem (E1). In this case the string curve intersects the axis of rotation ($\xi=0$); the constant B in Eq. (4.3—6) must then be zero and we get $\varphi_z=p_2=$ $=$const. β is a small parameter, so we conclude, as for problem (I2) in section 4.23, that A is a constant such that ξ'_z, regarded as a function of ξ' and β, is physically significant for $\xi'\in[-\xi'_2, \xi'_2]$. At the end points of the interval $\xi'_z=0$. Without loss of general validity we can take $\xi'_2=1$, and we then let the characteristical length a be the greatest distance between any string point and the axis of rotation. We also let the origin of our cylindrical coordinate system be a point of intersection between the string curve and the axis of rotation, and the conditions Eqs. (4.3—10) and (4.3—11) are then satisfied. After some calculation we get the following system from Eqs. (4.3—4) and (4.3—6), a system which (*in case of small strain*) is valid for *any steady rotatory motion of an elastic string intersecting the axis of rotation*:

$$\begin{aligned}
&(\xi'\xi'_z)^2=p_1^2(\xi')^2(1-(\xi')^2)(2Q-(\xi')^2)(q+\beta(\xi')^2)(1+\beta^2G_4),\\
&\varphi'_z=p_2, \qquad\qquad (4.3—21)\\
&\gamma'=(1+p_2^2)^{1/2}-p_1((\xi')^2-1)[1-\tfrac{1}{2}\beta(2Q-(\xi')^2)]+\beta^2G_2,
\end{aligned}$$

where $(\xi')^2\in[-1,1]$; $G_2=G_2(\xi', \beta)$ and $G_4=G_4(\xi', \beta)$ are analytic in ξ' and β. The parameters Q and q are determined in β, p_1, p_2 by

$$\begin{aligned}
2Q&=1+\left(\frac{p_2}{p_1}\right)^2+\frac{2}{p_1}(1+p_2^2)^{1/2},\\
q&=1-\beta\left(1+\frac{1}{p_1}(1+p_2^2)^{1/2}\right).
\end{aligned} \qquad (4.3—22)$$

It should be observed that $2Q>1$, since by definition $p_1>0$, *cf.* Eq. (3.1—9). Then, for small values of β, $\xi'_z\neq 0$ for $\xi'\in(-1,1)$, and we may take $\xi'_z>0$ in that interval. According to Eq. (4.3—10) we put $z=0$ for $\xi'=0$, and we can obtain $z=z(\xi', \beta)$ from Eq. (4.3—21) by means of quadrature. $z(\xi', \beta)$ would be strictly increasing, and it consequently does possess an inverse $\xi'=\xi'(z, \beta)$. From the last of the equations (4.3—21) we obtain $\gamma'=\gamma'(z, \beta)$. $\varphi'=\varphi'(z, \beta)$ is readily derived from

Eq. (4.3—21) and the condition Eq. (4.3—10). Since the evaluation of $\xi'(z, \beta)$ in this way would be a laborious procedure, we restrict ourselves to search for such solutions of the problem (E2) as specified in section 4.31.

As a solution of problem (E2) *we choose functions* $\xi(z, \beta)$, $\varphi(z, \beta)$, $\gamma(z, \beta)$ *such that they satisfy*

a) *The system*

$$\begin{aligned} &\xi^2 = t, \\ &t_z^2 = 4p_1^2 t(1-t)(2Q-t)(q+\beta t), \\ &\varphi_z = p_2, \\ &\gamma = (1+p_2^2)^{1/2} - p_1(t-1)\left[1-\tfrac{1}{2}\beta(2Q-t)\right], \end{aligned} \tag{4.3—23}$$

where $t \in [0,1]$ *and the parameters* Q *and* q *are the same as in* Eq. (4.3—22).

b) *The initial values*

$$t(0, \beta) = \varphi(0, \beta) = 0. \tag{4.3—24}$$

In order to constitute a solution of the problem (E2) the functions ξ, φ, γ must also satisfy the conditions Eq. (4.3—9). The proof might be carried out in the same way as for problem (E1), *cf.* Appendix A1.

After some calculation we get *the following solution of problem* (E2):

$$\begin{aligned} &\xi(z, \beta) = \mathrm{sn}(\lambda z)\left[1-\beta\,\mathrm{cn}^2(\lambda z)\right]^{1/2}, \\ &\varphi(z, \beta) = p_2 z, \\ &\gamma(z, \beta) = (1+p_2^2)^{1/2} + p_1(1-\xi^2)\left[1-\tfrac{1}{2}\beta(2Q-\xi^2)\right], \end{aligned} \tag{4.3—25}$$

where $z \in [a, b] \subset (-\infty, \infty)$.

The elliptic modulus k and the parameters λ and Q are determined by

$$\begin{aligned} &k = (2Q)^{-1/2}[1+\beta(2Q-1)]^{1/2}, \\ &\lambda = p_1(2Q)^{1/2}\left[1-\frac{1}{2}\,\frac{\beta}{p_1}\,(1+p_2^2)^{1/2}\right], \\ &2Q = 1 + \left(\frac{p_2}{p_1}\right)^2 + \frac{2}{p_1}\,(1+p_2^2)^{1/2}. \end{aligned} \tag{4.3—26}$$

If the three parameters β, p_1, p_2 are given, then the solution Eq. (4.3—25) of the problem (E2) is completely determined.

For $\beta=0$ the solution of (E2) turns into the solution of (I1), as might be expected.

4.34. A Special Case of Problem (E1)

On excluding condition Eq. (4.3—8), the problem (E1) may have a solution for which the string curve is a *circular helix* at certain values of the parameters p_1, p_2 and b.

Introducing the initial values Eq. (4.3—7), we find that the string curve is a helix, if p_1, p_2 and b are such that the equation

$$\tfrac{1}{2}f(b)(1+b^2)^{-1/2}+\beta\left[(1+b^2)^{1/2}+\tfrac{1}{2}p_2^2p_1^{-1}\right]- \\ -\beta^2G_1((1+b^2)^{1/2},\beta)=0 \qquad (4.3-27)$$

holds true. In Eq. (4.3—27) $f(b)$ is the same function as in Eq. (4.3—8), and $G_1(\gamma, \beta)$ is the same function as in Eq. (4.3—4), *i. e.* G_1 is analytic in γ and β, and $G_1=O(1)$, $\beta\to 0$. The equation (4.3—27) means that $\xi_{zz}=0$ for $\xi=1$, $\varphi_z=b$, $\gamma=(1+b^2)^{1/2}$, and it is easily derived from the first of the equations (3.2—4) and also from Eq. (4.3—1). Eq. (4.3—27) may also be considered to be the condition for $k_2n_2'=1$ in Eq. (4.3—12). After some elementary calculations we find

$$b=b_0+\beta b_1+O(\beta^2),\quad \beta\to 0, \qquad (4.3-28)$$

where b_0 and b_1 satisfy

$$f(b_0)=0, \\ b_1=-\frac{1+b_0^2}{4b_0}\,[b_0^2+2p_1(1+b_0^2)^{1/2}]^2\,[2p_1+b_0^2(1+b_0^2)^{1/2}]^{-1}. \qquad (4.3-29)$$

$f(b_0)=0$ always has one positive and one negative root (*i. e.* for any p_2 and $p_1>0$), *cf.* section 4.23.[35]

There is one value of b_1, of course, for each of the roots of $f(b_0)=0$, and Eq. (4.3—29) shows that they are both finite. In the case of an elastic string we may regard b_1 as a linear correction to the values $b=b_0$ which we obtained for an inextensible string. There is no difficulty in

[35] It should be noted that b_1 is not the same constant here as in section 4.23, where it is the positive root of $f(b)=0$.

calculating higher order terms in the series expansion of b after powers of β (the elastic parameter).

To sum up, we may claim that the string curve may be a helix for any pair of permissible values of p_1 and p_2, if b assumes either of the two values satisfying Eq. (4.3—27). These two values of b are different in sign, and if b is positive the curve will be a right-handed helix and *vice versa*. b is the cotangent of the helix angle.

4.35. Discussion of the Solutions of Problems (E1) and (E2)

The expressions Eq. (4.3—18) and Eq. (4.3—25) give the solutions to the basic problems (E1) and (E2) for the steady rotatory motion of an elastic string in the case of small strain and the absence of external forces. The solutions are correct up to terms linear in the elastic parameter β. The four parameters, β, p_1, p_2, b, constitute the solution of (E1); the parameter b does not appear in the solution of (E2).

Compared with the solutions of the corresponding problems for the inextensible string, (I1) and (I2), the solutions of (E1) and (E2) are more complicated. In principle, however, the difficulties connected with the determination of the string properties starting from physical boundary conditions are the same regarding both the elastic string and the inextensible string.

The evaluation of the physical string properties may be carried out in a manner similar to that used in section 4.24. The determination of the total string mass in an arbitrary interval on the string curve, *i. e.* the evaluation of the integral $\int m_0 a \varrho \gamma dz$, will cause no difficulties; $\varrho = \varrho(\gamma)$ is given by Eq. (4.3—4) and $\gamma = \gamma(\xi)$ by Eq. (4.3—15) or (4.3—23). (This particular problem does not occur in case of an inextensible string since in that case the string mass is proportional to the length of the string curve.)

The physical string properties will be completely determined if we are able to determine the parameters p_1, p_2, b (or, equivalently, a, T_0, b) on condition that the *a priori* constants m_0, ω, H, k are known. It should be observed that the elastic parameter β can be determined in p_1 and p_2 (and the *a priori* constants) by means of Eq. (4.3—3); the latter equation also gives the connection between a, T_0 on the one hand and p_1, p_2 on the other. In the case the string curve intersects the axis of rotation, problem (E4), parameter b disappears.

There are consequently *three* parameters (*two* in the case of intersection) to be determined from the boundary conditions and other available physical conditions. This situation is quite similar to that occurring for the inextensible string. It is also to be expected that for a particular string problem we can obtain a first order approximation of the solution, if we regard the string as inextensible. It must be pointed out, however, that the parameter values p_1, p_2, b, obtained in this way, must be modified when elasticity is taken into account.

4.4. The Steady Rotatory Motion Without Tangential Velocity of an Elastic String

When the string rotates without tangential velocity, an arbitrary string point describes a circle which has its centre on the z axis. It is evident from section 3.1 that $u=0$, $\overline{T}=T$ hold true everywhere on the string, or, in dimensionless variables, $\mu=0$, $\bar{\sigma}=\sigma$. When no external forces act on the string, it follows from section 3.2 that $\bar{\sigma}=\sigma=\text{const.}$ and as previously we can take $\bar{\sigma}=1$. This means that the string parameter T_0 is the component of the string tension vector on the z axis. From Eqs. (3.2—4) and (3.2—7) the following equations then result

$$\begin{aligned}
&\exp\left(\frac{\beta}{p_1}\gamma\right)+\beta\xi^2=A;\\
&\xi^2\varphi_z=B, \qquad\qquad (4.4-1)\\
&\xi_z^2+(\xi\varphi_z)^2+1=\gamma^2.
\end{aligned}$$

In these equations $A>0$ and B are arbitrary real constants and p_1, β are parameters, *cf.* Eq. (4.3—3). It should be noted that no restriction is laid upon the magnitude of the strain in the string. Two different cases now appear, corresponding to (E1) and (E2) (the equivalent problems in case of small strain and non-zero tangential velocity).

1) $B\neq 0$; the string curve has no point of intersection with the axis of rotation. We conclude from Eq. (4.4—1) that in this case A and B must be such that

$$\begin{aligned}
&\xi_z=0 \quad \text{if } \xi=\xi_1 \quad \text{or } \xi=\xi_2,\\
&\xi_z^2>0 \quad \text{if } \xi\in(\xi_1,\,\xi_2),
\end{aligned}$$

where $\xi_2 > \xi_1 > 0$. We can take $\xi_1 = 1$, as we did in problems (I1) and (E1), and it then implies that the characteristical length a in the problem is the smallest distance between the string curve and the axis of rotation. The basic one-point problem which results from $B \neq 0$ will be called (E3).

2) $B = 0$; the string curve intersects the axis of rotation in one or more points. We conclude from Eq. (4.4—1) that in this case $\varphi = \text{const.}$, *i. e.* the string describes a plane curve. Except for the trivial solution $\xi \equiv 0$, a real solution of Eq. (4.4—1) can only exist, if A is such that

$$\xi_z^2 > 0 \quad \text{for } \xi \in (-\xi_1, \xi_1),$$
$$\xi_z = 0 \quad \text{for } |\xi| = \xi_1.$$

We put $\xi_1 = 1$ and the characteristical length a is then the greatest distance between the string curve and the axis of rotation. The basic one-point problem connected with this kind of motion will be called (E4).

4.41. Problem (E3)

(E3) Problem: *To find functions* $\xi(z)$, $\varphi(z)$, $\gamma(z)$ *defined on some interval* (a_1, a_2), $a_1 < 0 < a_2$ *and constants A and B such that the following conditions obtain*

a) *The system Eq.* (4.4—1) *is satisfied.*

b) *The initial conditions*

$$\xi(0) = 1, \quad \varphi(0) = 0, \quad \varphi_z(0) = b, \quad \gamma(0) = (1 + b^2)^{1/2} \qquad (4.4-2)$$

are satisfied. b is a real constant which satisfies the inequality[36]

$$f(b) \equiv b^2 - 2p_1(1 + b^2)^{1/2} \exp\left(-\frac{\beta}{p_1}(1 + b^2)^{1/2}\right) > 0. \qquad (4.4-3)$$

Otherwise b is an arbitrary quantity. p_1 and β are parameters of the system connected with the *a priori* constants m_0, ω, k and the characteristical quantities a and T_0 according to the expressions Eq. (4.3—3).

---

[36] This inequality proves that $\xi(0) = 1$ is a local minimum. The helix case $f(b) = 0$ is treated separately.

The solution of problem (E3) is easily obtained. Eqs. (4.4—1), (4.4—2) imply that $t(z)\equiv(\xi(z))^2$ must satisfy a separable differential equation

$$t_z^2=4g(t),$$

where

$$\begin{aligned} g(t)&=t\left[\frac{p_1}{\beta}\log(C-\beta(t-1))\right]^2-t-b^2, \\ C&=\exp\left(\frac{\beta}{p_1}(1+b^2)^{1/2}\right). \end{aligned} \tag{4.4—4}$$

It will be evident from previous discussions that $g(t)>0$ for $t\in(1, t_2)$, where t_2 depends on the parameters p_1, β and the initial value b. In the neighbourhood of $t=1$ and $t=t_2$ $g(t)$ satisfies

$$g(1+\varepsilon)=\varepsilon g_t(1)+O(\varepsilon^2),\quad \varepsilon\to+0,$$

$$g(t_2-\varepsilon)=-\varepsilon g_t(t_2)+O(\varepsilon^2),\quad \varepsilon\to+0,$$

where $g_t(1)>0$, $g_t(t_2)<0$. $z=z(t)$ can consequently be obtained by quadrature. Taking $z(1)=0$, *cf.* Eq. (4.4—2), we get

$$z(t)=\frac{1}{2}\int_1^t(g(\tau))^{-1/2}d\tau. \tag{4.4—5}$$

$z(t)$ is defined on $[1,t_2]$. We now introduce $L\equiv z(t_2)$, *i. e.*

$$2L=\int_1^{t_2}(g(\tau))^{-1/2}d\tau. \tag{4.4—6}$$

Now, since $z(t)$ is strictly increasing, it possesses an inverse $F(z)\equiv t(z)\equiv(\xi(z))^2$ which is defined on $[0, L]$. By the introduction of $F(-z)=F(z)$, $F(z+2L)=F(z)$, $F(z)$ is extended to be an even periodic function with the period $2L$. As a *solution of the problem* (E3) we then obtain

$$\xi(z)=(F(z))^{1/2},$$

$$\varphi(z)=b\int_0^z(F(\zeta))^{-1}d\zeta, \tag{4.4—7}$$

$$\gamma(z)=\frac{p_1}{\beta}\log[C-\beta(F(z)-1)],$$

where C is a constant determined by Eq. (4.4—4) and $F(z)=t(z)$ is the inverse of $z(t)$ which is defined by Eqs. (4.4—5) and (4.4—4).

The period $2L$ of $\xi(z)$ is determined by Eq. (4.4—6), where $t=t_2$ is the second zero of $g(t)$ (the first is $t=1$). The physical distance $x^1(z)=a\xi(z)$ between the string points and the axis of rotation is immediately obtained from Eq. (4.4—7), and the derivation of the string tension $T(z)$, and the arc length $s(z)$ of the string curve can easily be carried out, *cf.* section 4.24. It must be observed, however, that the integral defining $z(t)$ (the inverse of $F(z)$) is not an elementary one.

We now turn to the *helix case* which has been exluded so far by the condition Eq. (4.4—3). It is clear that the equation $f(b)=0$ has two solutions, $b=\pm b_1$.[37] If $b=\pm b_1$, we have

$$\begin{aligned} &\xi(z)\equiv 1, \\ &\varphi(z)=\pm b_1 z, \qquad\qquad (4.4-8) \\ &\gamma(z)\equiv(1+b_1^2)^{1/2}. \end{aligned}$$

In this case the string curve will obviously be a circular helix.

4.42. Problem (E4)

(E4) Problem: *To find functions $\xi(z)$, $\varphi(z)$, $\gamma(z)$, defined on some interval (a_1, a_2), $a_1<0<a_2$, and constants A and B such that the following conditions obtain:*

a) *The system (4.4—1) is satisfied.*

b) *The initial conditions*

$$\xi(0)=\varphi(0)=0 \qquad (4.4-9)$$

are satisfied.

c) *The condition*

$$\xi_z=0 \quad \text{for } \xi=1 \qquad (4.4-10)$$

is satisfied.

The solution of this problem is obtained by straight-forward calculation. As mentioned previously, the constant B must be zero

[37] If the string has a non-vanishing tangential velocity, the negative and positive roots are different in magnitude, owing to the acceleration of Coriolis.

and the string curve is a plane one, *i. e.* $\varphi=\varphi(z)=\text{const}$. *The solution of problem* (E4) *may be written*

$$\xi(z)=F_0(z),$$

$$\varphi(z)\equiv 0, \tag{4.4—11}$$

$$\gamma(z)=[1+F_0^2(z)]^{1/2},$$

where $F_0(z)$ is the inverse function of $z(\xi)$; the latter is defined by

$$z(\xi)=\int_0^\xi (g_0(\tau))^{-1/2}\,d\tau,$$

$$g_0(\tau)=\left[\frac{p_1}{\beta}\log\left(\exp\frac{\beta}{p_1}+\beta(1-\tau^2)\right)\right]^2-1. \tag{4.4—12}$$

By the extension $F_0(-z)=-F_0(z)$, $F_0(2L_0-z)=F_0(z)$, where

$$L_0=z(1)=\int_0^1 (g_0(\tau))^{-1/2}\,d\tau, \tag{4.4—13}$$

we may take the domain of definition of the solution Eq. (4.4—11) to be the whole z axis. $\xi(z)=F_0(z)$ is obviously an odd, periodic function with the period $4L_0$.

4.43. **Discussion of the Solutions of Problems (E3) and (E4)**

Fundamental theorems on ordinary differential systems indicate that the solutions of problems (E3) and (E4) are unique and analytic in the independent variable z, the elastic parameter β, the parameter p_1, and the initial value b. The solutions are of general validity in the sense that, if a problem concerning a steady rotatory motion without tangential velocity of a string does have a solution, the latter is then contained in the solution of either problem (E3) or problem (E4).[38]

No restrictions have been imposed with regard to the magnitude of the strain in the string, but the expressions obtained are naturally only valid for a string which is linearly elastic in our sense of the term, see section 1.25.

Evidently, other kinds of elasticity will give rise to different expres-

[38] On condition that no external forces act on the string in the domain in question.

sions, but if the strain in a point of the string is completely determined by the string tension in that point, then a system essentially identical to Eq. (4.4—1) is valid. It is only the function $\exp(\beta p_1^{-1}\gamma)$ in the first equation that has to be changed for another function of the variable γ. The function $\exp\left(\frac{\beta}{p_1}\gamma\right)=\frac{p_1}{\beta}\int\exp\left(\frac{\beta}{p_1}\gamma\right)\gamma_z dz$ comes from the third equation in the system Eq. (3.2—4) and change of the elastic condition simply leads to another function of γ (other than the exponential one) under the integral sign. The ultimate consequence of a change to another kind of elasticity (than the linear one) is that the functions $g(t)$ and $g_0(t)$ that occur in the solutions of problems (E3) and (E4) have to be changed for other related functions.

The practical difficulties in finding a solution of a particular string problem connected with problem (E3) or (E4), *i. e.* to find the characteristical quantities a, T_0 (or alternatively the parameters p_1, β, *cf.* Eq. (4.3—3)) and the initial value b, are related to those discussed in section 4.24. An additional complication is caused by the presence of non-elementary functions.

There must exist a close connection between the solutions of the problems (E1) and (E2) on the one hand, and (E3) and (E4) on the other, since, putting $p_2=0$ in the former case, we get approximations linear in β for the latter. It is also possible, of course, to obtain these approximations directly from the solutions of (E3) and (E4). We cannot, however, expand the functions $(g(\tau))^{-1/2}$ or $(g_0(\tau))^{-1/2}$ after powers of β and then integrate termwise, since the intervals of integration contain singular points. The solutions of (E3) and (E4), in the form stated here, are consequently not suitable in the case of problems involving small strain.

5. The Steady Rotatory Motion of an Inextensible String in the Case of Negative Reduced String Tension and in the Absence of External Forces

The symbols and notations used in this chapter are essentially the same as those in Chapter 4.

5.1. Introduction

There are two different kinds of steady rotatory string motion possible, as shown in section 3.22. The first one is well-known in textile mechanics, and is characterized by the fact the so-called reduced string tension $T - mu^2q^2$ is positive. This type of steady motion was studied in some detail in chapter 4. The second type, characterized by negative reduced string tension, has apparently not been observed or discussed earlier.

It will become clear that *motion with negative reduced string tension is possible only if the tangential velocity uq* (*cf.* section 1.22) *is higher than the highest circumferential velocity occurring in any real string point*, see Fig. VIII. By a real string point we mean a

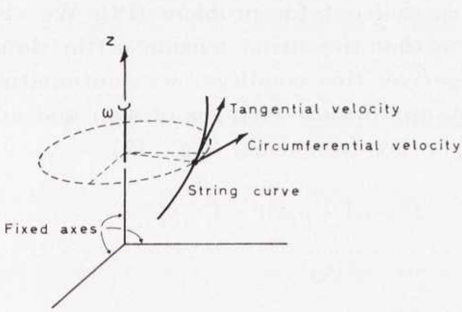

Fig. VIII. The broken curve (circle) is the path of a point on the string curve, fixed with respect to a co-rotating frame. It should be noted that this path does not coincide with the path of any string point. The circumferential velocity is tangential to the above curve.

point, where the string tension is not negative, *i. e.* $T \geq 0$. (String points with $T < 0$ may occur in formal solutions of string problems, but the solution has no physical significance in such points.)

The author suggests that motions where the above condition is satisfied do not occur in industrial practice, although they are easy to generate. As no theoretical objections have been advanced against the existence of the steady rotatory motion with negative reduced tension, the author has considered it proper briefly to investigate this type of motion in spite of the above-mentioned suggestion. We will only discuss motions analogous to those of problem (I2), *i. e.* motions of an inextensible string such that the string curve intersects the axis of rotation. For this kind of motion we must distinguish between two cases, as will become apparent later on. If we had to consider motions in which the string does not intersect the axis of rotation, *i. e.* problems analogous to (I1), matters would become more complicated. Moreover, problems for an elastic string with negative reduced string tension would involve tedious calculations.

5.2. The String Equations. The Two Cases

We shall start from the string equations in the form of Eq. (3.2—4). Since we shall deal solely with an inextensible string, we put $\varrho = \varrho(z) \equiv 1$, and just as in Chapter 4, we introduce $p_2 = 2p_1 h = 2p_1 \gamma \mu$. In accordance with remark (3) in section 3.22 we take $\bar{\sigma} = \text{const.} = -1$. Observing that a change of sign for $\bar{\sigma}$ in Eq. (3.2—4) has the same effect as a change of sign for p_1 and p_2, it becomes evident that we can apply the deductions carried out for problem (I2). We also have to take into consideration that the string tension σ (the dimensionless form) must not be negative; this condition was automatically satisfied in problem (I2). By appropriate changes of sign and adding the condition $\sigma \geq 0$, we then get from Eqs. (4.2—17)

$$\begin{aligned} \xi_z^2 &= (A + p_1 \xi^2)^2 - 1 - p_2^2 \xi^2, \\ \varphi_z &= -p_2, \\ \gamma &= A + p_1 \xi^2, \\ \sigma &= 2p_1 \gamma \mu^2 - 1 = p_2^2 \frac{1}{2p_1 \gamma} - 1 \geq 0, \end{aligned} \tag{5.2—1}$$

where $A>1$ is a constant of integration. It is true that *any* steady rotatory motion of an inextensible string must satisfy Eq. (5.2—1) provided that the string intersects the axis of rotation and that the external forces are neglected. We now have to consider two different cases, depending on the magnitude of the constant A. If A is sufficiently small compared to p_1 and p_2, ξ_z will become zero for two positive values of ξ^2, and we then get a situation similar to that in problem (I2). For large values of A ξ_z^2 will be positive for any (real) ξ^2, and we then get entirely different solutions of Eq. (5.2—1). A further complication is caused by the condition $\sigma \geq 0$.

The "critical" value of A will be called A_0; it is determined by

$$A_0 = \frac{p_1}{p_2^2} + \frac{1}{4}\frac{p_2^2}{p_1}. \tag{5.2—2}$$

$\sigma \geq 0$ implies $p_2^2 > 2p_1$, hence $A_0 > 1$. We now have the two cases:

a) $A_0 > A > 1$. In this case there are two values ξ_1 and ξ_2, $\xi_2 > \xi_1 > 0$, depending on p_1, p_2 and A, and such that $\xi_z^2 = 0$ for $|\xi| = \xi_1$, and $|\xi| = \xi_2$. It turns out that $\sigma > 0$ holds for $|\xi| \leq \xi_1$, and Eq. (5.2—1) is consequently physically significant for $|\xi| \leq \xi_1$, if $A_0 > A$.

b) $A > A_0$. In this case $\xi_z^2 > 0$ for every real ξ, but $\sigma \geq 0$ only holds for $|\xi| \leq \xi_3$ where ξ_3 depends on p_1, p_2 and A.

We call the basic boundary value problems connected with these two cases (IN2a) and (IN2b). I means inextensible string, N negative reduced string tension, and 2 that the string intersects the axis of rotation, *cf.* (I2). On considering the two cases for given values of p_1 and p_2 we find that at the point of intersection γ is smaller in case (a) than in case (b). Now, $\dfrac{1}{\gamma}$ is the direction cosine of the string tangent with respect to the z axis, and the string consequently intersects the z axis at a smaller angle in case (a) than in case (b).

5.21. **Problem (IN2a)**

We can without specialization take the smallest zero of ξ_z to be $\xi_1 = 1$. This means that the characteristical length a of the problem is chosen as the greatest distance between any string point and the axis of rotation.

(IN2a) Problem: *To find functions $\xi(z)$, $\varphi(z)$, $\gamma(z)$, $\sigma(z)$ defined on some interval $(-b, b)$, a constant A, and the parameters p_1, p_2 such that*

a) *The system Eq.* (5.2—1) *is satisfied.*

b) *The initial values*

$$\xi(0)=\varphi(0)=0 \tag{5.2—3}$$

are satisfied.

c) *The condition*

$$\xi_z=0 \quad \text{for} \quad \xi=1 \tag{5.2—4}$$

is satisfied.

The existence and uniqueness of the solution follows from basic theorems on differential systems, *cf.* section 4.22. The solution is readily obtained. From Eqs. (5.2—1) and (5.2—4) we get

$$A=(1+p_2^2)^{1/2}-p_1 \tag{5.2—5}$$

The inequality $\sigma\geq 0$ is satisfied for every ξ of interest, if

$$p_2^2\geq 2p_1(A+p_1)=2p_1(1+p_2^2)^{1/2}.$$

On solving this inequality with respect to p_2 we get

$$p_2^2\geq 2p_1[p_1+(1+p_1^2)^{1/2}]>4p_1^2. \tag{5.2—6}$$

A necessary condition for $\sigma\geq 0$ to hold is consequently that $|p_2|>2p_1$, or, expressed in physical quantities, it is necessary that the tangential velocity uq satisfies

$$|uq|=a\omega\,|h|=\frac{a\omega}{2p_1}\,|p_2|>a\omega.$$

$a\omega$ is the highest circumferential velocity of *any* point on the string, and this velocity must be lower than the tangential velocity (which is the same for every string point). Factoring the expression for ξ_z^2 in Eq. (5.2—1) we get by means of Eq. (5.2—5)

$$\xi_z^2=p_1^2\,(1-\xi^2)\left[1+\left(\frac{p_2}{p_1}\right)^2-\frac{2}{p_1}\,(1+p_2^2)^{1/2}-\xi^2\right].$$

It is evident that $\xi_z^2>0$ for $|\xi|<1$, and also that ξ_z^2 has a simple zero for $|\xi|=1$, if we exclude equality in Eq. (5.2—6). Using the results from problem (I2) in section 4.23 we get the following *solution of problem* (IN2a):

$$
\begin{aligned}
&\xi(z)=\operatorname{sn}\left(z\,\frac{p_1}{k}\right),\\
&\varphi(z)=-p_2 z,\\
&\gamma(z)=(1+p_2^2)^{1/2}-p_1\operatorname{cn}^2\left(z\,\frac{p_1}{k}\right),\\
&\sigma(z)=p_2^2\,\frac{1}{2p_1\gamma(z)}-1,
\end{aligned}
\tag{5.2—7}
$$

where $z\in(-\infty,\infty)$ and the modulus k of the elliptic functions is determined by

$$
k=\left[1+\left(\frac{p_2}{p_1}\right)^2-\frac{2}{p_1}\,(1+p_2^2)^{1/2}\right]^{-1/2}. \tag{5.2—8}
$$

The restriction Eq. (5.2—6) must be satisfied by p_1 and p_2. If equality holds in Eq. (5.2—6) then $\xi\to 1\Rightarrow z\to\infty$ as then the elliptic modulus $k\to 1$. The solution Eq. (5.2—7) is closely related to the solution of problem (I2), but has an important change of sign for the last term in the expression for k. We can transform the solution into physical variables without difficulty and also determine the length of the string curve, as we did in section 4.24 for problem (I2).

5.22. Problem (IN2b)

We can without specialization take $\xi_3=1$. This means that the characteristical length a of the problem is the distance between the axis of rotation and that point on the string, where the string tension T (or σ in the dimensionless form) becomes *zero*.

(IN2b) Problem: *To find functions $\xi(z)$, $\varphi(z)$, $\gamma(z)$, $\sigma(z)$, defined on some interval $(-b, b)$, a constant A, and the parameters p_1 and p_2 such that*

a) *The system Eq.* (5.2—1) *is satisfied.*

b) *The initial values*

$$
\xi(0)=\varphi(0)=0 \tag{5.2—3}
$$

are satisfied.

c) *The conditions*

$$\sigma=0 \text{ and } \xi_z=\varepsilon>0 \text{ for } \xi=1 \tag{5.2—9}$$

are satisfied.

The condition $\xi_z=\varepsilon>0$ in Eq. (5.2—9) has been introduced in order to guarantee that $\xi(z)$ has no turning point or, equivalently, that $A>A_0$. The solution of the problem is readily obtained. The constants A and ε can be determined in terms of the parameters p_1 and p_2.

From Eqs. (5.2—1) and (5.2—9) we get

$$\begin{aligned} A&=p_2^2\frac{1}{2p_1}-p_1, \\ \varepsilon^2&=p_2^4\frac{1}{4p_1^2}-p_2^2-1. \end{aligned} \tag{5.2—10}$$

From the last expression we have

$$p_2^2=2p_1[p_1+(1+p_1^2+\varepsilon^2)^{1/2}]>2p_1[p_1+(1+p_1^2)^{1/2}]>4p_1^2. \tag{5.2—11}$$

A necessary condition for (IN2b) to have a solution is then that the tangential velocity of the string is higher than the highest circumferential velocity occurring in any real point of the string (*i. e.* any point with $\sigma\geq 0$). This condition also appeared in problem (IN2a). After some calculation we get *the following solution of problem* (IN2b)

$$\begin{aligned} &\xi(z)=C\left[\frac{1-\operatorname{cn}\zeta(z)}{1+\operatorname{cn}\zeta(z)}\right]^{1/2}, \quad z>0, \\ &\xi(-z)=-\xi(z), \\ &\varphi(z)=-p_2 z, \\ &\gamma(z)=p_2^2\frac{1}{2p_1}-p_1[1-\xi^2(z)], \\ &\sigma(z)=p_2^2\frac{1}{2p_1\gamma(z)}-1, \\ &\zeta(z)=2p_1Cz, \quad z\in[-b, b]. \end{aligned} \tag{5.2—12}$$

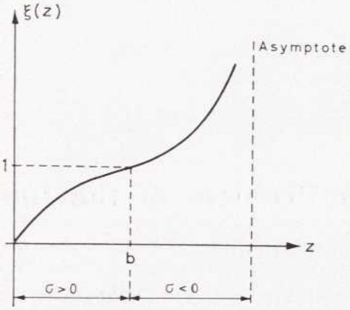

Fig. IX. Graph showing the main feature of the string curve in the case (IN2b). The vertical asymptote of $\xi(z)$ is $z=(p_1C)^{-1}K(k)$. The point $z=b$, $\xi=1$ is a point of inflection.

In these expressions we have the elliptic modulus k. The constants k, C, and b are determined by

$$\begin{aligned} k&=[\tfrac{1}{2}(1+C^{-2})]^{1/2}, \\ C^4&=1+\left(\frac{\varepsilon}{p_1}\right)^2,\ C>1, \\ b&=\frac{1}{2p_1C}F(\varphi_1), \\ \varphi_1&=\arccos\left(\frac{C^2-1}{C^2+1}\right), \end{aligned} \tag{5.2—13}$$

where $F(\varphi_1)=F(\varphi_1, k)$ is the incomplete elliptic integral of the first kind. The constant ε is determined in terms of p_1, p_2 by Eq. (5.2—10), and p_1, p_2 are restricted by the fact that ε must be real. The transformation of the solution into physical variables is easily carried out, *cf.* section 4.24. It is not difficult to show that the second derivative ξ_{zz} is negative for $z\in(0, b)$ and zero at $z=b$; the delineation of the function $\xi(z)$ is given by Fig. IX.

6. A Vibration Problem of the Inextensible String

Symbols and conventions:

y_i, $i=1, \ldots, 7$ Dimensionless variables describing a small time-dependent motion of the string and defined as in Chapter 3 for $i=1, \ldots, 5$ and by Eq. (6.2—3) for $i=6,7$.

z Dimensionless, independent variable, cylindrical coordinate defined by Eq. (3.1—6).

τ Independent variable, dimensionless time, defined by Eq. (3.1—6).

ξ Variable of a steady rotatory motion, dimensionless radial distance, defined by Eq. (3.1—8).

γ Variable of a steady rotatory motion. $\frac{1}{\gamma}$ is the direction cosine of the string tangent with respect to the axis of rotation (the z axis).

$p_1=2p$ String parameter, defined as in Chapter 3. p is used as perturbation parameter for the vibration problem.

x Independent variable used instead of z and defined by Eq. (6.2—2).

$g_\nu(x,p)$, $\nu=1, \ldots, 6$ Functions defined by Eqs. (6.2—6) and (6.2—7).

$g_{\nu,i}(x)$, $i=1, 2, 3, \ldots$ Coefficients in the series expansion of $g_\nu(x, p)$ after powers of p.

$\alpha=\alpha_\nu=\alpha_\nu(p)$, $\nu=1, 2, 3, \ldots$ Dimensionless natural, circular frequency of the vibratory motion, defined by Eq. (6.2—9). Ordinal number ν usually omitted.

$\alpha_0 \equiv \alpha_{1,0}$ First term in the series expansion of $\alpha(p)$ after powers of p, zero order approximation.

α_i, $i=1, 2, 3, \ldots$ General term of the above expansion, pertubation of the order i.

a_1, a_2 End points of the basic interval of the x axis. $[a_1, a_2]$ is the interval of definition of the vibratory motion.

$A_\mu(x)$, $B_\mu(x)$, $\mu=1, 3, 6, 7$ Dependent variables of the (separated) vibratory motion, see Eq. (6.2—9).

$Y_\mu(x)=Y_\mu(x, p)$, $\mu=1, 3, 6, 7$ Dependent variable used instead of A_μ and B_μ.

$Y_{\mu,0}(x)$ First term in the series expansion of $Y_\mu(x, p)$ after powers of p, zero order approximation.

$Y_{\mu,i}(x)$, $i=1, 2, 3, \ldots$ General term of the above expansion, pertubation of the order i.

D, L, M Differential operators, depending on p, defined by Eq. (6.2—12).

$L_0=M_0$ Differential operator obtained from L by putting $p=0$, first order 'approximation' of L and M.

L_i, M_i, $i=1, 2, 3, \ldots$ 'Pertubation' of the order i of the differential operators L and M. L_i, M_i are first order differential operators.

C_0 Boundary value, see Eq. (6.2—14).

λ_n, $n=1, 2, 3, \ldots$ Eigenvalue of a Sturm-Liouville eigenvalue problem, defined by Eq. (6.3—13).

$u_n=u_n(x)$ Orthonormal eigenfunctions of the above problem.

$a_{0,n}, b_{0,n}$ Coefficients in the series expansions of $Y_{1,0}(x)$ and $Y_{6,0}(x)$ after the eigenfunctions $u_n(x)$, see Eq. (6.3——15).

d_n Expansion coefficient, defined by Eq. (6.3—16).

E_n, H_n Functions of λ_n and α_0, defined by Eq. (6.3—16).

$(f, g)=\int_{a_1}^{a_2} fg dx$ Inner product of the functions $f(x)$, $g(x)$.

$S(\alpha_0)$ Function of α_0, defined by Eq. (6.3—20).

$\Phi_{\eta,0}$, $\eta=1, \ldots, 4$ Functions of x and α_0, see Eq. (6.3—23).

$Y'_{\mu,1}(x)$, $Y''_{\mu,1}(x)$, $\mu=1, 6$ Functions defined by Eq. (6.3—25).

$a'_{1,n}$ Expansion coefficient, defined by Eq. (6.3—30).

$T(\alpha_0)$ Function of α_0 defined by Eq. (6.3—33).

$\Phi_{\eta,i}$, $\eta=1, \ldots, 4$; $i=1, 2, 3, \ldots$ Functions of x and of perturbations α_k up to the order i, see Eq. (6.3—34).

Symbols not included in the above list will occur now and then in the text.

Greek subscripts are used to denote numbering of various functions, Latin subscripts are used to denote ordinal numbers of approximations, perturbations, and also for eigenvalues, eigenfunctions and expansion coefficients. Latin superscripts denote powers.

The summation convention is used for Latin but not for Greek indices.

6.1. Introduction

In this chapter we shall study a problem of vibration in connection with the linear differential system Eq. (3.2—10). In this system the independent variables are z (the dimensionless axial distance of a point on the string curve) and τ (the dimensionless time). There are five dependent variables $y_i(z, \tau)$ and the system is valid for a small time-dependent motion superimposed on a special kind of steady rotatory motion of an inextensible string. The coefficients of the system are constituted by the periodic functions $\xi(z)$, $\gamma(z)$, and a real, positive parameter p_1, they consequently do not depend on the time τ.

The functions $\xi(z)$ and $\gamma(z)$ are determined by the steady rotatory motion on which the time-dependent motion in question is superimposed. The steady motion was studied in a more general form in Chapter 4, and the functions $\xi(z)$ and $\gamma(z)$ may be taken from Eq. (4.2—20) if we put $p_2=0$; in that equation it means that the string has no tangential velocity at the steady motion. The string curve of the steady motion is a plane one and it intersects the axis of rotation at equidistant points.

The system Eq. (3.2—10) is linear and homogeneous and has coefficients with simple properties; yet most problems of technical interest connected with that system are quite complicated. This will be evident, if we recall that the system is not hyperbolic (the string is inextensible) and that curves $\tau=$const. are characteristics, *cf.* Chapter 2. We shall therefore confine ourselves to a simple vibration problem, which we will describe more precisely in section 6.2.

In order to be able to obtain results of practical interest we must restrict our investigations to the case of small values of the parameter p_1. It will be clear from Eq. (3.1—9) that a small value of p_1 means that the string tension T is large everywhere compared to the quantity

$m_0 a^2 \omega^2$.[39] This is actually true in many problems of practical interest and our specialization is therefore justified. The treatment of the vibration problem for small values of the parameter p_1 is characterized by the use of the method of *separation of variables* and then the application of *a pertubation method* to the resulting system of ordinary, linear differential equations.

The system Eq. (3.2—10) must have a meaning for $p_1=0$, if the perturbation method is to work. It should be observed that the functions $\xi(z)$ and $\gamma(z)$, which constitute the coefficients of the system, depend explicitely on p_1, *cf.* Eq. (4.2—19). It can easily be verified that the (real) period of the functions $\xi(z)$ and $\gamma(z)$ is $4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind. A simple investigation of Eqs. (4.2—19) and (4.2—20) shows that for $p_2=0$, $K=O(p_1^{-1/2})$, $p_1 \to 0$. This property of K has the physical significance that if the string tension tends to infinity, then the ratio between the period and the amplitude (which is the characteristical length) of the string curve (in the steady motion) also tends to infinity. In order to overcome this difficulty we shall introduce $x=\mathrm{am}\left(z\dfrac{p_1}{k}\right)$ as a new independent variable instead of z in the system Eq. (3.2—10). $\mathrm{am}(u)=\mathrm{am}(u, k)$ is the elliptic amplitude function, *cf.* [B2]. It will readily be seen that ξ, γ, and the coefficients of the system will be trigonometric functions of x with the period 2π.

We shall not concern ourselves with questions regarding the existence and uniqueness of the solution of the vibration problem, neither shall we discuss the convergence of the perturbation method applied. Consequently, we cannot affirm that the problem is properly stated; the results obtained are, of course, merely formal.

It may be remarked, however, that circumstances are favourable in so far as the coefficients of the ordinary differential system to be treated by the perturbation method will be analytic in the perturbation parameter $p=\frac{1}{2}p_1$ and the independent variable x, provided that p is sufficiently small.

---

[39] It will readily be observed that small values of p_1 correspond to large values of T_0, and by Eq. (3.1—8) to large values of T; it should be noted that $\dfrac{q}{a}=\gamma \geq 1$ and that a is the greatest distance between the z axis and any point on the string curve (in the steady motion).

6.2. Transformation of the Basic System The Vibration Problem

The differential system Eq. (3.2—10) shall now be transformed into a more suitable form. Let us introduce the parameter p defined by

$$p=\tfrac{1}{2}p_1, \tag{6.2—1}$$

the independent variable x defined by

$$x=am\left(z\frac{p_1}{k}\right), \tag{6.2—2}$$

and the dependent variables $y_6(x,\tau)$ and $y_7(x,\tau)$ defined by

$$\begin{aligned} y_6&=y_2\xi,\\ y_7&=y_5\gamma\sqrt{p}. \end{aligned} \tag{6.2—3}$$

The following condition must hold in order to ensure the existence of $y_5(x,\tau)$ when $p\to 0$

$$y_7=O\left(\sqrt{p}\right),\quad p\to 0. \tag{6.2—4}$$

Keeping in mind that the parameter p_2 of Eq. (4.2—20) is zero (the string has no tangential velocity in the steady motion on which the time-dependent motion is superimposed), we get the following system after some elementary calculation

$$\begin{aligned} &y_{1xx}+g_1(y_1-y_{1\tau\tau}+2y_{6\tau})+3pg_2\sin x\cos x\, y_{1x}-g_1\sin x\, y_3=0,\\ &y_{6xx}+g_1(y_6-y_{6\tau\tau}-2y_{1\tau})-pg_3\sin x\cos x\, y_{6x}-4g_4\cos x\, y_7=0,\\ &y_{7x}+2pg_5\cos x\, y_{1x\tau}=0,\\ &y_{3x}-2g_4y_{7\tau}=0,\\ &y_4-4pg_6\cos x\, y_{1x}=0. \end{aligned} \tag{6.2—5}$$

In this system $g_\nu=g_\nu(x,p)$, $\nu=1,\ldots,6$ are functions of the independent variable x and the parameter p. They are defined by the following expressions

$$\begin{aligned} &g_1=G_1^{-1}G_2, && g_2=(G_1G_2)^{-1}(1+\tfrac{2}{3}p\cos^2 x),\\ &g_3=G_1^{-1}, && g_4=(G_1)^{-1/2},\\ &g_5=G_1^{1/2}G_2^{-1}, && g_6=G_1G_2^{-1}, \end{aligned} \tag{6.2—6}$$

where

$$G_1 = G_1(x, p) = 1 + p \cos^2 x,$$
$$G_2 = G_2(x, p) = 1 + 2p \cos^2 x. \tag{6.2—7}$$

The functions $g_\nu(x, p)$ are analytic in x and p for every x, if p is sufficiently small. It is then clear that all the functions g_ν may be expanded after powers of p, and that the series obtained in this way are absolutely and uniformly convergent and posses a common, finite radius of convergence for every x. Moreover, the functions g_ν satisfy

$$g_\nu(x, p) = 1 + O(p), \ p \to 0, \ \nu = 1, \ldots, 6. \tag{6.2—8}$$

The first four equations of the system (6.2—5) constitute a system of linear, homogeneous partial differential equations for the functions $y_\mu(x, \tau)$, $\mu = 1,3,6,7$.

The last equation of the system (6.2—5) defines the function $y_4(x, \tau)$ and can be excluded from the system. The coefficients of the differential system are trigonometric functions and (regular) analytic in the whole complex x plane, if p is sufficiently small. As mentioned previously the differential system is not hyperbolic. The unknown functions $y_\mu(x, \tau)$ represent small, time-dependent motions superimposed on a steady rotatory motion of the string. It will be recognized from section 3.21 and Eq. (6.2—3) that y_1 and y_6 correspond to the time-dependent changes of the radial and angular distances of the string points. Further, y_3 determines the string tension, and y_7 the tangential velocity. Finally, y_4 determines the direction cosine of the string tangent vector with respect to the axis of rotation, hence its connection with y_{1x}, *cf.* the last equation of Eq. (6.2—5).

We now turn to the vibration problem which we will formulate in the following way:

Problem: *To find real constants* α *and functions* $A_\mu(x)$, $B_\mu(x)$, $\mu = 1,3,6,7$, *defined on an interval* $[a_1, a_2]$, *and such that*

a) *The differential system* Eq. (6.2—5) *is satisfied, for* $x \in (a_1, a_2)$ *by*

$$y_\mu(x, \tau) = A_\mu(x) \cos(\alpha\tau) + B_\mu(x) \sin(\alpha\tau). \tag{6.2—9}$$

b) *The homogeneous boundary conditions*

$$y_\nu(a_1, \tau) = y_\nu(a_2, \tau) \equiv 0, \ \nu = 1,6,7 \tag{6.2—10}$$

are satisfied.

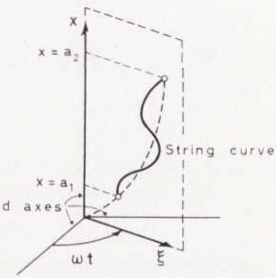

Fig. X. The figure illustrates the situation of the vibration problem. The end points of the string are fixed in the co-rotating plane at $x=a_1$ and $x=a_2$. The broken curve contained in the co-rotating plane is the string curve of the underlying steady motion.

The problem which we have set just now implies that we study an inextensible string on the interval $a_1 \leq x \leq a_2$. The end-points of the string are fixed in a plane rotating with constant angular velocity ω round the x axis which is contained in the plane and fixed in space, *cf.* Fig. X. The end-points of the string then move in two fixed circles which are assumed to be non-coplanar. The string has a small vibratory motion with the dimensionless period $\dfrac{2\pi}{\alpha}$ around its steady motion curve (the broken curve in the figure). We seek *the possible modes of vibration and the natural frequencies* α.

A few preliminary remarks may be made concerning the problem. The string curve of the steady motion is normalized so as to have the maximum distance a to the axis of rotation (it may happen, of course, that no turning point of the string curve falls in the interval (a_1, a_2)), and the curve intersects the axis of rotation at $x=0$. The effects of gravity and air resistance are disregarded. The length and shape of the string in steady motion is determined by Eq. (4.2—25), where we have to put $p_2=0$, $p_1=2p$ and $\operatorname{sn}\left(z\ \dfrac{p_1}{k}\right)=\sin x$ as in Eq. (6.2—2). By varying the magnitudes of a, p, a_1, a_2, we can cover any possible type of the described motion. Later on we shall confine ourselves to small values of p, *i. e.* to flat string curves.

It should be observed that no boundary condition is imposed on the function $y_3(x, \tau)$ which corresponds to the string tension. This appears reasonable from the physical point of view.

We now return to the system Eq. (6.2—5) and substitute the expressions Eq. (6.2—9) into that system. Studying the cos $(\alpha\tau)$ terms we conclude that $A_1(x)$, $A_3(x)$, $B_6(x)$, $B_7(x)$ must satisfy the ordinary differential system

$$\begin{aligned} &LY_1+2\alpha g_1 Y_6-g_1 \sin x\, Y_3=0,\\ &MY_6+2\alpha g_1 Y_1-4g_4 \cos x\; Y_7=0,\\ &DY_7-2\alpha p g_5 \cos x\; DY_1=0,\\ &DY_3-2\alpha g_4 Y_7=0, \end{aligned} \qquad (6.2-11)$$

where we have used the symbols $Y_\mu = Y_\mu(x)$ for the unknown functions, and D, L, M symbolize the following differential operators

$$\begin{aligned} &D=\frac{d}{dx},\\ &L=D^2+3pg_2 \sin x \cos x\; D+g_1(1+\alpha^2),\\ &M=D^2-pg_3 \sin x \cos x\; D+g_1(1+\alpha^2). \end{aligned} \qquad (6.2-12)$$

The functions $g_\nu = g_\nu(x, p)$ are the same as in Eq. (6.2—6).

From the boundary conditions Eq. (6.2—10) we obtain

$$Y_\mu(a_1)=Y_\mu(a_2)=0,\quad \mu=1,6,7. \qquad (6.2-13)$$

We have now to consider the sin $(\alpha\tau)$ terms which are obtained by the substitution of the expressions Eq. (6.2—9) in the system Eq. (6.2—5). It can easily be verified that the functions $B_1(x)$, $B_3(x)$, $-A_6(x)$, $-A_7(x)$ must satisfy the ordinary differential equations (6.2—11) and naturally the boundary conditions Eq. (6.2—13), too. *The essential properties of the vibratory motion in question are consequently determined by the solution of the boundary value problem constituted by the system Eq.* (6.2—11) *and the boundary conditions Eq.* (6.2—13). It will prove convenient, however, *to add one more boundary condition to the problem.* We take this condition to be

$$Y_3(a_1)=C_0, \qquad (6.2-14)$$

where $C_0 \neq 0$ is an arbitrary constant not depending on the parameter p.

We conclude from section 3.21 that this condition implies that the dimensionless reduced string tension $\bar{\sigma}$ at the end point $x=a_1$ of the string oscillates around its steady-state value $\bar{\sigma}_0=1$ according to the expression $\varepsilon C_0 \cos(\alpha\tau)$. (A $\sin(\alpha\tau)$ term may also be present, of course.) We may normalize the oscillation and take $C_0=1$, and the small parameter ε is then interpreted as the amplitude of the oscillation of the dimensionless reduced string tension at $x=a_1$. It will become clear from the perturbation method, as applied to the boundary value problem in the case of small parameter values p, that the functions $Y_\mu(x)$ become proportional to C_0, while the natural frequencies α do not depend on C_0. These circumstances are not surprising, if we recall that we are concerned with a small, linearized motion.

It may be expected that our boundary value problem will only have a solution, if α belongs to a particular set of real numbers which we will call the set of natural frequencies, or the Nf set. The determination of this set is the most important part of the solution, while the functions $Y_\mu(x)$ are of secondary interest. At this stage we know very little about the Nf set, but may expect it to be enumerably infinite with no point of accumulation other than infinity. It can easily be verified that if a number α belongs to the Nf set, then $-\alpha$ also does so. The Nf set must depend on the parameter p, too, but in what manner we do not know.

6.3. Outline of a Perturbation Method

6.31. Preliminary Considerations

In the following considerations we shall understand it as implied that α is a natural frequency; if necessary we shall indicate a particular natural frequency by means of a Greek subscript, *e. g.* α_ν. We now turn to the boundary value problem constituted by the differential system Eq. (6.2—11) and the boundary conditions Eqs. (6.2—13), (6.2—14); p being a small (real) parameter. *We shall assume that the natural frequencies* $\alpha=\alpha(p)$ *and the functions* $Y_\mu(x)=Y_\mu(x, p)$ *have power series in* p such that they permit all the subsequent derivations for $x\in(a_1, a_2)$ and p in a neighbourhood of $p=0$. We may consequently write

$$\begin{aligned} \alpha(p)&=p^i\alpha_i, \\ Y_\mu(x, p)&=p^i Y_{\mu, i}(x), \end{aligned} \qquad (6.3-1)$$

where we have $\mu=1,3,6,7$; $i=0,1,2,3,\ldots$, and summation on i.[40] α_i and $Y_{\mu, i}(x)$ do not depend on p. Then, for the differential operators

[40] p^i of course means the ith power of p.

L, M and the coefficient functions $g_\nu(x, p)$ of the system Eq. (6.2—11), we get

$$L = p^i L_i, \quad M = p^i M_i, \quad g_\nu(x, p) = p^i g_{\nu, i}(x), \tag{6.3—2}$$

where $\nu = 1, \ldots, 5$; $i = 0, 1, 2, 3, \ldots$ L_i, M_i are differential operators of the first order; L_i, M_i and $g_{\nu, i}(x)$ can be calculated from Eqs. (6.2—6), and (6.2—12), and they may consequently be regarded as known. We shall call α_n and $Y_{\mu, n}(x)$ *perturbations* of the order n for $\alpha(p)$ and $Y_\mu(x, p)$, respectively. The finite sums $p^i \alpha_i$ and $p^i Y_{\mu, i}$, $i = 0, 1, \ldots, n$, are called *approximations* of the order n for $\alpha(p)$ and $Y_\mu(x, p)$. The same denominations are used for L, M and $g_\nu(x, p)$. It should be observed that Greek subscripts are used for the numbering of the functions Y_1, Y_3, $\ldots$; g_1, g_2, $\ldots$, while Latin subscripts indicate the ordinal numbers of the perturbations of these functions, of α, and of the differential operators. The summation convention is applied to Latin subscripts and superscripts.

The following expressions for the zero order approximations are easily obtained:

$$\begin{aligned} &L_0 = M_0 = D^2 + 1 + \alpha_0^2, \\ &g_{\nu, 0}(x) \equiv 1, \quad \nu = 1, \ldots, 5. \end{aligned} \tag{6.3—3}$$

For the perturbations of the order i we have

$$\begin{aligned} L_i &= L'_{i-1} + (1 + \alpha_0^2) g_{1, i} + 2\alpha_0 \alpha_i, \\ M_i &= M'_{i-1} + (1 + \alpha_0^2) g_{1, i} + 2\alpha_0 \alpha_i, \end{aligned} \tag{6.3—4}$$

where L'_{i-1} and M'_{i-1} are (linear) differential operators of the first order with coefficients which are constituted by α_{i-1} and lower order perturbations of α and of the known functions $g_{\nu, k}(x)$, $k = 0, 1, \ldots, i-1$. L_i and M_i are consequently linear in α_i, which is an important property.

Substituting our series expansions in the system Eq. (6.2—11) and arranging the terms in suitable order we get

$$\begin{aligned} &p^i [L_{i-k} Y_{1, k} + 2\alpha_k g_{1, i-l-k} Y_{6, l} - g_{1, i-k} \sin x \, Y_{3, k}] = 0, \\ &p^i [M_{i-k} Y_{6, k} + 2\alpha_k g_{1, i-l-k} Y_{1, l} - 4 g_{4, i-k} \cos x \, Y_{7, k}] = 0, \\ &p^i [D Y_{7, i} - 2\alpha_k g_{5, i-l-k-1} \cos x \, D Y_{1, l}] = 0, \\ &p^i [D Y_{3, i} - 2\alpha_k g_{4, i-l-k} Y_{7, l}] = 0, \end{aligned} \tag{6.3—5}$$

where we have summation on i from 0 to ∞, on k, l from zero to i, and where functions $g_{\nu, n}(x) \equiv 0$ for $n < 0$; $\nu = 1, \ldots, 5$.

From the boundary conditions Eq. (6.2—13) we have

$$p^i Y_{\mu,\,i}(a_1) = p^i Y_{\mu,\,i}(a_2) = 0, \quad \mu = 1,6,7. \tag{6.3—6}$$

From the boundary condition Eq. (6.2—14) we finally obtain

$$\begin{aligned} &Y_{3,\,0}(a_1) = C_0, \\ &Y_{3,\,i}\,(a_1) = 0, \quad i \geq 1. \end{aligned} \tag{6.3—7}$$

The last equation is a consequence of our assumption that C_0 does not depend on p. The boundary value problem now consists in finding the perturbations α_i and $Y_{\mu,\,i}(x)$, $\mu=1,3,6,7$, of every order i. As implied previously, it should be understood that the functions $Y_{\mu,\,i}(x)$ shall be defined for $x \in [a_1, a_2]$. We now let our conditions be satisfied in such a way that the coefficients for p^i in the system Eq. (6.3—5) and the boundary conditions Eq. (6.3—6) are zero for every value of i. We then get a boundary value problem for every i.

6.32. The Zero Order Approximation

Putting $i=0$ in the system Eq. (6.3—5) we get

$$\begin{aligned} &L_0 Y_{1,\,0} + 2\alpha_0 Y_{6,\,0} = \sin x\, Y_{3,\,0}, \\ &L_0 Y_{6,\,0} + 2\alpha_0 Y_{1,\,0} = 4 \cos x\, Y_{7,\,0}, \\ &D Y_{7,\,0} = 0, \\ &D Y_{3,\,0} = 2\alpha_0 Y_{7,\,0}, \end{aligned} \tag{6.3—8}$$

where the second order differential operator L_0 is determined by Eq. (6.3—3). The boundary conditions become

$$\begin{aligned} &Y_{\mu,\,0}(a_1) = Y_{\mu,\,0}(a_2) = 0, \quad \mu = 1,6,7, \\ &Y_{3,\,0}(a_1) = C_0. \end{aligned} \tag{6.3—9}$$

It is evident that $Y_{3,0}$ and $Y_{7,0}$ satisfy[41]

$$\begin{aligned} &Y_{3,\,0}(x) \equiv C_0, \\ &Y_{7,\,0}(x) \equiv 0. \end{aligned} \tag{6.3—10}$$

---

[41] The condition Eq. (6.2—4) is obviously satisfied, if $Y_{7,\,0} \equiv 0$, since then $Y_7(x, p) = O(p)$, $p \to 0$.

It remains to find α_0, $Y_{1,0}(x)$, $Y_{6,0}(x)$ such that they satisfy the differential system

$$\begin{aligned} &L_0 Y_{1,0} + 2\alpha_0 Y_{6,0} = C_0 \sin x, \\ &L_0 Y_{6,0} + 2\alpha_0 Y_{1,0} = 0, \end{aligned} \tag{6.3—11}$$

and the boundary conditions

$$Y_{\mu,0}(a_1) = Y_{\mu,0}(a_2) = 0, \quad \mu = 1,6. \tag{6.3—12}$$

This problem can be treated as a Sturm-Liouville problem. The eigenvalues λ_n and the normalized eigenfunctions $u_n(x)$, of the related Sturm-Liouville eigenvalue problem

$$\begin{aligned} &D^2 u + \lambda u = 0, \\ &u(a_1) = u(a_2) = 0 \end{aligned} \tag{6.3—13}$$

are easily found to be

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{a_2 - a_1}\right)^2, \\ u_n(x) &= \left(\frac{2}{a_2 - a_1}\right)^{1/2} \sin\left(\sqrt{\lambda_n}\,(x - a_1)\right), \end{aligned} \tag{6.3—14}$$

where $n = 1, 2, 3, \ldots$ As shown in Appendix A2, the functions $Y_{1,0}(x)$ and $Y_{6,0}(x)$ can be expanded in series of the orthonormal functions $u_n(x)$.[42]

After some elementary calculations we obtain the following results for *the zero order approximations* α_0, $Y_{\mu,0}(x)$, $\mu = 1,3,6,7$:

$$\begin{aligned} Y_{1,0}(x) &= C_0 \sum_1^\infty a_{0,n} u_n(x), \\ Y_{6,0}(x) &= C_0 \sum_1^\infty b_{0,n} u_n(x), \\ Y_{3,0}(x) &\equiv C_0, \\ Y_{7,0}(x) &\equiv 0, \end{aligned} \tag{6.3—15}$$

---

[42] These series become Fourier sine series. The same method applies also in the case of a general Sturm-Liouville operator instead of L_0.

where the expansion coefficients $a_{0,n}$, $b_{0,n}$ are determined by

$$
\begin{aligned}
&a_{0,n}=d_n E_n H_n,\\
&b_{0,n}=-2\alpha_0 d_n H_n,\\
&d_n=(u_n, \sin x)=\int_{a_1}^{a_2} u_n(x)\sin x dx, \qquad (6.3-16)\\
&E_n=1+\alpha_0^2-\lambda_n,\\
&H_n=(E_n^2-4\alpha_0^2)^{-1}.
\end{aligned}
$$

λ_n and $u_n(x)$ are determined by Eq. (6.3—14). The computation of d_n is not carried out here; it is, however, an elementary operation. $(H_n)^{-1}=E_n^2-4\alpha_0^2\neq 0$ constitutes a necessary condition for the existence of $Y_{1,0}(x)$ and $Y_{6,0}(x)$. This condition can be written

$$(1+\alpha_0^2-\lambda_n)^2-4\alpha_0^2\neq 0. \qquad (6.3-17)$$

The results can be summarized as follows:

1) No conditions except those of Eq. (6.3—17) are obtained for the set of zero order approximations α_0 of the natural frequencies α.

2) The functions $Y_{\mu,0}(x)$, $\mu=1,3,6,7$ are uniquely determined for given α_0. $Y_{1,0}$ and $Y_{6,0}$ are obtained as sums of Fourier sine series and become proportional to C_0. The latter series converge absolutely and uniformly, *cf.* [C4], p. 293.

3) It can immediately be concluded from Eq. (6.3—15) that neither $Y_{1,0}(x)\equiv 0$ nor $Y_{6,0}(x)\equiv 0$, $x\in[a_1, a_2]$, is possible. The vibration therefore cannot possibly take place in a plane, neither can it be purely circumferential.

We shall make no further comments on these results at present, but turn instead to the first order perturbation $Y_{7,1}(x)$. Putting $i=1$ we get from the third equation of the system Eq. (6.3—5) and the boundary conditions Eq. (6.3—6)[43]

$$
\begin{aligned}
&DY_{7,1}-2\alpha_0 \cos x \; DY_{1,0}=0,\\
&Y_{7,1}(a_1)=Y_{7,1}(a_2)=0.
\end{aligned} \qquad (6.3-18)
$$

These equations determine $Y_{7,1}(x)$ and give *a further condition.*

---

[43] The third equation of the system simply implies conservation of string mass, *i.e.* string length, in the interval $[a_1, a_2]$.

Integrating the first expression over the basic interval $[a_1, a_2]$ and regarding $Y_{1,0}(a_1)=Y_{1,0}(a_2)=0$ we get[44]

$$(Y_{1,0}, \sin x)=0. \tag{6.3—19}$$

Substituting the series for $Y_{1,0}(x)$ in Eq. (6.3—19), reversing the order of summation and integration, and employing the notation d_n, we get the following equation, expressing *a necessary condition for the convergence of our perturbation method*

$$S(\alpha_0)=\sum_{n=1}^{\infty} d_n^2 E_n H_n=$$

$$=\sum_{n=1}^{\infty} d_n^2(1+\alpha_0^2-\lambda_n)[(1+\alpha_0^2-\lambda_n)^2-4\alpha_0^2]^{-1}=0. \tag{6.3—20}$$

This equation can be regarded as a condition for α_0, and consequently every α_0 satisfying Eq. (6.3—20) is the zero order approximation of a natural frequency α. The coefficients d_n are easily obtained from their defining equation in Eq. (6.3—16). There exist two different cases depending on the length of the basic interval $[a_1, a_2]$:

a) If the basic interval is such that $a_2-a_1\neq k\pi$, k being an arbitrary, positive integer, we have

$$d_n=\sqrt{2\lambda_n}(a_2-a_1)^{-1/2}(\lambda_n-1)^{-1}(\sin a_1-(-1)^n \sin a_2), \tag{6.3—21}$$

where, from Eq. (6.3—14), $\lambda_n=\left(\frac{n\pi}{a_2-a_1}\right)^2$.

b) If we have $a_2-a_1=k\pi$ we get

$$\begin{aligned} d_n&=\sqrt{2\lambda_n}\,(k\pi)^{-1/2}\,(\lambda_n-1)^{-1}\sin a_1(1-(-1)^{n+k}),\quad n\neq k,\\ d_k&=\tfrac{1}{2}\sqrt{2k\pi}\cos a_1, \end{aligned} \tag{6.3—22}$$

where $\lambda_n=\dfrac{n}{k}$.

Equation $S(\alpha_0)=0$ gives rise to the following comments:

1) The infinite series $S(\alpha_0)$ converges as $\Sigma\, n^{-4}$.

2) $S(\alpha_0)$ is an even function; the set of zero order approximations of the natural frequencies is therefore symmetric with respect to zero.

---

[44] The case $\alpha_0=0$ which provides $Y_{7,1}\equiv 0$ does not interest us, since it implies steady motion, *cf.* Eq. (6.2—9).

3) The condition Eq. (6.3—17) is automatically satisfied for those α_0 which satisfy $S(\alpha_0)=0$.

4) The sum of the series defining $S(\alpha_0)$ can be obtained by a method known from the theory of functions of a complex variable, *cf.* [T1], p. 114. The result is a finite sum of products of trigonometric functions (of α_0, a_1 and a_2) and algebraic functions (of α_0).

5) The author would advance the conjecture that the set of real α_0 satisfying $S(\alpha_0)=0$ is enumerably infinite and has no other point of accumulation than infinity. At least this comes true in special cases of case (b). In the general case (a) the evaluation of $S(\alpha_0)$ is rather tedious and it will not be reproduced here.

We have now succeeded in finding the zero order approximations α_0, $Y_{\mu,0}(x)$, $\mu=1,3,6,7$ of our boundary value problem, at least in principle. We now turn to the problems involved in the determination of the pertubations of the first and higher orders.

6.33. Pertubations of the First and Higher Orders

We shall first approach the determination of the first order perturbations α_1, $Y_{\mu,1}(x)$, $\mu=1,3,6$.

The perturbation $Y_{7,1}(x)$ is already known from Eq. (6.3—18) and it is clear that it is proportional to C_0, since this is the case of $Y_{1,0}(x)$. In the system Eq. (6.3—5) we put $i=1$ in the first, second and fourth equations. In the third equation of the system we put $i=2$. The following system results:

$$\begin{aligned} L_0Y_{1,1}+2\alpha_0Y_{6,1}&=-2\alpha_1(\alpha_0Y_{1,0}+Y_{6,0})+\sin x\,Y_{3,1}+\Phi_{1,0},\\ L_0Y_{6,1}+2\alpha_0Y_{1,1}&=-2\alpha_1(\alpha_0Y_{6,0}+Y_{1,0})+4\cos x\,Y_{7,1}+\Phi_{2,0},\\ DY_{3,1}&=2\alpha_0Y_{7,1}+2\alpha_1Y_{7,0}+\Phi_{3,0},\\ DY_{7,2}&=2\cos x(\alpha_1DY_{1,0}+\alpha_0DY_{1,1})+\Phi_{4,0}, \end{aligned} \tag{6.3—23}$$

where we let $Y_{7,0}(x)=0$ conformably to Eq. (6.3—10).

The following boundary conditions have to be satisfied:

$$\begin{aligned} &Y_{\mu,1}(a_1)=Y_{\mu,1}(a_2)=0,\ \mu=1,6,\\ &Y_{3,1}(a_1)=0,\\ &Y_{7,2}(a_1)=Y_{7,2}(a_2)=0. \end{aligned} \tag{6.3—24}$$

The functions $\Phi_{\eta, 0}$, $\eta=1, \ldots, 4$ in the system Eq. (6.3—23) are known functions of x and α_0 and proportional to C_0, they consequently do not contain α_1 or the unknown functions $Y_{7, 2}(x)$ and $Y_{\mu, 1}(x)$, $\mu=1,3,6$. Evidently we have one system of this kind for every α_0 in the set of zero order approximations of the natural frequencies. The differential operators occurring in the left members of the differential system Eq. (6.3—23) are wholly identical to those in the system Eq. (6.3—8) which governs the zero order approximations.

Regarding α_1 as an arbitrary parameter the system can be integrated step by step. The right member of the third equation in the system (6.3—23) contains only known quantities. Taken together with the boundary condition $Y_{3, 1}(a_1)=0$ we are able to obtain a unique determination of $Y_{3, 1}(x)$ which becomes proportional to C_0 and does not depend on α_1. Now, the right members of the first two equations of the differential system Eq. (6.3—23) are known (if we consider α_1 as known). We can therefore consider these two equations and the boundary conditions as a Sturm-Liouville problem for the unknown functions $Y_{1, 1}(x)$ and $Y_{6, 1}(x)$. We can write the solutions in the form

$$\begin{aligned} Y_{1, 1}(x) &= \alpha_1 Y'_{1, 1} + Y''_{1, 1}, \\ Y_{6, 1}(x) &= \alpha_1 Y'_{6, 1} + Y''_{6, 1}, \end{aligned} \qquad (6.3-25)$$

where the functions $Y'_{\mu, 1}=Y'_{\mu, 1}(x)$ and $Y''_{\mu, 1}=Y''_{\mu, 1}(x)$, $\mu=1,6$, do not depend on α_1. These functions can be expressed in terms of absolutely and uniformly convergent series of the eigenfunctions $u_n(x)$ of the problem Eq. (6.3—13). At the moment we shall take no interest in the explicit computation of these series. It now remains to determine α_1. The last equation of the system (6.3—23) together with the boundary conditions $Y_{7, 2}(a_1)=Y_{7, 2}(a_2)=0$ determine the functions $Y_{7, 2}(x)$ and also provide a further condition which is a linear algebraic equation for α_1. Integrating the differential equation over the basic interval $[a_1, a_2]$ and taking into account the boundary conditions, we obtain[45]

$$\begin{aligned} & 2\alpha_1(Y_{1, 0}, \sin x) + 2\alpha_0\alpha_1(Y'_{1, 1}, \sin x) + \\ & \qquad + 2\alpha_0(Y''_{1, 1}, \sin x) + (\Phi_{4, 0}, 1) = 0. \end{aligned} \qquad (6.3-26)$$

[45] It should be observed that $Y'_{1,1}$ and $Y''_{1,1}$ are both zero at the end points of the basic interval, a_1 and a_2.

The first term is zero according to Eq. (6.3—19), and Eq. (6.3—26) is a linear equation for α_1 since none of the inner products depend on α_1. We get a value α_1 for every particular α_0 in the set of zero order approximations of the natural frequencies, provided that the following condition is satisfied:

$$(Y'_{1,1}, \sin x) \neq 0. \tag{6.3—27}$$

This condition is extremely important to our perturbation method, and we shall therefore try to give it a more explicit formulation. First we must determine the function $Y'_{1,1}(x)$. Substituting the expressions of Eq. (6.3—25) in the first two equations of the system Eq. (6.3—23) and extracting the terms depending on α_1 (observe that $Y_{3,1}(x)$ and $Y_{7,1}(x)$ do not depend on α_1), we get the following equations for $Y'_{1,1}(x)$ and $Y'_{6,1}(x)$:

$$\begin{aligned} L_0 Y'_{1,1} + 2\alpha_0 Y'_{6,1} &= -2(\alpha_0 Y_{1,0} + Y_{6,0}), \\ L_0 Y'_{6,1} + 2\alpha_0 Y'_{1,1} &= -2(\alpha_0 Y_{6,0} + Y_{1,0}). \end{aligned} \tag{6.3—28}$$

The boundary conditions to be satisfied are

$$Y'_{\mu,1}(a_1) = Y'_{\mu,1}(a_2) = 0, \quad \mu = 1,6. \tag{6.3—29}$$

This Sturm-Liouville problem is essentially the same as the previous one which determined $Y_{1,0}(x)$ and $Y_{6,0}(x)$. After some calculations we get according to the results elaborated in Appendix A2

$$Y'_{1,1}(x) = \sum_{n=1}^{\infty} a'_{1,n} u_n(x), \tag{6.3—30}$$

where

$$\begin{aligned} a'_{1,n} &= H_n(E_n d'_{1,n} - 2\alpha_0 d'_{2,n}), \\ d'_{1,n} &= -2\alpha_0 (Y_{1,0}, u_n) - 2(Y_{6,0}, u_n), \\ d'_{2,n} &= -2\alpha_0 (Y_{6,0}, u_n) - 2(Y_{1,0}, u_n). \end{aligned} \tag{6.3—31}$$

In Eq. (6.3—31) E_n and H_n are determined by Eq. (6.3—16). The coefficients $d'_{1,n}$ and $d'_{2,n}$ are the expansion coefficients of the right members of Eq. (6.3—28) with respect to u_n. It can be concluded from Eqs. (6.3—15) and (6.3—16) that

$$(Y_{1,0}, u_n) = C_0 a_{0,n} = C_0 E_n H_n d_n,$$

$$(Y_{6,0}, u_n) = C_0 b_{0,n} = -2C_0 \alpha_0 H_n d_n.$$

After some calculation we get

$$a'_{1,n} = -2C_0 \alpha_0 H_n^2 d_n (E_n^2 - 4E_n + 4\alpha_0^2), \qquad (6.3-32)$$

where $d_n = (u_n, \sin x)$ according to Eq. (6.3—16). From the condition Eq. (6.3—27) and from Eq. (6.3—30) we obtain by termwise integration

$$\sum_{n=1}^{\infty} a'_{1,n} (\sin x, u_n) = \sum_{n=1}^{\infty} a'_{1,n} d_n \neq 0.$$

Substituting Eq. (6.3—32) in the last expression and deleting some non-essential factors we obtain

$$T(\alpha_0) \equiv \sum_{n=1}^{\infty} d_n^2 H_n^2 (E_n^2 - 4E_n + 4\alpha_0^2) \neq 0 \qquad (6.3-33)$$

as *the condition for the existence of the first order perturbations* α_1. In the inequality (6.3—33) E_n, H_n are given by Eq. (6.3—16), and d_n is given by Eq. (6.3—21) or Eq. (6.3—22).

The following comments are relevant to the above considerations:

1) The infinite series in Eq. (6.3—33) converges as Σn^{-6}.

2) The sum $T(\alpha_0)$ of the series can be obtained in the same way as $S(\alpha_0)$, *cf.* section 6.32. When $T(\alpha_0)$ is obtained in this way it has the form of a finite sum of products of trigonometric functions (of α_0, a_1 and a_2) and algebraic functions (of α_0).

3) It is a necessary condition for our perturbation method to work in the case of a particular value of α_0 that $T(\alpha_0) \neq 0$. If it happens that the set of real zeros of $S(\alpha_0)$ and the set of real zeros of $T(\alpha_0)$ are disjoint (for given boundary coordinates a_1 and a_2), then it is possible to determine all the first order perturbations α_1. It is the author's conjecture that this comes true, since it seems most unlikely that the two functions $S(\alpha_0)$ and $T(\alpha_0)$ should possess coincidental zeros. The proof of this conjecture seems to constitute a difficult problem.

Summarizing our results regarding the first order perturbations, we assert that α_1, $Y_{\mu,1}(x)$, $\mu=1,3,6$ do exist and are uniquely determinable, if the condition $T(\alpha_0)\neq 0$ is satisfied. Then the second order perturbation $Y_{7,2}(x)$ also does exist and is uniquely determinable.

We now turn to the perturbations of the arbitrary order i for α, $Y_\mu(x)$, $\mu=1,3,6$, and the order $i+1$ for $Y_7(x)$. Assuming that all lower order perturbations are determined, we get from system Eq. (6.3—5)

$$\begin{aligned}
L_0 Y_{1,i}+2\alpha_0 Y_{6,i}&=-2\alpha_i(\alpha_0 Y_{1,0}+Y_{6,0})+\sin x\, Y_{3,i}+\Phi_{1,i-1},\\
L_0 Y_{6,i}+2\alpha_0 Y_{1,i}&=-2\alpha_i(\alpha_0 Y_{6,0}+Y_{1,0})+4\cos x\, Y_{7,i}+\Phi_{2,i-1},\\
DY_{3,i}&=2\alpha_0 Y_{7,i}+\Phi_{3,i-1},\\
DY_{7,i+1}&=2\cos x(\alpha_i DY_{1,0}+\alpha_0 DY_{1,i})+\Phi_{4,i-1},
\end{aligned}\qquad (6.3-34)$$

where the functions $\Phi_{\eta,i-1}$, $\eta=1,\ldots,4$ are known functions of x and of perturbations α_k up to and including the order $i-1$. The boundary conditions to be satisfied are the same as those for the first order perturbations, *i. e.* the conditions Eq. (6.3—24) are valid with obvious changes of the second subscripts.

The system Eq. (6.3—34) is essentially the same as Eq. (6.3—23), and it is not difficult to show that all the important results obtained for the first order perturbations also apply to perturbations of arbitrary order. In particular, the condition $T(\alpha_0)\neq 0$ is the condition of existence of the perturbation α_i. The latter fact is easy to prove from the system Eq. (6.3—34), since the term depending on α_i in the function $Y_{1,i}(x)$ will become exactly the same as the term depending on α_1 in the function $Y_{1,1}(x)$, *i. e.* equal to $Y'_{1,1}(x)$.

We now conclude our discussion of the perturbation method applied to the vibration problem as set in section 6.2 with the following summary:

It is possible to obtain perturbations of arbitrary order of a particular natural frequency α and the functions $Y_\mu(x)$, $\mu=1,3,6,7$ provided that the condition $T(\alpha_0)\neq 0$ is valid for the zero order approximation α_0 of the natural frequency in question. The zero order approximations α_0 are the real roots of $S(\alpha_0)=0$. $S(\alpha_0)$ is defined by Eq. (6.3—20) and $T(\alpha_0)$ by Eq. (6.3—33). The zero order approximations and the first order perturbations are not difficult to calculate for given boundary coordinates a_1 and a_2. The explicit calculation of higher order perturbations will be laborious. We do not know whether the perturbation method converges or not. Our results are consequently only formal although the perturbations are determined without ambiguity. It is to be expected that the perturbation method discussed here will apply to more general problems. One might, for instance, study a small time-dependent motion of an inextensible string which is initially in a state of steady rotatory motion with no tangential velocity and initially possessing a plane string curve. The perturbations causing the time-dependent motion may be acting on one end of the string or along the string and may depend on the time, they may, for instance, be periodic functions of the time. In its initial state the string may be deflected from its steady state curve and possess a velocity in relation to a steady rotating frame. Starting from the system Eq. (3.2—10), as we did in the vibration problem, we can apply the Laplace transformation to the time variable (instead of separating the variables), and then apply the perturbation method to the resulting ordinary differential system.

Our choice to study a vibratory motion as an illustration of this kind of motions is based on the fact that it is the simplest time-dependent motion to be found.

Appendix A1

In section 4.32 we solved the problem (E1). The solution was regarded as an approximation, linear in the elastic constant β and valid for small values of β, of the solution of the basic one-point problem for the steadily rotating and linearly elastic string, when the string does not intersect the axis of rotation. The functions constituting the approximate solution were denoted $\xi(z, \beta)$, $\varphi(z, \beta)$, $\gamma(z, \beta)$ and the functions to be approximated were denoted $\xi'(z, \beta)$ $\varphi'(z, \beta)$, $\gamma'(z, \beta)$. In this appendix we shall prove that the conditions Eq. (4.3—9) are satisfied, *i. e.* that

$$\xi-\xi',\ \varphi-\varphi',\ \gamma-\gamma'=O(\beta^2), \quad \beta\to 0 \tag{A1—1}$$

hold true if z occurs in an arbitrary, bounded interval. The functions ξ, φ, γ satisfy the differential system Eq. (4.3—15) and the initial values $\xi(0, \beta)=1$, $\varphi(0, \beta)=0$. The functions ξ', φ', γ' satisfy the differential system Eq. (4.3—12) and the same initial values. It was shown that ξ and ξ' are periodic in z and have positive upper and lower bounds. Furthermore, all the functions discussed are analytic in z and in the parameters of the systems for every z on the real axis, on condition that β is small enough. This is true, because the system Eq. (4.3—1) from which we started the derivation of both the systems Eqs. (4.3—12) and (4.3—15) is analytic in its variables and parameters for $\xi>0$ and sufficiently small values of β. For the moment we assume that $\xi(z, \beta)$ and $\xi'(z, \beta)$ satisfy the condition in Eq. (A1—1) for some arbitrary z interval (a_1, a_2). We then obtain

$$\xi^2-(\xi')^2=(\xi+\xi')(\xi-\xi')=O(\beta^2), \quad \beta\to 0 \tag{A1—2}$$

and from Eqs. (4.3—12), (4.3—15)

$$|(\varphi-\varphi')_z|=|b-p_2|\,|\xi^{-2}(\xi')^{-2}|\,|\xi^2-(\xi')^2|<M_1\beta^2. \tag{A1—3}$$

M_1 is a positive number which does not depend on z, and the inequality holds if β is sufficiently small. By the initial conditions $\varphi(0, \beta)=$

$=\varphi'(0, \beta)=0$ we have at once $|\varphi-\varphi'|<|z|M_1\beta^2$ or, equivalently, $\varphi-\varphi'=O(\beta^2)$, $\beta\to 0$ for $z\in(a_1, a_2)$. From Eqs. (4.3—12), (4.3—15) we also get $\gamma-\gamma'=O(\beta^2)$, $\beta\to 0$.

It remains to show that $\xi-\xi'=O(\beta^2)$, $\beta\to 0$, is valid. The first equations of the two systems (4.3—12) and (4.3—15) may be written

$$(\xi'\xi'_z)^2=[k_2(n_2+\lambda K)-(\xi')^2][1+\delta G_3]Q(\xi', \beta), \qquad \text{(A1—4)}$$

$$(\xi\xi_z)^2=(k_2n_2-\xi^2)Q(\xi, \beta), \qquad \text{(A1—5)}$$

where $\lambda=\delta=\beta^2$. $K=K(\beta)$ is analytic in β, and $G_3=G_3(\xi, \beta)$ is analytic in z and β. Moreover, $K=O(1)$, $G_3=O(1)$, $\beta\to 0$. $Q(\xi', \beta)$ is a polynomial of the third degree in $(\xi')^2$ and analytic in β. Regarding λ and δ as parameters of the equation and adding the initial value $\xi'(0, \beta, \lambda, \delta)=1$ to Eq. (A1—4) we have an initial value problem which determines the function $\xi'=\xi'(z, \beta, \lambda, \delta)$. This function becomes periodic in z; analytic in z and in the parameters in the neighbourhood of $\beta=\lambda=\delta=0$, for every real z. We can consequently expand ξ' in a power series of β, λ, δ with coefficients which are analytic in z. We then obtain

$$\begin{aligned}\xi'(z, \beta, \lambda, \delta)=f(z)+\beta g_1(z)+\lambda g_2(z)+\delta g_3(z)+\\ +O(\beta^2)+O(\beta\lambda)+O(\beta\delta), \quad \beta, \lambda, \delta\to 0,\end{aligned} \qquad \text{(A1—6)}$$

where f, g_1, g_2, g_3 are analytic. With $\xi(0, \beta)=1$, we get from Eq. (A1—5)

$$\xi(z, \beta)=f(z)+\beta g_1(z)+O(\beta^2), \quad \beta\to 0. \qquad \text{(A1—7)}$$

Evidently the functions f and g_1 in Eq. (A1—7) are the same as in Eq. (A1—6), since for $\lambda=\delta=0$ $\xi'=\xi$ must hold for every z and β of interest. (The differential equations (A1—4) and (A1—5) coincide for $\lambda=\delta=0$, and ξ and ξ' satisfy the same initial value.)

From Eqs. (A1—6) and (A1—7) we obtain

$$\xi-\xi'=-\lambda g_2(z)-\delta g_3(z)+O(\beta^2)+O(\beta\lambda)+O(\beta\delta).$$

Recalling that $\lambda=\delta=\beta^2$ we readily get the desired expression $\xi-\xi'=O(\beta^2)$, $\beta\to 0$, which holds for every real z.

It is also concluded that the derivatives (with respect to z), of arbitrary orders, of the functions $\xi-\xi'$, $\varphi-\varphi'$, $\gamma-\gamma'$ are of the order $O(\beta^2)$.

Appendix A2

In section 6.32 we were faced with the problem of solving the following boundary value problem for two functions $y_1(x)$ and $y_2(x)$:

$$
\begin{aligned}
&L_0 y_1 + 2\alpha y_2 = C \sin x, \\
&L_0 y_2 + 2\alpha y_1 = 0, \qquad\qquad (A2-1)\\
&y_i(a_1) = y_i(a_2) = 0, \; i = 1,2,
\end{aligned}
$$

where α and C are real constants and the differential operator L_0 is defined by

$$
L_0 = D^2 + 1 + \alpha^2,
$$

$$
D = \frac{d}{dx}.
$$

Problem (A2—1) is a special case of the problem

$$
\begin{aligned}
&L y_1 + \alpha_1 \varrho y_2 = f_1, \\
&L y_2 + \alpha_2 \varrho y_1 = f_2, \qquad\qquad (A2-2)\\
&y_i(a) = y_i(b) = 0, \; i = 1,2.
\end{aligned}
$$

α_1, α_2 are real constants, $f_1 = f_1(x)$ and $f_2 = f_2(x)$ are arbitrary, continuous functions of x, and L is a Sturm-Liouville operator defined by

$$
L = D(pD) + \lambda \varrho, \qquad\qquad (A2-3)
$$

where $p = p(x)$ and $\varrho = \varrho(x)$ are positive and continuous for $x \in [a, b]$. λ is a real parameter. Essentially, the following derivations remain valid, if we change L for the general Sturm-Liouville operator $D(pD) + \lambda\varrho - q$, where $q = q(x)$ is a continuous function. For obvious reasons, we may call the problem (A2—2) a Sturm-Liouville problem. The author has not been able to find any treatment of such a problem in the literature, although it is quite possible that such a treatment in fact does exist, and we therefore find it worth-while to discuss it in some detail. Let $u_n(x)$ and λ_n, $n=1, 2, 3, \ldots$, be the normalized eigenfunctions and the eigenvalues of the related problem

$$\begin{aligned} &Lu=0, \\ &u(a)=u(b)=0. \end{aligned} \tag{A2—4}$$

With the notation $(f, g) \equiv \int_a^b \varrho(x) f(x) g(x) dx$ we then get by the orthonormality of the eigenfunctions

$$(u_n, u_k)=\begin{cases} 0 \text{ if } n \neq k, \\ 1 \text{ if } n=k. \end{cases} \tag{A2—5}$$

Let $c_{1,n}$ and $c_{2,n}$ be the wanted expansion coefficients, with respect to the eigenfunctions, for the unknown functions $y_1(x)$ and $y_2(x)$, *i. e.* let us write

$$\begin{aligned} &y_i=c_{i,n} u_n, \\ &c_{i,n}=(y_i, u_n), \end{aligned} \tag{A2—6}$$

where $i=1,2$ and we have summation on n from one to infinity. Multiplying the differential equations in Eq. (A2—2) by u_n and integrating over the basic interval we get

$$\begin{aligned} &\int u_n[D(pDy_1)+\lambda\varrho y_1]dx+\alpha_1\int\varrho y_2 u_n dx=\int f_1 u_n dx, \\ &\int u_n[D(pDy_2)+\lambda\varrho y_2]dx+\alpha_2\int\varrho y_1 u_n dx=\int f_2 u_n dx. \end{aligned} \tag{A2—7}$$

Now, from Eq. (A2—4) and the boundary conditions $y_i(a)=y_i(b)=0$, and by repeated integration by parts, we obtain

$$\int u_n D(pDy_i)dx=-\lambda_n(y_i, u_n), \; i=1,2.$$

With the notation

$$\int f_i u_n dx=d_{i,n}, \; i=1,2 \tag{A2—8}$$

we have from (A2—7)

$$\begin{aligned} &(\lambda-\lambda_n)c_{1,n}+\alpha_1 c_{2,n}=d_{1,n}, \\ &(\lambda-\lambda_n)c_{2,n}+\alpha_2 c_{1,n}=d_{2,n}. \end{aligned} \tag{A2—9}$$

From (A2—9) we get immediately

$$c_{1,n}=H_n[d_{1,n}(\lambda-\lambda_n)-\alpha_1 d_{2,n}],$$
$$c_{2,n}=H_n[d_{2,n}(\lambda-\lambda_n)-\alpha_2 d_{1,n}], \tag{A2—10}$$

where

$$H_n=[(\lambda-\lambda_n)^2-\alpha_1\alpha_2]^{-1}. \tag{A2—11}$$

The equations (A2—10) determine the expansion coefficients $c_{1,n}$ and $c_{2,n}$ if H_n is finite, *i. e.* if

$$(\lambda-\lambda_n)^2-\alpha_1\alpha_2\neq 0. \tag{A2—12}$$

The method can without difficulty be extended to the following Sturm-Liouville problem for n functions $y_i(x)$:

$$Ly_i+\varrho\alpha_{i,k}y_k=f_i,$$
$$y_i(a)=y_i(b)=0,$$

where $i=1,2,\ldots,n$; $k=1,2,\ldots,n$; $k\neq i$, and we have summation on k. $\alpha_{i,k}$ are arbitrary real constants and L, ϱ, f_i are defined as for problem (A2—2).

Returning to problem (A2—1) we find that this special case of problem (A2—2) is characterized by

$$L=L_0=D^2+1+\alpha^2,\ \lambda=1+\alpha^2,\ \alpha_1=\alpha_2=2\alpha,$$
$$\varrho(x)\equiv 1,\ f_1(x)=C\sin x,\ f_2(x)\equiv 0. \tag{A2—13}$$

The basic interval is $[a_1, a_2]$. The eigenvalues λ_n and eigenfunctions $u_n(x)$ of the corresponding problem Eq. (A2—4) are easily obtained, and the determination of the expansion coefficients $c_{1,n}$ and $c_{2,n}$ result from straight-forward calculations.

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