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Inclusions of Waterman–Shiba spaces into generalized Wiener classes

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ABSTRACT

The characterization of the inclusion of Waterman–Shiba spaces $\Lambda BV^{(p)}$ into generalized Wiener classes of functions $BV(q;\delta)$ is given. It uses a new and shorter proof and extends an earlier result of U. Goginava.

Let $\Lambda = (\lambda_i)$ be a Λ-sequence, that is, a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_i} = +\infty$ and let p be a number greater than or equal to 1. A function $f : [0,1] \to \mathbb{R}$ is said to be of bounded p-Λ-variation if

$$V(f) := \sup\left(\sum_{i=1}^{n} \frac{|f(I_i)|^p}{\lambda_i}\right)^{\frac{1}{p}} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping subintervals of $[0,1]$ and where $f(I_i) := f(\sup I_i) - f(\inf I_i)$ is the change of the function f over the interval I_i. The symbol $\Lambda BV^{(p)}$ denotes the linear space of all functions of bounded p-Λ-variation with domain $[0,1]$. The Waterman–Shiba space $\Lambda BV^{(p)}$ was introduced in 1980 by M. Shiba [21]. When $p = 1$, $\Lambda BV^{(p)}$ is the well-known Waterman space ΛBV (see e.g. [25] and [26]). Some of the properties and applications of functions of class $\Lambda BV^{(p)}$ were discussed in [8,9,11,12,16–20,22–24]. $\Lambda BV^{(p)}$ equipped with the norm $\|f\|_{\Lambda,p} := |f(0)| + V(f)$ is a Banach space.

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H. Kita and K. Yoneda introduced a new function space which is a generalization of Wiener classes [14] (see also [13] and [1]). The concept was further extended by T. Akhobadze in [2] who studied many properties of the generalized Wiener classes $BV(q,\delta)$ thoroughly (see [3–7]).

Definition 1. Let $q = (q(n))_{n=1}^{\infty}$ be an increasing positive sequence and let $\delta = (\delta(n))_{n=1}^{\infty}$ be an increasing and unbounded positive sequence. We say that a function $f : [0,1] \to \mathbb{R}$ belongs to the class $BV(q;\delta)$ if

$$V(f,q;\delta) := \sup_{n\geq 1}\sup_{\{I_k\}}\left\{\left(\sum_{k=1}^{s}\left|f(I_k)\right|^{q(n)}\right)^{\frac{1}{q(n)}} : \inf_k |I_k| \geq \frac{1}{\delta(n)}\right\} < \infty,$$

where $\{I_k\}_{k=1}^{s}$ are non-overlapping subintervals of $[0,1]$.

If $\delta(n)^{1/q(n)}$ is a bounded sequence, then $BV(q;\delta)$ is simply the space of all bounded functions. This follows from the estimate

$$\left(\sum_{k=1}^{s}\left|f(I_k)\right|^{q(n)}\right)^{\frac{1}{q(n)}} \leq 2Cs^{\frac{1}{q(n)}} \leq 2C\delta(n)^{\frac{1}{q(n)}}, \quad \left|f(x)\right| \leq C.$$

The following statement regarding inclusions of Waterman spaces into generalized Wiener classes has been presented in [10]: if $\lim_{n\to\infty} q(n) = \infty$ and $\delta(n) = 2^n$ the inclusion $\Lambda BV \subset BV(q,\delta)$ holds if and only if

$$\limsup_{n\to\infty}\left\{\max_{1\leq k\leq 2^n}\frac{k^{\frac{1}{q(n)}}}{(\sum_{i=1}^{k}\frac{1}{\lambda_i})}\right\} < +\infty. \tag{1}$$

Our result formulated below extends the above theorem of Goginava essentially and furnishes a new and much shorter proof.

Theorem 1. *For $p \in [1,\infty)$ and q and δ sequences satisfying the conditions in Definition 1, the inclusion $\Lambda BV^{(p)} \subset BV(q;\delta)$ holds if and only if*

$$\limsup_{n\to\infty}\left\{\max_{1\leq k\leq \delta(n)}\frac{k^{\frac{1}{q(n)}}}{(\sum_{i=1}^{k}\frac{1}{\lambda_i})^{\frac{1}{p}}}\right\} < +\infty. \tag{2}$$

Before we present a relatively short proof of Theorem 1, we give an example showing that it provides a non-trivial extension of (1) even for $p = 1$. The example – provided by the referee kindly instead of our more complicated one – is obtained by taking $\lambda_n = n$, $q(n) = \sqrt{n}$ and $\delta(n) = 2^{\sqrt{n}}$. With those choices, it follows immediately that the Goginava indicator (1) is infinite while (2) holds.

Proof of Theorem 1. To show that (2) is a *sufficiency* condition for the inclusion $\Lambda BV^{(p)} \subset BV(q;\delta)$, we will prove the inequality

$$V(f,q,\delta) \leq V_{\Lambda BV^p}(f)\sup_n\left\{\max_{1\leq k\leq \delta(n)}\frac{k}{(\sum_{i=1}^{k}1/\lambda_i)^{\frac{q(n)}{p}}}\right\}^{\frac{1}{q(n)}}. \tag{3}$$

This is a consequence of the numerical inequality

$$\sum_{j=1}^{s}x_j^q \leq \left(\sum_{j=1}^{s}x_jy_j\right)^q \max_{1\leq k\leq s}\frac{k}{(\sum_{j=1}^{k}y_j)^q}, \tag{4}$$

which is valid for $q \geq 0$ and

$$\begin{aligned} x_1 \geq x_2 \geq \ldots \geq x_s \geq 0, \\ y_1 \geq y_2 \geq \ldots \geq y_s \geq 0. \end{aligned} \tag{5}$$

In the cases $q \geq 1$ and $0 \leq q < 1$, (4) is a reformulation of [15, Lemma] and [11, Lemma 2.5], respectively.

To prove (3), consider a non-overlapping family $(I_k)_{k=1}^s$ with $\inf |I_k| \geq 1/\delta(n)$. In particular, $s \leq \delta(n)$. Apply (4) with q replaced by $q(n)/p$, $x_j = |f(I_j)|^p$ and $y_j = 1/\lambda_j$, where we reorder the intervals so that (5) holds. This gives

$$\begin{aligned} \left(\sum_{j=1}^{s} \left|f(I_j)\right|^{q(n)}\right)^{\frac{1}{q(n)}} &\leq \left(\sum_{j=1}^{s} \frac{|f(I_j)|^p}{\lambda_j}\right)^{\frac{1}{p}} \max_{1\leq k\leq s} \frac{k^{\frac{1}{q(n)}}}{(\sum_{j=1}^{k} \frac{1}{\lambda_j})^{\frac{1}{p}}} \\ &\leq V_{\Lambda BV^p}(f) \max_{1\leq k\leq \delta(n)} \frac{k^{\frac{1}{q(n)}}}{(\sum_{j=1}^{k} \frac{1}{\lambda_j})^{\frac{1}{p}}}. \end{aligned}$$

Taking the supremum over n yields (3).

Necessity. Suppose (2) doesn't hold. Then there is an increasing sequence (n_k) of positive integers such that for all indices k

$$\delta(n_k) \geq 2^{k+2} \tag{6}$$

and

$$\max_{1\leq n\leq \delta(n_k)} \frac{n^{\frac{1}{q(n_k)}}}{(\sum_{i=1}^{n} \frac{1}{\lambda_i})^{\frac{1}{p}}} > 2^{2k+\frac{k+1}{q(1)}}. \tag{7}$$

Let (m_k) be a sequence of positive integers such that

$$1 \leq m_k \leq \delta(n_k), \tag{8}$$

and

$$\max_{1\leq n\leq \delta(n_k)} \frac{n^{\frac{1}{q(n_k)}}}{(\sum_{i=1}^{n} \frac{1}{\lambda_i})^{\frac{1}{p}}} = \frac{m_k^{\frac{1}{q(n_k)}}}{(\sum_{i=1}^{m_k} \frac{1}{\lambda_i})^{\frac{1}{p}}}. \tag{9}$$

Denote

$$\Phi_k := \frac{1}{\sum_{i=1}^{m_k} 1/\lambda_i}.$$

Consider

$$g_k(y) := \begin{cases} 2^{-k}\Phi_k^{1/p}, & y \in [\frac{1}{2^k} + \frac{2j-2}{\delta(n_k)}, \frac{1}{2^k} + \frac{2j-1}{\delta(n_k)});\ 1 \leq j \leq N_k, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$N_k = \min\{m_k, s_k\}, \quad s_k = \max\left\{j \in \mathbb{N}:\ 2j \leq \frac{\delta(n_k)}{2^k} + 1\right\}.$$

Applying the fact that $2(s_k+1) \geq \frac{\delta(n_k)}{2^k} + 1$ and (6), we have

$$\frac{2s_k - 1}{\delta(n_k)} \geq 2^{-k-1}. \tag{10}$$

The functions g_k have disjoint support and the series $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly to a function g. Thus,

$$\|g\|_{\Lambda,p} \leq \sum_{k=1}^{\infty} \|g_k\|_{\Lambda,p} = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{2N_k} \frac{(2^{-k}\Phi_k^{1/p})^p}{\lambda_j} \right)^{1/p} \leq \sum_{k=1}^{\infty} 2^{-k} \left(2 \sum_{j=1}^{N_k} \frac{\Phi_k}{\lambda_j} \right)^{1/p}$$
$$\leq \sum_{k=1}^{\infty} 2^{-k} \left(2 \sum_{j=1}^{m_k} \frac{\Phi_k}{\lambda_j} \right)^{1/p} = \sum_{k=1}^{\infty} 2^{-k} \left(2 \frac{\sum_{j=1}^{m_k} \frac{1}{\lambda_j}}{\sum_{j=1}^{m_k} \frac{1}{\lambda_j}} \right)^{1/p} < +\infty,$$

that is, $g \in \Lambda BV^{(p)}$.

If $N_k = m_k$, then $2N_k - 1 \geq m_k$, and if $N_k = s_k$, then

$$2N_k - 1 \overset{(10)}{\geq} \frac{\delta(n_k)}{2^{k+1}} \overset{(8)}{\geq} \frac{m_k}{2^{k+1}}.$$

Hence

$$2N_k - 1 \geq \frac{m_k}{2^{k+1}} \quad \text{for all } k. \tag{11}$$

Now, given any positive integer k, all intervals $[\frac{1}{2^k} + \frac{j-1}{\delta(n_k)}, \frac{1}{2^k} + \frac{j}{\delta(n_k)}]$ for $j = 1, .., 2N_k - 1$, have length $\frac{1}{\delta(n_k)}$ and thus

$$V(g,q,\delta) \geq \left(\sum_{j=1}^{2N_k-1} \left| g\left(\frac{1}{2^k} + \frac{j-1}{\delta(n_k)}\right) - g\left(\frac{1}{2^k} + \frac{j}{\delta(n_k)}\right) \right|^{q(n_k)} \right)^{\frac{1}{q(n_k)}}$$
$$= \left((2N_k - 1)\left(2^{-k}\Phi_k^{\frac{1}{p}}\right)^{q(n_k)} \right)^{\frac{1}{q(n_k)}} = \frac{1}{2^k} \left(\frac{2N_k - 1}{\left(\sum_{i=1}^{m_k} \frac{1}{\lambda_i}\right)^{\frac{q(n_k)}{p}}} \right)^{\frac{1}{q(n_k)}}$$
$$\overset{(11)}{\geq} \frac{1}{2^k} \left(\frac{1}{2^{k+1}} \cdot \frac{m_k}{\left(\sum_{i=1}^{m_k} \frac{1}{\lambda_i}\right)^{\frac{q(n_k)}{p}}} \right)^{\frac{1}{q(n_k)}} \overset{(7),(9)}{\geq} \frac{2^{k+(k+1)/q(1)}}{2^{(k+1)/q(n_k)}} \geq 2^k.$$

Since k was arbitrary, $V(g,q,\delta)$ must be infinite which shows that $g \notin BV(q;\delta)$. □

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References

[1] T. Akhobadze, On Fourier coefficients of functions of generalized Wiener class, Georgian Math. J. 7 (1) (2000) 1–10.
[2] T. Akhobadze, Functions of generalized Wiener classes $BV(p(n) \uparrow \infty, \phi)$ and their Fourier coefficients, Georgian Math. J. 7 (3) (2000) 401–416.
[3] T. Akhobadze, On Fourier coefficients of functions of $BV(p(n) \uparrow \infty, \phi)$ class, Bull. Georgian Acad. Sci. 162 (3) (2000) 415–417.
[4] T. Akhobadze, Generalized $BV(p(n) \uparrow \infty, \phi)$ class of bounded variation, Bull. Georgian Acad. Sci. 163 (3) (2001) 426–428.

[5] T. Akhobadze, $B\Lambda(p(n) \uparrow \infty, \phi)$ classes of functions of bounded variation, Bull. Georgian Acad. Sci. 164 (1) (2001) 18–20.

[6] T. Akhobadze, Relations between H^ω, $V[v]$ and $BV(p(n) \uparrow \infty, \phi)$ classes of functions, Bull. Georgian Acad. Sci. 164 (3) (2001) 433–435.

[7] T. Akhobadze, A generalization of bounded variation, Acta Math. Hungar. 97 (3) (2002) 223–256.

[8] W.W. Breckner, T. Trif, On the singularities of certain families of nonlinear mappings, Pure Math. Appl. 6 (1995) 121–137.

[9] W.W. Breckner, T. Trif, C. Varga, Some applications of the condensation of the singularities of families of nonnegative functions, Anal. Math. 25 (1999) 12–32.

[10] U. Goginava, Relations between ΛBV and $BV(p(n) \uparrow \infty)$ classes of functions, Acta Math. Hungar. 101 (4) (2003) 264–272.

[11] M. Hormozi, Inclusion of $\Lambda BV^{(p)}$ spaces in the classes H_ω^q, J. Math. Anal. Appl. 404 (2) (2013) 195–200.

[12] M. Hormozi, A.A. Ledari, F. Prus-Wisniowski, On p-Λ-bounded variation, Bull. Iranian Math. Soc. 37 (4) (2011) 29–43.

[13] H. Kita, Convergence of Fourier series of a function on generalized Wiener's class $BV(p(n) \to \infty)$, Acta Math. Hungar. 57 (3) (1991) 233–243.

[14] H. Kita, K. Yoneda, A generalization of bounded variation, Acta Math. Hungar. 56 (3–4) (1990) 229–238.

[15] Y.E. Kuprikov, Moduli of continuity of functions from Waterman classes, Moscow Univ. Math. Bull. 52 (5) (1997) 46–49.

[16] L. Leindler, A note on embedding of classes H^ω, Anal. Math. 27 (2001) 71–76.

[17] M. Lind, On functions of bounded Λ-variation and integral smoothness, Forum Math., http://dx.doi.org/10.1515/forum-2012-0186, ahead of print.

[18] P.B. Pierce, D. Waterman, On the invariance of classes ΦBV, ΛBV under composition, Proc. Amer. Math. Soc. 132 (2003) 755–760.

[19] M. Schramm, D. Waterman, On the magnitude of Fourier coefficients, Proc. Amer. Math. Soc. 85 (1982) 407–410.

[20] M. Schramm, D. Waterman, Absolute convergence of Fourier series of functions of $\Lambda BV^{(p)}$ and $\Phi\Lambda BV$, Acta Math. Hungar. 40 (1982) 273–276.

[21] M. Shiba, On the absolute convergence of Fourier series of functions of class $\Lambda BV^{(p)}$, Sci. Rep. Fukushima Univ. 30 (1980) 7–10.

[22] R.G. Vyas, On the absolute convergence of small gaps Fourier series of functions of $\Lambda BV^{(p)}$, JIPAM. J. Inequal. Pure Appl. Math. 6 (1) (2005), Article 23.

[23] R.G. Vyas, On the convolution of functions of generalized bounded variation, Georgian Math. J. 13 (2006) 193–197.

[24] R.G. Vyas, Properties of functions of generalized bounded variation, Mat. Vesnik 58 (2006) 91–96.

[25] D. Waterman, On convergence of Fourier series of functions of bounded generalized variation, Studia Math. 44 (1972) 107–117.

[26] D. Waterman, On Λ-bounded variation, Studia Math. 57 (1976) 33–45.