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Local operator multipliers and positivity

N.M. Steen [a], I.G. Todorov [a,*], L. Turowska [b]

[a] *Pure Mathematics Research Centre, Queen's University Belfast, Belfast BT7 1NN, United Kingdom*
[b] *Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Sweden*

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ABSTRACT

We establish an unbounded version of Stinespring's Theorem and a lifting result for Stinespring representations of completely positive modular maps defined on the space of all compact operators. We apply these results to study positivity for Schur multipliers. We characterise positive local Schur multipliers, and provide a description of positive local Schur multipliers of Toeplitz type. We introduce local operator multipliers as a non-commutative analogue of local Schur multipliers, and characterise them extending both the characterisation of operator multipliers from [16] and that of local Schur multipliers from [27]. We provide a description of the positive local operator multipliers in terms of approximation by elements of canonical positive cones.

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1. Introduction

A bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is called a Schur multiplier if $(\varphi(i,j)a_{i,j})$ is the matrix of a bounded linear operator on ℓ^2 whenever $(a_{i,j})$ is such. The study of Schur multipliers was initiated by I. Schur in the early 20th century, and a characterisation of

* Corresponding author.
E-mail addresses: nsteen01@qub.ac.uk (N.M. Steen), i.todorov@qub.ac.uk (I.G. Todorov), turowska@chalmers.se (L. Turowska).

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these objects was given by A. Grothendieck in his *Résumé* [10] (see also [23]). A measurable version of Schur multipliers was developed by M.S. Birman and M.Z. Solomyak (see [3] and the references therein) and V.V. Peller [22]. More concretely, given standard measure spaces (X,μ) and (Y,ν) and a function $\varphi : X \times Y \to \mathbb{C}$, one defines a linear transformation S_φ on the space of all Hilbert–Schmidt operators from $H_1 = L^2(X,\mu)$ to $H_2 = L^2(Y,\nu)$ by multiplying their integral kernels by φ; if S_φ is bounded in the operator norm (in which case φ is called a *measurable Schur multiplier*), one extends it to the space $\mathcal{K}(H_1,H_2)$ of all compact operators from H_1 into H_2 by continuity. The map S_φ is defined on the space $\mathcal{B}(H_1,H_2)$ of all bounded linear operators from H_1 into H_2 by taking the second dual of the constructed map on $\mathcal{K}(H_1,H_2)$. A characterisation of measurable Schur multipliers, extending Grothendieck's result, was obtained in [22] (see also [18] and [29]). Namely, a function $\varphi \in L^\infty(X \times Y)$ was shown to be a Schur multiplier if and only if φ coincides almost everywhere with a function of the form $\sum_{k=1}^\infty a_k(x)b_k(y)$, where $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are families of essentially bounded measurable functions such that $\operatorname{esssup}_{x\in X} \sum_{k=1}^\infty |a_k(x)|^2 < \infty$ and $\operatorname{esssup}_{y\in Y} \sum_{k=1}^\infty |b_k(y)|^2 < \infty$.

A local version of Schur multipliers was defined and studied in [27]. Local Schur multipliers are, in general, unbounded, but necessarily closable, densely defined linear transformations on $\mathcal{B}(L^2(X,\mu), L^2(Y,\nu))$. A measurable function $\varphi : X \times Y \to \mathbb{C}$ was shown in [27] to be a local Schur multiplier if and only if it agrees almost everywhere with a function of the form $\sum_{k=1}^\infty a_k(x)b_k(y)$, where $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are families of measurable functions such that $\sum_{k=1}^\infty |a_k(x)|^2 < \infty$ for almost all $x \in X$ and $\sum_{k=1}^\infty |b_k(y)|^2 < \infty$ for almost all $y \in Y$.

In [19], a quantised version of Schur multipliers, called universal operator multipliers, was introduced. Universal operator multipliers are defined as elements of C*-algebras satisfying certain boundedness conditions, and hence are non-commutative versions of continuous Schur multipliers. A characterisation of universal operator multipliers, generalising Grothendieck–Peller's results, was obtained in [15].

In the present paper, we introduce and study local operator multipliers. Due to their spatial nature, the natural setting here is that of von Neumann algebras. Pursuing the analogue with the commutative setting, where local multipliers are measurable (not necessarily bounded) functions of two variables, we define local operator multipliers as operators affiliated with the tensor product of two von Neumann algebras. We characterise local operator multipliers, extending both the description of local Schur multipliers from [27] and the description of universal operator multipliers from [15]. We further characterise the positive local Schur multipliers (Section 4), as well as the positive local operator multipliers (Section 6). We describe positive local Schur multipliers of Toeplitz type, and consider local Schur multipliers that are divided differences, that is, functions of the form $\frac{f(x)-f(y)}{x-y}$. We show that such a divided difference is a positive local Schur multiplier with respect to every standard Borel measure if and only if f is an operator monotone function.

Our main tool for characterising positivity of multipliers is an unbounded version of Stinespring's Theorem (Section 2). In the literature, there are a number of versions of

Stinespring's Theorem for completely positive, not necessarily bounded, maps, defined on $*$-algebras, see e.g. [13,25,26]. Our version differs from the existing ones in that the domain is a non-unital pre-C*-algebra, and a partial boundedness of the map is assumed – as a result, we are able to obtain more specific conclusions regarding the (closable) operator implementing the Stinespring representation. In Section 3, we prove a lifting result for Stinespring representations of completely positive maps defined on the space of compact operators (Theorem 3.4). The result, which we believe is interesting in its own right, is used in Section 4 to obtain a lifting result for positive Schur multipliers, and provides an alternative approach to the unbounded Stinespring Theorem from Section 2.

All Hilbert spaces appearing in the paper will be assumed to be separable. The inner product of a Hilbert space H is denoted by $(\cdot,\cdot)_H$, if H needs to be emphasised. We let I_H denote the identity operator acting on H, and write I when H is clear from the context. For Hilbert spaces H and K, we denote by $\mathcal{B}(H,K)$ (resp. $\mathcal{K}(H,K)$, $\mathcal{C}_2(H,K)$) the space of all bounded linear (resp. compact, Hilbert–Schmidt) operators from H into K, and let $\mathcal{B}(H)=\mathcal{B}(H,H)$, $\mathcal{K}(H)=\mathcal{K}(H,H)$ and $\mathcal{C}_2(H)=\mathcal{C}_2(H,H)$. The operator norm is denoted by $\|\cdot\|_{\mathrm{op}}$. We often use the weak* topology of $\mathcal{B}(H,K)$, which arises from the identification of this space with the dual of the space of all nuclear operators from K into H. The weak* continuous linear maps on $\mathcal{B}(H,K)$ will be referred to as normal maps. If α is a cardinal number, we let H^α denote the direct sum of α copies of H, and for $x\in\mathcal{B}(H)$, we let $x\otimes 1_\alpha$ be the ampliation of x acting on H^α. We let ℓ^2_α be the Hilbert space of square summable sequences of length α.

Throughout the paper, we will use notions and results from Operator Space Theory; we refer the reader to the monographs [4,8,21,24]. If $\mathcal{A}\subseteq\mathcal{B}(H)$ is a C*-algebra, we denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$, and by $M_{n,m}(\mathcal{A})$ the set of all $n\times m$ matrices with entries in $\mathcal{A}$ which define bounded operators (here n or m may be ∞). For a matrix $a\in M_{n,m}(\mathcal{A})$, we denote by a^t its transposed matrix. If $\mathcal{M}\subseteq\mathcal{B}(H)$ is a von Neumann algebra, we will denote by $\operatorname{Aff}\mathcal{M}$ the set of all densely defined operators on H that are affiliated with $\mathcal{M}$; thus, $T\in\operatorname{Aff}\mathcal{M}$ if and only if the spectral measure of the operator $|T|=(T^*T)^{1/2}$ takes values in $\mathcal{M}$ and the partial isometry in the polar decomposition of T belongs to $\mathcal{M}$. The domain of an (unbounded) operator T will be denoted by $\operatorname{dom}(T)$.

If $\mathcal{A}$ and $\mathcal{B}$ are linear spaces, we will denote by $\mathcal{A}\odot\mathcal{B}$ their algebraic tensor product; if $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, their weak* spatial tensor product will be denoted by $\mathcal{A}\bar{\otimes}\mathcal{B}$. The linear span of a subset $\mathcal{X}$ of a vector space will be denoted by $[\mathcal{X}]$.

2. An unbounded version of Stinespring's Theorem

The classical Stinespring Representation Theorem for completely positive maps states that if $\mathcal{A}$ is a unital C*-algebra and $\Phi:\mathcal{A}\to\mathcal{B}(H)$ is a completely positive map, then there exists a Hilbert space K, a unital $*$-homomorphism $\pi:\mathcal{A}\to\mathcal{B}(K)$ and a bounded operator $V:H\to K$ with $\|\Phi(1)\|=\|V\|^2$ such that $\Phi(a)=V^*\pi(a)V$, $a\in\mathcal{A}$. In the case where $\overline{[\pi(\mathcal{A})VH]}=K$, we say that (π,V,K) is a *minimal Stinespring representation* for Φ (see [21]). Our aim in this section is to prove an unbounded version of Stinespring's

Theorem for maps defined on pre-C*-algebras, and apply it in the special case where the C*-completion of the domain coincides with the C*-algebra of all compact operators acting on a Hilbert space.

Let $\mathcal{A}$ be a $*$-algebra, $\mathcal{X}$ be a linear (not necessarily closed) subspace of a Hilbert space H and $\mathcal{L}(\mathcal{X})$ be the space of all linear mappings on $\mathcal{X}$. A linear mapping $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{X})$ will be called *completely positive* if

$$\sum_{k,l=1}^{n} \big(\Phi\big(a_k^* a_l\big)\xi_l, \xi_k\big) \geq 0$$

for arbitrary $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathcal{A}$ and $\xi_1, \ldots, \xi_n \in \mathcal{X}$.

We denote by $M(\mathcal{B})$ the multiplier algebra of a C*-algebra $\mathcal{B}$. We recall that, if $\mathcal{B}$ is identified with a subalgebra of its enveloping von Neumann algebra (that is, its second dual) $\mathcal{B}^{**}$, then $M(\mathcal{B}) \cong \{x \in \mathcal{B}^{**} : x\mathcal{B} \subseteq \mathcal{B}, \mathcal{B}x \subseteq \mathcal{B}\}$.

Theorem 2.1. *Let $\mathcal{B}$ be a C*-algebra and $\mathcal{A} \subseteq \mathcal{B}$ be a dense $*$-subalgebra of the form $\mathcal{A} = \bigcup_{k=1}^{\infty} p_k \mathcal{A} p_k$, where $(p_k)_{k\in\mathbb{N}} \subseteq M(\mathcal{B})$ is an increasing sequence of projections with $p_n\mathcal{A} \subseteq \mathcal{A}$ and $\mathcal{A}p_n \subseteq \mathcal{A}$, $n \in \mathbb{N}$. Let H be a Hilbert space, $(q_k)_{k\in\mathbb{N}}$ be an increasing sequence of projections on H with strong limit I, and $\mathcal{X} = \bigcup_{k=1}^{\infty} q_k H$. Assume that $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{X})$ is a completely positive map such that $\Phi(p_k a p_m) = q_k \Phi(a) q_m$, $k, m \in \mathbb{N}$, $a \in \mathcal{A}$. The following are equivalent:*

(i) the restriction $\Phi|_{p_k \mathcal{A} p_k}$ is bounded for each $k \in \mathbb{N}$;

(ii) there exist a Hilbert space K, a bounded $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(K)$ and an operator $V : \mathcal{X} \to K$, such that $V|_{q_k H}$ is bounded for every $k \in \mathbb{N}$, and

$$\big(\Phi(a)\xi, \eta\big) = \big(\pi(a)V\xi, V\eta\big), \quad a \in \mathcal{A},\ \xi, \eta \in \mathcal{X}.$$

Moreover, the operator V appearing in (ii) can be chosen to be closable.

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{A}_k = p_k \mathcal{A} p_k$, $H_k = q_k H$, and write Φ_k for the restriction of Φ to $\mathcal{A}_k$. Since Φ_k is bounded, it can be extended to a bounded map from $\mathcal{B}_k \stackrel{\text{def}}{=} \overline{\mathcal{A}_k}$ into $\mathcal{B}(H_k)$ (where the closure is taken in the norm topology of $\mathcal{B}$), which we also denote by Φ_k. Let $\Phi_k^{**} : \mathcal{B}_k^{**} \to \mathcal{B}(H_k)^{**}$ be the second dual of Φ_k and $\mathcal{E}_k : \mathcal{B}(H_k)^{**} \to \mathcal{B}(H_k)$ be the canonical projection (we point out that $\mathcal{E}_k$ is the dual of the inclusion of $\mathcal{B}(H_k)_*$ into $\mathcal{B}(H_k)^*$ and is hence weak* continuous). We note that if $k \leq m$ then $p_k \in M(\mathcal{B}_m) \subseteq \mathcal{B}_m^{**}$. Set $\widetilde{\Phi}_k = \mathcal{E}_k \circ \Phi_k^{**}$; thus, $\widetilde{\Phi}_k$ is a weak* continuous completely positive map from $\mathcal{B}_k^{**}$ into $\mathcal{B}(H_k)$ extending Φ_k. We note that if $k \leq m$ then $\widetilde{\Phi}_m|_{\mathcal{B}_k^{**}} = \widetilde{\Phi}_k$. By weak* continuity, $\widetilde{\Phi}_k(p_l x p_m) = q_l \widetilde{\Phi}_k(x) q_m$ whenever $l, m \leq k$ and $x \in \mathcal{B}_k^{**}$.

We now modify the well-known Stinespring construction. Let $\mathcal{L}$ be the linear space generated by $\mathcal{A} \otimes \mathcal{X}$ and all vectors of the form $1 \otimes \xi$, $\xi \in \mathcal{X}$. We define a sesquilinear form $\langle \cdot, \cdot \rangle_1$ on $\mathcal{L}$: if $\zeta = \sum_{i=1}^{s} a_i \otimes \xi_i$ and $\theta = \sum_{j=1}^{t} b_j \otimes \eta_j$, set

$$\langle \theta, \zeta \rangle_1 = \sum_{i=1}^{s} \sum_{j=1}^{t} \big(\widetilde{\Phi}_k\big(p_k a_i^* b_j p_k\big)\eta_j, \xi_i\big),$$

where k is such that $\xi_i,\ \eta_j \in H_k$, for all $i = 1, \ldots, s$ and all $j = 1, \ldots, t$. We note that $\langle \cdot, \cdot \rangle_1$ is well-defined; indeed, if ξ_i and $\eta_j \in H_m$ for some m, assuming that $m \leq k$, we have

$$\begin{aligned}
\big(\widetilde{\Phi}_k\big(p_k a_i^* b_j p_k\big)\eta_j, \xi_i\big) &= \big(q_m \widetilde{\Phi}_k\big(p_k a_i^* b_j p_k\big) q_m \eta_j, \xi_i\big) \\
&= \big(\widetilde{\Phi}_k\big(p_k p_m a_i^* b_j p_m p_k\big)\eta_j, \xi_i\big) \\
&= \big(\widetilde{\Phi}_k\big(p_m a_i^* b_j p_m\big)\eta_j, \xi_i\big) \\
&= \big(\widetilde{\Phi}_m\big(p_m a_i^* b_j p_m\big)\eta_j, \xi_i\big).
\end{aligned}$$

Since Φ is completely positive, $\langle \cdot, \cdot \rangle_1$ is a semidefinite inner product on $\mathcal{L}$.

Let $N := \{\zeta \in \mathcal{L} : \langle \zeta, \zeta \rangle_1 = 0\}$ and let

$$\rho(a)\left(\sum_{k=1}^{n} a_k \otimes \varphi_k\right) = \sum_{k=1}^{n} a a_k \otimes \varphi_k, \quad a \in \mathcal{A};$$

ρ is a well-defined $*$-homomorphism. A standard application of the Cauchy–Schwartz inequality shows that N is invariant under $\rho(\mathcal{A})$. Let $i : \mathcal{L} \to \mathcal{L}/N$ be the quotient map. Then $(i(\xi), i(\eta)) \stackrel{\text{def}}{=} \langle \xi, \eta \rangle_1$ defines a scalar product on $\mathcal{L}/N$ and $\pi(a)i(\eta) = i(\rho(a)\eta)$ is a well-defined $*$-representation of $\mathcal{A}$ on $\mathcal{L}/N$. Let $\mathcal{H}$ be the Hilbert space completion of $\mathcal{L}/N$.

We claim that π is bounded. In fact, for $\zeta = \sum_{i=1}^{n} a_i \otimes \xi_i \in \mathcal{L}$, there exists k such that $\xi_i \in H_k$ and if $a_i \neq 1$, $a_i \in \mathcal{A}_k$, for all $i = 1, \ldots, n$. Since $\widetilde{\Phi}_k$ is completely positive, we obtain, for $a \in \mathcal{A}_k$,

$$\begin{aligned}
\big\|\pi(a)i(\zeta)\big\|^2 = \big\langle \rho(a)\zeta, \rho(a)\zeta \big\rangle_1 &= \left\langle \sum_{i=1}^{n} a a_i \otimes \xi_i, \sum_{i=1}^{n} a a_i \otimes \xi_i \right\rangle_1 \\
&= \sum_{i,j=1}^{n} \big(\widetilde{\Phi}_k\big(p_k a_i^* a^* a a_j p_k\big)\xi_i, \xi_j\big) \\
&\leq \big\|a^* a\big\| \sum_{i,j=1}^{n} \big(\widetilde{\Phi}_k\big(p_k a_j^* a_i p_k\big)\xi_i, \xi_j\big) \\
&= \|a\|^2 \left\langle \sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{i=1}^{n} a_i \otimes \xi_i \right\rangle_1 = \|a\|^2 \big\|i(\zeta)\big\|^2,
\end{aligned}$$

giving the statement.

Define $V : \mathcal{X} \to \mathcal{H}$ by $V\xi = i(1 \otimes \xi)$. If $\xi, \eta \in H_k$ then

$$\|V\xi\|^2 = \big|\big(\widetilde{\Phi}_k(p_k)\xi, \xi\big)\big| \leq \|\Phi_k\| \|\xi\|^2,$$

showing that $V|_{H_k}$ is bounded for each k. Moreover,

$$\begin{aligned}\big(\Phi(a)\zeta,\theta\big) &= \big\langle \rho(a)(1\otimes\zeta), 1\otimes\theta\big\rangle_1 = \big(\pi(a)i(1\otimes\zeta), i(1\otimes\theta)\big)\\ &= \big(\pi(a)V\zeta, V\theta\big). \end{aligned} \tag{1}$$

We show that V is closable. If $\eta_n \to 0$ and $i(1\otimes\eta_n)\to f$ then, for each $a\otimes\xi\in\mathcal{L}$ with $a\in\mathcal{A}\cup\{1\}$, $\xi\in\bigcup_{k=1}^{\infty} H_k$, we have that

$$\big(i(a\otimes\xi), i(1\otimes\eta_n)\big)\to\big(i(a\otimes\xi), f\big).$$

On the other hand, if $\xi\in H_k$ and $\eta_n\in H_{l_n}$, $l_n\geq k$, then

$$\begin{aligned}\big(i(a\otimes\xi), i(1\otimes\eta_n)\big) &= \big(\widetilde{\Phi}_{l_n}(p_{l_n}ap_{l_n})\xi,\eta_n\big) = \big(\widetilde{\Phi}_{l_n}(p_kap_k)\xi,\eta_n\big)\\ &= \big(\widetilde{\Phi}_k(p_kap_k)q_k\xi, q_k\eta_n\big)\to_{n\to\infty} 0.\end{aligned}$$

It follows that $(i(a\otimes\xi), f) = 0$, for all $a\otimes\xi\in\mathcal{L}$. As $i(\mathcal{L})$ is dense in $\mathcal{H}$, we conclude that $f=0$.

(ii) $\Rightarrow$ (i) is trivial. □

Theorem 2.2. *Let H be a Hilbert space, $(p_n)_{n=1}^{\infty}$ be an increasing sequence of projections on H such that $\bigvee_{n=1}^{\infty} p_n = I$ and let $H_n = p_nH$, $n\in\mathbb{N}$. Let $\mathcal{A} := \bigcup_{n=1}^{\infty}\mathcal{C}_2(H_n)$ and $\mathcal{X} = \bigcup_{n=1}^{\infty} H_n$. Assume that $\Phi:\mathcal{A}\to\mathcal{L}(\mathcal{X})$ is a completely positive map such that $\Phi(p_nxp_m) = p_n\Phi(x)p_m$ for all $n,m\in\mathbb{N}$, and $\Phi_n = \Phi|_{\mathcal{C}_2(H_n)}$ is bounded with respect to the operator norm on $\mathcal{B}(H_n)$.*

(i) There exists a family $(V_i)_{i=1}^{\infty}$ of closable linear maps from $\mathcal{X}$ into H such that

$$\big(\Phi(a)\xi,\eta\big) = \sum_{i=1}^{\infty}(aV_i\xi, V_i\eta),\quad a\in\mathcal{A},\ \xi,\eta\in\mathcal{X}. \tag{2}$$

(ii) If $\mathcal{D}\subseteq\mathcal{B}(H)$ is a unital C^*-subalgebra, $(p_n)_{n=1}^{\infty}\subseteq\mathcal{D}$ and Φ is $\mathcal{D}$-bimodular, then V_i, $i\in\mathbb{N}$, can be chosen to be closable operators affiliated with $\mathcal{D}'$.

Proof. In the notation of Theorem 2.1, $\mathcal{B} = \mathcal{K}(H)$, and hence the representation π arising in Theorem 2.1 is unitarily equivalent to an ampliation of the identity representation. At the expense of changing the operator V, we may thus assume that $\pi(a) = a\otimes 1$, $a\in\bigcup_{k=1}^{\infty}\mathcal{C}_2(H_k)$. Let

$$V = (V_1,\ldots,V_n,\ldots)^t$$

be the corresponding matrix of V; identity (2) follows now trivially from Theorem 2.1. The fact that the operators V_i are closable follows easily from the closability of V.

We show that we can choose V_i to be affiliated with $\mathcal{D}'$ when Φ satisfies condition (ii). The arguments that follow are similar to the ones in [7, Corollary 5.9]. The modularity of Φ gives

$$\big((a\otimes 1)V\xi,(r\otimes 1)V\eta\big)=\big((a\otimes 1)V\xi,Vr\eta\big)\quad\text{for all } r\in\mathcal{D},\ a\in\mathcal{A},\ \xi,\eta\in\mathcal{X}.$$

Let e_a be the projection onto $\overline{(a\otimes 1)V\mathcal{X}}$, $a\in\mathcal{A}$. Then $e_a(r\otimes 1)V\eta=e_aVr\eta$, $r\in\mathcal{D}$, $\eta\in\mathcal{X}$. If $e=\bigvee_{a\in\mathcal{A}}e_a$ then

$$e(r\otimes 1)V\eta=eVr\eta,\quad r\in\mathcal{D},\ \eta\in\mathcal{X}.$$

As $(r\otimes 1)(a\otimes 1)V\eta=(ra\otimes 1)V\eta$ and $ra\in\mathcal{A}$ ($r\in\mathcal{D}$, $a\in\mathcal{A}$), we obtain that $e(r\otimes 1)e=(r\otimes 1)e$ for each $r\in\mathcal{D}$ and hence e commutes with $r\otimes 1$, $r\in\mathcal{D}$. This implies

$$(r\otimes 1)(eV)\eta=(eV)r\eta,\quad \eta\in\mathcal{X},\ r\in\mathcal{D},$$

and

$$\big(\Phi(a)\xi,\eta\big)=\big((a\otimes 1)eV\xi,eV\eta\big),\quad \xi,\eta\in\mathcal{X}.$$

The only thing that is left to prove is that if $eV=(V_1',\dots,V_n',\dots)^t$ then V_i' is closable and its closure is affiliated with $\mathcal{D}'$, for every i.

In order to show that V_i' is closable, it suffices to show that eV is closable. Suppose that $\xi_n\to 0$ and $eV\xi_n\to y$, for some $y\in H^\infty$. Then $(\Phi(a)\xi,\xi_n)\to 0$. On the other hand, given $a\in\mathcal{A}$, $\xi\in\mathcal{X}$,

$$\big(\Phi(a)\xi,\xi_n\big)=\big((a\otimes 1)V\xi,V\xi_n\big)=\big((a\otimes 1)V\xi,eV\xi_n\big)\to\big((a\otimes 1)V\xi,y\big).$$

Thus $((a\otimes 1)V\xi,y)=0$ and hence $(e\eta,y)=0$ for any $\eta\in H$, giving $ey=0$. As $eV\xi_n\to ey$, we have $ey=y=0$, showing that eV is closable.

We have $rV_i'\eta=V_i'r\eta$, for all $r\in\mathcal{D}$ and $\eta\in\mathcal{X}$. By the previous paragraph, the operator V_i' is closable as an operator defined on $\mathcal{X}$; let us denote its closure again by V_i'. Then clearly $r\,\mathrm{dom}(V_i')\subseteq\mathrm{dom}(V_i')$ and $rV_i'=V_i'r$. As V_i' is closed, the equality holds for each r in the strong closure of $\mathcal{D}$. Since $\bigvee_{n=1}^{\infty}p_n=I$ and $p_n\in\mathcal{D}$, the C*-algebra $\mathcal{D}$ is non-degenerate and hence $rV_i'=V_i'r$ for all $r\in\mathcal{D}''$. Thus, V_i' is affiliated with $\mathcal{D}'$. □

3. Lifting for Stinespring's representations

Our aim in this section is to obtain a lifting for Stinespring's representations of maps defined on the algebra of compact operators. The result will be used in Section 4 to obtain a lifting for positive Schur multipliers, but we believe that it is interesting in its own right as well. It also provides an independent route to Theorem 2.2.

Let H be a Hilbert space and $\Phi : \mathcal{K}(H) \to \mathcal{B}(H)$ be a completely positive map. Then there exists a Stinespring representation

$$\Phi(x) = V^*(x \otimes 1_\alpha)V = \sum_{i=1}^{\alpha} a_i^* x a_i, \quad x \in \mathcal{K}(H), \tag{3}$$

where $1 \leq \alpha \leq \infty$ and $V = (a_1, a_2, \ldots)^t \in M_{\alpha,1}(\mathcal{B}(H))$. We will say in this case that V *implements* the representation (3). If Φ is moreover modular over a C*-algebra $\mathcal{A} \subseteq \mathcal{B}(H)$, then the entries of V can be chosen from $\mathcal{D}'$.

We start by characterising the minimal representations of the map Φ in terms of the operator V (Lemma 3.1). Note that given any element $(\lambda_i)_{i=1}^{\alpha} \in \ell^2_\alpha$, the series $\sum_{i=1}^{\alpha} \lambda_i a_i$ is norm convergent in $\mathcal{B}(H)$. Following S.D. Allen, A.M. Sinclair and R.R. Smith [1], we say that the set $\{a_i\}_{i=1}^{\alpha}$ is *strongly independent* if $\sum_{i=1}^{\alpha} \lambda_i a_i = 0$ implies that $\lambda_i = 0$, for all i. It was established in [1, Lemma 2.2] that, for the case $\alpha = \infty$, the family $\{a_i\}_{i=1}^{\alpha}$ is strongly independent if and only if the set $\mathcal{K} = \{(\omega(a_1), \omega(a_2), \ldots) : \omega \in \mathcal{B}(H)^*\}$ is norm dense in ℓ^2_α.

Lemma 3.1. *Let $\Phi : \mathcal{K}(H) \to \mathcal{B}(H)$ be a completely positive map and $V = (a_i)_{i=1}^{\alpha} \in M_{\alpha,1}(\mathcal{B}(H))$, where α is an at most countable cardinal, implement a representation of Φ. The following are equivalent:*

(i) The operator V implements a minimal representation of Φ;

(ii) The set

$$\mathcal{F} = \big\{\big(\omega(a_1), \omega(a_2), \ldots\big) : \omega \text{ is a vector functional on } \mathcal{B}(H)\big\}$$

has norm dense linear span in ℓ^2_α;

(iii) The set $\mathcal{J} = \{(\omega(a_1), \omega(a_2), \ldots) : \omega \in \mathcal{B}(H)_*\}$ is norm dense in ℓ^2_α;

(iv) The set $\{a_i\}_{i=1}^{\alpha}$ is strongly independent.

Proof. (i) $\Rightarrow$ (ii) Suppose that the linear span of $\mathcal{F}$ is not dense and let $0 \neq (\lambda_i)_{i=1}^{\alpha} \in \mathcal{F}^\perp$. Then

$$0 = \sum_{i=1}^{\alpha} \overline{\lambda_i}\big(a_i\xi, x^*\eta\big) = \sum_{i=1}^{\alpha} (x a_i \xi, \lambda_i \eta), \quad \xi, \eta \in H,\ x \in \mathcal{K}(H).$$

Thus, if η is a non-zero vector in H, then $(\lambda_i \eta)_{i=1}^{\alpha} \in H^\alpha$ is non-zero and orthogonal to $(x a_i \xi)_{i=1}^{\alpha}$ for all ξ in H and all x in $\mathcal{K}(H)$. As a result, the representation implemented by V cannot be minimal.

(ii) $\Rightarrow$ (i) Suppose that the representation is not minimal. Then $\mathcal{E} = \overline{[(\mathcal{K}(H) \otimes 1_\alpha)VH]} \neq H^\alpha$. Let Q be the projection from H^α onto $\mathcal{E}$; since $\mathcal{E}$ is invariant for $\mathcal{K}(H) \otimes 1_\alpha$, we have that $Q \in (\mathcal{K}(H) \otimes 1_\alpha)'$ and hence $Q = I_H \otimes Q_0$, where Q_0 is a projection in $\mathcal{B}(\ell^2_\alpha)$. Thus we have that $\mathcal{E}^\perp = H \otimes (Q_0^\perp \ell^2_\alpha)$. Choose a non-zero $(\lambda_i)_{i=1}^{\alpha} \in Q_0^\perp \ell^2_\alpha$. Then for every $\eta \in H$ we have $\eta \otimes (\lambda_i)_{i=1}^{\alpha} \in \mathcal{E}^\perp$ and hence

$$0 = \sum_{i=1}^{\alpha} (x^* a_i \xi, \lambda_i \eta) = \sum_{i=1}^{\alpha} \overline{\lambda}_i (a_i \xi, x\eta) = \left(\sum_{i=1}^{\alpha} \overline{\lambda}_i a_i \xi, x\eta \right),$$

for all $x \in \mathcal{K}(H)$ and all $\xi \in H$. It follows that $\sum_{i=1}^{\alpha} \overline{\lambda}_i (a_i \xi, \eta) = 0$ for every $\eta \in H$. Hence $(\overline{\lambda_i})_{i=1}^{\alpha}$ is orthogonal to $((a_i \xi, \eta))_{i=1}^{\alpha}$, for all $\xi, \eta \in H$. It follows that the linear span of $\mathcal{F}$ is not dense in ℓ^2_α.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii) follows from the inclusion $\mathcal{J} \subseteq \overline{[\mathcal{F}]}^{\|\cdot\|}$ whose verification is straightforward.

(iii) $\Leftrightarrow$ (iv) The set $\mathcal{J}$ is not dense in ℓ^2_α if and only if there exists a non-zero element $(\lambda_i)_{i=1}^{\alpha} \in \ell^2_\alpha$ lying in the orthogonal complement of $\mathcal{J}$; that is, such that

$$\omega \left(\sum_{i=1}^{\alpha} \lambda_i a_i \right) = \sum_{i=1}^{\alpha} \lambda_i \omega(a_i) = 0, \quad \omega \in \mathcal{B}(H)_*.$$

This is equivalent to the existence of a non-zero $(\lambda_i)_{i=1}^{\alpha} \in \ell^2_\alpha$ such that $\sum_{i=1}^{\alpha} \lambda_i a_i = 0$, that is, to $\{a_i\}_{i=1}^{\alpha}$ not being strongly independent. □

Lemma 3.2. *Let H be a separable Hilbert space, $\Psi : \mathcal{K}(H) \to \mathcal{B}(H)$ be a completely positive map, and suppose that $A \in M_{\infty,1}(\mathcal{B}(H))$ implements a representation of Ψ. Then there exists an at most countable cardinal α and $\Lambda \in \mathcal{B}(\ell^2, \ell^2_\alpha)$, such that the operator $(I_H \otimes \Lambda)A$ implements a minimal representation of Ψ.*

Proof. Let $\mathcal{E} = \overline{(\mathcal{K}(H) \otimes 1)AH}$. As in the proof of Lemma 3.1, the projection Q from H^∞ onto $\mathcal{E}$ is of the form $I_H \otimes Q_0$, where Q_0 is a projection in $\mathcal{B}(\ell^2)$. Since $\mathcal{E}$ is reducing for $\mathcal{K}(H) \otimes 1$, the map $\rho : \mathcal{K}(H) \to \mathcal{B}(\mathcal{E})$ (resp. $\rho' : \mathcal{K}(H) \to \mathcal{B}(\mathcal{E}^\perp)$) given by $\rho(x) = x \otimes 1|_{\mathcal{E}}$ (resp. $\rho'(x) = x \otimes 1|_{\mathcal{E}^\perp}$) is a $*$-representation of $\mathcal{K}(H)$. Thus, there exists an (at most countable) cardinal α and a unitary operator $S : \mathcal{E} \to H^\alpha$ such that $\rho(x) = S^*(x \otimes 1_\alpha)S$. Consider the operator $T : H^\infty \to H^\alpha$ given by $T\xi = SQ\xi$, $\xi \in H^\infty$. Then

$$T(x \otimes 1)T^* = \begin{pmatrix} S & 0 \end{pmatrix} \begin{pmatrix} \rho(x) & 0 \\ 0 & \rho'(x) \end{pmatrix} \begin{pmatrix} S^* \\ 0 \end{pmatrix} = S\rho(x)S^* = x \otimes 1_\alpha.$$

In addition,

$$T^*T = \begin{pmatrix} S^* \\ 0 \end{pmatrix} \begin{pmatrix} S & 0 \end{pmatrix} = \begin{pmatrix} S^*S & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{\mathcal{E}} & 0 \\ 0 & 0 \end{pmatrix} = I_H \otimes Q_0.$$

Thus, $T(x \otimes 1)(I_H \otimes Q_0) = (x \otimes 1_\alpha)T$. Also,

$$T(x \otimes 1)\big(I_H \otimes Q_0^\perp\big) = \begin{pmatrix} S & 0 \end{pmatrix} \begin{pmatrix} \rho(x) & 0 \\ 0 & \rho'(x) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{E}^\perp} \end{pmatrix} = 0;$$

so

$$T(x \otimes 1) = T(x \otimes 1)(I_H \otimes Q_0) = (x \otimes 1_\alpha)T, \quad x \in \mathcal{K}(H).$$

It follows that $T = I_H \otimes \Lambda$, for some $\Lambda \in \mathcal{B}(\ell^2, \ell^2_\alpha)$. We thus have

$$\Psi(x) = A^*(x \otimes 1)A = A^*\rho(x)A = A^*S^*(x \otimes 1_\alpha)SA = A^*T^*(x \otimes 1_\alpha)TA.$$

Moreover,

$$\begin{aligned} H^\alpha = S\mathcal{E} &= \overline{S\big(\mathcal{K}(H) \otimes 1\big)AH} = \overline{S\big(\mathcal{K}(H) \otimes 1\big)|_{\mathcal{E}}AH} \\ &= \overline{\big(\mathcal{K}(H) \otimes 1_\alpha\big)SAH} = \overline{\big(\mathcal{K}(H) \otimes 1_\alpha\big)TAH} \end{aligned}$$

and thus the representation of Ψ implemented by TA is minimal. □

Remark 3.3. As part of the proof of Lemma 3.2, it was shown that $T^*T = I_H \otimes Q_0$. This fact will be used in the sequel.

The main result of this section is the following.

Theorem 3.4. *Let H_2 be a separable Hilbert space, $\mathcal{D}_2 \subseteq \mathcal{B}(H_2)$ be a unital* C*-*subalgebra, $H_1 \subseteq H_2$ be a closed subspace such that the projection p from H_2 onto H_1 belongs to $\mathcal{D}_2$, and $\mathcal{D}_1 = p\mathcal{D}_2p$. Let $\Phi : \mathcal{K}(H_2) \to \mathcal{B}(H_2)$ be a completely positive $\mathcal{D}_2$-bimodule map, and let $\Psi : \mathcal{K}(H_1) \to \mathcal{B}(H_1)$ be the map given by $\Psi(x) = \Phi(x)|_{H_1}$, $x \in \mathcal{K}(H_1)$. Suppose that the operator $V \in M_{\beta,1}(\mathcal{D}_1')$ implements a minimal representation of Ψ. Then there exist an at most countable cardinal $\gamma \geq \beta$ and an operator $W \in M_{\gamma,1}(\mathcal{D}_2')$, which implements a minimal representation of Φ, such that $W|_{H_1} = V$.*

We note that throughout the statement and the proof of the theorem, we identify H_1^β with $H_1^\beta \oplus 0 \subseteq H_1^\gamma$; under this assumption, it will be shown that $W|_{H_1}$ has range in H_1^β and is equal to V.

Proof. Let $\Phi(x) = A^*(x \otimes 1)A$, $x \in \mathcal{K}(H_2)$, be any representation of Φ, where $A \in M_{\infty,1}(\mathcal{D}_2')$. Then $\Psi(x) = A^*(x \otimes 1)A|_{H_1}$, $x \in \mathcal{K}(H_1)$. By Lemma 3.2, there exists a minimal representation

$$\Psi(x) = A^*T^*(x \otimes 1_\alpha)TA|_{H_1}, \quad x \in \mathcal{K}(H_1),$$

where $T = I_{H_1} \otimes T_0 = (\lambda_{ij} I_{H_1})$ for some $T_0 = (\lambda_{ij}) \in \mathcal{B}(\ell^2, \ell^2_\alpha)$. Since the representation $\Psi(x) = V^*(x \otimes 1_\beta)V$ is also minimal, there exists [21] a unitary operator $U : H_1^\alpha \to H_1^\beta$ such that $UTA|_{H_1} = V$ and $U(x \otimes 1_\alpha) = (x \otimes 1_\beta)U$, $x \in \mathcal{K}(H_1)$. Hence $U = I_{H_1} \otimes U_0$, where U_0 is a unitary operator in $\mathcal{B}(\ell^2_\alpha, \ell^2_\beta)$, and thus $\alpha = \beta$. Let $\tilde{U} = I_{H_2} \otimes U_0$, $\tilde{T} = I_{H_2} \otimes T_0$ and $B = \tilde{U}\tilde{T}A \in \mathcal{B}(H_2, H_2^\alpha)$. Then

$$B|_{H_1} = \tilde{U}\tilde{T}A|_{H_1} = \tilde{U}\tilde{T}(p \otimes 1)A|_{H_1} = UTA|_{H_1} = V.$$

Let $\Phi' : \mathcal{K}(H_2) \to \mathcal{B}(H_2)$ be given by

$$\Phi'(x) = B^*(x \otimes 1_\alpha)B, \quad x \in \mathcal{K}(H_2), \tag{4}$$

and let $\mathcal{E} = \overline{(\mathcal{K}(H_1) \otimes 1)AH_1}$. As in Lemma 3.1, the projection Q from H_1^∞ onto $\mathcal{E}$ has the form $I_{H_1} \otimes Q_0$, where Q_0 is a projection in $\mathcal{B}(\ell_2)$. Since $Ap = (p \otimes 1)Ap$, it follows that

$$\big(I_{H_2} \otimes Q_0^\perp\big)Ap = \big(p \otimes Q_0^\perp\big)Ap = \big(I_{H_1} \otimes Q_0^\perp\big)Ap = 0,$$

where the last equality follows from the fact that $ApH_2 \subseteq \mathcal{E}$. We thus have

$$\big(I_{H_2} \otimes Q_0^\perp\big)A = \big(I_{H_2} \otimes Q_0^\perp\big)Ap^\perp. \tag{5}$$

Define $\Phi'' : \mathcal{K}(H_2) \to \mathcal{B}(H_2)$ by

$$\Phi''(x) = A^*\big(I_{H_2} \otimes Q_0^\perp\big)(x \otimes 1)\big(I_{H_2} \otimes Q_0^\perp\big)A, \quad x \in \mathcal{K}(H_2). \tag{6}$$

Then for every x in $\mathcal{K}(H_2)$, we have that

$$\begin{aligned}
&\Phi'(x) + \Phi''(x)\\
&\quad = A^*\tilde{T}^*\tilde{U}^*(x \otimes 1_\alpha)\tilde{U}\tilde{T}A + A^*\big(I_{H_2} \otimes Q_0^\perp\big)(x \otimes 1)\big(I_{H_2} \otimes Q_0^\perp\big)A\\
&\quad = A^*(x \otimes 1)\tilde{T}^*\tilde{U}^*\tilde{U}\tilde{T}A + A^*(x \otimes 1)\big(I_{H_2} \otimes Q_0^\perp\big)A\\
&\quad = A^*(x \otimes 1)(I_{H_2} \otimes Q_0)A + A^*(x \otimes 1)\big(I_{H_2} \otimes Q_0^\perp\big)A\\
&\quad = A^*(x \otimes 1)A = \Phi(x),
\end{aligned}$$

where the third equality follows from Remark 3.3 and the fact that $\tilde{U}$ is unitary.

Now let Φ''' be the restriction of Φ'' to $\mathcal{K}(H_2 \ominus H_1)$ so that, for $x \in \mathcal{K}(H_2 \ominus H_1)$, we have

$$\begin{aligned}
\Phi'''(x) &= A^*\big(I_{H_2} \otimes Q_0^\perp\big)(x \otimes 1)\big(I_{H_2} \otimes Q_0^\perp\big)A|_{H_2 \ominus H_1}\\
&= A^*\big(I_{H_2} \otimes Q_0^\perp\big)(x \otimes 1)\big(I_{H_2} \otimes Q_0^\perp\big)A.
\end{aligned}$$

Using Lemma 3.2, find a minimal representation

$$\Phi'''(x) = A^*\big(I_{H_2} \otimes Q_0^\perp\big)R^*(x \otimes 1_\delta)R\big(I_{H_2} \otimes Q_0^\perp\big)A,$$

where R is a bounded operator from $(H_2 \ominus H_1)^\infty$ into $(H_2 \ominus H_1)^\delta$ that can be expressed as $R = I_{H_2} \otimes R_0$ for some R_0 in $\mathcal{B}(\ell^2, \ell^2_\delta)$, δ being an at most countable cardinal. By construction, $R(x \otimes 1)R^* = x \otimes 1_\delta$. Letting $C = R(I_{H_2} \otimes Q_0^\perp)A$, Eq. (5) implies that

$$Cp^\perp = C \tag{7}$$

and hence we have that for each $x \in \mathcal{K}(H_2)$,

$$\begin{aligned} C^*(x\otimes 1_\delta)C &= p^\perp C^*(x\otimes 1_\delta)Cp^\perp = C^*\big(p^\perp xp^\perp\otimes 1_\delta\big)C \\ &= \Phi''\big(p^\perp xp^\perp\big) = A^*\big(I_{H_2}\otimes Q_0^\perp\big)\big(p^\perp xp^\perp\otimes 1\big)\big(I_{H_2}\otimes Q_0^\perp\big)A \\ &= p^\perp A^*\big(I_{H_2}\otimes Q_0^\perp\big)(x\otimes 1)\big(I_{H_2}\otimes Q_0^\perp\big)Ap^\perp \\ &\overset{\text{by (5)}}{=} A^*\big(I_{H_2}\otimes Q_0^\perp\big)(x\otimes 1)\big(I_{H_2}\otimes Q_0^\perp\big)A = \Phi''(x). \end{aligned}$$

The operators $B : H_2 \to H_2^\alpha$ and $C : H_2 \to (H_2 \ominus H_1)^\delta$ can be expressed as columns of length α and δ, respectively; say, $B = (b_1, b_2, \ldots)^t$ and $C = (c_1, c_2, \ldots)^t$, where $b_i, c_j \in \mathcal{D}_2'$ for all i, j. Let U be the column operator with entries B and C, that is,

$$U = \big(B^t, C^t\big)^t. \tag{8}$$

Suppose that $\sum_{i=1}^{\alpha}\lambda_i b_i + \sum_{j=1}^{\delta}\mu_j c_j = 0$ for some $(\lambda_i)_{i=1}^{\alpha} \in \ell^2_\alpha$, $(\mu_j)_{j=1}^{\delta} \in \ell^2_\delta$. Then $\sum_{i=1}^{\alpha}\lambda_i pb_i + \sum_{j=1}^{\delta}\mu_j pc_j = 0$. Since $Cp = 0$, we have that $\sum_{i=1}^{\alpha}\lambda_i pb_i = 0$ and since $B|_{H_1} = V$ implements a minimal representation, we have by Lemma 3.1 that the entries of Bp are strongly independent and hence $\lambda_i = 0$ for all i. Consequently, $\sum_{j=1}^{\delta}\mu_j c_j = 0$ and the minimality of the representation associated with C implies, again by Lemma 3.1, that $\mu_j = 0$ for all j.

Let

$$b_i' = \begin{cases} b_i, & 1 \le i \le \alpha, \\ 0, & i > \alpha, \end{cases} \qquad c_i' = \begin{cases} c_i, & 1 \le i \le \delta, \\ 0, & i > \delta, \end{cases}$$

and set $W = (b_1', c_1', b_2', c_2', \ldots)^t$ – note that in the case in which both cardinals are finite the sequence has finitely many non-zero terms. In the case where both α and δ are infinite, the series $\sum_{i=1}^{\infty} b_i^* x b_i + c_i^* x c_i$ is easily seen to converge weak* to $\sum_{i=1}^{\infty} b_i^* x b_i + \sum_{i=1}^{\infty} c_i^* x c_i$. It now follows that $\Phi(x) = W^*(x \otimes 1_{\alpha+\delta})W$, $x \in \mathcal{K}(H_2)$. By Lemma 3.1, the representation of Φ implemented by W is minimal. We note that

$$W|_{H_1} = \big(b_1', c_2', b_2', c_2', \ldots\big)^t|_{H_1} = (b_1, 0, b_2, 0, \ldots)|_{H_1} = B|_{H_1} = V,$$

where we used (7) to obtain the second equality and the third equality follows as a result of the identification made at the start of the proof. □

Next we show how Theorem 3.4 can be applied to obtain a result closely related to Theorem 2.2.

Theorem 3.5. *Let $\mathcal{D} \subseteq \mathcal{B}(H)$ be a unital C*-subalgebra, $(p_n)_{n=1}^{\infty} \subseteq \mathcal{D}$ be an increasing sequence of projections such that $\bigvee_{n\in\mathbb{N}} p_n = I_H$, $H_n = p_n H$ and $\mathcal{D}_n = p_n \mathcal{D} p_n$. Let $\Phi : \bigcup_{n=1}^{\infty}\mathcal{K}(H_n) \to \bigcup_{n=1}^{\infty}\mathcal{B}(H_n)$ be a map such that $\Phi|_{\mathcal{K}(H_n)}$ is completely positive and $\mathcal{D}_n$-bimodular. Then there exists a family $(a_i)_{i=1}^{\infty}$ of closed operators affiliated with $\mathcal{D}'$ such that $H_n \subseteq \operatorname{dom}(a_i)$, $n, i \in \mathbb{N}$ and*

$$\Phi(x) = \sum_{i=1}^{\infty} a_i^* x a_i, \quad x \in \bigcup_{n=1}^{\infty} \mathcal{K}(H_n),$$

where the series converges in the weak-topology.*

In the course of the proof we will encounter operators $V_n : H_n \to (H_n)^{\alpha_n}$ and $V_{n+1} : H_{n+1} \to (H_{n+1})^{\alpha_{n+1}}$ for cardinals $\alpha_n \leq \alpha_{n+1}$; we will identify throughout the proof $(H_n)^{\alpha_n}$ with the subspace $(H_n)^{\alpha_n} \oplus 0$ of $(H_n)^{\alpha_{n+1}}$.

Proof. Let $H_0 = \bigcup_{n=1}^{\infty} H_n$ and $\Phi_n : \mathcal{K}(H_n) \to \mathcal{B}(H_n)$ be the map given by $\Phi_n(x) = \Phi(x)|_{H_n}$ for $x \in \mathcal{K}(H_n)$. Using analogous notation and repeating the process used to obtain the operator U in identity (8), we can form an operator $U_n : H_n \to \bigoplus_{i=1}^{n}(H_n \ominus H_{i-1})^{\delta_i}$, where δ_i is at most countable, such that $\Phi_n(x) = U_n^*((x \otimes 1_{\delta_1}) \oplus (x \otimes 1_{\delta_2}) \oplus \cdots \oplus (x \otimes 1_{\delta_n}))U_n$, $x \in \mathcal{K}(H_n)$, and $U_n|_{H_m} = U_m$ for all $m \leq n$. By construction, U_n is equal to

$$\big(b_{1,1}^{(n)}, b_{1,2}^{(n)}, \cdots, b_{2,1}^{(n)}, b_{2,2}^{(n)}, \cdots, \cdots, b_{n,1}^{(n)}, b_{n,2}^{(n)}, \cdots\big)^t,$$

where $b_{r,s}^{(t)} \in \mathcal{D}' p_t$. Since $b_{r,s}^{(n)}|_{H_{n-1}} = b_{r,s}^{(n-1)}$, we can form a densely defined operator $b_{r,s}$ by letting $b_{r,s}|_{H_n} = b_{r,s}^{(n)}$. Standard arguments show that $b_{r,s}$ is closable and its closure is affiliated with $\mathcal{D}'$. Define an operator $V : H_0 \to (H_0)^{\infty}$ by

$$V = (a_1, a_2, \ldots)^t \stackrel{\text{def}}{=} (b_{1,1}, b_{1,2}, b_{2,1}, b_{1,3}, b_{2,2}, b_{3,1}, b_{1,4}, \ldots)^t,$$

where we have, if necessary, extended V to an infinite column operator. From this we can obtain an operator $V_n : H_n \to (H_n)^{\alpha_n}$ by observing that when restricted to H_n, all terms $b_{i,j}$ for which $i > n$ vanish, and each of the remaining α_n terms is such that $b_{i,j}|_{H_n} = b_{i,j}^n$. This also implies that $V_n|_{H_m} = V_m$ for all $m \leq n$ (in the sense described before the start of the proof). It can easily be seen that V_n is a bounded column operator and

$$V_n^*(x \otimes 1_{\alpha_n})V_n = U_n^*(x \otimes 1_{\alpha_n})U_n.$$

Finally, given $x \in \bigcup_{n=1}^{\infty} \mathcal{K}(H_n)$, fix n such that $x \in \mathcal{K}(H_n)$ and notice that

$$\Phi(x) = \Phi_n(x) = U_n^*(x \otimes 1_{\alpha_n})U_n = V_n^*(x \otimes 1_{\alpha_n})V_n = V^*(x \otimes 1)V = \sum_{i=1}^{\infty} a_i^* x a_i.$$

In particular, since V_n is bounded, this series indeed converges in the weak* topology. □

Remark 3.6. In the course of the proof of Theorem 3.5 it was shown that $b_{r,s}p_n = b_{r,s}^{(n)} \in \mathcal{D}' p_n$ for all $r, s \in \mathbb{N}$, $n \geq r$; thus, $a_i p_n \in \mathcal{D}' p_n$ for all $i, n \in \mathbb{N}$. It is easily observed that $H_n \subseteq \operatorname{dom} a_i^*$ and that $a_i^* p_n = (a_i p_n)^*$. This will be used in the sequel.

4. Positive local Schur multipliers

In this section we examine positive local Schur multipliers. The main results of the section are the characterisation contained in Theorem 4.4, the lifting established in Corollary 4.7 and the characterisation result of Theorem 4.11.

We begin by recalling some definitions, known results and notational conventions. We assume throughout the section that (X, μ) and (Y, ν) are standard (σ-finite) measure spaces, equip $X \times Y$ with the product measure and denote by $\mathfrak{B}(X)$ the linear space of all measurable functions on X. If $f \in L^\infty(X)$, we write M_f for the operator of multiplication by f acting on $L^2(X)$, and let $\mathcal{D} = \{M_f : f \in L^\infty(X)\}$. The characteristic function of a measurable subset $\alpha \subseteq X$ will be denoted by χ_α. For each $k \in L^2(X \times Y)$, we let $T_k \in \mathcal{C}_2(L^2(X), L^2(Y))$ be the operator given by

$$(T_k\xi)(y) = \int_X k(x,y)\xi(x)d\mu(x), \quad \xi \in L^2(X)$$

and $S_\varphi : \mathcal{C}_2(L^2(X), L^2(Y)) \to \mathcal{C}_2(L^2(X), L^2(Y))$ be the map given by $S_\varphi(T_k) = T_{\varphi k}$, $k \in L^2(X \times Y)$. Recall that the map φ is called a *(measurable) Schur multiplier* if S_φ is bounded in $\|\cdot\|_{\mathrm{op}}$.

We next recall some notions from [2] and [27]. Two subsets $E, F \subseteq X$ are called *equivalent* (written $E \sim F$) if their symmetric difference is a null set. A subset of $X \times Y$ is said to be a *rectangle* if it has the form $\alpha \times \beta$, where $\alpha \subseteq X$ and $\beta \subseteq Y$ are measurable. A subset $E \subseteq X \times Y$ is called *marginally null* if $E \subseteq (X_0 \times Y) \cup (X \times Y_0)$, where $\mu(X_0) = \nu(Y_0) = 0$. We call two subsets $E, F \subseteq X \times Y$ *marginally equivalent* (and write $E \simeq F$) if their symmetric difference is marginally null. A measurable function $\varphi : X \times Y \to \mathbb{C}$ is called *ω-continuous* if $\varphi^{-1}(U)$ is marginally equivalent to a countable union of rectangles, for every open subset $U \subseteq \mathbb{C}$. A countable family of rectangles is called a *covering family for* $X \times Y$ if its union is marginally equivalent to $X \times Y$. We say that a function $\varphi \in \mathfrak{B}(X \times Y)$ is a *local Schur multiplier* if there exists a covering family $\{\kappa_m\}_{m=1}^\infty$ of rectangles in $X \times Y$ such that $\varphi|_{\kappa_m}$ is a Schur multiplier, for all $m \in \mathbb{N}$.

Proposition 4.1. *For a function $\varphi \in \mathfrak{B}(X \times X)$, the following are equivalent:*

(i) φ is a local Schur multiplier;
(ii) there exists an increasing sequence $(X_n)_{n=1}^\infty$ of measurable subsets of X such that $X \setminus (\bigcup_{n=1}^\infty X_n)$ is null and $\varphi|_{X_n \times X_n}$ is a Schur multiplier for each n.

Proof. (i) ⇒ (ii) Suppose that φ is a local Schur multiplier and let $\{\kappa_m\}_{m\in\mathbb{N}}$ be a covering family for $X \times X$ such that $\varphi|_{\kappa_m}$ is a Schur multiplier. By [27, Lemma 3.4], there exists a pairwise disjoint family $\{Y_i\}_{i=1}^\infty \subseteq X$ such that $\bigcup_{i=1}^\infty Y_i$ is equivalent to X and each $Y_i \times Y_j$ is contained in a finite union of sets from $\{\kappa_m\}_{m=1}^\infty$. By [27, Lemma 2.4 (ii)], $\varphi|_{Y_i \times Y_j}$ is a Schur multiplier. Let $X_n = \bigcup_{i=1}^n Y_i$; then $(X_n)_{n\in\mathbb{N}}$ is an increasing sequence,

$X_n \times X_n = \bigcup_{i,j=1}^{n} Y_i \times Y_j$ and $\bigcup_{n=1}^{\infty} X_n \sim \bigcup_{i=1}^{\infty} Y_i \sim X$. Again by [27, Lemma 2.4 (ii)], $\varphi|_{X_n \times X_n}$ is a Schur multiplier.

(ii) $\Rightarrow$ (i) The collection $(X_n \times X_n)_{n=1}^{\infty}$ is a covering family and thus φ is a local Schur multiplier. □

We will denote by $\mathcal{C}_2(H)^+$ the cone of all positive operators in $\mathcal{C}_2(H)$. In view of Proposition 4.1, it is natural to introduce the following notions.

Definition 4.2. Let (X, μ) be a standard measure space and φ be a measurable function on $X \times X$. We say that φ is a

(i) *positive Schur multiplier* if the map S_φ is bounded in $\|\cdot\|_{\mathrm{op}}$ and leaves $\mathcal{C}_2(L^2(X))^+$ invariant;

(ii) *positive local Schur multiplier* if there exists an increasing sequence $(X_n)_{n=1}^{\infty}$ of measurable subsets of X such that $X \setminus (\bigcup_{n=1}^{\infty} X_n)$ is null and $\varphi|_{X_n \times X_n}$ is a positive Schur multiplier for every n.

It is immediate that every positive Schur multiplier is a Schur multiplier and that every positive local Schur multiplier is a local Schur multiplier.

R.R. Smith has established an automatic complete boundedness result for maps, modular over C*-algebras with a cyclic vector [28, Theorem 2.1]. We will need the following automatic complete positivity result; we omit its proof since it follows closely the ideas in Smith's proof.

Lemma 4.3. *Let H be a Hilbert space, $\mathcal{E} \subseteq \mathcal{B}(H)$ be an operator system, and $\mathcal{B} \subseteq \mathcal{B}(H)$ be a C*-algebra with a cyclic vector such that $\mathcal{B}\mathcal{E}\mathcal{B} \subseteq \mathcal{E}$. Then every positive $\mathcal{B}$-bimodule map $\Phi : \mathcal{E} \to \mathcal{B}(H)$ is completely positive.*

We can now formulate and prove one of the main results of this section.

Theorem 4.4. *A function $\varphi \in \mathfrak{B}(X \times X)$ is a positive local Schur multiplier if and only if there exists a measurable function $a : X \to \ell^2$ such that $\varphi(x_1, x_2) = (a(x_1), a(x_2))_{\ell^2}$ almost everywhere on $X \times X$.*

Proof. We let $H_0 = \bigcup_{n=1}^{\infty} L^2(X_n)$. Suppose that φ is a positive local Schur multiplier and let $(X_n)_{n=1}^{\infty}$ be the sequence of subsets of X from Definition 4.2(ii). We can, moreover, assume that $\mu(X_n) < \infty$. Recall that $\mathcal{D} = \{M_f : f \in L^\infty(X)\}$. The projection p_n from $L^2(X)$ onto $L^2(X_n)$, $n \in \mathbb{N}$, is given by $p_n = M_{\chi_{X_n}}$. We identify $\mathcal{C}_2(L^2(X_n))$ with a subspace of $\mathcal{C}_2(L^2(X_{n+1}))$ in the natural way. Let

$$S_\varphi : \bigcup_{n=1}^{\infty} \mathcal{C}_2\big(L^2(X_n)\big) \to \bigcup_{n=1}^{\infty} \mathcal{C}_2\big(L^2(X_n)\big)$$

be the map given by $S_\varphi(T_k) = T_{\chi_{X_n\times X_n}\varphi k}$, $k \in L^2(X_n \times X_n)$. We have that the restriction $S_\varphi|_{\mathcal{C}_2(L^2(X_n))}$ is positive, bounded and $\mathcal{D}_n$-bimodular. Hence S_φ satisfies the conditions of Theorem 2.2 (or those of Theorem 3.5), and thus there exists a linear operator $V : H_0 \to H_0^\infty$ of the form $V = (M_{a_1}, M_{a_2}, \ldots)^t$, where $a_i \in \mathfrak{B}(X)$, $i \in \mathbb{N}$, such that

$$S_\varphi(T_k) = V^*(T_k \otimes 1)V = \sum_{i=1}^{\infty} M_{a_i}^* T_k M_{a_i} \quad \text{for all } T_k \in \bigcup_{n=1}^{\infty} \mathcal{C}_2\big(L^2(X_n)\big).$$

Fix $n \in \mathbb{N}$. We have that $\operatorname{esssup}_{x\in X_n} \sum_{i=1}^\infty |a_i(x)|^2 = \|\sum_{i=1}^\infty M_{|a_i\chi_{X_n}|^2}\| = \|Vp_n\|^2$. It follows that $\sum_{i=1}^\infty |a_i(x)|^2 < \infty$ for almost all $x \in X$. Thus the function $a : X \to \ell^2$ given by $a(x) = (a_i(x))_{i=1}^\infty$, $x \in X$, is well-defined up to a null set.

Let $\psi = \sum_{i=1}^\infty a_i \otimes \overline{a_i}$. Then

$$T_{\varphi k} = S_\varphi(T_k) = \sum_{i=1}^{\infty} M_{a_i}^* T_k M_{a_i} = T_{\psi k}, \quad k \in L^2(X_n \times X_n),\ n \in \mathbb{N}.$$

This implies $\varphi = \psi$ almost everywhere on $X_n \times X_n$; as a consequence,

$$\varphi(x_1, x_2) = \sum_{i=1}^{\infty} a_i(x_1)\overline{a_i(x_2)} = \big(a(x_1), a(x_2)\big)_{\ell^2},$$

for almost all $(x_1, x_2) \in \bigcup_{n=1}^\infty (X_n \times X_n)$, and hence for almost all $(x_1, x_2) \in X \times X$.

Conversely, suppose that there exists a function $a : X \to \ell^2$, say $a(x) = (a_i(x))_{i\in\mathbb{N}}$, $x \in X$, such that $\varphi(x_1, x_2) = (a(x_1), a(x_2))_{\ell^2}$ almost everywhere on $X \times X$. Let $X_n = \{x \in X : \|a(x)\|_2^2 \le n\}$ and observe that $\bigcup_{i=1}^\infty X_n \sim X$. For $k \in L^2(X_n \times X_n)$, we have that $S_\varphi(T_k) = \sum_{i=1}^\infty M_{a_i\chi_{X_n}}^* T_k M_{a_i\chi_{X_n}}$ and hence $S_\varphi|_{\mathcal{C}_2(L^2(X_n))}$ is a bounded positive map. Consequently, φ is a positive local Schur multiplier. □

Remark. Let (X, μ) be a standard measure space and μ' be a measure defined on the same σ-algebra and absolutely continuous with respect to μ. If φ is a positive local Schur multiplier with respect to μ then it is so with respect to μ'. Indeed, this follows immediately from the representation given in Theorem 4.4.

Following the proof of Theorem 4.4 and using Stinespring's Theorem, we also note the following, rather well-known, description of positive Schur multipliers.

Corollary 4.5. *A function $\varphi \in L^\infty(X \times X)$ is a positive Schur multiplier if and only if there exists a measurable function $a : X \to \ell^2$ such that $\operatorname{esssup}_{x\in X} \|a(x)\| < \infty$ and $\varphi(x_1, x_2) = (a(x_1), a(x_2))_{\ell^2}$ almost everywhere on $X \times X$.*

Let $\varphi \in L^\infty(X \times X)$ be a positive Schur multiplier. Then there are potentially many functions $a : X \to \ell^2$ satisfying the conclusion of Theorem 4.5; we call them *representing*

functions for φ. For each such function, say, $a(x) = (a_i(x))_{i\in\mathbb{N}}$ $(x \in X)$, we have a corresponding Stinespring representation of the completely positive map S_φ:

$$S_\varphi(T) = \sum_{i=1}^{\infty} M_{a_i}^* T M_{a_i}, \quad T \in \mathcal{K}\big(L^2(X)\big). \tag{9}$$

Let us call a a *minimal representing function for* φ if the representation (9) of S_φ is minimal.

Proposition 4.6. *Let* $\varphi \in L^\infty(X \times X)$ *be a positive Schur multiplier and* $a : X \to \ell^2$ *be a representing function for* φ. *The following are equivalent:*

(i) a *is minimal;*

(ii) for each null set $M \subseteq X$, *the set* $\{a(x) : x \in X \setminus M\}$ *has dense linear span in* ℓ^2.

Proof. Suppose that a_i, $i \in \mathbb{N}$, are the coordinate functions of a. By Lemma 3.1, a is not a minimal representing function for φ if and only if $\{M_{a_i}\}_{i\in\mathbb{N}}$ is not strongly independent, if and only if there exists $0 \neq (\lambda_i)_{i\in\mathbb{N}} \in \ell^2$ such that $\sum_{i=1}^{\infty} \lambda_i M_{a_i} = 0$, if and only if there exists a null set $M \subseteq X$ such that $\sum_{i=1}^{\infty} \lambda_i a_i(x) = 0$ for all $x \in X \setminus M$, if and only if there exists a null set $M \subseteq X$ such that the linear span of $\{a(x) : x \in X \setminus M\}$ is not dense in ℓ^2. □

The following lifting result for positive Schur multipliers now follows from Theorem 3.4.

Corollary 4.7. *Let* $\varphi \in L^\infty(X \times X)$ *be a positive Schur multiplier, and let* $Y \subseteq X$ *be a measurable subset. Suppose that* $a : Y \to \ell^2$ *is a minimal representing function for* $\varphi|_{Y\times Y}$. *Then there exists a minimal representing function* $b : X \to \ell^2 \oplus \ell^2$ *for* φ *such that* $b(x) = a(x) \oplus 0$ *for all* $x \in Y$.

Let X be a set and $\varphi : X \times X \to \mathbb{C}$ be a function. We recall that φ is called *positive definite* if $(\varphi(x_i, x_j))_{i,j=1}^N$ is a positive matrix for all $x_1, x_2, \ldots, x_N \in X$ and all $N \in \mathbb{N}$. In the proof of the following proposition, we will use the following well-known fact: If X is a locally compact Hausdorff space, μ is a regular Borel measure on X, $K \subseteq X$ is compact and $k \in L^2(X \times X)$ is continuous and positive definite on $K \times K$ then the (Hilbert–Schmidt) operator T_k is positive on $L^2(K, \mu)$.

The motivation behind part (ii) of the next theorem is [19, Theorem 9.3], where a relation between ω-continuous measurable Schur multipliers and classical Schur multipliers is established.

Theorem 4.8. *Let* $\varphi \in \mathfrak{B}(X \times X)$.

(i) φ *is a positive local Schur multiplier if and only if* φ *is a local Schur multiplier and* φ *is equivalent to a positive definite function.*

(ii) Suppose that φ is ω-continuous. Then φ is a positive local Schur multiplier if and only if there exist a null set X_0 and an increasing sequence $\{Y_k\}$ of measurable subsets of X such that $X \setminus X_0 = \bigcup_{k=1}^{\infty} Y_k$ and $\varphi|_{Y_k \times Y_k}$ is a positive Schur multiplier with respect to the counting measure on Y_k, for every k.

Proof. (i) Suppose that φ is a positive local Schur multiplier. By Theorem 4.4, there exists a measurable function $a : X \to \ell^2$ such that $\varphi(x_1, x_2) = (a(x_1), a(x_2))_{\ell^2}$ almost everywhere on $X \times X$. On the other hand, a straightforward verification shows that the function $(x_1, x_2) \to (a(x_1), a(x_2))_{\ell^2}$ is positive definite.

Conversely, suppose that φ is a local Schur multiplier and a positive definite function. We may assume that X is a σ-compact metric space and μ is a regular Borel measure. Let $(X_n)_{n\in\mathbb{N}}$ be the increasing sequence of measurable sets arising from Proposition 4.1. Let $k = \sum_{i=1}^{m} f_i \otimes \overline{f_i}$ for some $f_1, \ldots, f_m \in L^2(X_n)$, so that the operator T_k is a positive and of finite rank. Fix $\epsilon > 0$. A successive application of Lusin's Theorem shows that there exists a compact subset $K_\epsilon \subseteq X_n$, whose complement in X_n has measure less than ϵ, such that $f_i|_{K_\epsilon}$ is continuous, for each $i = 1, \ldots, m$. On the other hand, by [27, Proposition 3.2], φ is ω-continuous and by Lusin's Theorem for ω-continuous functions [19, Theorem 8.3], there exists a compact set $L_\epsilon \subseteq X$, whose complement has measure less than ϵ, such that φ is continuous on $L_\epsilon \times L_\epsilon$. It follows that φk is positive definite on $(K_\epsilon \cap L_\epsilon) \times (K_\epsilon \cap L_\epsilon)$, and by the result stated before the statement of the theorem, $P_\epsilon T_{\varphi k} P_\epsilon \geq 0$, where P_ϵ is the projection of multiplication by the characteristic function of $K_\epsilon \cap L_\epsilon$. Letting ϵ tend to zero, we get that $T_{\varphi k} \geq 0$. Since $\varphi|_{X_n \times X_n}$ is a Schur multiplier, it follows that $\varphi|_{X_n \times X_n}$ is a positive Schur multiplier. Thus, φ is a positive local Schur multiplier.

(ii) Let $a : X \to \ell^2$ be the measurable function from Theorem 4.4 such that $\varphi(x, y) = (a(x), a(y))_{\ell^2}$ almost everywhere. Since both functions in the last equation are ω-continuous, we have [27] that they are equal marginally almost everywhere. Hence, there exists a null set Z_0 such that φ is positive definite on $(X \setminus Z_0) \times (X \setminus Z_0)$. Letting $Z_k = \{x \in X \setminus Z_0 : \|a(x)\|_2 \leq k\}$, by [19, Theorem 9.3] we can find null sets $Z_k^0 \subseteq Z_k$ such that $\varphi|_{(Z_k \setminus Z_k^0) \times (Z_k \setminus Z_k^0)}$ is a Schur multiplier with respect to the counting measure. Then, setting $Y_0 = \bigcup_{k=1}^{\infty} Z_k^0$, $Y_k = Z_k \setminus Y_0$ and $X_0 = Z_0 \cup Y_0$, the sequence $(Y_k)_{k\in\mathbb{N}}$ is increasing with union $X \setminus X_0$. Finally, by part (i) of the present theorem, $\varphi|_{Y_k \times Y_k}$ is a positive Schur multiplier when Y_k is equipped with the counting measure.

The converse follows from part (i) and [19, Theorem 9.3]. □

Example 4.9. Let $\varphi(x, y) = 1/(x + y)$, $x, y \in \mathbb{R}^+$, $(x, y) \neq (0, 0)$. Then $\varphi \in \mathfrak{B}(\mathbb{R}^+ \times \mathbb{R}^+, \lambda \times \lambda)$, where λ is the Lebesque measure. We have

$$\varphi(x, y) = \int_0^{+\infty} e^{-sx} e^{-sy} ds = \left(e^{-\cdot x}, e^{-\cdot y}\right)_{L^2(\mathbb{R}^+)}$$

and $\|e^{-sx}\|^2_{L^2(\mathbb{R}^+)} = 1/(2x)$. Expressing the function $e^{-\cdot x}$ in terms of an orthonormal basis of $L^2(\mathbb{R}^+)$, we can find a measurable function $a : \mathbb{R}^+ \to \ell^2$ such that $\varphi(x,y) = (a(x), a(y))_{\ell^2}$ almost everywhere and $\|a(x)\|_{\ell^2} < \infty$ for almost all $x \in \mathbb{R}^+$. Hence $\varphi(x,y)$ is a positive local Schur multiplier. It is not a Schur multiplier since $\varphi \notin L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$.

The previous example is taken from the theory of operator monotone functions. It is known that if a function f is continuously differentiable on an interval (a,b) then f is operator monotone on (a,b) if and only if the divided difference $\check{f}$, given by $\check{f}(x,y) = (f(x)-f(y))/(x-y)$, $x \neq y$ and $\check{f}(x,x) = f'(x)$, $x \in (a,b)$, is positive definite on $(a,b)\times(a,b)$ (see, for example, [12], where a proof of this fact is given using Schur multiplier techniques). Operator monotonicity is related to positivity of local Schur multipliers in the following way.

Proposition 4.10. *Let $f : (a,b) \to \mathbb{C}$ be a continuously differentiable function. The divided difference $\check{f}$ is a positive local Schur multiplier on $(a,b)\times(a,b)$ with respect to any choice of a standard Borel measure on (a,b) if and only if f is operator monotone.*

Proof. Suppose that $\check{f}$ is a positive local Schur multiplier on $(a,b)\times(a,b)$ with respect to every standard Borel measure. Let $F \subseteq (a,b)$ be a finite set and let μ_F be the measure given by $\mu_F(\alpha) = |\alpha \cap F|$ for a Borel set α. Our assumption implies that there exists a Borel set $Y \subseteq (a,b)$ with $F \subseteq Y$ such that $\check{f}|_{Y\times Y}$ is a positive Schur multiplier with respect to μ_F. It follows that $\check{f}|_{F\times F}$ is a positive Schur multiplier (with respect to μ_F), and hence $\check{f}|_{F\times F}$ is a positive matrix. Since this is true for all finite sets $F \subseteq (a,b)$, we have that $\check{f}$ is a positive definite function. By [12], f is operator monotone.

Conversely, suppose that f is operator monotone and let μ be a standard Borel measure on (a,b); by [12], $\check{f}$ is positive definite. Let $U_n = \{x \in (a,b) : f'(x) < n\}$; then $\bigcup_{n\in\mathbb{N}} U_n = (a,b)$. Let $n \in \mathbb{N}$ and $F \subseteq U_n$ be a finite subset. Since $\check{f}|_{F\times F}$ is a positive matrix, the norm of its corresponding Schur multiplication is bounded by $\max_{x\in F} f'(x)$, which does not exceed n. It follows that $\check{f}|_{U_n\times U_n}$ is a Schur multiplier with respect to the counting measure. By [19, Theorem 9.3], $\check{f}|_{U_n\times U_n}$ is a Schur multiplier with respect to μ. Hence, $\check{f}$ is a local Schur multiplier with respect to μ. Now Theorem 4.8 shows that $\check{f}$ is a positive local Schur multiplier with respect to μ. □

4.1. Positive multipliers of Toeplitz type

We conclude this section by considering positive multipliers of Toeplitz type. Let G be a locally compact group equipped with left Haar measure and N be the map sending a measurable function $f : G \to \mathbb{C}$ to the function $Nf : G \times G \to \mathbb{C}$, given by $Nf(s,t) = f(st^{-1})$; we call the functions of the form Nf *functions of Toeplitz type*. It was shown in [6] that if $f \in L^\infty(G)$ then Nf is a Schur multiplier if and only if f is equivalent to an element of $M^{\mathrm{cb}}A(G)$ (the latter being the set of all completely bounded multipliers of the Fourier algebra $A(G)$ of G). On the other hand, it was proved in [27] that if G is

abelian then Nf is a local Schur multiplier if and only if f is equivalent to a function that belongs locally to $A(G)$ at every point of the group G. In particular, examples were given of local Schur multipliers φ of Toeplitz type that are not Schur multipliers. The following proposition shows that this cannot happen with the additional assumption that φ is positive, provided G is amenable.

Theorem 4.11. *Let G be an amenable locally compact group, $f : G \to \mathbb{C}$ be a measurable function and $\varphi = Nf$. The following are equivalent:*

(i) φ is a positive Schur multiplier;

(ii) φ is a positive local Schur multiplier;

(iii) f is equivalent to a positive definite function from the Fourier–Stiltjes algebra $B(G)$.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) By [27, Corollary 4.5], φ is equivalent to an ω-continuous function. By [27, Proposition 7.3], f is equivalent to a continuous function $g : G \to \mathbb{C}$. We may thus assume that f is itself continuous.

Let $T(G) = L^2(G) \hat{\otimes} L^2(G)$, where by $\hat{\otimes}$ we denote the projective tensor product. The space $T(G)$ can be naturally identified with the trace class on $L^2(G)$. We let $T(G)^+$ be the cone in $T(G)$ corresponding to the positive trace class operators under this identification; we have that $T(G)^+$ consists of all elements of the form $\sum_{i=1}^{\infty} \xi_i \otimes \overline{\xi_i}$, with $\sum_{i=1}^{\infty} \|\xi_i\|_2^2 < \infty$. Let $P : T(G) \to A(G)$ be the contraction given by $P(\xi \otimes \eta)(s) = (\lambda_s \xi, \overline{\eta})$, $\xi, \eta \in L^2(G)$.

Since Nf is a positive local Schur multiplier, there exists an increasing sequence $(X_n)_{n\in\mathbb{N}}$ of measurable subsets of G such that the set $G \setminus (\bigcup_{n\in\mathbb{N}} X_n)$ is null and $Nf|_{X_n \times X_n}$ is a positive Schur multiplier. Clearly, $\bigcup_{n\in\mathbb{N}} L^2(X_n)$ is dense in $L^2(G)$. Since G is amenable, [20, Lemma 7.2] shows that there exists a net $(u_\alpha)_\alpha$, with $u_\alpha = P(\xi_\alpha \otimes \overline{\xi_\alpha})$, $\|\xi_\alpha\|_2 \leq 1$, which converges to the constant function 1 uniformly on compact subsets. Since P is contractive and the uniform norm is dominated by the norm of $A(G)$, we can replace u_α by a function of the form $v_\alpha = P(\eta_\alpha \otimes \overline{\eta_\alpha})$, with η_α having support in some X_n, $n \in \mathbb{N}$.

We have that $(Nf)(\eta_\alpha \otimes \overline{\eta_\alpha}) \in^{\mu\times\mu} T(G)^+$ for each α. Applying the mapping P, we obtain that $fv_\alpha \in A(G)^+$ for each α. Let $K = \{s_1, \ldots, s_n\} \subseteq G$. We have that

$$\big(f\big(s_i s_j^{-1}\big) v_\alpha\big(s_i s_j^{-1}\big)\big)_{i,j} \to_\alpha \big(f\big(s_i s_j^{-1}\big)\big)_{i,j}.$$

Since the matrix $(f(s_i s_j^{-1}) v_\alpha(s_i s_j^{-1}))_{i,j}$ is positive for each α, it follows that $(f(s_i s_j^{-1}))_{i,j}$ is positive as well. Thus, f is a positive definite function. Since f is continuous, we have that $f \in B(G)$ (see [9]).

(iii) $\Rightarrow$ (i) Since G is amenable, $B(G)$ coincides with the algebra of all completely bounded multipliers of $A(G)$. The fact that Nf is a Schur multiplier follows from [6] (see also [29]). The proof of Theorem 4.8 now shows that Nf is a positive Schur multiplier. □

Corollary 4.12. *(i) The space of all local Schur multipliers coincides with the linear span of the cone of all positive local Schur multipliers.*

(ii) The space of all local Schur multipliers of Toeplitz type is strictly larger than the linear span of the cone of all positive local Schur multipliers of Toeplitz type.

Proof. (i) follows from [27, Theorem 3.6], Theorem 4.4 and a standard polarisation argument.

(ii) follows from Theorem 4.11 and the fact that the space of local Schur multipliers of Toeplitz type is strictly larger than that of Schur multipliers of Toeplitz type (see [27, Remark 7.11]). □

It was shown in [14] that if a continuous function φ of Toeplitz type, defined on the direct product $G \times G$, where G is a locally compact group, is a Schur multiplier, then the functions $a, b : G \to \ell^2$ in the representation $\varphi(x, y) = (a(x), b(y))_{\ell^2}$, can be chosen to be continuous. It is thus natural to ask the following questions:

Question 4.13. Let X be a locally compact topological space equipped with a regular Borel measure. Suppose that $\varphi : X \times X \to \mathbb{C}$ is a continuous Schur multiplier.

(i) Do there exist continuous bounded functions $a, b : X \to \ell^2$ such that $\varphi(x, y) = (a(x), b(y))_{\ell^2}$ for all $x, y \in X$?

(ii) If φ is moreover positive, can one choose a continuous bounded function $a : X \to \ell^2$ such that $\varphi(x, y) = (a(x), a(y))_{\ell^2}$ for all $x, y \in X$?

(iii) Assuming that φ is a local (resp. positive local) Schur multiplier, can a similar choice be made with a and b (resp. a) not necessarily bounded?

5. Local operator multipliers

In this section we introduce local operator multipliers, a non-commutative version of local Schur multipliers, and characterise them, generalising the characterisation of local Schur multipliers given in [27]. The suitable setting for local operator multipliers is that of von Neumann algebras, as opposed to the setting of C*-algebras, which was used to define and study universal multipliers in [19] and [16]. We therefore start by collecting some notions and results from [16] in a form convenient for our purposes.

Let H and K be Hilbert spaces and let H^{d} be the dual Banach space of H; note that H^{d} is conjugate linear isometric to H via the map $\partial : H \to H^{\mathrm{d}}$ sending $x \in H$ to the element $x^{\mathrm{d}} \in H^{\mathrm{d}}$ given by $x^{\mathrm{d}}(y) = (y, x)$, $y \in H$. If $T \in \mathcal{B}(H, K)$, we let $T^{\mathrm{d}} \in \mathcal{B}(K^{\mathrm{d}}, H^{\mathrm{d}})$ be the dual operator of T. If $\mathcal{M} \subseteq \mathcal{B}(H)$ is a von Neumann algebra, we denote by $\mathcal{M}^o$ the opposite von Neumann algebra of $\mathcal{M}$; we have that $\mathcal{M}^o \subseteq \mathcal{B}(H^{\mathrm{d}})$ consists of the elements of the form a^{d}, where $a \in \mathcal{M}$. In particular, $\mathcal{B}(H)^o = \mathcal{B}(H^{\mathrm{d}})$. By $H \otimes K$ we denote the Hilbert space tensor product of H and K. If $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, we denote by $\mathcal{M} \bar{\otimes} \mathcal{N}$ the (spatial weak* closed) von Neumann algebra tensor product. Thus, $\mathcal{B}(H^{\mathrm{d}} \otimes K) = \mathcal{B}(H)^o \bar{\otimes} \mathcal{B}(K)$.

We let $\theta : H^{\mathrm{d}} \otimes K \to \mathcal{C}_2(H,K)$ be the canonical isomorphism sending an elementary tensor $x^{\mathrm{d}} \otimes y$ to the rank one operator given by $\theta(x^{\mathrm{d}} \otimes y)(z) = (z,x)y$, $z \in H$. This allows us to equip $H^{\mathrm{d}} \otimes K$ with an "operator" norm:

$$\|\xi\|_{\mathrm{op}} \stackrel{\mathrm{def}}{=} \big\|\theta(\xi)\big\|_{\mathrm{op}}, \quad \xi \in H^{\mathrm{d}} \otimes K.$$

For $\varphi \in \mathcal{B}(H^{\mathrm{d}} \otimes K)$, we define $S_\varphi : \mathcal{C}_2(H,K) \to \mathcal{C}_2(H,K)$ to be the mapping given by $S_\varphi(\theta(\xi)) = \theta(\varphi\xi)$, $\xi \in H^{\mathrm{d}} \otimes K$. We call φ an *operator multiplier* if there exists $C > 0$ such that $\|S_\varphi(\theta(\xi))\|_{\mathrm{op}} \leq C\|\theta(\xi)\|_{\mathrm{op}}$, for every $\xi \in H^{\mathrm{d}} \otimes K$. If φ is an operator multiplier, then the mapping S_φ extends by continuity to a mapping (denoted in the same way) $S_\varphi : \mathcal{K}(H,K) \to \mathcal{K}(H,K)$ and, by taking the second dual, to a mapping $S_\varphi^{**} : \mathcal{B}(H,K) \to \mathcal{B}(H,K)$. An element $\varphi \in \mathcal{B}(H^{\mathrm{d}} \otimes K)$ will be called a *completely bounded operator multiplier*, or a *c.b. operator multiplier*, if S_φ is completely bounded with respect to the operator space structure arising from the inclusion $\mathcal{C}_2(H,K) \subseteq \mathcal{K}(H,K)$. If $\mathcal{M} \subseteq \mathcal{B}(H)$ and $\mathcal{N} \subseteq \mathcal{B}(K)$ are von Neumann algebras, we will denote by $\mathrm{M}^{\mathrm{cb}}(\mathcal{M},\mathcal{N})$ the collection of all c.b. operator multipliers in $\mathcal{M}^o \bar{\otimes} \mathcal{N}$ and call its elements *completely bounded $\mathcal{M}$, $\mathcal{N}$-multipliers*, or *c.b. $\mathcal{M}$, $\mathcal{N}$-multipliers*. We note that $\mathrm{M}^{\mathrm{cb}}(\mathcal{M},\mathcal{N})$ is a subalgebra of $\mathcal{M}^o \bar{\otimes} \mathcal{N}$.

We next recall [5] that the *extended Haagerup tensor product* $\mathcal{B}(K) \otimes_{\mathrm{eh}} \mathcal{B}(H)$ consists of the sums of the form $\sum_{i=1}^{\infty} b_i \otimes a_i$, where $(b_i)_{i\in\mathbb{N}}$ (resp. $(a_i)_{i\in\mathbb{N}}$) is a bounded row (resp. column) operator. There exists a one-to-one correspondence between the elements of $\mathcal{B}(K) \otimes_{\mathrm{eh}} \mathcal{B}(H)$ and the normal completely bounded maps on $\mathcal{B}(H,K)$: to the element $u = \sum_{i=1}^{\infty} b_i \otimes a_i \in \mathcal{B}(K) \otimes_{\mathrm{eh}} \mathcal{B}(H)$, there corresponds the map Φ_u given by $\Phi_u(x) = \sum_{i=1}^{\infty} b_i x a_i$, $x \in \mathcal{B}(H,K)$.

Let $\varphi \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M},\mathcal{N})$. The mapping S_φ^{**} is normal and completely bounded; by the previous paragraph, there exists a (unique) element $u_\varphi \in \mathcal{B}(K) \otimes_{\mathrm{eh}} \mathcal{B}(H)$, called the *symbol* of φ [15], such that $S_\varphi^{**} = \Phi_{u_\varphi}$. Moreover, [15, Proposition 5.5] shows that $u_\varphi \in \mathcal{N} \otimes_{\mathrm{eh}} \mathcal{M}$. In particular, the map S_φ^{**} is $\mathcal{N}'$, $\mathcal{M}'$-modular.

In the next proposition, we describe the elements $u \in \mathcal{N} \otimes_{\mathrm{eh}} \mathcal{M}$ that are symbols of c.b. operator multipliers.

Proposition 5.1. *The mapping $\Lambda : \varphi \to \Phi_{u_\varphi}$ is a bijective homomorphism from $\mathrm{M}^{\mathrm{cb}}(\mathcal{B}(H), \mathcal{B}(K))$ onto the space of all normal completely bounded maps on $\mathcal{B}(H,K)$ which leave $\mathcal{C}_2(H,K)$ invariant.*

Proof. Suppose that $\varphi \in \mathrm{M}^{\mathrm{cb}}(\mathcal{B}(H),\mathcal{B}(K))$. The map Φ_{u_φ} is the unique normal extension of $S_\varphi : \mathcal{C}_2(H,K) \to \mathcal{C}_2(H,K)$ to $\mathcal{B}(H,K)$. It follows that Φ_{u_φ} preserves $\mathcal{C}_2(H,K)$.

Conversely, suppose that Φ is a normal completely bounded map which leaves $\mathcal{C}_2(H,K)$ invariant. Let $\varphi : H^{\mathrm{d}} \otimes K \to H^{\mathrm{d}} \otimes K$ be the map given by $\varphi\xi = \theta^{-1}(\Phi(\theta(\xi)))$. Clearly, φ is a linear map. We show that it has a closed graph. Suppose $\xi_k \to 0$ and $\varphi\xi_k \to \eta$ in the norm of $H^{\mathrm{d}} \otimes K$. It follows that $\|\theta(\xi_k)\|_{\mathrm{op}} \to 0$ and hence $\|\Phi(\theta(\xi_k))\|_{\mathrm{op}} \to 0$. Thus, $\|\theta(\varphi\xi_k)\|_{\mathrm{op}} \to 0$ and hence $\eta = 0$.

It follows from the Closed Graph Theorem that $\varphi \in \mathcal{B}(H^{\mathrm{d}} \otimes K)$. By its definition, $S_\varphi = \Phi|_{\mathcal{C}_2(H,K)}$ and it follows that $\varphi \in \mathrm{M}^{\mathrm{cb}}(\mathcal{B}(H), \mathcal{B}(K))$ and $\Phi_{u_\varphi} = \Phi$. The fact that Λ is a homomorphism is immediate from its definition. □

We recall that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are C*-algebras, then the Haagerup norm of an element ω of $\mathcal{A}_1 \odot \mathcal{A}_2$ is defined by

$$\|\omega\|_{\mathrm{h}} = \inf\Big\{ \Big\| \sum a_i a_i^* \Big\|^{\frac{1}{2}} \Big\| \sum b_i^* b_i \Big\|^{\frac{1}{2}} : \omega = \sum a_i \otimes b_i \Big\}.$$

We also let [19]

$$\|\omega\|_{\mathrm{ph}} = \inf\Big\{ \Big\| \sum a_i a_i^* \Big\|^{\frac{1}{2}} \Big\| \sum b_i b_i^* \Big\|^{\frac{1}{2}} : \omega = \sum a_i \otimes b_i \Big\}.$$

Let $(\varphi_\nu)_\nu \subseteq \mathcal{B}(H^{\mathrm{d}}) \odot \mathcal{B}(K)$ be a net and $\varphi \in \mathcal{B}(H^{\mathrm{d}} \otimes K)$. We write $\varphi = \mathrm{m}\text{–}\lim_\nu \varphi_\nu$ if the net $(\varphi_\nu)_\nu$ converges semi-weakly to φ (that is, $\langle \varphi_\nu(h_1 \otimes k_1), h_2 \otimes k_2 \rangle \to \langle \varphi(h_1 \otimes k_1), h_2 \otimes k_2 \rangle$ for every $h_1, h_2 \in H^{\mathrm{d}}$ and $k_1, k_2 \in K$), and there exists $C > 0$ such that $\|\varphi_\nu\|_{\mathrm{ph}} \leq C$ for all ν.

We first note the following fact, whose proof is straightforward.

Remark 5.2. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ and $\mathcal{N} \subseteq \mathcal{B}(K)$ be von Neumann algebras. If $(\varphi_\nu)_{\nu \in J} \subseteq \mathcal{M} \odot \mathcal{N}$ and $\varphi = \mathrm{m}\text{–}\lim_\nu \varphi_\nu$, then $\varphi \in \mathcal{M} \mathbin{\overline{\otimes}} \mathcal{N}$.

The following characterisation of c.b. $\mathcal{M}, \mathcal{N}$-multipliers follows from [16] and [15]:

Theorem 5.3. *An element $\varphi \in \mathcal{M}^o \mathbin{\bar{\otimes}} \mathcal{N}$ is a c.b. $\mathcal{M}, \mathcal{N}$-multiplier if and only if there exists a net $(\varphi_\nu)_\nu \subseteq \mathcal{M}^o \odot \mathcal{N}$ such that $\varphi = \mathrm{m}\text{–}\lim_\nu \varphi_\nu$.*

We now introduce local operator multipliers as a non-commutative version of local Schur multipliers. To motivate our definition, recall that, in the commutative case, local Schur multipliers are defined within the class of all measurable, in general unbounded, functions on the direct product of two measure spaces. The natural non-commutative analogue of this algebra is the set of all operators affiliated with the tensor product of two von Neumann algebras. On the other hand, the non-commutative analogues of measurable subsets are projections. We are thus naturally led to define an *$\mathcal{M}', \mathcal{N}'$-covering family* (where $\mathcal{M} \subseteq \mathcal{B}(H)$ and $\mathcal{N} \subseteq \mathcal{B}(K)$ are von Neumann algebras) as a family $\{p_n \otimes q_m\}_{n,m \in \mathbb{N}}$, where $\{p_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}'$ and $\{q_m\}_{m \in \mathbb{N}} \subseteq \mathcal{N}'$ are families of pairwise commuting projections, such that $\bigvee_{n \in \mathbb{N}} p_n = I$ and $\bigvee_{m \in \mathbb{N}} q_m = I$.

Suppose that $\mathcal{M} \subseteq \mathcal{B}(H)$ is a von Neumann algebra, p is a projection in the commutant of $\mathcal{M}$, and $a : \mathrm{dom}(a) \to H$ is a densely defined operator. We will consider pa as the operator with domain $\mathrm{dom}(a)$ given by $(pa)(\xi) = p(a(\xi))$, $\xi \in \mathrm{dom}(a)$. By writing $pa \in \mathcal{M}p$, we will mean that the operator pa is bounded on $\mathrm{dom}(a)$ and its extension to H, which will again be denoted by pa, belongs to the von Neumann algebra $\mathcal{M}p$.

Definition 5.4. Given a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and projections $\{p_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}'$, we say that a densely defined operator $a : H \to H$ is *associated with* $\mathcal{M}$ *with respect to* $\{p_i\}$ if $p_i H \subseteq \mathrm{dom}(a)$ and $p_i a, a p_i \in \mathcal{M} p_i$ for all $i \in \mathbb{N}$.

The set of all such operators will be denoted by $\mathrm{Assoc}\,\mathcal{M}_{\{p_i\}}$. It is not difficult to see that, under the assumptions in Definition 5.4, $p_i a = a p_i$, $i \in \mathbb{N}$.

Definition 5.5. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ and $\mathcal{N} \subseteq \mathcal{B}(K)$ be von Neumann algebras. An element $\varphi \in \mathrm{Aff}(\mathcal{M}^o \bar{\otimes} \mathcal{N})$ will be called a *local* $\mathcal{M}, \mathcal{N}$*-multiplier* if there exists an $\mathcal{M}', \mathcal{N}'$-covering family $\{p_n \otimes q_m\}_{n,m\in\mathbb{N}}$ such that $\varphi(p_n^{\mathrm{d}} \otimes q_m) \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M}p_n, \mathcal{N}q_m)$ for all $n, m \in \mathbb{N}$.

Lemma 5.6. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ and $\mathcal{N} \subseteq \mathcal{B}(K)$ be von Neumann algebras and $\varphi \in \mathcal{M}^o \bar{\otimes} \mathcal{N}$.*

(i) If $p_1, p_2 \in \mathcal{M}'$ and $q_1, q_2 \in \mathcal{N}'$ are projections, $p_1 \leq p_2$, $q_1 \leq q_2$, and $\varphi(p_2^{\mathrm{d}} \otimes q_2) \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M}p_2, \mathcal{N}q_2)$ then $\varphi(p_1^{\mathrm{d}} \otimes q_1) \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M}p_1, \mathcal{N}q_1)$.

(ii) If $(e_i)_{i=1}^n \subseteq \mathcal{M}'$, $(f_j)_{i=1}^m \subseteq \mathcal{N}'$ are sequences of pairwise orthogonal projections such that $\bigvee_{i=1}^n e_i = I$, $\bigvee_{j=1}^m f_j = I$ and $\varphi(e_i^{\mathrm{d}} \otimes f_j) \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M}e_i, \mathcal{N}f_j)$ for each i, j, then $\varphi \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M}, \mathcal{N})$.

Proof. (i) follows from the fact that $S_{\varphi(p_1^{\mathrm{d}} \otimes q_1)} = S_{\varphi(p_2^{\mathrm{d}} \otimes q_2)}|_{\mathcal{C}_2(p_1 H, q_1 K)}$.

(ii) Since $S_\varphi = \sum_{i,j} S_{\varphi(p_i^{\mathrm{d}} \otimes q_j)}$, we have that $\|S_\varphi\|_{\mathrm{cb}} \leq \sum_{i,j} \|S_{\varphi(p_i^{\mathrm{d}} \otimes q_j)}\|_{\mathrm{cb}}$. □

Proposition 5.7. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ and $\mathcal{N} \subseteq \mathcal{B}(K)$ be von Neumann algebras and suppose that $\{e_i\} \subseteq \mathcal{M}'$ and $\{f_j\} \subseteq \mathcal{N}'$ are at most countable families of pairwise orthogonal projections. Let $\mathcal{E}$ be the linear span of $\bigcup_{i,j} f_j \mathcal{B}(H,K) e_i$, and $\Phi : \mathcal{E} \to \mathcal{B}(H,K)$ be a linear map. The following are equivalent:*

(i) Φ leaves $f_j \mathcal{B}(H,K) e_i$ invariant, and the restriction $\Phi_{i,j} \overset{\mathrm{def}}{=} \Phi|_{f_j \mathcal{B}(H,K) e_i} : f_j \mathcal{B}(H,K) e_i \to f_j \mathcal{B}(H,K) e_i$ is $f_j \mathcal{N}' f_j$, $e_i \mathcal{M}' e_i$-modular, normal and completely bounded;

(ii) there exist families $\{a_k\}_{k\in\mathbb{N}} \subseteq \mathrm{Assoc}\,\mathcal{M}_{\{e_i\}}$ and $\{b_k\}_{k\in\mathbb{N}} \subseteq \mathrm{Assoc}\,\mathcal{N}_{\{f_j\}}$ such that $(e_i a_k)_{k\in\mathbb{N}}$ defines a bounded column operator for each i, $(b_k f_j)_{k\in\mathbb{N}}$ defines a bounded row operator for each j, and $\Phi(x) = \sum_{k=1}^\infty b_k x a_k$, for all $x \in \mathcal{E}$.

Proof. (ii) ⇒ (i) Suppose that $x = f_j x e_i$ for some $i, j \in \mathbb{N}$. We have that $a_k = \sum_{i=1}^\infty a_k e_i$, where the sum converges pointwise on $\bigcup_{i=1}^\infty e_i H$. A similar formula holds for b_k. By assumption, for every i (resp. j) and for every k, there exists $\tilde{a}_{k,i} \in \mathcal{M}$ (resp. $\tilde{b}_{k,j} \in \mathcal{N}$) such that $e_i a_k = e_i \tilde{a}_{k,i}$ (resp. $b_k f_j = \tilde{b}_{k,j} f_j$). Let $c \in e_i \mathcal{M}' e_i$ and $d \in f_j \mathcal{N}' f_j$. We have that

$$\begin{aligned}\Phi(dxc) &= \sum_{k=1}^\infty b_k (f_j d f_j x e_i c e_i) a_k = \sum_{k=1}^\infty \tilde{b}_{k,j} (f_j d f_j x e_i c e_i) \tilde{a}_{k,i} \\ &= \sum_{k=1}^\infty f_j d f_j \tilde{b}_{k,j} f_j x e_i \tilde{a}_{k,i} (e_i c e_i) = d \left(\sum_{k=1}^\infty b_k x a_k \right) c.\end{aligned}$$

These identities show that Φ leaves $f_j\mathcal{B}(H,K)e_i$ invariant, and that its restriction to $f_j\mathcal{B}(H,K)e_i$ is a normal completely bounded $f_j\mathcal{N}'f_j$, $e_i\mathcal{M}'e_i$-modular map.

(i) $\Rightarrow$ (ii) For each i and j, let $A_{i,j} = (a_{i,j}^k)_{k\in\mathbb{N}} \in M_{\infty,1}(\mathcal{M}e_i)$ be a bounded column operator and $B_{i,j} = (b_{i,j}^k)_{k\in\mathbb{N}} \in M_{1,\infty}(\mathcal{N}f_j)$ be a bounded row operator such that

$$\Phi(x) = \sum_{k=1}^{\infty} b_{i,j}^k x a_{i,j}^k = B_{i,j}(x\otimes 1)A_{i,j}, \quad x \in f_j\mathcal{B}(H,K)e_i.$$

Assume, without loss of generality, that $\|A_{i,j}\| = \|B_{i,j}\|$, and let $\alpha_{i,j} = \|A_{i,j}\|^2$. By [27, Lemma 3.5], there exist vectors $r_i = (r_i(l))_{l\in\mathbb{N}}$, $s_j = (s_j(l))_{l\in\mathbb{N}} \in \ell^2$ such that $(r_i, \overline{s}_j)_{\ell^2} = \alpha_{i,j}$.

Let $\mathbb{N}_1$, $\mathbb{N}_2$, $\mathbb{N}_3$ and $\mathbb{N}_4$ be copies of $\mathbb{N}$, set $\Lambda = \mathbb{N}_1\times\mathbb{N}_2\times\mathbb{N}_3\times\mathbb{N}_4$ and equip Λ with the lexicographic order, where each $\mathbb{N}_s$, $s = 1,2,3,4$, is given its natural order.

Let A (resp. B) be the column (resp. row) operator given by

$$A = \left(\frac{ir_i(l)}{j\sqrt{\alpha_{i,j}}}a_{i,j}^k\right)_{(i,j,k,l)\in\Lambda} \quad \left(\text{resp. } B = \left(\frac{js_j(l)}{i\sqrt{\alpha_{i,j}}}b_{i,j}^k\right)_{(i,j,k,l)\in\Lambda}\right).$$

We note that A (resp. B) does not necessarily define a bounded column (resp. row) operator, but it can be regarded as a linear operator densely defined on $[\bigcup_{i\in\mathbb{N}} e_iH]$ (resp. $[\bigcup_{j\in\mathbb{N}} f_jK]$). Note that each entry of A (resp. B) is a bounded operator on H (resp. K).

We have that the non-zero entries of $e_{i_0}A$ are the elements of the family $(\frac{i_0 r_{i_0}(l)}{j\sqrt{\alpha_{i_0,j}}}a_{i_0,j}^k)_{j,k,l\in\mathbb{N}}$, and hence

$$\|e_{i_0}A\| \le i_0\|r_{i_0}\|_2\left(\sum_{j=1}^{\infty}\frac{1}{j^2}\right)^{1/2}.$$

Similarly, each non-zero entry of $B(f_{j_0}\otimes 1)$ lies in $\mathcal{N}f_{j_0}$ and

$$\big\|B(f_{j_0}\otimes 1)\big\| \le j_0\|s_{j_0}\|_2\left(\sum_{j=1}^{\infty}\frac{1}{j^2}\right)^{1/2}.$$

Suppose that $x = f_{j_0}xe_{i_0}$. Then

$$\begin{aligned}\Phi(x) &= \sum_{k=1}^{\infty} b_{i_0,j_0}^k x a_{i_0,j_0}^k = B_{i_0,j_0}(x\otimes 1)A_{i_0,j_0}\\ &= \sum_{i,j=1}^{\infty}\sum_{k=1}^{\infty} ij\alpha_{i,j}\frac{1}{i\sqrt{\alpha_{i,j}}}b_{i,j}^k x\frac{1}{j\sqrt{\alpha_{i,j}}}a_{i,j}^k\\ &= \sum_{(i,j,k,l)\in\Lambda}\frac{js_j(l)}{i\sqrt{\alpha_{i,j}}}b_{i,j}^k x\frac{ir_i(l)}{j\sqrt{\alpha_{i,j}}}a_{i,j}^k.\end{aligned}$$

The claim now follows by choosing any enumeration of Λ. □

Lemma 5.8. *Let H and K be Hilbert spaces and $\varphi \in \mathcal{B}(H^{\mathrm{d}} \otimes K)$. Then*

$$\big(S_\varphi\big(\theta(\xi)\big)h_1, h_2\big)_K = \big(\varphi\xi, h_1^{\mathrm{d}} \otimes h_2\big)_{H^{\mathrm{d}}\otimes K}, \quad \xi \in H^{\mathrm{d}} \otimes K,\ h_1^{\mathrm{d}} \in H^{\mathrm{d}},\ h_2 \in K.$$

Proof. Using the identity

$$\mathrm{tr}\big(T\theta\big(h_1^{\mathrm{d}} \otimes h_2\big)\big) = (Th_1, h_2)_K, \quad T \in \mathcal{C}_2(H, K),$$

we see that

$$\begin{aligned}\big(\varphi\xi, h_1^{\mathrm{d}} \otimes h_2\big)_{H^{\mathrm{d}}\otimes K} &= \big(\theta(\varphi\xi), \theta\big(h_1^{\mathrm{d}} \otimes h_2\big)\big)_{\mathcal{C}_2} = \mathrm{tr}\big(\theta(\varphi\xi)\theta\big(h_1^{\mathrm{d}} \otimes h_2\big)\big)\\ &= \big(\theta(\varphi\xi)h_1, h_2\big)_K = \big(S_\varphi\big(\theta(\xi)\big)h_1, h_2\big)_K. \qquad \square\end{aligned}$$

We will need the following lemma; since the statement is rather well-known, the proof is omitted.

Lemma 5.9. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra. If $\varphi \in \mathrm{Aff}(\mathcal{M})$ and $p \in \mathcal{M}'$ is a projection such that φp is bounded and $p\varphi p = \varphi p$, then (the closure of the operator φp which is again denoted by) φp belongs to $\mathcal{M}p$.*

Theorem 5.10. *Let $\varphi \in \mathrm{Aff}(\mathcal{M}^o \bar{\otimes} \mathcal{N})$. Then the following are equivalent:*

(i) φ is a local $\mathcal{M}, \mathcal{N}$-multiplier;

(ii) there exist increasing sequences $(p_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}'$ and $(q_n)_{n\in\mathbb{N}} \subseteq \mathcal{N}'$ of projections such that $\bigvee p_n \otimes q_n = I$ and $\varphi(p_n^{\mathrm{d}} \otimes q_n) \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M}p_n, \mathcal{N}q_n)$ for every $n \in \mathbb{N}$;

(iii) there exist families $(e_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}'$ and $(f_j)_{j\in\mathbb{N}} \subseteq \mathcal{N}'$ of mutually orthogonal projections such that $\bigvee_{i\in\mathbb{N}} e_i = I$ and $\bigvee_{j\in\mathbb{N}} f_j = I$, and a net $(\varphi_\nu)_\nu \subseteq \mathrm{Assoc}\,\mathcal{M}^o \bar{\otimes} \mathcal{N}_{\{e_i^{\mathrm{d}}\otimes f_j\}}$ such that $\varphi_\nu(e_i^{\mathrm{d}} \otimes f_j) \in \mathcal{M}^o e_i^{\mathrm{d}} \odot \mathcal{N}f_j$ and $\varphi(e_i^{\mathrm{d}} \otimes f_j) = \mathrm{m}\text{-}\lim_\nu \varphi_\nu(e_i^{\mathrm{d}} \otimes f_j)$, for all i, j.

Proof. (i) ⇒ (iii) Let $\{p_n \otimes q_m\}_{n,m\in\mathbb{N}} \subseteq \mathcal{M}' \otimes \mathcal{N}'$ be a covering family of projections such that $\varphi(p_n^{\mathrm{d}} \otimes q_m) \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M}p_n, \mathcal{N}q_m)$ for all n and m. Let $e_1 = p_1$ and $e_{i+1} = p_{i+1}(I - e_i)(I - e_{i-1})\dots(I - e_1)$, $i \geq 1$. Define the projections f_j, $j \in \mathbb{N}$, similarly. We have that $\bigvee_{i\in\mathbb{N}} e_i = I$ and $\bigvee_{j\in\mathbb{N}} f_j = I$.

Fix i and j; then $e_i \otimes f_j \leq p_i \otimes q_j$ and by Lemma 5.6(i), $\varphi(e_i^{\mathrm{d}} \otimes f_j) \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M}e_i, \mathcal{N}f_j)$ for all i, j. Let $\mathcal{E} = [\bigcup_{i,j} f_j\mathcal{B}(H, K)e_i]$ and let $\Phi : \mathcal{E} \to \mathcal{E}$ be the map whose restriction to $f_j\mathcal{B}(H, K)e_i$ coincides with $S^{**}_{\varphi(e_i^{\mathrm{d}}\otimes f_j)}$. By Proposition 5.7, there exist operators $(a_k)_{k\in\mathbb{N}} \subseteq \mathrm{Assoc}\,\mathcal{M}_{\{e_i\}_{i\in\mathbb{N}}}$ and $(b_k)_{k\in\mathbb{N}} \subseteq \mathrm{Assoc}\,\mathcal{N}_{\{f_j\}_{j\in\mathbb{N}}}$ such that $(e_ia_k)_{k\in\mathbb{N}}$ defines a bounded column operator for each i, $(b_kf_j)_{k\in\mathbb{N}}$ defines a bounded row operator for each j, and $\Phi(x) = \sum_{k=1}^\infty b_kxa_k$, for all $x \in \mathcal{E}$.

Let $\varphi_N = \sum_{k=1}^N a_k^{\mathrm{d}} \otimes b_k$. Then $\varphi_N(e_i^{\mathrm{d}} \otimes f_j)$ belongs to $(\mathcal{M}^o e_i^{\mathrm{d}}) \odot (\mathcal{N}f_j)$ for all i, j, and

$$\sup_{N\in\mathbb{N}} \big\|\varphi_N\big(e_i^{\mathrm{d}} \otimes f_j\big)\big\|_{\mathrm{ph}} \leq \sup_{N\in\mathbb{N}} \Big\|\sum_{k=1}^N b_kf_j \otimes e_ia_k\Big\|_{\mathrm{h}} \leq \big\|(e_ia_k)_{k\in\mathbb{N}}\big\|\big\|(b_kf_j)_{k\in\mathbb{N}}\big\|.$$

For all $\xi \in (e_i^{\mathrm{d}} H^{\mathrm{d}}) \otimes (f_j K)$ and all $h^{\mathrm{d}} \in H^{\mathrm{d}}$, $k \in K$, we have, by Lemma 5.8, that

$$\big(\Phi\big(\theta(\xi)\big)h, k\big)_K = \big(\varphi\big(e_i^{\mathrm{d}} \otimes f_j\big)\xi, h^{\mathrm{d}} \otimes k\big)_{H^{\mathrm{d}} \otimes K}. \tag{10}$$

On the other hand, if $\Phi_N = S^{**}_{\varphi_N}$, then we have that

$$\Phi_N\big(\theta(\xi)\big) \to_{N\to\infty} \Phi\big(\theta(\xi)\big) \text{ weakly, for all } \xi \in \big(e_i^{\mathrm{d}} H^{\mathrm{d}}\big) \otimes (f_j K). \tag{11}$$

It follows from (10) and (11) that $(\varphi_N(e_i^{\mathrm{d}} \otimes f_j))_{N=1}^{\infty}$ converges semi-weakly to $\varphi(e_i^{\mathrm{d}} \otimes f_j)$, for all i, j. Thus, $\varphi(e_i^{\mathrm{d}} \otimes f_j) = \mathrm{m}\text{-}\lim_N \varphi_N(e_i^{\mathrm{d}} \otimes f_j)$ and (iii) is established.

(iii) ⇒ (ii) By Remark 5.2 and Theorem 5.3, $\varphi(e_i^{\mathrm{d}} \otimes f_j)$ is a c.b. $\mathcal{M}e_i, \mathcal{N}f_j$-multiplier for all $i, j \in \mathbb{N}$. Let $p_n = \bigvee_{i=1}^n e_i$ and $q_n = \bigvee_{j=1}^n f_j$. The claim now follows from Lemma 5.9 and Lemma 5.6(ii).

(ii) ⇒ (i) We have that $(p_n \otimes q_m)_{n,m\in\mathbb{N}}$ is a covering family. By Lemma 5.6(i), $\varphi(p_n^{\mathrm{d}} \otimes q_m) \in \mathrm{M}^{\mathrm{cb}}(\mathcal{M}p_n, \mathcal{N}q_m)$ for all $n, m \in \mathbb{N}$. □

We next include analogous versions of some of the previous results for the case where the respective projections are central. The first one is a "local" version of the well-known representation theorem for completely bounded bimodular maps (see [11,28]).

Proposition 5.11. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ and $\mathcal{N} \subseteq \mathcal{B}(K)$ be von Neumann algebras, $(P_n)_{n\in\mathbb{N}} \subseteq \mathcal{M} \cap \mathcal{M}'$ and $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{N} \cap \mathcal{N}'$ be increasing sequences of projections, $\mathcal{E} = \bigcup_{n\in\mathbb{N}} Q_n\mathcal{B}(H,K)P_n$ and $\Phi : \mathcal{E} \to \mathcal{B}(H,K)$ be a linear map. The following are equivalent:*

(i) the restriction of Φ to $Q_n\mathcal{B}(H,K)P_n$ is completely bounded, normal and $(\mathcal{N}Q_n)'$, $(\mathcal{M}P_n)'$-modular;

(ii) there exist families $(a_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}$ and $(b_k)_{k\in\mathbb{N}} \subseteq \mathcal{N}$ such that $(P_n a_k)_{k\in\mathbb{N}}$ (resp. $(b_k Q_n)_{k\in\mathbb{N}}$) is a bounded column (resp. row) operator for every n and $\Phi(x) = \sum_{k=1}^{\infty} b_k x a_k$, for every $x \in \mathcal{E}$.

Proof. (i) ⇒ (ii) Let $e_1 = P_1$ (resp. $f_1 = Q_1$) and $e_i = P_{i+1} - P_i$ (resp. $f_j = Q_{j+1} - Q_j$), $i \geq 2$ (resp. $j \geq 2$). It is clear that $\Phi|_{f_j\mathcal{B}(H,K)e_i} : f_j\mathcal{B}(H,K)e_i \to \mathcal{B}(H,K)$ is completely bounded and $\mathcal{N}'f_j$, $\mathcal{M}'e_i$-modular. Let

$$A = \left(\frac{ir_i(l)}{j\sqrt{\alpha_{i,j}}} a_{i,j}^k\right)_{(i,j,k,l)\in\Lambda} \quad \text{and} \quad B = \left(\frac{js_j(l)}{i\sqrt{\alpha_{i,j}}} b_{i,j}^k\right)_{(i,j,k,l)\in\Lambda}$$

be the operators from the proof of Proposition 5.7. Since the projections P_n and Q_n, $n \in \mathbb{N}$, are central, we have that the entries of A (resp. B) belong to $\mathcal{M}$ (resp. $\mathcal{N}$). The estimates from the proof of Proposition 5.7 show that

$$\|P_n A\|^2 \leq \sum_{d=1}^{n} d^2 \|r_d\|_2^2 \sum_{j=1}^{\infty} \frac{1}{j^2} \quad \text{and} \quad \big\|B(Q_n \otimes 1)\big\|^2 \leq \sum_{d=1}^{n} d^2 \|s_d\|_2^2 \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

The conclusion follows.

(ii) ⇒ (i) is immediately obtained, via the discussion prior to Proposition 5.1, from the observation that $a_k P_n \in \mathcal{M}P_n$ and $b_k Q_n \in \mathcal{N}Q_n$, and consequently $\sum_{k=1}^{\infty}(b_k Q_n) \otimes (a_k P_n) \in (\mathcal{N}Q_n) \otimes_{\mathrm{eh}} (\mathcal{M}P_n)$. □

Call a local $\mathcal{M}, \mathcal{N}$-multiplier *central* if the covering family $\{p_n \otimes q_m\}_{n,m\in\mathbb{N}}$ associated with φ as in Definition 5.5 can be chosen from the centre of $\mathcal{M}' \bar{\otimes} \mathcal{N}'$.

Corollary 5.12. *Let $\varphi \in \operatorname{Aff} \mathcal{M}^o \overline{\otimes} \mathcal{N}$. Then the following are equivalent:*

(i) φ is a central local $\mathcal{M}, \mathcal{N}$-multiplier;

(ii) there exist a net $(\varphi_\nu) \subseteq \operatorname{Aff} \mathcal{M}^o \bar{\otimes} \mathcal{N}$ and increasing sequences of projections $(P_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}\cap\mathcal{M}'$ and $(Q_n)_{n\in\mathbb{N}} \subseteq \mathcal{N}\cap\mathcal{N}'$ such that $\bigvee_{n\in\mathbb{N}} P_n \otimes Q_n = I$, $\varphi_\nu(P_n^{\mathrm{d}} \otimes Q_n) \in \mathcal{M}^o P_n^{\mathrm{d}} \odot \mathcal{N}Q_n$ and $\varphi(P_n^{\mathrm{d}} \otimes Q_n) = \mathrm{m}-\lim_\nu \varphi_\nu(P_n^{\mathrm{d}} \otimes Q_n)$ for every $n \in \mathbb{N}$.

Proof. (ii) ⇒ (i) follows from Theorem 5.3 and Lemma 5.6.

(i) ⇒ (ii) It is easy to see that the operators φ_N from the proof of Theorem 5.10 can, under the assumption of the corollary, be chosen from $\mathcal{M}^o \odot \mathcal{N}$. The conclusion follows by letting $P_n = \bigvee_{i=1}^n e_i$ and $Q_n = \bigvee_{j=1}^n f_j$, where (e_i) and (f_j) are the sequences of projections from Theorem 5.10. □

6. Positive local operator multipliers

In this section, we study completely positive local operator multipliers. The main result is the characterisation Theorem 6.4. Throughout this section, we fix a von Neumann algebra $\mathcal{M}$.

Definition 6.1. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and $\varphi \in \mathcal{M}^o \overline{\otimes} \mathcal{M}$. We say that φ is a *completely positive $\mathcal{M}$-multiplier* if the map $S_\varphi : \mathcal{C}_2(H) \to \mathcal{C}_2(H)$, given by

$$S_\varphi\big(\theta(\xi)\big) = \theta(\varphi\xi), \quad \xi \in H^{\mathrm{d}} \otimes H,$$

is completely positive and bounded in $\|\cdot\|_{\mathrm{op}}$.

Let

$$\mathcal{P}(\mathcal{M}) = \left\{ \sum_{k=1}^{N} b_k^{\mathrm{d}} \otimes b_k^* : b_k \in \mathcal{M}, N \in \mathbb{N}. \right\} \subseteq \mathcal{M}^o \odot \mathcal{M}.$$

It is clear that $\mathcal{P}(\mathcal{M})$ is a cone, and it is easy to verify that if $\psi \in \mathcal{P}(\mathcal{M})$ then the map S_ψ is bounded and completely positive; thus, every element of $\mathcal{P}(\mathcal{M})$ is a completely positive $\mathcal{M}$-multiplier. In the next theorem, we show that completely positive $\mathcal{M}$-multipliers can be approximated by elements of $\mathcal{P}(\mathcal{M})$.

Theorem 6.2. *Let $\varphi \in \mathcal{M}^o \overline{\otimes} \mathcal{M}$. Then φ is a completely positive $\mathcal{M}$-multiplier if and only if there exists a net $(\varphi_\nu)_{\nu\in J} \subseteq \mathcal{P}(\mathcal{M})$ such that $\varphi = \mathrm{m}-\lim_\nu \varphi_\nu$.*

Proof. Suppose that φ is a completely positive $\mathcal{M}$-multiplier. By definition, S_φ is bounded and completely positive and thus, by the remarks before Proposition 5.1, it is $\mathcal{M}'$-bimodular. There exists a family $(a_i)_{i=1}^\infty \subseteq \mathcal{M}$ of operators that defines a bounded column operator V such that

$$S_\varphi(x) = \sum_{i=1}^{\infty} a_i^* x a_i, \quad x \in \mathcal{K}(H),$$

where the series converges in the weak* topology. Let $\varphi_N = \sum_{i=1}^N a_i^{\mathrm{d}} \otimes a_i^* \in \mathcal{P}(\mathcal{M})$, $N \in \mathbb{N}$. Now

$$S_{\varphi_N}\big(\theta(\xi)\big) \to_{N\to\infty} S_\varphi\big(\theta(\xi)\big) \tag{12}$$

weakly for all $\xi \in H^{\mathrm{d}} \otimes H$. It follows from Lemma 5.8 that $(\varphi_N)_{N\in\mathbb{N}}$ converges semi-weakly to φ. A standard estimate shows that

$$\sup_{N\in\mathbb{N}} \|\varphi_N\|_{\mathrm{ph}} \leq \|V\|^2.$$

Thus, $\varphi = \mathrm{m}\text{-}\lim_N \varphi_\nu$.

Conversely, suppose that there exists a net $(\varphi_\nu)_{\nu\in J} \subseteq \mathcal{P}(\mathcal{M})$ such that $(\varphi_\nu)_{\nu\in J}$ converges semi-weakly to φ and $D = \sup_\nu \|\varphi_\nu\|_{\mathrm{ph}} < +\infty$. As in the proof of the implication (iii) $\Rightarrow$ (ii) of Theorem 5.10, we can see that $\|S_\varphi\| \leq D$.

To obtain the complete positivity, suppose that $(\theta(\xi_{ij}))_{i,j=1}^l$ is a positive element of $\mathcal{C}_2(H^l)$, where $\xi_{ij} \in H^{\mathrm{d}} \odot H$, $i,j = 1,\ldots,l$. If $h = (h_1,\ldots,h_l) \in H^l$ then

$$\begin{aligned}
0 &\leq \big(S_{\varphi_\nu}^{(l)}\big(\big(\theta(\xi_{ij})\big)_{i,j=1}^l\big)h,h\big)_{H^n} \\
&= \sum_{i=1}^{l}\Bigg(\sum_{j=1}^{l}\sum_{k=1}^{N_\nu}\big(a_k^\nu\big)^*\theta(\xi_{ij})a_k^\nu h_j, h_i\Bigg)_H \\
&= \sum_{i,j=1}^{l}\Bigg(\theta\Bigg(\sum_{k=1}^{N_\nu}\big(a_k^\nu\big)^{\mathrm{d}} \otimes \big(a_k^\nu\big)^*\xi_{ij}\Bigg)h_j, h_i\Bigg)_H \\
&= \sum_{i,j=1}^{l}\Bigg(\theta\Bigg(\sum_{k=1}^{N_\nu}\big(a_k^\nu\big)^{\mathrm{d}} \otimes \big(a_k^\nu\big)^*\xi_{ij}\Bigg), \theta\big(h_j^{\mathrm{d}} \otimes h_i\big)\Bigg)_{\mathcal{C}_2(H)} \\
&= \sum_{i,j=1}^{l}\big(\varphi_\nu(\xi_{ij}), h_j^{\mathrm{d}} \otimes h_i\big)_{H^{\mathrm{d}}\otimes H} \\
&\to \sum_{i,j=1}^{l}\big(\varphi(\xi_{ij}), h_j^{\mathrm{d}} \otimes h_i\big)_{H^{\mathrm{d}}\otimes H} = \big(S_\varphi^{(l)}\big(\big(\theta(\xi_{ij})\big)_{i,j=1}^l\big)h,h\big)_H
\end{aligned}$$

and hence the map S_φ is completely positive on the $*$-algebra $\mathcal{F}(H)$ of all finite rank operators.

If $(\theta(\xi_{ij}))_{i,j=1}^{l} \in M_l(\mathcal{C}_2(H))^+$ then $(\theta(\xi_{ij}))$ can be approximated by positive matrices of finite rank operators in the Hilbert–Schmidt norm, and hence in the operator norm. It follows from the previous arguments that S_φ is completely positive on $\mathcal{C}_2(H)$. Thus φ is a completely positive $\mathcal{M}$-multiplier. □

We next introduce the non-commutative version of positive local Schur multipliers.

Definition 6.3. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and $\varphi \in \mathrm{Aff}(\mathcal{M}^o \mathbin{\overline{\otimes}} \mathcal{M})$. We say that φ is a *completely positive local $\mathcal{M}$-multiplier* if

(i) there exists an increasing sequence $(p_n)_{n=1}^{\infty} \subseteq \mathcal{M}'$ of projections such that $\bigvee_{n\in\mathbb{N}} p_n = I$ and $\varphi \in \mathrm{Assoc}\, \mathcal{M}^o \mathbin{\overline{\otimes}} \mathcal{M}_{\{p_n^{\mathrm{d}} \otimes p_n\}_{n\in\mathbb{N}}}$;

(ii) $\varphi(p_n^{\mathrm{d}} \otimes p_n)$ is a completely positive $\mathcal{M}p_n$-multiplier.

We note that every completely positive local $\mathcal{M}$-multiplier is a local $\mathcal{M}$-multiplier. We will call the sequence $(p_n)_{n\in\mathbb{N}}$ of projections from Definition 6.3 an *implementing sequence* for φ.

Suppose that $T : \mathrm{dom}(T) \to H$ is a densely defined, closed operator affiliated with a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$. We again let $T^{\mathrm{d}} : \mathrm{dom}(T^{\mathrm{d}}) \to H^{\mathrm{d}}$ be the Banach space dual of T; note that T^{d} is closed and affiliated with the opposite von Neumann algebra $\mathcal{M}^o$, $\mathrm{dom}(T^{\mathrm{d}}) = (\mathrm{dom}(T^*))^{\mathrm{d}}$ and $T^{\mathrm{d}}\xi^{\mathrm{d}} = (T^*\xi)^{\mathrm{d}}$, $\xi \in \mathrm{dom}(T)$.

Given two densely defined, closable operators S and T, we denote by $S \odot T$ the linear operator defined on the algebraic tensor product $\mathrm{dom}(S) \odot \mathrm{dom}(T)$ by

$$(S \odot T)(h \otimes k) = Sh \otimes Tk, \quad h \in \mathrm{dom}(S),\ k \in \mathrm{dom}(T).$$

The operator $S \odot T$ is closable [17]; by abuse of notation, we denote the closure again by $S \odot T$. If $P, Q \in \mathcal{B}(H)$ are projections such that $PH \subseteq \mathrm{dom}(S)$ (resp. $QH \subseteq \mathrm{dom}(T)$), then

$$(S \odot T)(P \odot Q) = SP \odot TQ$$

is bounded and coincides with the usual tensor product $SP \otimes TQ$ of bounded operators.

For von Neumann algebras $\mathcal{M}, \mathcal{N}$, we define

$$\mathrm{Aff}\,\mathcal{M} \odot \mathrm{Aff}\,\mathcal{N} = \left\{ \sum_{k=1}^{m} S_k \odot T_k : S_k \in \mathrm{Aff}\,\mathcal{M}, T_k \in \mathrm{Aff}\,\mathcal{N}, m \in \mathbb{N} \right\}.$$

Theorem 6.4. *Let $(p_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}'$ be an increasing sequence of projections such that $\bigvee_{n\in\mathbb{N}} p_n = I_H$. An operator $\varphi \in \mathrm{Aff}(\mathcal{M}^o \mathbin{\overline{\otimes}} \mathcal{M})$ is a completely positive local $\mathcal{M}$-multiplier with implementing sequence $(p_n)_{n\in\mathbb{N}}$ if and only if there exists a net*

$(\varphi_\nu)_{\nu\in J} \subseteq \operatorname{Aff}\mathcal{M}^o \odot \operatorname{Aff}\mathcal{M}$ *such that, for each* $n \in \mathbb{N}$, $\varphi_\nu(p_n^{\mathrm{d}} \odot p_n) \in \mathcal{P}(\mathcal{M}p_n)$ *and* $\varphi(p_n^{\mathrm{d}} \otimes p_n) = \mathrm{m}-\lim_\nu \varphi_\nu(p_n^{\mathrm{d}} \odot p_n)$.

Proof. Suppose that φ is a completely positive local $\mathcal{M}$-multiplier with an implementing sequence $(p_n)_{n\in\mathbb{N}}$. By definition, $\bigvee_{n\in\mathbb{N}} p_n = I$ and, if $H_n = p_nH$, the map $S_\varphi|_{\mathcal{C}_2(H_n)}$ is bounded, completely positive and $p_n\mathcal{M}'p_n$-bimodular.

Since the map is bounded on $\mathcal{C}_2(H_n)$, it can be extended to $\mathcal{K}(H_n)$, and this extension preserves the bimodularity and complete positivity. Thus, by Theorem 3.5, there exists a family $\{a_k\}_{k=1}^\infty$ of closable operators affiliated with $\mathcal{M}$ that, as noted in Remark 3.6, are such that $a_kp_n, a_k^*p_n \in \mathcal{M}p_n$ and $S_\varphi(x) = \sum_{k=1}^\infty a_k^*xa_k$, for $x \in \bigcup_{n=1}^\infty \mathcal{K}(H_n)$. Recall that $(a_1p_n, a_2p_n, \ldots)^t$ is a bounded column operator, say V_n. We define $\varphi_N = \sum_{k=1}^N a_k^{\mathrm{d}} \odot a_k^*$, $n \in \mathbb{N}$. Clearly, $\varphi_N \in \operatorname{Aff}\mathcal{M}^o \odot \operatorname{Aff}\mathcal{M}$ for each $N \in \mathbb{N}$ and since $a_kp_n \in \mathcal{M}p_n$, it follows that $\varphi_N(p_n^{\mathrm{d}} \odot p_n) \in \mathcal{P}(\mathcal{M}p_n)$.

Analogously to (12), we see that the sequence $(S_{\varphi_N}|_{\mathcal{C}_2(H_n)}(\theta(\xi)))_{N=1}^\infty$ converges weakly to $S_\varphi|_{\mathcal{C}_2(H_n)}(\theta(\xi))$. By Lemma 5.8, $(\varphi_N(p_n^{\mathrm{d}} \odot p_n))_{N\in\mathbb{N}}$ converges semi-weakly to $\varphi(p_n^{\mathrm{d}} \otimes p_n)$. As before, one can easily see that

$$\sup_{N\in\mathbb{N}} \left\|\varphi_N\left(p_n^{\mathrm{d}} \odot p_n\right)\right\|_{\mathrm{ph}} \leq \|V_n\|^2.$$

To prove the converse, observe that Theorem 6.2 and Remark 5.2 show that, under the stated assumptions, $\varphi(p_n^{\mathrm{d}} \otimes p_n)$ is a completely positive $\mathcal{M}p_n$-multiplier for each $n \in \mathbb{N}$. Since $\bigvee_{n\in\mathbb{N}} p_n = I$, we have that φ is a completely positive local $\mathcal{M}$-multiplier. □

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