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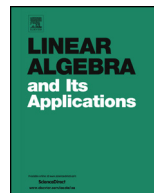


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Evaluation modules for the q -tetrahedron algebra



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ABSTRACT

Let \mathbb{F} denote an algebraically closed field, and fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. We consider the q -tetrahedron algebra \mathfrak{X}_q over \mathbb{F} . It is known that each finite-dimensional irreducible \mathfrak{X}_q -module of type 1 is a tensor product of evaluation modules. This paper contains a comprehensive description of the evaluation modules for \mathfrak{X}_q . This description includes the following topics. Given an evaluation module V for \mathfrak{X}_q , we display 24 bases for V that we find attractive. For each basis we give the matrices that represent the \mathfrak{X}_q -generators. We give the transition matrices between certain pairs of bases among the 24. It is known that the cyclic group \mathbb{Z}_4 acts on \mathfrak{X}_q as a group of automorphisms. We describe what happens when V is twisted via an element of \mathbb{Z}_4 . We discuss how evaluation modules for \mathfrak{X}_q are related to Leonard pairs of q -Racah type.

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1. Introduction

The q -tetrahedron algebra \boxtimes_q was introduced in [10]. This algebra is associative, non-commutative, and infinite-dimensional. It is defined by generators and relations. There are eight generators, and it is natural to identify each of these with an orientation on an edge in a tetrahedron. From this point of view the generating set looks as follows. In the tetrahedron, a pair of opposite edges are each oriented in both directions. The four remaining edges are each oriented in one direction, to create a directed 4-cycle. Thus the cyclic group \mathbb{Z}_4 acts transitively on the vertex set of the tetrahedron, in a manner that preserves the set of edge-orientations. The relations in \boxtimes_q are described as follows. For each doubly oriented edge of the tetrahedron, the product of the two edge-orientations is 1. For each pair of edge-orientations that create a directed 2-path involving three distinct vertices, these edge-orientations satisfy a q -Weyl relation. For each pair of edges in the tetrahedron that are opposite and singly oriented, the two associated edge-orientations satisfy the cubic q -Serre relations. By construction, the \mathbb{Z}_4 action on the tetrahedron induces a \mathbb{Z}_4 action on \boxtimes_q as a group of automorphisms.

We will be discussing the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$, the loop algebra $U_q(L(\mathfrak{sl}_2))$ [10, Section 8], and an algebra \mathcal{A}_q called the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$ [10, Definition 9.1]. These algebras are related to \boxtimes_q in the following way. Each face of the tetrahedron is surrounded by three edges, of which two are singly oriented and one is doubly oriented. The resulting four edge-orientations generate a subalgebra of \boxtimes_q that is isomorphic to $U_q(\mathfrak{sl}_2)$ [10, Proposition 7.4], [21, Proposition 4.3]. Upon removing one doubly oriented edge from the tetrahedron, the remaining six edge-orientations generate a subalgebra of \boxtimes_q that is isomorphic to $U_q(L(\mathfrak{sl}_2))$ [21, Proposition 4.3]. For each pair of edges in the tetrahedron that are opposite and singly oriented, the two associated edge-orientations generate a subalgebra of \boxtimes_q that is isomorphic to \mathcal{A}_q [21, Proposition 4.1].

The above containments reveal a close relationship between the representation theories of \boxtimes_q , $U_q(L(\mathfrak{sl}_2))$, and \mathcal{A}_q . Before discussing the details, we comment on $U_q(L(\mathfrak{sl}_2))$. In [1], Chari and Pressley classify up to isomorphism the finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$ -modules. This classification involves a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$ -modules of type 1; (ii) the polynomials in one variable that have constant coefficient 1. The polynomial is called the Drinfel'd polynomial.

The representation theories for \boxtimes_q and $U_q(L(\mathfrak{sl}_2))$ are related as follows. Let V denote a \boxtimes_q -module. Earlier we mentioned a subalgebra of \boxtimes_q that is isomorphic to $U_q(L(\mathfrak{sl}_2))$. Upon restricting the \boxtimes_q action on V to this subalgebra, V becomes a $U_q(L(\mathfrak{sl}_2))$ -module. The restriction procedure yields a map from the set of \boxtimes_q -modules to the set of $U_q(L(\mathfrak{sl}_2))$ -modules. By [7, Remark 1.8] and [10, Remark 10.5], this map induces a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible \boxtimes_q -modules of type 1; (ii) the isomorphism classes of finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$ -modules of type 1 whose associated Drinfel'd polynomial does not vanish at 1. (We follow the normalization conventions from [21].) In [21], Miki extends

the above bijective correspondence to include finite-dimensional modules that are not necessarily irreducible.

The representation theories for \boxtimes_q and \mathcal{A}_q are related as follows. A finite-dimensional \mathcal{A}_q -module is called *NonNil* whenever the two \mathcal{A}_q -generators are not nilpotent on the module [7, Definition 1.3]. Let V denote a \boxtimes_q -module. Earlier we mentioned a subalgebra of \boxtimes_q that is isomorphic to \mathcal{A}_q . Upon restricting the \boxtimes_q action on V to this subalgebra, V becomes an \mathcal{A}_q -module. This yields a map from the set of \boxtimes_q -modules to the set of \mathcal{A}_q -modules. By [10, Remark 10.5] this map induces a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible \boxtimes_q -modules of type 1; (ii) the isomorphism classes of NonNil finite-dimensional irreducible \mathcal{A}_q -modules of type (1, 1).

We just related the representation theories of \boxtimes_q and \mathcal{A}_q . To illuminate this relationship we bring in the concept of a Leonard pair [25–30,33] and tridiagonal pair [5,6,9,15]. Roughly speaking, a Leonard pair consists of two diagonalizable linear transformations of a finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other one [25, Definition 1.1]. The Leonard pairs are classified [25,29] and correspond to the orthogonal polynomials that make up the terminating branch of the Askey scheme [20]. A tridiagonal pair is a mild generalization of a Leonard pair [5, Definition 1.1]. Let V denote a NonNil finite-dimensional irreducible \mathcal{A}_q -module of type (1, 1). Then the two \mathcal{A}_q -generators act on V as a tridiagonal pair [5, Example 1.7]. A tridiagonal pair obtained in this way is said to be q -geometric [15, Section 1]. Now the bijection from the previous paragraph amounts to the following. Let V denote a finite-dimensional irreducible \boxtimes_q -module of type 1. Then for any pair of edges in the tetrahedron that are opposite and singly oriented, the two associated edge-orientations act on V as a tridiagonal pair of q -geometric type. Moreover, every tridiagonal pair of q -geometric type is obtained in this way. For more details on this correspondence see [7, Section 2] and [10, Section 10]. In [2], Funk-Neubauer obtains a similar correspondence between finite-dimensional irreducible \boxtimes_q -modules and certain tridiagonal pairs of q -Hahn type.

We mention some other results concerning \boxtimes_q . In [11], Ito and Terwilliger characterize the finite-dimensional irreducible \boxtimes_q -modules using q -inverting pairs of linear transformations. In [12], Ito and Terwilliger display an action of \boxtimes_q on the standard module of any distance-regular graph that is self-dual and has classical parameters with base q^2 . In [19], Joohyung Kim provides more details about this \boxtimes_q action. In [13], Ito and Terwilliger obtain a similar \boxtimes_q action for certain distance-regular graphs of q -Racah type.

Turning to the present paper, our topic is a family of finite-dimensional irreducible \boxtimes_q -modules of type 1, called evaluation modules. These modules are important for the following reason. In [1], Chari and Pressley show that each finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$ -module of type 1 is a tensor product of evaluation modules. Earlier we described how each finite-dimensional irreducible \boxtimes_q -module of type 1 corresponds to a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$ -module of type 1. The tensor product structure survives the correspondence [21, Theorem 8.1] and consequently each finite-dimensional

irreducible \boxtimes_q -module of type 1 is a tensor product of evaluation modules [21, Sections 8, 9].

This paper contains a comprehensive description of the evaluation modules for \boxtimes_q . Hoping to keep this description accessible, we avoid Hopf algebra theory and use only linear algebra. Our description is roughly analogous to the description given in [8] concerning the evaluation modules for the tetrahedron algebra.

In our description, each evaluation module for \boxtimes_q gets a notation of the form $\mathbf{V}_d(t)$. Here d is a positive integer, and t is a nonzero scalar in the underlying field that is not among $\{q^{d-2n+1}\}_{n=1}^d$. The \boxtimes_q -module $\mathbf{V}_d(t)$ has dimension $d + 1$. On $\mathbf{V}_d(t)$, each of the eight \boxtimes_q -generators is diagonalizable with eigenvalues $\{q^{d-2n}\}_{n=0}^d$. The \boxtimes_q -module $\mathbf{V}_d(t)$ is determined up to isomorphism by d and t . We obtain several polynomial identities that hold on $\mathbf{V}_d(t)$; these identities involve the eight \boxtimes_q -generators and also t .

We display 24 bases for $\mathbf{V}_d(t)$ that we find attractive. These bases are described as follows. For each permutation i, j, k, ℓ of the vertices of the tetrahedron, we define a basis for $\mathbf{V}_d(t)$ denoted $[i, j, k, \ell]$. This basis diagonalizes each \boxtimes_q -generator involving the vertices k and ℓ . Moreover, the sum of the basis vectors is an eigenvector for each \boxtimes_q -generator involving the vertex j . We display the matrices that represent the eight \boxtimes_q -generators with respect to $[i, j, k, \ell]$. We also give the transition matrices from the basis $[i, j, k, \ell]$ to each of the bases

$$[j, i, k, \ell], \quad [i, k, j, \ell], \quad [i, j, \ell, k].$$

The first transition matrix is diagonal, the second one is lower triangular, and the third one is the identity matrix reflected about a vertical axis.

Recall that the group \mathbb{Z}_4 acts on \boxtimes_q as a group of automorphisms. We show that if $\mathbf{V}_d(t)$ is twisted via a generator for \mathbb{Z}_4 , then the resulting \boxtimes_q -module is isomorphic to $\mathbf{V}_d(t^{-1})$. Consider the element of \mathbb{Z}_4 that has order 2. If $\mathbf{V}_d(t)$ is twisted via this element, then the resulting \boxtimes_q -module is isomorphic to $\mathbf{V}_d(t)$. A corresponding isomorphism of \boxtimes_q -modules is called an exchanger. We describe how these exchangers act on the 24 bases for $\mathbf{V}_d(t)$. We also characterize the exchangers in various ways.

Near the end of the paper we discuss how Leonard pairs of q -Racah type are related to evaluation modules for \boxtimes_q . Given a Leonard pair of q -Racah type, we consider a certain basis for the underlying vector space, called the compact basis, with respect to which the matrices representing the pair are each tridiagonal with attractive entries. Using the Leonard pair we turn the underlying vector space into an evaluation module for \boxtimes_q . On this \boxtimes_q -module, each element of the Leonard pair acts as a linear combination of two \boxtimes_q -generators; the associated edges in the tetrahedron are adjacent and singly oriented. We show that the compact basis diagonalizes a pair of \boxtimes_q -generators that correspond to a doubly oriented edge of the tetrahedron.

The paper is organized as follows. In Section 2 we review some preliminaries and fix our notation. In Section 3 we recall some facts about $U_q(\mathfrak{sl}_2)$ that will be used throughout the paper. In Sections 4–6 we describe the finite-dimensional irreducible \boxtimes_q -modules of

type 1. In Sections 7, 8 we further describe these \boxtimes_q -modules, bringing in the dual space and \mathbb{Z}_4 -twisting. In Section 9 we define an evaluation module for \boxtimes_q called $\mathbf{V}_d(t)$, and we obtain several polynomial identities that hold on this module. In Section 10 we describe 24 bases for $\mathbf{V}_d(t)$. In Section 11 we describe how the eight \boxtimes_q -generators act on the 24 bases. In Sections 12, 13 we describe the transition matrices between certain pairs of bases among the 24. In Section 14 we obtain some identities that involve $\mathbf{V}_d(t)$ and its dual space. Section 15 is about exchangers. In Sections 16, 17 we describe how the evaluation modules for \boxtimes_q are related to Leonard pairs of q -Racah type. Appendices A and B contain some matrix definitions and related identities.

2. Preliminaries

Our conventions are as follows. Throughout the paper \mathbb{F} denotes an algebraically closed field. An algebra is meant to be associative and have a 1. A subalgebra has the same 1 as the parent algebra. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. For the duration of this paragraph fix $d \in \mathbb{N}$. Let $\{u_n\}_{n=0}^d$ denote a sequence. We call u_n the n th component of the sequence. We call d the diameter of the sequence. By the inversion of the sequence $\{u_n\}_{n=0}^d$ we mean the sequence $\{u_{d-n}\}_{n=0}^d$. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let $\text{End}(V)$ denote the \mathbb{F} -algebra consisting of the \mathbb{F} -linear maps from V to V . An element $A \in \text{End}(V)$ is called diagonalizable whenever V is spanned by the eigenspaces of A . The map A is called multiplicity-free whenever A is diagonalizable and each eigenspace of A has dimension 1. Let $\text{Mat}_{d+1}(\mathbb{F})$ denote the \mathbb{F} -algebra consisting of the $d + 1$ by $d + 1$ matrices that have all entries in \mathbb{F} . We index the rows and columns by $0, 1, \dots, d$. Let $\{v_n\}_{n=0}^d$ denote a basis for V . For $A \in \text{End}(V)$ and $M \in \text{Mat}_{d+1}(\mathbb{F})$, we say that M represents A with respect to $\{v_n\}_{n=0}^d$ whenever $Av_n = \sum_{i=0}^d M_{in}v_i$ for $0 \leq n \leq d$. For $M \in \text{Mat}_{d+1}(\mathbb{F})$, M is called upper bidiagonal whenever each nonzero entry lies on the diagonal or the superdiagonal. The matrix M is called lower bidiagonal whenever the transpose M^t is upper bidiagonal. The matrix M is called tridiagonal whenever each nonzero entry lies on the diagonal, the subdiagonal, or the superdiagonal. Assume that M is tridiagonal. Then M is said to be irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Definition 2.1. (See [25, Definition 1.1].) Let V denote a vector space over \mathbb{F} with finite positive dimension. By a Leonard pair on V , we mean an ordered pair A, B of elements in $\text{End}(V)$ that satisfy the following conditions:

- (i) there exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is irreducible tridiagonal;
- (ii) there exists a basis for V with respect to which the matrix representing B is diagonal and the matrix representing A is irreducible tridiagonal.

The above Leonard pair is said to be over \mathbb{F} . We call V the underlying vector space.

Definition 2.2. Let A, B denote a Leonard pair over \mathbb{F} . Let A', B' denote a Leonard pair over \mathbb{F} . By an *isomorphism of Leonard pairs* from A, B to A', B' we mean an \mathbb{F} -linear bijection μ from the vector space underlying A, B to the vector space underlying A', B' such that $\mu A = A' \mu$ and $\mu B = B' \mu$. The Leonard pairs A, B and A', B' are called *isomorphic* whenever there exists an isomorphism of Leonard pairs from A, B to A', B' .

Lemma 2.3. (See [27, Corollary 5.5].) Let A, B denote a Leonard pair on V , as in Definition 2.1. Then the algebra $\text{End}(V)$ is generated by A, B .

We refer the reader to [25–27,33] for background information on Leonard pairs.

3. The equitable presentation for $U_q(\mathfrak{sl}_2)$

In this section we recall the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$. For background information on $U_q(\mathfrak{sl}_2)$, we refer the reader to the books by Jantzen [17] and Kassel [18]. We will work with the equitable presentation of $U_q(\mathfrak{sl}_2)$, which was introduced in [16].

Throughout the paper, fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. For $n \in \mathbb{Z}$ define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

and for $n \geq 0$ define

$$[n]_q! = [n]_q [n - 1]_q \cdots [2]_q [1]_q.$$

We interpret $[0]_q! = 1$.

Definition 3.1. (See [16, Theorem 2.1].) For the \mathbb{F} -algebra $U_q(\mathfrak{sl}_2)$ the equitable presentation has generators $x, y^{\pm 1}, z$ and relations $yy^{-1} = 1, y^{-1}y = 1,$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1. \tag{1}$$

We call $x, y^{\pm 1}, z$ the *equitable generators* for $U_q(\mathfrak{sl}_2)$.

In the next three lemmas we comment on the relations (1).

Lemma 3.2. Let u, v denote elements in any \mathbb{F} -algebra, such that

$$\frac{quv - q^{-1}vu}{q - q^{-1}} = 1.$$

Then

$$q(1 - uv) = q^{-1}(1 - vu) = \frac{[u, v]}{q - q^{-1}}, \tag{2}$$

where $[u, v]$ means $uv - vu$.

Proof. Routine. \square

Lemma 3.3. *Let u, v denote elements in any \mathbb{F} -algebra, such that*

$$\frac{quv - q^{-1}vu}{q - q^{-1}} = 1.$$

(i) *Assume u^{-1} exists. Then*

$$[u^{-1}, [u, v]] = (q - q^{-1})^2(u^{-1} - v). \tag{3}$$

(ii) *Assume v^{-1} exists. Then*

$$[[u, v], v^{-1}] = (q - q^{-1})^2(v^{-1} - u). \tag{4}$$

Proof. (i) Using Lemma 3.2,

$$\frac{[u^{-1}, [u, v]]}{q - q^{-1}} = qu^{-1}(1 - uv) - q^{-1}(1 - vu)u^{-1} = (q - q^{-1})(u^{-1} - v).$$

(ii) Similar to the proof of (i) above. \square

Lemma 3.4. *Let u, v, w denote elements in any \mathbb{F} -algebra, such that both*

$$\frac{quv - q^{-1}vu}{q - q^{-1}} = 1, \quad \frac{qvw - q^{-1}wv}{q - q^{-1}} = 1.$$

Then both

$$[v, uw] = q(q - q^{-1})(u - w), \quad [v, wu] = q^{-1}(q - q^{-1})(u - w). \tag{5}$$

Moreover

$$[v, [u, w]] = (q - q^{-1})^2(u - w). \tag{6}$$

Proof. To obtain (5), observe

$$\frac{[v, uw]}{q - q^{-1}} = qu \frac{qvw - q^{-1}wv}{q - q^{-1}} - q \frac{quv - q^{-1}vu}{q - q^{-1}} w = q(u - w),$$

and

$$\frac{[v, wu]}{q - q^{-1}} = q^{-1} \frac{qvw - q^{-1}wv}{q - q^{-1}} u - q^{-1} w \frac{quv - q^{-1}vu}{q - q^{-1}} = q^{-1}(u - w).$$

We have obtained (5), and (6) follows. \square

In the literature on $U_q(\mathfrak{sl}_2)$ there is a certain central element called the Casimir element [17, Section 2.7], [18, p. 130]. We now recall how the Casimir element looks from the equitable point of view.

Definition 3.5. (See [31, Lemma 2.15].) Let Λ denote the following element in $U_q(\mathfrak{sl}_2)$:

$$\Lambda = qx + q^{-1}y + qz - qxyz. \tag{7}$$

We call Λ the (normalized) Casimir element.

Note 3.6. The element $\Lambda(q - q^{-1})^{-2}$ is equal to the Casimir element of $U_q(\mathfrak{sl}_2)$ discussed in [17, Section 2.7].

Lemma 3.7. (See [17, Lemma 2.7, Proposition 2.18].) The elements $\{\Lambda^n\}_{n \in \mathbb{N}}$ form a basis for the center of $U_q(\mathfrak{sl}_2)$.

Lemma 3.8. (See [31, Lemma 2.15].) The element Λ is equal to each of the following:

$$\begin{aligned} & qx + q^{-1}y + qz - qxyz, & q^{-1}x + qy + q^{-1}z - q^{-1}zyx, \\ & qy + q^{-1}z + qx - qyzx, & q^{-1}y + qz + q^{-1}x - q^{-1}xzy, \\ & qz + q^{-1}x + qy - qzxy, & q^{-1}z + qx + q^{-1}y - q^{-1}yxz. \end{aligned}$$

The finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -modules are described in [17, Section 2]. We now recall how these modules look from the equitable point of view [16,32].

Lemma 3.9. (See [16, Lemma 4.2], [17, Theorem 2.6].) There exists a family of finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -modules

$$\mathbf{V}_{d,\varepsilon} \quad \varepsilon \in \{1, -1\}, \quad d \in \mathbb{N} \tag{8}$$

with the following property: $\mathbf{V}_{d,\varepsilon}$ has a basis with respect to which the matrices representing x, y, z are

$$\begin{aligned} x: & \quad \varepsilon \begin{pmatrix} q^{-d} & q^d - q^{-d} & & & \mathbf{0} \\ & q^{2-d} & q^d - q^{2-d} & & \\ & & q^{4-d} & \ddots & \\ & & & \ddots & q^d - q^{d-2} \\ \mathbf{0} & & & & q^d \end{pmatrix}, \\ y: & \quad \varepsilon \operatorname{diag}(q^d, q^{d-2}, q^{d-4}, \dots, q^{-d}), \end{aligned}$$

$$z: \varepsilon \begin{pmatrix} q^{-d} & & & & & & & & \mathbf{0} \\ q^{-d} - q^{2-d} & q^{2-d} & & & & & & & \\ & q^{-d} - q^{4-d} & q^{4-d} & & & & & & \\ & & & \ddots & & & & & \\ & & & & \ddots & & & & \\ \mathbf{0} & & & & & q^{-d} - q^d & q^d & & \end{pmatrix}.$$

Every finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$ -module is isomorphic to exactly one of the modules (8).

Note 3.10. For $\text{Char}(\mathbb{F}) = 2$ we interpret $\{1, -1\}$ to have a single element.

Note 3.11. The dimension of $\mathbf{V}_{d,\varepsilon}$ is $d + 1$.

Definition 3.12. For $\mathbf{V}_{d,\varepsilon}$ the parameter d is called the *diameter*. The parameter ε is called the *type*. We sometimes abbreviate $\mathbf{V}_d = \mathbf{V}_{d,1}$.

Note 3.13. For each of $x, y^{\pm 1}, z$ the action on $\mathbf{V}_{d,\varepsilon}$ is multiplicity-free with eigenvalues $\{\varepsilon q^{d-2n} \mid 0 \leq n \leq d\}$.

Note 3.14. In Lemma 3.9 the matrix representing x (resp. z) has constant row sum εq^d (resp. εq^{-d}). This reflects the fact that $xv = \varepsilon q^d v$ (resp. $zv = \varepsilon q^{-d} v$), where v denotes the sum of the basis vectors.

Note 3.15. In Appendix A we define some matrices in $\text{Mat}_{d+1}(\mathbb{F})$ called E_q, K_q, Z . The displayed matrices from Lemma 3.9 that represent x, y, z for \mathbf{V}_d are $E_q, K_q, ZE_{q^{-1}}Z$ respectively.

Lemma 3.16. (See [17, Section 2.7].) The Casimir element Λ acts on \mathbf{V}_d as $(q^{d+1} + q^{-d-1})I$.

Lemma 3.17. (See [32, Lemma 9.8].) Pick $\xi \in \{x, y, z\}$. There exists a basis $\{v_n\}_{n=0}^d$ for \mathbf{V}_d such that

- (i) $\xi v_n = q^{d-2n} v_n$ for $0 \leq n \leq d$;
- (ii) $\sum_{n=0}^d v_n$ is a common eigenvector for the two elements among x, y, z other than ξ .

Definition 3.18. (See [32, Definition 9.5].) Pick $\xi \in \{x, y, z\}$. By a $[\xi]_{\text{row}}$ -basis for \mathbf{V}_d we mean a basis for \mathbf{V}_d from Lemma 3.17.

Note 3.19. The basis for \mathbf{V}_d in Lemma 3.9 is a $[y]_{\text{row}}$ -basis.

We comment on the uniqueness of the bases in Definition 3.18.

Lemma 3.20. (See [32, Lemma 9.12].) Pick $\xi \in \{x, y, z\}$ and let $\{v_n\}_{n=0}^d$ denote a $[\xi]_{\text{row}}$ -basis for \mathbf{V}_d . Let $\{v'_n\}_{n=0}^d$ denote any vectors in \mathbf{V}_d . Then the following are equivalent:

- (i) the sequence $\{v'_n\}_{n=0}^d$ is a $[\xi]_{\text{row}}$ -basis for \mathbf{V}_d ;
- (ii) there exists $0 \neq \alpha \in \mathbb{F}$ such that $v'_n = \alpha v_n$ for $0 \leq n \leq d$.

Definition 3.21. (See [32, Definition 9.5].) Pick $\xi \in \{x, y, z\}$. By a $[\xi]_{\text{row}}^{\text{inv}}$ -basis for \mathbf{V}_d we mean the inversion of a $[\xi]_{\text{row}}$ -basis for \mathbf{V}_d .

In Definition 3.18 and Definition 3.21 we gave the following six bases for \mathbf{V}_d :

$$[x]_{\text{row}}, \quad [y]_{\text{row}}, \quad [z]_{\text{row}}, \tag{9}$$

$$[x]_{\text{row}}^{\text{inv}}, \quad [y]_{\text{row}}^{\text{inv}}, \quad [z]_{\text{row}}^{\text{inv}}. \tag{10}$$

Lemma 3.22. (See [32, Theorem 10.12].) Consider the elements x, y, z of $U_q(\mathfrak{sl}_2)$. In the table below we display the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent these elements with respect to the six bases (9), (10) for \mathbf{V}_d .

Basis	x	y	z
$[x]_{\text{row}}$	K_q	$ZE_{q^{-1}}Z$	E_q
$[x]_{\text{row}}^{\text{inv}}$	$K_{q^{-1}}$	$E_{q^{-1}}$	ZE_qZ
$[y]_{\text{row}}$	E_q	K_q	$ZE_{q^{-1}}Z$
$[y]_{\text{row}}^{\text{inv}}$	ZE_qZ	$K_{q^{-1}}$	$E_{q^{-1}}$
$[z]_{\text{row}}$	$ZE_{q^{-1}}Z$	E_q	K_q
$[z]_{\text{row}}^{\text{inv}}$	$E_{q^{-1}}$	ZE_qZ	$K_{q^{-1}}$

For more background information on the bases (9), (10) we refer the reader to [32].

4. The q -tetrahedron algebra \boxtimes_q

In this section we recall the q -tetrahedron algebra \boxtimes_q and review some of its properties. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4.

Definition 4.1. (See [10, Definition 6.1].) Let \boxtimes_q denote the \mathbb{F} -algebra defined by generators

$$\{x_{ij} \mid i, j \in \mathbb{Z}_4, j - i = 1 \text{ or } j - i = 2\} \tag{11}$$

and the following relations:

- (i) For $i, j \in \mathbb{Z}_4$ such that $j - i = 2$,

$$x_{ij}x_{ji} = 1. \tag{12}$$

(ii) For $i, j, k \in \mathbb{Z}_4$ such that $(j - i, k - j)$ is one of $(1, 1), (1, 2), (2, 1),$

$$\frac{qx_{ij}x_{jk} - q^{-1}x_{jk}x_{ij}}{q - q^{-1}} = 1. \tag{13}$$

(iii) For $i, j, k, \ell \in \mathbb{Z}_4$ such that $j - i = k - j = \ell - k = 1,$

$$x_{ij}^3x_{k\ell} - [3]_q x_{ij}^2x_{k\ell}x_{ij} + [3]_q x_{ij}x_{k\ell}x_{ij}^2 - x_{k\ell}x_{ij}^3 = 0. \tag{14}$$

We call \boxtimes_q the q -tetrahedron algebra. The elements (11) are called the *standard generators* for \boxtimes_q .

We have some comments.

Note 4.2. We find it illuminating to view \boxtimes_q as follows. Identify \mathbb{Z}_4 with the vertex set of a tetrahedron. View each standard generator x_{ij} as an orientation $i \rightarrow j$ of the edge in the tetrahedron that involves vertices i and j .

Lemma 4.3. *There exists an automorphism ρ of \boxtimes_q that sends each standard generator x_{ij} to $x_{i+1,j+1}$. Moreover $\rho^4 = 1$.*

Lemma 4.4. *There exists an automorphism σ of \boxtimes_q that sends each standard generator x_{ij} to $-x_{ij}$. We have $\sigma^2 = 1$ if $\text{Char}(\mathbb{F}) \neq 2$ and $\sigma = 1$ if $\text{Char}(\mathbb{F}) = 2$.*

Lemma 4.5. (See [21, Proposition 4.3].) *For $i \in \mathbb{Z}_4$ there exists an \mathbb{F} -algebra homomorphism $\kappa_i : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ that sends*

$$x \mapsto x_{i+2,i+3}, \quad y \mapsto x_{i+3,i+1}, \quad y^{-1} \mapsto x_{i+1,i+3}, \quad z \mapsto x_{i+1,i+2}.$$

This homomorphism is injective.

Recall the Casimir element Λ of $U_q(\mathfrak{sl}_2)$, from Definition 3.5.

Definition 4.6. For $i \in \mathbb{Z}_4$ let Υ_i denote the image of Λ under the injection κ_i from Lemma 4.5.

The elements Υ_i from Definition 4.6 are not central in \boxtimes_q . However we do have the following.

Lemma 4.7. *For $i \in \mathbb{Z}_4$ the element Υ_i commutes with each of*

$$x_{i+2,i+3}, \quad x_{i+3,i+1}, \quad x_{i+1,i+3}, \quad x_{i+1,i+2}.$$

Proof. By Lemma 4.5 and since Λ is central in $U_q(\mathfrak{sl}_2)$. \square

5. Comparing \boxtimes_q and $\boxtimes_{q^{-1}}$

In this section we compare the algebras \boxtimes_q and $\boxtimes_{q^{-1}}$. For both algebras we use the same notation x_{ij} for the standard generators.

Lemma 5.1. *There exists an \mathbb{F} -algebra isomorphism $\vartheta : \boxtimes_q \rightarrow \boxtimes_{q^{-1}}$ that sends*

$$\begin{aligned} x_{01} &\mapsto x_{01}, & x_{12} &\mapsto x_{30}, & x_{23} &\mapsto x_{23}, & x_{30} &\mapsto x_{12}, \\ x_{02} &\mapsto x_{31}, & x_{13} &\mapsto x_{20}, & x_{20} &\mapsto x_{13}, & x_{31} &\mapsto x_{02}. \end{aligned}$$

Proof. Routine. \square

We recall the notion of antiisomorphism. Given \mathbb{F} -algebras \mathcal{A}, \mathcal{B} a map $\gamma : \mathcal{A} \rightarrow \mathcal{B}$ is called an *antiisomorphism of \mathbb{F} -algebras* whenever γ is an isomorphism of \mathbb{F} -vector spaces and $(ab)^\gamma = b^\gamma a^\gamma$ for all $a, b \in \mathcal{A}$. An antiisomorphism can be interpreted as follows. The \mathbb{F} -vector space \mathcal{B} has an \mathbb{F} -algebra structure \mathcal{B}^{opp} such that for all $a, b \in \mathcal{B}$ the product ab (in \mathcal{B}^{opp}) is equal to ba (in \mathcal{B}). A map $\gamma : \mathcal{A} \rightarrow \mathcal{B}$ is an antiisomorphism of \mathbb{F} -algebras if and only if $\gamma : \mathcal{A} \rightarrow \mathcal{B}^{opp}$ is an isomorphism of \mathbb{F} -algebras.

Proposition 5.2. *There exists an \mathbb{F} -algebra antiisomorphism $\tau : \boxtimes_q \rightarrow \boxtimes_{q^{-1}}$ that sends each standard generator x_{ij} to x_{ij} .*

Proof. In Definition 4.1 we gave a presentation for \boxtimes_q by generators and relations. We now modify this presentation by adjusting the relations as follows. For each relation in the presentation, replace q by q^{-1} and invert the order of multiplication. In each case, the adjusted relation coincides with the original one. Now on one hand, the modified presentation is a presentation for $(\boxtimes_{q^{-1}})^{opp}$ by generators and relations. On the other hand, the modified presentation coincides with the original one. Therefore there exists an \mathbb{F} -algebra isomorphism $\tau : \boxtimes_q \rightarrow (\boxtimes_{q^{-1}})^{opp}$ that sends each standard generator x_{ij} to x_{ij} . The result follows by the sentence prior to the theorem statement. \square

6. Finite-dimensional irreducible \boxtimes_q -modules

In this section we review some basic facts and notation concerning finite-dimensional irreducible \boxtimes_q -modules. This material is summarized from [10].

Let V denote a vector space over \mathbb{F} with finite positive dimension. Let $\{s_n\}_{n=0}^d$ denote a sequence of positive integers whose sum is the dimension of V . By a *decomposition of V of shape $\{s_n\}_{n=0}^d$* we mean a sequence $\{V_n\}_{n=0}^d$ of subspaces for V such that V_n has dimension s_n for $0 \leq n \leq d$ and $V = \sum_{n=0}^d V_n$ (direct sum). For notational convenience define $V_{-1} = 0$ and $V_{d+1} = 0$.

Now let V denote a finite-dimensional irreducible \boxtimes_q -module. By [10, Theorem 12.3] each standard generator x_{ij} of \boxtimes_q is diagonalizable on V . Also by [10, Theorem 12.3]

there exist $d \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$ such that for each x_{ij} the set of distinct eigenvalues on V is $\{\varepsilon q^{d-2n} \mid 0 \leq n \leq d\}$. We call d the *diameter* of V . We call ε the *type* of V . Replacing each x_{ij} by εx_{ij} the type becomes 1. So without loss of generality we may assume that V has type 1. From now on we adopt this assumption. For distinct $i, j \in \mathbb{Z}_4$ we now define a decomposition of V called $[i, j]$. First assume $j - i = 1$ or $j - i = 2$, so that x_{ij} exists. The decomposition $[i, j]$ has diameter d , and for $0 \leq n \leq d$ the n th component of $[i, j]$ is the eigenspace of x_{ij} with eigenvalue q^{d-2n} . Next assume $j - i = 3$. In this case the decomposition $[i, j]$ is defined as the inversion of $[j, i]$. By the construction and Definition 4.1(i), for distinct $i, j \in \mathbb{Z}_4$ the decomposition $[i, j]$ is the inversion of $[j, i]$. By [10, Proposition 13.3], for distinct $i, j \in \mathbb{Z}_4$ the shape of $[i, j]$ is independent of the choice of i, j . Denote this shape by $\{\rho_n\}_{n=0}^d$ and note that $\rho_n = \rho_{d-n}$ for $0 \leq n \leq d$. By the *shape of V* we mean the sequence $\{\rho_n\}_{n=0}^d$ [10, Definition 13.4].

One feature of the shape $\{\rho_n\}_{n=0}^d$ is that $\rho_{n-1} \leq \rho_n$ for $1 \leq n \leq d/2$. This feature is obtained as follows. Pick $i \in \mathbb{Z}_4$ and consider the homomorphism $\kappa_i : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ from Lemma 4.5. Using κ_i we pull back the \boxtimes_q -module structure on V to obtain a $U_q(\mathfrak{sl}_2)$ -module structure on V . The $U_q(\mathfrak{sl}_2)$ -module V is completely reducible; this means that V is a direct sum of irreducible $U_q(\mathfrak{sl}_2)$ -submodules [17, Theorem 2.9]. For this sum consider the summands. Each summand has type 1. For each summand the diameter is at most d and has the same parity as d . For $0 \leq n \leq d/2$ the following coincide: (i) the multiplicity with which the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_{d-2n} appears as a summand; (ii) the integer $\rho_n - \rho_{n-1}$, where $\rho_{-1} = 0$. Therefore $\rho_{n-1} \leq \rho_n$ for $1 \leq n \leq d/2$. We just mentioned some multiplicities. These multiplicities are independent of the $i \in \mathbb{Z}_4$ that we initially picked. Therefore, up to isomorphism of $U_q(\mathfrak{sl}_2)$ -modules the $U_q(\mathfrak{sl}_2)$ -module V is independent of i .

Returning to the \boxtimes_q -module V , for distinct $i, j \in \mathbb{Z}_4$ and each standard generator x_{rs} we now describe the action of x_{rs} on the decomposition $[i, j]$ of V . Denote this decomposition by $\{V_n\}_{n=0}^d$. For the case $s - r = 1$, by [10, Theorem 14.1] the action of x_{rs} on V_n is described in the table below:

Decomposition	Action of $x_{r,r+1}$ on V_n
$[r, r + 1]$	$(x_{r,r+1} - q^{d-2n}I)V_n = 0$
$[r + 1, r]$	$(x_{r,r+1} - q^{2n-d}I)V_n = 0$
$[r + 1, r + 2]$	$(x_{r,r+1} - q^{2n-d}I)V_n \subseteq V_{n-1}$
$[r + 2, r + 1]$	$(x_{r,r+1} - q^{d-2n}I)V_n \subseteq V_{n+1}$
$[r + 2, r + 3]$	$x_{r,r+1}V_n \subseteq V_{n-1} + V_n + V_{n+1}$
$[r + 3, r + 2]$	$x_{r,r+1}V_n \subseteq V_{n-1} + V_n + V_{n+1}$
$[r + 3, r]$	$(x_{r,r+1} - q^{2n-d}I)V_n \subseteq V_{n+1}$
$[r, r + 3]$	$(x_{r,r+1} - q^{d-2n}I)V_n \subseteq V_{n-1}$
$[r, r + 2]$	$(x_{r,r+1} - q^{d-2n}I)V_n \subseteq V_{n-1}$
$[r + 2, r]$	$(x_{r,r+1} - q^{2n-d}I)V_n \subseteq V_{n+1}$
$[r + 1, r + 3]$	$(x_{r,r+1} - q^{2n-d}I)V_n \subseteq V_{n-1}$
$[r + 3, r + 1]$	$(x_{r,r+1} - q^{d-2n}I)V_n \subseteq V_{n+1}$

For the case $s - r = 2$, by [10, Theorem 14.2] the action of x_{rs} on V_n is described in the table below:

Decomposition	Action of $x_{r,r+2}$ on V_n
$[r, r + 1]$	$(x_{r,r+2} - q^{d-2n}I)V_n \subseteq V_0 + \cdots + V_{n-1}$
$[r + 1, r]$	$(x_{r,r+2} - q^{2n-d}I)V_n \subseteq V_{n+1} + \cdots + V_d$
$[r + 1, r + 2]$	$(x_{r,r+2} - q^{d-2n}I)V_n \subseteq V_{n+1} + \cdots + V_d$
$[r + 2, r + 1]$	$(x_{r,r+2} - q^{2n-d}I)V_n \subseteq V_0 + \cdots + V_{n-1}$
$[r + 2, r + 3]$	$(x_{r,r+2} - q^{2n-d}I)V_n \subseteq V_{n-1}$
$[r + 3, r + 2]$	$(x_{r,r+2} - q^{d-2n}I)V_n \subseteq V_{n+1}$
$[r + 3, r]$	$(x_{r,r+2} - q^{2n-d}I)V_n \subseteq V_{n+1}$
$[r, r + 3]$	$(x_{r,r+2} - q^{d-2n}I)V_n \subseteq V_{n-1}$
$[r, r + 2]$	$(x_{r,r+2} - q^{d-2n}I)V_n = 0$
$[r + 2, r]$	$(x_{r,r+2} - q^{2n-d}I)V_n = 0$
$[r + 1, r + 3]$	$x_{r,r+2}V_n \subseteq V_{n-1} + \cdots + V_d$
$[r + 3, r + 1]$	$x_{r,r+2}V_n \subseteq V_0 + \cdots + V_{n+1}$

We recall the notion of a flag. For the moment let V denote a vector space over \mathbb{F} with finite positive dimension and let $\{s_n\}_{n=0}^d$ denote a sequence of positive integers whose sum is the dimension of V . By a *flag on V of shape $\{s_n\}_{n=0}^d$* we mean a sequence $\{U_n\}_{n=0}^d$ of subspaces for V such that $U_{n-1} \subseteq U_n$ for $1 \leq n \leq d$ and U_n has dimension $s_0 + \cdots + s_n$ for $0 \leq n \leq d$. Observe that $U_d = V$. The following construction yields a flag on V . Let $\{V_n\}_{n=0}^d$ denote a decomposition of V of shape $\{s_n\}_{n=0}^d$. Define $U_n = V_0 + \cdots + V_n$ for $0 \leq n \leq d$. Then the sequence $\{U_n\}_{n=0}^d$ is a flag on V of shape $\{s_n\}_{n=0}^d$. This flag is said to be *induced* by the decomposition $\{V_n\}_{n=0}^d$. We now recall what it means for two flags to be opposite. Suppose we are given two flags on V with the same diameter: $\{U_n\}_{n=0}^d$ and $\{U'_n\}_{n=0}^d$. These flags are called *opposite* whenever there exists a decomposition $\{V_n\}_{n=0}^d$ of V that induces $\{U_n\}_{n=0}^d$ and whose inversion induces $\{U'_n\}_{n=0}^d$. In this case $V_n = U_n \cap U'_{d-n}$ for $0 \leq n \leq d$, and also $U_r \cap U'_s = 0$ if $r + s < d$ ($0 \leq r, s \leq d$) [26, p. 846].

We now return our attention to \boxtimes_q -modules. Let V denote a finite-dimensional irreducible \boxtimes_q -module of type 1 and diameter d . By [10, Theorem 16.1] there exists a collection of flags on V , denoted $[i]$, $i \in \mathbb{Z}_4$, such that for distinct $i, j \in \mathbb{Z}_4$ the decomposition $[i, j]$ induces $[i]$. By construction, the shape of the flag $[i]$ coincides with the shape of V . By construction, the flags $[i]$, $i \in \mathbb{Z}_4$ are mutually opposite. Also by construction, for distinct $i, j \in \mathbb{Z}_4$ and $0 \leq n \leq d$ the n th component of $[i, j]$ is the intersection of the following two sets:

- (i) component n of flag $[i]$;
- (ii) component $d - n$ of flag $[j]$.

We comment on a very special case. Up to isomorphism there exists a unique \boxtimes_q -module of dimension one, on which each standard generator acts as the identity.

This \boxtimes_q -module is said to be *trivial*. For a \boxtimes_q -module V the following are equivalent: (i) V is trivial; (ii) V is finite-dimensional and irreducible, with type 1 and diameter 0.

7. The dual space

Throughout this section V denotes a finite-dimensional irreducible \boxtimes_q -module of type 1 and diameter d . By definition, the dual space V^* is the vector space over \mathbb{F} consisting of the \mathbb{F} -linear maps from V to \mathbb{F} . The vector spaces V and V^* have the same dimension. In this section we will turn V^* into a $\boxtimes_{q^{-1}}$ -module, and discuss how this module is related to the original \boxtimes_q -module V .

Definition 7.1. Define a bilinear form $(\cdot, \cdot) : V \times V^* \rightarrow \mathbb{F}$ such that $(u, f) = f(u)$ for all $u \in V$ and $f \in V^*$. The form (\cdot, \cdot) is nondegenerate.

Vectors $u \in V$ and $v \in V^*$ are called *orthogonal* whenever $(u, v) = 0$.

We recall the adjoint map [22, p. 227]. Let $A \in \text{End}(V)$. The *adjoint* of A , denoted A^{adj} , is the unique element of $\text{End}(V^*)$ such that $(Au, v) = (u, A^{adj}v)$ for all $u \in V$ and $v \in V^*$. The adjoint map $\text{End}(V) \rightarrow \text{End}(V^*)$, $A \mapsto A^{adj}$ is an antiisomorphism of \mathbb{F} -algebras.

Recall the antiisomorphism $\tau : \boxtimes_q \rightarrow \boxtimes_{q^{-1}}$ from Proposition 5.2.

Proposition 7.2. *There exists a unique $\boxtimes_{q^{-1}}$ -module structure on V^* such that*

$$(\zeta u, v) = (u, \zeta^\tau v) \quad u \in V, v \in V^*, \zeta \in \boxtimes_q. \tag{15}$$

Proof. The action of \boxtimes_q on V induces an \mathbb{F} -algebra homomorphism $\boxtimes_q \rightarrow \text{End}(V)$. Call this homomorphism ψ . The composition

$$\boxtimes_{q^{-1}} \xrightarrow{\tau^{-1}} \boxtimes_q \xrightarrow{\psi} \text{End}(V) \xrightarrow{adj} \text{End}(V^*)$$

is an \mathbb{F} -algebra homomorphism. This homomorphism gives V^* a $\boxtimes_{q^{-1}}$ -module structure. By construction the $\boxtimes_{q^{-1}}$ -module V^* satisfies (15). We have shown that the desired $\boxtimes_{q^{-1}}$ -module structure exists. One routinely checks that this structure is unique. \square

Proposition 7.3. *For all $\zeta \in \boxtimes_q$, ζ^τ acts on V^* as the adjoint of the action of ζ on V .*

Proof. By (15) and the definition of adjoint from above Proposition 7.2. \square

Proposition 7.4. *For each standard generator x_{ij} ,*

$$(x_{ij}u, v) = (u, x_{ij}v) \quad u \in V, v \in V^*.$$

Proof. Evaluate (15) using Proposition 5.2. \square

Given a subspace W of V (resp. V^*) let W^\perp denote the set of vectors in V^* (resp. V) that are orthogonal to everything in W . The space W^\perp is called the *orthogonal complement* of W . For W, W^\perp the sum of the dimensions is equal to the common dimension of V, V^* . Note that $(W^\perp)^\perp = W$.

Lemma 7.5. *For a subspace $W \subseteq V$ and $\zeta \in \boxtimes_q$, W is ζ -invariant if and only if W^\perp is ζ^τ -invariant.*

Proof. Use (15). \square

Lemma 7.6. *The $\boxtimes_{q^{-1}}$ -module V^* is irreducible.*

Proof. Let W denote a $\boxtimes_{q^{-1}}$ -submodule of V^* . We show that $W = 0$ or $W = V^*$. Consider the orthogonal complement $W^\perp \subseteq V$. By Lemma 7.5 W^\perp is a \boxtimes_q -submodule of V . The \boxtimes_q -module V is irreducible so $W^\perp = V$ or $W^\perp = 0$. It follows that $W = 0$ or $W = V^*$. \square

Lemma 7.7. *For $\zeta \in \boxtimes_q$ the following coincide:*

- (i) *the minimal polynomial for the action of ζ on V ;*
- (ii) *the minimal polynomial for the action of ζ^τ on V^* .*

Proof. Use (15). \square

Proposition 7.8. *The $\boxtimes_{q^{-1}}$ -module V^* has type 1 and diameter d .*

Proof. By assumption the \boxtimes_q -module V has type 1 and diameter d . Therefore each x_{ij} is diagonalizable on V with eigenvalues $\{q^{d-2n} \mid 0 \leq n \leq d\}$. Now by Lemma 7.7 and since $x_{ij}^\tau = x_{ij}$, the element x_{ij} is diagonalizable on V^* with eigenvalues $\{q^{d-2n} \mid 0 \leq n \leq d\}$. The result follows. \square

Recall that V is an irreducible \boxtimes_q -module of type 1 and diameter d . We described V in Section 6. In view of Lemma 7.6 and Proposition 7.8, this description (with q replaced by q^{-1}) applies to V^* . In particular, for distinct $i, j \in \mathbb{Z}_4$ we may speak of the decomposition $[i, j]$ of V^* , and for $i \in \mathbb{Z}_4$ we may speak of the flag $[i]$ on V^* .

Suppose we are given a decomposition of V and a decomposition of V^* . These decompositions are said to be *dual* whenever (i) they have the same diameter δ ; (ii) for distinct $0 \leq i, j \leq \delta$ the i th component of the one is orthogonal to the j th component of the other. Each decomposition of V (resp. V^*) is dual to a unique decomposition of V^* (resp. V). Dual decompositions have the same shape.

Proposition 7.9. For distinct $i, j \in \mathbb{Z}_4$ the decomposition $[i, j]$ of V is dual to the decomposition $[j, i]$ of V^* .

Proof. First assume $j - i = 1$ or $j - i = 2$, so that x_{ij} exists. Pick distinct integers r, s ($0 \leq r, s \leq d$). Let u denote a vector in component r of the decomposition $[i, j]$ of V , and let v denote a vector in component s of the decomposition $[j, i]$ of V^* . We show that $(u, v) = 0$. By Proposition 7.4 $(x_{ij}u, v) = (u, x_{ij}v)$. By construction $x_{ij}u = q^{d-2r}u$ and $x_{ij}v = q^{d-2s}v$. Observe that $q^{d-2r} \neq q^{d-2s}$ since q is not a root of unity. By these comments $(u, v) = 0$. Next assume that $j - i = 3$. By the first part of this proof, the decomposition $[j, i]$ of V is dual to the decomposition $[i, j]$ of V^* . Invert both decompositions to find that the decomposition $[i, j]$ of V is dual to the decomposition $[j, i]$ of V^* . \square

Proposition 7.10. The \mathbb{F}_q -module V and the $\mathbb{F}_{q^{-1}}$ -module V^* have the same shape.

Proof. Pick distinct $i, j \in \mathbb{Z}_4$. By definition, the shape of the \mathbb{F}_q -module V is equal to the shape of the decomposition $[i, j]$ of V . Similarly the shape of the $\mathbb{F}_{q^{-1}}$ -module V^* is equal to the shape of the decomposition $[j, i]$ of V^* . All these shapes are the same, by Proposition 7.9 and the sentence prior to it. \square

Proposition 7.11. For $i \in \mathbb{Z}_4$ and $0 \leq n \leq d - 1$ the following are orthogonal complements with respect to the bilinear form $(,)$:

- (i) component n of the flag $[i]$ on V ;
- (ii) component $d - n - 1$ of the flag $[i]$ on V^* .

Proof. Pick $j \in \mathbb{Z}_4$ with $j \neq i$. By construction, the decomposition $[i, j]$ of V (resp. V^*) induces the flag $[i]$ of V (resp. V^*). The decomposition $[i, j]$ of V^* is the inversion of the decomposition $[j, i]$ of V^* . The result follows from these comments and Proposition 7.9. \square

8. Twisting

Recall the automorphism ρ of \mathbb{F}_q from Lemma 4.3. In this section we investigate what happens when we twist a \mathbb{F}_q -module via ρ .

Definition 8.1. Let V denote a \mathbb{F}_q -module and let π denote an automorphism of \mathbb{F}_q . Then there exists a \mathbb{F}_q -module structure on V , called V twisted via π , that behaves as follows: for all $\zeta \in \mathbb{F}_q$ and $v \in V$, the vector ζv computed in V twisted via π coincides with the vector $\pi^{-1}(\zeta)v$ computed in the original \mathbb{F}_q -module V . Sometimes we abbreviate ${}^\pi V$ for V twisted via π .

Lemma 8.2. Referring to [Definition 8.1](#), the \boxtimes_q -module V is irreducible if and only if the \boxtimes_q -module ${}^\pi V$ is irreducible.

Proof. Immediate from [Definition 8.1](#). \square

Referring to [Definition 8.1](#), we now consider the case $\pi = \rho$.

Lemma 8.3. Let V denote a \boxtimes_q -module. For each standard generator x_{ij} of \boxtimes_q , the following are the same:

- (i) the action of x_{ij} on the \boxtimes_q -module V ;
- (ii) the action of $x_{i+1,j+1}$ on the \boxtimes_q -module ${}^\rho V$.

Proof. By [Lemma 4.3](#) and [Definition 8.1](#). \square

Lemma 8.4. Let V denote a finite-dimensional irreducible \boxtimes_q -module of type 1 and diameter d . Then the \boxtimes_q -module ${}^\rho V$ is irreducible, with type 1 and diameter d .

Proof. By [Lemma 8.2](#) and [Lemma 8.3](#), along with the meaning of type and diameter. \square

For the following three lemmas the proof is routine and left to the reader.

Lemma 8.5. Let V denote a finite-dimensional irreducible \boxtimes_q -module of type 1. Then for distinct $i, j \in \mathbb{Z}_4$ the following coincide:

- (i) the decomposition $[i, j]$ of V ;
- (ii) the decomposition $[i + 1, j + 1]$ of ${}^\rho V$.

Lemma 8.6. Let V denote a finite-dimensional irreducible \boxtimes_q -module of type 1. Then for $i \in \mathbb{Z}_4$ the following coincide:

- (i) the flag $[i]$ on V ;
- (ii) the flag $[i + 1]$ on ${}^\rho V$.

Lemma 8.7. Let V denote a finite-dimensional irreducible \boxtimes_q -module of type 1. Then the following coincide:

- (i) the shape of V ;
- (ii) the shape of ${}^\rho V$.

9. Evaluation modules

We have been discussing the finite-dimensional irreducible \boxtimes_q -modules. We now restrict our attention to a special case, called an evaluation module.

Definition 9.1. By an *evaluation module* for \boxtimes_q we mean a nontrivial, finite-dimensional, irreducible \boxtimes_q -module, of type 1 and shape $(1, 1, \dots, 1)$.

Let V denote an evaluation module for \boxtimes_q . Since V is nontrivial the diameter $d \geq 1$. Each standard generator x_{ij} is multiplicity-free on V , with eigenvalues $\{q^{d-2n} \mid 0 \leq n \leq d\}$. In Section 6, for each $i \in \mathbb{Z}_4$ we used the homomorphism $\kappa_i : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ to turn V into a $U_q(\mathfrak{sl}_2)$ -module. Each of these $U_q(\mathfrak{sl}_2)$ -modules is isomorphic to \mathbf{V}_d . In order to recover the \boxtimes_q -module V from \mathbf{V}_d , we add extra structure involving a parameter t . This is done as follows.

Proposition 9.2. *Pick an integer $d \geq 1$ and a nonzero $t \in \mathbb{F}$ that is not among $\{q^{d-2n+1}\}_{n=1}^d$. Then there exists an evaluation module $\mathbf{V}_d(t)$ for \boxtimes_q with the following property: $\mathbf{V}_d(t)$ has a basis with respect to which the standard generators x_{ij} are represented by the following matrices in $\text{Mat}_{d+1}(\mathbb{F})$:*

Generator	x_{01}	x_{12}	x_{23}	x_{30}
Representing matrix	$ZS_{q^{-1}}(t^{-1})Z$	E_q	K_q	$ZG_{q^{-1}}(t)Z$
Generator	x_{02}	x_{13}	x_{20}	x_{31}
Representing matrix	$L_q(t)$	$(ZE_{q^{-1}}Z)^{-1}$	$(L_q(t))^{-1}$	$ZE_{q^{-1}}Z$

(The above matrices are defined in [Appendix A](#).)

Proof. It is routine (but tedious) to check that the above matrices satisfy the defining relations for \boxtimes_q . This makes $\mathbf{V}_d(t)$ a \boxtimes_q -module. The \boxtimes_q -module $\mathbf{V}_d(t)$ is irreducible by [Note 3.15](#) and since the matrices $E_q, K_q, ZE_{q^{-1}}Z$ are included in the above tables. By construction the \boxtimes_q -module $\mathbf{V}_d(t)$ is nontrivial, with type 1 and shape $(1, 1, \dots, 1)$. \square

We have a comment.

Lemma 9.3. *On the evaluation module $\mathbf{V}_d(t)$ from [Proposition 9.2](#), the actions of the standard generators (11) are mutually distinct.*

Proof. The eight representing matrices in the tables of [Proposition 9.2](#) are mutually distinct. This is checked using the definitions in [Appendix A](#). \square

Consider the \boxtimes_q -module $\mathbf{V}_d(t)$ from [Proposition 9.2](#). This module has dimension $d+1$. Now let V denote any evaluation module for \boxtimes_q that has dimension $d+1$. Shortly we will show that the \boxtimes_q -module V is isomorphic to $\mathbf{V}_d(t)$ for a unique t .

Lemma 9.4. *On the \boxtimes_q -module $\mathbf{V}_d(t)$,*

$$t(x_{01} - x_{23}) = \frac{[x_{30}, x_{12}]}{q - q^{-1}}, \quad t^{-1}(x_{12} - x_{30}) = \frac{[x_{01}, x_{23}]}{q - q^{-1}}. \tag{16}$$

Proof. To get the equation on the right, represent the standard generators by matrices as in Proposition 9.2. To get the equation on the left, start with the equation on the right. In this equation take the commutator of x_{12} with each side, and evaluate the result using Lemma 3.4 (with $u = x_{01}$, $v = x_{12}$, $w = x_{23}$). \square

Lemma 9.5. *On the \boxtimes_q -module $\mathbf{V}_d(t)$,*

$$t(x_{01} - x_{02}) = \frac{[x_{30}, x_{02}]}{q - q^{-1}}, \quad t^{-1}(x_{12} - x_{13}) = \frac{[x_{01}, x_{13}]}{q - q^{-1}}, \tag{17}$$

$$t(x_{23} - x_{20}) = \frac{[x_{12}, x_{20}]}{q - q^{-1}}, \quad t^{-1}(x_{30} - x_{31}) = \frac{[x_{23}, x_{31}]}{q - q^{-1}}, \tag{18}$$

and also

$$t^{-1}(x_{30} - x_{20}) = \frac{[x_{20}, x_{01}]}{q - q^{-1}}, \quad t(x_{01} - x_{31}) = \frac{[x_{31}, x_{12}]}{q - q^{-1}}, \tag{19}$$

$$t^{-1}(x_{12} - x_{02}) = \frac{[x_{02}, x_{23}]}{q - q^{-1}}, \quad t(x_{23} - x_{13}) = \frac{[x_{13}, x_{30}]}{q - q^{-1}}. \tag{20}$$

Proof. We first verify the equations on the left in (17)–(20). Call these equations (17L)–(20L). To obtain (20L), represent the standard generators by matrices as in Proposition 9.2. To obtain (17L), in (20L) take the commutator of x_{30} with each side, and evaluate the result using Lemma 3.4 (with $u = x_{23}$, $v = x_{30}$, $w = x_{02}$) along with the equation on the left in (16). To obtain (19L), in (17L) take the commutator of each side with x_{20} , and evaluate the result using Lemma 3.3(ii). To obtain (18L), in (19L) take the commutator of x_{12} with each side, and evaluate the result using Lemma 3.4 (with $u = x_{01}$, $v = x_{12}$, $w = x_{20}$) along with the equation on the left in (16). We have verified the equations on the left in (17)–(20).

We now verify the equations on the right in (17)–(20). Call these equations (17R)–(20R). To obtain (18R), represent the standard generators by matrices as in Proposition 9.2. To obtain (20R), in (18R) take the commutator of each side with x_{13} , and evaluate the result using Lemma 3.3(ii). To obtain (17R), in (20R) take the commutator of x_{01} with each side, and evaluate the result using Lemma 3.4 (with $u = x_{30}$, $v = x_{01}$, $w = x_{13}$) along with the equation on the right in (16). To obtain (19R), in (17R) take the commutator of each side with x_{31} , and evaluate the result using Lemma 3.3(ii). We have verified the equations on the right in (17)–(20). The result follows. \square

Proposition 9.6. *Let V denote an evaluation module for \boxtimes_q that has diameter d . Then there exists a unique $t \in \mathbb{F}$ such that:*

- (i) t is nonzero and not among $\{q^{d-2n+1}\}_{n=1}^d$;
- (ii) the \boxtimes_q -module V is isomorphic to $\mathbf{V}_d(t)$.

Proof. We first show that t exists. In Section 6, above the first table, for each $i \in \mathbb{Z}_4$ we used the homomorphism $\kappa_i : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ to turn V into a $U_q(\mathfrak{sl}_2)$ -module isomorphic to \mathbf{V}_d . Let us take $i = 0$. For this $U_q(\mathfrak{sl}_2)$ -module V let $\{v_n\}_{n=0}^d$ denote an $[x]_{row}$ -basis from Lemma 3.17 and Definition 3.18. By Lemma 3.22 and the construction, with respect to $\{v_n\}_{n=0}^d$ the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent x_{12}, x_{23}, x_{31} are $E_q, K_q, ZE_{q^{-1}}Z$ respectively. Let G (resp. S) (resp. L) denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents x_{30} (resp. x_{01}) (resp. x_{02}) with respect to the basis $\{v_n\}_{n=0}^d$. Using the tables in Section 6 we find that (i) G is lower bidiagonal with (n, n) -entry q^{2n-d} for $0 \leq n \leq d$; (ii) S is tridiagonal; (iii) L is upper bidiagonal with (n, n) -entry q^{2n-d} for $0 \leq n \leq d$. On the above matrices we impose the defining relations for \boxtimes_q , and solve for G, S, L . After a brief calculation we find that there exists a nonzero $t \in \mathbb{F}$ not among $\{q^{d-2n+1}\}_{n=1}^d$ such that

$$G = ZG_{q^{-1}}(t)Z, \quad S = ZS_{q^{-1}}(t^{-1})Z, \quad L = L_q(t).$$

Therefore the \boxtimes_q -module V is isomorphic to $\mathbf{V}_d(t)$. We have shown that t exists. The scalar t is unique by Lemma 9.3 and either equation in (16). \square

Definition 9.7. Let V denote an evaluation module for \boxtimes_q . By the *evaluation parameter* of V we mean the scalar t in Proposition 9.6.

Lemma 9.8. *Two evaluation modules for \boxtimes_q are isomorphic if and only if they have the same diameter and same evaluation parameter.*

Proof. Consider two isomorphic evaluation modules for \boxtimes_q . They have the same diameter, since they have the same dimension and the dimension is one more than the diameter. They have the same evaluation parameter by Proposition 9.6 and Definition 9.7. \square

Recall the elements Υ_i of \boxtimes_q , from Definition 4.6.

Lemma 9.9. *Let V denote an evaluation module for \boxtimes_q that has diameter d . Then for $i \in \mathbb{Z}_4$ the element Υ_i acts on V as $(q^{d+1} + q^{-d-1})I$.*

Proof. Using the homomorphism $\kappa_i : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ we give V a $U_q(\mathfrak{sl}_2)$ -module structure as in Section 6. By Definition 4.6 and the construction, on V the element Υ_i agrees with the Casimir element of $U_q(\mathfrak{sl}_2)$. The result now follows via Lemma 3.16 and since the $U_q(\mathfrak{sl}_2)$ -module V is isomorphic to \mathbf{V}_d . \square

Definition 9.10. Let V denote an evaluation module for \boxtimes_q that has diameter d . By Lemma 9.9, for $i \in \mathbb{Z}_4$ the action of \mathcal{Y}_i on V is independent of i . Denote this common action by \mathcal{Y} . Thus on V ,

$$\mathcal{Y} = (q^{d+1} + q^{-d-1})I. \tag{21}$$

We now give some identities that involve \mathcal{Y} .

Lemma 9.11. Let V denote an evaluation module for \boxtimes_q , with evaluation parameter t . Then on V ,

$$\begin{aligned} \mathcal{Y} &= t(x_{01}x_{23} - 1) + qx_{30} + q^{-1}x_{12}, & \mathcal{Y} &= t^{-1}(x_{12}x_{30} - 1) + qx_{01} + q^{-1}x_{23}, \\ \mathcal{Y} &= t(x_{23}x_{01} - 1) + qx_{12} + q^{-1}x_{30}, & \mathcal{Y} &= t^{-1}(x_{30}x_{12} - 1) + qx_{23} + q^{-1}x_{01}. \end{aligned}$$

Proof. Pick $i \in \mathbb{Z}_4$. In (7) we defined the Casimir element Λ for $U_q(\mathfrak{sl}_2)$. In Lemma 4.5 we described the map $\kappa_i : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$. Apply κ_i to each side of (7), and evaluate the result using Definition 4.6 and Definition 9.10. This shows that on V ,

$$\begin{aligned} \mathcal{Y} &= qx_{i+2,i+3} + q^{-1}x_{i+3,i+1} + qx_{i+1,i+2} - qx_{i+2,i+3}x_{i+3,i+1}x_{i+1,i+2} \\ &= qx_{i+2,i+3} + q^{-1}x_{i+3,i+1} + q(1 - x_{i+2,i+3}x_{i+3,i+1})x_{i+1,i+2}. \end{aligned} \tag{22}$$

By (2) and (17), (18) the following hold on V :

$$\begin{aligned} q(1 - x_{i+2,i+3}x_{i+3,i+1}) &= \frac{[x_{i+2,i+3}, x_{i+3,i+1}]}{q - q^{-1}} \\ &= t^s(x_{i+3,i} - x_{i+3,i+1}), \end{aligned}$$

where $s = (-1)^{i+1}$. Evaluating (22) using these comments we find that on V ,

$$\begin{aligned} \mathcal{Y} &= qx_{i+2,i+3} + q^{-1}x_{i+3,i+1} + t^s(x_{i+3,i} - x_{i+3,i+1})x_{i+1,i+2} \\ &= qx_{i+2,i+3} + q^{-1}x_{i+3,i+1} + t^s(x_{i+3,i}x_{i+1,i+2} - 1) + t^s(1 - x_{i+3,i+1}x_{i+1,i+2}). \end{aligned} \tag{23}$$

By (2) and (19), (20) the following hold on V :

$$\begin{aligned} q(1 - x_{i+3,i+1}x_{i+1,i+2}) &= \frac{[x_{i+3,i+1}, x_{i+1,i+2}]}{q - q^{-1}} \\ &= t^{-s}(x_{i,i+1} - x_{i+3,i+1}). \end{aligned}$$

Evaluating (23) using these comments we find that on V ,

$$\begin{aligned} \mathcal{Y} &= qx_{i+2,i+3} + q^{-1}x_{i+3,i+1} + t^s(x_{i+3,i}x_{i+1,i+2} - 1) + q^{-1}(x_{i,i+1} - x_{i+3,i+1}) \\ &= t^s(x_{i+3,i}x_{i+1,i+2} - 1) + qx_{i+2,i+3} + q^{-1}x_{i,i+1}. \end{aligned}$$

The result follows. \square

Lemma 9.12. *Let V denote an evaluation module for \boxtimes_q , with evaluation parameter t . Then on V ,*

$$\begin{aligned} \Upsilon &= (q + q^{-1})x_{30} + t \left(\frac{qx_{01}x_{23} - q^{-1}x_{23}x_{01}}{q - q^{-1}} - 1 \right), \\ \Upsilon &= (q + q^{-1})x_{01} + t^{-1} \left(\frac{qx_{12}x_{30} - q^{-1}x_{30}x_{12}}{q - q^{-1}} - 1 \right), \\ \Upsilon &= (q + q^{-1})x_{12} + t \left(\frac{qx_{23}x_{01} - q^{-1}x_{01}x_{23}}{q - q^{-1}} - 1 \right), \\ \Upsilon &= (q + q^{-1})x_{23} + t^{-1} \left(\frac{qx_{30}x_{12} - q^{-1}x_{12}x_{30}}{q - q^{-1}} - 1 \right). \end{aligned} \tag{24}$$

Proof. For each displayed equation, evaluate the parenthetical expression using Lemma 9.11 and simplify the result. \square

Note 9.13. Let V denote an evaluation module for \boxtimes_q . Among the standard generators for \boxtimes_q consider the following two pairs: (i) x_{01} and x_{23} ; (ii) x_{12} and x_{30} . Each pair acts on V as a Leonard pair; see [5, Example 1.7] and [10, Theorem 10.3]. Lemma 9.12 shows how each Leonard pair determines the other.

Lemma 9.14. *Let V denote an evaluation module for \boxtimes_q , with evaluation parameter t . Then x_{01}, x_{23} satisfy the following on V :*

$$\begin{aligned} &x_{01}^2x_{23} - (q^2 + q^{-2})x_{01}x_{23}x_{01} + x_{23}x_{01}^2 \\ &= -(q - q^{-1})^2(1 + t^{-1}\Upsilon)x_{01} + (q - q^{-1})(q^2 - q^{-2})t^{-1}, \\ &x_{23}^2x_{01} - (q^2 + q^{-2})x_{23}x_{01}x_{23} + x_{01}x_{23}^2 \\ &= -(q - q^{-1})^2(1 + t^{-1}\Upsilon)x_{23} + (q - q^{-1})(q^2 - q^{-2})t^{-1}. \end{aligned}$$

Moreover x_{12}, x_{30} satisfy the following on V :

$$\begin{aligned} &x_{12}^2x_{30} - (q^2 + q^{-2})x_{12}x_{30}x_{12} + x_{30}x_{12}^2 \\ &= -(q - q^{-1})^2(1 + t\Upsilon)x_{12} + (q - q^{-1})(q^2 - q^{-2})t, \\ &x_{30}^2x_{12} - (q^2 + q^{-2})x_{30}x_{12}x_{30} + x_{12}x_{30}^2 \\ &= -(q - q^{-1})^2(1 + t\Upsilon)x_{30} + (q - q^{-1})(q^2 - q^{-2})t. \end{aligned}$$

Proof. To obtain the first equation, compute (24) times qx_{01} minus $q^{-1}x_{01}$ times (24), and simplify the result using

$$\frac{qx_{30}x_{01} - q^{-1}x_{01}x_{30}}{q - q^{-1}} = 1.$$

The remaining equations are similarly obtained. \square

Note 9.15. The equations in Lemma 9.14 are the Askey–Wilson relations [33, Theorem 1.5] for the Leonard pairs in Note 9.13.

We end this section with some comments about the evaluation parameter.

Lemma 9.16. *Let V denote an evaluation module for \boxtimes_q , with evaluation parameter t . Then the $\boxtimes_{q^{-1}}$ -module V^* is an evaluation module with evaluation parameter t .*

Proof. The $\boxtimes_{q^{-1}}$ -module V^* is an evaluation module by Proposition 7.8 and Proposition 7.10. Let t' denote the corresponding evaluation parameter. We show that $t' = t$. Applying Lemma 9.4 to V we find that on V ,

$$t(x_{01} - x_{23}) = \frac{[x_{30}, x_{12}]}{q - q^{-1}}. \tag{25}$$

Applying Lemma 9.4 to the $\boxtimes_{q^{-1}}$ -module V^* we find that on V^* ,

$$t'(x_{01} - x_{23}) = \frac{[x_{30}, x_{12}]}{q^{-1} - q}. \tag{26}$$

Let Δ denote the left-hand side of (25) minus the right-hand side of (26). Then $\Delta V = 0$, so $\Delta^\tau V^* = 0$ in view of (15). By this and Proposition 5.2 we find that on V^* ,

$$t(x_{01} - x_{23}) = \frac{[x_{12}, x_{30}]}{q - q^{-1}}. \tag{27}$$

We now compare (26) and (27). For these equations the right-hand sides are the same, so the left-hand sides agree on V^* . In other words $(t - t')(x_{01} - x_{23})V^* = 0$. But $(x_{01} - x_{23})V^* \neq 0$ by Lemma 9.3 and since V^* is an evaluation module. Therefore $t = t'$. \square

Lemma 9.17. *Let V denote an evaluation module for \boxtimes_q , with evaluation parameter t . Then the \boxtimes_q -module ${}^\rho V$ is an evaluation module with evaluation parameter t^{-1} .*

Proof. The \boxtimes_q -module ${}^\rho V$ is an evaluation module by Lemma 8.4 and Lemma 8.7. To see that ${}^\rho V$ has evaluation parameter t^{-1} , compare the two equations in Lemma 9.4, and use Lemma 9.3. \square

Corollary 9.18. *Let V denote an evaluation module for \boxtimes_q . Then the following \boxtimes_q -modules are isomorphic: (i) V ; (ii) V twisted via ρ^2 .*

Proof. Apply Lemma 9.17 twice to V , and use Lemma 9.8. \square

10. 24 bases for an evaluation module

Let V denote an evaluation module for \boxtimes_q that has diameter d . In this section we display 24 bases for V that we find attractive. In Section 11 we show how the standard generators for \boxtimes_q act on these bases.

For $i \in \mathbb{Z}_4$ consider the flag $[i]$ on V . We will be discussing component 0 of this flag. This component has dimension one. It is an eigenspace for each standard generator listed in the table below. In each case the corresponding eigenvalue is given.

Generator	$x_{i,i+1}$	$x_{i,i+2}$	$x_{i-1,i}$	$x_{i+2,i}$
Eigenvalue	q^d	q^d	q^{-d}	q^{-d}

Definition 10.1. Let V denote an evaluation module for \boxtimes_q that has diameter d . Pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 . A basis $\{v_n\}_{n=0}^d$ for V is called an $[i, j, k, \ell]$ -basis whenever:

- (i) for $0 \leq n \leq d$ the vector v_n is contained in component n of the decomposition $[k, \ell]$ of V ;
- (ii) $\sum_{n=0}^d v_n$ is contained in component 0 of the flag $[j]$ on V .

We will discuss the existence and uniqueness of the bases in Definition 10.1. We start with uniqueness.

Lemma 10.2. Let V denote an evaluation module for \boxtimes_q that has diameter d . Pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 and let $\{v_n\}_{n=0}^d$ denote an $[i, j, k, \ell]$ -basis for V . Let $\{v'_n\}_{n=0}^d$ denote any vectors in V . Then the following are equivalent:

- (i) the sequence $\{v'_n\}_{n=0}^d$ is an $[i, j, k, \ell]$ -basis for V ;
- (ii) there exists $0 \neq \alpha \in \mathbb{F}$ such that $v'_n = \alpha v_n$ for $0 \leq n \leq d$.

Proof. (i) \Rightarrow (ii) By assumption V has shape $(1, 1, \dots, 1)$.
 (ii) \Rightarrow (i) Immediate from Definition 10.1. \square

Let V denote an evaluation module for \boxtimes_q that has diameter d . In Section 6, for $i \in \mathbb{Z}_4$ we used the homomorphism $\kappa_i : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ to turn V into a $U_q(\mathfrak{sl}_2)$ -module isomorphic to \mathbf{V}_d . Six bases for this $U_q(\mathfrak{sl}_2)$ -module were displayed in (9), (10).

Lemma 10.3. Let V denote an evaluation module for \boxtimes_q . Pick $i \in \mathbb{Z}_4$ and consider the corresponding $U_q(\mathfrak{sl}_2)$ -module V as above. In each row of the table below we display a basis for the $U_q(\mathfrak{sl}_2)$ -module V from (9), (10) and a basis for the \boxtimes_q -module V from Definition 10.1. These two bases are the same.

Basis for V from (9), (10)	Basis for V from Definition 10.1
$[x]_{row}$	$[i, i + 1, i + 2, i + 3]$
$[x]_{row}^{inv}$	$[i, i + 1, i + 3, i + 2]$
$[y]_{row}$	$[i, i + 2, i + 3, i + 1]$
$[y]_{row}^{inv}$	$[i, i + 2, i + 1, i + 3]$
$[z]_{row}$	$[i, i + 3, i + 1, i + 2]$
$[z]_{row}^{inv}$	$[i, i + 3, i + 2, i + 1]$

Proof. By Lemma 4.5 and the construction. \square

Lemma 10.4. *Let V denote an evaluation module for \boxtimes_q , and pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 . Then there exists an $[i, j, k, \ell]$ -basis for V .*

Proof. Immediate from Lemma 10.3. \square

Note 10.5. The basis for $\mathbf{V}_d(t)$ given in Proposition 9.2 is a $[0, 1, 2, 3]$ -basis.

Let V denote an evaluation module for \boxtimes_q . In Definition 10.1 we gave 24 bases for V . In Section 11 we will compute the matrices that represent the standard generators with respect to these bases. We now mention some results that will facilitate this computation.

Lemma 10.6. *Let V denote an evaluation module for \boxtimes_q , and pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 . Then each $[i, j, k, \ell]$ -basis for V is the inversion of an $[i, j, \ell, k]$ -basis for V .*

Proof. By Definition 10.1 and the meaning of inversion. \square

Corollary 10.7. *Let V denote an evaluation module for \boxtimes_q , and pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 . For each standard generator x_{rs} consider the following two matrices:*

- (i) *the matrix that represents x_{rs} with respect to an $[i, j, k, \ell]$ -basis for V ;*
- (ii) *the matrix that represents x_{rs} with respect to an $[i, j, \ell, k]$ -basis for V .*

Each of these matrices is obtained from the other via conjugation by Z .

Proof. By Lemma 10.6 and linear algebra. \square

Lemma 10.8. *Let V denote an evaluation module for \boxtimes_q , and pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 . Then the following are the same:*

- (i) *an $[i, j, k, \ell]$ -basis for V ;*
- (ii) *an $[i + 1, j + 1, k + 1, \ell + 1]$ -basis for ${}^{\rho}V$.*

Proof. Use Lemma 8.5 and Lemma 8.6. \square

Lemma 10.9. *Consider the \boxtimes_q -module $\mathbf{V}_d(t)$. Pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 . Then for each standard generator x_{rs} the following are the same:*

- (i) the matrix that represents x_{rs} with respect to an $[i, j, k, \ell]$ -basis for $\mathbf{V}_d(t)$;
- (ii) the matrix that represents $x_{r+1, s+1}$ with respect to an $[i + 1, j + 1, k + 1, \ell + 1]$ -basis for $\mathbf{V}_d(t^{-1})$.

Proof. Use Lemma 8.3, Lemma 9.17, and Lemma 10.8. \square

Corollary 10.10. Let V denote an evaluation module for \boxtimes_q . Pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 . Then for each standard generator x_{rs} the following are the same:

- (i) the matrix that represents x_{rs} with respect to an $[i, j, k, \ell]$ -basis for V ;
- (ii) the matrix that represents $x_{r+2, s+2}$ with respect to an $[i + 2, j + 2, k + 2, \ell + 2]$ -basis for V .

Proof. Apply Lemma 10.9 twice, or use Corollary 9.18 along with Lemma 10.8. \square

11. The action of the standard generators on the 24 bases

Let V denote an evaluation module for \boxtimes_q . In Definition 10.1 we defined 24 bases for V . In this section we display the matrices that represent the standard generators x_{ij} with respect to these bases.

The matrices displayed in the following theorem are defined in Appendix A.

Theorem 11.1. Let V denote an evaluation module for \boxtimes_q , with diameter d and evaluation parameter t . In the tables below, we display the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent the standard generators x_{ij} with respect to the 24 bases for V from Definition 10.1. Pick $r \in \mathbb{Z}_4$, and first assume that r is even. Then

Basis	$x_{r, r+1}$	$x_{r+1, r+2}$	$x_{r+2, r+3}$	$x_{r+3, r}$
$[r, r + 1, r + 2, r + 3]$	$ZS_{q^{-1}}(t^{-1})Z$	E_q	K_q	$ZG_{q^{-1}}(t)Z$
$[r + 1, r, r + 2, r + 3]$	$S_q(t^{-1})$	$G_q(t)$	K_q	$ZE_{q^{-1}}Z$
$[r, r + 1, r + 3, r + 2]$	$S_{q^{-1}}(t^{-1})$	ZE_qZ	$K_{q^{-1}}$	$G_{q^{-1}}(t)$
$[r + 1, r, r + 3, r + 2]$	$ZS_q(t^{-1})Z$	$ZG_q(t)Z$	$K_{q^{-1}}$	$E_{q^{-1}}$
$[r, r + 2, r + 1, r + 3]$	$F_q(t^{-1})$	$E_{q^{-1}}$	ZE_qZ	$ZF_{q^{-1}}(t)Z$
$[r + 2, r, r + 1, r + 3]$	E_q	$F_{q^{-1}}(t)$	$ZF_q(t^{-1})Z$	$ZE_{q^{-1}}Z$

Basis	$x_{r, r+2}$	$x_{r+1, r+3}$	$x_{r+2, r}$	$x_{r+3, r+1}$
$[r, r + 1, r + 2, r + 3]$	$L_q(t)$	$(ZE_{q^{-1}}Z)^{-1}$	$(L_q(t))^{-1}$	$ZE_{q^{-1}}Z$
$[r + 1, r, r + 2, r + 3]$	E_q	$(ZL_{q^{-1}}(t)Z)^{-1}$	$(E_q)^{-1}$	$ZL_{q^{-1}}(t)Z$
$[r, r + 1, r + 3, r + 2]$	$ZL_q(t)Z$	$(E_{q^{-1}})^{-1}$	$(ZL_q(t)Z)^{-1}$	$E_{q^{-1}}$
$[r + 1, r, r + 3, r + 2]$	ZE_qZ	$(L_{q^{-1}}(t))^{-1}$	$(ZE_qZ)^{-1}$	$L_{q^{-1}}(t)$
$[r, r + 2, r + 1, r + 3]$	$ZM_{q^{-1}}(t^{-1})Z$	K_q	$M_q(t)$	$K_{q^{-1}}$
$[r + 2, r, r + 1, r + 3]$	$ZM_q(t)Z$	K_q	$M_{q^{-1}}(t^{-1})$	$K_{q^{-1}}$

Next assume that r is odd. Then in the above tables replace t by t^{-1} .

Proof. By Lemma 10.9 we may assume that $r = 0$. We now consider six cases. In the following discussion all bases mentioned are for V .

Basis $[0, 1, 2, 3]$. By Proposition 9.2 and Note 10.5.

Basis $[1, 0, 2, 3]$. We first verify the data for x_{23}, x_{30}, x_{02} . Using the map $\kappa_1 : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ we turn V into a $U_q(\mathfrak{sl}_2)$ -module isomorphic to \mathbf{V}_d . By Lemma 10.3 the bases $[1, 0, 2, 3]$ and $[z]_{row}$ are the same. By Lemma 4.5 the equations $x_{23} = z, x_{30} = x, x_{02} = y$ hold on V . By Lemma 3.22 the matrices representing z, x, y with respect to $[z]_{row}$ are $K_q, ZE_{q^{-1}}Z, E_q$ respectively. By these comments the matrices representing x_{23}, x_{30}, x_{02} with respect to $[1, 0, 2, 3]$ are $K_q, ZE_{q^{-1}}Z, E_q$ respectively. By Definition 4.1(i) the generator x_{20} is the inverse of x_{02} . Therefore the matrix representing x_{20} with respect to $[1, 0, 2, 3]$ is $(E_q)^{-1}$. To get the matrix representing x_{01} with respect to $[1, 0, 2, 3]$ use the equation on the left in (17). To get the matrix representing x_{12} with respect to $[1, 0, 2, 3]$ use the equation on the left in (20). We now verify the data for x_{31} and x_{13} . Let L denote the matrix representing x_{31} with respect to $[1, 0, 2, 3]$. By the second table of Section 6, L is lower bidiagonal with (n, n) -entry q^{2n-d} for $0 \leq n \leq d$. On this matrix we impose the defining relations for \boxtimes_q . After a brief calculation we obtain $L = ZL_{q^{-1}}(t)Z$. By Definition 4.1(i) x_{13} is the inverse of x_{31} . Therefore the matrix representing x_{13} with respect to $[1, 0, 2, 3]$ is $(ZL_{q^{-1}}(t)Z)^{-1}$.

Basis $[0, 1, 3, 2]$. Use Corollary 10.7.

Basis $[1, 0, 3, 2]$. Use Corollary 10.7.

Basis $[0, 2, 1, 3]$. We first verify the data for x_{12}, x_{23}, x_{31} . Using the map $\kappa_0 : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ we turn V into a $U_q(\mathfrak{sl}_2)$ -module isomorphic to \mathbf{V}_d . By Lemma 10.3 the bases $[0, 2, 1, 3]$ and $[y]_{row}^{inv}$ are the same. By Lemma 4.5 the equations $x_{12} = z, x_{23} = x, x_{31} = y$ hold on V . By Lemma 3.22 the matrices representing z, x, y with respect to $[y]_{row}^{inv}$ are $E_{q^{-1}}, ZE_qZ, K_{q^{-1}}$ respectively. By these comments the matrices representing x_{12}, x_{23}, x_{31} with respect to $[0, 2, 1, 3]$ are $E_{q^{-1}}, ZE_qZ, K_{q^{-1}}$ respectively. By Definition 4.1(i) the generator x_{13} is the inverse of x_{31} . Therefore the matrix representing x_{13} with respect to $[0, 2, 1, 3]$ is $(K_{q^{-1}})^{-1}$, which is equal to K_q . To get the matrix representing x_{01} with respect to $[0, 2, 1, 3]$ use the equation on the right in (19). To get the matrix representing x_{30} with respect to $[0, 2, 1, 3]$ use the equation on the right in (18). We now verify the data for x_{20} and x_{02} . Let M (resp. M') denote the matrix representing x_{20} (resp. x_{02}) with respect to $[0, 2, 1, 3]$. On this matrix we impose the defining relations for \boxtimes_q . After a brief calculation we obtain $M = M_q(t)$ and $M' = ZM_{q^{-1}}(t^{-1})Z$.

Basis $[2, 0, 1, 3]$. Use Corollary 10.7 and Corollary 10.10. \square

12. A normalization of the 24 bases

Let V denote an evaluation module for \boxtimes_q . In Definition 10.1 we gave 24 bases for V . In Section 13 we will give the transition matrices between certain pairs of bases among the 24. Before doing this, we first normalize our bases.

Definition 12.1. Let V denote an evaluation module for \boxtimes_q . For $i \in \mathbb{Z}_4$ let η_i (resp. η_i^*) denote a nonzero vector in component 0 of the flag $[i]$ on V (resp. V^*).

Lemma 12.2. Let V denote an evaluation module for \boxtimes_q that has diameter d . Then the following (i), (ii) hold for $i \in \mathbb{Z}_4$.

- (i) The vector η_i is an eigenvector for each standard generator listed in the table below. In each case the corresponding eigenvalue is given.

Generator	$x_{i,i+1}$	$x_{i,i+2}$	$x_{i-1,i}$	$x_{i+2,i}$
Eigenvalue	q^d	q^d	q^{-d}	q^{-d}

- (ii) The vector η_i^* is an eigenvector for each standard generator listed in the table below. In each case the corresponding eigenvalue is given.

Generator	$x_{i,i+1}$	$x_{i,i+2}$	$x_{i-1,i}$	$x_{i+2,i}$
Eigenvalue	q^{-d}	q^{-d}	q^d	q^d

Proof. (i) By Definition 12.1 and the paragraph above Definition 10.1.

- (ii) Apply (i) above to the $\boxtimes_{q^{-1}}$ -module V^* . \square

Lemma 12.3. Referring to Definition 12.1 the following (i), (ii) hold.

- (i) For distinct $i, j \in \mathbb{Z}_4$ we have $(\eta_i, \eta_j^*) \neq 0$.
- (ii) For $i \in \mathbb{Z}_4$ we have $(\eta_i, \eta_i^*) = 0$.

Proof. (i) The vector η_i is a basis for component 0 of the decomposition $[i, j]$ for V . The vector η_j^* is a basis for component 0 of the decomposition $[j, i]$ for V^* . By Proposition 7.9 the decomposition $[i, j]$ for V is dual to the decomposition $[j, i]$ for V^* . Therefore $(\eta_i, \eta_j^*) \neq 0$.

(ii) Use Proposition 7.11 with $n = 0$. Recall that V is nontrivial so it has diameter $d \geq 1$. \square

Lemma 12.4. Let V denote an evaluation module for \boxtimes_q that has diameter d . Given mutually distinct $i, j, k, \ell \in \mathbb{Z}_4$ there exists a unique basis $\{u_n\}_{n=0}^d$ for V such that (i), (ii) hold below:

- (i) for $0 \leq n \leq d$ the vector u_n is contained in component n of the decomposition $[k, \ell]$ of V ;
- (ii) $\eta_j = \sum_{n=0}^d u_n$.

We denote this basis by $[i, j, k, \ell]$.

Proof. We first show that the basis $[i, j, k, \ell]$ exists. By Lemma 10.4 there exists an $[i, j, k, \ell]$ -basis for V . Denote this basis by $\{v_n\}_{n=0}^d$. Define $v = \sum_{n=0}^d v_n$ and note that $v \neq 0$. By Definition 10.1 v is contained in component 0 of the flag $[j]$ for V . By construction η_j is a basis for component 0 of the flag $[j]$ for V . Therefore there exists $0 \neq \alpha \in \mathbb{F}$ such that $v = \alpha\eta_j$. Define $u_n = \alpha^{-1}v_n$ for $0 \leq n \leq d$. Then $\{u_n\}_{n=0}^d$ is a basis for V that satisfies the requirements (i), (ii). One checks using Lemma 10.2 that the basis $[i, j, k, \ell]$ is unique. \square

Lemma 12.5. *Let V denote an evaluation module for \boxtimes_q , and pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 . Then the bases $[i, j, k, \ell]$ and $[i, j, \ell, k]$ for V are the inversion of each other.*

Proof. By Lemma 12.4 and the meaning of inversion. \square

Proposition 12.6. *Let V denote an evaluation module for \boxtimes_q that has diameter d . Pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 and consider the basis $[i, j, k, \ell]$ of V . For this basis the components 0 and d are given below.*

Component 0	Component d
$\frac{(\eta_j, \eta_k^*)}{(\eta_k, \eta_\ell^*)} \eta_k$	$\frac{(\eta_j, \eta_\ell^*)}{(\eta_\ell, \eta_k^*)} \eta_\ell$

Proof. Denote the basis by $\{u_n\}_{n=0}^d$. Recall that for $0 \leq n \leq d$ the vector u_n is contained in component n of the decomposition $[k, \ell]$ of V . By construction η_k (resp. η_ℓ) is a basis for component 0 (resp. d) of the decomposition $[k, \ell]$ of V . Similarly η_k^* (resp. η_ℓ^*) is a basis for component 0 (resp. d) of the decomposition $[k, \ell]$ of V^* . By Proposition 7.9 the decomposition $[k, \ell]$ of V is dual to decomposition $[\ell, k]$ of V^* . Therefore $(u_n, \eta_\ell^*) = 0$ for $1 \leq n \leq d$ and $(u_n, \eta_k^*) = 0$ for $0 \leq n \leq d - 1$. Moreover there exist $\alpha, \beta \in \mathbb{F}$ such that $u_0 = \alpha\eta_k$ and $u_d = \beta\eta_\ell$. Now using Lemma 12.4(ii),

$$(\eta_j, \eta_\ell^*) = \sum_{n=0}^d (u_n, \eta_\ell^*) = (u_0, \eta_\ell^*) = \alpha(\eta_k, \eta_\ell^*)$$

so $\alpha = (\eta_j, \eta_\ell^*) / (\eta_k, \eta_\ell^*)$. Similarly

$$(\eta_j, \eta_k^*) = \sum_{n=0}^d (u_n, \eta_k^*) = (u_d, \eta_k^*) = \beta(\eta_\ell, \eta_k^*)$$

so $\beta = (\eta_j, \eta_k^*) / (\eta_\ell, \eta_k^*)$. The result follows. \square

13. Transition matrices between the 24 bases

Let V denote an evaluation module for \boxtimes_q that has diameter d . Recall the 24 bases for V from Lemma 12.4. In this section we compute the transition matrices between certain pairs of bases among the 24. First we clarify a few terms.

Suppose we are given two bases for V , denoted $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$. By the *transition matrix from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$* we mean the matrix $S \in \text{Mat}_{d+1}(\mathbb{F})$ such that $v_n = \sum_{r=0}^d S_{rn}u_r$ for $0 \leq n \leq d$. Let S denote the transition matrix from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$. Then S^{-1} exists and equals the transition matrix from $\{v_n\}_{n=0}^d$ to $\{u_n\}_{n=0}^d$.

Let $\{w_n\}_{n=0}^d$ denote a basis for V and let T denote the transition matrix from $\{v_n\}_{n=0}^d$ to $\{w_n\}_{n=0}^d$. Then ST is the transition matrix from $\{u_n\}_{n=0}^d$ to $\{w_n\}_{n=0}^d$.

Let $A \in \text{End}(V)$ and let M denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents A with respect to $\{u_n\}_{n=0}^d$. Then the matrix $S^{-1}MS$ represents A with respect to $\{v_n\}_{n=0}^d$.

The matrix Z is defined in [Appendix A](#). Let $\{v_n\}_{n=0}^d$ denote a basis for V and consider the inverted basis $\{v_{d-n}\}_{n=0}^d$. Then Z is the transition matrix from $\{v_n\}_{n=0}^d$ to $\{v_{d-n}\}_{n=0}^d$.

Pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 and consider the basis $[i, j, k, \ell]$ of V . We will display the transition matrix from this basis to each of the bases

$$[j, i, k, \ell], \quad [i, k, j, \ell], \quad [i, j, \ell, k].$$

As we will see, the first transition matrix is diagonal, the second is lower triangular, and the third one is equal to Z .

We now consider the transitions of type $[i, j, k, \ell] \rightarrow [j, i, k, \ell]$. We will be discussing the matrices $D_q(t)$ and $\mathcal{D}_q(t)$ defined in [Appendix A](#).

Theorem 13.1. *Let V denote an evaluation module for \boxtimes_q , with diameter d and evaluation parameter t . In the table below we display some transition matrices between the 24 bases for V from [Lemma 12.4](#). Each transition matrix is diagonal. Pick $r \in \mathbb{Z}_4$, and first assume that r is even.*

Transition	Transition matrix
$[r, r + 1, r + 2, r + 3] \rightarrow [r + 1, r, r + 2, r + 3]$	$D_q(t) \frac{(\eta_r, \eta_{r+3}^*)}{(\eta_{r+1}, \eta_{r+3}^*)}$
$[r + 1, r, r + 2, r + 3] \rightarrow [r, r + 1, r + 2, r + 3]$	$(D_q(t))^{-1} \frac{(\eta_{r+1}, \eta_{r+3}^*)}{(\eta_r, \eta_{r+3}^*)}$
$[r, r + 1, r + 3, r + 2] \rightarrow [r + 1, r, r + 3, r + 2]$	$(D_{q^{-1}}(t))^{-1} \frac{(\eta_r, \eta_{r+2}^*)}{(\eta_{r+1}, \eta_{r+2}^*)}$
$[r + 1, r, r + 3, r + 2] \rightarrow [r, r + 1, r + 3, r + 2]$	$D_{q^{-1}}(t) \frac{(\eta_{r+1}, \eta_{r+2}^*)}{(\eta_r, \eta_{r+2}^*)}$
$[r, r + 2, r + 1, r + 3] \rightarrow [r + 2, r, r + 1, r + 3]$	$\mathcal{D}_q(t) \frac{(\eta_r, \eta_{r+3}^*)}{(\eta_{r+2}, \eta_{r+3}^*)}$
$[r + 2, r, r + 1, r + 3] \rightarrow [r, r + 2, r + 1, r + 3]$	$(\mathcal{D}_q(t))^{-1} \frac{(\eta_{r+2}, \eta_{r+3}^*)}{(\eta_r, \eta_{r+3}^*)}$

Next assume that r is odd. Then in the above table replace t by t^{-1} .

Proof. By [Lemma 9.17](#) and the construction, we may assume that $r = 0$. We consider six cases. In the following discussion all bases mentioned are for V .

$[0, 1, 2, 3] \rightarrow [1, 0, 2, 3]$. Let $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$ denote the bases $[0, 1, 2, 3]$ and $[1, 0, 2, 3]$ respectively. Let $D \in \text{Mat}_{d+1}(\mathbb{F})$ denote the transition matrix from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$. For $0 \leq n \leq d$ the vectors u_n, v_n are contained in component n of the decomposition $[2, 3]$. Therefore D is diagonal. By [Theorem 11.1](#) the matrix representing

x_{12} with respect to $\{u_n\}_{n=0}^d$ is equal to E_q , and the matrix representing x_{12} with respect to $\{v_n\}_{n=0}^d$ is equal to $G_q(t)$. Therefore $E_q D = D G_q(t)$. Comparing this with (51) we find that there exists $0 \neq \alpha \in \mathbb{F}$ such that $D = \alpha D_q(t)$. We now find α . The $(0, 0)$ -entry of $D_q(t)$ is 1, so the $(0, 0)$ -entry of D is α . Therefore $v_0 = \alpha u_0$. By Proposition 12.6, both

$$u_0 = \frac{(\eta_1, \eta_3^*)}{(\eta_2, \eta_3^*)} \eta_2, \quad v_0 = \frac{(\eta_0, \eta_3^*)}{(\eta_2, \eta_3^*)} \eta_2.$$

Therefore $\alpha = (\eta_0, \eta_3^*) / (\eta_1, \eta_3^*)$.

$[1, 0, 2, 3] \rightarrow [0, 1, 2, 3]$. This transition matrix is the inverse of the transition matrix for $[0, 1, 2, 3] \rightarrow [1, 0, 2, 3]$.

$[0, 1, 3, 2] \rightarrow [1, 0, 3, 2]$. Let $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$ denote the bases $[0, 1, 3, 2]$ and $[1, 0, 3, 2]$ respectively. Let $D \in \text{Mat}_{d+1}(\mathbb{F})$ denote the transition matrix from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$. For $0 \leq n \leq d$ the vectors u_n, v_n are contained in component n of the decomposition $[3, 2]$. Therefore D is diagonal. By Theorem 11.1 the matrix representing x_{30} with respect to $\{u_n\}_{n=0}^d$ is equal to $G_{q^{-1}}(t)$, and the matrix representing x_{30} with respect to $\{v_n\}_{n=0}^d$ is equal to $E_{q^{-1}}$. Therefore $G_{q^{-1}}(t) D = D E_{q^{-1}}$. Comparing this with (51) we find that there exists $0 \neq \alpha \in \mathbb{F}$ such that $D = \alpha (D_{q^{-1}}(t))^{-1}$. We now find α . The matrix $D_{q^{-1}}(t)$ is diagonal with $(0, 0)$ -entry 1, so D has $(0, 0)$ -entry α . Therefore $v_0 = \alpha u_0$. By Proposition 12.6, both

$$u_0 = \frac{(\eta_1, \eta_2^*)}{(\eta_3, \eta_2^*)} \eta_3, \quad v_0 = \frac{(\eta_0, \eta_2^*)}{(\eta_3, \eta_2^*)} \eta_3.$$

Therefore $\alpha = (\eta_0, \eta_2^*) / (\eta_1, \eta_2^*)$.

$[1, 0, 3, 2] \rightarrow [0, 1, 3, 2]$. This transition matrix is the inverse of the transition matrix for $[0, 1, 3, 2] \rightarrow [1, 0, 3, 2]$.

$[0, 2, 1, 3] \rightarrow [2, 0, 1, 3]$. Let $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$ denote the bases $[0, 2, 1, 3]$ and $[2, 0, 1, 3]$ respectively. Let $D \in \text{Mat}_{d+1}(\mathbb{F})$ denote the transition matrix from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$. For $0 \leq n \leq d$ the vectors u_n, v_n are contained in component n of the decomposition $[1, 3]$. Therefore D is diagonal. By Theorem 11.1 the matrix representing x_{01} with respect to $\{u_n\}_{n=0}^d$ is equal to $F_q(t^{-1})$, and the matrix representing x_{01} with respect to $\{v_n\}_{n=0}^d$ is equal to E_q . Therefore $F_q(t^{-1}) D = D E_q$. Comparing this with (50) we find that there exists $0 \neq \alpha \in \mathbb{F}$ such that $D = \alpha \mathcal{D}_q(t)$. We now find α . The matrix $\mathcal{D}_q(t)$ has $(0, 0)$ -entry 1, so D has $(0, 0)$ -entry α . Therefore $v_0 = \alpha u_0$. By Proposition 12.6, both

$$u_0 = \frac{(\eta_2, \eta_3^*)}{(\eta_1, \eta_3^*)} \eta_1, \quad v_0 = \frac{(\eta_0, \eta_3^*)}{(\eta_1, \eta_3^*)} \eta_1.$$

Therefore $\alpha = (\eta_0, \eta_3^*) / (\eta_2, \eta_3^*)$.

$[2, 0, 1, 3] \rightarrow [0, 2, 1, 3]$. This transition matrix is the inverse of the transition matrix for $[0, 2, 1, 3] \rightarrow [2, 0, 1, 3]$. \square

We now consider the transitions of type $[i, j, k, \ell] \rightarrow [i, k, j, \ell]$. We will be discussing the matrix T_q defined in Appendix A.

Theorem 13.2. *Let V denote an evaluation module for \boxtimes_q . In the table below we display some transition matrices between the 24 bases for V from Lemma 12.4. Each transition matrix is lower triangular. Pick $r \in \mathbb{Z}_4$.*

Transition	Transition matrix
$[r, r + 1, r + 2, r + 3] \rightarrow [r, r + 2, r + 1, r + 3]$	$T_q \frac{(\eta_{r+2}, \eta_{r+3}^*)}{(\eta_{r+1}, \eta_{r+3}^*)}$
$[r + 1, r, r + 2, r + 3] \rightarrow [r + 1, r + 2, r, r + 3]$	$T_q \frac{(\eta_{r+2}, \eta_{r+3}^*)}{(\eta_r, \eta_{r+3}^*)}$
$[r, r + 1, r + 3, r + 2] \rightarrow [r, r + 3, r + 1, r + 2]$	$T_{q^{-1}} \frac{(\eta_{r+3}, \eta_{r+2}^*)}{(\eta_{r+1}, \eta_{r+2}^*)}$
$[r + 1, r, r + 3, r + 2] \rightarrow [r + 1, r + 3, r, r + 2]$	$T_{q^{-1}} \frac{(\eta_{r+3}, \eta_{r+2}^*)}{(\eta_r, \eta_{r+2}^*)}$
$[r, r + 2, r + 1, r + 3] \rightarrow [r, r + 1, r + 2, r + 3]$	$T_{q^{-1}} \frac{(\eta_{r+1}, \eta_{r+3}^*)}{(\eta_{r+2}, \eta_{r+3}^*)}$
$[r + 2, r, r + 1, r + 3] \rightarrow [r + 2, r + 1, r, r + 3]$	$T_q \frac{(\eta_{r+1}, \eta_{r+3}^*)}{(\eta_r, \eta_{r+3}^*)}$

Proof. These transition matrices were found in [32, Theorem 15.4]. In that article the notation is a bit different from what we are presently using. To translate between the notations use Lemma 10.3 and [32, Definition 13.4]. \square

We now consider the transitions of type $[i, j, k, \ell] \rightarrow [i, j, \ell, k]$.

Lemma 13.3. *Let V denote an evaluation module for \boxtimes_q , and pick mutually distinct i, j, k, ℓ in \mathbb{Z}_4 . Then the transition matrix from the basis $[i, j, k, \ell]$ to the basis $[i, j, \ell, k]$ is equal to Z .*

Proof. By Lemma 12.5. \square

14. Comments on the bilinear form

Throughout this section V denotes an evaluation module for \boxtimes_q , with diameter d and evaluation parameter t . Recall from Definition 12.1 the vectors $\{\eta_i\}_{i \in \mathbb{Z}_4}$ in V , and the vectors $\{\eta_i^*\}_{i \in \mathbb{Z}_4}$ in V^* . We now consider how the scalars

$$(\eta_i, \eta_j^*) \quad i, j \in \mathbb{Z}_4, \quad i \neq j \tag{28}$$

are related.

Proposition 14.1. *With the above notation,*

$$\frac{(\eta_0, \eta_1^*)(\eta_2, \eta_3^*)}{(\eta_2, \eta_1^*)(\eta_0, \eta_3^*)} = t^d q^{d(d-1)}, \quad \frac{(\eta_1, \eta_2^*)(\eta_3, \eta_0^*)}{(\eta_3, \eta_2^*)(\eta_1, \eta_0^*)} = t^{-d} q^{d(d-1)}.$$

Proof. Throughout this proof all bases mentioned are for V . Pick $r \in \mathbb{Z}_4$. Let $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$ denote the bases $[r, r + 2, r + 1, r + 3]$ and $[r + 2, r, r + 1, r + 3]$, respectively. We relate u_d, v_d in two ways. By the fifth row of the table in [Theorem 13.1](#),

$$v_d = t^{de} q^{d(d-1)} \frac{(\eta_r, \eta_{r+3}^*)}{(\eta_{r+2}, \eta_{r+3}^*)} u_d, \tag{29}$$

where $e = (-1)^r$. Observe that $\{u_{d-n}\}_{n=0}^d$ is the basis $[r, r + 2, r + 3, r + 1]$ and $\{v_{d-n}\}_{n=0}^d$ is the basis $[r + 2, r, r + 3, r + 1]$. By the sixth row of the table in [Theorem 13.1](#) (with r replaced by $r + 2$),

$$v_d = \frac{(\eta_r, \eta_{r+1}^*)}{(\eta_{r+2}, \eta_{r+1}^*)} u_d. \tag{30}$$

Comparing (29), (30) we obtain

$$\frac{(\eta_r, \eta_{r+1}^*)}{(\eta_{r+2}, \eta_{r+1}^*)} \frac{(\eta_{r+2}, \eta_{r+3}^*)}{(\eta_r, \eta_{r+3}^*)} = t^{de} q^{d(d-1)}.$$

The result follows. \square

Proposition 14.2. *With the above notation,*

$$\begin{aligned} \frac{(\eta_0, \eta_2^*)(\eta_1, \eta_3^*)}{(\eta_1, \eta_2^*)(\eta_0, \eta_3^*)} &= (1 - tq^{d-1})(1 - tq^{d-3}) \dots (1 - tq^{1-d}), \\ \frac{(\eta_1, \eta_3^*)(\eta_2, \eta_0^*)}{(\eta_2, \eta_3^*)(\eta_1, \eta_0^*)} &= (1 - t^{-1}q^{d-1})(1 - t^{-1}q^{d-3}) \dots (1 - t^{-1}q^{1-d}), \\ \frac{(\eta_2, \eta_0^*)(\eta_3, \eta_1^*)}{(\eta_3, \eta_0^*)(\eta_2, \eta_1^*)} &= (1 - tq^{d-1})(1 - tq^{d-3}) \dots (1 - tq^{1-d}), \\ \frac{(\eta_3, \eta_1^*)(\eta_0, \eta_2^*)}{(\eta_0, \eta_1^*)(\eta_3, \eta_2^*)} &= (1 - t^{-1}q^{d-1})(1 - t^{-1}q^{d-3}) \dots (1 - t^{-1}q^{1-d}). \end{aligned}$$

Proof. Throughout this proof all bases mentioned are for V . Pick $r \in \mathbb{Z}_4$. Let $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$ denote the bases $[r, r + 1, r + 2, r + 3]$ and $[r + 1, r, r + 2, r + 3]$, respectively. We relate u_d, v_d in two ways. By the first row of the table in [Theorem 13.1](#),

$$v_d = (t^e q^{1-d}; q^2)_d \frac{(\eta_r, \eta_{r+3}^*)}{(\eta_{r+1}, \eta_{r+3}^*)} u_d, \tag{31}$$

where $e = (-1)^r$. Observe that $\{u_{d-n}\}_{n=0}^d$ is the basis $[r, r + 1, r + 3, r + 2]$ and $\{v_{d-n}\}_{n=0}^d$ is the basis $[r + 1, r, r + 3, r + 2]$. By the third row of the table in [Theorem 13.1](#),

$$v_d = \frac{(\eta_r, \eta_{r+2}^*)}{(\eta_{r+1}, \eta_{r+2}^*)} u_d. \tag{32}$$

Comparing (31), (32) we obtain

$$\frac{(\eta_r, \eta_{r+2}^*)}{(\eta_{r+1}, \eta_{r+2}^*)} \frac{(\eta_{r+1}, \eta_{r+3}^*)}{(\eta_r, \eta_{r+3}^*)} = (t^e q^{1-d}; q^2)_d.$$

The result follows. \square

We view the following result as a \boxtimes_q -analog of [32, Proposition 13.11].

Corollary 14.3. *With the above notation,*

$$\begin{aligned} \frac{(\eta_0, \eta_1^*)(\eta_1, \eta_2^*)(\eta_2, \eta_0^*)}{(\eta_1, \eta_0^*)(\eta_2, \eta_1^*)(\eta_0, \eta_2^*)} &= (-1)^d q^{d(d-1)}, & \frac{(\eta_1, \eta_2^*)(\eta_2, \eta_3^*)(\eta_3, \eta_1^*)}{(\eta_2, \eta_1^*)(\eta_3, \eta_2^*)(\eta_1, \eta_3^*)} &= (-1)^d q^{d(d-1)}, \\ \frac{(\eta_2, \eta_3^*)(\eta_3, \eta_0^*)(\eta_0, \eta_2^*)}{(\eta_3, \eta_2^*)(\eta_0, \eta_3^*)(\eta_2, \eta_0^*)} &= (-1)^d q^{d(d-1)}, & \frac{(\eta_3, \eta_0^*)(\eta_0, \eta_1^*)(\eta_1, \eta_3^*)}{(\eta_0, \eta_3^*)(\eta_1, \eta_0^*)(\eta_3, \eta_1^*)} &= (-1)^d q^{d(d-1)}. \end{aligned}$$

Proof. We verify the first equation; the others are similarly verified. Consider the first two equations in Proposition 14.2. The right-hand side of the first one is equal to $(-1)^{dt^d}$ times the right-hand side of the second one. Therefore, the left-hand side of the first one is equal to $(-1)^{dt^d}$ times the left-hand side of the second one. In this equality eliminate t^d using the first equation in Proposition 14.1. \square

Note 14.4. By Propositions 14.1, 14.2 the scalars (28) are determined by the sequence

$$(\eta_0, \eta_1^*), (\eta_0, \eta_2^*), (\eta_0, \eta_3^*), (\eta_1, \eta_2^*), (\eta_2, \eta_1^*), (\eta_3, \eta_0^*), (\eta_3, \eta_1^*). \tag{33}$$

The scalars (33) are “free” in the following sense. Given a sequence θ of seven nonzero scalars in \mathbb{F} , there exist vectors η_i, η_i^* ($i \in \mathbb{Z}_4$) as in Definition 12.1 such that the sequence (33) is equal to θ .

15. Exchangers

Throughout this section V denotes an evaluation module for \boxtimes_q , with diameter d and evaluation parameter t . We investigate a type of map in $\text{End}(V)$ called an exchanger.

Lemma 15.1. *For each standard generator x_{ij} of \boxtimes_q and each $X \in \text{End}(V)$, the following are equivalent:*

- (i) X is invertible and $Xx_{ij}X^{-1} = x_{i+2, j+2}$ holds on V ;
- (ii) X sends the decomposition $[i, j]$ of V to the decomposition $[i + 2, j + 2]$ of V .

Proof. Recall from Section 6 that for $0 \leq n \leq d$, component n of the decomposition $[i, j]$ (resp. $[i + 2, j + 2]$) of V is the eigenspace of x_{ij} (resp. $x_{i+2, j+2}$) with eigenvalue q^{d-2n} . The result follows from this and linear algebra. \square

Definition 15.2. By an *exchanger* for V we mean an invertible $X \in \text{End}(V)$ such that on V the equation

$$Xx_{ij}X^{-1} = x_{i+2,j+2}$$

holds for each standard generator x_{ij} of \boxtimes_q .

We now consider the existence and uniqueness of the exchangers. We start with existence.

Lemma 15.3. *For $X \in \text{End}(V)$ the following are equivalent:*

- (i) X is an exchanger for V ;
- (ii) X is an isomorphism of \boxtimes_q -modules from V to V twisted via ρ^2 .

Proof. Recall that ρ^2 sends each standard generator $x_{ij} \mapsto x_{i+2,j+2}$. \square

Lemma 15.4. *There exists an exchanger for V .*

Proof. By [Corollary 9.18](#) and [Lemma 15.3](#). \square

We now consider the uniqueness of the exchangers.

Lemma 15.5. *Let Ψ denote an exchanger for V . Then for all $X \in \text{End}(V)$ the following are equivalent:*

- (i) X is an exchanger for V ;
- (ii) there exists $0 \neq \alpha \in \mathbb{F}$ such that $X = \alpha\Psi$.

Proof. (i) \Rightarrow (ii) The composition $G = X\Psi^{-1}$ commutes with each standard generator of \boxtimes_q , and therefore everything in \boxtimes_q . Since \mathbb{F} is algebraically closed and V has finite positive dimension, there exists an eigenspace $W \subseteq V$ for G . Let $\alpha \in \mathbb{F}$ denote the corresponding eigenvalue. Then $\alpha \neq 0$ since G is invertible. Since G commutes with everything in \boxtimes_q , we see that W is a \boxtimes_q -submodule of V . The \boxtimes_q -module V is irreducible so $W = V$. Therefore $G = \alpha I$ so $X = \alpha\Psi$.

(ii) \Rightarrow (i) Routine. \square

We now characterize the exchangers in various ways.

Lemma 15.6. *For $X \in \text{End}(V)$ the following are equivalent:*

- (i) X is an exchanger for V ;
- (ii) for all $i \in \mathbb{Z}_4$, X sends the flag $[i]$ for V to the flag $[i + 2]$ for V ;

(iii) for all distinct $i, j \in \mathbb{Z}_4$, X sends the decomposition $[i, j]$ of V to the decomposition $[i + 2, j + 2]$ of V .

Proof. (i) \Rightarrow (ii) Let $i \in \mathbb{Z}_4$ be given. By Lemma 15.3, X sends the flag $[i]$ for V to the flag $[i]$ for V twisted via ρ^2 . By Lemma 8.6, the flag $[i]$ for V twisted via ρ^2 is the same thing as the flag $[i + 2]$ for V . The result follows.

(ii) \Rightarrow (iii) For the flags and decompositions under discussion, their relationship is described near the end of Section 6.

(iii) \Rightarrow (i) By Lemma 15.1 and Definition 15.2. \square

Lemma 15.7. For $X \in \text{End}(V)$ the following are equivalent:

- (i) X is an exchanger for V ;
- (ii) for all mutually distinct i, j, k, ℓ in \mathbb{Z}_4 , X sends each $[i, j, k, \ell]$ -basis for V to an $[i + 2, j + 2, k + 2, \ell + 2]$ -basis for V ;
- (iii) there exist mutually distinct i, j, k, ℓ in \mathbb{Z}_4 such that X sends the basis $[i, j, k, \ell]$ for V to an $[i + 2, j + 2, k + 2, \ell + 2]$ -basis for V .

Proof. (i) \Rightarrow (ii) Let mutually distinct i, j, k, ℓ in \mathbb{Z}_4 be given. By Lemma 15.3, X sends each $[i, j, k, \ell]$ -basis for V to an $[i, j, k, \ell]$ -basis for V twisted via ρ^2 . By Lemma 10.8, an $[i, j, k, \ell]$ -basis for V twisted via ρ^2 is the same thing as an $[i + 2, j + 2, k + 2, \ell + 2]$ -basis for V . The result follows.

(ii) \Rightarrow (iii) Clear.

(iii) \Rightarrow (i) By Lemma 15.4 there exists an exchanger Ψ for V . By the implication (i) \Rightarrow (ii) above, Ψ sends the basis $[i, j, k, \ell]$ of V to an $[i + 2, j + 2, k + 2, \ell + 2]$ -basis of V . By assumption, X also sends the basis $[i, j, k, \ell]$ of V to an $[i + 2, j + 2, k + 2, \ell + 2]$ -basis of V . Now by Lemma 10.2, there exists $0 \neq \alpha \in \mathbb{F}$ such that $X = \alpha\Psi$. Now by Lemma 15.5, X is an exchanger for V . \square

Lemma 15.8. For $X \in \text{End}(V)$ the following are equivalent:

- (i) X is an exchanger for V ;
- (ii) X is an exchanger for ${}^\rho V$.

Proof. Use Lemma 8.6 and Lemma 15.6(i), (ii). \square

Lemma 15.9. Let X denote an exchanger for V . Then X^{-1} is an exchanger for V .

Proof. Use Lemma 15.6(i), (ii). \square

We will return to exchangers shortly.

Lemma 15.10. Consider the 24 bases for V from Lemma 12.4. In the table below we display some transition matrices between these bases. For each transition matrix we give the (i, j) -entry for $0 \leq i, j \leq d$. Pick $r \in \mathbb{Z}_4$, and first assume that r is even.

Transition matrix	(i, j) -entry
$[r, r + 2, r + 1, r + 3] \rightarrow [r + 2, r, r + 3, r + 1]$	$\delta_{i+j, d} t^i q^{i(d-1)} \frac{(\eta_r, \eta_{r+3}^*)}{(\eta_{r+2}, \eta_{r+3}^*)}$
$[r + 2, r, r + 1, r + 3] \rightarrow [r, r + 2, r + 3, r + 1]$	$\delta_{i+j, d} t^{-i} q^{i(1-d)} \frac{(\eta_{r+2}, \eta_{r+3}^*)}{(\eta_r, \eta_{r+3}^*)}$

Next assume that r is odd. Then in the above table replace t by t^{-1} .

Proof. To find the transition matrix $[r, r + 2, r + 1, r + 3] \rightarrow [r + 2, r, r + 3, r + 1]$, compute the product of transition matrices

$$[r, r + 2, r + 1, r + 3] \rightarrow [r + 2, r, r + 1, r + 3] \rightarrow [r + 2, r, r + 3, r + 1].$$

In this product the first transition matrix is from Theorem 13.1 and the second one is equal to Z . To find the transition matrix $[r + 2, r, r + 1, r + 3] \rightarrow [r, r + 2, r + 3, r + 1]$, compute the product of transition matrices

$$[r + 2, r, r + 1, r + 3] \rightarrow [r, r + 2, r + 1, r + 3] \rightarrow [r, r + 2, r + 3, r + 1].$$

In this product the first transition matrix is from Theorem 13.1 and the second one is equal to Z . \square

Theorem 15.11. There exists an exchanger \mathcal{X} for V that is described as follows. In the table below, each row contains a basis for V from Lemma 12.4, and the entries of a matrix in $\text{Mat}_{d+1}(\mathbb{F})$. The matrix represents \mathcal{X} with respect to the basis.

Basis	(i, j) -entry for $0 \leq i, j \leq d$
$[0, 2, 1, 3]$	$\delta_{i+j, d} t^i q^{i(d-1) - \binom{d}{2}}$
$[0, 2, 3, 1]$	$\delta_{i+j, d} t^{d-i} q^{\binom{d}{2} - i(d-1)}$
$[2, 0, 3, 1]$	$\delta_{i+j, d} t^i q^{i(d-1) - \binom{d}{2}}$
$[2, 0, 1, 3]$	$\delta_{i+j, d} t^{d-i} q^{\binom{d}{2} - i(d-1)}$
$[1, 3, 2, 0]$	$\delta_{i+j, d} (-1)^d t^{d-i} q^{i(d-1) - \binom{d}{2}}$
$[1, 3, 0, 2]$	$\delta_{i+j, d} (-1)^d t^i q^{\binom{d}{2} - i(d-1)}$
$[3, 1, 0, 2]$	$\delta_{i+j, d} (-1)^d t^{d-i} q^{i(d-1) - \binom{d}{2}}$
$[3, 1, 2, 0]$	$\delta_{i+j, d} (-1)^d t^i q^{\binom{d}{2} - i(d-1)}$

Proof. The above table has 8 rows. For $1 \leq h \leq 8$, row h of the table contains a basis and the entries of a matrix which we call X_h . Define $\mathcal{X}_h \in \text{End}(V)$ such that \mathcal{X}_h is represented by X_h with respect to the basis in row h . We claim that \mathcal{X}_h is an exchanger for V . To prove the claim, first assume that $h = 1$. Compare row 1 of the above table with row 1 of the table in Lemma 15.10 (with $r = 0$). The comparison shows that the

matrix X_1 is a scalar multiple of the transition matrix from the basis $[0, 2, 1, 3]$ to the basis $[2, 0, 3, 1]$. Therefore \mathcal{X}_1 sends the basis $[0, 2, 1, 3]$ to a $[2, 0, 3, 1]$ -basis. Now \mathcal{X}_1 is an exchanger for V , in view of Lemma 15.7(i), (iii). The claim is proven for $h = 1$, and for $2 \leq h \leq 8$ the argument is similar. We now show that \mathcal{X}_h is independent of h for $1 \leq h \leq 8$.

We show $\mathcal{X}_1 = \mathcal{X}_2$. By construction the matrix X_1 represents \mathcal{X}_1 with respect to $[0, 2, 1, 3]$. The transition matrix from $[0, 2, 1, 3]$ to $[0, 2, 3, 1]$ is equal to Z . Therefore the matrix ZX_1Z represents \mathcal{X}_1 with respect to $[0, 2, 3, 1]$. By construction X_2 represents \mathcal{X}_2 with respect to $[0, 2, 3, 1]$. One checks $X_1Z = ZX_2$ so $ZX_1Z = X_2$. Therefore $\mathcal{X}_1 = \mathcal{X}_2$. By a similar argument

$$\mathcal{X}_3 = \mathcal{X}_4, \quad \mathcal{X}_5 = \mathcal{X}_6, \quad \mathcal{X}_7 = \mathcal{X}_8.$$

We show $\mathcal{X}_2 = \mathcal{X}_3$. By Theorem 13.1, the transition matrix from $[2, 0, 3, 1]$ to $[0, 2, 3, 1]$ is a scalar multiple of $\mathcal{D}_q(t)$. One checks $X_3\mathcal{D}_q(t) = \mathcal{D}_q(t)X_2$. Therefore $\mathcal{X}_2 = \mathcal{X}_3$. We show $\mathcal{X}_5 = \mathcal{X}_8$. By Theorem 13.1, the transition matrix from $[1, 3, 2, 0]$ to $[3, 1, 2, 0]$ is a scalar multiple of $\mathcal{D}_q(t^{-1})$. One checks $X_5\mathcal{D}_q(t^{-1}) = \mathcal{D}_q(t^{-1})X_8$. Therefore $\mathcal{X}_5 = \mathcal{X}_8$. We show $\mathcal{X}_1 = \mathcal{X}_6$. Since each of $\mathcal{X}_1, \mathcal{X}_6$ is an exchanger, by Lemma 15.5 there exists $0 \neq \alpha \in \mathbb{F}$ such that $\mathcal{X}_1 = \alpha\mathcal{X}_6$. We show $\alpha = 1$. Let T denote the transition matrix from $[0, 2, 1, 3]$ to $[1, 3, 0, 2]$. By construction $X_1T = \alpha TX_6$. We now find T . To this end, compute the product of transition matrices

$$[0, 2, 1, 3] \rightarrow [0, 1, 2, 3] \rightarrow [1, 0, 2, 3] \rightarrow [1, 0, 3, 2] \rightarrow [1, 3, 0, 2].$$

In this product each factor is given in Section 13. The computation shows that T is a scalar multiple of $T_{q^{-1}}\mathcal{D}_q(t)ZT_{q^{-1}}$. Therefore

$$X_1T_{q^{-1}}\mathcal{D}_q(t)ZT_{q^{-1}} = \alpha T_{q^{-1}}\mathcal{D}_q(t)ZT_{q^{-1}}X_6. \tag{34}$$

Using the fact that $T_{q^{-1}}$ is lower triangular and $\mathcal{D}_q(t)$ is diagonal, we routinely compute the $(0, d)$ -entry for each side of (34). Comparing these entries we find $\alpha = 1$. Therefore $\mathcal{X}_1 = \mathcal{X}_6$. We have shown that \mathcal{X}_h is independent of h for $1 \leq h \leq 8$. The result follows. \square

Definition 15.12. By the *standard exchanger* for V , we mean the exchanger \mathcal{X} from Theorem 15.11.

Theorem 15.13. For the standard exchanger \mathcal{X} of V ,

$$\mathcal{X}^2 = t^d I.$$

Proof. Let M denote the matrix that represents \mathcal{X} with respect to the basis $[0, 2, 1, 3]$. The matrix M is given in the first row of the table in Theorem 15.11. By matrix multiplication $M^2 = t^d I$. The result follows. \square

Proposition 15.14. *The following coincide:*

- (i) *the standard exchanger for ${}^{\rho}V$;*
- (ii) *$(-1)^d$ times the inverse of the standard exchanger for V .*

Proof. Referring to [Theorem 15.11](#), compare the matrices that represent \mathcal{X} with respect to the bases $[0, 2, 1, 3]$ and $[1, 3, 2, 0]$. Recall from [Lemma 9.17](#) that t^{-1} is the evaluation parameter for ${}^{\rho}V$. Also, by [Theorem 15.13](#) $\mathcal{X}^{-1} = t^{-d}\mathcal{X}$. \square

Corollary 15.15. *The following coincide:*

- (i) *the standard exchanger for V twisted via ρ^2 ;*
- (ii) *the standard exchanger for V .*

Proof. Apply [Proposition 15.14](#) twice. \square

Let \mathcal{X} denote the standard exchanger for V . We now describe what \mathcal{X} does to the vectors $\{\eta_i\}_{i \in \mathbb{Z}_4}$ from [Definition 12.1](#).

Proposition 15.16. *Let \mathcal{X} denote the standard exchanger for V . Then*

$$\begin{aligned} \mathcal{X}\eta_0 &= q^{-\binom{d}{2}} \frac{(\eta_0, \eta_1^*)}{(\eta_2, \eta_1^*)} \eta_2, & \mathcal{X}\eta_1 &= (-1)^d q^{\binom{d}{2}} \frac{(\eta_1, \eta_0^*)}{(\eta_3, \eta_0^*)} \eta_3, \\ \mathcal{X}\eta_2 &= q^{-\binom{d}{2}} \frac{(\eta_2, \eta_3^*)}{(\eta_0, \eta_3^*)} \eta_0, & \mathcal{X}\eta_3 &= (-1)^d q^{\binom{d}{2}} \frac{(\eta_3, \eta_2^*)}{(\eta_1, \eta_2^*)} \eta_1. \end{aligned}$$

Proof. We verify the first equation. Let $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$ denote the bases $[2, 0, 3, 1]$ and $[0, 2, 1, 3]$ for V , respectively. Let T denote the transition matrix from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$; this matrix is given in [Lemma 15.10](#). Let M denote the matrix that represents \mathcal{X} with respect to $\{u_n\}_{n=0}^d$; this matrix is given in [Theorem 15.11](#). Comparing M and T we obtain

$$M = \alpha T, \quad \alpha = q^{-\binom{d}{2}} \frac{(\eta_0, \eta_1^*)}{(\eta_2, \eta_1^*)}.$$

Therefore $\mathcal{X}u_n = \alpha v_n$ for $0 \leq n \leq d$. By [Lemma 12.4\(ii\)](#), $\eta_0 = \sum_{n=0}^d u_n$ and $\eta_2 = \sum_{n=0}^d v_n$. Consequently $\mathcal{X}\eta_0 = \alpha\eta_2$. The first equation is verified. The remaining equations are similarly verified. \square

Let \mathcal{X} denote the standard exchanger for V . We now describe what \mathcal{X} does to the 24 bases for V from [Lemma 12.4](#).

Theorem 15.17. *Let \mathcal{X} denote the standard exchanger for V . For mutually distinct i, j, k, ℓ in \mathbb{Z}_4 , consider the bases $[i, j, k, \ell]$ and $[i + 2, j + 2, k + 2, \ell + 2]$ of V . The map \mathcal{X} sends $[i, j, k, \ell]$ to a scalar multiple of $[i + 2, j + 2, k + 2, \ell + 2]$. The scalar is given in the tables below.*

i	j	k	ℓ	Scalar
0	2	1	3	$q^{-\binom{d}{2}} \frac{(\eta_2, \eta_2^*)}{(\eta_0, \eta_2^*)}$
0	2	3	1	$q^{-\binom{d}{2}} \frac{(\eta_2, \eta_2^*)}{(\eta_0, \eta_2^*)}$
2	0	3	1	$q^{-\binom{d}{2}} \frac{(\eta_0, \eta_1^*)}{(\eta_2, \eta_1^*)}$
2	0	1	3	$q^{-\binom{d}{2}} \frac{(\eta_0, \eta_1^*)}{(\eta_2, \eta_1^*)}$

i	j	k	ℓ	Scalar
1	3	2	0	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_3, \eta_3^*)}{(\eta_1, \eta_3^*)}$
1	3	0	2	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_3, \eta_3^*)}{(\eta_1, \eta_3^*)}$
3	1	0	2	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_1, \eta_0^*)}{(\eta_3, \eta_0^*)}$
3	1	2	0	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_1, \eta_0^*)}{(\eta_3, \eta_0^*)}$

i	j	k	ℓ	Scalar
0	1	2	3	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_1, \eta_0^*)}{(\eta_3, \eta_0^*)}$
0	1	3	2	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_1, \eta_0^*)}{(\eta_3, \eta_0^*)}$
1	0	3	2	$q^{-\binom{d}{2}} \frac{(\eta_0, \eta_1^*)}{(\eta_2, \eta_1^*)}$
1	0	2	3	$q^{-\binom{d}{2}} \frac{(\eta_0, \eta_1^*)}{(\eta_2, \eta_1^*)}$

i	j	k	ℓ	Scalar
1	2	3	0	$q^{-\binom{d}{2}} \frac{(\eta_2, \eta_3^*)}{(\eta_0, \eta_3^*)}$
1	2	0	3	$q^{-\binom{d}{2}} \frac{(\eta_2, \eta_3^*)}{(\eta_0, \eta_3^*)}$
2	1	0	3	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_1, \eta_0^*)}{(\eta_3, \eta_0^*)}$
2	1	3	0	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_1, \eta_0^*)}{(\eta_3, \eta_0^*)}$

i	j	k	ℓ	Scalar
2	3	0	1	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_3, \eta_2^*)}{(\eta_1, \eta_2^*)}$
2	3	1	0	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_3, \eta_2^*)}{(\eta_1, \eta_2^*)}$
3	2	1	0	$q^{-\binom{d}{2}} \frac{(\eta_2, \eta_3^*)}{(\eta_0, \eta_3^*)}$
3	2	0	1	$q^{-\binom{d}{2}} \frac{(\eta_2, \eta_3^*)}{(\eta_0, \eta_3^*)}$

i	j	k	ℓ	Scalar
3	0	1	2	$q^{-\binom{d}{2}} \frac{(\eta_0, \eta_1^*)}{(\eta_2, \eta_1^*)}$
3	0	2	1	$q^{-\binom{d}{2}} \frac{(\eta_0, \eta_1^*)}{(\eta_2, \eta_1^*)}$
0	3	2	1	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_3, \eta_2^*)}{(\eta_1, \eta_2^*)}$
0	3	1	2	$(-1)^d q^{\binom{d}{2}} \frac{(\eta_3, \eta_2^*)}{(\eta_1, \eta_2^*)}$

Proof. Let $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$ denote the bases $[i, j, k, \ell]$ and $[i + 2, j + 2, k + 2, \ell + 2]$, respectively. By Lemma 10.2 and Lemma 15.7, there exists $0 \neq \alpha \in \mathbb{F}$ such that $\mathcal{X}u_n = \alpha v_n$ for $0 \leq n \leq d$. By Lemma 12.4(ii), $\eta_j = \sum_{n=0}^d u_n$ and $\eta_{j+2} = \sum_{n=0}^d v_n$. Therefore $\mathcal{X}\eta_j = \alpha\eta_{j+2}$. Evaluating this using Proposition 15.16 we find that α is as shown in the tables. \square

16. Leonard pairs of q -Racah type

We now turn our attention to Leonard pairs. There is a general family of Leonard pairs said to have q -Racah type [4, Section 5], [29, Example 5.3]. In this section, we show that for any Leonard pair of q -Racah type, the underlying vector space becomes an evaluation module for \boxtimes_q in a natural way.

Let a, b, c denote nonzero scalars in \mathbb{F} . Recall the equitable generators x, y, z for $U_q(\mathfrak{sl}_2)$.

Definition 16.1. Using a, b, c we define some elements in $U_q(\mathfrak{sl}_2)$:

$$\begin{aligned} \mathbf{A} &= ax + a^{-1}y + bc^{-1} \frac{xy - yx}{q - q^{-1}}, \\ \mathbf{B} &= by + b^{-1}z + ca^{-1} \frac{yz - zy}{q - q^{-1}}, \\ \mathbf{C} &= cz + c^{-1}x + ab^{-1} \frac{zx - xz}{q - q^{-1}}. \end{aligned}$$

The sequence $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is called the *Askey–Wilson triple* for a, b, c .

Lemma 16.2. (See [31, Proposition 1.1].) We have

$$\begin{aligned} \mathbf{A} + \frac{q\mathbf{BC} - q^{-1}\mathbf{CB}}{q^2 - q^{-2}} &= \frac{(a + a^{-1})\Lambda + (b + b^{-1})(c + c^{-1})}{q + q^{-1}}, \\ \mathbf{B} + \frac{q\mathbf{CA} - q^{-1}\mathbf{AC}}{q^2 - q^{-2}} &= \frac{(b + b^{-1})\Lambda + (c + c^{-1})(a + a^{-1})}{q + q^{-1}}, \\ \mathbf{C} + \frac{q\mathbf{AB} - q^{-1}\mathbf{BA}}{q^2 - q^{-2}} &= \frac{(c + c^{-1})\Lambda + (a + a^{-1})(b + b^{-1})}{q + q^{-1}}. \end{aligned}$$

Note 16.3. The equations in Lemma 16.2 are a variation on the \mathbb{Z}_3 -symmetric Askey–Wilson relations [4, Theorem 10.1], [14, Section 1].

Definition 16.4. Using a, b, c we define some more elements in $U_q(\mathfrak{sl}_2)$:

$$\begin{aligned} \mathbf{A}' &= ay + a^{-1}z + cb^{-1}\frac{yz - zy}{q - q^{-1}}, \\ \mathbf{B}' &= bx + b^{-1}y + ac^{-1}\frac{xy - yx}{q - q^{-1}}, \\ \mathbf{C}' &= cz + c^{-1}x + ba^{-1}\frac{zx - xz}{q - q^{-1}}. \end{aligned}$$

The sequence $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ is called the *dual Askey–Wilson triple* for a, b, c .

Lemma 16.5. Let $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ denote the dual Askey–Wilson triple for a, b, c . Then $\mathbf{B}', \mathbf{A}', \mathbf{C}'$ is the Askey–Wilson triple for b, a, c .

Proof. Compare the equations in Definition 16.1 and Definition 16.4. \square

Lemma 16.6. Let $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ denote the dual Askey–Wilson triple for a, b, c . Then

$$\begin{aligned} \mathbf{A}' + \frac{q\mathbf{C}'\mathbf{B}' - q^{-1}\mathbf{B}'\mathbf{C}'}{q^2 - q^{-2}} &= \frac{(a + a^{-1})\Lambda + (b + b^{-1})(c + c^{-1})}{q + q^{-1}}, \\ \mathbf{B}' + \frac{q\mathbf{A}'\mathbf{C}' - q^{-1}\mathbf{C}'\mathbf{A}'}{q^2 - q^{-2}} &= \frac{(b + b^{-1})\Lambda + (c + c^{-1})(a + a^{-1})}{q + q^{-1}}, \\ \mathbf{C}' + \frac{q\mathbf{B}'\mathbf{A}' - q^{-1}\mathbf{A}'\mathbf{B}'}{q^2 - q^{-2}} &= \frac{(c + c^{-1})\Lambda + (a + a^{-1})(b + b^{-1})}{q + q^{-1}}. \end{aligned}$$

Proof. By Lemma 16.5 the triple $\mathbf{B}', \mathbf{A}', \mathbf{C}'$ is the Askey–Wilson triple for b, a, c . Apply Lemma 16.2 to this triple. \square

Throughout this section and the next, fix an integer $d \geq 1$. Using a, b, c, d we now define some parameters in \mathbb{F} .

Definition 16.7. Define

$$\theta_n = aq^{2n-d} + a^{-1}q^{d-2n}, \quad \theta_n^* = bq^{2n-d} + b^{-1}q^{d-2n}$$

for $0 \leq n \leq d$, and

$$\begin{aligned} \varphi_n &= a^{-1}b^{-1}q^{d+1}(q^n - q^{-n})(q^{n-d-1} - q^{d-n+1})(q^{-n} - abcq^{n-d-1}) \\ &\quad \times (q^{-n} - abc^{-1}q^{n-d-1}), \\ \phi_n &= ab^{-1}q^{d+1}(q^n - q^{-n})(q^{n-d-1} - q^{d-n+1})(q^{-n} - a^{-1}bcq^{n-d-1}) \\ &\quad \times (q^{-n} - a^{-1}bc^{-1}q^{n-d-1}) \end{aligned}$$

for $1 \leq n \leq d$.

In order to avoid degenerate situations in Definition 16.7, we sometimes impose restrictions on how a, b, c, d are related. We now describe these restrictions.

Definition 16.8. The sequence (a, b, c, d) is called *feasible* whenever the following (i), (ii) hold:

- (i) neither of a^2, b^2 is among $q^{2d-2}, q^{2d-4}, \dots, q^{2-2d}$;
- (ii) none of $abc, a^{-1}bc, ab^{-1}c, abc^{-1}$ is among $q^{d-1}, q^{d-3}, \dots, q^{1-d}$.

Lemma 16.9. *The following are equivalent:*

- (i) *the sequence (a, b, c, d) is feasible;*
- (ii) *the $\{\theta_n\}_{n=0}^d$ are mutually distinct, the $\{\theta_n^*\}_{n=0}^d$ are mutually distinct, and the $\{\varphi_n\}_{n=1}^d, \{\phi_n\}_{n=1}^d$ are all nonzero.*

Proof. This is routinely checked. \square

The literature on Leonard pairs contains the notion of a parameter array [28,29]. For our purpose we do not need the full definition; just the following feature.

Lemma 16.10. (See [4, Lemma 7.3].) *The following are equivalent:*

- (i) *the sequence (a, b, c, d) is feasible;*
- (iii) *the sequence $(\{\theta_n\}_{n=0}^d; \{\theta_n^*\}_{n=0}^d; \{\varphi_n\}_{n=1}^d; \{\phi_n\}_{n=1}^d)$ is a parameter array.*

Lemma 16.11. *Assume that (a, b, c, d) is feasible. Then each of the following sequences is feasible:*

$$(a^{-1}, b, c, d); \quad (a, b^{-1}, c, d); \quad (a, b, c^{-1}, d); \quad (b, a, c, d). \tag{35}$$

Proof. By Definition 16.8. \square

In Lemma 16.11 we gave some feasible sequences. We now consider how their parameter arrays are related.

Lemma 16.12. Assume that (a, b, c, d) is feasible. In the table below, each row contains a feasible sequence and the corresponding parameter array.

Feasible sequence	Corresponding parameter array
(a, b, c, d)	$(\{\theta_n\}_{n=0}^d; \{\theta_n^*\}_{n=0}^d; \{\varphi_n\}_{n=1}^d; \{\phi_n\}_{n=1}^d)$
(a^{-1}, b, c, d)	$(\{\theta_{d-n}\}_{n=0}^d; \{\theta_n^*\}_{n=0}^d; \{\phi_n\}_{n=1}^d; \{\varphi_n\}_{n=1}^d)$
(a, b^{-1}, c, d)	$(\{\theta_n\}_{n=0}^d; \{\theta_{d-n}^*\}_{n=0}^d; \{\phi_{d-n+1}\}_{n=1}^d; \{\varphi_{d-n+1}\}_{n=1}^d)$
(a, b, c^{-1}, d)	$(\{\theta_n\}_{n=0}^d; \{\theta_n^*\}_{n=0}^d; \{\varphi_n\}_{n=1}^d; \{\phi_n\}_{n=1}^d)$
(b, a, c, d)	$(\{\theta_n^*\}_{n=0}^d; \{\theta_n\}_{n=0}^d; \{\varphi_n\}_{n=1}^d; \{\phi_{d-n+1}\}_{n=1}^d)$

Proof. Use Definition 16.7. \square

Lemma 16.13. (See [25, Theorem 1.9].) Assume that (a, b, c, d) is feasible. Then there exists a Leonard pair over \mathbb{F} that is described as follows. In one basis the pair is represented by

$$\begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}. \tag{36}$$

In another basis the pair is represented by

$$\begin{pmatrix} \theta_d & & & & \mathbf{0} \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \theta_2 & \\ & & & 1 & \theta_1 \\ \mathbf{0} & & & & 1 & \theta_0 \end{pmatrix}, \begin{pmatrix} \theta_0^* & \phi_1 & & & \mathbf{0} \\ & \theta_1^* & \phi_2 & & \\ & & \theta_2^* & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \phi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}. \tag{37}$$

Up to isomorphism the above Leonard pair is uniquely determined by (a, b, c, d) .

Definition 16.14. Assume that (a, b, c, d) is feasible. The Leonard pair from Lemma 16.13 is said to correspond to (a, b, c, d) .

Not every Leonard pair arises from the construction of Lemma 16.13. The ones that do are said to have *q-Racah type* [4, Section 5], [29, Example 5.3].

A Leonard pair of *q-Racah type* corresponds to more than one feasible sequence. This is explained in the next result.

Lemma 16.15. *Let A, B denote a Leonard pair of q -Racah type, with feasible sequence (a, b, c, d) . Then each of*

$$(a, b, c, d); \quad (a^{-1}, b, c, d); \quad (a, b^{-1}, c, d); \quad (a, b, c^{-1}, d); \tag{38}$$

$$(a, b^{-1}, c^{-1}, d); \quad (a^{-1}, b, c^{-1}, d); \quad (a^{-1}, b^{-1}, c, d); \quad (a^{-1}, b^{-1}, c^{-1}, d) \tag{39}$$

is a feasible sequence for A, B . The Leonard pair A, B has no other feasible sequence.

Proof. This follows from [28, Lemma 12.2] and Lemma 16.12. \square

Lemma 16.16. *Let A, B denote a Leonard pair of q -Racah type, with feasible sequence (a, b, c, d) . Then the Leonard pair B, A is of q -Racah type, with feasible sequence (b, a, c, d) .*

Proof. By [25, Theorem 1.11] and Lemma 16.12. \square

Proposition 16.17. (See [4, Theorem 10.1].) *Let A, B denote a Leonard pair over \mathbb{F} that has q -Racah type. Let V denote the underlying vector space. Then there exists a unique $C \in \text{End}(V)$ such that*

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{(a + a^{-1})(q^{d+1} + q^{-d-1}) + (b + b^{-1})(c + c^{-1})}{q + q^{-1}},$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{(b + b^{-1})(q^{d+1} + q^{-d-1}) + (c + c^{-1})(a + a^{-1})}{q + q^{-1}},$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{(c + c^{-1})(q^{d+1} + q^{-d-1}) + (a + a^{-1})(b + b^{-1})}{q + q^{-1}}.$$

Here (a, b, c, d) denotes a feasible sequence for the Leonard pair A, B .

Definition 16.18. Referring to Proposition 16.17, we call C the \mathbb{Z}_3 -symmetric completion of the Leonard pair A, B .

Definition 16.19. Let A, B denote a Leonard pair of q -Racah type. By the dual \mathbb{Z}_3 -symmetric completion of A, B we mean the \mathbb{Z}_3 -symmetric completion of the Leonard pair B, A .

Proposition 16.20. *Let A, B denote a Leonard pair of q -Racah type, with dual \mathbb{Z}_3 -symmetric completion C' . Then*

$$A + \frac{qC'B - q^{-1}BC'}{q^2 - q^{-2}} = \frac{(a + a^{-1})(q^{d+1} + q^{-d-1}) + (b + b^{-1})(c + c^{-1})}{q + q^{-1}},$$

$$B + \frac{qAC' - q^{-1}C'A}{q^2 - q^{-2}} = \frac{(b + b^{-1})(q^{d+1} + q^{-d-1}) + (c + c^{-1})(a + a^{-1})}{q + q^{-1}},$$

$$C' + \frac{qBA - q^{-1}AB}{q^2 - q^{-2}} = \frac{(c + c^{-1})(q^{d+1} + q^{-d-1}) + (a + a^{-1})(b + b^{-1})}{q + q^{-1}}.$$

Here (a, b, c, d) denotes a feasible sequence for the Leonard pair A, B .

Proof. By Definition 16.19, C' is the \mathbb{Z}_3 -symmetric completion of the Leonard pair B, A . By Lemma 16.16, the Leonard pair B, A has a feasible sequence (b, a, c, d) . Using these comments, apply Proposition 16.17 to the Leonard pair B, A . \square

Let A, B denote a Leonard pair that has q -Racah type. We now relate its \mathbb{Z}_3 -symmetric completion and dual \mathbb{Z}_3 -symmetric completion.

Lemma 16.21. *Let A, B denote a Leonard pair of q -Racah type, with \mathbb{Z}_3 -symmetric completion C and dual \mathbb{Z}_3 -symmetric completion C' . Then*

$$C' - C = \frac{AB - BA}{q - q^{-1}}.$$

Proof. Subtract the last equation in Proposition 16.17 from the last equation in Proposition 16.20. \square

Lemma 16.22. *(See [30, Theorem 5.8].) Let A, B denote a Leonard pair over \mathbb{F} . Let V denote the underlying vector space. Then there exists a unique antiautomorphism \dagger of $\text{End}(V)$ that fixes each of A, B . Moreover $\dagger^2 = 1$.*

Lemma 16.23. *With reference to Lemma 16.22, assume that A, B has q -Racah type. Then its \mathbb{Z}_3 -symmetric completion C and dual \mathbb{Z}_3 -symmetric completion C' are swapped by \dagger .*

Proof. In the last equation of Proposition 16.17, apply \dagger to each term. Compare the resulting equation with the last equation of Proposition 16.20. \square

We now use $U_q(\mathfrak{sl}_2)$ to construct Leonard pairs of q -Racah type.

Theorem 16.24. *Assume that (a, b, c, d) is feasible. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ denote the Askey–Wilson triple for a, b, c . Then the following (i)–(iii) hold.*

- (i) *The pair \mathbf{A}, \mathbf{B} acts on the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_d as a Leonard pair.*
- (ii) *This Leonard pair corresponds to (a, b, c, d) .*
- (iii) *The element \mathbf{C} acts on \mathbf{V}_d as the \mathbb{Z}_3 -symmetric completion of this Leonard pair.*

Proof. (i), (ii) The parameter array $(\{\theta_n\}_{n=0}^d; \{\theta_n^*\}_{n=0}^d; \{\varphi_n\}_{n=1}^d; \{\phi_n\}_{n=1}^d)$ for (a, b, c, d) is shown in Definition 16.7. Let $\{u_n\}_{n=0}^d$ denote a $[y]_{\text{row}}^{\text{inv}}$ -basis for \mathbf{V}_d . Consider the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent x, y, z with respect to $\{u_n\}_{n=0}^d$. These matrices are given in Lemma 3.22. The matrix representing x is lower bidiagonal, with (n, n) -entry

q^{d-2n} for $0 \leq n \leq d$ and $(n, n - 1)$ -entry $q^d - q^{d-2n}$ for $1 \leq n \leq d$. The matrix representing y is diagonal, with (n, n) -entry q^{2n-d} for $0 \leq n \leq d$. The matrix representing z is upper bidiagonal, with (n, n) -entry q^{d-2n} for $0 \leq n \leq d$ and $(n - 1, n)$ -entry $q^{-d} - q^{d-2n+2}$ for $1 \leq n \leq d$. Using this data and Definition 16.1 we compute the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent \mathbf{A}, \mathbf{B} with respect to $\{u_n\}_{n=0}^d$. The matrix representing \mathbf{A} is lower bidiagonal, with (n, n) -entry θ_{d-n} for $0 \leq n \leq d$ and $(n, n - 1)$ -entry

$$aq^d(q^n - q^{-n})(q^{-n} - a^{-1}bc^{-1}q^{n-d-1}) \tag{40}$$

for $1 \leq n \leq d$. The matrix representing \mathbf{B} is upper bidiagonal, with (n, n) -entry θ_n^* for $0 \leq n \leq d$ and $(n - 1, n)$ -entry

$$b^{-1}q(q^{n-d-1} - q^{d-n+1})(q^{-n} - a^{-1}bcq^{n-d-1}) \tag{41}$$

for $1 \leq n \leq d$. We now adjust the basis $\{u_n\}_{n=0}^d$. For $1 \leq n \leq d$ let α_n (resp. β_n) denote the scalar (40) (resp. (41)). We have $\alpha_n\beta_n = \phi_n$, so each of α_n, β_n is nonzero. Define $u'_n = \alpha_1\alpha_2 \cdots \alpha_n u_n$ for $0 \leq n \leq d$. By construction $\{u'_n\}_{n=0}^d$ is a basis for \mathbf{V}_d . With respect to this basis the matrices representing \mathbf{A}, \mathbf{B} are the ones shown in (37). Therefore the pair \mathbf{A}, \mathbf{B} acts on \mathbf{V}_d as a Leonard pair that corresponds to (a, b, c, d) .

(iii) Recall that A acts on \mathbf{V}_d as $q^{d+1} + q^{d-1}$ times the identity. Using this comment, compare the last equation in Lemma 16.2 with the last equation in Proposition 16.17. \square

Lemma 16.25. *Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type, with feasible sequence (a, b, c, d) . Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ denote the Askey–Wilson triple for a, b, c . Assume that there exists a $U_q(\mathfrak{sl}_2)$ -module structure on the underlying vector space V such that $A = \mathbf{A}$ and $B = \mathbf{B}$ on V . Then the $U_q(\mathfrak{sl}_2)$ -module V is irreducible.*

Proof. By Lemma 2.3. \square

Corollary 16.26. *Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type, with feasible sequence (a, b, c, d) . Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ denote the Askey–Wilson triple for a, b, c . Then the following (i), (ii) hold.*

- (i) *There exists a unique type 1 $U_q(\mathfrak{sl}_2)$ -module structure on the underlying vector space V such that $A = \mathbf{A}$ and $B = \mathbf{B}$ on V .*
- (ii) *The element \mathbf{C} acts on V as the \mathbb{Z}_3 -symmetric completion of A, B .*

Proof. The Leonard pair A, B corresponds to (a, b, c, d) . The Leonard pair in Theorem 16.24 also corresponds to (a, b, c, d) . Therefore these Leonard pairs are isomorphic. Let $\zeta : V \rightarrow \mathbf{V}_d$ denote an isomorphism of Leonard pairs from A, B to the Leonard pair in Theorem 16.24. Via ζ we transport the $U_q(\mathfrak{sl}_2)$ -module structure from \mathbf{V}_d to V . This turns V into a $U_q(\mathfrak{sl}_2)$ -module that is isomorphic to \mathbf{V}_d . By construction $A = \mathbf{A}$ and

$B = \mathbf{B}$ on V . Also by Theorem 16.24(iii), \mathbf{C} acts on V as the \mathbb{Z}_3 -symmetric completion of A, B . The uniqueness assertion in (i) follows from Lemma 2.3. \square

Theorem 16.27. *Assume that (a, b, c, d) is feasible. Let $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ denote the dual Askey–Wilson triple for a, b, c . Then the following (i)–(iii) hold.*

- (i) *The pair \mathbf{A}', \mathbf{B}' acts on the $U_q(\mathfrak{sl}_2)$ -module \mathbf{V}_d as a Leonard pair.*
- (ii) *This Leonard pair corresponds to (a, b, c, d) .*
- (iii) *The element \mathbf{C}' acts on \mathbf{V}_d as the dual \mathbb{Z}_3 -symmetric completion of this Leonard pair.*

Proof. Conceptually this proof is similar to the proof of Theorem 16.24, but as the details are different they will be displayed.

(i), (ii) The parameter array $(\{\theta_n\}_{n=0}^d; \{\theta_n^*\}_{n=0}^d; \{\varphi_n\}_{n=1}^d; \{\phi_n\}_{n=1}^d)$ for (a, b, c, d) is shown in Definition 16.7. Let $\{v_n\}_{n=0}^d$ denote a $[y]_{row}$ -basis for \mathbf{V}_d . Consider the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent x, y, z with respect to $\{v_n\}_{n=0}^d$. These matrices are given in Lemma 3.9. The matrix representing x is upper bidiagonal, with (n, n) -entry q^{2n-d} for $0 \leq n \leq d$ and $(n-1, n)$ -entry $q^d - q^{2n-d-2}$ for $1 \leq n \leq d$. The matrix representing y is diagonal, with (n, n) -entry q^{d-2n} for $0 \leq n \leq d$. The matrix representing z is lower bidiagonal, with (n, n) -entry q^{2n-d} for $0 \leq n \leq d$ and $(n, n-1)$ -entry $q^{-d} - q^{2n-d}$ for $1 \leq n \leq d$. Using this data and Definition 16.4 we compute the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent \mathbf{A}', \mathbf{B}' with respect to $\{v_n\}_{n=0}^d$. The matrix representing \mathbf{A}' is lower bidiagonal, with (n, n) -entry θ_{d-n} for $0 \leq n \leq d$ and $(n, n-1)$ -entry

$$b^{-1}cq(q^n - q^{-n})(q^{-n} - a^{-1}bc^{-1}q^{n-d-1}) \tag{42}$$

for $1 \leq n \leq d$. The matrix representing \mathbf{B}' is upper bidiagonal, with (n, n) -entry θ_n^* for $0 \leq n \leq d$ and $(n-1, n)$ -entry

$$ac^{-1}q^d(q^{n-d-1} - q^{d-n+1})(q^{-n} - a^{-1}bcq^{n-d-1}) \tag{43}$$

for $1 \leq n \leq d$. We now adjust the basis $\{v_n\}_{n=0}^d$. For $1 \leq n \leq d$ let α_n (resp. β_n) denote the scalar (42) (resp. (43)). We have $\alpha_n\beta_n = \phi_n$, so each of α_n, β_n is nonzero. Define $v'_n = \alpha_1\alpha_2 \cdots \alpha_n v_n$ for $0 \leq n \leq d$. By construction $\{v'_n\}_{n=0}^d$ is a basis for \mathbf{V}_d . With respect to this basis the matrices representing \mathbf{A}', \mathbf{B}' are the ones shown in (37). Therefore the pair \mathbf{A}', \mathbf{B}' acts on \mathbf{V}_d as a Leonard pair that corresponds to (a, b, c, d) .

(iii) Recall that A acts on \mathbf{V}_d as $q^{d+1} + q^{-d-1}$ times the identity. Using this comment, compare the last equation in Lemma 16.6 with the last equation in Proposition 16.20. \square

Lemma 16.28. *Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type, with feasible sequence (a, b, c, d) . Let $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ denote the dual Askey–Wilson triple for a, b, c . Assume that there exists a $U_q(\mathfrak{sl}_2)$ -module structure on the underlying vector space V such that $A = \mathbf{A}'$ and $B = \mathbf{B}'$ on V . Then the $U_q(\mathfrak{sl}_2)$ -module V is irreducible.*

Proof. By Lemma 2.3. \square

Corollary 16.29. Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type, with feasible sequence (a, b, c, d) . Let $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ denote the dual Askey–Wilson triple for a, b, c . Then the following (i), (ii) hold.

- (i) There exists a unique type 1 $U_q(\mathfrak{sl}_2)$ -module structure on the underlying vector space V such that $A = \mathbf{A}'$ and $B = \mathbf{B}'$ on V .
- (ii) The element \mathbf{C}' acts on V as the dual \mathbb{Z}_3 -symmetric completion of A, B .

Proof. Similar to the proof of Corollary 16.26. \square

We now bring in \boxtimes_q . Recall the injections $\kappa_i : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ from Lemma 4.4.

Definition 16.30. Referring to Definition 16.1, we identify $\mathbf{A}, \mathbf{B}, \mathbf{C}$ with their images under the injection $\kappa_2 : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$. Thus

$$\begin{aligned} \mathbf{A} &= ax_{01} + a^{-1}x_{13} + bc^{-1} \frac{[x_{01}, x_{13}]}{q - q^{-1}}, \\ \mathbf{B} &= bx_{13} + b^{-1}x_{30} + ca^{-1} \frac{[x_{13}, x_{30}]}{q - q^{-1}}, \\ \mathbf{C} &= cx_{30} + c^{-1}x_{01} + ab^{-1} \frac{[x_{30}, x_{01}]}{q - q^{-1}}. \end{aligned}$$

Definition 16.31. Referring to Definition 16.4, we identify $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ with their images under the injection $\kappa_0 : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$. Thus

$$\begin{aligned} \mathbf{A}' &= ax_{31} + a^{-1}x_{12} + cb^{-1} \frac{[x_{31}, x_{12}]}{q - q^{-1}}, \\ \mathbf{B}' &= bx_{23} + b^{-1}x_{31} + ac^{-1} \frac{[x_{23}, x_{31}]}{q - q^{-1}}, \\ \mathbf{C}' &= cx_{12} + c^{-1}x_{23} + ba^{-1} \frac{[x_{12}, x_{23}]}{q - q^{-1}}. \end{aligned}$$

Lemma 16.32. Assume that (a, b, c, d) is feasible, and define $t = abc^{-1}$. On the \boxtimes_q -module $\mathbf{V}_d(t)$,

$$\mathbf{A} = ax_{01} + a^{-1}x_{12} = \mathbf{A}', \tag{44}$$

$$\mathbf{B} = bx_{23} + b^{-1}x_{30} = \mathbf{B}'. \tag{45}$$

Proof. Use Lemma 9.5 (with $t = abc^{-1}$). \square

Theorem 16.33. Assume that (a, b, c, d) is feasible, and define $t = abc^{-1}$. Then the following (i)–(iv) hold.

(i) The pair

$$ax_{01} + a^{-1}x_{12}, \quad bx_{23} + b^{-1}x_{30} \tag{46}$$

acts on the \boxtimes_q -module $\mathbf{V}_d(t)$ as a Leonard pair of q -Racah type.

- (ii) This Leonard pair corresponds to (a, b, c, d) .
- (iii) The element \mathbf{C} from Definition 16.30 acts on $\mathbf{V}_d(t)$ as the \mathbb{Z}_3 -symmetric completion of this Leonard pair.
- (iv) The element \mathbf{C}' from Definition 16.31 acts on $\mathbf{V}_d(t)$ as the dual \mathbb{Z}_3 -symmetric completion of this Leonard pair.

Proof. We first obtain (i)–(iii). Using the homomorphism $\kappa_2 : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ we turn the \boxtimes_q -module $\mathbf{V}_d(t)$ into a $U_q(\mathfrak{sl}_2)$ -module isomorphic to \mathbf{V}_d . Apply Theorem 16.24 to this $U_q(\mathfrak{sl}_2)$ -module, and use the equations on the left in (44), (45). This yields (i)–(iii). Next we obtain (iv). Using the homomorphism $\kappa_0 : U_q(\mathfrak{sl}_2) \rightarrow \boxtimes_q$ we turn the \boxtimes_q -module $\mathbf{V}_d(t)$ into a $U_q(\mathfrak{sl}_2)$ -module isomorphic to \mathbf{V}_d . Apply Theorem 16.27 to this $U_q(\mathfrak{sl}_2)$ -module, and use the equations on the right in (44), (45). This yields (iv), along with a second proof of (i), (ii). \square

Corollary 16.34. Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type, with feasible sequence (a, b, c, d) . Define $t = abc^{-1}$. Then the following (i)–(iii) hold.

(i) The underlying vector space V supports a unique t -evaluation module for \boxtimes_q such that on V ,

$$A = ax_{01} + a^{-1}x_{12}, \quad B = bx_{23} + b^{-1}x_{30}. \tag{47}$$

- (ii) The element \mathbf{C} from Definition 16.30 acts on V as the \mathbb{Z}_3 -symmetric completion of A, B .
- (iii) The element \mathbf{C}' from Definition 16.31 acts on V as the dual \mathbb{Z}_3 -symmetric completion of A, B .

Proof. The Leonard pair A, B corresponds to (a, b, c, d) . The Leonard pair in Theorem 16.33 also corresponds to (a, b, c, d) . Therefore these Leonard pairs are isomorphic. Let $\partial : V \rightarrow \mathbf{V}_d(t)$ denote an isomorphism of Leonard pairs from A, B to the Leonard pair in Theorem 16.33. Via ∂ we transport the \boxtimes_q -module structure from $\mathbf{V}_d(t)$ to V . This turns V into a \boxtimes_q -module isomorphic to $\mathbf{V}_d(t)$. By construction and Theorem 16.33, Eqs. (47) hold on V , and the assertions (ii), (iii) are valid. The uniqueness assertion in (i) follows from Lemma 2.3. \square

Note 16.35. Let A, B denote a Leonard pair of q -Racah type. Using [Lemma 16.15](#) and [Corollary 16.34](#), we get eight \boxtimes_q -module structures on the underlying vector space.

17. The compact basis

Let A, B denote a Leonard pair over \mathbb{F} that has q -Racah type. In this section we introduce a certain basis for the underlying vector space, with respect to which the matrices representing A and B are tridiagonal with attractive entries. We call this basis the compact basis. We show how the compact basis is related to the \boxtimes_q -module structure discussed in [Corollary 16.34](#).

Proposition 17.1. *Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type, with feasible sequence (a, b, c, d) . Then there exists a basis for the underlying vector space V , with respect to which A and B are represented by the following tridiagonal matrices in $\text{Mat}_{d+1}(\mathbb{F})$:*

Element	$(n, n - 1)$ -entry	(n, n) -entry	$(n - 1, n)$ -entry
A	$c^{-1}(1 - q^{-2n})$	$(a + a^{-1})q^{d-2n}$	$c(1 - q^{2d-2n+2})$
B	$q^{-d-1}(1 - q^{2n})$	$(b + b^{-1})q^{2n-d}$	$q^{d+1}(1 - q^{2n-2d-2})$

Proof. Recall the \boxtimes_q -module structure on V from [Corollary 16.34](#). This is an evaluation module, with evaluation parameter $t = abc^{-1}$. Let $\{u_n\}_{n=0}^d$ denote a $[1, 3, 0, 2]$ -basis for V . Consider the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent $x_{01}, x_{12}, x_{23}, x_{30}$ with respect to $\{u_n\}_{n=0}^d$. These matrices are given in the last row of the first table in [Theorem 11.1](#) (with $r = 3$). Their entries are described as follows. The matrix representing x_{01} is upper bidiagonal, with (n, n) -entry q^{d-2n} for $0 \leq n \leq d$ and $(n - 1, n)$ -entry $q^{-1}t^{-1}(1 - q^{2d-2n+2})$ for $1 \leq n \leq d$. The matrix representing x_{12} is lower bidiagonal, with (n, n) -entry q^{d-2n} for $0 \leq n \leq d$ and $(n, n - 1)$ -entry $qt(1 - q^{-2n})$ for $1 \leq n \leq d$. The matrix representing x_{23} is lower bidiagonal, with (n, n) -entry q^{2n-d} for $0 \leq n \leq d$ and $(n, n - 1)$ -entry $q^{-d}(1 - q^{2n})$ for $1 \leq n \leq d$. The matrix representing x_{30} is upper bidiagonal, with (n, n) -entry q^{2n-d} for $0 \leq n \leq d$ and $(n - 1, n)$ -entry $q^d(1 - q^{2n-2d-2})$ for $1 \leq n \leq d$. Now consider the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent A and B with respect to $\{u_n\}_{n=0}^d$. Their entries are found using [\(47\)](#) and the above comments. The matrix representing A is tridiagonal, with $(n, n - 1)$ -entry $qbc^{-1}(1 - q^{-2n})$ for $1 \leq n \leq d$, (n, n) -entry $(a + a^{-1})q^{d-2n}$ for $0 \leq n \leq d$, and $(n - 1, n)$ -entry $q^{-1}b^{-1}c(1 - q^{2d-2n+2})$ for $1 \leq n \leq d$. The matrix representing B is tridiagonal, with $(n, n - 1)$ -entry $bq^{-d}(1 - q^{2n})$ for $1 \leq n \leq d$, (n, n) -entry $(b + b^{-1})q^{2n-d}$ for $0 \leq n \leq d$, and $(n - 1, n)$ -entry $b^{-1}q^d(1 - q^{2n-2d-2})$ for $1 \leq n \leq d$. We now adjust the basis $\{u_n\}_{n=0}^d$. Define $v_n = q^n b^n u_n$ for $0 \leq n \leq d$. Then $\{v_n\}_{n=0}^d$ is a basis for V . With respect to this basis the matrices representing A and B are as shown in the theorem statement. \square

Example 17.2. The matrices from [Proposition 17.1](#) look as follows for $d = 3$. The matrix representing A is

$$\begin{pmatrix} (a + a^{-1})q^3 & c(1 - q^6) & 0 & 0 \\ c^{-1}(1 - q^{-2}) & (a + a^{-1})q & c(1 - q^4) & 0 \\ 0 & c^{-1}(1 - q^{-4}) & (a + a^{-1})q^{-1} & c(1 - q^2) \\ 0 & 0 & c^{-1}(1 - q^{-6}) & (a + a^{-1})q^{-3} \end{pmatrix}.$$

The matrix representing B is

$$\begin{pmatrix} (b + b^{-1})q^{-3} & q^4(1 - q^{-6}) & 0 & 0 \\ q^{-4}(1 - q^2) & (b + b^{-1})q^{-1} & q^4(1 - q^{-4}) & 0 \\ 0 & q^{-4}(1 - q^4) & (b + b^{-1})q & q^4(1 - q^{-2}) \\ 0 & 0 & q^{-4}(1 - q^6) & (b + b^{-1})q^3 \end{pmatrix}.$$

Definition 17.3. The basis discussed in Proposition 17.1 is said to be *compact*.

Note 17.4. Our motivation for Definition 17.3 is that the entries shown in the table of Proposition 17.1 are rather concise.

Note 17.5. The existence of the compact basis was hinted at in [3], [23, Section 6], [24, Section 2.3].

We comment on the uniqueness of the compact basis.

Lemma 17.6. Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type, with feasible sequence (a, b, c, d) . Let $\{v_n\}_{n=0}^d$ denote a basis for the underlying vector space V that meets the requirements of Proposition 17.1. Let $\{v'_n\}_{n=0}^d$ denote any basis for V . Then the following are equivalent:

- (i) the basis $\{v'_n\}_{n=0}^d$ meets the requirements of Proposition 17.1;
- (ii) there exists a nonzero $\alpha \in \mathbb{F}$ such that $v'_n = \alpha v_n$ for $0 \leq n \leq d$.

Proof. (i) \Rightarrow (ii) Consider the map $\psi \in \text{End}(V)$ that sends $v_n \mapsto v'_n$ for $0 \leq n \leq d$. The matrix representing A with respect to $\{v_n\}_{n=0}^d$ is equal to the matrix representing A with respect to $\{v'_n\}_{n=0}^d$. Therefore ψ commutes with A . By a similar argument ψ commutes with B . Now by Lemma 2.3, ψ commutes with everything in $\text{End}(V)$. Consequently there exists $\alpha \in \mathbb{F}$ such that $\psi = \alpha I$. We have $\alpha \neq 0$ since $\psi \neq 0$. By construction $v'_n = \alpha v_n$ for $0 \leq n \leq d$.

(ii) \Rightarrow (i) Clear. \square

Theorem 17.7. Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type, with feasible sequence (a, b, c, d) . Let $\{v_n\}_{n=0}^d$ denote a basis for the underlying vector space V . Then the following are equivalent:

- (i) the basis $\{v_n\}_{n=0}^d$ meets the requirements of Proposition 17.1;

(ii) the basis $\{q^{-n}b^{-n}v_n\}_{n=0}^d$ is a $[1, 3, 0, 2]$ -basis of V , for the \boxtimes_q -module structure in Corollary 16.34.

Proof. This follows from the proof of Proposition 17.1, together with Lemma 17.6. \square

Proposition 17.8. Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type, with \mathbb{Z}_3 -symmetric completion C and dual \mathbb{Z}_3 -symmetric completion C' . Let (a, b, c, d) denote a feasible sequence for A, B . Consider the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent C and C' with respect to a basis from Proposition 17.1. The matrix representing C is upper triangular, with (n, n) -entry

$$cq^{2n-d} + c^{-1}q^{d-2n}$$

for $0 \leq n \leq d$, $(n - 1, n)$ -entry

$$(q^{d-n+1} - q^{n-d-1})((b + b^{-1})cq^n - (a + a^{-1})q^{d-n+1})$$

for $1 \leq n \leq d$, $(n - 2, n)$ -entry

$$cq^{d+1}(q^{d-n+1} - q^{n-d-1})(q^{d-n+2} - q^{n-d-2})$$

for $2 \leq n \leq d$, and all other entries 0. The matrix representing C' is lower triangular, with (n, n) -entry

$$cq^{d-2n} + c^{-1}q^{2n-d}$$

for $0 \leq n \leq d$, $(n, n - 1)$ -entry

$$(q^n - q^{-n})((a + a^{-1})q^{-n} - (b + b^{-1})c^{-1}q^{n-d-1})$$

for $1 \leq n \leq d$, $(n, n - 2)$ -entry

$$c^{-1}q^{-d-1}(q^n - q^{-n})(q^{n-1} - q^{1-n})$$

for $2 \leq n \leq d$, and all other entries 0.

Proof. The matrix representing C is obtained using Proposition 17.1 and the last equation in Proposition 16.17. The matrix representing C' is similarly obtained using the last equation in Proposition 16.20. \square

We now summarize Proposition 17.1 and Proposition 17.8.

Corollary 17.9. *Let A, B denote a Leonard pair over \mathbb{F} of q -Racah type. Consider a compact basis for this pair. In the table below, for each of the displayed maps we describe the matrix that represents it with respect to the basis.*

Map	Representing matrix
A	Irreducible tridiagonal
B	Irreducible tridiagonal
$qAB - q^{-1}BA$	Upper triangular
$qBA - q^{-1}AB$	Lower triangular

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Appendix A. Some matrix definitions

In this appendix we define and discuss the matrices that were used earlier in the paper.

Fix an integer $d \geq 1$ and a nonzero $t \in \mathbb{F}$ that is not among $\{q^{d-2n+1}\}_{n=1}^d$. Let I denote the identity matrix in $\text{Mat}_{d+1}(\mathbb{F})$.

Definition A.1. Let Z denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, j) -entry $\delta_{i+j, d}$ for $0 \leq i, j \leq d$. Note that $Z^2 = I$.

Example A.2. For $d = 3$,

$$Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For each lemma in this appendix, the proof is routine and left to the reader.

Lemma A.3. *For $B \in \text{Mat}_{d+1}(\mathbb{F})$ and $0 \leq i, j \leq d$ the following coincide:*

- (i) *the (i, j) -entry of ZBZ ;*
- (ii) *the $(d - i, d - j)$ -entry of B .*

We now consider the matrices used in Section 13. They are

$$D_q(t), \quad \mathcal{D}_q(t), \quad T_q. \tag{48}$$

Recall the notation

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \quad n = 0, 1, 2, \dots$$

We interpret $(a; q)_0 = 1$.

Definition A.4. Let $D_q(t)$ denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, i) -entry $(tq^{d-2i+1}; q^2)_i$ for $0 \leq i \leq d$.

Example A.5. For $d = 3$,

$$D_q(t) = \text{diag}(1, 1 - tq^2, (1 - tq^2)(1 - t), (1 - tq^2)(1 - t)(1 - tq^{-2})).$$

Lemma A.6. We have

$$(D_q(t))^{-1} = \frac{ZD_{q^{-1}}(t)Z}{(tq^{1-d}; q^2)_d}.$$

Definition A.7. Let $\mathcal{D}_q(t)$ denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, i) -entry $t^i q^{i(d-1)}$ for $0 \leq i \leq d$.

Example A.8. For $d = 3$,

$$\mathcal{D}_q(t) = \text{diag}(1, tq^2, t^2 q^4, t^3 q^6).$$

Lemma A.9. We have

$$(\mathcal{D}_q(t))^{-1} = \mathcal{D}_{q^{-1}}(t^{-1}) = t^{-d} q^{d(1-d)} Z\mathcal{D}_q(t)Z.$$

We recall some notation. For integers $n \geq i \geq 0$ define

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{[n]_q!}{[i]_q! [n-i]_q!}.$$

Definition A.10. Let T_q denote the lower triangular matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, j) -entry

$$(-1)^j q^{j(1-i)} \begin{bmatrix} i \\ j \end{bmatrix}_q$$

for $0 \leq j \leq i \leq d$.

Example A.11. For $d = 3$,

$$T_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -q^{-1}[2]_q & q^{-2} & 0 \\ 1 & -q^{-2}[3]_q & q^{-4}[3]_q & -q^{-6} \end{pmatrix}.$$

By [32, Theorem 15.4] we have $(T_q)^{-1} = T_{q^{-1}}$.

We are done discussing the matrices (48). We now consider the matrices used in Section 11. They are

$$K_q, \quad E_q, \quad F_q(t), \quad G_q(t), \quad L_q(t), \quad S_q(t), \quad M_q(t). \tag{49}$$

Definition A.12. Let K_q denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, i) -entry q^{d-2i} for $0 \leq i \leq d$.

Example A.13. For $d = 3$,

$$K_q = \text{diag}(q^3, q, q^{-1}, q^{-3}).$$

Lemma A.14. We have

$$(K_q)^{-1} = K_{q^{-1}} = ZK_qZ.$$

Definition A.15. Let E_q denote the upper bidiagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, i) -entry q^{2i-d} for $0 \leq i \leq d$ and $(i - 1, i)$ -entry $q^d - q^{2i-2-d}$ for $1 \leq i \leq d$. Note that E_q has constant row sum q^d .

Example A.16. For $d = 3$,

$$E_q = \begin{pmatrix} q^{-3} & q^3 - q^{-3} & 0 & 0 \\ 0 & q^{-1} & q^3 - q^{-1} & 0 \\ 0 & 0 & q & q^3 - q \\ 0 & 0 & 0 & q^3 \end{pmatrix}.$$

Lemma A.17. We have $E_q = X^{-1}K_qX$ where $X = T_qZ$.

In Section 11 we refer to $(E_q)^{-1}$.

Lemma A.18. The matrix $(E_q)^{-1}$ is upper triangular with (i, j) -entry

$$(q^{2(d-j+1)}; q^2)_{j-i} q^{d-2j}$$

for $0 \leq i \leq j \leq d$.

Example A.19. For $d = 3$,

$$(E_q)^{-1} = \begin{pmatrix} q^3 & (1 - q^6)q & (1 - q^4)(1 - q^6)q^{-1} & (1 - q^2)(1 - q^4)(1 - q^6)q^{-3} \\ 0 & q & (1 - q^4)q^{-1} & (1 - q^2)(1 - q^4)q^{-3} \\ 0 & 0 & q^{-1} & (1 - q^2)q^{-3} \\ 0 & 0 & 0 & q^{-3} \end{pmatrix}.$$

Definition A.20. Let $F_q(t)$ denote the upper bidiagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, i) -entry q^{2i-d} for $0 \leq i \leq d$ and $(i - 1, i)$ -entry $(q^d - q^{2i-2-d})q^{1-d}t$ for $1 \leq i \leq d$.

Example A.21. For $d = 3$,

$$F_q(t) = \begin{pmatrix} q^{-3} & (q^3 - q^{-3})q^{-2}t & 0 & 0 \\ 0 & q^{-1} & (q^3 - q^{-1})q^{-2}t & 0 \\ 0 & 0 & q & (q^3 - q)q^{-2}t \\ 0 & 0 & 0 & q^3 \end{pmatrix}.$$

Lemma A.22. We have

$$F_q(t) = \mathcal{D}_q(t^{-1})E_q(\mathcal{D}_q(t^{-1}))^{-1} \tag{50}$$

and also

$$F_q(t) = K_{q^{-1}} - t \frac{[E_{q^{-1}}, K_{q^{-1}}]}{q - q^{-1}}.$$

Definition A.23. Let $G_q(t)$ denote the upper bidiagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, i) -entry q^{2i-d} for $0 \leq i \leq d$ and $(i - 1, i)$ -entry $(q^d - q^{2i-2-d})(1 - tq^{d-2i+1})$ for $1 \leq i \leq d$.

Example A.24. For $d = 3$,

$$G_q(t) = \begin{pmatrix} q^{-3} & (q^3 - q^{-3})(1 - tq^2) & 0 & 0 \\ 0 & q^{-1} & (q^3 - q^{-1})(1 - t) & 0 \\ 0 & 0 & q & (q^3 - q)(1 - tq^{-2}) \\ 0 & 0 & 0 & q^3 \end{pmatrix}.$$

Lemma A.25. We have

$$G_q(t) = (D_q(t))^{-1}E_qD_q(t) \tag{51}$$

and also

$$G_q(t) = E_q - K_q + F_{q^{-1}}(t).$$

Definition A.26. Let $L_q(t)$ denote the upper bidiagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, i) -entry q^{2i-d} for $0 \leq i \leq d$ and $(i - 1, i)$ -entry $(q^d - q^{2i-2-d})(1 - tq^{d-2i+1})^{-1}$ for $1 \leq i \leq d$.

Example A.27. For $d = 3$,

$$L_q(t) = \begin{pmatrix} q^{-3} & \frac{q^3 - q^{-3}}{1 - tq^2} & 0 & 0 \\ 0 & q^{-1} & \frac{q^3 - q^{-1}}{1 - t} & 0 \\ 0 & 0 & q & \frac{q^3 - q}{1 - tq^{-2}} \\ 0 & 0 & 0 & q^3 \end{pmatrix}.$$

Lemma A.28. We have

$$L_q(t) = D_q(t)E_q(D_q(t))^{-1}.$$

In Section 11 we refer to $(L_q(t))^{-1}$.

Lemma A.29. The matrix $(L_q(t))^{-1}$ is upper triangular with (i, j) -entry

$$\frac{(q^{2(d-j+1)}; q^2)_{j-i}}{(tq^{d-2j+1}; q^2)_{j-i}} q^{d-2j}$$

for $0 \leq i \leq j \leq d$.

Example A.30. For $d = 3$,

$$(L_q(t))^{-1} = \begin{pmatrix} q^3 & \frac{(1-q^6)q}{1-tq^2} & \frac{(1-q^4)(1-q^6)q^{-1}}{(1-t)(1-tq^2)} & \frac{(1-q^2)(1-q^4)(1-q^6)q^{-3}}{(1-tq^{-2})(1-t)(1-tq^2)} \\ 0 & q & \frac{(1-q^4)q^{-1}}{1-t} & \frac{(1-q^2)(1-q^4)q^{-3}}{(1-tq^{-2})(1-t)} \\ 0 & 0 & q^{-1} & \frac{(1-q^2)q^{-3}}{1-tq^{-2}} \\ 0 & 0 & 0 & q^{-3} \end{pmatrix}.$$

Definition A.31. Let $S_q(t)$ denote the tridiagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with $(i-1, i)$ -entry $(q^d - q^{2i-d-2})(1 - tq^{2i-d-1})$ for $1 \leq i \leq d$, $(i, i-1)$ -entry $(q^{2i-d} - q^{-d})q^{2i-d-1}t$ for $1 \leq i \leq d$, and (i, i) -entry

$$q^d - (q^{2i-d} - q^{-d})q^{2i-d-1}t - (q^d - q^{2i-d})(1 - tq^{2i-d+1})$$

for $0 \leq i \leq d$. Note that $S_q(t)$ has constant row sum q^d .

Example A.32. For $d = 3$,

$$S_q(t) = E_q + t \begin{pmatrix} q - q^{-5} & q^{-5} - q & 0 & 0 \\ q^{-3} - q^{-5} & (q - q^{-1})(q^2 + 1 - q^{-4}) & q^{-1} - q^3 & 0 \\ 0 & q - q^{-3} & (q - q^{-1})(q^4 - 1 - q^{-2}) & q^3 - q^5 \\ 0 & 0 & q^5 - q^{-1} & q^{-1} - q^5 \end{pmatrix}.$$

Lemma A.33. We have

$$S_q(t) = T_q G_{q^{-1}}(t) T_{q^{-1}}$$

and also

$$S_q(t) = E_q - t \frac{[E_q, Z E_{q^{-1}} Z]}{q - q^{-1}}.$$

Definition A.34. We define $M_q(t) \in \text{Mat}_{d+1}(\mathbb{F})$ as follows. For $0 \leq i, j \leq d$ the (i, j) -entry of $M_q(t)$ is given in the table below.

Case	(i, j) -entry of $M_q(t)$
$i - j > 1$	0
$i - j = 1$	$\frac{q^{-1} - q^{2i-1}}{t^{-1} - q^{2i-d-1}}$
$j \geq i, i \neq 0, j \neq d$	$\frac{1-tq^{d+1}}{1-tq^{2j-d+1}} \frac{1-tq^{-d-1}}{1-tq^{2i-d-1}} \frac{(q^{2i-2d}; q^2)_{j-i}}{(tq^{2i-d+1}; q^2)_{j-i}} \frac{1}{q^{d-2j}}$
$i = 0, j \neq d$	$\frac{1-tq^{d+1}}{1-tq^{2j-d+1}} \frac{(q^{-2d}; q^2)_j}{(tq^{1-d}; q^2)_j} \frac{1}{q^{d-2j}}$
$i \neq 0, j = d$	$\frac{1-tq^{-d-1}}{1-tq^{2i-d-1}} \frac{(q^{2i-2d}; q^2)_{d-i}}{(tq^{2i-d+1}; q^2)_{d-i}} q^d$
$i = 0, j = d$	$\frac{(q^{-2d}; q^2)_d}{(tq^{1-d}; q^2)_d} q^d$

Example A.35. For $d = 3$,

$$M_q(t) = \begin{pmatrix} \frac{1-tq^4}{q^3(1-tq^{-2})} & \frac{(1-tq^4)(1-q^{-6})}{q(1-tq^{-2})(1-t)} & \frac{q(1-tq^4)(1-q^{-4})(1-q^{-6})}{(1-tq^{-2})(1-t)(1-tq^2)} & \frac{q^3(1-q^{-2})(1-q^{-4})(1-q^{-6})}{(1-tq^{-2})(1-t)(1-tq^2)} \\ \frac{q^{-1}-q}{t^{-1}-q^{-2}} & \frac{(1-tq^4)(1-tq^{-4})}{q(1-tq^{-2})(1-t)} & \frac{q(1-tq^4)(1-tq^{-4})(1-q^{-4})}{(1-tq^{-2})(1-t)(1-tq^2)} & \frac{q^3(1-tq^{-4})(1-q^{-2})(1-q^{-4})}{(1-tq^{-2})(1-t)(1-tq^2)} \\ 0 & \frac{q^{-1}-q^3}{t^{-1}-1} & \frac{q(1-tq^4)(1-tq^{-4})}{(1-t)(1-tq^2)} & \frac{q^3(1-tq^{-4})(1-q^{-2})}{(1-t)(1-tq^2)} \\ 0 & 0 & \frac{q^{-1}-q^5}{t^{-1}-q^2} & \frac{q^3(1-tq^{-4})}{1-tq^2} \end{pmatrix}.$$

Lemma A.36. We have

$$M_q(t) = T_{q^{-1}} (L_q(t))^{-1} T_q.$$

Appendix B. Some matrix identities

In this section we record some miscellaneous facts about the matrices listed in (48), (49).

Lemma B.1. We have

$$t(ZE_q Z - F_q(t^{-1})) = \frac{[E_{q^{-1}}, Z F_{q^{-1}}(t) Z]}{q - q^{-1}},$$

$$t(G_q(t^{-1}) - Z E_{q^{-1}} Z) = \frac{[S_q(t), K_q]}{q - q^{-1}},$$

$$t(K_q - S_q(t^{-1})) = \frac{[G_q(t), Z E_{q^{-1}} Z]}{q - q^{-1}}.$$

Proof. These equations express how the relations from Lemma 9.4 look in the 24 bases for $\mathbf{V}_d(t)$ from Lemma 12.4. \square

Lemma B.2. We have

$$\begin{aligned}
 t(ZE_qZ - M_q(t)) &= \frac{[E_{q^{-1}}, M_q(t)]}{q - q^{-1}}, & t^{-1}(F_{q^{-1}}(t) - K_q) &= \frac{[E_q, K_q]}{q - q^{-1}}, \\
 t(E_{q^{-1}} - K_q) &= \frac{[F_q(t), K_q]}{q - q^{-1}}, \\
 t^{-1}(ZG_{q^{-1}}(t)Z - (L_q(t))^{-1}) &= \frac{[(L_q(t))^{-1}, ZS_{q^{-1}}(t^{-1})Z]}{q - q^{-1}}, \\
 t(K_q - (E_q)^{-1}) &= \frac{[G_q(t), (E_q)^{-1}]}{q - q^{-1}}, & t^{-1}(S_q(t) - E_q) &= \frac{[ZE_{q^{-1}}Z, E_q]}{q - q^{-1}}, \\
 t(S_q(t^{-1}) - ZL_{q^{-1}}(t)Z) &= \frac{[ZL_{q^{-1}}(t)Z, G_q(t)]}{q - q^{-1}}, & t(G_q(t^{-1}) - E_q) &= \frac{[E_q, K_q]}{q - q^{-1}}, \\
 t(ZE_{q^{-1}}Z - (E_q)^{-1}) &= \frac{[(E_q)^{-1}, S_q(t)]}{q - q^{-1}}, & t(K_q - (L_q(t))^{-1}) &= \frac{[E_q, (L_q(t))^{-1}]}{q - q^{-1}}, \\
 t^{-1}(E_q - L_q(t)) &= \frac{[L_q(t), K_q]}{q - q^{-1}}, & t^{-1}(ZF_{q^{-1}}(t)Z - M_q(t)) &= \frac{[M_q(t), F_q(t^{-1})]}{q - q^{-1}}.
 \end{aligned}$$

Proof. These equations express how the relations from Lemma 9.5 look in the 24 bases for $\mathbf{V}_d(t)$ from Lemma 12.4. \square

Lemma B.3. We have

$$\begin{aligned}
 T_q Z T_q Z T_q Z &= (-1)^d q^{-d(d-1)} I, \\
 T_q \mathcal{D}_q(t) T_q \mathcal{D}_{q^{-1}}(t^{-1}) T_{q^{-1}} (\mathcal{D}_q(t))^{-1} &= I.
 \end{aligned}$$

Proof. For $n \in \mathbb{N}$, let B_0, B_1, \dots, B_n denote a sequence of bases for $\mathbf{V}_d(t)$ such that $B_0 = B_n$. For $1 \leq i \leq n$ let T_i denote the transition matrix from B_{i-1} to B_i . Then $T_1 T_2 \cdots T_n = I$. Both equations in the lemma statement are obtained in this way, using an appropriate sequence of bases from the 24 given in Lemma 12.4. \square

Lemma B.4. We have

$$\begin{aligned}
 ZT_q ZG_q(t) &= F_q(t) ZT_q Z, & E_{q^{-1}} &= T_{q^{-1}} E_q T_q, \\
 L_q(t) ZT_{q^{-1}} ZL_{q^{-1}}(t^{-1}) ZT_q Z &= I, & S_q(t) &= ZT_{q^{-1}} F_{q^{-1}}(t) T_q Z, \\
 \mathcal{D}_q(t^{-1}) S_q(t) (\mathcal{D}_q(t^{-1}))^{-1} &= ZS_{q^{-1}}(t) Z, & M_q(t) &= ZT_q L_{q^{-1}}(t^{-1}) T_{q^{-1}} Z, \\
 M_{q^{-1}}(t^{-1}) &= (\mathcal{D}_q(t))^{-1} M_q(t) \mathcal{D}_q(t), & M_q(t) ZM_{q^{-1}}(t^{-1}) Z &= I.
 \end{aligned}$$

Proof. These are all applications of the linear algebra principle from the fourth paragraph of Section 13. \square

Lemma B.5. *In the table below we list some 3-tuples u, v, w of matrices in $\text{Mat}_{d+1}(\mathbb{F})$. For each case*

$$\frac{quv - q^{-1}vu}{q - q^{-1}} = I, \quad \frac{qvw - q^{-1}wv}{q - q^{-1}} = I, \quad \frac{qwu - q^{-1}uw}{q - q^{-1}} = I.$$

u	v	w
E_q	K_q	$ZE_{q^{-1}}Z$
$L_q(t)$	K_q	$ZG_{q^{-1}}(t)Z$
$S_{q^{-1}}(t)$	$(E_{q^{-1}})^{-1}$	$G_{q^{-1}}(t^{-1})$
E_q	$(L_q(t^{-1}))^{-1}$	$ZS_{q^{-1}}(t)Z$
$ZG_q(t)Z$	$K_{q^{-1}}$	$L_{q^{-1}}(t)$
$G_q(t^{-1})$	$(E_q)^{-1}$	$S_q(t)$
$ZS_q(t)Z$	$(L_{q^{-1}}(t^{-1}))^{-1}$	$E_{q^{-1}}$
$F_q(t^{-1})$	$E_{q^{-1}}$	$M_q(t)$
$M_{q^{-1}}(t)$	E_q	$F_{q^{-1}}(t^{-1})$
$F_q(t^{-1})$	K_q	$ZF_{q^{-1}}(t)Z$

Proof. The relations (13) hold in \boxtimes_q . The equations in the present lemma express how these relations look in the 24 bases for $\mathbf{V}_d(t)$ from Lemma 12.4. \square

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