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Inclusions of Waterman–Shiba spaces into generalized Wiener classes

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ABSTRACT

The characterization of the inclusion of Waterman–Shiba spaces $\Lambda BV^p$ into generalized Wiener classes of functions $BV(q; \delta)$ is given. It uses a new and shorter proof and extends an earlier result of U. Goginava.

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Let $A = (\lambda_i)$ be a $A$-sequence, that is, a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_i} = +\infty$ and let $p$ be a number greater than or equal to 1. A function $f : [0, 1] \rightarrow \mathbb{R}$ is said to be of bounded $p$-$A$-variation if

$$V(f) := \sup \left( \sum_{i=1}^{n} \frac{|f(I_i)|^p}{\lambda_i} \right)^{\frac{1}{p}} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^{n}$ of nonoverlapping subintervals of $[0, 1]$ and where $f(I_i) := f(\text{sup } I_i) - f(\text{inf } I_i)$ is the change of the function $f$ over the interval $I_i$. The symbol $ABV^p$ denotes the linear space of all functions of bounded $p$-$A$-variation with domain $[0, 1]$. The Waterman–Shiba space $ABV^p$ was introduced in 1980 by M. Shiba [21]. When $p = 1$, $ABV^p$ is the well-known Waterman space $ABV$ (see e.g. [25] and [26]). Some of the properties and applications of functions of class $ABV^p$ were discussed in [8,9,11,12,16–20,22–24]. $ABV^p$ equipped with the norm $\|f\|_{A,p} := |f(0)| + V(f)$ is a Banach space.

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H. Kita and K. Yoneda introduced a new function space which is a generalization of Wiener classes \([14]\) (see also \([13]\) and \([1]\)). The concept was further extended by T. Akhobadze in \([2]\) who studied many properties of the generalized Wiener classes \(BV(q, \delta)\) thoroughly (see \([3–7]\)).

**Definition 1.** Let \(q = (q(n))_{n=1}^{\infty}\) be an increasing positive sequence and let \(\delta = (\delta(n))_{n=1}^{\infty}\) be an increasing and unbounded positive sequence. We say that a function \(f : [0, 1] \rightarrow \mathbb{R}\) belongs to the class \(BV(q; \delta)\) if

\[
V(f, q; \delta) := \sup_{n \geq 1} \sup_{\{I_k\}} \left\{ \left( \sum_{k=1}^{s} |f(I_k)|^{q(n)} \right)^{\frac{1}{q(n)}} : \inf_k |I_k| \geq \frac{1}{n \delta(n)} \right\} < \infty,
\]

where \(\{I_k\}_{k=1}^{s}\) are non-overlapping subintervals of \([0, 1]\).

If \(\delta(n)^{1/q(n)}\) is a bounded sequence, then \(BV(q; \delta)\) is simply the space of all bounded functions. This follows from the estimate

\[
\left( \sum_{k=1}^{s} |f(I_k)|^{q(n)} \right)^{\frac{1}{q(n)}} \leq 2Cs^{\frac{1}{q(n)}} \leq 2C\delta(n)^{\frac{1}{q(n)}}, \quad |f(x)| \leq C.
\]

The following statement regarding inclusions of Waterman spaces into generalized Wiener classes has been presented in \([10]\): if \(\lim_{n \to \infty} q(n) = \infty\) and \(\delta(n) = 2^n\) the inclusion \(ABV \subset BV(q, \delta)\) holds if and only if

\[
\limsup_{n \to \infty} \left\{ \max_{1 \leq k \leq 2^n} \frac{k^{\frac{1}{q(n)}}}{\left( \sum_{i=1}^{k} \frac{1}{\lambda_i} \right)^{\frac{1}{q(n)}}} \right\} < +\infty. \quad (1)
\]

Our result formulated below extends the above theorem of Goginava essentially and furnishes a new and much shorter proof.

**Theorem 1.** For \(p \in [1, \infty)\) and \(q\) and \(\delta\) sequences satisfying the conditions in Definition 1, the inclusion \(ABV^{(p)} \subset BV(q; \delta)\) holds if and only if

\[
\limsup_{n \to \infty} \left\{ \max_{1 \leq k \leq \delta(n)} \frac{k^{\frac{1}{q(n)}}}{\left( \sum_{i=1}^{k} \frac{1}{\lambda_i} \right)^{\frac{1}{q(n)}}} \right\} < +\infty. \quad (2)
\]

Before we present a relatively short proof of Theorem 1, we give an example showing that it provides a non-trivial extension of \((1)\) even for \(p = 1\). The example – provided by the referee kindly instead of our more complicated one – is obtained by taking \(\lambda_n = n\), \(q(n) = \sqrt{n}\) and \(\delta(n) = 2\sqrt{n}\). With those choices, it follows immediately that the Goginava indicator \((1)\) is infinite while \((2)\) holds.

**Proof of Theorem 1.** To show that \((2)\) is a *sufficiency* condition for the inclusion \(ABV^{(p)} \subset BV(q; \delta)\), we will prove the inequality

\[
V(f, q, \delta) \leq V_{ABV^{(p)}}(f) \sup_n \left\{ \max_{1 \leq k \leq \delta(n)} \frac{k}{\left( \sum_{i=1}^{k} 1/\lambda_i \right)^{\frac{2q(n)}{p}}} \right\}^{\frac{1}{q(n)}}. \quad (3)
\]

This is a consequence of the numerical inequality

\[
\sum_{j=1}^{s} x_j^q \leq \left( \sum_{j=1}^{s} x_j y_j \right)^q \max_{1 \leq k \leq s} \frac{k}{\left( \sum_{j=1}^{k} y_j \right)^q}, \quad (4)
\]
which is valid for $q \geq 0$ and

$$
x_1 \geq x_2 \geq \ldots \geq x_s \geq 0, \\
y_1 \geq y_2 \geq \ldots \geq y_s \geq 0.
$$

(5)

In the cases $q \geq 1$ and $0 \leq q < 1$, (4) is a reformulation of [15, Lemma] and [11, Lemma 2.5], respectively.

To prove (3), consider a non-overlapping family $(I_k)_{k=1}^s$ with $\inf |I_k| \geq 1/\delta(n)$. In particular, $s \leq \delta(n)$.

Apply (4) with $q$ replaced by $q(n)/p$, $x_j = |f(I_j)|^p$ and $y_j = 1/\lambda_j$, where we reorder the intervals so that (5) holds. This gives

$$
\left(\sum_{j=1}^s |f(I_j)|^{q(n)}\right)^{\frac{1}{p(n)}} \leq \left(\sum_{j=1}^s \frac{|f(I_j)|^p}{\lambda_j}\right)^{\frac{1}{p}} \max_{1 \leq k \leq s} \left(\frac{k^{\frac{1}{p(n)}}}{\sum_{j=1}^k \frac{1}{\lambda_j}}\right)^{\frac{1}{p}} \\
\leq V_{ABV^p}(f) \max_{1 \leq k \leq \delta(n)} \left(\frac{k^{\frac{1}{p(n)}}}{\sum_{j=1}^k \frac{1}{\lambda_j}}\right)^{\frac{1}{p}}.
$$

Taking the supremum over $n$ yields (3).

**Necessity.** Suppose (2) doesn’t hold. Then there is an increasing sequence $(n_k)$ of positive integers such that for all indices $k$

$$
\delta(n_k) \geq 2^{k+2}
$$

(6)

and

$$
\max_{1 \leq n \leq \delta(n_k)} \frac{n^{\frac{1}{p(n)}}}{\left(\sum_{i=1}^n \frac{1}{\lambda_i}\right)^{\frac{1}{p}}} > 2^{2k + \frac{k+1}{p(n)}}.
$$

(7)

Let $(m_k)$ be a sequence of positive integers such that

$$
1 \leq m_k \leq \delta(n_k),
$$

(8)

and

$$
\max_{1 \leq n \leq \delta(n_k)} \frac{n^{\frac{1}{q(n_k)}}}{\left(\sum_{i=1}^n \frac{1}{\lambda_i}\right)^{\frac{1}{p}}} = \frac{m_k^{\frac{1}{q(n_k)}}}{\left(\sum_{i=1}^{m_k} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}}.
$$

(9)

Denote

$$
\Phi_k := \frac{1}{\sum_{i=1}^{m_k} \frac{1}{\lambda_i}}.
$$

Consider

$$
g_k(y) := \begin{cases} 
2^{-k} \Phi_k^{1/p}, & y \in \left[\frac{1}{2^k}, \frac{2j-2}{\delta(n_k)}\right], \frac{1}{2^k} + \frac{2j-1}{\delta(n_k)}; \\
0, & \text{otherwise},
\end{cases} \\
1 \leq j \leq N_k,
$$

where

$$
N_k = \min\{m_k, s_k\}, \quad s_k = \max\{j \in \mathbb{N}: 2j \leq \frac{\delta(n_k)}{2^k} + 1\}.
$$
Applying the fact that \(2(s_k + 1) \geq \frac{\delta(n_k)}{2^k} + 1\) and \((6)\), we have

\[
\frac{2s_k - 1}{\delta(n_k)} \geq 2^{-k-1}.
\]

(10)

The functions \(g_k\) have disjoint support and the series \(\sum_{k=1}^{\infty} g_k(x)\) converges uniformly to a function \(g\). Thus,

\[
\|g\|_{A,p} \leq \sum_{k=1}^{\infty} \|g_k\|_{A,p} = \sum_{k=1}^{\infty} \left( \sum_{j=1}^{2N_k} \left( \frac{2-k}{\lambda_j} \right)^{1/p} \right)^{1/p} \leq \sum_{k=1}^{\infty} 2^{-k} \left( \frac{N_k}{\sum_{j=1}^{\infty} \lambda_j} \right)^{1/p}
\]

\[
\leq \sum_{k=1}^{\infty} 2^{-k} \left( \frac{m_k}{\sum_{j=1}^{\infty} \lambda_j} \right)^{1/p} = \sum_{k=1}^{\infty} 2^{-k} \left( \frac{\sum_{j=1}^{m_k} \frac{1}{\lambda_j}}{\sum_{j=1}^{\infty} \lambda_j} \right)^{1/p} < +\infty,
\]

that is, \(g \in ABV^{(p)}\).

If \(N_k = m_k\), then \(2N_k - 1 \geq m_k\), and if \(N_k = s_k\), then

\[
2N_k - 1 \geq \frac{\delta(n_k)}{2^{k+1}} \geq \frac{m_k}{2^{k+1}}.
\]

Hence

\[
2N_k - 1 \geq \frac{m_k}{2^{k+1}} \quad \text{for all } k.
\]

(11)

Now, given any positive integer \(k\), all intervals \([\frac{1}{2^k} + \frac{j-1}{\delta(n_k)}, \frac{1}{2^k} + \frac{j}{\delta(n_k)}]\) for \(j = 1, \ldots, 2N_k - 1\), have length \(\frac{1}{\delta(n_k)}\) and thus

\[
V(g, q, \delta) \geq \left( \sum_{j=1}^{2N_k-1} \left| g\left( \frac{1}{2^k} + \frac{j-1}{\delta(n_k)} \right) - g\left( \frac{1}{2^k} + \frac{j}{\delta(n_k)} \right) \right| q(n_k) \right) \frac{1}{\varphi(n_k)}^{1/q(n_k)}
\]

\[
= \left( (2N_k - 1)(2-k) \frac{\varphi(n_k)}{\delta(n_k)} \right) \frac{1}{\varphi(n_k)}^{1/q(n_k)} = \frac{1}{2^k} \left( \frac{2N_k - 1}{\sum_{i=1}^{m_k} \frac{1}{\lambda_i}} \right) \frac{1}{\varphi(n_k)}^{1/q(n_k)}
\]

\[
\geq \frac{1}{2^k} \left( \frac{1}{2^{k+1}} \cdot \frac{m_k}{\left( \sum_{i=1}^{m_k} \frac{1}{\lambda_i} \right)^{\frac{1}{q(n_k)}}} \right) \frac{1}{\varphi(n_k)}^{1/q(n_k)} \geq \frac{2^{k+(k+1)/q(n_k)}}{2^{(k+1)/q(n_k)}\varphi(n_k)} \geq 2^k.
\]

(11)

Since \(k\) was arbitrary, \(V(g, q, \delta)\) must be infinite which shows that \(g \notin BV(q; \delta)\). \(\square\)

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**References**


