



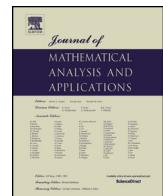
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Inclusions of Waterman–Shiba spaces into generalized Wiener classes



Mahdi Hormozi ^{a,*}, Franciszek Prus-Wiśniowski ^b, Hjalmar Rosengren ^c

^a Department of Mathematical Sciences, Division of Mathematics, University of Gothenburg, Gothenburg 41296, Sweden

^b Institute of Mathematics, Szczecin University, ul. Wielkopolska 15, PL-70-451 Szczecin, Poland

^c Department of Mathematical Sciences, Division of Mathematics, Chalmers University of Technology, Gothenburg 41296, Sweden

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ABSTRACT

The characterization of the inclusion of Waterman–Shiba spaces $\Lambda BV^{(p)}$ into generalized Wiener classes of functions $BV(q; \delta)$ is given. It uses a new and shorter proof and extends an earlier result of U. Goginava.

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Let $\Lambda = (\lambda_i)$ be a Λ -sequence, that is, a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_i} = +\infty$ and let p be a number greater than or equal to 1. A function $f : [0, 1] \rightarrow \mathbb{R}$ is said to be of bounded p - Λ -variation if

$$V(f) := \sup \left(\sum_{i=1}^n \frac{|f(I_i)|^p}{\lambda_i} \right)^{\frac{1}{p}} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping subintervals of $[0, 1]$ and where $f(I_i) := f(\sup I_i) - f(\inf I_i)$ is the change of the function f over the interval I_i . The symbol $\Lambda BV^{(p)}$ denotes the linear space of all functions of bounded p - Λ -variation with domain $[0, 1]$. The Waterman–Shiba space $\Lambda BV^{(p)}$ was introduced in 1980 by M. Shiba [21]. When $p = 1$, $\Lambda BV^{(p)}$ is the well-known Waterman space ΛBV (see e.g. [25] and [26]). Some of the properties and applications of functions of class $\Lambda BV^{(p)}$ were discussed in [8,9,11,12,16–20,22–24]. $\Lambda BV^{(p)}$ equipped with the norm $\|f\|_{\Lambda,p} := |f(0)| + V(f)$ is a Banach space.

* Corresponding author.

E-mail addresses: hormozi@chalmers.se (M. Hormozi), wisniows@univ.szczecin.pl (F. Prus-Wiśniowski), hjalmar@chalmers.se (H. Rosengren).

H. Kita and K. Yoneda introduced a new function space which is a generalization of Wiener classes [14] (see also [13] and [1]). The concept was further extended by T. Akhobadze in [2] who studied many properties of the generalized Wiener classes $BV(q, \delta)$ thoroughly (see [3–7]).

Definition 1. Let $q = (q(n))_{n=1}^\infty$ be an increasing positive sequence and let $\delta = (\delta(n))_{n=1}^\infty$ be an increasing and unbounded positive sequence. We say that a function $f : [0, 1] \rightarrow \mathbb{R}$ belongs to the class $BV(q; \delta)$ if

$$V(f, q; \delta) := \sup_{n \geq 1} \sup_{\{I_k\}} \left\{ \left(\sum_{k=1}^s |f(I_k)|^{q(n)} \right)^{\frac{1}{q(n)}} : \inf_k |I_k| \geq \frac{1}{\delta(n)} \right\} < \infty,$$

where $\{I_k\}_{k=1}^s$ are non-overlapping subintervals of $[0, 1]$.

If $\delta(n)^{1/q(n)}$ is a bounded sequence, then $BV(q; \delta)$ is simply the space of all bounded functions. This follows from the estimate

$$\left(\sum_{k=1}^s |f(I_k)|^{q(n)} \right)^{\frac{1}{q(n)}} \leq 2C s^{\frac{1}{q(n)}} \leq 2C \delta(n)^{\frac{1}{q(n)}}, \quad |f(x)| \leq C.$$

The following statement regarding inclusions of Waterman spaces into generalized Wiener classes has been presented in [10]: if $\lim_{n \rightarrow \infty} q(n) = \infty$ and $\delta(n) = 2^n$ the inclusion $ABV \subset BV(q, \delta)$ holds if and only if

$$\limsup_{n \rightarrow \infty} \left\{ \max_{1 \leq k \leq 2^n} \frac{k^{\frac{1}{q(n)}}}{(\sum_{i=1}^k \frac{1}{\lambda_i})} \right\} < +\infty. \quad (1)$$

Our result formulated below extends the above theorem of Goginava essentially and furnishes a new and much shorter proof.

Theorem 1. For $p \in [1, \infty)$ and q and δ sequences satisfying the conditions in Definition 1, the inclusion $ABV^{(p)} \subset BV(q; \delta)$ holds if and only if

$$\limsup_{n \rightarrow \infty} \left\{ \max_{1 \leq k \leq \delta(n)} \frac{k^{\frac{1}{q(n)}}}{(\sum_{i=1}^k \frac{1}{\lambda_i})^{\frac{1}{p}}} \right\} < +\infty. \quad (2)$$

Before we present a relatively short proof of Theorem 1, we give an example showing that it provides a non-trivial extension of (1) even for $p = 1$. The example – provided by the referee kindly instead of our more complicated one – is obtained by taking $\lambda_n = n$, $q(n) = \sqrt{n}$ and $\delta(n) = 2\sqrt{n}$. With those choices, it follows immediately that the Goginava indicator (1) is infinite while (2) holds.

Proof of Theorem 1. To show that (2) is a *sufficiency* condition for the inclusion $ABV^{(p)} \subset BV(q; \delta)$, we will prove the inequality

$$V(f, q, \delta) \leq V_{ABV^p}(f) \sup_n \left\{ \max_{1 \leq k \leq \delta(n)} \frac{k}{(\sum_{i=1}^k 1/\lambda_i)^{\frac{q(n)}{p}}} \right\}^{\frac{1}{q(n)}}. \quad (3)$$

This is a consequence of the numerical inequality

$$\sum_{j=1}^s x_j^q \leq \left(\sum_{j=1}^s x_j y_j \right)^q \max_{1 \leq k \leq s} \frac{k}{(\sum_{j=1}^k y_j)^q}, \quad (4)$$

which is valid for $q \geq 0$ and

$$\begin{aligned} x_1 &\geq x_2 \geq \dots \geq x_s \geq 0, \\ y_1 &\geq y_2 \geq \dots \geq y_s \geq 0. \end{aligned} \quad (5)$$

In the cases $q \geq 1$ and $0 \leq q < 1$, (4) is a reformulation of [15, Lemma] and [11, Lemma 2.5], respectively.

To prove (3), consider a non-overlapping family $(I_k)_{k=1}^s$ with $\inf |I_k| \geq 1/\delta(n)$. In particular, $s \leq \delta(n)$. Apply (4) with q replaced by $q(n)/p$, $x_j = |f(I_j)|^p$ and $y_j = 1/\lambda_j$, where we reorder the intervals so that (5) holds. This gives

$$\begin{aligned} \left(\sum_{j=1}^s |f(I_j)|^{q(n)} \right)^{\frac{1}{q(n)}} &\leq \left(\sum_{j=1}^s \frac{|f(I_j)|^p}{\lambda_j} \right)^{\frac{1}{p}} \max_{1 \leq k \leq s} \frac{k^{\frac{1}{q(n)}}}{(\sum_{j=1}^k \frac{1}{\lambda_j})^{\frac{1}{p}}} \\ &\leq V_{ABV^p}(f) \max_{1 \leq k \leq \delta(n)} \frac{k^{\frac{1}{q(n)}}}{(\sum_{j=1}^k \frac{1}{\lambda_j})^{\frac{1}{p}}}. \end{aligned}$$

Taking the supremum over n yields (3).

Necessity. Suppose (2) doesn't hold. Then there is an increasing sequence (n_k) of positive integers such that for all indices k

$$\delta(n_k) \geq 2^{k+2} \quad (6)$$

and

$$\max_{1 \leq n \leq \delta(n_k)} \frac{n^{\frac{1}{q(n_k)}}}{(\sum_{i=1}^n \frac{1}{\lambda_i})^{\frac{1}{p}}} > 2^{2k+\frac{k+1}{q(1)}}. \quad (7)$$

Let (m_k) be a sequence of positive integers such that

$$1 \leq m_k \leq \delta(n_k), \quad (8)$$

and

$$\max_{1 \leq n \leq \delta(n_k)} \frac{n^{\frac{1}{q(n_k)}}}{(\sum_{i=1}^n \frac{1}{\lambda_i})^{\frac{1}{p}}} = \frac{m_k^{\frac{1}{q(n_k)}}}{(\sum_{i=1}^{m_k} \frac{1}{\lambda_i})^{\frac{1}{p}}}. \quad (9)$$

Denote

$$\varPhi_k := \frac{1}{\sum_{i=1}^{m_k} 1/\lambda_i}.$$

Consider

$$g_k(y) := \begin{cases} 2^{-k} \varPhi_k^{1/p}, & y \in [\frac{1}{2^k} + \frac{2j-2}{\delta(n_k)}, \frac{1}{2^k} + \frac{2j-1}{\delta(n_k)}); 1 \leq j \leq N_k, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$N_k = \min\{m_k, s_k\}, \quad s_k = \max \left\{ j \in \mathbb{N}: 2j \leq \frac{\delta(n_k)}{2^k} + 1 \right\}.$$

Applying the fact that $2(s_k + 1) \geq \frac{\delta(n_k)}{2^k} + 1$ and (6), we have

$$\frac{2s_k - 1}{\delta(n_k)} \geq 2^{-k-1}. \quad (10)$$

The functions g_k have disjoint support and the series $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly to a function g . Thus,

$$\begin{aligned} \|g\|_{A,p} &\leq \sum_{k=1}^{\infty} \|g_k\|_{A,p} = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{2N_k} \frac{(2^{-k}\varPhi_k^{1/p})^p}{\lambda_j} \right)^{1/p} \leq \sum_{k=1}^{\infty} 2^{-k} \left(2 \sum_{j=1}^{N_k} \frac{\varPhi_k}{\lambda_j} \right)^{1/p} \\ &\leq \sum_{k=1}^{\infty} 2^{-k} \left(2 \sum_{j=1}^{m_k} \frac{\varPhi_k}{\lambda_j} \right)^{1/p} = \sum_{k=1}^{\infty} 2^{-k} \left(2 \frac{\sum_{j=1}^{m_k} \frac{1}{\lambda_j}}{\sum_{j=1}^{m_k} \frac{1}{\lambda_j}} \right)^{1/p} < +\infty, \end{aligned}$$

that is, $g \in ABV^{(p)}$.

If $N_k = m_k$, then $2N_k - 1 \geq m_k$, and if $N_k = s_k$, then

$$2N_k - 1 \stackrel{(10)}{\geq} \frac{\delta(n_k)}{2^{k+1}} \stackrel{(8)}{\geq} \frac{m_k}{2^{k+1}}.$$

Hence

$$2N_k - 1 \geq \frac{m_k}{2^{k+1}} \quad \text{for all } k. \quad (11)$$

Now, given any positive integer k , all intervals $[\frac{1}{2^k} + \frac{j-1}{\delta(n_k)}, \frac{1}{2^k} + \frac{j}{\delta(n_k)}]$ for $j = 1, \dots, 2N_k - 1$, have length $\frac{1}{\delta(n_k)}$ and thus

$$\begin{aligned} V(g, q, \delta) &\geq \left(\sum_{j=1}^{2N_k-1} \left| g\left(\frac{1}{2^k} + \frac{j-1}{\delta(n_k)}\right) - g\left(\frac{1}{2^k} + \frac{j}{\delta(n_k)}\right) \right|^{q(n_k)} \right)^{\frac{1}{q(n_k)}} \\ &= ((2N_k - 1)(2^{-k}\varPhi_k^{\frac{1}{p}})^{q(n_k)})^{\frac{1}{q(n_k)}} = \frac{1}{2^k} \left(\frac{2N_k - 1}{(\sum_{i=1}^{m_k} \frac{1}{\lambda_i})^{\frac{q(n_k)}{p}}} \right)^{\frac{1}{q(n_k)}} \\ &\stackrel{(11)}{\geq} \frac{1}{2^k} \left(\frac{1}{2^{k+1}} \cdot \frac{m_k}{(\sum_{i=1}^{m_k} \frac{1}{\lambda_i})^{\frac{q(n_k)}{p}}} \right)^{\frac{1}{q(n_k)}} \stackrel{(7), (9)}{\geq} \frac{2^{k+(k+1)/q(1)}}{2^{(k+1)/q(n_k)}} \geq 2^k. \end{aligned}$$

Since k was arbitrary, $V(g, q, \delta)$ must be infinite which shows that $g \notin BV(q; \delta)$. \square

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