CHALMERS | UNIVERSITY OF GOTHENBURG

PREPRINT 2013:1

Approximate globally convergent algorithm with applications in electrical prospecting

JOHN BONDESTAM MALMBERG
LARISA BEILINA

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GOTHENBURG
Gothenburg Sweden 2013

Preprint 2013:1

Approximate globally convergent algorithm with applications in electrical prospecting

John Bondestam Malmberg and Larisa Beilina

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Gothenburg, Sweden
Gothenburg, January 2013

Preprint 2013:1
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2013

Approximate globally convergent algorithm with applications in electrical prospecting

John Bondestam Malmberg* and Larisa Beilina**

Abstract In this paper we present at the first time an approximate globally convergent method for the reconstruction of an unknown conductivity function from backscattered electric field measured at the boundary of geological medium under assumptions that dielectric permittivity and magnetic permeability functions are known. This is the typical case of an coefficient inverse problem in electrical prospecting. We consider a simplified mathematical model assuming that geological medium is isotropic and non-dispersive.

1 Introduction

In this work we consider a Coefficient Inverse Problem (CIP) for Maxwell equations in time domain and derive an approximate globally convergent method for reconstruction of an unknown conductivity function in space with data resulted from a single measurement. This means that our boundary data are generated by a single source location or a single direction of the propagation of an incident plane wave. We assume that we are working in isotropic medium with known values of electric permeability and magnetic permittivity functions. This is the typical case of electrical prospecting [4] and is of great interest in the geological community.

The first generation of globally convergent algorithms developed in [6, 7, 10] is called convexifixation algorithms. In this paper we use similar technique as in [1] to derive an approximate globally convergent method of second generation for finding

John Bondestam Malmberg*
Department of Mathematical Sciences, Chalmers University of Technology and Gothenburg University, SE-42196 Gothenburg, Sweden, e-mail: john.bondestam.malmberg@chalmers.se

Larisa Beilina**
Department of Mathematical Sciences, Chalmers University of Technology and Gothenburg University, SE-42196 Gothenburg, Sweden e-mail: larisa.beilina@chalmers.se

the conductivity function. This method, first appearing in [2], uses a layer stripping procedure with respect to the pseudo-frequency.

The main difficulty for solution of CIPs is ill-posedness and nonlinearity of these problems. Approximate globally convergent method of [1] gives an answer to the question: How to obtain an unknown coefficient function inside the domain of interest in a small neighborhood of the exact solution without a priori knowledge of any information about this solution? The approximate globally convergent method of [1] is experimentally verified in recent works [3, 8]. Using our recent numerical experience [1, 3, 8] we can conclude that approximate globally convergent method is reliable tool for solution of CIPs using a single measurement data.

In the current work we derive an approximate globally convergent method for explicit computation of the conductivity function using iterative layer stripping procedure in pseudo-frequency. We also present an approximate mathematical model for computation of the so-called 'tail' function which is crucial for reliable reconstruction of an unknown conductivity function. Finally, we formulate an approximate globally convergent algorithm which can be used in real computations for reconstruction of an unknown function from backscattered data collected at the boundary.

2 The Maxwell equations in time domain

We consider the Maxwell equations in an isotropic, non-dispersive medium (see for instance [5])

$$\frac{\partial B(x,t)}{\partial t} = -\nabla \times E(x,t) - M(x,t) \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,T), \tag{1}$$

$$\frac{\partial D(x,t)}{\partial t} = \nabla \times H(x,t) - J(x,t) \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,T). \tag{2}$$

Here, $n = 2, 3$, $T > 0$, $B = \mu H$ is the magnetic flux density, $D = \varepsilon E$ is the electric flux density, $H = (H_1, H_2, H_3)$ is the magnetic field, $E = (E_1, E_2, E_3)$ is the electric field, μ and ε are the magnetic permeability and the dielectric permittivity of the medium, respectively, and J and M are electric and magnetic current densities, respectively. The electric and magnetic fields satisfy the relations

$$\nabla \cdot (\varepsilon E) = \rho, \ \nabla \cdot (\mu H) = 0 \quad \text{ in } \mathbb{R}^n \times (0,T), \tag{3}$$

where $\rho(x,t)$ is a given charge density.

As it suffices for our purposes we consider the case when μ and ε are constants, $M(x,t) \equiv 0$, $\rho(x,t) \equiv 0$, and J is generated by the electric field such that $J = \sigma E$, where σ is the conductivity of the medium. We assume that $\sigma = \sigma(x)$ is dependent only on the spatial variable x.

Under these assumptions equations (1) and (2) are reduced to

$$\mu \frac{\partial H(x,t)}{\partial t} = -\nabla \times E(x,t) \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,T), \quad (4)$$

$$\varepsilon \frac{\partial E(x,t)}{\partial t} = \nabla \times H(x,t) - \sigma(x) E(x,t) \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,T), \quad (5)$$

and since μ and ε are positive constants Gauss's law (3) is reduced to

$$\nabla \cdot E = 0, \ \nabla \cdot H = 0 \quad \text{in } \mathbb{R}^n \times (0,T). \quad (6)$$

In addition to equations (4) and (5) we impose the following initial conditions on the magnetic and electric fields

$$H(x,0) = 0, \quad (7)$$

$$E(x,0) = (E_{0,1}, E_{0,2}, E_{0,3}) \delta(x - x_0) =: E_0 \delta(x - x_0), \quad (8)$$

where $E_{0,k}$, $k = 1, 2, 3$ are constants, δ is the three dimensional Dirac delta, and x_0 is some specific point in $\mathbb{R}^n$. This corresponds to the initialisation of an electric pulse at the point x_0 at time $t = 0$.

The problem described by equations (4), (5), (7), and (8) is similar to those considered in [4]. It describes the electric and magnetic fields generated in response to an electric pulse initiated at $x_0 \in \mathbb{R}^n$ and propagating through 'the ground.'

Further we will consider the inverse problem when $\sigma(x)$ is included in the equation for the electric field. Hence we eliminate the dependence on the magnetic field from the Cauchy problem described in equations (4), (5), (7), and (8).

Applying the curl operator to equation (4) yields

$$\nabla \times \frac{\partial H(x,t)}{\partial t} = -\frac{1}{\mu} \nabla \times \nabla \times E(x,t) \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,T).$$

Using above equation and differentiating (5) with respect to t gives

$$\begin{aligned} \varepsilon \frac{\partial^2 E(x,t)}{\partial t^2} &= \nabla \times \frac{\partial H(x,t)}{\partial t} - \sigma(x) \frac{\partial E(x,t)}{\partial t} \\ &= -\frac{1}{\mu} \nabla \times \nabla \times E(x,t) - \sigma(x) \frac{\partial E(x,t)}{\partial t} \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,T). \end{aligned}$$

Hence, after some rearrangement of the terms, and by applying the identity $\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \Delta E$ together with Gauss' law (6) we get

$$\mu\varepsilon \frac{\partial^2 E(x,t)}{\partial t^2} + \mu \sigma(x) \frac{\partial E(x,t)}{\partial t} - \Delta E(x,t) = 0 \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,T). \quad (9)$$

Letting t go to zero in (5) and using (7) and (8) we get

$$\frac{\partial E(x,0)}{\partial t} = -\frac{1}{\varepsilon} \sigma(x) E_0 \delta(x - x_0). \quad (10)$$

Noting that the equations for each component of the electric field in equations (8), (9) and (10) are decoupled we may write the following componentwise Cauchy problems, for $k = 1, 2, 3$:

$$
\begin{aligned}
&\mu\varepsilon\frac{\partial^2 E_k(x,t)}{\partial t^2} + \mu\sigma(x)\frac{\partial E_k(x,t)}{\partial t} - \Delta E_k(x,t) = 0, && \text{for } (x,t) \in \mathbb{R}^n \times (0,T),\\
&E_k(x,0) = E_{0,k}\delta(x-x_0), && \text{for } x \in \mathbb{R}^n,\\
&\frac{\partial E_k(x,0)}{\partial t} = -\frac{1}{\varepsilon}\sigma(x)E_{0,k}\delta(x-x_0), && \text{for } x \in \mathbb{R}^n.
\end{aligned}
\tag{11}
$$

Further we will assume that only the first component E_1 of the electric field $E = (E_1, E_2, E_2)$ is initialized by the function $E_0 = (E_{0,1}, 0, 0)$ and thus, by (6), the other two components E_2 and E_3 are zero. This yields that the problem (11) is reduced to the solution of the following Cauchy problem

$$
\begin{aligned}
&\mu\varepsilon\frac{\partial^2 E_1(x,t)}{\partial t^2} + \mu\sigma(x)\frac{\partial E_1(x,t)}{\partial t} - \Delta E_1(x,t) = 0, && \text{for } (x,t) \in \mathbb{R}^n \times (0,T),\\
&E_1(x,0) = E_{0,1}\delta(x-x_0), && \text{for } x \in \mathbb{R}^n,\\
&\frac{\partial E_1(x,0)}{\partial t} = -\frac{1}{\varepsilon}\sigma(x)E_{0,1}\delta(x-x_0), && \text{for } x \in \mathbb{R}^n.
\end{aligned}
\tag{12}
$$

To reduce notations we will in the following drop the index on E_1, writing $E(x,t) = E_1(x,t)$.

A coefficient inverse problem (CIP)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary Γ such that $x_0 \notin \overline{\Omega}$. Let $L^2(\Omega)$ be the space of square integrable functions on Ω, and define $\Omega_T := \Omega \times (0,T)$, and $\Gamma_T := \Gamma \times (0,T)$. Suppose that $\sigma(x)$ satisfies the Cauchy problem (12), restricted to Ω_T, for known coefficients μ and ε, and that $\sigma(x) \in C_{d,\Omega}$, where

$$
C_{d,\Omega} := \{u \in C^2(\mathbb{R}^n) : 1 \le u(x) \le d,\, x \in \mathbb{R}^n,\, u \equiv 1 \text{ in } \mathbb{R}^n \setminus \Omega\} \tag{13}
$$

for some given $d > 1$. We then seek to determine $\sigma(x)$, $x \in \Omega$, under assumption that the function

$$
g(x,t) = E(x,t)\Big|_{\Gamma_T} \tag{14}
$$

is known. In other words, σ and E satisfy the following initial boundary value problem:

$$
\begin{aligned}
&\mu\varepsilon \frac{\partial^2 E(x,t)}{\partial t^2} + \mu\sigma(x)\frac{\partial E(x,t)}{\partial t} - \Delta E(x,t) = 0, && \text{for } (x,t) \in \Omega_T, \\
&E(x,t) = g(x,t) && \text{for } (x,t) \in \Gamma_T \\
&E(x,0) = \frac{\partial E(x,0)}{\partial t} = 0, && \text{for } x \in \Omega.
\end{aligned} \tag{15}
$$

3 Approximately globally convergent method

In this section we develop an approximately globally convergent method for the coefficient inverse problem **CIP** to reconstruct the conductivity function. The method uses the Laplace transform of the problem (15), and hence we start by deriving some properties thereof.

3.1 Laplace transformation of the initial boundary value problem

Let

$$\mathbf{E}(x,s) := \mathscr{L}\Big[E(x,\cdot)\Big](s) := \int_0^\infty E(x,t)\mathrm{e}^{-st}\,\mathrm{d}t, \quad s \geq \underline{s}$$

for some fixed $\underline{s} > 0$ be the Laplace transform of the electric field $E(x,t)$ given by (15). Applying the Laplace transform to the differential equation in (15) and using the two well-known properties $\mathscr{L}[f'](s) = s\mathscr{L}[f](s) - f(0)$ and $\mathscr{L}[f''](s) = s^2\mathscr{L}[f](s) - sf(0) - f'(0)$ of the Laplace transform we get

$$\Delta\mathbf{E}(x,s) - (\mu\varepsilon s^2 + \mu s\sigma(x))\mathbf{E}(x,s) = -\mu\varepsilon s E_0\delta(x-x_0). \tag{16}$$

Similarly with Theorems 2.7.1 and 2.7.2 of [1] it can be proved that $\mathbf{E}(x,s) \to 0$ as $|x| \to \infty$ and that $\mathbf{E}(x,s) > 0$, hence the following function

$$v(x,s) := \frac{\ln(\mathbf{E}(x,s))}{s}, \qquad x \in \Omega,\ s \in [\underline{s},\overline{s}] \tag{17}$$

for some $\overline{s} > \underline{s} > 0$ is well-defined.

Next, we assume that the following asymptotic behavior for the function $\mathbf{E}(x,s)$ holds

$$D_x^\alpha \frac{\partial^n}{\partial s^n}\mathbf{E}(x,s) = D_x^\alpha \frac{\partial^n}{\partial s^n}\left(\frac{\mathrm{e}^{-sl(x,x_0)}}{f(x,x_0)}\left(1 + O\left(\frac{1}{s}\right)\right)\right), \quad s \to \infty \tag{18}$$

where $|\alpha| \leq 3$, $n = 0, 1$, $l(x,x_0)$ is the length of a geodesic line, generated by the eikonal equation corresponding to the function σ, connecting points x and $x_0, x \neq x_0$, and $f(x,x_0)$ is a certain function, nonzero for $x \in \overline{\Omega}$. This Lemma follows from Theorem 4.1 of [9]. We note that asymptotic behavior (18) is fulfilled for the general hyperbolic equation of the second order under the assumption that the geodesic lines are regular, see Remarks 2.3.1 of [1].

Using (18) we get the following asymptotic behavior for the function $v(x,s)$ of (17):

$$\left\| \frac{\partial^n v(\cdot,s)}{\partial s^n} \right\|_{C^{2+\alpha}(\overline{\Omega})} = O\left(\frac{1}{s^n}\right), \quad s \to \infty,\ n = 0,\ 1, \tag{19}$$

where $C^{2+\alpha}(\overline{\Omega})$ is the Hölder space of order $2+\alpha$, $0 \leq \alpha < 1$.

3.2 The transformation procedure

In this section we will show how to reduce the inverse problem **CIP** to the solution of a nonlinear integral differential equation. First, we write $\mathbf{E}(x,s) = \mathrm{e}^{sv(x,s)}$ with v defined in (17). Substituting this into equation (16) and noting that $x_0 \notin \overline{\Omega}$, we get the equation

$$\Delta v(x,s) + s(\nabla v(x,s))^2 = \mu \varepsilon s + \mu \sigma(x). \tag{20}$$

Given knowledge of the functions v, as well as the coefficients μ and ε, we calculate $\sigma(x)$ explicitly from (20):

$$\sigma(x) = \frac{1}{\mu}(\Delta v(x,s) + s(\nabla v(x,s))^2) - \varepsilon s \tag{21}$$

for any $s \geq \underline{s}$. However, v is at this point unknown. Next, denoting

$$q(x,s) = \frac{\partial v(x,s)}{\partial s} \tag{22}$$

and differentiating equation (20) with respect to s yields

$$\Delta q(x,s) + (\nabla v(x,s))^2 + 2s\nabla v(x,s) \cdot \nabla q(x,s) = \mu \varepsilon. \tag{23}$$

Using asymptotic behavior (19) in (22) we get

$$v(x,s) = -\int_s^\infty q(x,s) ds. \tag{24}$$

Next, we define the so-called 'tail function' $V(x,\overline{s})$ as

$$V(x,\overline{s}) := \int_{\overline{s}}^\infty q(x,\tau)\,\mathrm{d}\tau = v(x,s) + \int_s^{\overline{s}} q(x,\tau)\,\mathrm{d}\tau, \tag{25}$$

allowing us to rewrite (23) on the form

$$\begin{aligned}A(q)(x,s) :=&\Delta q(x,s) + (\nabla V(x,\bar{s}))^2 + \left(\int_s^{\bar{s}} \nabla q(x,\tau)\,\mathrm{d}\tau\right)^2 \\ &- 2\nabla V(x,\bar{s}) \cdot \int_s^{\bar{s}} \nabla q(x,\tau)\,\mathrm{d}\tau + 2s\nabla V(x,\bar{s}) \cdot \nabla q(x,s) \\ &- 2s\int_s^{\bar{s}} \nabla q(x,\tau)\,\mathrm{d}\tau \cdot \nabla q(x,s) = \mu\varepsilon.\end{aligned} \tag{26}$$

In view of (14), we may write

$$q(x,s)\Big|_{\Gamma} = \frac{\partial}{\partial s}\frac{\ln\left(\mathscr{L}\left[g(x,\cdot)\right](s)\right)}{s} =: \varphi(x,s), \tag{27}$$

which, together with (26), constitutes a non-linear problem for the unknown function q, given knowledge of the tail function $V(x,\bar{s})$. Under assumption that $V(x,\bar{s})$ or some approximation thereof is known we now derive a frequency discretised analogue of the problem (26), (27).

Define a partition $\underline{s} = s_N < s_{N-1} < \ldots < s_1 = \bar{s}$ with $s_n - s_{n+1} = h$ for $n = 1,\ldots,N-1$. We assume that q is a constant function of s on each interval $(s_{n+1}, s_n]$ and require that it satisfies equations (26) and (27) in weighted average on each such interval. That is, $q(x,s) = q_n(x)$, $s \in (s_{n+1}, s_n]$,

$$\int_{s_{n+1}}^{s_n} w_{1,n}(s)A(q)(x,s)\,\mathrm{d}s = \mu\varepsilon \int_{s_{n+1}}^{s_n} w_{1,n}(s)\,\mathrm{d}s, \qquad n = 1\ldots,N-1, \tag{28}$$

and

$$\int_{s_{n+1}}^{s_n} w_{2,n} q_n(x)\,\mathrm{d}s = \int_{s_{n+1}}^{s_n} w_{2,n}\varphi(x,s)\,\mathrm{d}s, \qquad n = 1,\ldots,N-1, \tag{29}$$

where $w_{1,n}$ and $w_{2,n}$ are some weight functions.

Similarly with [1] we define so-called Carleman Weight Functions in pseudo-frequency s, $w_{1,n}(s) = \mathrm{e}^{\lambda(s-s_n)}$, for some parameter $\lambda \gg 1$. This will 'reduce' the non-linearity of the equation (26). We take $w_{2,n}(s) \equiv 1$ for simplicity.

With these weight functions, and noting that

$$\int_s^{\bar{s}} \nabla q(x,s)\,\mathrm{d}s = (s_n - s)\nabla q_n(x) + \sum_{j=0}^{n-1} h\nabla q_j(x) \text{ for } s \in (s_{n+1}, s_n],$$

where we set $q_0 \equiv 0$, we can now use (26) and (28) to get

$$\begin{aligned}&\Delta q_n(x) + B_n(\lambda,h)\left(\nabla V(x,\bar{s}) - \sum_{j=0}^{n-1} h\nabla q_j(x)\right)\nabla q_n(x) \\ &\quad = \mu\varepsilon - C_n(\lambda,h)(\nabla q_n(x))^2 - \left(\nabla V(x,\bar{s}) - \sum_{j=0}^{n-1} h\nabla q_j(x)\right)^2,\end{aligned} \tag{30}$$

where

$$B_n(\lambda,h) = 4\frac{I_1(\lambda,h)}{I_0(\lambda,h)} + 2s_n, \tag{31}$$

$$C_n(\lambda,h) = 3\frac{I_2(\lambda,h)}{I_0(\lambda,h)} + 2s_n\frac{I_1(\lambda,h)}{I_0(\lambda,h)}, \tag{32}$$

$$I_k(\lambda,h) = \int_{-h}^{0} \tau^k \mathrm{e}^{\lambda\tau}\,\mathrm{d}\tau = (-1)^k \frac{k! - \mathrm{e}^{-\lambda h}\sum_{j=0}^{k}\frac{k!}{j!}(\lambda h)^j}{\lambda^{k+1}}. \tag{33}$$

It should be noted that as $\lambda \to \infty$

$$\frac{I_k(\lambda,h)}{I_l(\lambda,h)} = O(\lambda^{l-k}),$$

so that in particular the coefficient $C_n(\lambda,h)$ becomes small, $O(\lambda^{-1})$, for large λ. Thus, for sufficiently large values of λ we may neglect the first, non-linear, term of the right hand side of (30).

Similarly, from (29) with $w_{n,2}(s) \equiv 1$ we get

$$q_n(x)\Big|_{\Gamma} = \frac{1}{h}\int_{s_n}^{s_{n-1}} \varphi(x,s)\,\mathrm{d}s =: \overline{\varphi}_n(x). \tag{34}$$

If $V(x,\overline{s})$ or some approximation thereof is known, we can use the boundary value problem (30), (34) to successively compute q_n for $n = 1, 2, \ldots, N$.

3.3 *Modeling of the tail function* $V(x,\overline{s})$

Let the function $\sigma^*(x)$ be the exact solution of our **CIP** for the exact data g^* in (14) with the known exact functions μ and ε, and let $\mathbf{E}^*(x,s)$ be the Laplace transform of the corresponding solution to (15). We define the exact tail function

$$V^*(x,\overline{s}) = \frac{\ln(\mathbf{E}^*(x,\overline{s}))}{\overline{s}}. \tag{35}$$

Let $q^*(x,s)$ and $\varphi^*(x,s)$ be the exact functions corresponding to $q(x,s)$ and $\varphi(x,s)$ in (26), respectively, defined from the following nonlinear integral differential equation

$$\begin{aligned} A(q^*)(x,s) :=& \Delta q^*(x,s) + (\nabla V^*(x,\overline{s}))^2 + \left(\int_s^{\overline{s}} \nabla q^*(x,\tau)\,\mathrm{d}\tau\right)^2 \\ &- 2\nabla V^*(x,\overline{s}) \cdot \int_s^{\overline{s}} \nabla q^*(x,\tau)\,\mathrm{d}\tau + 2s\nabla V^*(x,\overline{s}) \cdot \nabla q^*(x,s) \\ &- 2s\int_s^{\overline{s}} \nabla q^*(x,\tau)\,\mathrm{d}\tau \cdot \nabla q^*(x,s) = \mu\varepsilon \end{aligned} \tag{36}$$

with

$$q^*(x,s)\Big|_{\Gamma} = \varphi^*(x,s). \tag{37}$$

Using (19) assume that the functions V^* and q^* have the following asymptotic behavior

$$\begin{aligned} V^*(x,\bar{s}) &= p^*(x) + \frac{f^*(x)}{\bar{s}} + O\left(\frac{1}{\bar{s}^2}\right) \approx p^*(x) + \frac{f^*(x)}{\bar{s}},\ \bar{s} \to \infty, \\ q^*(x,\bar{s}) &= \partial_{\bar{s}} V^*(x,\bar{s}) = -\frac{f^*(x)}{\bar{s}^2} + O\left(\frac{1}{\bar{s}^3}\right) \approx -\frac{f^*(x)}{\bar{s}^2},\ \bar{s} \to \infty. \end{aligned} \tag{38}$$

We take $s = \bar{s}$ in (36)-(37) to get

$$\begin{aligned} \Delta q^* + 2\bar{s}\nabla q^* \nabla V^* + (\nabla V^*)^2 &= \mu\varepsilon \qquad \text{in } \Omega, \\ q^*(x,\bar{s}) &= \psi^*(x,\bar{s}) \qquad \text{for } x \in \Gamma. \end{aligned} \tag{39}$$

Then we use the first two terms in the asymptotic behavior (38) for the exact tail $V^*(x,\bar{s}) = p^*(x) + \frac{f^*(x)}{\bar{s}}$ and for the exact function $q^*(x,\bar{s}) = -\frac{f^*(x)}{\bar{s}^2}$ to obtain

$$\begin{aligned} -\frac{\Delta f^*}{\bar{s}^2} - 2\bar{s}\left(\nabla p^* + \frac{\nabla f^*}{\bar{s}}\right) \cdot \frac{\nabla f^*}{\bar{s}^2} + \left(\nabla p^* + \frac{(\nabla f^*)}{\bar{s}}\right)^2 &= \mu\varepsilon \qquad \text{in } \Omega, \\ -\frac{f^*(x)}{\bar{s}^2} &= \psi^*(x,\bar{s}) \quad \text{for } x \in \Gamma. \end{aligned}$$

Multiplying the above equation by $-\bar{s}^2$ we obtain the following *approximate* Dirichlet boundary value problem for the functions $p^*, f^* \in C^{2+\alpha}$

$$\Delta f^* + (\nabla f^*)^2 - \bar{s}^2 (\nabla p^*)^2 = -\bar{s}^2 \mu\varepsilon \qquad \text{in } \Omega, \tag{40}$$

$$f^*(x) = -\bar{s}^2 \psi^*(x,\bar{s}) \qquad \text{for } x \in \Gamma. \tag{41}$$

The function $p^*(x)$ in (40) can be determined by taking only the first term in the asymptotic behavior in (38) assuming that

$$\begin{aligned} V^*(x,\bar{s}) &= p^*(x), \\ q^*(x,\bar{s}) &= 0. \end{aligned} \tag{42}$$

Then substituting (42) in (39) we get the following equation for the function $p^*(x)$:

$$\begin{aligned} (\nabla p^*)^2 &= \mu\varepsilon \quad \text{in } \Omega, \\ p^* &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{43}$$

where the boundary condition is obtained from the asymptotics for the function $q^*(x,\bar{s}) = 0$.

3.4 New Approximate Mathematical Model

In this subsection we will present the new approximate mathematical model for solution of our **CIP** using a new representation of the tail function $V(x,\bar{s})$. Let conditions (13) and (14) hold. Then there exists functions $p^*(x), f^*(x) \in C^{2+\alpha}\left(\overline{\Omega}\right)$ such that the exact tail function $V^*(x)$ has the form

$$V^*(x,s) := p^*(x) + \frac{f^*(x)}{s} \tag{44}$$

for $s \geq \bar{s}$. Here we used assumption that

$$V^*(x,\bar{s}) = p^*(x) + \frac{f^*(x)}{\bar{s}} = \frac{\ln(\mathbf{E}^*(x,\bar{s}))}{\bar{s}^2}. \tag{45}$$

Using definition $q^*(x,s) = \partial_s V^*(x,s)$ for $s \geq \bar{s}$, we get from (44)

$$q^*(x,\bar{s}) = -\frac{f^*(x)}{\bar{s}^2}. \tag{46}$$

Then we can obtain the following explicit formula for reconstruction of the coefficient $\sigma^*(x)$

$$\sigma^*(x) = \frac{1}{\mu}(\Delta v^*(x,s) + s(\nabla v^*(x,s))^2) - \varepsilon s, \tag{47}$$

where

$$v^* = -\int_s^{\bar{s}} q^*(x,\tau)\,d\tau + p^*(x) + \frac{f^*(x)}{\bar{s}}.$$

Using the new mathematical model above we can obtain the first guess for the tail function $V(x,\bar{s})$ in (26) as

$$V_{0,0}(x) := p(x) + \frac{f(x)}{\bar{s}}. \tag{48}$$

Here, the function $p(x)$ is determined by solution of the problem (43), and the function $f(x)$ is the solution of the problem (40), (41) with the computed function $p(x)$.

3.5 The algorithm

We are now ready to present an approximately globally convergent algorithm for **CIP**.

Step 0. Construct the initial approximation $V_{0,0}$ of the tail function $V(x,\bar{s})$. This can be done by first solving (15) with $\sigma \equiv 1$, or applying (48) using the new mathematical model of section 3.4. Set $q_0 \equiv 0$, and set counters n and i to 1, and i_0 and m to 0.

Step 1. Calculate an approximation $q_{n,i}^m$ of q_n from (30), (34) with $V = V_{n,i-1}$ if $i > 1$ or $V = V_{n-1,i_{n-1}}$ if $i = 1$, and $(\nabla q_n)^2 = (\nabla q_{n,i}^{m-1})^2$ if $m > 0$ or $(\nabla q_n)^2 = 0$ if $m = 0$.

Step 2. If $m = 0$, set $m = 1$ and return to Step 1. Otherwise, calculate

$$d_{n,i}^m = \frac{\|q_{n,i}^m - q_{n,i}^{m-1}\|_{L^2(\Omega)}}{\|q_{n,i}^{m-1}\|_{L^2(\Omega)}}.$$

If either $d_{n,i}^m < \eta_1$ for some predefined tolerance η_1, or $d_{n,i}^m > d_{n,i}^{m-1}$, set $q_{n,i} = q_{n,i}^m$ and $m = 0$, then proceed to Step 3. Otherwise, increase m by 1 and return to Step 1.

Step 3. Calculate $v_{n,i} = -hq_{n,i} - h\sum_{j=0}^{n-1} q_j$, then $\sigma_{n,i}$ using (21) with $v = v_{n,i}$ and $s = s_n$, and extend $\sigma_{n,i}$ to all of $\mathbb{R}^n$ so that $\sigma_{n,i} \in C_{d,\Omega}$. Compute $E_{n,i}$ by solving (15) with $\sigma = \sigma_{n,i}$, then $\mathbf{E}_{n,i}$ by applying the Laplace transform to $E_{n,i}$ for $s = \bar{s}$. Update the approximation of the tail function V by setting

$$V_{n,i} = \frac{\ln(\mathbf{E}_{n,i})}{\bar{s}}.$$

Step 4. If $i = 1$ set $i = 2$ and return to Step 1. Otherwise, calculate

$$e_{n,i} = \frac{\|\sigma_{n,i} - \sigma_{n,i-1}\|_{L^2(\Omega)}}{\|\sigma_{n,i-1}\|_{L^2(\Omega)}}.$$

If either $e_{n,i} < \eta_2$ for some predefined tolerance η_2, or $e_{n,i} > e_{n,i-1}$, set $q_n = q_{n,i}$, $V_{n+1,0} = V_{n,i}$, $\sigma_n = \sigma_{n,i}$, $i_n = i$, then set $i = 0$ and proceed to Step 5. Otherwise, increase i by 1 and return to Step 1.

Step 5. If $n = 1$, return to Step 1. Otherwise, compute

$$f_n = \frac{\|\sigma_n - \sigma_{n-1}\|_{L^2(\Omega)}}{\|\sigma_{n-1}\|_{L^2(\Omega)}}.$$

If either $f_n < \eta_3$ for some predefined tolerance η_3, $f_n > f_{n-1}$, or $n = N - 1$ we accept $\sigma = \sigma_n$ as an approximate solution of **CIP** and stop the calculations. Otherwise, we increase n by 1 and return to Step 1.

Acknowledgements This research was supported by the Swedish Research Council, the Swedish Foundation for Strategic Research (SSF) through the Gothenburg Mathematical Modelling Centre (GMMC), and the Swedish Institute, Visby Program.

References

1. L. Beilina and M. V. Klibanov, *Approximate global convergence and adaptivity for Coefficient Inverse Problems*, Springer, New York, 2012.
2. L. Beilina and M. V. Klibanov, A globally convergent numerical method for a coefficient inverse problem, *SIAM J. Sci. Comp.* **31** (2008), 478-509.
3. L. Beilina and M.V. Klibanov, Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive inverse algorithm, *Inverse Problems*, 26, 125009, 2010.
4. V. P. Gubatenko, *On the formulation of inverse problem in electrical prospecting*, Springer Conference Proceedings in Mathematics, to appear.
5. P. Hammond and J. K. Sykulski, *Engineering electromagnetism : physical processes and computation*, Oxford University Press, Oxford, 1994.
6. M.V. Klibanov and A. Timonov, *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*, VSP, Utrecht, 2004.
7. M. V. Klibanov and A. Timonov, A unified framework for constructing the globally convergent algorithms for multidimensional coefficient inverse problems, *Applicable Analysis*, 83, 933-955, 2004.
8. M. V. Klibanov, M. A. Fiddy, L. Beilina, N. Pantong and J. Schenk, Picosecond scale experimental verification of a globally convergent numerical method for a coefficient inverse problem, *Inverse Problems*, 26, 045003, 2010.
9. V. G. Romanov, *Inverse Problems of Mathematical Physics*, VNU, Utrecht, 1986.
10. J. Xin and M. V. Klibanov, Numerical solution of an inverse problem of imaging of antipersonnel land mines by the globally convergent convexification algorithm, *SIAM J. Sci. Comp.*, 30, 3170-3196, 2008.