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Inclusions of Waterman–Shiba spaces into generalized Wiener classes

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ABSTRACT

The characterization of the inclusion of Waterman–Shiba spaces $\Lambda BV^{(p)}$ into generalized Wiener classes of functions $BV(q;\delta)$ is given. It uses a new and shorter proof and extends an earlier result of U. Goginava.

Let $\Lambda = (\lambda_i)$ be a Λ-sequence, that is, a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_i} = +\infty$ and let p be a number greater than or equal to 1. A function $f : [0,1] \to \mathbb{R}$ is said to be of bounded p-Λ-variation if

$$V(f) := \sup\left(\sum_{i=1}^{n} \frac{|f(I_i)|^p}{\lambda_i}\right)^{\frac{1}{p}} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping subintervals of $[0,1]$ and where $f(I_i) := f(\sup I_i) - f(\inf I_i)$ is the change of the function f over the interval I_i. The symbol $\Lambda BV^{(p)}$ denotes the linear space of all functions of bounded p-Λ-variation with domain $[0,1]$. The Waterman–Shiba space $\Lambda BV^{(p)}$ was introduced in 1980 by M. Shiba [21]. When $p = 1$, $\Lambda BV^{(p)}$ is the well-known Waterman space ΛBV (see e.g. [25] and [26]). Some of the properties and applications of functions of class $\Lambda BV^{(p)}$ were discussed in [8,9,11,12,16–20,22–24]. $\Lambda BV^{(p)}$ equipped with the norm $\|f\|_{\Lambda,p} := |f(0)| + V(f)$ is a Banach space.

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H. Kita and K. Yoneda introduced a new function space which is a generalization of Wiener classes [14] (see also [13] and [1]). The concept was further extended by T. Akhobadze in [2] who studied many properties of the generalized Wiener classes $BV(q,\delta)$ thoroughly (see [3–7]).

Definition 1. Let $q = (q(n))_{n=1}^{\infty}$ be an increasing positive sequence and let $\delta = (\delta(n))_{n=1}^{\infty}$ be an increasing and unbounded positive sequence. We say that a function $f : [0,1] \to \mathbb{R}$ belongs to the class $BV(q;\delta)$ if

$$V(f,q;\delta) := \sup_{n\geq 1}\sup_{\{I_k\}}\left\{\left(\sum_{k=1}^{s}\left|f(I_k)\right|^{q(n)}\right)^{\frac{1}{q(n)}} : \inf_k |I_k| \geq \frac{1}{\delta(n)}\right\} < \infty,$$

where $\{I_k\}_{k=1}^{s}$ are non-overlapping subintervals of $[0,1]$.

If $\delta(n)^{1/q(n)}$ is a bounded sequence, then $BV(q;\delta)$ is simply the space of all bounded functions. This follows from the estimate

$$\left(\sum_{k=1}^{s}\left|f(I_k)\right|^{q(n)}\right)^{\frac{1}{q(n)}} \leq 2Cs^{\frac{1}{q(n)}} \leq 2C\delta(n)^{\frac{1}{q(n)}}, \quad \left|f(x)\right| \leq C.$$

The following statement regarding inclusions of Waterman spaces into generalized Wiener classes has been presented in [10]: if $\lim_{n\to\infty} q(n) = \infty$ and $\delta(n) = 2^n$ the inclusion $\Lambda BV \subset BV(q,\delta)$ holds if and only if

$$\limsup_{n\to\infty}\left\{\max_{1\leq k\leq 2^n}\frac{k^{\frac{1}{q(n)}}}{(\sum_{i=1}^{k}\frac{1}{\lambda_i})}\right\} < +\infty. \tag{1}$$

Our result formulated below extends the above theorem of Goginava essentially and furnishes a new and much shorter proof.

Theorem 1. *For $p \in [1,\infty)$ and q and δ sequences satisfying the conditions in Definition 1, the inclusion $\Lambda BV^{(p)} \subset BV(q;\delta)$ holds if and only if*

$$\limsup_{n\to\infty}\left\{\max_{1\leq k\leq \delta(n)}\frac{k^{\frac{1}{q(n)}}}{(\sum_{i=1}^{k}\frac{1}{\lambda_i})^{\frac{1}{p}}}\right\} < +\infty. \tag{2}$$

Before we present a relatively short proof of Theorem 1, we give an example showing that it provides a non-trivial extension of (1) even for $p = 1$. The example – provided by the referee kindly instead of our more complicated one – is obtained by taking $\lambda_n = n$, $q(n) = \sqrt{n}$ and $\delta(n) = 2^{\sqrt{n}}$. With those choices, it follows immediately that the Goginava indicator (1) is infinite while (2) holds.

Proof of Theorem 1. To show that (2) is a *sufficiency* condition for the inclusion $\Lambda BV^{(p)} \subset BV(q;\delta)$, we will prove the inequality

$$V(f,q,\delta) \leq V_{\Lambda BV^p}(f)\sup_n\left\{\max_{1\leq k\leq \delta(n)}\frac{k}{(\sum_{i=1}^{k}1/\lambda_i)^{\frac{q(n)}{p}}}\right\}^{\frac{1}{q(n)}}. \tag{3}$$

This is a consequence of the numerical inequality

$$\sum_{j=1}^{s}x_j^q \leq \left(\sum_{j=1}^{s}x_jy_j\right)^q \max_{1\leq k\leq s}\frac{k}{(\sum_{j=1}^{k}y_j)^q}, \tag{4}$$

which is valid for $q \geq 0$ and

$$\begin{aligned} x_1 \geq x_2 \geq \ldots \geq x_s \geq 0, \\ y_1 \geq y_2 \geq \ldots \geq y_s \geq 0. \end{aligned} \tag{5}$$

In the cases $q \geq 1$ and $0 \leq q < 1$, (4) is a reformulation of [15, Lemma] and [11, Lemma 2.5], respectively.

To prove (3), consider a non-overlapping family $(I_k)_{k=1}^s$ with $\inf |I_k| \geq 1/\delta(n)$. In particular, $s \leq \delta(n)$. Apply (4) with q replaced by $q(n)/p$, $x_j = |f(I_j)|^p$ and $y_j = 1/\lambda_j$, where we reorder the intervals so that (5) holds. This gives

$$\begin{aligned} \left(\sum_{j=1}^{s} \left|f(I_j)\right|^{q(n)}\right)^{\frac{1}{q(n)}} &\leq \left(\sum_{j=1}^{s} \frac{|f(I_j)|^p}{\lambda_j}\right)^{\frac{1}{p}} \max_{1\leq k\leq s} \frac{k^{\frac{1}{q(n)}}}{(\sum_{j=1}^{k} \frac{1}{\lambda_j})^{\frac{1}{p}}} \\ &\leq V_{ABV^p}(f) \max_{1\leq k\leq \delta(n)} \frac{k^{\frac{1}{q(n)}}}{(\sum_{j=1}^{k} \frac{1}{\lambda_j})^{\frac{1}{p}}}. \end{aligned}$$

Taking the supremum over n yields (3).

Necessity. Suppose (2) doesn't hold. Then there is an increasing sequence (n_k) of positive integers such that for all indices k

$$\delta(n_k) \geq 2^{k+2} \tag{6}$$

and

$$\max_{1\leq n\leq \delta(n_k)} \frac{n^{\frac{1}{q(n_k)}}}{(\sum_{i=1}^{n} \frac{1}{\lambda_i})^{\frac{1}{p}}} > 2^{2k+\frac{k+1}{q(1)}}. \tag{7}$$

Let (m_k) be a sequence of positive integers such that

$$1 \leq m_k \leq \delta(n_k), \tag{8}$$

and

$$\max_{1\leq n\leq \delta(n_k)} \frac{n^{\frac{1}{q(n_k)}}}{(\sum_{i=1}^{n} \frac{1}{\lambda_i})^{\frac{1}{p}}} = \frac{m_k^{\frac{1}{q(n_k)}}}{(\sum_{i=1}^{m_k} \frac{1}{\lambda_i})^{\frac{1}{p}}}. \tag{9}$$

Denote

$$\Phi_k := \frac{1}{\sum_{i=1}^{m_k} 1/\lambda_i}.$$

Consider

$$g_k(y) := \begin{cases} 2^{-k}\Phi_k^{1/p}, & y \in [\frac{1}{2^k} + \frac{2j-2}{\delta(n_k)}, \frac{1}{2^k} + \frac{2j-1}{\delta(n_k)});\ 1 \leq j \leq N_k, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$N_k = \min\{m_k, s_k\}, \quad s_k = \max\left\{j \in \mathbb{N}:\ 2j \leq \frac{\delta(n_k)}{2^k} + 1\right\}.$$

Applying the fact that $2(s_k+1) \geq \frac{\delta(n_k)}{2^k}+1$ and (6), we have

$$\frac{2s_k-1}{\delta(n_k)} \geq 2^{-k-1}. \tag{10}$$

The functions g_k have disjoint support and the series $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly to a function g. Thus,

$$\begin{aligned}\|g\|_{\Lambda,p} &\leq \sum_{k=1}^{\infty}\|g_k\|_{\Lambda,p} = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{2N_k}\frac{(2^{-k}\Phi_k^{1/p})^p}{\lambda_j}\right)^{1/p} \leq \sum_{k=1}^{\infty}2^{-k}\left(2\sum_{j=1}^{N_k}\frac{\Phi_k}{\lambda_j}\right)^{1/p} \\ &\leq \sum_{k=1}^{\infty}2^{-k}\left(2\sum_{j=1}^{m_k}\frac{\Phi_k}{\lambda_j}\right)^{1/p} = \sum_{k=1}^{\infty}2^{-k}\left(2\frac{\sum_{j=1}^{m_k}\frac{1}{\lambda_j}}{\sum_{j=1}^{m_k}\frac{1}{\lambda_j}}\right)^{1/p} < +\infty,\end{aligned}$$

that is, $g \in \Lambda BV^{(p)}$.

If $N_k = m_k$, then $2N_k - 1 \geq m_k$, and if $N_k = s_k$, then

$$2N_k - 1 \overset{(10)}{\geq} \frac{\delta(n_k)}{2^{k+1}} \overset{(8)}{\geq} \frac{m_k}{2^{k+1}}.$$

Hence

$$2N_k - 1 \geq \frac{m_k}{2^{k+1}} \quad \text{for all } k. \tag{11}$$

Now, given any positive integer k, all intervals $[\frac{1}{2^k}+\frac{j-1}{\delta(n_k)}, \frac{1}{2^k}+\frac{j}{\delta(n_k)}]$ for $j=1,..,2N_k-1$, have length $\frac{1}{\delta(n_k)}$ and thus

$$\begin{aligned}V(g,q,\delta) &\geq \left(\sum_{j=1}^{2N_k-1}\left|g\left(\frac{1}{2^k}+\frac{j-1}{\delta(n_k)}\right)-g\left(\frac{1}{2^k}+\frac{j}{\delta(n_k)}\right)\right|^{q(n_k)}\right)^{\frac{1}{q(n_k)}} \\ &= \left((2N_k-1)\left(2^{-k}\Phi_k^{\frac{1}{p}}\right)^{q(n_k)}\right)^{\frac{1}{q(n_k)}} = \frac{1}{2^k}\left(\frac{2N_k-1}{(\sum_{i=1}^{m_k}\frac{1}{\lambda_i})^{\frac{q(n_k)}{p}}}\right)^{\frac{1}{q(n_k)}} \\ &\overset{(11)}{\geq} \frac{1}{2^k}\left(\frac{1}{2^{k+1}}\cdot\frac{m_k}{(\sum_{i=1}^{m_k}\frac{1}{\lambda_i})^{\frac{q(n_k)}{p}}}\right)^{\frac{1}{q(n_k)}} \overset{(7),(9)}{\geq} \frac{2^{k+(k+1)/q(1)}}{2^{(k+1)/q(n_k)}} \geq 2^k.\end{aligned}$$

Since k was arbitrary, $V(g,q,\delta)$ must be infinite which shows that $g \notin BV(q;\delta)$. □

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