Chalmers Tekniska Högskola
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DIACOPTICS AND CODIACOPTICS

Two dual mixed methods for analysing elastic structures

by

Nils-Erik Wiberg

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Göteborg 1967

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Synopsis

Two direct methods for analysing elastic structures are presented. The methods are called diacoptics and codiacoptics and they are each other's duals. The prescribed load may consist of external forces, temperature changes and misfits. Forces and deformations are used simultaneously as unknowns in the fundamental equations. The well-known pure deformation and force methods are special cases of the mixed methods diacoptics and codiacoptics, respectively. The number of unknowns in the fundamental equations of a mixed method may be reduced under the number required in each of the pure methods. The solution of the fundamental equations is left factorized which offers computational advantages and facilitates modifications of the solution required by modifications of the given structure.

Göteborg 1967

PREFACE

The aim of the investigation reported here is to give methods for analyzing elastic structures in which the fundamental equations are established with a more expedient choice of unknowns than in the pure force and deformation methods. The aim also is to obtain a solution in factorized form which is easy to modify when the structure is modified and which is suited to programming for a digital computer.

My investigation into the subject started in connection with my "examination work" in 1964. Many of the ideas underlying the present paper have their origin in my study of the solution of corresponding problems in electrical engineering. The American engineer Gabriel Kron is the most important contributor in this field.

The research was carried out during 1964 to 1967 at the Department of Structural Mechanics at Chalmers University of Technology in Gothenburg. I have been Assistant to the Head of the Department, Professor Sven Olof Asplund, Tekn. D., and I am most grateful to him for his support.

I wish to express my deep gratitude to Laborator Alf Samuelsson, Tekn. D., for introducing me into Kron's work and for his stimulating interest and proposed improvements. I also want to thank Universitetslektor Bengt Å. Åkesson, Tekn. D., for reading the final manuscript, and Miss Lisbeth Renhult and Miss Lisbeth Trygg for the typing.

Gothenburg, June 1967
Nils-Erik Wiberg

CONTENTS

INTRODUCTION

The presented solution methods for linearly elastic structures

Two direct methods, diacoptics and its dual codiacoptics, for analyzing linearly elastic structures are presented. They are direct in the meaning that the solution is not given by some iterative procedure. The methods mentioned are mixed in the meaning that the used variables are of two kinds, forces and displacements. The methods contain the well-known force and displacement methods as special cases. The information from the structural network is believed to be taken care of in a better way than in the pure force- and displacement methods, because the number of variables used in the fundamental equations may be reduced and the inverse be left factorized, which gives possibilities for modifications in the solution. The methods are particularly well suited for calculation of large structures which are an assembly of identical structural parts, substructures, because the calculation of the substructures can be made once for all. These part solutions can then be used for all combinations of the substructures. Therefore the methods are well adapted for the calculation of structures built up from prefabricated elements which usually occur in few types.

Development in structural analysis

Increased size and complexity of elastic structures and requirements as to more precise design and performance have caused a demand for more rigorous methods of analysis. Analysis is made more mechanical so that the engineer can concentrate on the design. Recently, the digital computer has become an indispensable tool of the structural engineer and it has entirely altered the approach to structural analysis. Furthermore, many recent developments are derived from pertinent domains in mathematics.

Solution methods in structural analysis

The fundamental problem in the analysis of elastic structures is the determination of the distribution of internal forces and external displacements under prescribed loads and constraints. For certain types of structures this problem can be solved by a direct solution of the system of differential equations with boundary conditions describing the elastic behaviour of the structure under prescribed load. In other cases numerical methods, for example difference calculus, must be used. It can be used either for only the solution of the differential equations, or for only the boundary conditions, or for both of them.

In the method which will be reported here, the structure is idealized into an assembly of discrete structural elements for which the differential equation has been solved, by an exact or approximate method. This solution gives the connection between forces and deformations at the boundaries of the elements. The complete solution is then obtained by combining these individual forces and deformations at the element boundaries in a manner which satisfies the force equilibrium and displacement compatibility at the boundaries of these elements. The problem is then transferred to a problem of algebra. In this paper the behaviour of the structural element will not be treated but only the interconnection of the elements.

The force and displacement methods

Methods based on discrete element idealization have been used extensively in the recent years for the analysis of structures. Such methods may be classified broadly into two groups: Displacement methods (stiffness methods) in which geometrically compatible states in individual elements are combined to give equilibrium, and force methods (flexibility methods) in which equilibrium states in individual elements are combined to give geometrical compatibility. The optimal choice of method of analysis depends mainly on the type of the considered structure.

Recently, many authors, Samuelsson [1], Fenves and Branin [2] and Spillers [3], have studied the structural network problem and have given network formulations of the force and displacement methods.

The structural network and the equations derived from it

The physical problem is described by a fundamental equation system. By variable transformations the fundamental equation system can be transformed to an equation system which may contain fewer variables, and which may be easier to solve. Such a suitable variable transformation is often very difficult to find. It is much easier to study the physical problem itself to find the most suitable fundamental equations.

The pure displacement or force method will not always give the simplest solution. The information of the structural network can often be utilized in a better way, in the meaning that the number of variables can be reduced, by mixing the force and displacement variables.

The American electrical engineer G. Kron says: "Give me not only the equations, but the model itself, from which the equations have been derived and I can solve the problem more satisfactorily". This means that the equations do not fully represent the model from which they have been derived. The model contains more information than the equations.

The idea of tearing and interconnecting by G. Kron

G. Kron has introduced a very useful method for solving large physical problems which he calls diacoptics. Most of his work is based on the idea of "tearing and interconnecting". This consists in removing "tearing branches" (members) from the structure in such a manner that the original structure is broken up in a set of independent wellbehaved substructures. They should be wellbehaved in the meaning that they should be grounded so that their stiffness matrices become nonsingular. The substructures contain all the joints (nodes) of the original structure. First the substructures are calculated separately by the displacement method and then they are interconnected by use of the tearing branches to a so called intersection network by the force method and then the grounds must be released. The number of variables is here increased compared with the usual displacement method by the number of the force variables in the intersection network and the variables of the temporary grounds, but other computational advantages are gained, see Kron [4], [5], [6], [7], Roth [8], [9], as seen in a later chapter. Kron himself has published solutions to electrical networks but also to a couple of structural problems in the way mentioned above.

The same method for calculating elastic structures has been used by Spillers [3], Fenves [10] and Kjellberg-Wiberg [11].

The diacoptical and codiacoptical methods

The number of variables is increased when Kron's method, diacoptics, is used in the original manner. An improvement of the method which may result in a decrease of the number of variables is possible, see papers on electrical networks by Weinzweig [12], Amari [13].

This improvement is here adapted to elastic structures and gives the solution in a mixed form by using both force and displacement variables which can be chosen quite freely. By special choices the number of variables may be reduced below the number used in displacement and force methods. In Kron's original method the substructures are solved with displacement variables, and they are interconnected by beams by use of force variables. In the improvement of the method, the substructures can be interconnected by an arbitrary structure. By use of self-equilibrating forces for nongrounded substructures, an improvement is gained compared with Kron's use of temporary grounds, which must be released, in order to get the real force distribution. There is also a dual method, codiacoptics, in which the substructures are solved by force variables and the intersection network is solved by displacement variables. The usual force and displacement methods are two special cases of diacoptics and its dual codiacoptics. A mixed method in structural analysis has earlier been presented by Asplund [14].

Here the exposition is restricted to frames, but by a simple generalization the methods can be used for structures composed of more general members as plate and shell members.

TOPOLOGICAL MODEL OF THE FRAME STRUCTURE

Graphs, branches, nodes, loops

A topological graph or short graph X, Fig. 1, see Lefschetz [15], is a geometrical representation of a finite set $\{x\}$ of elements x^0 or x_0 and of oriented elements x^1 or x_1, with a one-one correspondence to an oriented open line element. The numbers 0 and 1 denote the dimension. The dimension of x is denoted dim x or by an index as x^p, $p = 0, 1$. Between elements of different dimensions an incidence number is defined. The incidence number of x_j^1 and x_i^0 is denoted by $[x_j^1 : x_i^0]$ or E_{ij}.

The 0-dimensional elements x^0, x_0 are called nodes and the 1-dimensional elements x^1, x_1 are called branches.

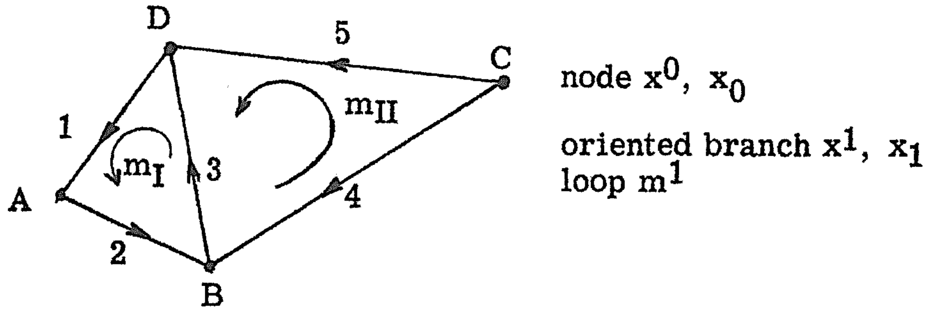


Fig. 1. Graph X

The orientation of the branches is given by defining the first and second ends of the branch and is visualized by an arrow, see Fig. 2. The choice of orientation is trivial.

Fig. 2. Oriented branch

A branch x_j^1 is said to be positively incident to a node x_i^0 if it begins there, and negatively incident if it ends there, see Fig. 2. The incident number gets the respective values

(1) $$[x_j^1 : x_i^0] = E_{ij} = 1, \ -1 \text{ or } 0$$

In the examination of the graph oriented closed branch successions of nodes and branches called loops, denoted by m^1 play an important role. The orientation of a loop is defined by a cyclic order of the nodes.

The orientation of a loop can also be chosen arbitrarily and is shown by a curved arrow.

Incidence matrices

The incidence relation for the graph given by the incidence numbers (1) can be represented by a matrix E which contains elements

(2) $$E_{ij} = +1,\ -1,\ \text{or } 0$$

when branch x_j^1 is positive, negative or not incident to the node x_i^0.

The incidence relation between a loop and its branches can be given by an incidence matrix Z with elements

(3) $$Z_{ij} = +1,\ -1,\ \text{or } 0$$

when the branch x_i^1 is positively, negatively or not included in the loop m_j. For the graph, Fig. 1, the following incidence matrices are defined

(4) $$E = \begin{array}{c} \text{node} \\ \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{branch} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} -1 & 1 & & & \\ & -1 & 1 & -1 & \\ & & & 1 & 1 \\ 1 & & -1 & & -1 \end{array}\right] \end{array}, \quad Z = \begin{array}{c} \text{branch} \\ \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \text{loop} \\ \begin{array}{cc} m_I & m_{II} \end{array} \\ \left[\begin{array}{cc} 1 & \\ 1 & \\ 1 & -1 \\ & -1 \\ & -1 \end{array}\right] \end{array}$$

Open and closed subgraphs

Hitherto the graph has been studied as a whole, but now parts of it will be examined.

Define as an <u>open graph</u>, see Fig. 3, any subgraph U of X such that if any element $x_i^0 \in U$ and $E_{ij} \neq 0$ it implies that $x_j^1 \in U$ or short

(5) $$U = \{x_i^0 \in X\} \cup \{x_j^1 \in X \mid E_{ij} \neq 0\}$$

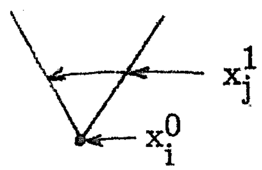


Fig. 3. Open graph

Define as a <u>closed graph</u>, see Fig. 4, any subgraph V of X such that if $x_j^1 \in V$ and $E_{ij} \neq 0$ it implies that $x_i^0 \in V$ or short

(6) $$V = \{x_j^1 \in X\} \cup \{x_i^0 \in X \mid E_{ij} \neq 0\}$$

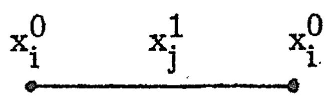


Fig. 4. Closed graph

Stars, closures, boundaries

There are three important sets associated with any element $x \in X$.

(7) The star of $x_i^0 = St\,x_i^0 = x_i^0 \cup \{x_j^1 \mid E_{ij} \neq 0\}$

(8) The closure of $x_j^1 = Cl\,x_j^1 = x_j^1 \cup \{x_i^0 \mid E_{ij} \neq 0\}$

(9) The boundary of $x_j^1 = B\,x_j^1 = Cl\,x_j^1 - x_j^1$

The notion star, closure and boundary can be generalized as follows. Let $Y = \{x^0\}$ be any subset of nodes of X, then the star of Y, written St Y, is the union of the stars of all elements x^0, see Fig. 5. Thus

(10) $St\{x^0\} = \{St\,x^0\}$

In a similar way, see Fig. 6,

(11) $Cl\{x^1\} = \{Cl\,x^1\}$

The boundary of an open subgraph Y is

(12) $BY = Cl\,Y - Y$

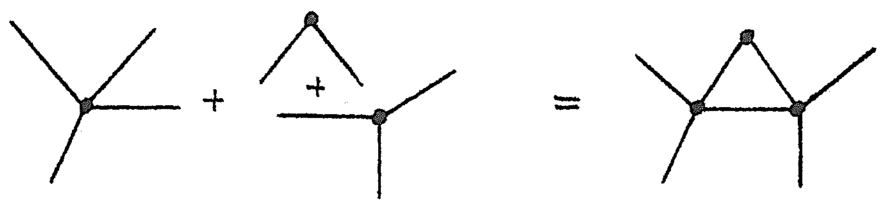


Fig. 5. $St\{x^0\} = \{St\,x^0\}$

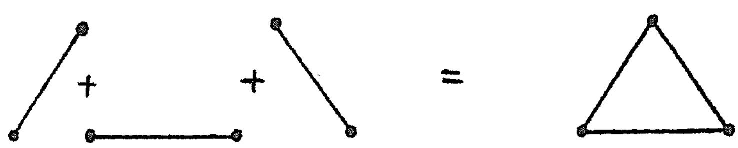


Fig. 6. $Cl\{x^1\} = \{Cl\,x^1\}$

A subgraph Y of X is said to be open whenever

(13) $St\,Y = Y$

and closed whenever

(14) $Cl\,Y = Y$

If one of the graphs Y and the complement of Y in X denoted X - Y is an open subgraph, the other is a closed subgraph.

DEFORMATIONS AND FORCES

Chains and cochains

The graph X is made use of in the following investigation of the frame structure. Forces and deformations are associated to the graph, see Samuelsson [1] and [16].

The load on a member $x_j^1 \in X$ is defined as a vector N_j of a vector space V^6 over the field R of real numbers, because any load can be obtained by a linear combination of three independent forces and three independent moments. (In special cases, a vector space of lower dimension than 6 will suffice.) As member loads we choose the forces and moments at the positive end of the oriented member, see Fig. 2. The state of internal forces can be represented by help of functions c^1 called chains:

(15) $$c^1 : \{x^1\} \rightarrow V^6$$

which are defined as

(16) $$c^1 = x^1 N$$

where x^1 is a row matrix of the basis elements x_j^1 for the branches and N is a column matrix of the α_1 "associated" member loads N_j. The functions c^1 are now considered as vectors of a vector space C of dimension $6\,\alpha_1$. Such a vector space is called a chain group.

In the same way the deformation of a member $x_1 \in X$ can be represented by functions c_1 called cochains. The member deformation is uniquely obtained by a linear combination of 6 independent deformation components:

(17) $$c_1 : \{x_1\} \rightarrow V^6$$

which are defined as

(18) $$c_1 = x_1 n$$

where x_1 is a row matrix of the basis elements x_{1j} for the branches and n is a column matrix of the α_1 member deformations n_j.

The cochains c_1 considered as vectors form a vector space C of dimension $6\,\alpha_1$ called a cochain group.

The load and deformations at one node x_j^0, $x_{0j} \in X$, can be obtained by linearly independent components P_j and p_j and are represented by chains and cochains

(19) $$c^0 : \{x^0\} \rightarrow V^6$$

(20) $$c_0 : \{x_0\} \rightarrow V^6$$

which are defined by

(21) $$c^0 = x^0 P$$

where x^0 is a row matrix of basis elements x_j^0 and P is a column matrix of the α_0 "associated" node loads P_j, and

(22) $\qquad c_0 = x_0 p$

where x_0 is a row matrix of the basis elements x_{0j} and p is a column matrix of the α_0 associated node deformations p_j.

The chains c^0, c_0 considered as vectors form chain groups C^0, C_0 of dimension $6\alpha_0$, where α_0 is the number of nodes in X.

Chain-boundaries and 1-cycles

The boundary of a closed branch x_j^1, see Fig. 7, is defined by

(23) $$\partial x_j^1 = x_2^0 - x_1^0 = x_2^0 E_{2j} + x_1^0 E_{1j} = \sum_{i=1}^{2} x_i^0 E_{ij} = x^0 E_{.j}$$

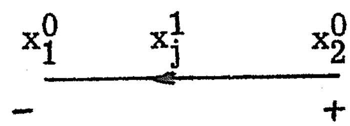


Fig. 7. The boundary of x_j^1

Now the chain-boundary or merely the boundary of a 1-chain, see (16), by use of (23) is defined as a 0-chain

(24) $\qquad \partial c^1 = \partial x^1 N = x^0 EN$

A chain c^1 which satisfies

(25) $\qquad \partial c^1 = 0$

is said to be a 1-cycle on X or short a cycle and is denoted by h^1.

Continuity conditions for forces

All combinations of vectors of C^1, C^0 and C_1, C_0 are not physically realizable but only those which satisfy defined continuity conditions, see Samuelsson [1].

The relation between the chains in C^1 and C^0, is formulated by a linear transformation

(26) $\partial : C^1 \rightarrow C^0$

called the boundary operator, which is defined in (24) as

(27) $\partial c^1 = x^0 EN = x^0 P = c^0$

We observe that the member forces N_j must be written relative the same coordinate system with a common origin.

It follows from (27) that

(28) $EN = P$

Of all elements of C^0 only those are of interest which satisfy 6 independent conditions of statical compatibility for the structure as a whole. These elements constitute a subspace B^0. Any element of B^0, called structure load, can be prescribed if the structure is stable. For a stable structure holds

(29) $\mathrm{Im}\,\partial = B^0$

(30) $\dim B^0 = 6\alpha_0 - 6 = 6(\alpha_0 - 1)$

A set c^1 of inner forces is an element of $\mathrm{Ker}\,\partial = H^1$, with $\dim H^1 = \mu$, if $\partial c^1 = 0$. These elements, called cycle forces are denoted h^1. The α_1 member loads N_h in a frame with zero structure load can be expressed as a linear combination of μ independent loads R_j. The basis of the μ cycles can be chosen such that it can be split into sets of oriented loops m^1. Thus we write an element $h^1 \epsilon H$ as

(31) $h^1 = m^1 R$

where

(32) $m^1 = x^1 Z$

is a basis row matrix of the μ cycles, and R is a 6 μ-column matrix of the R_j cycle loads.

In a formalistic way we write the projection

(33) $j : H^1 \rightarrow C^1$

where j is defined as an inclusion transformation. Thus

(34) $c^1 = jh^1 = j(m^1 R) = x^1 ZR = x^1 N_h$

(35) $0 = \partial c^1 = \partial jh^1 = \partial x^1 ZR = x^0 EZR$

It follows from (34) that

(36) $ZR = N_h$

For the dimensions hold

(37) $\dim C^1 = \dim B^0 + \dim H^1$

(38) $\dim H^1 = \mu = 6\alpha_1 - 6(\alpha_0 - 1) = 6(\alpha_1 - \alpha_0 + 1)$

Continuity conditions for deformations

In order to study the kinematical configuration, the dual operator to ∂, the coboundary operator δ is defined by regarding the cochains as linear functionals of the chains. See Samuelsson [16], Halmos [17], Ghenzi [18].

(39) $\delta : C_0 \rightarrow C_1$

For arbitrary values of $c^1 \in C^1$ and $c^0 \in C^0$ the scalar product

(40) $[c_0, \partial c^1] = [\delta c_0, c^1]$

is defined, see Halmos [17].

The value of (40) is a scalar and corresponds physically to virtual work. Thus (40) is the conceptual counterpart to the generalized work equation of Maxwell-Mohr, see Asplund [14].

The elements of C_0 are linearly dependent because of the rigid motion, composed of 6 linearly independent displacements of the frame which do not cause any deformation of the members. This corresponds to the Kernel of δ denoted $\mathrm{Ker}\,\delta$ with $\dim(\mathrm{Ker}\,\delta) = 6$.

We now define a quotient space

(41) $B_0 = C_0/\mathrm{Ker}\,\delta$

(42) with $\dim B_0 = 6(\alpha_0 - 1)$

The elements b_0 in B_0 are called structure deformations.

Chains h_1 of member deformations which differ by chains of member deformations kinematically compatible with chains of joint displacement are considered as vectors, called cycle deformations denoted by r, in a quotient space

(43) $H_1 = C_1/\mathrm{Im}\,\delta$

(44) $\dim H_1 = 6(\alpha_1 - \alpha_0 + 1)$

The continuity condition for deformation is now written

(45) $j^* : C_1 \rightarrow H_1$

The operator j^* is the dual operator to j because

(46) $[j^*c_1, h^1] = [c_1 + \delta c_0, h^1] = [c_1, h^1] + [\delta c_0, h^1] = [c_1, h^1] +$

$+ [c_0, \partial h^1] = [c_1, h^1] + [c_0, 0] = [c_1, h^1] + 0 = [c_1, jh^1]$

The formula (46) is the conceptual formulation of the generalized work equation of Maxwell-Mohr for an auxiliary structure loaded with "gap loads" here called cycle loads.

If $h_1 = 0$ then

(47) $\quad 0 = h_1 = j^* c_1 = j^* \delta b_0$

We now have the following sequences:

(48) $$H^1 \xrightarrow{j} C^1 \xrightarrow{\partial} B^0$$

$$\dim \; 6(\alpha_1 - \alpha_0 + 1) \qquad 6\alpha_1 \qquad 6(\alpha_0 - 1)$$

(49) $$H_1 \xleftarrow{j^*} C_1 \xleftarrow{\delta} B_0$$

It follows from (35) and (47) that for operators ∂, j and its duals δ, j^* hold

(50) $\quad \partial j = 0$

(51) $\quad j^* \delta = 0$

The sequences (50) and (51) are said to be exact.

If we choose dual basis for corresponding spaces the dual transformations are represented by transposed matrices, see Halmos [17].

DISSECTION OF A GRAPH

Subgraphs X(0) and X(1) of a graph X. Dissection (X(0), X(1))

Let X(1) be a closed subgraph of X, then according to (11)

(52) $\quad \mathrm{Cl}\, X(1) = X(1)$

The other part of X called X(0) is an open subgraph, the complement of X(1) in X, which is written

(53) $\quad X(0) = X - X(1)$

We observe that

(54) $\quad X(0) \cup X(1) = X$

(55) $\quad X(0) \cap X(1) = 0$

We also observe that

(56) $\quad \mathrm{St}\, X(0) = X(0)$

The pair (X(0), X(1)) is called a dissection of the graph X, see Lefschetz [15].

Physically the dissection of the graph X, see Fig. 8 where the dissection is visualized by a marked line, means that if X(1) contains a branch (loop) it also contains the nodes (branches) incident to it, because X(1) is closed. If X(0) contains a node it also contains all the branches incident to it because X(0) is open. Compare Figs. 5 and 6.

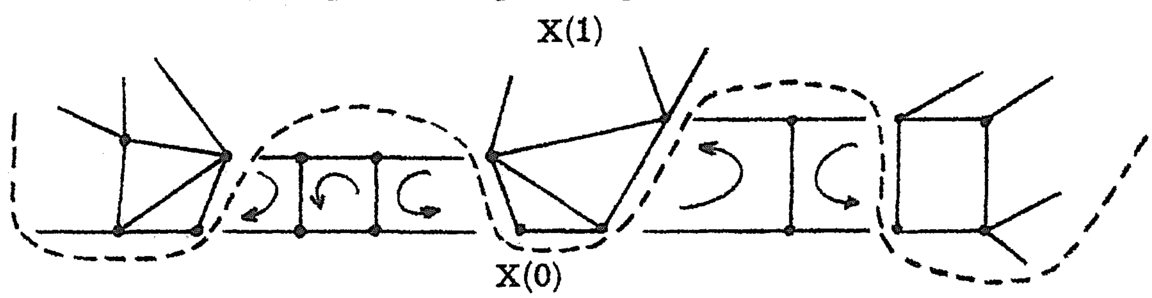


Fig. 8. Part of a dissected graph

Dissection of a chain

The dissection of the graph induces a dissection of an arbitrary chain d in chain group D, which can be C^1, C_1 or B^0, B_0 on X. The dissection of a chain is made by the projection and the injection operations.

The projection $\pi(i)$ of D(X) into D(X(i)) or short

(57) $\quad \pi(i) : D \rightarrow D(i); \quad i = 0,\ 1$

is defined by regarding the X(i) part of the chain d in D as a chain d(i) in D(i), see Fig. 9. We write shortly

(58) $\pi(i)\,d = d(i)$

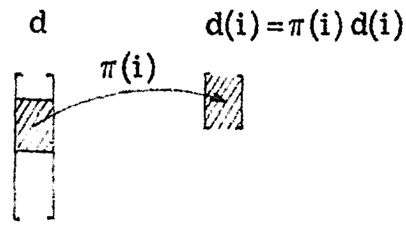


Fig. 9. Projection $\pi(i)$

We observe that

(59) $d = d(0) \oplus d(1)$

We can write

(60) $d = yY = \begin{bmatrix} d(0) \\ d(1) \end{bmatrix} = [y(0),\ y(1)] \begin{bmatrix} Y(0) \\ Y(1) \end{bmatrix}$

$y = x^1,\ x_1,\ x^0$ or x_0; $\quad Y = N,\ n,\ P$ or p

where $[y(0),\ y(1)]$ is a partitioned row matrix of element basis (the elements in $\{y\} \in X$ are called an element basis) and $[Y(0)^*,\ Y(1)^*]^*$ is the partitioned column matrix with "associated" matrix representatives.

In the analysis of the dissected structure mixed vectors are needed and are defined as

(61) $\begin{bmatrix} c_1(0) \\ c^1(1) \end{bmatrix} = [x_1(0),\ x^1(1)] \begin{bmatrix} n(0) \\ N(1) \end{bmatrix}$

and

(62) $\begin{bmatrix} c^1(0) \\ c_1(1) \end{bmatrix} = [x^1(0),\ x_1(1)] \begin{bmatrix} N(0) \\ n(1) \end{bmatrix}$

The <u>injection $\sigma(i)$</u> of $D(X(i))$ into $D(X)$ or short

(63) $\sigma(i): D(i) \to D; \quad i = 0,\ 1$

is defined by regarding a chain on $X(i)$ as a chain on X itself, see Fig. 10, and we write shortly

(64) $\sigma(i)\,d(i) = d; \quad i = 0,\ 1$

We write by use of matrix formulation the injection operator

(65) $\sigma = [\sigma(0),\ \sigma(1)]$

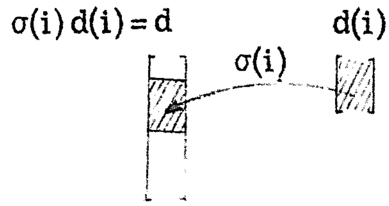


Fig. 10. Injection $\sigma(i)$

and the projection operator

(66) $$\pi = \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix}$$

It is easily seen that

(67) $$\sigma\pi = [\,\sigma(0),\ \sigma(1)\,] \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} = \sigma(0)\pi(0) + \sigma(1)\pi(1) = I$$

where I is the identity operator.

Properties of the groups on X(0) and X(1)

Now the groups on X(i) have to be compared with those on X itself. By use of the identity operator (67) we can write the boundary of a 1-chain on X

(68) $$\partial c^1 = \partial(I c^1) = \partial\sigma\pi c^1 = \partial\,[\sigma(0),\ \sigma(1)] \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} c^1$$

Since X(1) is closed a boundary of a 1-chain on X(1)

(69) $$\partial\sigma(1)\pi(1)\,c^1 = \partial\sigma(1)\,c^1(1)$$

has no part on X(0).

It follows that a cycle on X(1) is also a cycle on X.

Since X(0) is the open complement of a closed subgraph X(1) in X, the boundary of a chain on X(0)

(70) $$\partial\sigma(0)\pi(0)\,c^1 = \partial\sigma(0)\,c^1(0)$$

when regarded as a chain on X may lie partly on X(1).

Definition: A relative 1-cycle on X modulo X(1) is a 1-chain c^1 on X(0) the boundary of which (a 0-chain) is on X(1).

We now also define cycle spaces for the dissection:

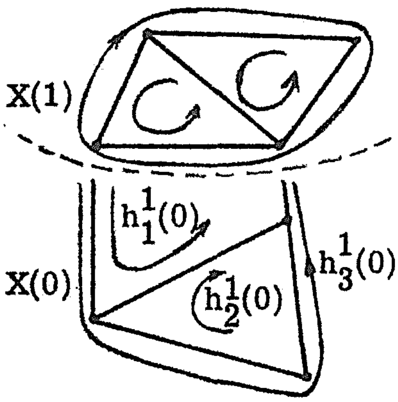
H^1 is the space of all the cycles and relative cycles $h^1(0)$ on X(0)

and

$H^1(1)$ is the space of all cycles $h^1(1)$ on X(1) so that

(71) $$H^1 = H^1(0) \oplus H^1(1)$$

The cycles on X and X(1) are sometimes called absolute cycles.
In Fig. 11 cycles and relative cycles in a dissected graph are shown.



cycles on X(1)

$h_1^1(0)$ and $h_1^3(0)$ are relative cycles on X(0)
$h_2^1(0)$ is a cycle on X(0)

Fig. 11. Cycles on X(1) and cycles and relative cycles on X(0)

Dissection of the boundary operator

We need boundary operators for the dissected parts. Four such operators, two pure and two mixed, can be formally obtained with the aid of matrix algebra.

From (27) and by use of (68) we obtain

(72) $$b^0 = \partial c^1 = \partial[\sigma(0),\ \sigma(1)] \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} c^1$$

This equation is now operated on by the projection operator (66) from the left which gives

(73) $$b^0 = \pi b^0 = \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} b^0 = \begin{bmatrix} b^0(0) \\ b^0(1) \end{bmatrix} = \pi\partial\, c^1 = \pi\partial I c^1 = \pi\partial\sigma\pi\, c^1 =$$

$$= \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} \partial\,[\sigma(0),\ \sigma(1)] \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} c^1 = \begin{bmatrix} \pi(0)\partial\sigma(0) & \pi(0)\partial\sigma(1) \\ \pi(1)\partial\sigma(0) & \pi(1)\partial\sigma(1) \end{bmatrix} \begin{bmatrix} c^1(0) \\ c^1(1) \end{bmatrix}$$

$$= \begin{bmatrix} \partial(0) & \partial(01) \\ \partial(10) & \partial(1) \end{bmatrix} \begin{bmatrix} c^1(0) \\ c^1(1) \end{bmatrix} = \partial\, c^1$$

where new boundary operators are defined as

(74) $$\pi(i)\partial\sigma(k) = \begin{cases} \partial(i) & \text{when } i = k \\ \partial(ik) & \text{when } i \neq k \end{cases} \qquad i,\ k = 0,\ 1$$

Since X(1) is closed the boundary, $\partial\sigma(1)\,c^1(1)$ has no part on X(0). Thus

(75) $$\pi(0)\partial\sigma(1) = \partial(01) = 0$$

The subgraph X(0) is open which implies that the boundary $\partial\sigma(0)c^1(0)$ has a part on X(1). Thus

(76) $$\pi(1)\partial\sigma(0) = \partial(10) \neq 0$$

The dissected boundary operator can be written

(77) $$\partial = \pi\partial\sigma = \begin{bmatrix} \partial(0) & 0 \\ \partial(10) & \partial(1) \end{bmatrix}$$

The operator $\partial(10)$ transforms $c^1(0)$ into $b^0(1)$, which means that it yields the X(1) part of the boundary $\partial\sigma(0)$ of $c^1(0)$ on X. Physically this means that $\partial(10)$ represents the connection relation of the boundary operator between X(1) and X(0). The dissection of ∂ is shown in the diagram of Fig. 12.

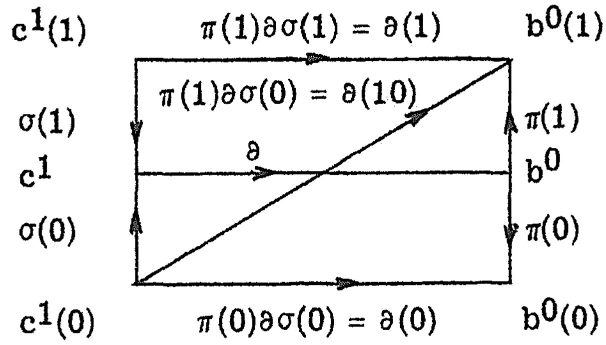


Fig. 12. Diagram representation of the dissection of ∂

The dissected boundary operator ∂ is represented by the incidence matrix E, see (2), which is partitioned according to

(78) $$b^0 = x^0P = [x^0(0),\ x^0(1)] \begin{bmatrix} P(0) \\ P(1) \end{bmatrix} = \partial c^1 = \partial x^1 N = x^0 E N =$$

$$= [x^0(0),\ x^0(1)]\, E \begin{bmatrix} N(0) \\ N(1) \end{bmatrix} = [x^0(0),\ x^0(1)] \begin{bmatrix} E(0) & 0 \\ E(10) & E(1) \end{bmatrix} \begin{bmatrix} N(0) \\ N(1) \end{bmatrix}$$

The partitioning of the matrix E should be compared with the definitions of the open and closed graphs, see formulas (5) and (6). From (78) we find

(79) $$\begin{bmatrix} P(0) \\ P(1) \end{bmatrix} = \begin{bmatrix} E(0) & 0 \\ E(10) & E(1) \end{bmatrix} \begin{bmatrix} N(0) \\ N(1) \end{bmatrix}$$

The dissected boundary operator ∂ for the dissected structure in Fig. 13 is represented by the matrix

(80)
$$E = \begin{array}{c} \\ B \\ C \\ A \\ D \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & -1 & & & & \\ & 1 & -1 & & & \\ \hline -1 & & & -1 & 1 & \\ & & 1 & 1 & & 1 \end{array}\right] \end{array} = \begin{bmatrix} E(0) & 0 \\ E(10) & E(1) \end{bmatrix}$$

which is a topological matrix with coefficients 1, -1, 0, see (2).

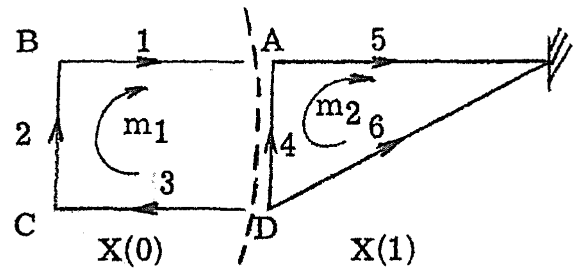


Fig. 13. Example of a dissected structure

In the same way as above the coboundary operator δ is dissected and new coboundary operators $\delta(0)$, $\delta(1)$ and $\delta(01)$, $\delta(10)$ are defined.

(81)
$$\delta = \pi \delta \sigma = \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} \delta[\sigma(0),\ \sigma(1)] = \begin{bmatrix} \delta(0) & \delta(01) \\ \delta(10) & \delta(1) \end{bmatrix} = \begin{bmatrix} \delta(0) & \delta(01) \\ 0 & \delta(1) \end{bmatrix}$$

The dissected coboundary operator is shown in the diagram of Fig. 14.

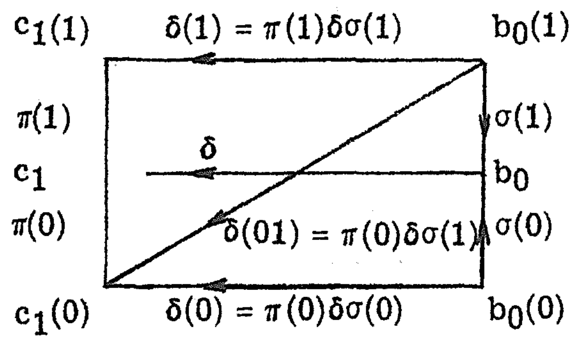


Fig. 14. Diagram representation of the dissection of δ

Dissection of cycles

For cycles h^1 on X we define projection and injection operations. The projection

(82) $\pi(1) : H^1 \longrightarrow H^1(1)$

transforms cycles on X in the X(1) part to cycles on X(1) and

(83) $\pi(0) : H^1 \longrightarrow H^1(0)$

transforms cycles on X in the X(0) part to cycles or relative cycles on X(0).

The injection

(84) $\sigma(1) : H^1(1) \dashrightarrow H^1$

is obtained by regarding the cycles on X(1) as cycles on X itself and

(85) $\sigma(0) : H^1(0) \dashrightarrow H^1$

is obtained by regarding cycles and relative cycles on X(0) as cycles on X itself.

Dissection of the inclusion operator j

The continuity condition (34) for the member forces when the joint loads are equal to zero, is

(86) $j\,h^1 = c^1$

We use matrix algebra to dissect j into four parts, two pure and two mixed ones.

(87) $$\pi\, j\, h^1 = \pi\, c^1 = \begin{bmatrix} c^1(0) \\ c^1(1) \end{bmatrix} = \pi\, j\, I\, h^1 = \pi\, j\, \sigma\pi\, h^1 = \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} j[\sigma(0),\ \sigma(1)] \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} h^1 =$$

$$= \begin{bmatrix} \pi(0)\, j\sigma(0) & \pi(0)\, j\sigma(1) \\ \pi(1)\, j\sigma(0) & \pi(1)\, j\sigma(1) \end{bmatrix} \begin{bmatrix} h^1(0) \\ h^1(1) \end{bmatrix} = \begin{bmatrix} j(0) & j(01) \\ j(10) & j(1) \end{bmatrix} \begin{bmatrix} h^1(0) \\ h^1(1) \end{bmatrix}$$

Since X(1) is closed a cycle $j\sigma(1)\, h^1(1)$ on X(1) is a cycle on X and thus it has no part on X(0), which gives

(88) $\pi(0)\, j\sigma(1)\, h^1(1) = 0$ or $\pi(0)\, j\sigma(1) = j(01) = 0$

A cycle on X which partly lies on X(0) is a relative cycle and contains 1-chains both on X(0) and X(1). Thus

(89) $\pi(1)\, j\sigma(0)\, h^1(0) \neq 0$ and $\pi(1)\, j\sigma(0) = j(10) \neq 0$

The operator j(10) takes the X(1) part of a cycle, which is a relative cycle on X(0), on X, see Fig. 15.

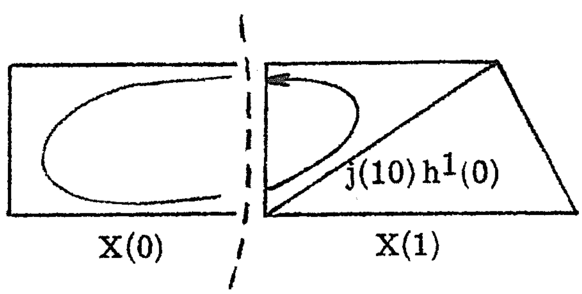


Fig. 15. The operator j(10)

Now the dissected operator j is written

$$(90) \qquad j = \begin{bmatrix} j(0) & 0 \\ j(10) & j(1) \end{bmatrix}$$

The operator j is shown in dissected form in the diagram of Fig. 16.

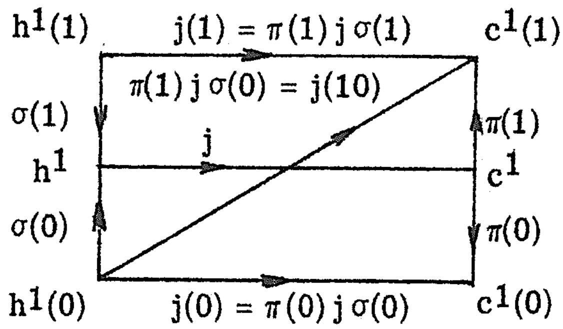


Fig. 16. Diagram representation of the operator j

The matrix representation of the dissected operator j is defined by the matrix Z in partitioned form in

$$(91) \qquad c^1 = \begin{bmatrix} c^1(0) \\ c^1(1) \end{bmatrix} = [x^1(0),\ x^1(1)] \begin{bmatrix} N(0) \\ N(1) \end{bmatrix} = jh^1 = jm^1 Z$$

$$= x^1 Z R = [x^1(0),\ x^1(1)] \begin{bmatrix} Z(0) & 0 \\ Z(10) & Z(1) \end{bmatrix} \begin{bmatrix} R(0) \\ R(1) \end{bmatrix}$$

We obtain form (91)

$$(92) \qquad \begin{bmatrix} N(0) \\ N(1) \end{bmatrix} = \begin{bmatrix} Z(0) & 0 \\ Z(10) & Z(1) \end{bmatrix} \begin{bmatrix} R(0) \\ R(1) \end{bmatrix}$$

For the dissected structure in Fig. 13 the operator j is represented by the partitioned matrix

$$(93) \qquad Z = \begin{bmatrix} Z(0) & 0 \\ Z(10) & Z(1) \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c|c} m_1 & m_2 \\ 1 & \\ 1 & \\ 1 & \\ \hline -1 & 1 \\ & 1 \\ & -1 \end{array}$$

Exact sequence $\partial j = 0$ in dissected form

The exact sequence $\partial j = 0$ is now dissected in a formal way by use of matrix algebra.

(94) $$0 = \partial j = \begin{bmatrix} \partial(0) & 0 \\ \partial(10) & \partial(1) \end{bmatrix} \begin{bmatrix} j(0) & 0 \\ j(10) & j(1) \end{bmatrix} = \begin{bmatrix} \partial(0)\,j(0) & 0 \\ \partial(10)\,j(0) + \partial(1)\,j(10) & \partial(1)\,j(1) \end{bmatrix} =$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This implies that

(95) $\partial(0)\,j(0) = 0$

(96) $\partial(1)\,j(1) = 0$

(97) $\partial(10)\,j(0) + \partial(1)\,j(10) = 0$

In X(0) a relative cycle is a chain on X(0) the boundary $\partial(10)\,j(0)$ of which lies on X(1). Compare Fig. 17.

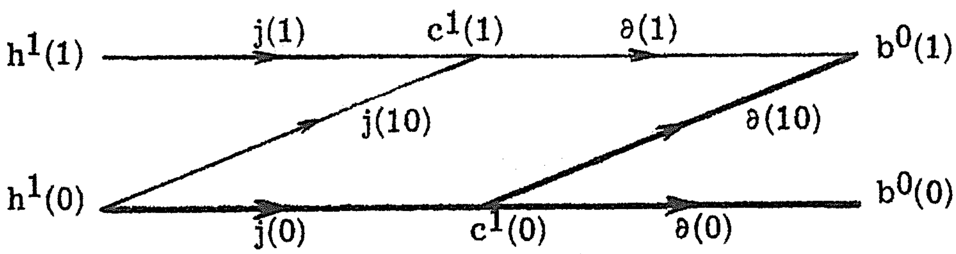


Fig. 17. Diagram representation of the exact sequence $\partial j = 0$. Operators for the relative cycle with heavy lines.

We now want to verify the formulas (95), (96) and (97) for the dissected structure in Fig. 13.

According to (80) and (93) we obtain by use of matrix representation

$$E = \begin{bmatrix} E(0) & 0 \\ E(10) & E(1) \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & -1 & & & & \\ & 1 & -1 & & & \\ \hline -1 & & & -1 & 1 & \\ & & 1 & 1 & & 1 \end{array}\right]$$

$$Z = \begin{bmatrix} Z(0) & 0 \\ Z(10) & Z(1) \end{bmatrix} = \left[\begin{array}{c|c} 1 & \\ 1 & \\ 1 & \\ \hline -1 & 1 \\ & 1 \\ & -1 \end{array}\right]$$

We get

$$E(0)\,Z(0) = \begin{bmatrix} 1 & -1 & \\ & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad E(1)\,Z(1) = \begin{bmatrix} -1 & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$E(10)\,Z(0) + E(1)\,Z(10) = \begin{bmatrix} -1 & & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & \\ 1 & & 1 \end{bmatrix} [-1] = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A relative cycle is a chain on X(0) the boundary of which, a 0-chain, lies on X(1). By use of (77) and (90) we find

(98) $\quad c^0(1) = \partial(10)\,c^1(0) = \partial(10)\,j(0)\,h^1(0)$

The operators $\partial(10)$ and $j(0)$ are found in ∂ and j but the product $\partial(10)\,j(0)$ can be established at once by help of a so called equivalent structure of X(0), see Fig. 18, which only describes the connection relation to X(1).

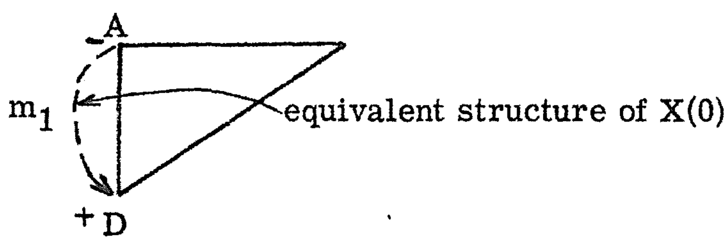


Fig. 18. Equivalent structure

Thus we find by help of the equivalent structure in Fig. 18

$$E(10)\,Z(0) = \begin{matrix} & m_1 \\ A & \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ D & \end{matrix}$$

The operator j(10) takes the X(1) part of a cycle, which is a relative cycle on X(0), on X.

The boundary

(99) $\quad c^0(1) = \partial(1)\,j(10)\,h^1(0)$

is the boundary of the X(1) part of a cycle, which is a relative cycle on X(0), on X. The operator $\partial(1)\,j(10)$ can be established at once, see Fig. 19. Thus

$$E(1)\,Z(10) = \begin{matrix} & m_1 \\ A & \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ D & \end{matrix}$$

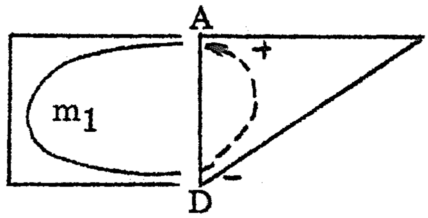


Fig. 19. The operator j(10) takes the X(1) part of a cycle on X

<u>Dissection of the exact sequence $j^*\delta = 0$</u>

In a similar way as for $\partial j = 0$ the exact sequence $j^*\delta = 0$ is formally dissected

(100) $$0 = j^*\delta = \pi j^* I\, \delta\, \sigma = \pi\, j^*\, \sigma \pi \delta\, \sigma = \begin{bmatrix} j^*(0)\,\delta(0) & j^*(0)\,\delta(01) + j^*(01)\,\delta(1) \\ 0 & j^*(1)\,\delta(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This means that

(101) $$j^*(0)\,\delta(0) = 0$$

(102) $$j^*(1)\,\delta(1) = 0$$

(103) $$j^*(0)\,\delta(01) + j^*(01)\,\delta(1) = 0$$

In Fig. 20 the exact sequence $j^*\delta = 0$ is given in diagram form.

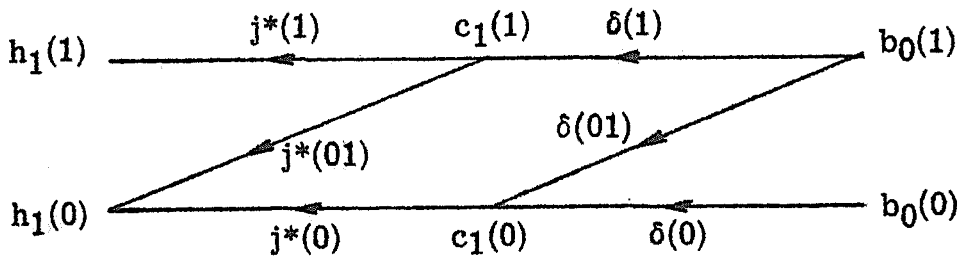


Fig. 20. Diagram representation of the exact sequence $j^*\delta = 0$

THE ELASTICITY CONDITIONS. HOOKE'S LAW

The elasticity condition in mixed form

The elasticity condition for a structure gives the connection between all its member forces and member deformations. It is described by functions called the member flexibility isomorfism

(104) $\quad f : C^1 \rightarrow C_1$

or the member stiffness isomorfism

(105) $\quad F : C_1 \rightarrow C^1$

The flexibility isomorfism is represented by the member flexibility matrix f according to

(106) $\quad x_1 n = f x^1 N = x^1 f N$

where x_1, x^1 are row matrices of element basis and n, N are corresponding column matrices with associated matrix representatives and $f = [f_j]^D$ is a matrix with member flexibility matrices f_j along the diagonal. From (106) we obtain

(107) $\quad n = f N$

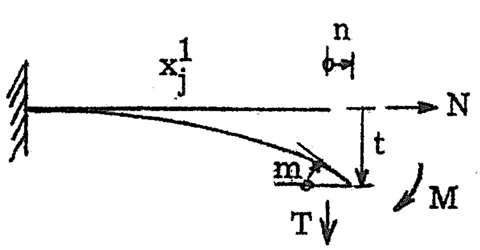
We state that the matrices f_j should be positive definite matrices which implies the member stiffness matrices

(108) $\quad F_j = f_j^{-1}$

We may split the vectors n_j and N_j into $n_j = [n(0)_j^*, n(1)_j^*]^*$ and $N_j = [N_j(0)^*, N_j(1)^*]^*$. We may regard the member deformations $n_j(0)$ and member forces $N_j(1)$ as a mixed vector $[n(0)_j^*, N_j(1)]^*$ and member forces $N(0)_j^*$ and member deformations $n(1)_j$ as another mixed vector $[N(0)_j^*, n(1)_j^*]^*$. We can then write the elasticity condition for the member in the mixed form

(109) $$\begin{bmatrix} n(0)_j \\ N(1)_j \end{bmatrix} = \begin{bmatrix} f(0)_j & \alpha_j \\ \beta_j & F(1)_j \end{bmatrix} \begin{bmatrix} N(0)_j \\ n(1)_j \end{bmatrix}$$

It is mixed in the meaning that it contains both flexibility and stiffness coefficients. Compare Fig. 21.



$$\begin{bmatrix} n(0)_j \\ N(1)_j \end{bmatrix} = \begin{bmatrix} m \\ N \\ T \end{bmatrix} = \begin{bmatrix} f(0)_j & \alpha_j \\ \beta_j & F(1)_j \end{bmatrix} \begin{bmatrix} M \\ n \\ t \end{bmatrix}$$

where $$\begin{bmatrix} M \\ n \\ t \end{bmatrix} = \begin{bmatrix} N(0)_j \\ n(1)_j \end{bmatrix}$$

Fig. 21. Beam with mixed variables

For a dissected structure (X(0), X(1)) we regard a deformation vector $c_1(0)$ on X(0) together with a force vector $c^1(1)$ on X(1) as a mixed vector, see (61), represented by a column matrix$[n(0)^*, N(1)^*]^*$. The set of associated forces $c^1(0)$ and deformations $c_1(1)$ regarded as a mixed vector is represented by a column matrix $[N(0)^*, n(1)^*]^*$. We define column matrices $n(0) = [n(0)_j]$, $N(0) = [N(0)_j]$, $n(1) = [n(1)_j]$ and $N(1) = [N(1)_j]$.
The elasticity condition for the dissected structure can now be written

(110) $$\begin{bmatrix} n(0) \\ N(1) \end{bmatrix} = \begin{bmatrix} f(0) & \alpha \\ \beta & F(1) \end{bmatrix} \begin{bmatrix} N(0) \\ n(1) \end{bmatrix}$$

where

$$f(0) = [f(0)_j]^D, \quad F(1) = [F(1)_j]^D, \quad \alpha = [\alpha_j]^D \text{ and } \beta = [\beta_j]^D$$

D denotes diagonal

We now define for the dissected structure a <u>mixed isomorfism (f, F)</u> represented by

(111) $$(f, F) = \begin{bmatrix} f(0) & \alpha \\ \beta & F(1) \end{bmatrix}$$

We observe that $\alpha = \beta = 0$ means that the variables in a branch x_j^1 either are merely forces $N(0)_j$ or merely deformations $n(1)_j$.
In Fig. 22 the mixed isomorfism (f, F) is shown in a diagram.

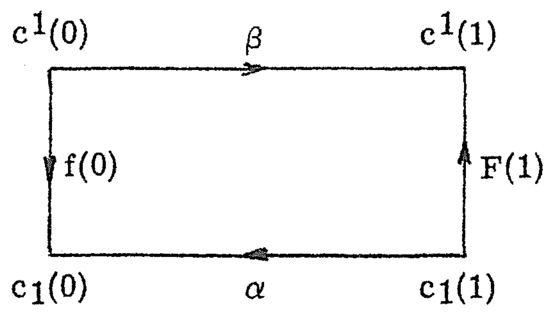


Fig. 22. Diagram representation of the mixed isomorfism (f, F)

Here we restrict our exposition ot the special case when (f, F) is a linear function. A sufficient condition for this is that the member material obeys Hooke's law and that a first order theory member analysis is undertaken, see Åkesson [19]. It is possible to extend the treatment to the case when (f, F) is given as a differential operator. Our frame problem is then translated into an initial value problem. To get a solution to this problem an initial value is needed and the solution can then be obtained by some suitable method of numerical integration, for example by the Runge-Kutta integration scheme, see Richard and Goldberg [20], [21].

Transformation of a pure isomorfism into a mixed one

The isomorfisms f or F given for the whole structure can easily be transformed to a mixed isomorfism (f, F) consistent with a chosen dissection (X(0), X(1)). We find by formal expansion of

$$(112) \qquad c^1 = F c_1$$

$$(113) \qquad \pi c^1 = \begin{bmatrix}\pi(0)\\ \pi(1)\end{bmatrix} c^1 = \begin{bmatrix}c^1(0)\\ c^1(1)\end{bmatrix} = \pi F c_1 = \pi F I c_1 = \pi F \sigma \pi c_1 =$$

$$= \begin{bmatrix}\pi(0)\\ \pi(1)\end{bmatrix} F [\sigma(0), \ \sigma(1)] \begin{bmatrix}\pi(0)\\ \pi(1)\end{bmatrix} c_1 = \begin{bmatrix}\pi(0) F\sigma(0) & \pi(0) F\sigma(1)\\ \pi(1) F\sigma(0) & \pi(1) F\sigma(1)\end{bmatrix} \begin{bmatrix}c_1(0)\\ c_1(1)\end{bmatrix} =$$

$$= \begin{bmatrix}F(0) & F(01)\\ F(10) & F(1)\end{bmatrix} \begin{bmatrix}c_1(0)\\ c_1(1)\end{bmatrix}$$

In (113) new member stiffness matrices are defined according to the dissection (X(0), X(1)):

$$(114) \qquad \pi(i) F\sigma(i) = \begin{cases} F(i) & \text{if } i = k \\ F(ik) & \text{if } i \neq k \end{cases} \qquad i, \ k = 0, \ 1$$

Partiversion of (113) yields

$$(115) \qquad \begin{bmatrix}c_1(0)\\ c^1(1)\end{bmatrix} = \begin{bmatrix} F(0)^{-1} & -F(0)^{-1} F(01) \\ -F(10) F(0)^{-1} & F(1) - F(10) F(0)^{-1} F(01)\end{bmatrix} \begin{bmatrix}c^1(0)\\ c_1(1)\end{bmatrix}$$

In the same way a pure flexibility isomorfism can be transformed into a mixed one.

If $\alpha = \beta = 0$ the transformation (115) is simple because $F(01) = F(10) = 0$. The inversion of F(0) is only an inversion of block matrices:

$$F(0)^{-1} = ([F(0)_j]^D)^{-1} = [F(0)_j^{-1}]^D = [f(0)_j]^D$$

THE SOLUTION OF THE FRAME PROBLEM IN DISSECTED FORM

The partitioned system of equations

The continuity condition for the force distribution can be written

(116) $$c^1 = c_p^1 + j\,h^1$$

where

(117) $$c_p^1 = \partial^{-1} b^0$$

is a particular solution of the inhomogeneous continuity equation

(118) $$\partial c^1 = b^0$$

The continuity equation (116) is dissected and we get by use of (90) and the notion (58)

(119) $$\pi c^1 = \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} c^1 = \begin{bmatrix} \pi(0)\, c_p^1 \\ \pi(1)\, c_p^1 \end{bmatrix} + \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} j\,h^1 = \begin{bmatrix} c^1(0) \\ c^1(1) \end{bmatrix} = \begin{bmatrix} c^1(0)_p \\ c^1(1)_p \end{bmatrix} + \begin{bmatrix} j(0) & 0 \\ j(10) & j(1) \end{bmatrix} \begin{bmatrix} h^1(0) \\ h^1(1) \end{bmatrix}$$

The continuity condition for the deformation configuration can be written

(120) $$c_1 = c_{1p} + \delta\, b_0$$

where

(121) $$c_{1p} = (j^*)^{-1} h_1$$

is a particular solution of the inhomogeneous continuity equation

(122) $$j^* c_1 = h_1$$

The continuity equation(120) is dissected and by use of the notion (58) we obtain

(123) $$\pi c_1 = \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} c_1 = \begin{bmatrix} \pi(0) c_{1p} \\ \pi(1) c_{1p} \end{bmatrix} + \begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix} \delta b_0 = \begin{bmatrix} c_1(0) \\ c_1(1) \end{bmatrix} = \begin{bmatrix} c_1(0)_p \\ c_1(1)_p \end{bmatrix} + \begin{bmatrix} \delta(0) & \delta(01) \\ 0 & \delta(1) \end{bmatrix} \begin{bmatrix} b_0(0) \\ b_0(1) \end{bmatrix}$$

In addition to the continuity conditions we have the elasticity conditions given by the mixed isomorfism (f, F) in (111).

(124) $$\begin{bmatrix} c_1(0) \\ c^1(1) \end{bmatrix} = \begin{bmatrix} f(0) & \alpha \\ \beta & F(1) \end{bmatrix} \begin{bmatrix} c^1(0) \\ c_1(1) \end{bmatrix}$$

Insertion of (119) and (123) into (124) gives

(125) $$\begin{bmatrix} c_1(0)_p \\ c^1(1)_p \end{bmatrix} + \begin{bmatrix} \delta(0) & \delta(01) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_0(0) \\ b_0(1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ j(10) & j(1) \end{bmatrix} \begin{bmatrix} h^1(0) \\ h^1(1) \end{bmatrix} =$$
$$= \begin{bmatrix} f(0) & \alpha \\ \beta & F(1) \end{bmatrix} \left(\begin{bmatrix} c^1(0)_p \\ c_1(1)_p \end{bmatrix} + \begin{bmatrix} j(0) \\ 0 \end{bmatrix} h^1(0) + \begin{bmatrix} 0 \\ \delta(1) \end{bmatrix} b_0(1) \right)$$

or

(126) $$\begin{bmatrix} f(0) & \alpha \\ \beta & F(1) \end{bmatrix} \begin{bmatrix} j(0) & 0 \\ 0 & \delta(1) \end{bmatrix} \begin{bmatrix} h^1(0) \\ b_0(1) \end{bmatrix} - \begin{bmatrix} \delta(0) & \delta(01) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_0(0) \\ b_0(1) \end{bmatrix} -$$
$$- \begin{bmatrix} 0 & 0 \\ j(10) & j(1) \end{bmatrix} \begin{bmatrix} h^1(0) \\ h^1(1) \end{bmatrix} = \begin{bmatrix} c_1(0)_p \\ c^1(1)_p \end{bmatrix} - \begin{bmatrix} f(0) & \alpha \\ \beta & F(1) \end{bmatrix} \begin{bmatrix} c^1(0)_p \\ c_1(1)_p \end{bmatrix}$$

We multiply the equation (126) with $\begin{bmatrix} j^*(0) & 0 \\ 0 & \partial(1) \end{bmatrix}$ from the left:

(127) $$\begin{bmatrix} j^*(0)\,f(0)\,j(0) & j^*(0)\alpha\,\delta(1) \\ \partial(1)\beta\,j(0) & \partial(1)\,F(1)\,\delta(1) \end{bmatrix} \begin{bmatrix} h^1(0) \\ b_0(1) \end{bmatrix} - \begin{bmatrix} j^*(0)\,\delta(0) & j^*(0)\,\delta(01) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_0(0) \\ b_0(1) \end{bmatrix} -$$
$$- \begin{bmatrix} 0 & 0 \\ \partial(1)\,j(10) & \partial(1)\,j(1) \end{bmatrix} \begin{bmatrix} h^1(0) \\ h^1(1) \end{bmatrix} = \begin{bmatrix} j^*(0) & 0 \\ 0 & \partial(1) \end{bmatrix} \left(\begin{bmatrix} c_1(0)_p \\ c^1(1)_p \end{bmatrix} - \begin{bmatrix} f(0) & \alpha \\ \beta & F(1) \end{bmatrix} \begin{bmatrix} c^1(0)_p \\ c_1(1)_p \end{bmatrix} \right)$$

By use of (96), (97) and (101) we get the <u>fundamental equation system</u> of the dissected elastic structure

(128) $$\begin{bmatrix} j^*(0)\,f(0)\,j(0) & -j^*(0)\,\delta(01) + j^*(0)\,\alpha\,\delta(1) \\ \partial(10)\,j(0) + \partial(1)\,\beta\,j(0) & \partial(1)\,F(1)\,\delta(1) \end{bmatrix} \begin{bmatrix} h^1(0) \\ b_0(1) \end{bmatrix} = \begin{bmatrix} h_1(0,1) \\ b^0(1,0) \end{bmatrix}$$

where

(129) $$\begin{bmatrix} h_1(0,1) \\ b^0(1,0) \end{bmatrix} = \begin{bmatrix} j^*(0) & 0 \\ 0 & \partial(1) \end{bmatrix} \left(\begin{bmatrix} c_1(0)_p \\ c^1(1)_p \end{bmatrix} - \begin{bmatrix} f(0) & \alpha \\ \beta & F(1) \end{bmatrix} \begin{bmatrix} c^1(0)_p \\ c_1(1)_p \end{bmatrix} \right)$$

The fundamental equations are built up in such a way that we can prescribe structure loads b^0 and cycle deformations h_1, which implies particular solutions c_p^1 and c_{1p} according (117) and (121). These particular solutions enter the right hand side of the fundamental equations (128). The term $h_1(0,1)$, in (128), is an equivalent cycle deformation on X(0). It depends on cycle deformations on X(0) and influence from cycle deformations and structure loads on X(1). The term $b^0(1,0)$ is an equivalent structure load on X(1). It depends on structure loads on X(1) and influence from structure loads and cycle deformations on X(0).

From the fundamental equation system (128) we can solve cycle loads $h^1(0)$ and structure deformations $b_0(1)$ in two special ways. In the diacoptical procedure cycle forces $h^1(0)$ are first solved and in the codiacoptical procedure structure deformations $b_0(1)$ are first solved.

The dissection of the particular solutions

The particular solution of $\partial c^1 = b^0$, see (118), which may be written $c_p^1 = \partial^{-1} b^0$ where ∂^{-1} is determined from the condition

(130) $\partial \partial^{-1} = I$

is now dissected, see (77).

(131) $$\partial c^1 = \begin{bmatrix} \partial(0) & 0 \\ \partial(10) & \partial(1) \end{bmatrix} \begin{bmatrix} c^1(0) \\ c^1(1) \end{bmatrix} = \begin{bmatrix} b^0(0) \\ b^0(1) \end{bmatrix} = b^0$$

From the first and second rows in (131) we obtain

(132) $c^1(0)_p = \partial(0)^{-1} b^0(0)$

where $\partial(0)^{-1}$ is determined from

(133) $\partial(0) \partial(0)^{-1} = I$

and

(134) $\partial(1) c^1(1)_p = -\partial(10) c^1(0)_p + b^0(1)$

By studying the prescribed vector (129) we observe that the term $\partial(1) c^1(1)_p$ is directly used. Thus a particular solution c_p^1 for forces is only needed in the X(0) part of the structure. This particular solution $c^1(0)_p$ must be chosen so that the structure X(1) is in equilibrium, see Fig. 23, or the forces (134) on X(1) must be selfequilibrating, if the structure X(1) has no ground. Between these selfequilibrating forces there exists a condition of statical compatibility.

By using these selfequilibrating forces a great advantage is gained relative to earlier work by Kron [6], [7], because in Kron's papers the structure parts corresponding to X(1) had to be grounded, and then the temporary grounds had to be released which resulted in extra variables and computational work.

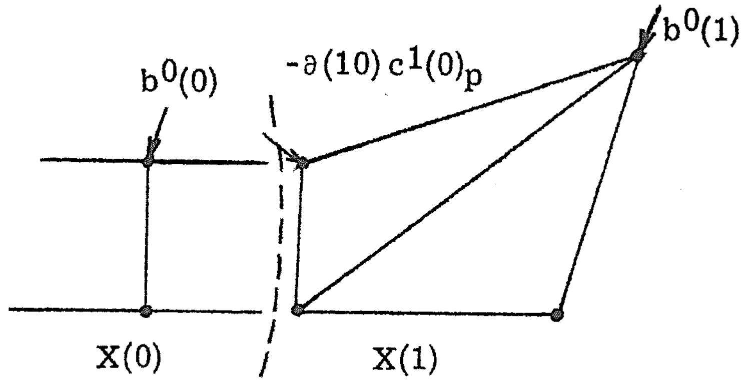


Fig. 23. Part of dissected structure X where X(1) is ungrounded

The particular solution of $j^* c_1 = h_1$, see (122) which may be written $c_{1p} = (j^*)^{-1} h_1$ where $(j^*)^{-1}$ is determined from

(135) $\quad j^*(j^*)^{-1} = I$

is now dissected:

(136) $$j^*c^1 = \begin{bmatrix} j^*(0) & j^*(01) \\ 0 & j^*(1) \end{bmatrix} \begin{bmatrix} c_1(0) \\ c_1(1) \end{bmatrix} = \begin{bmatrix} h_1(0) \\ h_1(1) \end{bmatrix} = h_1$$

The first and second rows in (136) give

(137) $\quad c_1(1)_p = j^*(1)^{-1} h_1(1)$

where $j^*(1)^{-1}$ is determined from

(138) $\quad j^*(1)\, j^*(1)^{-1} = I$

and

(139) $\quad j^*(0)\, c_1(0)_p = -j^*(01)\, c_1(1)_p + h_1(0)$

By studying the prescribed vector (129) we observe that a particular solution c_{1p} of deformations is needed only in the X(1) part of the structure. This particular solution $c_1(1)_p$ must be chosen so that these deformations are compatible with those of X(0) which is described by equation (139).

Now a couple of examples on the determination of particular solutions will be given.

Example 1. The structure in Fig. 24a is acted on by structure load only on the part X(1) of the dissected structure, Fig. 24b.

The particular solution $N(0)_p$ is determined by (132) and (134) which are here written by matrix representation

$$E(0)\, N(0)_p = P(0) = 0$$

$$E(1)\, N(1)_p = -E(10)\, N(0)_p + P(1)$$

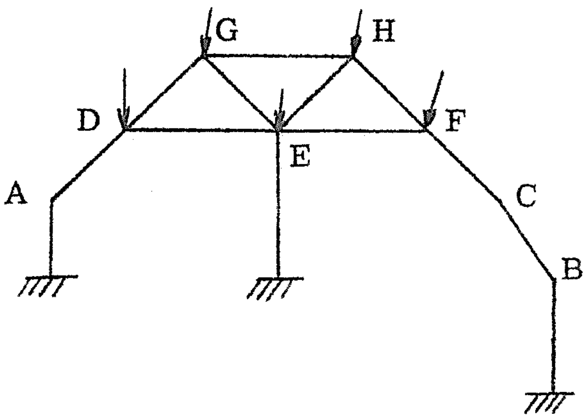


Fig. 24a. Loaded structure

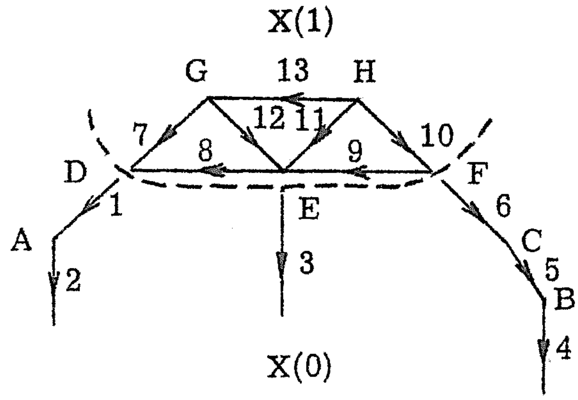


Fig. 24b. Dissection of the structure

By studying Fig. 24b we can array the incidence matrices

$$E(0) = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & & & & \\ & & & 1 & -1 & \\ & & & & & -1 \end{bmatrix} \end{array}, \quad E(10) = \begin{array}{c} \\ D \\ E \\ F \\ G \\ H \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & & & & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

If $N(0)_p = [N_{1p}\ N_{2p}\ N_{3p}\ N_{4p}\ N_{5p}\ N_{6p}]^*$ we find that $N(0)_p = [0\ 0\ N_{3p}\ 0\ 0\ 0]^*$ satisfies the equation $E(0)N(0)_p = 0$. It must also make the forces

$$E(1)N(1)_p = -E(10)N(0)_p + P(1) = \begin{bmatrix} \\ \\ E(10) \\ \\ \\ \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ N_{3p} \\ 0 \\ 0 \end{bmatrix} + \begin{vmatrix} P_D \\ P_E \\ P_F \\ P_G \\ P_H \end{vmatrix} = \begin{vmatrix} P_D \\ P_E - N_{3p} \\ P_F \\ P_G \\ P_H \end{vmatrix}$$

selfequilibrating or $P_D + P_E - N_{3p} + P_F + P_G + P_H = 0$, because the structure X(1) is non-grounded. This gives the value of N_{3p}. Thus $N(0)_p$ is a particular solution for forces in X(0).

The loaded structure in Fig. 24a is now dissected so that X(1) is grounded according Fig. 25.

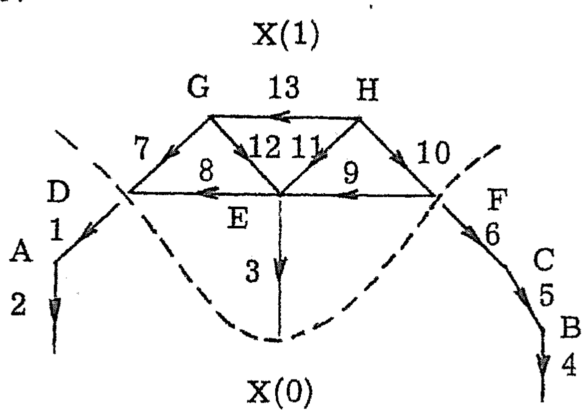


Fig. 25. Dissection of structure in Fig. 24a

We obtain the same incidence matrices E(0) and E(10) as above.

In this case when X(1) is grounded the forces $E(1)N(1)_p = -E(10)N(0)_p + P(1)$ are selfequilibrating. A particular solution is for example $N(0)_p = 0$, which means that a particular solution in X(0) need not be determined.

Example 2. The structure in Fig. 26a has a prescribed cycle deformation $h_1(0)$ in the X(0) part of the dissected structure in Fig. 26b.

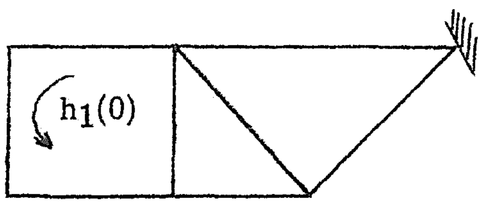


Fig. 26a. Loaded structure

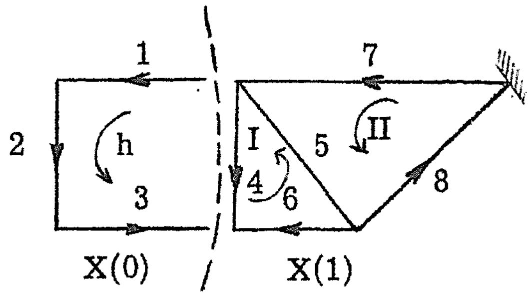


Fig. 26b. Dissected structure

The particular solution according (137) and (139) written with matrix representation is

$$Z(1)^* n(1)_p = r(1) = 0$$

$$Z(0)^* n(0)_p = -Z(10)^* n(1)_p + r(0)$$

The structure in Fig. 26b has the following incidence matrices

$$Z(1)^* = \begin{matrix} & 4 & 5 & 6 & 7 & 8 \\ \mathrm{I} & 1 & & -1 & & \\ \mathrm{II} & & -1 & & 1 & 1 \end{matrix} \; ; \quad Z(01)^* = \begin{matrix} & 4 & 5 & 6 & 7 & 8 \\ h & -1 & & & & \end{matrix} \; ;$$

The particular solution $n(1)_p = [n_{4p}, n_{5p}, n_{7p}, n_{8p}]^*$ having the simplest form is $n(1)_p = 0$, which implies that $Z(0)^* n(0)_p = h_1(0)$.

Another particular solution is for example $n(1)_p = [n_{4p} \;\; 0 \;\; n_{4p} \;\; 0 \;\; 0]^*$. If we choose to have no member deformation in X(0), $n(0)_p = 0$, then we obtain $n_{4p} = -h_1(0)$.

Another particular solution is for example $n(1)_p = [n_{p4} \;\; -n_{p4} \;\; 0 \;\; -n_{p4} \;\; 0]^*$, which cause disturbances in both the cycles I and II, which is often not desirable because the particular solution ought to be as simple as possible.

The diacoptical solution

The fundamental equation system (128) of the dissected structure is solved in partitioned form. We first solve for cycle forces $h^1(0)$:

(140) $\quad b_0(1) = F(1,1)^{-1}[-(\partial(10)\,j(0) + \partial(1)\beta\, j(0))\,h^1(0) + b^0(1,0)]$

(141) $\quad h^1(0) = f(0,0,0)^{-1}\, h_1(1,0)$

where

(142) $f(0,0,0) = (j^*(0)\,\delta(01) - j^*(0)\,\alpha\,\delta(1))\,F(1,1)^{-1}(\partial(10)\,j(0) + \partial(1)\,\beta\,j(1)) + f(0,0)$

(143) $F(1,1) = \partial(1)\,F(1)\,\delta(1)$

(144) $f(0,0) = j^*(0)\,f(0)\,j(0)$

(145) $h_1(0,1,0) = h_1(0,1) + (j^*(0)\,\delta(01) - j(0)\,\alpha\,\delta(1))\,F(1,1)^{-1}\,b^0(1,0)$

If X is dissected in such a way that X(1) contains a number of disjoint parts $X(1)_i$ the matrix representing $\partial(1)\,F(1)\,\delta(1)$ will consist of block matrices along the diagonal. If any disjoint part $X(1)_i$ has no ground, the loads on the part are chosen selfequilibrating according to (134). In every such non-grounded part one point is chosen as reference point to which we refer the structure deformations. By deleting the corresponding rows and columns in F(1,1), we make F(1,1) invertable.

Note that if X = X(1) the diacoptical method will be the same as the well-known displacement method. If X(0) contains only "interconnecting branches" the diacoptical method will be the same as Kron's method, used in the original manner.

The solution can be described by space diagrams, see Fig. 27, introduced by Roth [8] and developed for structures by for example Samuelsson [1], [16] and Åkesson [22]. Compare Figs. 17, 20 and 22.

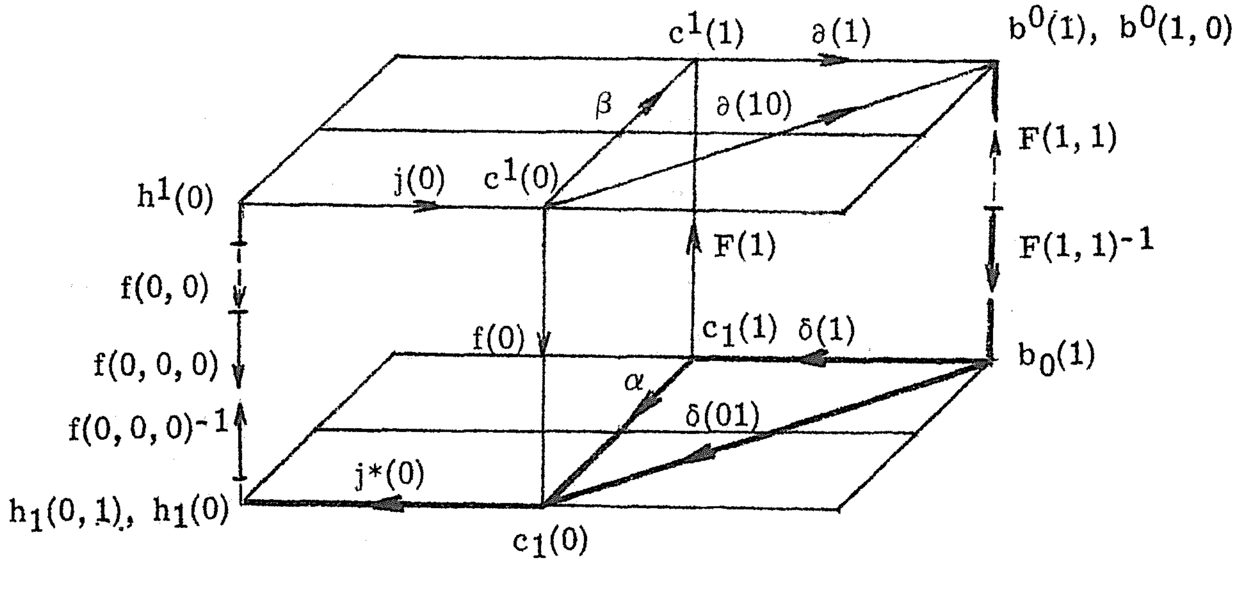


Fig. 27. Diagram representation of diacoptical solution

The codiacoptical solution

The partitioned system of equations (128) is solved in partitioned form and is first solved for structure deformations $b^0(1)$:

(146) $h^1(0) = f(0,0)^{-1}[(j^*(0)\,\delta(01) - j^*(0)\,\alpha\,\delta(1))\,b^0(1) + h_1(0,1)]$

(147) $b^0(1) = F(1,1,1)^{-1}\,b^0(1,0,1)$

where

(148) $F(1,1,1) = (\partial(10)\,j(0) + \partial(1)\beta\, j(0))\, f(0,0)^{-1}(j^*(0)\,\delta(01) - j^*(0)\,\alpha\,\delta(1)) + F(1,1)$

(149) $b^0(1,0,1) = b^0(1,0) - (\partial(10)\,j(0) + \partial(1)\beta\, j(0))\, f(0,0)^{-1}\, h_1(0,1)$

and $F(1,1)$, $f(0,0)$ are according (143), (144).

If $X(0)$ is dissected into a number of disjoint parts the matrix representing $j^*(0)\, f(0)\, j(0)$ will consist of block matrices along the diagonal. Observe that if $X = X(0)$ the codiacoptical procedure is the well-known force method.

The codiacoptical solution is described in the diagram of Fig. 28.

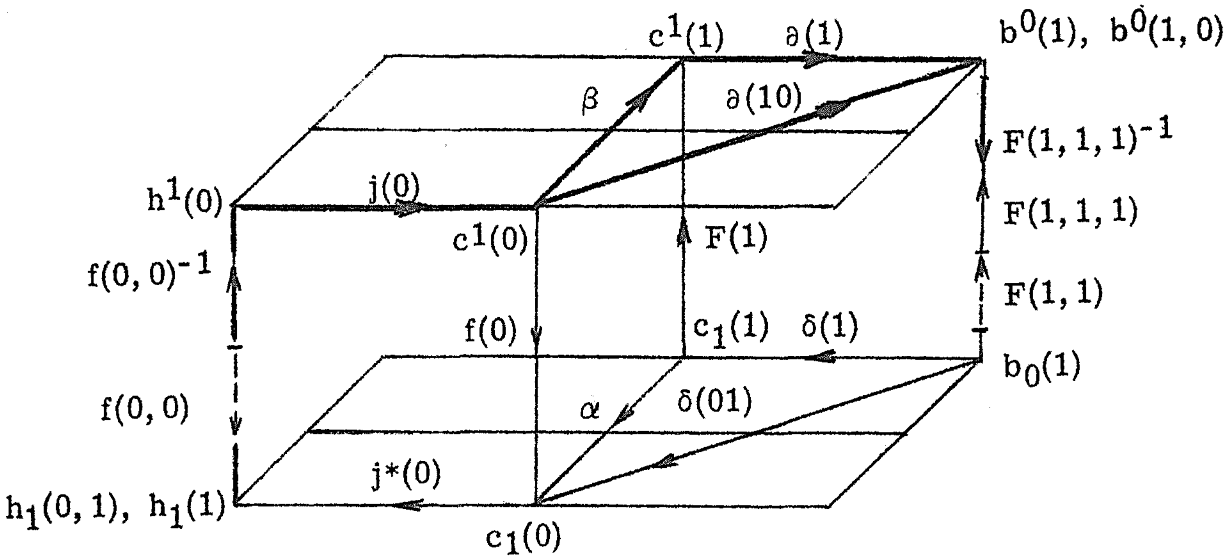


Fig. 28. Diagram representation of codiacoptical solution

<u>The vectors $h_1(0,1)$ and $b^0(1,0)$ with diagram</u>

From (129) we obtain the prescribed equivalent cycle deformation $h_1(0,1)$, and the equivalent structure load $b^0(1,0)$

(150) $h_1(0,1) = j^*(0)\, c_1(0)_p - j^*(0)\, f(0)\, c^1(0)_p - j^*(0)\,\alpha\, c_1(1)_p$

(151) $b^0(1,0) = \partial(1)\, c^1(1)_p - \partial(1)\beta\, c^1(0)_p - \partial(1)\, F(1)\, c_1(1)_p$

in which c_{1p} and c^1_p are obtained from (130) to (139). The vectors $h_1(0,1)$ and $b^0(1,0)$ are represented in the diagram in Figs. 29 and 30.

<u>Calculation of member forces and member deformations</u>

When the unknown cycle forces $h^1(0)$ and structure deformations $b_0(1)$ have been calculated by either the diacoptical or codiacoptical method, we know the behaviour of the whole structure.

By use of formulas (119) and (123) the member forces $c^1(0)$ and member deformations $c_1(1)$ can be calculated:

(152) $$\begin{bmatrix} c^1(0) \\ c_1(1) \end{bmatrix} = \begin{bmatrix} c^1(0)_p + j(0)\, h^1(0) \\ c_1(1)_p + \delta(1)\, b_0(1) \end{bmatrix}$$

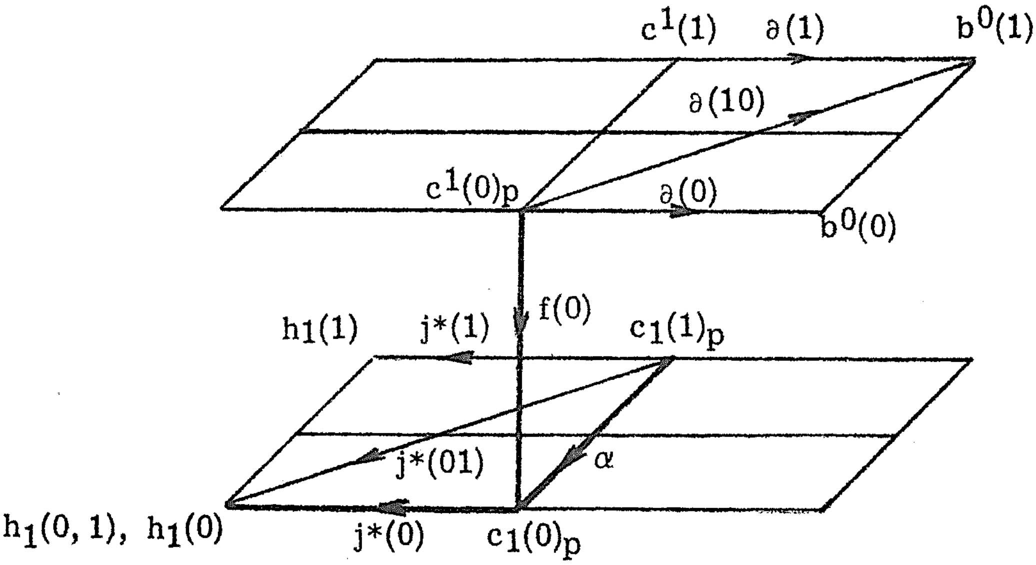


Fig. 29. Diagram representation of the equivalent cycle deformation $h_1(0,1)$

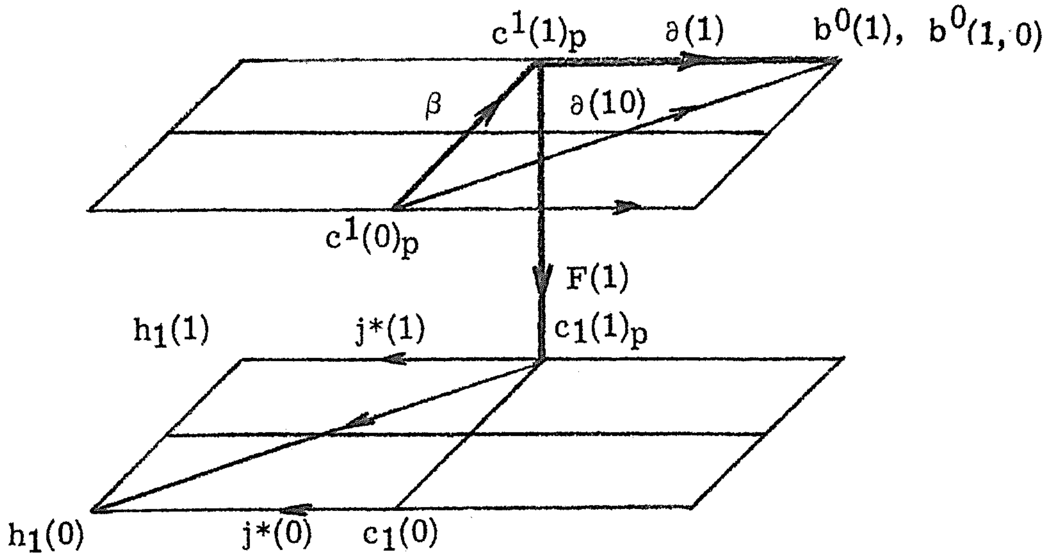


Fig. 30. Diagram representation of the equivalent structure load $b^0(1,0)$

The member deformations $c_1(0)$ and member forces $c^1(1)$ can be calculated by (124) as

(153) $$\begin{bmatrix} c_1(0) \\ c^1(1) \end{bmatrix} = \begin{bmatrix} f(0) & \alpha \\ \beta & F(1) \end{bmatrix} \begin{bmatrix} c^1(0) \\ c_1(1) \end{bmatrix}$$

Calculation of structure deformations

When the member deformations have been obtained by (152) and (153), the structure deformations b_0 can be determined by use of (123) or

(154) $$\begin{bmatrix} c_1(0) - c_1(0)_p \\ c_1(1) - c_1(1)_p \end{bmatrix} = \begin{bmatrix} \delta(0) & \delta(01) \\ 0 & \delta(1) \end{bmatrix} \begin{bmatrix} b_0(0) \\ b_0(1) \end{bmatrix}$$

We observe that if the structure deformations are required the whole particular solution c_{1p} must be determined. For solving the member forces, only $c_1(1)_p$ is needed. Thus we find from (139) that

(155) $$c_1(0)_p = j^*(0)^{-1}(-j^*(01)\,c_1(1)_p + h_1(0))$$

where $j^*(0)^{-1}$ is determined by

(156) $$j^*(0)\,j^*(0)^{-1} = I$$

We shall study two special cases. In the first case the structure X(1) is grounded and in the second case not grounded.

If X(1) is grounded, see Fig. 31, the structure deformations in X(1) are

(157) $b_0(1)$ according to (138) or (145)

By part solution of (154) we find the structure deformations

(158) $$b_0(0) = \delta(0)^{-1}\left[c_1(0) - c_1(0)_p - \delta(01)\,b_0(1)\right]$$

We observe that $\delta(0)$ and $\partial(0)$ are dual operators so $\delta(0)^{-1}$ and $\partial(0)^{-1}$ are dual operators. The operator $\partial(0)^{-1}$ has been determined already in (133) in order to get a particular solution of forces.

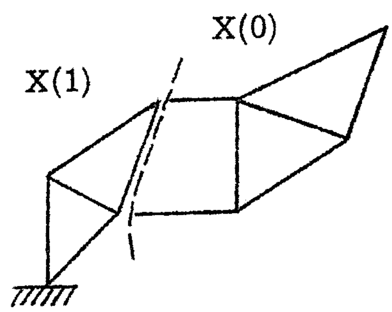


Fig. 31. Structure with part X(1) grounded

In the second case the part X(1) is not grounded but X(0) is, see Fig. 32. This implies that the structure deformations $b_0(1)$ calculated by (138) or (145) are structure deformations, relative to a chosen point in X(1).

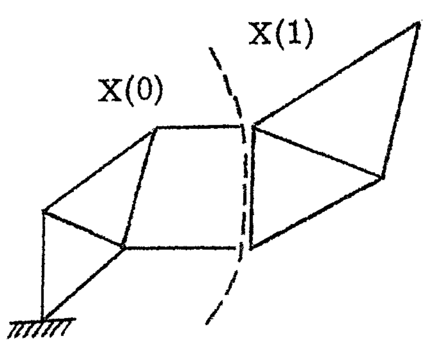


Fig. 32. Structure with part X(0) grounded

In the equation (154) $b_0(1)$ is split into two parts $b_0(1)_Q$ and $b_0(1)$. The structure deformations $b_0(1)_Q$ are the real structure deformations of point Q. This point can then be chosen as reference point for the structure deformations in X(1). This implies that $\delta(01)$ and $\delta(1)$ are dissected into two parts according to

$$(159) \qquad \begin{bmatrix} c_1(0) - c_1(0)_p \\ c_1(1) - c_1(1)_p \end{bmatrix} = \begin{bmatrix} \delta(0) & \delta(01)_Q & \delta(01) \\ 0 & \delta(1)_Q & \delta(1) \end{bmatrix} \begin{bmatrix} b_0(0) \\ b_0(1)_Q \\ b_0(1) \end{bmatrix}$$

The structure deformations $b_0(0)$ for the grounded structure X(0) are determined by (159). They are uniquely determined by member deformations in X(0), which means that the term $\delta(0)^{-1}\delta(01)b_0(1)$ in (158) is zero

$$(160) \qquad b_0(0) = \delta(0)^{-1}[c_1(0) - c_1(0)_p]$$

Now let $b_0(1)$ in (159) be the real structure deformations in X(1) and denote it by $b_0(1)_r$. From the second row in (139) it is found that

$$(161) \qquad b_0(1)_r = \delta(1)^{-1}(c_1(1) - c_1(1)_p) - \delta(1)^{-1}\delta(1)_Q b_0(1)_Q$$

where $\delta(1)^{-1}$ is determined from

$$(162) \qquad \delta(1)\delta(1)^{-1} = I$$

We observe that in (161) $\delta(1)^{-1}c_1(1) = b_0(1)$ are the relative structure deformations referred to point Q found in the diacoptical or codiacoptical solution. Thus it is not necessary to resolve the whole deformation picture. The first row in (159) gives the value of $b_0(1)_r$ at the boundaries of X(0). The equation (161) contains as many equations to solve $b_0(1)_Q$ as there are connection points between X(0) and X(1). The term $-\delta(1)^{-1}\delta(1)_Q b_0(1)_Q$ in (161) is the rigid motion of part X(1) to fit to the boundaries of X(0).

If X(0) or X(1) consists of disjoint parts and some of them are grounded, and some of them are not, then the calculation of structure deformations can be made by use of a combination of the two cases developed above.

ACTIONS ON THE STRUCTURE

The solution of the differential equation describing the load deformation relation in a member

The main problem in the structural analysis is to find member forces and structure deformations when external forces, temperature changes and misfits are prescribed. When a structure is to be analyzed, it is divided into discrete elements, members, for which the differential equation describing the load deformation relation is fairly easy to solve. The solution of the differential equation can be divided into two parts. The first part gives the connection between forces and deformations at the boundaries which is described by flexibility and stiffness transformations, see the diacoptical and codiacoptical solutions. The second part, a particular solution of member forces and member deformations, is dependent on the non-boundary action on the segment.

The choice of particular solution

When a non-boundary action on a segment is prescribed the particular solution of member forces c_1^s and member deformations c_{1s} can be chosen in such a way that half of the number of boundary values can be chosen arbitrarily. Then the continuity condition for forces and deformations are not satisfied, which implies that in order to obtain continuity we must add forces

(163) $\quad b^0 = -\Sigma\, c_s^1$

at the nodes and deformations

(164) $\quad h_1 = -j^* c_{1s}$

round each cycle. We can for example choose, compare Samuelsson [1], the pure cases in Fig. 33a or Fig. 33b as particular solutions. Such particular solutions for beams can be found in for example Bygg [23] or Åkesson [24].

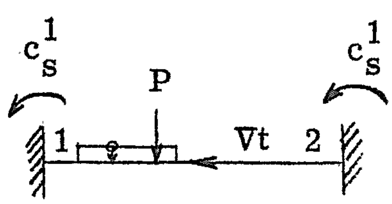


Fig. 33a. Particular solution when $b_0 = 0$. Load: forces P and temperature gradient Vt

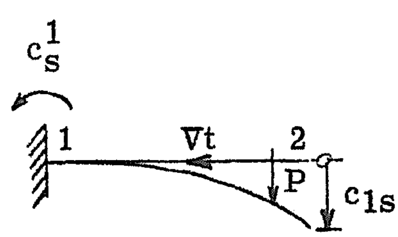


Fig. 33b. Particular solution when $b_{01} = 0$; $c_2^1 = 0$

The total member forces and member deformations

The total member forces c_r^1 and member deformations c_{1r} are obtained by superposition

(165) $$c_r^1 = c_s^1 + c^1$$

(166) $$c_{1r} = c_{1s} + c_1$$

where c^1 and c_1 are obtained from (152) and (153).

Matrix representation of the elasticity condition

In order to calculate the flexibility and stiffness matrices for the straight elastic beam we choose linearly independent forces and moments at end 2 of the oriented straight beam in Fig. 34a. The load components, collected in a column matrix $N_i^{\prime}$, are written relative to a chosen coordinate system (x', y', z') according to Fig. 34b. The corresponding deformations of the member are written as a column matrix $n_i^{\prime} = p_2^{\prime} - p_1^{\prime}$. If we assume that $p_1^{\prime} = 0$ we get the simple cantilever problem in Fig. 34b.

Fig. 34a. Oriented straight beam

Fig. 34b. Cantilever with chosen vector basis for N

The flexibility and stiffness matrices are defined in (107) and (108) where

(166) $n_i^{\prime} = f_i^{\prime} N_i^{\prime}$; $F_i^{\prime} = f_i^{\prime -1}$

The flexibility and stiffness matrices f' and F' for the cantilever in Fig. 35 are directly cited from Samuelsson [1], pp. 66-67.

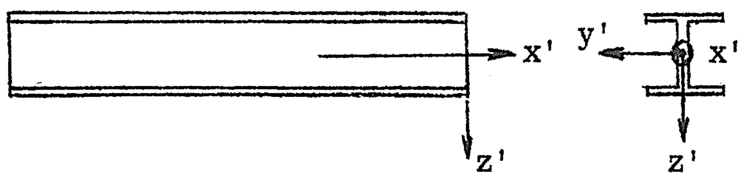


Fig. 35. Straight beam

(167) $f' = \begin{bmatrix} \alpha & \beta \\ \beta^* & \gamma \end{bmatrix}$, $F' = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$

where α, β, γ, a, b, c are 3 · 3-matrices.

For a beam with constant size and form of cross-section, elementary theory of elastic beams yields

(168) $\alpha = \text{diag}\,[1/\mu, \quad 4\omega(1+\lambda_z/2), \quad 4(1+\lambda_y/2)\,]\, L^3/12\,EI_y$

$\gamma = \text{diag}\,[1/\kappa, \quad 12, \quad 12\omega\,]\, L/12\,EI_y$

$a = \text{diag}\,[\,\mu, \quad 1/(1+2\lambda_z)\omega, \quad 1/(1+2\lambda_y)\,]\, 12\,EI_y/L^3$

$c = \text{diag}\,[\,\kappa, \quad (1+\lambda_y/2)/3(1+2\lambda_y), \quad (1+\lambda_z/2)/3\omega(1+2\lambda_z)\,]\, 12\,EI_y/L$

$$\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 6\omega \\ 0 & -6 & 0 \end{bmatrix} L^2/12\,EI_y, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/2\omega(1+2\lambda_z) \\ 0 & 1/2(1+2\lambda_y) & 0 \end{bmatrix} 12\,EI_y/L^3$$

where L is the length of the beam, E is Young's modulus, I_y, I_z are the moments of inertia and $\omega = I_y/I_z$. Further, $\mu = A\,L^2/12\,I_y$, and $\lambda_i = 6\beta_i\,EI_i/L^2\,GA$, $i = y, z$, express the influence of normal and shear deformations where A is the area of the cross-section, G the modulus of elasticity in shear, and β_i a constant that depends upon the form of the cross-section. Finally, $\kappa = \gamma GI_v/2EI_y$ where GI_v is S:t Venant's torsional rigidity, and γ is the warping constant.

For the case of plane frames and grids the matrices f' and F' are reduced, see Samuelsson [1], pp. 69-71.

The vector basis for the load deflection characteristics N_q and n_q for the member can be chosen quite arbitrarily at a point q. The connection between two vector basis is a pure coordinate transformation, see Samuelsson [1] or Fenves and Branin [2]. Physically this transformation can be interpreted by insering a stiff beam, which transmits the forces and deformations between the points 2 and q, see Fig. 36.

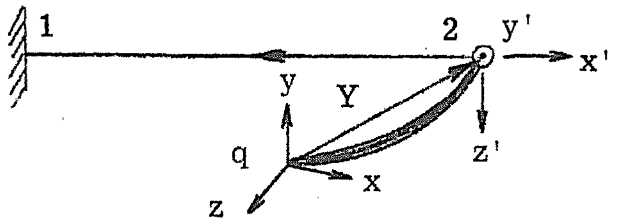


Fig. 36. Load deformation characteristics for the member are given relative an arbitrary vector basis at q

The connection between the forces in the two vector basis is written

(169) $N' = T_q N_q$

where T_q is a $6 \cdot 6$-matrix

(170) $$T_q = \begin{bmatrix} \gamma & 0 \\ \gamma Y & \gamma \end{bmatrix} = KG$$

with

(171) $$K = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}, \quad G = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$$

(172) $$\gamma = [\gamma_{i1}, \gamma_{i2}, \gamma_{i3}] ; \quad [\gamma_{ik}] = 1 \text{ for } k = x, y, z$$

The term γ_{ik} is the direction cosine of the i - th(i = x', y', z') local axis relative to the k - th(k = x, y, z) axis at q, and

(173) $$Y = \begin{bmatrix} 0 & Y_z & -Y_y \\ -Y_z & 0 & Y_x \\ Y_y & -Y_x & 0 \end{bmatrix}$$

where Y_i are the components of the vector Y, which is the location vector of beam end 2 in the coordinate system at q.

For the associated deformations then

(174) $$n_q = T_q^* n'$$

where T_q^* is the transposed matrix of T_q. Dual transformations are represented by transposed matrices, see Halmos [17].

By use of (166), (169) and (174) we get the flexibility matrix f_q relative the coordinate system (x, y, z) at q

(175) $$n_q = T_q^* f' T_q N_q = f_q N_q$$

The corresponding stiffness relation is found by inversion of (175)

(176) $$N_q = f_q^{-1} n_q$$

where we find by use of (166)

(177) $$f_q^{-1} = T_q^{-1} f'^{-1} (T_q^*)^{-1} = T_q^{-1} F' (T_q^*)^{-1} = F_q$$

By observing that γ is a unitary matrix which implies that $\gamma^* = \gamma^{-1}$, and that Z is a scew-symmetric matrix, or $Y^* = -Y$, we find that T_q^{-1} can easily be obtained from the block matrices in T_q, see (170).

(178) $$T_q^{-1} = (KG)^{-1} = G^{-1} K^{-1} = \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix} \begin{bmatrix} \gamma^{-1} & 0 \\ 0 & \gamma^{-1} \end{bmatrix} =$$

$$= \begin{bmatrix} I & 0 \\ Y^* & I \end{bmatrix} \begin{bmatrix} \gamma^* & 0 \\ 0 & \gamma^* \end{bmatrix} = \begin{bmatrix} \gamma^* & 0 \\ Y^* \gamma^* & \gamma^* \end{bmatrix}$$

For the structure part X(0) we now collect the member deformations n_i' in a column matrix $n' = [n_i']$ and the member forces N_i' in a column matrix $N' = [N_i']$, see (60). In the same way we get $n_q = [n_{qi}]$, $N_q = [N_{qi}]$, $n' = [f']^D N'$, $N' = [T_q]^D N_q$ and $n_q = [T_q^*]^D n'$.

We now get

(179) $n(0) = n_q = [T_q^*]^D [f']^D [T_q]^D N_q = [T_q^* f' T_q]^D N_q = [f_q]^D N_q = f(0) N(0)$

For the structure part X(1) we get in the corresponding manner

(180) $N(1) = [T_q^{-1}]^D [F']^D [(T_q^*)^{-1}]^D n_q = [T_q^{-1} F'(T_q^*)^{-1}]^D n_q = [F_q]^D n_q =$

$= F(1) n(1)$

Thus we have obtained the block matrices, f(0) and F(1) in (111).

Representation of the boundary operator ∂ by the matrix E

The equilibrium of a loaded joint will be studied. The member forces N' and the associated deformations n' are given relative a local coordinate system. The structure loads b^0 are represented by a column matrix denoted by $P = [P(0)^*, P(1)^*]^*$, according to the structural dissection, see (78), and the corresponding structure deformations b_0 by a column matrix $p = [p(0)^*, p(1)^*]^*$.

The continuity condition for forces on a part of a structure, the joints A and D, are written in coordinate systems with origins at arbitrary points q and s, see Fig. 37.

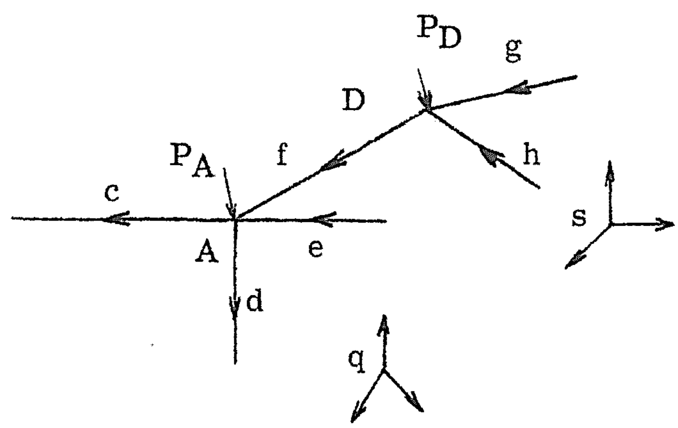


Fig. 37. Part of a structure

By inversion of (169) we get

(181) $N_q = T_q^{-1} N'$

By summing up all vector components N_c to N_f after suitable geometric transformations to the joints A and D, and noticing that the forces N_e and N_f affect A with negative sign, we get the equilibrium equations

(182) $$\begin{bmatrix} P_A \\ P_D \end{bmatrix} = \begin{bmatrix} T_{qc}^{-1} & T_{qd}^{-1} & -T_{qe}^{-1} & -T_{qf}^{-1} & 0 & 0 \\ 0 & 0 & 0 & T_{sf}^{-1} & -T_{sg}^{-1} & T_{sh}^{-1} \end{bmatrix} \begin{bmatrix} N_c \\ N_d \\ N_e \\ N_f \\ N_g \\ N_h \end{bmatrix}$$

or short

(183) $$P = EN$$

(Observe that this E represents the boundary operator ∂ as the incidence matrix E, see (78), does, but this is not the same E)

Now we can write the structure stiffness relation

(184) $$P = EFE^*p$$

where E^* is the transposed matrix of E.

(185) $$F = [F_i']^D \; ; \quad i = cdefgh$$

(186) $$EFE^* = \begin{bmatrix} \sum_{i = cdef} T_{qi}^{-1} F_i' (T_{qi}^*)^{-1} & -T_{qf} F_f' (T_{sf}^*)^{-1} \\ -T_{sf} F_f' (T_{qf}^*)^{-1} & \sum_{k = fgh} T_{sk} F_k' (T_{sk}^*)^{-1} \end{bmatrix}$$

If the coordinate systems at q and s are located at A and D, we get, if we observe the factorization (170) of T_q,

(187) $$E = \begin{bmatrix} K_{Ac}^{-1} & K_{Ad}^{-1} & -T_{Ae}^{-1} & -T_{Af}^{-1} & 0 & 0 \\ 0 & 0 & 0 & K_{Df}^{-1} & -T_{Dg}^{-1} & -T_{Dh}^{-1} \end{bmatrix}$$

If every joint is chosen as an origin for the equilibrium equation at the respective joint and if the member stiffness is written for every member relative to a coordinate system at beam end 2, and if the direction of axes are the same for the whole structure, then we obtain

(188) $$E = \begin{bmatrix} I & I & -G_{qe}^{-1} & -G_{qf}^{-1} & 0 & 0 \\ 0 & 0 & 0 & I & -G_{sg}^{-1} & -G_{sh}^{-1} \end{bmatrix}$$

(189) $$F = [K_i^{-1} F_i' K_i]^D ; \quad i = cdefgh$$

If the same origin q and the same direction of axes are chosen for the whole structure we get

(190) $$E = \begin{bmatrix} I & I & -I & -I & 0 & 0 \\ 0 & 0 & 0 & I & -I & -I \end{bmatrix}$$

(191) $$F = [F_{qi}]^D = [T_{qi}^{-1} F_i' (F_{qi}^*)^{-1}]^D; \quad i = cdefgh$$

Comparing the result in (190) with that earlier obtained in (188) we observe that network theory is directly applicable, by simply interpreting the elements of the incidence matrix (2) as identity matrices of appropriate order. The topological information, in E, is separated from metrics and mechanical properties in F according to (191). Samuelsson [1] and Fenves-Branin [2] have written the structural equations relative to such a global coordinate system.

The choice of global basis may lead to ill-conditioning of the set of equations because the forces, the deformations and the mixed isomorfism must be transformed to a common origin and their influences studied there. This will imply that the errors are correspondingly larger for influences from parts of the structure, which are remote from the origin. If the structure is large, these errors will be too large for the choice of a common origin in practical calculations. The choice (187) or (188) of boundary operator will make the matrix EFE* better conditioned, see Fenves-Branin [2].

We observe that we can get (188) if we modify the topological transformation ∂ in such a way that all positive elements in the incidence matrix E are interpreted as identity matrices of appropriate order and all negative elements are replaced by geometric transformations G from positive to negative end of the members.

The dissection of the structure implies a dissection of the boundary operator, see (77). For the corresponding matrix E the dissection implies a partitioning of the matrix, see (79).

Representation of the operator j by the matrix Z

The equilibrium of a structure with zero joint loads will now be studied. The member forces N' and the corresponding deformation n' are given relative a local coordinate system. To every cycle in a chosen basis, we associate a cycle force with at most six components. All cycle forces h^1 in a structure are represented by a column matrix $R = [R(0)^*, R(1)^*]^*$, see (92), and corresponding deformations h_1 called cycle deformations or gaps, are denoted by a column matrix $r = [r(0)^*, r(1)^*]^*$. The transformation between cycle forces R and member forces N are represented by a matrix Z, defined by

(192) $$N = ZR$$

For the structure part in Fig. 38 we find, if we choose a separate vector basis for each cycle,

$$(193)\qquad \begin{bmatrix} N_c \\ N_d \\ N_e \\ N_f \\ N_g \\ N_h \end{bmatrix} = \begin{bmatrix} T_{qc} & 0 \\ -T_{qd} & 0 \\ -T_{qe} & 0 \\ T_{qf} & -T_{sf} \\ 0 & T_{sg} \\ 0 & T_{sh} \end{bmatrix} \begin{bmatrix} R_q \\ R_s \end{bmatrix}$$

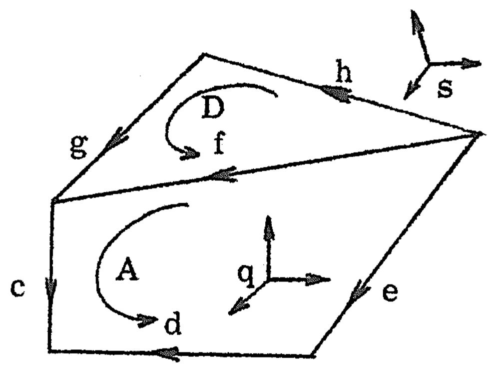


Fig. 38. Part of a structure

According to (144) we get the structure flexibility relation in matrix form as

$$(194)\qquad r = Z^* f\, Z R$$

where

$$(195)\qquad f = [f_i']^D; \quad i = cdefgh$$

$$(196)\qquad Z^* f Z = \begin{bmatrix} \sum_{i=cdef} T^*_{qi} f_i' T_{qi} & -T^*_{gf} f_f' T_{sf} \\ -T^*_{sf} f_f' T_{gf} & \sum_{k=fgh} T^*_{sk} f_k' T_{sk} \end{bmatrix}$$

We can choose one origin for each cycle and let the direction of the axes be the same for the whole structure. Then the member flexibility f' for every member is written relative to a coordinate system with origin at beam end 2. We then get by use of separation (170) of T

$$(197)\qquad Z^* = \begin{bmatrix} G^*_{qc} & -G^*_{qd} & -G^*_{qe} & G^*_{qf} & 0 & 0 \\ 0 & 0 & 0 & -G^*_{sf} & G^*_{sg} & G^*_{sh} \end{bmatrix}$$

$$(198)\qquad f = [K_i^* f_i' K_i]^D$$

The matrix (196) can also be written with components

$$(199) \qquad Z^* = \begin{bmatrix} I & -I & -I & I & 0 & 0 \\ 0 & 0 & 0 & -T^*_{qs} & I & I \end{bmatrix}$$

where T_{qs} is the geometrical transformation between q and s, and

$$(200) \qquad f = [f_{ki}]^D = [T^*_{ki} f'_i T_{ki}]^D; \quad \begin{matrix} k = q; & i = cdef \\ k = s; & i = gh \end{matrix}$$

If the same vector base is chosen for the whole structure we get

$$(201) \qquad Z^* = \begin{bmatrix} I & -I & -I & I & 0 & 0 \\ 0 & 0 & 0 & -I & I & I \end{bmatrix}$$

$$(202) \qquad f = [T_{qi}]^D = [T^*_{qi} f'_i T_{qi}]^D; \qquad i = cdefgh$$

Here we get the direct counterpart to network theory by interpreting the elements of the incidence matrix, see (3), as identity matrices.

The vector basis for the cycle forces should be chosen, so that the matrix Z^*fZ will be as well conditioned as possible. As far as the conditioning of the flexibility matrix is concerned the main aim should be to make the leading diagonal elements as large as possible in comparison with the off-diagonal elements. Physically this corresponds to choosing a system of cycles, a cycle basis, in which the direct flexibilities of the cycles are large compared with the flexibilities of the members which couple the members together. If the cycles have no member in common, the corresponding off-diagonal elements are zero. It follows that the cycle forces in a network should be chosen in such a way that the cycles are as independent of each other as possible. By a suitable choice and ordering of the cycles it is possible to get the matrix Z^*fZ in a banded form. The set of equations is then correspondingly easy to solve, see Henderson [25].

The dissection of the structure implies a dissection of the operator j, see (90). For the corresponding matrix Z the dissection implies a partitioning of the matrix, see (92).

Elimination of constrained axes

Hitherto the formulation is based on the assumption that each member in space transmits six independent force components. In the general case, there exist constraints among the forces and moments which means that one, two or three of the forces or moments (or any combination of them) may be zero at the end of the beam in a particular direction or plane. To take care of the constraints the solution procedure must be modified. As a first step a special coordinate system is chosen for every constrained beam with an orientation so assumed that one or two of the axes are directed along the directions or planes of constraints. Constraints can also exist among deformations.

The stiffness formulation of elimination of constraints

The modified procedure because of the constraints implies that the stiffness matrix must be changed.

Assume that there exists a constraint among the loads N' on a beam, see Kron [5]. This implies that in a coordinate system with a special direction, the load vector N^c has some zero components. The relation between the forces N' and N^c are described by a transformation matrix M

$$(203) \qquad N' = MN^c = M\begin{bmatrix} N_1^c \\ 0 \end{bmatrix}, \text{ or } N^c = M^{-1}N'$$

For the members the stiffness relation

$$(204) \qquad N' = F'n'$$

holds. By use of (203) and (204) we write the equation system in the partitioned form

$$(205) \qquad N^c = M^{-1}N' = M^{-1}F'(M^{-1})^* n^c = \begin{bmatrix} N_1^c \\ 0 \end{bmatrix} = \begin{bmatrix} F_{11}^c & F_{12}^c \\ F_{21}^c & F_{22}^c \end{bmatrix} \begin{bmatrix} n_1^c \\ n_2^c \end{bmatrix}$$

Part-inversion of (205) yields

$$(206) \qquad N_1^c = (F_{11}^c - F_{12}^c(F_{22}^c)^{-1}F_{21}^c)n_1^c = F_1^c n_1^c$$

The continuity equation for forces $P = EN$ can now be written in partitioned form as

$$(207) \qquad P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} E_{11}^c & E_{12}^c \\ E_{21}^c & E_{22}^c \end{bmatrix} \begin{bmatrix} N_1^c \\ 0 \end{bmatrix} = \begin{bmatrix} E_{11}^c \\ E_{21}^c \end{bmatrix} N_1^c = E_1^c N_1^c$$

The relation between structure deformations and loads then is

$$(208) \qquad P = E_1^c F_1^c E_1^{c*} p$$

In the case that all beams which connect at a point have the same release, then $P = [P_1^*, P_2^*]^* = [P_1^*, 0]^*$. From (207) and (208) follows that

$$(209) \qquad P_1 = E_{11}^c F_1^c E_{11}^{c*}$$

Assume now that there exists a constraint among the deformations. Then in a coordinate system with a special orientation the deformation vector p has zero components and the equation system of stiffness form can be written in a partitioned form as

$$(210) \qquad \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} F_{11}^c & F_{12}^c \\ F_{21}^c & F_{22}^c \end{bmatrix} \begin{bmatrix} p_1 \\ 0 \end{bmatrix}$$

from which

$$(211)\qquad P_1 = F^c_{11}\, p_1$$

is obtained.

<u>The flexibility formulation of the elimination of constraints</u>

Assume that for a cycle there exists a constraint among the cycle forces R. Let R^c_1 be the linearly independent cycle forces in a coordinate system of the special orientation mentioned above. Thus

$$(212)\qquad R^c = \begin{bmatrix} R^c_1 \\ R^c_2 \end{bmatrix} = \begin{bmatrix} R^c_1 \\ 0 \end{bmatrix}$$

The relation between the cycle forces R^c and the cycle forces R_q written at point q can be written

$$(213)\qquad R_q = MR^c$$

The beam flexibilities are written at point q

$$(214)\qquad n_q = f_q N_q$$

and the continuity condition for the forces are

$$(215)\qquad N_q = ZR_q$$

From (213) and (214) we find

$$(216)\qquad N_q = ZMR^c = Z^c R^c$$

The structure flexibility equations are now written in partitioned form

$$(217)\qquad r = Z^{c*} f\, Z^c R^c = \begin{bmatrix} r^c_1 \\ r^c_2 \end{bmatrix} = \begin{bmatrix} f^c_{11} & f^c_{12} \\ f^c_{21} & f^c_{22} \end{bmatrix} \begin{bmatrix} R^c_1 \\ 0 \end{bmatrix}$$

from which we get

$$(218)\qquad r^c_1 = f^c_{11} R^c_1$$

Assume that there exists a constraint among the deformations. Then in a special coordinate system, r has some zero components and we can write the structure flexibility equations

(219) $$\begin{bmatrix} r_1^c \\ 0 \end{bmatrix} = \begin{bmatrix} f_{11}^c & f_{12}^c \\ f_{21}^c & f_{22}^c \end{bmatrix} \begin{bmatrix} R_1^c \\ R_2^c \end{bmatrix}$$

Part-inversion of (219) yields

(220) $$r_1^c = (f_{11}^c - f_{12}^c (f_{22}^c)^{-1} f_{21}^c) R_1^c$$

STABILITY CONDITIONS. DISCUSSION OF THE DIACOPTICAL AND CODIACOPTICAL METHODS

Stability conditions

In order to get a stability condition the coefficient matrix for the diacoptic or codiacoptic solution is studied. Assume that the dissection of X is made in such a way that $\alpha = \beta = 0$.

The codiacoptic solution, compare (147)

$$F(1,1,1)\, b_0(1) = b^0(1,0,1) \tag{221}$$

is studied here.

To get a condition of stability the eigenvalues of the coefficient matrix F(1, 1, 1) are studied by a method developed by Wahlström [26]. A necessary and sufficient condition for the frame to be stable, is that the quadratic form $b_0(1)^* F(1,1,1)\, b_0(1)$ is positive definite, or that the matrix F(1, 1, 1) of the quadratic form has only positive eigenvalues. Such an investigation consists in a study of the sign of the eigenvalues of F(1, 1, 1) which is carried out by use of congruent transformations of F(1, 1, 1) to diagonal form. The sign of the eigenvalues are invariant under such transformations.

In the part X(0) of the structure we obtain the flexibility matrix $f(0) = [f_q]^D$ according to (179). According to (180) we obtain for the part X(1) the stiffness matrix $F(1) = [F_q]^D$.

We assume that all f_q and F_q are positive definite matrices and thus f(0) and F(1) are positive definite.

It holds, see Zurmühl, [27], p. 133:

> If U is an $n \cdot p$-matrix with the rank r and V is a symmetric positive definite $n \cdot n$-matrix, then the $p \cdot p$-matrix U^*VU is positive definite with the rank r.

The coefficient matrix F(1, 1, 1) given by (148) is now studied. If we use (97) and (103), if we represent $\partial(1)$ by a $6\alpha_1(1) \cdot 6(\alpha_0(1) - 1)$-matrix E(1) with the rank $6(\alpha_0(1) - 1)$, and if we represent j(0) by a $6\alpha_1(0) \cdot 6(\alpha_1(0) - \alpha_0(0) + 1)$-matrix with the rank $6(\alpha_1(0) - \alpha_0(0) + 1)$. Then we obtain

$$F(1,1,1) = E(1)\, Z(10)\, (Z(0)^* f(0)\, Z(0))^{-1}\, Z(10)^* E(1)^* + E(1)\, F(1)\, E(1)^* \tag{222}$$

We thus find that because F(1) and f(0) are positive definite matrices then $F(1,1) = E(1)\, F(1)\, E(1)^*$, $f(0,0) = Z(0)^* f(0)\, Z(0)$ and $f(0,0)^{-1}$ are positive definite matrices. We also find that the matrix

$$E(1)\, Z(10)\, (Z(0)^* f(0)\, Z(0))^{-1}\, Z(10)^* E(1)^* = (Z(10)^* E(1)^*)^* f(0,0)^{-1}\, Z(10)^* E(1)^* \tag{223}$$

is positive definite with a rank which is 6 times the number of the relative cycles on X(0). The sum F(1, 1, 1), (222), of the two positive definite matrices is positive definite.

Thus $F(1, 1, 1)$ is positive definite if $f(0)$ and $F(1)$ are positive definite. The reverse statement does not hold.

Discussion

In the following the investigation is restricted to diacoptics and the case that $\alpha = \beta = 0$, but what is here said about diacoptics can directly be used in codiacoptics in the corresponding manner.

From (140) to (145) we obtain

(224) $$p(1) = F(1,1)^{-1} [-E(10)\, Z(0) R(0) + P(1,0)]$$

(225) $$R(0) = f(0,0,0)^{-1} r(0,1,0)$$

where

(226) $$f(0,0,0) = Z(0)^* E(01)^* F(1,1)^{-1} E(10)\, Z(0) + f(0,0)$$

(227) $$r(0,1,0) = r(0,1) + Z(0)^* E(01)^* F(1,1)^{-1} P(1,0)$$

The equations (225) and (227) are inserted into (224) which yields

(228) $$\begin{aligned} p(1) = {} & F(1,1)^{-1} P(1,0) - \\ & - F(1,1)^{-1} E(10)\, Z(0) f(0,0,0)^{-1} Z(0)^* E(10)^* F(1,1)^{-1} P(1,0) - \\ & - F(1,1)^{-1} E(10)\, Z(0) f(0,0,0)^{-1} r(0,1) \end{aligned}$$

The constituent elements of the solution written in the factorized form (228) can be given in a scheme (not a matrix) of four matrices, called the <u>factorized inverse</u> due to Kron [5].

(229) $$\left| \begin{matrix} f(0,0,0)^{-1} & Z(0)^* E(10)^* \\ F(1,1)^{-1} E(10)\, Z(0) & F(1,1)^{-1} \end{matrix} \right|$$

These four matrices take the same storage capacity as the non-factorized inverse. The solution is for each loading case determined by vector multiplication from the right, in (228).

It is not necessary to invert the matrix $f(0,0,0)$ in order to calculate $R(0)$. As an alternative, $R(0)$ can be calculated by for example Gaussian elimination applied to

(230) $$f(0,0,0) R(0) = r(0,1,0)$$

where the right hand side is a vector obtained by vector multiplication of $r(0,1,0)$ from the right. One solution of a system of linear equations requires about one third as much work as the inversion of the corresponding matrix of coefficients. By the Gaussian elimination procedure the coefficient matrix $f(0,0,0)$ is transformed to triangular form. By studying the signs of the diagonal terms a stability test is obtained as an additional result. The elements in the diagonal of the triangular matrix are the same as the elements obtained by congruent transformation to diagonal form by the method of Wahlström [26].

If by dissection X(1) consists of a number of disjoint parts $X(1)_i$, the matrix F(1, 1) will contain block matrices $F(1,1)_i$ along the diagonal, and we get

$$F(1,1) = [F(1,1)_i]^D \tag{231}$$

The inverse

$$F(1,1)^{-1} = [F(1,1)_i^{-1}]^D \tag{232}$$

will also have block matrices along the diagonal. The numerical work is reduced to a great extent if X(1) consists of identical substructures having identical stiffness matrices $F(1,1)_i$.

When the solution is given in factorized form it is easy to add new parts $X(1)_i$ to the structure or change the flexibility or stiffness of its members. The solution of the modified structure is found as the solution of the original structure and an additional solution due to the influence of the added parts.

From a practical viewpoint diacoptics can with special advantage be used for solving structures, which is built from identical parts, for instance for structures built from few different (prefabricated) substructures. The elastic behaviour of the substructures can be studied once for all by some suitable method, and then diacoptics can be used to get a rapid solution for the connected structure.

Comparison with existing mathematical algoritms

The mentioned factorized inverse is closely related to the recursion formula called K-partitioning, or elimination and backsubstitution, which is frequently used in literature, see for example Kron [5], Spillers [3], Jenkins [28], Franklin and Branin [29] and Edlund [30].

The recursion formula is obtained from the solution of an equation system

$$Ax = y \tag{233}$$

in partitioned form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{234}$$

$$x_1 = A_{11}^{-1}(y_1 - A_{12}x_2) \tag{235}$$

$$x_2 = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}(y_2 - A_{21}A_{11}^{-1}y_1) \tag{236}$$

If the matrix A is partitioned into more parts than four the K-partitioning will be computationally more effective. In the limit when the partitioning is made along every row and column it coincides with the well-known Gaussian elimination, which thus is the most efficient way of K-partitioning. The elimination gets its most effective form if the variables are ordered in such a way that the coefficient matrix A gets a banded form. It is a combinatorial problem to order the variables suitably. This is shown by Spillers [31], who has studied the special diacoptical problem $X = X(1)$.

Henderson [25] gives methods to make the best selection of redundant forces with respect to the conditioning of the equations of compatibility by use of topology. He thus studies the special codiacoptical problem $X = X(0)$.

In such combinatorial procedures we cannot take care of the advantage of fewer variables, identical parts and the possibility to store parts of the solution as is done by use of diacoptics and codiacoptics.

Diacoptics and codiacoptics can on a pure computational basis be compared with a direct method of inverting a matrix by use of successive modifications. This method was first described by Sherman and Morrison [32]. They showed how to get the inverse of a matrix, when in the original matrix one element at a time is modified. Householder [33] gave this method a more general formulation, and it can be described as follows:

> If the inverse of a matrix A is known, and if a new matrix differs from A additively, then the inverse of the new matrix can be obtained as a result of a modification of the inverse A^{-1} in the specific manner:

(237) $$(A - USV^*)^{-1} = A^{-1} + A^{-1}U(S^{-1} - V^*A^{-1}U)^{-1}V^*A^{-1}$$

where S must be nonsingular.

The modifications can be made for one element at a time. The method is then called a "link at a time" (LAT) algoritm. It can with appreciation be used for small modifications in a structure, see Spillers [3].

The modification formula (237) has a direct counterpart in the diacoptical solution (228), where the inverse $F(1,1)^{-1}$ is modified. The first two terms in (228) has a direct counterpart in the modification formula. The structure part X(1) is first studied by use of its stiffness matrix F(1, 1). It is then modified in order to take care of the elastic conditions in the part X(0) of the structure. The modification formula is only a computational tool and says nothing about how to dissect and modify. The latter information is obtained through diacoptics which thus is more useful than (237).

NOTES ON THE SELECTION OF AN EFFECTIVE DISSECTION

The deformation and force methods

The number of independent joint displacements, used as unknowns in the pure deformation method of structural analysis, is not related to the number of independent cycle forces. The number of independent cycle forces, used as unknowns in the pure force method, is not related to the number of independent joint displacements. In the diacoptical and codiacoptical methods both the number of independent cycle forces and independent number of joint displacements and their location in the structure are essential. In diacoptics and codiacoptics we can choose our equations in such a way that the total number of variables can be reduced under the number used in each of the force and deformation methods.

The computational labour is not always reduced in diacoptics or codiacoptics in comparison with the classical methods mentioned. The form of the graph and the manner of dissection are of importance in this respect.

The completely connected structure

If every two joints in the structure are connected by a member, the structure is said to be completely connected, then dissection is of no use. The stiffness matrix will in this case be full (have no zero elements).

In a completely connected structure with α independent joints there are $(\alpha+1)/2$ members and $\alpha(\alpha-1)/2 = v_l$ independent loops. The number of joints is less than the number of loops

$$\alpha \leqq \frac{\alpha(\alpha-1)}{2}; \quad 0 \leqq \alpha(\alpha-3) \tag{238}$$

when $\alpha \geqq 3$.

Thus a completely connected structure should be solved by the displacement method if $\alpha \geqq 3$, and by the force method if $\alpha \leqq 3$, see Fig. 39.

In a structure with less members than the number which corresponds to the completely connected structure, the loops are fewer than said above. When the graph has a suitable form, the structure can with advantage be solved by use of mixed node and loop variables.

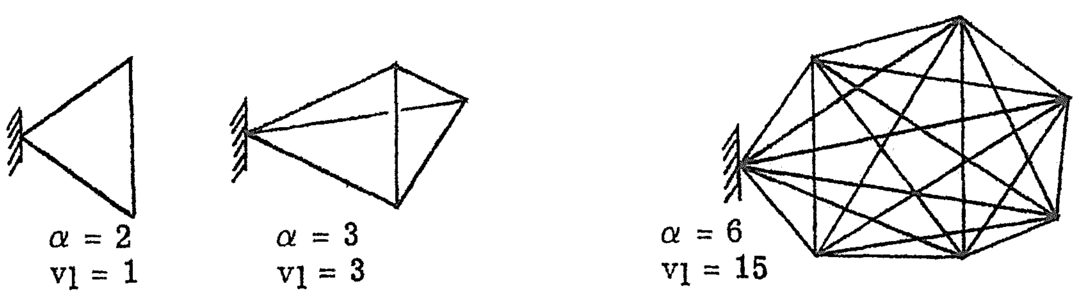


Fig. 39. Some completely connected structures

Loop-node incidence matrix

An appropriate dissection can be found by studying the loop-node incidence matrix L which describes the topological properties of the structure.

The matrix L is defined according to

$$(239) \qquad L_{ij} = \begin{cases} 1 & \text{if node i is contained in loop j} \\ 0 & \text{if node i is not contained in loop j} \end{cases}$$

The matrix L can also be obtained by use of the incidence matrices E and Z in the following way. Delete all negative signs in E and Z which gives the new matrices E_1 and Z_1. By establishing the matrix product $E_1 Z_1$ and replacing all nonzero elements in $E_1 Z_1$ by the element 1, we get a matrix $(E_1 Z_1)_1$. The matrix $(E_1 Z_1)_1$ coincides with L.

At dissection (X(0), X(1)) of the structure X, the total number α of nodes and m of loops are invariant, or

$$(240) \qquad \alpha(0) + \alpha(1) = \alpha$$

$$(241) \qquad m(0) + m(1) = m$$

The total number of unknowns in diacoptics and codiacoptics is

$$(242) \qquad \phi = m(0) + \alpha(1)$$

For the dissected structure we establish the matrix $E_1 Z_1$ in partitioned form as

$$(243) \qquad E_1 Z_1 = \begin{bmatrix} E(0)_1 & 0 \\ E(10)_1 & E(1)_1 \end{bmatrix} \begin{bmatrix} Z(0)_1 & 0 \\ Z(10)_1 & Z(1)_1 \end{bmatrix} =$$

$$= \begin{bmatrix} E(0)_1 Z(0)_1 & 0 \\ E(10)_1 Z(0)_1 + E(1)_1 Z(10)_1 & E(1)_1 Z(1)_1 \end{bmatrix}$$

and

$$(244) \qquad L = (E_1 Z_1)_1 = \begin{matrix} & m(0) & m(1) \\ \alpha(0) & L(0) & 0 \\ \alpha(1) & L(10) & L(1) \end{matrix}$$

We observe that the minimum number of unknowns is obtained when the rows and columns of L are ordered in such a way that the sum $\phi = m(0) + \alpha(1)$ gets its minimum value and the dissection is chosen accordingly.

A main principal of dissection

By studying the matrix L we find that a main principal of dissection is that X should be dissected so that X(1) contains a part of the structure with many loops compared with the number of joints, and so that X(0) contains a part of the structure with many nodes compared with the number of loops.

Example of a selection of an effective dissection

We want to dissect the structure in Fig. 40 in such a way that we get as few unknowns as possible in the fundamental equations.

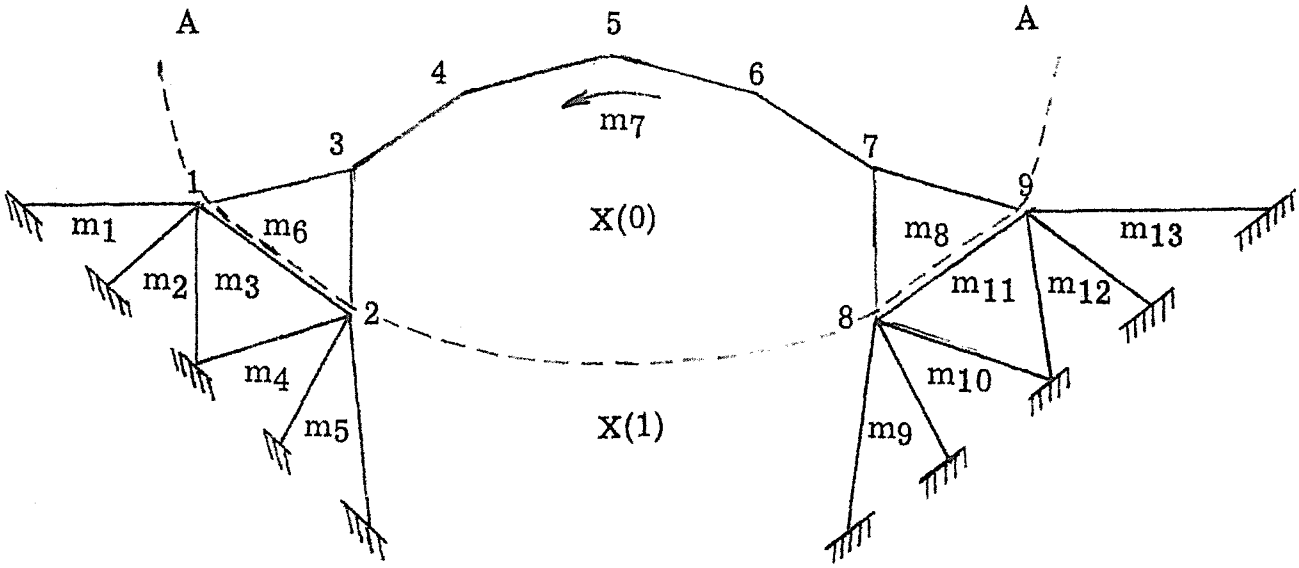


Fig. 40. Structure X, with joints 1 to 9 and independent loops m_1 to m_{13}

In order to find the least possible number of diacoptical variables we study the loop-node incidence matrix L according to (244). For the structure in Fig. 40 we get, after suitable rearrangements of rows and columns,

(245) $$L = \begin{bmatrix} L(0) & 0 \\ L(10) & L(1) \end{bmatrix} =$$

node / loop	6	7	8	1	2	3	4	5	9	10	11	12	13	Σ_m
3	1	1												2
4		1												1
5		1												1
6		1												1
7		1	1											2
1	1			1	1	1								4
2	1	1				1	1	1						5
8		1	1						1	1	1			5
9			1								1	1	1	4
Σ_α	3	7	3	1	1	2	1	1	1	1	2	1	1	

Rows 1, 2, 8, 9: Parts with high loop density. Node variables should be used

Columns 6, 7, 8: Parts with high node density. Loop variables should be used

The sum Σ_α in (245) is the number of nodes in loop j and the sum Σ_m in (245) is the number of loops incident to node i. By studying these sums we easy find the parts of the structure which have low (or high) node or loop density.

We observe that if X(0) contains a node it also contains all the loops incident to it, and if X(1) contains a loop it also contains all the nodes incident to it, see the partitioning lines in the matrix L. By studying the matrix L we easily see how the numbers of loops and nodes vary when the partitioning is changed.

From the matrix L we also find that the dissection AA, see Fig. 40, is one of the (two) dissections which give the minimum number

$$\phi = \alpha(1) + m(0) = 4 + 3 = 7$$

of unknowns in a solution by diacoptics or codiacoptics. This number 7 should be compared with 9 which is the number of unknowns in a pure deformation method solution, and with 13 which is the number of unknowns in a pure force method solution.

DESCRIPTION OF DIACOPTICS AND CODIACOPTICS BY USE OF EQUIVALENT STRUCTURES

The diacoptical method. Equivalent network of node type

The same example as in the preceding chapter is studied here. In the diacoptical method a cut is made in the structure, see Figs. 41a, b.

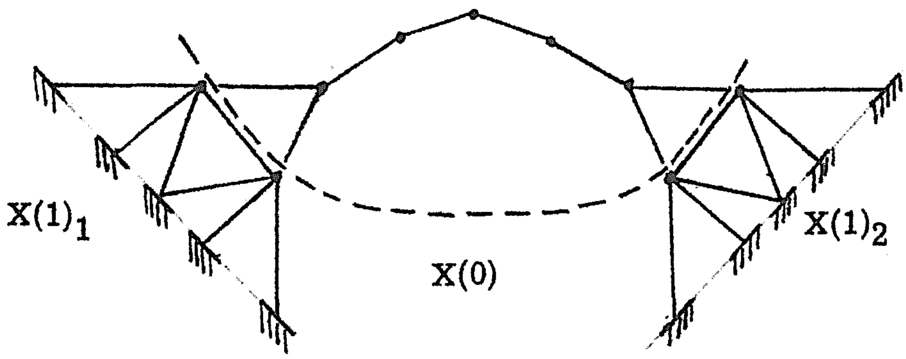


Fig. 41a. Dissected structure X

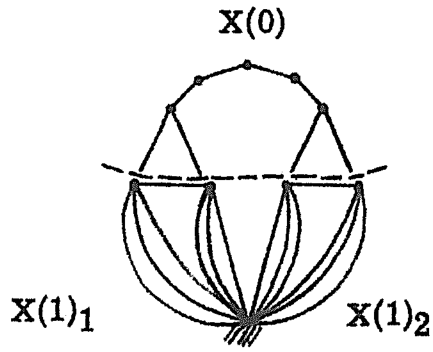


Fig. 41b. The graph X

First the stiffness matrix for each subnetwork $X(1)_i$ is computed relative to the reference point of that substructure. Each subnetwork is then replaced by its equivalent radial tree (radiating from the reference point). The radial tree is called the equivalent subnetwork of node type. It has the same number of joints, see Figs. 42a, b, as the original subnetwork.

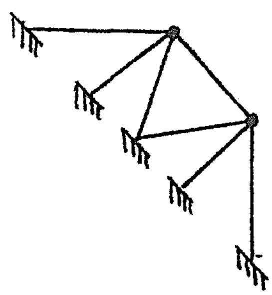


Fig. 42a. Subnetwork $X(1)_1$

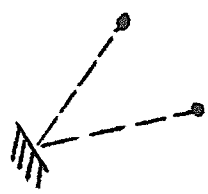


Fig. 42b. Equivalent subnetwork $X(1)_1$ of node type

Connecting the equivalent subnetworks of node type to X(0), we obtain an equivalent network of node type, see Figs. 43a, b. It has the same number of nodes but less loops than the original network. It is therefore convenient to use cycle forces as variables in the remaining analysis.

The codiacoptical method. Equivalent network of loop type

In the codiacoptical method a short-circuiting is made in the structure in the meaning that some node displacements are set to zero, see Fig. 44.

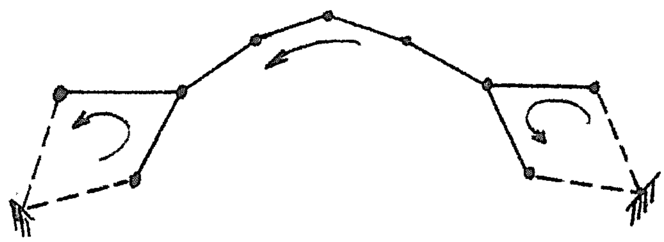


Fig. 43a. Equivalent network of node type

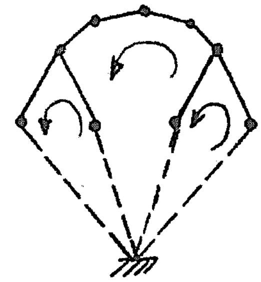


Fig. 43b. Equivalent graph

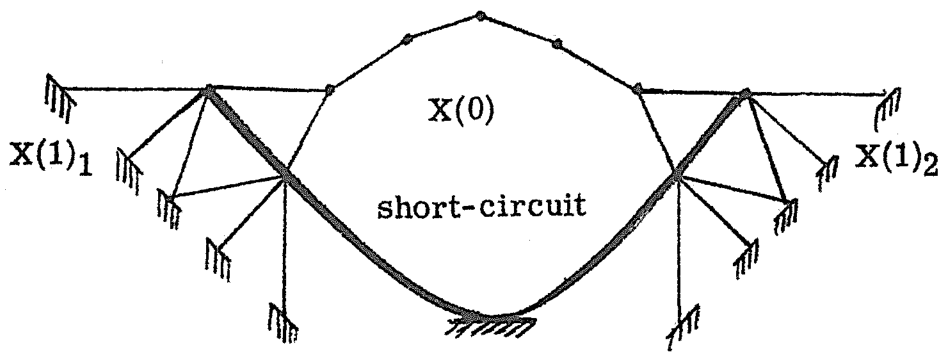


Fig. 44. Dissected structure. Part X(1) is short-circuited

The subnetworks $X(0)_i$ (here only one) are analyzed by use of loop variables. An equivalent subnetwork of loop type is obtained, see Figs. 45a, b.

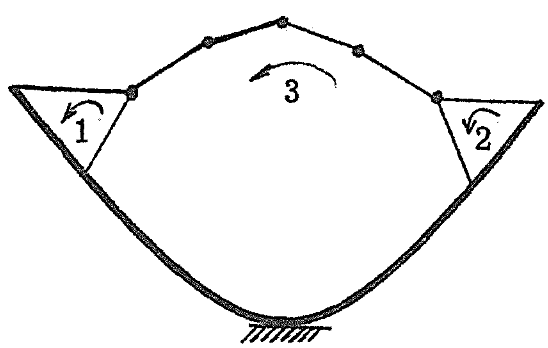


Fig. 45a. Subnetwork X(0)

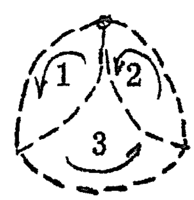


Fig. 45b. Equivalent subnetwork X(0) of loop type

Connecting the equivalent subnetworks of loop type to X(1), we obtain an equivalent network of loop type, see Figs. 46a, b. It has the same number of loops but less number of nodes than the original network. It is therefore convenient to use node displacements as unknowns in the remaining solution.

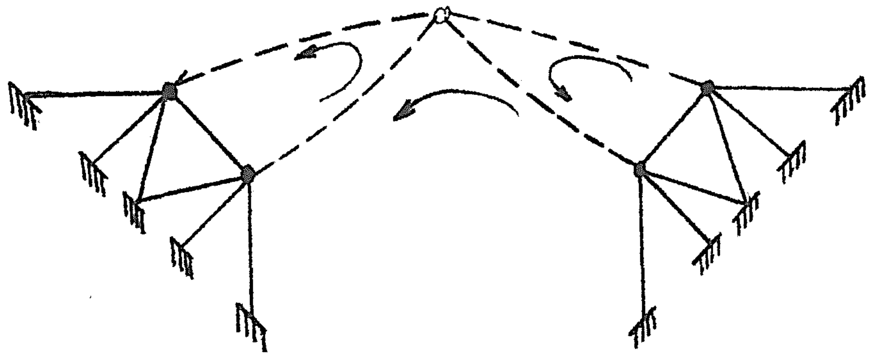


Fig. 46a. Equivalent network X solved by node variables

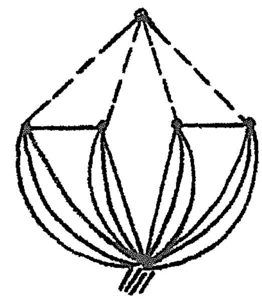


Fig. 46b. Equivalent graph

Choice of method

In the diacoptical method, we obeserve that X(1) consists of two disjoint parts whose solution requires the inversion of two 2 • 2-matrices, and that the interconnection of the two parts requires the inversion of a 3 • 3-matrix. In the codiacoptical method, the solution of X(0) requires the inversion of a 3 • 3-matrix and the interconnection requires the inversion of a 4 • 4-matrix. We observe that the number of variables is the same in the diacoptical and codiacoptical methods, but that their occurrence in the steps of the solution is different.

The analyst's choice between the methods should be guided by the required amount of numerical work. In this example the diacoptical method is preferable, see Fig. 47.

Number of	given network	equivalent network of node type	equivalent network of loop type
independent nodes	9	9	4
nodes in the equivalent network X(1)		2 • 2	
branches	22	12	16
independent loops	13	3	13
loops in the equivalent network X(0)			3
diacoptical variables		2 • 2 + 3 = 7	
codiacoptical variables			4 + 3 = 7

Fig. 47. Table of characteristic numbers for the structure X

The topological invariants

In the diacoptical solution the topological invariant is a set of points and in the codiacoptics it is a set of independent loops, compare Fig. 40 with Fig. 43a, and Fig. 40 with Fig. 46a.

RECAPITULATION OF FORMULAS

The main formulas of the diacoptical and codiacoptical methods are given below with matrix representation in order to facilitate their application in the solution of practical problems. The calculations are subdivided into eight parts which we denote by capital letters A to H.

For $\alpha = \beta = 0$, the fundamental equation system of the dissected elastic structure the solution of which is required, is

(126) $$\begin{bmatrix} Z(0)^* f(0)\, Z(0) & -Z(0)^* E(10)^* \\ E(10)\, Z(0) & E(1)\, F(1)\, E(1)^* \end{bmatrix} \begin{bmatrix} R(0) \\ p(0) \end{bmatrix} = \begin{bmatrix} r(0,1) \\ P(1,0) \end{bmatrix}$$

Structure loads $P = [P(0)^*, \; P(1)^*]^*$ and cycle deformations $r = [r(0)^*, r(1)^*]^*$ are considered in (126).

A. Choose a dissection that makes the total number of unknowns

(242) $$\phi = m(0) + \alpha(1)$$

a minimum. Use the node-loop incidence matrix

(244) $$L = (E_1 Z_1)_1 = \begin{array}{c} \\ \alpha(1)\{ \end{array} \overset{m(0)}{\begin{bmatrix} L(0) & 0 \\ L(10) & L(1) \end{bmatrix}}$$

B. Choose vector basis for writing the structural continuity equations. Array the matrices

(92), (199) $Z(0), \; Z(1)$

(79), (187) $E(0), \; E(1), \; E(10), \; E(10)\, Z(0)$

(167) $f', \; F'$

(170) $$T_q = \begin{bmatrix} \gamma & 0 \\ \gamma Y & \gamma \end{bmatrix}$$

(175) $$f_q = T_q^* f' T_q$$

(179) $$f(0) = [f_q]^D$$

(144) $$f(0,0) = Z(0)^* f(0)\, Z(0)$$

(177) $$F_q = T_q^{-1} F' (T_q^*)^{-1}$$

(178) $$T_q^{-1} = \begin{bmatrix} \gamma^* & 0 \\ Y^* \gamma^* & \gamma^* \end{bmatrix}$$

(180) $F(1) = [F_q]^D$

(143) $F(1,1) = E(1)\,F(1)\,E(1)^*$

C. The structure action consists of cycle deformations

$$r = [r(0)^*,\ r(1)^*]^*$$

and structure loads

$$P = [P(0)^*,\ P(1)^*]^*$$

Transformation of structure loads and cycle deformations to an arbitrary coordinate system is made as follows:

(169) $N_q = T_q^{-1} N'$

(174) $n_q = T_q^* n'$

In case of non-boundary action, loads

(163) $P = -\Sigma N_s$

are added to the structure loads, and deformations

(164) $r = -Z^* n_s$

are added to the cycle deformations r.

D. Particular solutions for member forces are

(132) $N(0)_p = E(0)^{-1} P(0)$

(134) $E(1)\,N(1)_p = -E(10)\,N(0)_p + P(1)$

where

(133) $E(0)\,E(0)^{-1} = I$

If the structure part X(1) is not grounded, the forces $E(1)N(1)_p$ are selfequilibrating.

Particular solutions for member deformations are

(137) $n(1)_p = Z(1)^{*-1} r(1)$

(139) $Z(0)^* n(0)_p = -Z(10)^* n(1)_p + r(0)$

where

(138) $Z(1)^* Z(1)^{*-1} = I$

If structure deformations according to H, see below, are required we also calculate

(155) $n(0)_p = Z(0)^{*-1} (Z(10)^* n(1)_p + r(0))$

where

(156) $Z(0)^* Z(0)^{*-1} = I$

E. Calculate the equivalent cycle deformations r(0, 1) on X(0) and the equivalent structure loads P(1, 0) on X(1):

(150) $r(0, 1) = Z(0)^* n(0)_p - Z(0)^* f(0) N(0)_p$

(151) $P(1, 0) = E(1) N(1)_p - E(1) F(1) n(1)_p$

Fa. The diacoptical solution of (126) is

(140) $p(1) = F(1, 1)^{-1} (-E(10) Z(0) R(0) + P(1, 0))$

(141) $R(0) = f(0, 0, 0)^{-1} r(0, 1, 0)$

where

(142) $f(0, 0, 0) = f(0, 0) + Z(0)^* E(10)^* F(1, 1)^{-1} E(10) Z(0)$

(145) $r(0, 1, 0) = r(0, 1) + Z(0)^* E(10)^* F(1, 1)^{-1} P(1, 0)$

We observe that r(0, 1, 0) contains the term $Z(0)^* E(10)^* F(1, 1)^{-1}$ which has already been calculated in f(0, 0, 0).

A solution of (141) by use of Gaussian elimination gives a stability test.

Fb. The codiacoptical solution of (126) is

(146) $R(0) = f(0, 0)^{-1} (Z(0)^* E(10)^* p(1) + r(0, 1))$

(147) $p(1) = F(1, 1, 1)^{-1} P(1, 0, 1)$

where

(148) $F(1, 1, 1) = F(1, 1) + E(10) Z(0) f(0, 0)^{-1} Z(0)^* E(10)^*$

(149) $P(1, 0, 1) = P(1, 0) - E(10) Z(0) f(0, 0)^{-1} r(0, 1)$

G. The member forces and member deformations are

(152) $N(0) = N(0)_p + Z(0)R(0)$

$n(1) = n(1)_p + E(1)^* p(1)$

(153) $n(0) = f(0)N(0)$

$N(1) = F(1)n(1)$

Transformation of member forces and member deformations to local coordinate systems is made as follows:

(169) $N' = T_q N_q$

(174) $n' = (T_q^*)^{-1} n_q$

The total member forces N_r and member deformations n_r are

(165) $N_r = N_s + N$

(166) $n_r = n_s + n$

Ha. If X(1) is grounded the structure deformations are

(158) $p(0) = E(0)^{*-1}(n(0) - n(0)_p - E(10)^* p(1))$

with p(1) according (140) or (147), see F above, and with $E(0)^{*-1} = E(0)^{-1*}$ where $E(0)^{-1}$ is given by (131), see D above.

Hb. If X(0) is grounded and X(1) is not, the structure deformations are

(160) $p(0) = E(0)^{*-1}(n(0) - n(0)_p)$

(161) $p(1)_r = E(1)^{*-1}(n(1) - n(1)_p) + E(1)^{*-1} E(1)^*_Q p(1)_Q$

The value of $p(1)_r$ at the boundary between X(0) and X(1) is given by

(159) $n(0) = n(0)_p + E(0)^* p(0) + E(10)^* p(1)_Q + E(10)^* p(1)_r$

where

(162) $E(1)^* E(1)^{*-1} = I$

NUMERICAL EXAMPLES

The following simple Examples 1 and 2 will show the use of the established formulas of diacoptics and codiacoptics. In Example 3 the minimum number of unknowns is determined.

The examples are given only for instructive purposes. The best use of diacoptics and codiacoptics is in the solution of large structures and is not demonstrated here.

Example 1. A straight beam with uniform stiffness EI, see Fig. 48, is divided into four members and submitted to a vertical concentrated load P at joint A. We want to calculate member forces and structure deformations.

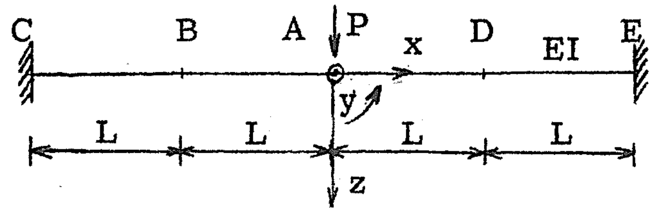


Fig. 48. Loaded beam

The diacoptical method will be applied. The subdivision of the calculations given in the preceding chapter will be followed.

A. The structural dissection (X(0), X(1)) in Fig. 49 is chosen.

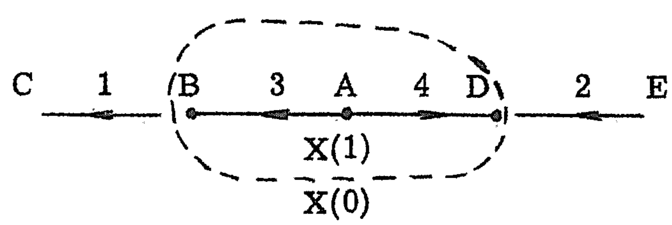


Fig. 49. Structural dissection

In the diacoptical procedure X(1) is first solved by deformation (node) variables. In X(1) the point A is used as reference point Q. We then get an equivalent structure of node type, see Fig. 50, which is solved by cycle forces.

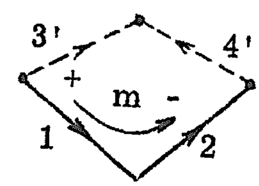


Fig. 50. Equivalent structure of node type

We choose a global coordinate system with origin at A according to Fig. 48.

B. We array the matrices

$$(92) \qquad Z(0)^* = m\begin{matrix} 1 & 2 \\ [I & I] \end{matrix}$$

$$(79) \qquad E(1) = \begin{matrix} \\ A \\ B \\ C \end{matrix}\begin{matrix} 3 & 4 \\ \left[\begin{matrix} I & I \\ -I & \\ & I \end{matrix}\right] \end{matrix}, \quad E(10) = \begin{matrix} \\ A \\ B \\ D \end{matrix}\begin{matrix} 1 & 2 \\ \left[\begin{matrix} & \\ I & \\ & -I \end{matrix}\right] \end{matrix}, \quad E(10)\,Z(0) = \begin{matrix} A \\ B \\ D \end{matrix}\left[\begin{matrix} 0 \\ I \\ -I \end{matrix}\right]$$

Because we have no axial force in the beam we may discard the axial force as a variable in the analysis. We thus have

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The load deformation characteristics given by f' and F' for the members are given relative to the local coordinate system at beam end 2. We have

$$(167) \qquad f' = \begin{bmatrix} 2L^2 & -3L \\ -3L & 6 \end{bmatrix}\frac{L}{6EI}, \quad F' = \begin{bmatrix} 1 & 2L \\ 2L & 3L^2 \end{bmatrix}\frac{12EI}{L^3}$$

The flexibilities of the members in X(0) relative to the global coordinate system are

$$(175) \qquad f(0)_{iA} = T^*_{iA}\, f'_i\, T_{iA}, \quad i = 1, 2$$

$$f(0)_{1A} = \begin{bmatrix} 1 & -L \\ 0 & 1 \end{bmatrix} f'_1 \begin{bmatrix} 1 & 0 \\ -L & 1 \end{bmatrix} = \begin{bmatrix} 14L^2 & -9L \\ -9L & 6 \end{bmatrix}\frac{L}{6EI}$$

$$f(0)_{2A} = \begin{bmatrix} 1 & 2L \\ 0 & 1 \end{bmatrix} f'_2 \begin{bmatrix} 1 & 0 \\ 2L & 1 \end{bmatrix} = \begin{bmatrix} 14L^2 & 9L \\ 9L & 6 \end{bmatrix}\frac{L}{6EI}$$

$$(179) \qquad f(0) = [f(0)_{iA}]^D, \quad i = 1, 2$$

The structure flexibility of X(0) is

$$(144) \qquad f(0,0) = Z(0)^* f(0)\, Z(0) = f(0)_{1A} + f(0)_{2A} = \begin{bmatrix} 14L^2 & 0 \\ 0 & 6 \end{bmatrix}\frac{L}{3EI}$$

The stiffnesses of the members in X(1) relative to the global coordinate system are

$$(177) \qquad F(1)_{iA} = T^{-1}_{iA}\, F'_i\, (T^*_{iA})^{-1}, \quad i = 3, 4$$

$$F(1)_{3A} = \begin{bmatrix} 12 & 6L \\ 6L & 4L^3 \end{bmatrix}\frac{EI}{L^3}$$

$$F(1)_{4A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} F'_4 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & -6L \\ -6L & 4L^3 \end{bmatrix} \frac{EI}{L^3}$$

(180) $F(1) = [F_{iA}]^D, \quad i = 3,4$

The structure stiffness of X(1) is (A is used as reference point which implies that the corresponding rows and columns in E(1) and E(1)* must be deleted)

(143) $$F(1,1) = E(1)F(1)E(1)^* = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} F(1)_{3A} & \\ & F(1)_{4A} \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} =$$

$$= \begin{bmatrix} F(1)_{3A} & 0 \\ 0 & F(1)_{4A} \end{bmatrix}$$

C. The structure action in Fig. 48 gives

$$N_s = n_s = 0, \; r = 0, \; P = \begin{bmatrix} P_A \\ P_B \\ P_D \end{bmatrix} = \begin{bmatrix} P_A \\ 0 \\ 0 \end{bmatrix} = P(1), \; P_A = \begin{bmatrix} P \\ 0 \end{bmatrix}$$

Note that the load P is situated at the reference point Q = A.

D. Because Z(0) and P(0) do not exist the equation (132) is irrelevant. The particular solution for X(1) is contained in

(134) $$E(1)N(1)_p = -E(10)N(0)_p + P(1) = -\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} + \begin{bmatrix} P_A \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P_A \\ -N_1 \\ N_2 \end{bmatrix}$$

These forces are selfequilibrating because X(1) is not grounded, that is

$$P_A - N_1 + N_2 = 0$$

We observe that this equation can be satisfied by infinitely many sets of values of N_1 and N_2. We choose $N_1 = P_A$, $N_2 = 0$.

From r = 0 follows that $n(0)_p = n(1)_p = 0$ according (137) and (139).

E. The equivalent cycle deformation in X(1), that is the vector r(0, 1), is obtained by vector multiplication from the right in (150):

(150) $$r(0,1) = -Z(0)^* f(0) N(0)_p = -[I \quad I]\,[f(0)_{1A}, \; f(0)_{2A}]^D \begin{bmatrix} I \\ 0 \end{bmatrix} P_A =$$

$$= \begin{bmatrix} -14L \\ 9 \end{bmatrix} \frac{PL^2}{6EI}$$

The equivalent structure loads P(1, 0) in X(1) are

(151) $$P(1,0) = E(1)N(1)_p = -\begin{bmatrix} I \\ 0 \end{bmatrix} P_A$$

Fa. The inverse of the structure stiffness of X(1) is

$$F(1,1)^{-1} = \begin{bmatrix} F(1)^{-1}_{3A} & 0 \\ 0 & F(1)^{-1}_{4A} \end{bmatrix} = \begin{bmatrix} 2L^2 & -3L & 0 & 0 \\ -3L & 1 & 0 & 0 \\ 0 & 0 & 2L^2 & 3L \\ 0 & 0 & 3L & 6 \end{bmatrix} \frac{L}{6EI}$$

Vector multiplication from the right in (145) gives

(145) $$r(0,1,0) = r(0,1) + Z(0)^* E(10)^* F(1,1)^{-1} P(1,0) =$$

$$= r(0,1) + [I \;\; -I] \begin{bmatrix} F(1)^{-1}_{3A} & 0 \\ 0 & F(1)^{-1}_{4A} \end{bmatrix} \begin{bmatrix} -I \\ 0 \end{bmatrix} P_A =$$

$$= r(0,1) - F(1)^{-1}_{3A} P_A = \begin{bmatrix} -8L \\ 6 \end{bmatrix} \frac{PL^2}{3EI}$$

Further

(142) $$f(0,0,0) = f(0,0) + Z(0)^* E(10)^* F(1,1) E(10) Z(0) = f(0,0) +$$

$$+ [I \;\; -I] \begin{bmatrix} F(1)^{-1}_{3A} & 0 \\ 0 & F(1)^{-1}_{4A} \end{bmatrix} \begin{bmatrix} I \\ -I \end{bmatrix} = f(0,0) + F(1)^{-1}_{3A} + F(1)^{-1}_{4A} =$$

$$= \begin{bmatrix} 16L^2 & 0 \\ 0 & 12 \end{bmatrix} \frac{L}{3EI}$$

The unknown cycle forces and structure deformations in (126) thus are

(141) $$R(0) = f(0,0,0)^{-1} r(0,1,0) = \begin{bmatrix} 3 & 0 \\ 0 & 4L^2 \end{bmatrix} \frac{EI}{16L^3} \begin{bmatrix} -8L \\ 6 \end{bmatrix} \frac{PL^2}{3EI} = \frac{1}{2} \begin{bmatrix} -P \\ PL \end{bmatrix}$$

and

(140) $$p(1) = F(1,1)^{-1}(-E(10)Z(0)R(0) + P(1,0)) = F(1,1)^{-1} \begin{bmatrix} -P \\ -PL \\ -P \\ PL \end{bmatrix} \frac{1}{2} = \begin{bmatrix} L \\ -3 \\ L \\ 3 \end{bmatrix} \frac{PL^2}{12EI}$$

G. The member forces are, in the global coordinate system,

$$(152) \qquad N(0) = N(0)_p + Z(0)R(0) = \begin{bmatrix} N_{1A} \\ N_{2A} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} P + \begin{bmatrix} I \\ I \end{bmatrix} \begin{bmatrix} -P \\ PL \end{bmatrix} \frac{1}{2} = \frac{1}{2} \begin{bmatrix} P \\ PL \\ -P \\ PL \end{bmatrix}$$

$$(153) \qquad N(1) = F(1)n(1) = F(1)E(1)^*p(1) = \begin{bmatrix} F_{3A} & 0 \\ 0 & F_{4A} \end{bmatrix} \begin{bmatrix} -I & \\ & -I \end{bmatrix} \begin{bmatrix} F_{3A}^{-1} & 0 \\ 0 & F_{4A}^{-1} \end{bmatrix} \cdot$$

$$\cdot \frac{1}{2} \begin{bmatrix} -P \\ -PL \\ -P \\ PL \end{bmatrix} = \frac{1}{2} \begin{bmatrix} P \\ PL \\ P \\ -PL \end{bmatrix}$$

Transformation to local coordinates yields

$$(169) \qquad N(0) = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} T_{1A} & T_{2A} \end{bmatrix}^D \begin{bmatrix} N_{1A} \\ N_{2A} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} P \\ 0 \\ -P \\ -PL \end{bmatrix}$$

$$N(1) = \begin{bmatrix} N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} T_{3A} & T_{4A} \end{bmatrix}^D \begin{bmatrix} N_{3A} \\ N_{4A} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} P \\ PL \\ -P \\ -PL \end{bmatrix}$$

The shear force and moment diagrams for the beam are drawn in Fig. 50'.

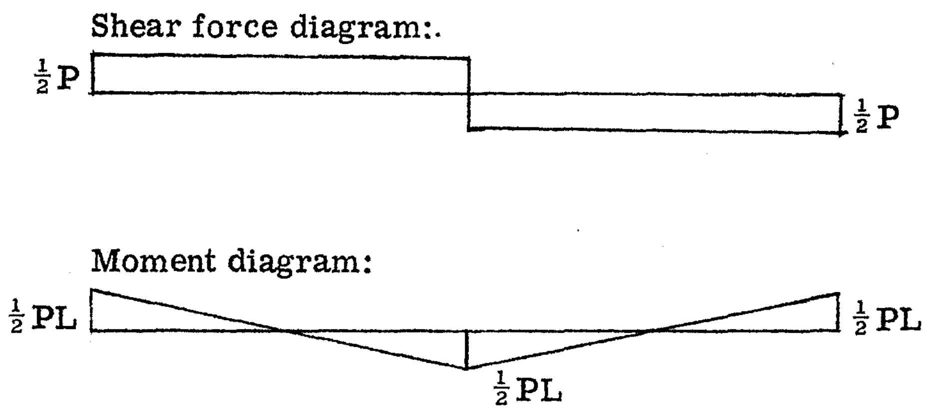


Fig. 50'. Section forces in beam CE

The member deformations are

$$(153) \qquad n(0) = f(0)N(0) = \begin{bmatrix} 5L \\ -3 \\ -5L \\ -3 \end{bmatrix} \frac{PL^2}{12EI}$$

$$(152)\qquad n(1) = E(1)^{*}p(1) = \begin{bmatrix} -I & \\ & -I \end{bmatrix} p(1) = \begin{bmatrix} -L \\ 3 \\ -L \\ -3 \end{bmatrix} \frac{PL^2}{12EI}$$

Hb. The relative structure deformations p(1) were obtained in (140), see F. The value of the real structure deformations at the boundary between X(0) and X(1) is given by

$$(159)\qquad n(0) = n(0)_p + E(0)^{*}p(0) + E(10)^{*}_{Q}p_Q + E(10)^{*}p(1)_r, \qquad Q = A$$

$$n(0) = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = 0 + 0 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} p_Q + \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} p_{Br} \\ p_{Dr} \end{bmatrix}$$

$$\begin{bmatrix} p_{Br} \\ p_{Dr} \end{bmatrix} = \begin{bmatrix} n_1 \\ -n_2 \end{bmatrix} = \begin{bmatrix} 5L \\ -3 \\ 5L \\ 3 \end{bmatrix} \frac{PL^2}{12EI}$$

The real structure deformations are

$$(161)\qquad p(1)_r = E(1)^{*-1}(n(1) - n(1)_p) - E(1)^{*-1}E(1)^{*}_{Q}p(1)_Q$$

$$p(1)_r = p(1) + 0 - E(1)^{*-1}E(1)^{*}_{Q}p(1)_Q$$

where

$$E(1)^{*-1}E(1)^{*}_{Q} = \begin{bmatrix} -I & \\ & -I \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} = \begin{bmatrix} -I \\ -I \end{bmatrix}$$

The equation (161) gives

$$\begin{bmatrix} 5L \\ -3 \\ 5L \\ 3 \end{bmatrix} \frac{PL^2}{12EI} = \begin{bmatrix} L \\ -3 \\ L \\ 3 \end{bmatrix} \frac{PL^2}{12EI} + \begin{bmatrix} I \\ I \end{bmatrix} p_Q$$

We have as many equations to solve p_Q as there are boundary points between X(0) and X(1). The two equations of the present problem have the solution

$$p_{Qr} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{PL^3}{3EI}$$

which is the real structure deformation at point Q = A. By use of (161) all other real structure deformations are found from

(161) $p(1)_r = p(1) - E(1)^{*-1} E(1)^*_Q p_Q$

We observe that the term $E(1)^{*-1} E(1)^*_Q p_Q$ is a rigid motion of the structure X(1) to fit X(1) and X(0) together at the connection points.

Example 2. The same beam, Fig. 51, as in Example 1, see Fig. 51, will be dissected in a different way. The same solution as in Example 1 will be here calculated by use of the codiacoptical method.

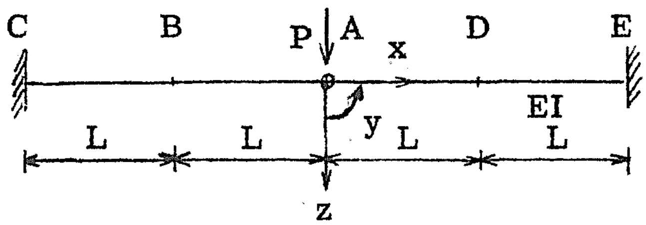


Fig. 51. Loaded beam

A. The structural dissection (X(0), X(1)) in Fig. 52 is chosen.

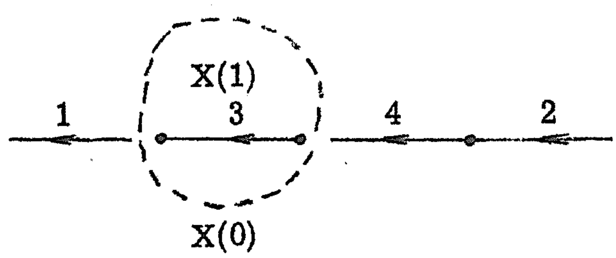


Fig. 52. Structural dissection

In the codiacoptical method X(0) is first solved by cycle forces. We then get an equivalent structure of loop type, see Fig. 53, which is solved by use of structure deformations.

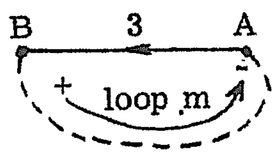


Fig. 53. Equivalent structure of loop type

B. We choose a global coordinate system with origin at A, see Fig. 51. Array the matrices

(92) $$Z(0)^* = m \begin{bmatrix} \overset{1}{I} & \overset{2}{I} & \overset{4}{I} \end{bmatrix}$$

(79) $$E(1) = \begin{matrix} \\ A \\ B \end{matrix}\begin{matrix} 3 \\ \begin{bmatrix} I \\ -I \end{bmatrix} \end{matrix}, \quad E(10) = \begin{matrix} \\ A \\ B \end{matrix}\begin{matrix} 1\ 2\ 4 \\ \begin{bmatrix} & & -I \\ I & & \end{bmatrix} \end{matrix}, \quad E(0) = D\begin{matrix} 1 \quad 2 \quad 4 \\ [\ \ -I \quad I] \end{matrix}$$

$$E(10)\,Z(0) = \begin{matrix} \\ A \\ B \end{matrix}\begin{matrix} m \\ \begin{bmatrix} -I \\ I \end{bmatrix} \end{matrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The member flexibility matrices $f(0)_{1A}$, $f(0)_{2A}$ and $F(1)_{3A}$ are found in Example 1. Further we get

(175) $$f(0)_{4A} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} f_4' \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} = \begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix} \frac{L}{6EI}$$

(179) $$f(0) = [f(0)_{iA}]^D; \quad i = 1,\ 2,\ 4$$

The structure flexibility of X(0) is

(144) $$f(0,0) = Z(0)^* f(0)\,Z(0) = f(0)_{1A} + f(0)_{2A} + f(0)_{4A} = \begin{bmatrix} 10L^2 & L \\ L & 6 \end{bmatrix} \frac{L}{2EI}$$

The stiffnesses of the members in X(1) relative to the global coordinate system are

(180) $$F(1) = [F_{iA}]^D; \quad i = 3$$

The point Q = B is used as reference point in X(1). The structure stiffness of X(1) is

(143) $$F(1,1) = E(1)\,F(1)\,E(1)^* = F(1)_{3A} = \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \frac{EI}{L^3}$$

C. The structure action in Fig. 51 gives

$$r(1) = 0; \quad P = \begin{bmatrix} P_A \\ P_B \\ \hline P_D \end{bmatrix} = \begin{bmatrix} P_A \\ 0 \\ \hline 0 \end{bmatrix} = \begin{bmatrix} P(1) \\ \\ P(0) \end{bmatrix}; \quad P_A = \begin{bmatrix} P \\ 0 \end{bmatrix}$$

D. The particular solution for X(1) is contained in

(134) $$E(1)\,N(1)_p = -E(10)\,N(0)_p + P(1) = -\begin{bmatrix} 0 & 0 & -I \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_4 \end{bmatrix} + \begin{bmatrix} P_A \\ 0 \end{bmatrix} = \begin{bmatrix} N_4 + P_A \\ -N_1 \end{bmatrix}$$

Because X(1) is not grounded the forces (134) are selfequilibrating or

$$N_4 + P_A - N_1 = 0$$

A solution to this equation is for example $N_1 = P_A$; $N_4 = 0$.
The solution $N(0)_p$ must also satisfy

(132) $$P(0) = E(0)N(0)_p = \begin{bmatrix} 0 & -I & I \end{bmatrix} \begin{bmatrix} P_A \\ N_2 \\ 0 \end{bmatrix} = -N_2 = 0$$

which gives $N_2 = 0$. Thus

$$N(0)_p = \begin{bmatrix} P_A \\ 0 \\ 0 \end{bmatrix}$$

E. The equivalent cycle deformations r(0, 1) in X(0), that is obtained by vector multiplication from the right in (150):

(150) $$r(0,1) = -Z(0)^* f(0) N(0)_p = -[I \quad I \quad I] [f(0)_{1A} \quad f(0)_{2A} \quad f(0)_{4A}]^D \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} P_A =$$

$$= \begin{bmatrix} -14L \\ 9 \end{bmatrix} \frac{PL^2}{6EI} .$$

The equivalent structure loads P(1, 0) in X(1) are

(151) $$P(1,0) = E(1)N(1)_p = [I] P_A$$

Fb. The inverse of the structure flexibility of X(0) is

$$f(0,0)^{-1} = \begin{bmatrix} 6 & -L \\ -L & 10L^2 \end{bmatrix} \frac{2EI}{59 \cdot L^3}$$

Further

(148) $$F(1,1,1) = F(1,1) + E(10) Z(0) f(0,0)^{-1} Z(0)^* E(10)^* = \begin{bmatrix} 45 & 22L \\ 22L & 16L^2 \end{bmatrix} \frac{16EI}{59L^3}$$

$$F(1,1,1)^{-1} = \begin{bmatrix} 16L^2 & -22L \\ -22L & 45 \end{bmatrix} \frac{L}{64EI}$$

Vector multiplication from the right in (149) gives

(149) $$P(1,0,1) = P(1,0) - E(10)Z(0)f(0,0)^{-1}r(0,1) = [\,I\,]\begin{bmatrix} P \\ 0 \end{bmatrix} -$$

$$-[-I]\begin{bmatrix} 6 & -L \\ -L & 10L^2 \end{bmatrix}\frac{2EI}{59L^3}\begin{bmatrix} -14L \\ 9 \end{bmatrix}\frac{PL^2}{6EI} = \frac{1}{177}\begin{bmatrix} 84P \\ 104PL \end{bmatrix}$$

The unknown structure deformations and cycle forces in (126) thus are

(147) $$p(1) = F(1,1,1)^{-1}P(1,0,1) = \begin{bmatrix} 16L^2 & -22L \\ -22L & 45 \end{bmatrix}\frac{L}{64EI}\cdot\frac{1}{177}\begin{bmatrix} 84P \\ 104PL \end{bmatrix} =$$

$$= \begin{bmatrix} -L \\ 3 \end{bmatrix}\frac{PL^2}{22EI}$$

and

(146) $$R(0) = f(0,0)^{-1}(Z(0)^* E(10)^* p(1) + r(0,1))$$

$$R(0) = \begin{bmatrix} 6 & -L \\ -L & 10L^2 \end{bmatrix}\frac{2EI}{59L^3}\left([-I]\begin{bmatrix} -L \\ 3 \end{bmatrix}\frac{PL^2}{12EI} + \begin{bmatrix} -14L \\ 9 \end{bmatrix}\frac{PL^2}{6EI}\right) = \frac{1}{2}\begin{bmatrix} -P \\ PL \end{bmatrix}$$

G. The member forces are in the global coordinate system

(152) $$N(0) = N(0)_p + Z(0)R(0) = \begin{bmatrix} P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} I \\ I \\ I \end{bmatrix}\frac{1}{2}\begin{bmatrix} -P \\ PL \end{bmatrix} = \frac{1}{2}\begin{vmatrix} P \\ PL \\ -P \\ PL \\ -P \\ PL \end{vmatrix}$$

(153) $$N(1) = F(1)E(1)^*p(1) = \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\frac{EI}{L^3}[I]\begin{bmatrix} -L \\ 3 \end{bmatrix}\frac{PL^2}{12EI} = \frac{1}{2}\begin{bmatrix} P \\ PL \end{bmatrix}$$

The member forces in local coordinates are

(169) $$N(0) = [T_{iA}]^D N(0)_A = \begin{bmatrix} 1 & 0 & & & & \\ -L & 1 & & & & \\ & & 1 & 0 & & \\ & & 2L & 1 & & \\ & & & & 1 & 0 \\ & & & & L & 1 \end{bmatrix}\frac{1}{2}\cdot\begin{bmatrix} P \\ PL \\ -P \\ PL \\ -P \\ PL \end{bmatrix} = \frac{1}{2}\begin{bmatrix} P \\ 0 \\ -P \\ -PL \\ -P \\ 0 \end{bmatrix}$$

$$N(1) = [T_{iA}]^D N(1)_A = [I]\;N(1) = \frac{1}{2}\begin{bmatrix} P \\ PL \end{bmatrix}$$

This is the same result as was obtained in Example 1.

We get the member deformations

(153) $$n(0) = f(0)N(0) = \begin{bmatrix} 14L^2 & -9L & & & & \\ -9L & 6 & & & & \\ & & 14L^2 & 9L & & \\ & & 9L & 6 & & \\ & & & & 2L^2 & 3L \\ & & & & 3L & 6 \end{bmatrix} \frac{L}{6EI} \frac{1}{2} \begin{bmatrix} P \\ PL \\ -P \\ PL \\ -P \\ PL \end{bmatrix} =$$

$$= \begin{bmatrix} 5L \\ -3 \\ -5L \\ -3 \\ L \\ 3 \end{bmatrix} \frac{PL^2}{12EI}$$

(152) $$n(1) = E(1)^* p(1) = \begin{bmatrix} -L \\ 3 \end{bmatrix} \frac{PL^2}{12EI}$$

Hb. The relative structure deformations p(1) were obtained in (147), see Fb. In order to calculate the structure deformations p(0) we first calculate

$$E(0)^{*-1} = E(0)^{-1*} = [0 \quad -I \quad 0]$$

The matrix $E(0)^{-1}$ satisfies

(133) $$E(0)\,E(0)^{-1} = [0 \quad -I \quad I] \begin{bmatrix} 0 \\ -I \\ 0 \end{bmatrix} = I$$

Thus we calculate the structure deformations

(160) $$p(0) = E(0)^{*-1}(n(0) - n(0)_p) = E(0)^{*-1} n(0) = [0 \quad -I \quad 0] \begin{bmatrix} 5L \\ -3 \\ -5L \\ -3 \\ -3L \\ 3 \end{bmatrix} \frac{PL^2}{12EI} =$$

$$= \begin{bmatrix} 5L \\ 3 \end{bmatrix} \frac{PL^2}{12EI}$$

The value of the real structure deformations at the boundary between X(0) and X(1) is given by

(159) $n(0) = E(0)^* p(0) + E(10)^*_Q p(1)_Q + E(10)^* p(1)_r$

$$\begin{bmatrix} n(0)_1 \\ n(0)_2 \\ n(0)_4 \end{bmatrix} = \begin{bmatrix} 5L \\ -3 \\ -5L \\ -3 \\ L \\ 3 \end{bmatrix} \frac{PL^2}{12EI} = \begin{bmatrix} 0 \\ -I \\ I \end{bmatrix} \begin{bmatrix} 5L \\ 3 \end{bmatrix} \frac{PL^2}{12EI} + \begin{bmatrix} 0 & I \\ 0 & 0 \\ -I & 0 \end{bmatrix} \begin{bmatrix} p(1)_r \\ p(1)_Q \end{bmatrix}$$

From the equation (159) we obtain the real structure deformations of point B (= Q) as

$$p(1)_Q = \begin{bmatrix} 5L \\ -3 \end{bmatrix} \frac{PL^3}{12EI}$$

We observe that the equation (159) gives the real structure deformations $p(1)_r$ for joints at the boundaries between X(0) and X(1). We obtain the real structure deformation at joint A as

$$p(1)_r = \begin{bmatrix} 5L & -L \\ 3 & -3 \end{bmatrix} \frac{PL^2}{12EI} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{PL^3}{3EI}$$

If we instead use the formula (161) in order to obtain the real structure deformations at A we find

(161) $p(1)_r = p(1) - E(1)^{*-1} E(1)^*_Q p(1)_Q = p(1) - I(-I)p(1)_Q = p(1) + p(1)_Q$

$$p(1)_r = \begin{bmatrix} -L \\ 3 \end{bmatrix} \frac{PL^2}{12EI} + \begin{bmatrix} 5L \\ -3 \end{bmatrix} \frac{PL^3}{12EI} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{PL^3}{3EI}$$

Thus we obtain the same result as in Example 1.

Example 2b. The beam in Fig. 51 with the height h and the temperature gradient α is now submitted to a temperature gradient Vt. We want to calculate member forces and structure deformations.

C. The temperature gradient is a non-boundary action. It implies for the members, that we use the particular solution in Fig. 33b. We thus obtain the member free deformations

$$n_s = \begin{bmatrix} n(0)_s \\ n(1)_s \end{bmatrix} = \begin{bmatrix} n(0)_{1s} \\ n(0)_{2s} \\ n(0)_{4s} \\ n(1)_{3s} \end{bmatrix} = \begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix} \begin{bmatrix} L \\ -2 \end{bmatrix} \frac{\alpha L Vt}{2h}$$

Transformation of the free deformations n_s to the global coordinate system at A yields

$$(174)\qquad \begin{bmatrix} n(0)_s \\ n(1)_s \end{bmatrix} = n_s = T^*_{iA}\, n_{is} = \begin{bmatrix} 1 & -L & & & & & & \\ 0 & 1 & & & & & & \\ & & 1 & 2L & & & & \\ & & 0 & 1 & & & & \\ & & & & 1 & L & & \\ & & & & 0 & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} = \begin{bmatrix} 3L \\ -2 \\ -3L \\ -2 \\ -L \\ -2 \\ L \\ -2 \end{bmatrix} \frac{\alpha L\nabla t}{2h}$$

D. In order to restate the deformation continuity conditions we add deformations round the cycle which yields

$$(164)\qquad r = -Z^* n_s = \begin{bmatrix} r(0) \\ r(1) \end{bmatrix} = -\begin{bmatrix} Z(0)^* & Z(10)^* \\ 0 & Z(1)^* \end{bmatrix} \begin{bmatrix} n(0)_s \\ n(1)_s \end{bmatrix} = \begin{bmatrix} Z(0)^* & Z(10)^* \\ 0 & Z(1)^* \end{bmatrix} \begin{bmatrix} -n(0)_s \\ -n(1)_s \end{bmatrix}$$

The equation (164) gives the particular solution

$$n_p = \begin{bmatrix} n(0)_p \\ n(1)_p \end{bmatrix} = \begin{bmatrix} -n(0)_s \\ -n(1)_s \end{bmatrix}$$

E. The equivalent structure loads P(1, 0) in X(1) are

$$(151)\qquad P(1,0) = -E(1)\,F(1)\,n(1)_p = [\,I\,] \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \frac{EI}{L^3} \begin{bmatrix} L \\ -2 \end{bmatrix} \frac{\alpha\nabla tL}{2h} = \begin{bmatrix} 0 \\ -L^2 \end{bmatrix} \frac{\alpha\nabla tEI}{hL^2}$$

Fb. We calculate

$$(149)\qquad P(1,0,1) = P(1,0) - E(10)\,Z(0)^* f(0,0,0)^{-1}\, r(0,1) =$$

$$= \begin{bmatrix} 0 \\ -L^2 \end{bmatrix} \frac{\alpha\nabla tEI}{hL^2} - [-I] \begin{bmatrix} 6 & -L \\ -L & 10L^2 \end{bmatrix} \frac{2EI}{59L^3} \begin{bmatrix} L \\ 6 \end{bmatrix} \frac{\alpha\nabla tL}{2h} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The unknown structure deformations and cycle forces thus are

$$(147)\qquad p(1) = F(1,1,1)^{-1}\, P(1,0,1) = 0$$

$$(146)\qquad R(0) = f(0,0)^{-1}\, Z(0)^* E(10)^* p(1) + r(0,1)$$

$$= \begin{bmatrix} 6 & -L \\ -L & 10L^2 \end{bmatrix} \frac{2EI}{59L^3} \cdot \begin{bmatrix} L \\ 6 \end{bmatrix} \frac{\alpha\nabla tL}{2h} = \begin{bmatrix} 0 \\ 59L^2 \end{bmatrix} \frac{2\alpha\nabla tLEI}{59\cdot 2L^3h} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\alpha\nabla tEI}{h}$$

G. The member forces and member deformations are in global coordinate system

(152) $$N(0) = N(0)_p + Z(0)R(0) = \begin{bmatrix} I \\ I \\ I \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\alpha \nabla t EI}{h}$$

(152) $$n(1) = n(1)_p + Z(1)^* p(1) = -n(1)_s + 0 = -n(1)_s = \begin{bmatrix} -L \\ 2 \end{bmatrix} \frac{\alpha \nabla t L}{2h}$$

(153) $$N(1) = F(1)\,n(1) = \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{bmatrix} -L \\ 2 \end{bmatrix} \frac{EI\alpha \nabla t L}{L^3 \cdot 2h} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\alpha \nabla t EI}{h}$$

(153) $$n(0) = f(0)\,N(0) = \begin{bmatrix} n(0)_1 \\ n(0)_2 \\ n(0)_4 \end{bmatrix} = \begin{bmatrix} 14L^2 & -9L & & & & \\ -9L & 6 & & & & \\ & & 14L^2 & 9L & & \\ & & 9L & 6 & & \\ & & & & 6L^2 & 3L \\ & & & & 3L & 6 \end{bmatrix} \cdot$$

$$\cdot \frac{L}{6EI} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \frac{\alpha \nabla t EI}{h} = \begin{bmatrix} -9L \\ 6 \\ 9L \\ 6 \\ 3L \\ 6 \end{bmatrix} \cdot \frac{\alpha \nabla t L}{6h}$$

The real member deformations n_r are

(166) $$n_r = n + n_s = \begin{bmatrix} n(0)_r \\ n(1)_r \end{bmatrix} = \begin{bmatrix} n(0) \\ n(1) \end{bmatrix} + \begin{bmatrix} n(0)_s \\ n(1)_s \end{bmatrix}$$

The equation (166) gives

$$n(0)_r = \begin{bmatrix} -3L \\ 2 \\ 3L \\ 2 \\ L \\ 2 \end{bmatrix} \frac{\alpha \nabla t L}{2h} + \begin{bmatrix} 3L \\ -2 \\ -3L \\ -2 \\ -L \\ -2 \end{bmatrix} \frac{\alpha \nabla t L}{2h} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$n(1)_r = \begin{bmatrix} -L \\ 2 \end{bmatrix} \frac{\alpha \nabla t L}{2h} + \begin{bmatrix} L \\ -2 \end{bmatrix} \frac{\alpha \nabla t L}{2h} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hb. The relative structure deformations $p(1) = 0$, see Fb, are

(140) $\quad p(0) = \delta(0)^{-1}(n(0) - n(0)_p) = \delta(0)^{-1}(n(0) + n(0)_s) = 0$

The equation (139) gives the values of $p(1)_r$ at joint A, Q = B.

(139) $\quad n(0) - n(0)_p = +\delta(0)p(0) + \delta(01)p_{1Q} + \delta(01)p(1)_r$

$$0 = 0 + \delta(01)_Q p_{1Q} + \delta(01)p(1)_r$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \\ -I & 0 \end{bmatrix} \begin{bmatrix} p(1)_r \\ p(1)_Q \end{bmatrix}$$

from (139) thus follows that

$$\begin{bmatrix} p(1)_r \\ p(1)_Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 3. A plane frame, see Fig. 54, submitted to joint loads is divided into 6 members. All the beams have the section DIMAX 20 with the area $A = 135{,}4\ \mathrm{cm}^2$, and moment of inertia $I = 10\,897\ \mathrm{cm}^4$ and $I/A = 8{,}048 \cdot 10^{-3}\ \mathrm{m}^2$.

We want to calculate member forces.

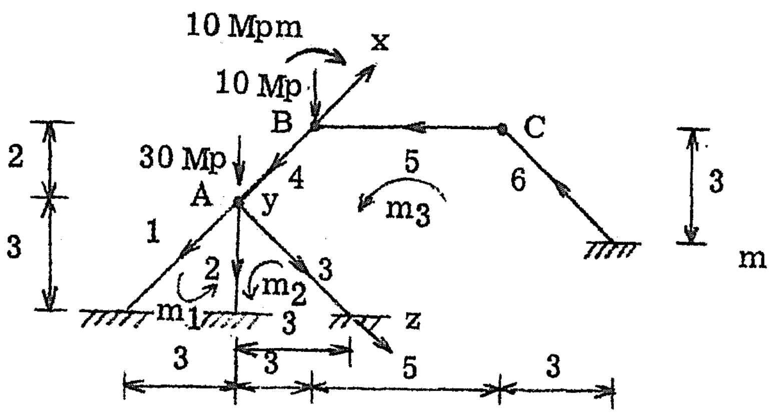


Fig. 54. Loaded plane frame

A. We choose a dissection that makes the number of unknowns $\phi = m(0) + \alpha(1)$ a minimum.

We array the matrices E_1 and Z_1 and make the matrix product

$$E_1 Z_1 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{bmatrix} \end{matrix} \begin{matrix} & \begin{matrix} m_1 & m_2 & m_3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \\ & & 1 \\ & & 1 \\ & & 1 \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} m_1 & m_2 & m_3 \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \end{matrix}$$

We now obtain the node-loop incidence matrix

$$L = (E_1 Z_1)_1 = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right] \end{array}$$

We change the ordering of the rows and columns in order to get an ordering according to

$$L = \begin{bmatrix} L(0) & 0 \\ L(10) & L(1) \end{bmatrix} = \begin{array}{r} \\ \\ B \\ C \\ \alpha(1)\{\, A \\ \Sigma_\alpha \end{array} \begin{array}{c|cc|c} \overbrace{m_3}^{m(0)} & m_2 & m_1 & \Sigma_m \\ \hline 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 3 \\ \hline 3 & 1 & 1 & \end{array}$$

We get $\alpha(1) + m(0) = 2$.

Thus we dissect the structure according to Fig. 55. Note that in the diacoptical or codiacoptical solution we need $2 \cdot 3 = 6$ variables, but in the force and displacement methods we need $3 \cdot 3 = 9$ variables.

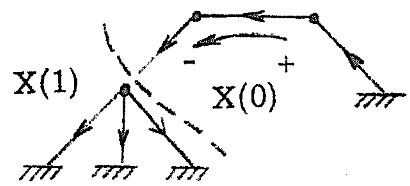


Fig. 55. Structural dissection

B. We choose a global coordinate system with the origin at A, see Fig. 54. We array the matrices

(92) $$Z(0)^* = m_3 \overset{\begin{array}{ccc} 4 & 5 & 6 \end{array}}{[\, I \;\; I \;\; I \,]}$$

(79) $$E(0) = \begin{array}{c} B \\ C \end{array} \overset{\begin{array}{ccc} 4 & 5 & 6 \end{array}}{\begin{bmatrix} I & -I & \\ & I & -I \end{bmatrix}}, \qquad E(1) = A \overset{\begin{array}{ccc} 1 & 2 & 3 \end{array}}{[\, I \;\; I \;\; I \,]}$$

$$E(10)\, Z(0) = A \overset{m_3}{[-I\,]}, \qquad E(10) = A \overset{\begin{array}{ccc} 4 & 5 & 6 \end{array}}{[-I \;\; 0 \;\; 0\,]}$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The flexibilities and stiffnesses are given relative to the local coordinate system at beam end 2

(167) $$f' = \begin{bmatrix} 6I/A & 0 & 0 \\ 0 & L^2 & -3L \\ 0 & -3L & 6L \end{bmatrix} \frac{L}{6EI}; \quad F' = \begin{bmatrix} A/12I & 0 & 0 \\ 0 & 1 & 2L \\ 0 & 2L & 3L^2 \end{bmatrix} \frac{12EI}{L^3}$$

The flexibilities of the members in X(0) relative to the global coordinate system are

(179) $$f(0)_{iA} = T_{iA}^* f_i' T_{iA} \qquad i = 4,\ 5,\ 6$$

where

(170) $$T_{iA} = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$$

$$f_4' = \begin{bmatrix} 2,276 \cdot 10^{-2} & 0 & 0 \\ 0 & 7,542 & -4,000 \\ 0 & -4,000 & 2,828 \end{bmatrix} \frac{1}{EI}, \quad T_{4A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2,8284 & 1 \end{bmatrix}$$

$$f(0)_{4A} = T_{4A}^* f_4' T_{4A} = \begin{bmatrix} 0,0228 & 0 & 0 \\ 0 & 7,542 & 4,000 \\ 0 & 4,000 & 2,828 \end{bmatrix}$$

$$f_5' = \begin{bmatrix} 0,0424 & 0 & 0 \\ 0 & 41,667 & -12,500 \\ 0 & -12,500 & 5,000 \end{bmatrix} \frac{1}{EI}, \quad T_{5A} = \begin{bmatrix} 0,7071 & 0,7071 & 0 \\ -0,7071 & 0,7071 & 0 \\ -3,5350 & 6,3640 & 1 \end{bmatrix}$$

$$f(0)_{5A} = T_{5A}^* f_5' T_{5A} = \begin{bmatrix} 20,844 & -45,790 & -8,836 \\ -45,790 & 110,803 & 22,975 \\ -8,836 & 22,975 & 5,000 \end{bmatrix} \frac{1}{EI}$$

$$f_6' = \begin{bmatrix} 0,0341 & 0 & 0 \\ 0 & 25,455 & -9,000 \\ 0 & -9,000 & 4,243 \end{bmatrix} \frac{1}{EI}, \quad T_{6A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -7,778 & 6,364 & 1 \end{bmatrix}$$

$$f(0)_{6A} = \begin{bmatrix} 142,126 & -152,734 & -24,000 \\ -152,734 & 171,825 & 27,000 \\ -24,000 & 27,000 & 4,243 \end{bmatrix} \frac{1}{EI}$$

The structure flexibility of X(0) is

(144) $$f(0,0) = Z(0)^* f(0) Z(0) = Z(0)^* \left[f(0)_{iA}\right]^D Z(0) = f(0)_{4A} + f(0)_{5A} + f(0)_{6A} =$$

$$= \begin{bmatrix} 162,993 & -198,524 & -32,836 \\ -198,524 & 299,006 & 53,975 \\ -32,836 & 53,975 & 12,071 \end{bmatrix} \frac{1}{EI}$$

The stiffnesses of the members in X(1) relative to the global coordinate system are

(177) $$F(1)_{iA} = T_{iA}^{-1} F_i (T_{iA}^{*})^{-1}$$

(178) $$T_{iA}^{-1} = \begin{bmatrix} \gamma^* & \\ Y^*\gamma^* & \gamma^* \end{bmatrix}$$

$$F(1)_{1A} = F_1' = \begin{bmatrix} 29,2872 & 0 & 0 \\ 0 & 0,1571 & 0,3333 \\ 0 & 0,3333 & 0,9428 \end{bmatrix} EI$$

$$F_3' = F_1', \; T_{3A} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F(1)_{3A} = (T_{3A}^{*})^{-1} F_3' (T_{3A}^{*})^{-1} = \begin{bmatrix} 0,1571 & 0 & 0,3333 \\ 0 & 29,2872 & 0 \\ 0,3333 & 0 & 0,9428 \end{bmatrix} EI$$

$$F_2' = \begin{bmatrix} 41,418 & 0 & 0 \\ 0 & 0,4444 & +0,6667 \\ 0 & +0,6667 & 1,3333 \end{bmatrix} EI, \; T_{2A} = \begin{bmatrix} 0,7071 & -0,7071 & 0 \\ 0,7071 & 0,7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F(1)_{2A} = (T_{2A}^{*})^{-1} F_2' (T_{2A}^{*})^{-1} = \begin{bmatrix} 20,9308 & -20,4866 & 0,4714 \\ -20,4866 & 20,9308 & 0,4714 \\ 0,4714 & 0,4714 & 1,3333 \end{bmatrix} EI$$

The structure stiffness of X(1) is

(143) $$F(1,1) = E(1)\,F(1)\,E(1)^* = E(1) \left[F(1)_{iA}\right]^D E(1)^* = F(1)_{1A} + F(1)_{2A} + F(1)_{3A}$$

$$F(1,1) = \begin{bmatrix} 50,3751 & -20,4876 & 0,8047 \\ -20,4866 & 50,3751 & 0,8047 \\ 0,8047 & 0,8047 & 3,2189 \end{bmatrix} EI$$

C. The structure action in Fig. 54 gives

$$r = \begin{bmatrix} r(0) \\ r(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} P(0) \\ \hline P(1) \end{bmatrix} = \begin{bmatrix} P_B \\ P_C \\ \hline P_A \end{bmatrix}; \; P_A = \begin{bmatrix} -15\sqrt{2}\,Mp \\ 15\sqrt{2}\,Mp \\ 0 \end{bmatrix}, \; P_B = \begin{bmatrix} -5\sqrt{2}\,Mp \\ 5\sqrt{2}\,Mp \\ -10 - 2 \cdot 10 Mpm \end{bmatrix} =$$

$$= \begin{bmatrix} -\;7,071\,Mp \\ 7,071\,Mp \\ -30,000\,Mpm \end{bmatrix}, \; P_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The load P(0) is written relative to the global coordinate system at A.

D. The particular solution for member forces are

(132) $$N(0)_p = E(0)^{-1}P(0) = \begin{bmatrix} I & I \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_B \\ P_C \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_B \\ 0 \end{bmatrix} = \begin{bmatrix} P_B \\ 0 \\ 0 \end{bmatrix}$$

(134) $$E(1)N(1)_p = -E(10)N(0)_p + P(1) = -\begin{bmatrix} -I & 0 & 0 \end{bmatrix} \begin{bmatrix} P_B \\ 0 \\ 0 \end{bmatrix} + P_A + P_B =$$

$$= \begin{bmatrix} -28,284 \text{ Mp} \\ 28,284 \text{ Mp} \\ -30,000 \text{ Mpm} \end{bmatrix}$$

where

(133) $$E(0)E(0)^{-1} = \begin{bmatrix} I & -I & 0 \\ 0 & I & -I \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

The load $r = 0$ implies the particular solution for cycle deformations

(137) $$n(1)_p = 0$$

(139) $$Z^*(0)\,n(0)_p = 0$$

E. The equivalent cycle deformation $r(0, 1)$ on $X(0)$ is obtained by vector multiplication from the right in

(150) $$r(0,1) = -Z(0)^* f(0) N(0)_p = -\begin{bmatrix} I & I & I \end{bmatrix} \begin{bmatrix} f(0)_{4A}, & f(0)_{5A}, & f(0)_{6A} \end{bmatrix}^D \begin{bmatrix} P_B \\ 0 \\ 0 \end{bmatrix} =$$

$$= -f(0)_{4A}P_B = \begin{bmatrix} 0,300 \\ 66,692 \\ 56,556 \end{bmatrix} \frac{1}{EI}$$

The equivalent structure deformations $P(1, 0)$ on $X(1)$ are

(151) $$P(1,0) = E(1)N(1)_p = \begin{bmatrix} -28,284 \\ 28,284 \\ -30,000 \end{bmatrix}$$

Fa. The inverse of the structure stiffness of X(1) is

$$F(1,1)^{-1} = \begin{bmatrix} 0,02401 & 0,00990 & -0,00848 \\ 0,00990 & 0,02401 & -0,00848 \\ -0,00848 & -0,00848 & 0,31490 \end{bmatrix} \frac{1}{EI}$$

Further

(142) $f(0,0,0) = f(0,0) + Z(0)^* E(10)^* F(1,1)^{-1} E(10) E(0) =$

$$= \begin{bmatrix} 163,017 & 198,514 & -32,845 \\ 198,514 & 299,030 & 53,906 \\ -32,845 & 53,906 & 12,385 \end{bmatrix} \cdot \frac{1}{EI}$$

(145) $r(0,1,0) = r(0,1) + Z(0)^* E(10)^* F(1,1)^{-1} P(1,0) = \begin{bmatrix} 0,300 \\ 66,692 \\ 56,556 \end{bmatrix} \frac{1}{EI} +$

$$+ \begin{bmatrix} 0,145 \\ -0,667 \\ 9,447 \end{bmatrix} \frac{1}{EI} = \begin{bmatrix} 0,445 \\ 66,025 \\ 66,003 \end{bmatrix} \frac{1}{EI}$$

The calculation of the unknown cycle forces in (141) is made by use of Gaussian elimination.

(141) $R(0) = f(0,0,0)^{-1} r(0,1,0) = \begin{bmatrix} -0,594 \text{ Mp} \\ -3,978 \text{ Mp} \\ 21,083 \text{ Mpm} \end{bmatrix}$

We calculate

$$-E(10)Z(0)R(0) + P(1,0) = -[-I]\, R(0) + P(1,0) = \begin{bmatrix} -0,594 \\ -3,978 \\ 21,083 \end{bmatrix} + \begin{bmatrix} -28,284 \\ 28,284 \\ -30,000 \end{bmatrix} =$$

$$= \begin{bmatrix} -28,878 \\ 24,306 \\ -8,917 \end{bmatrix}$$

We obtain the structure deformations p(1) in X(1) as

(140) $p(1) = F(1,1)^{-1}(-E(10)Z(0)R(0) + P(1,0)) =$

$$= \begin{bmatrix} 0,02401 & 0,00990 & -0,00848 \\ 0,00990 & 0,02401 & -0,00848 \\ -0,00848 & 0,00848 & 0,31490 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} -28,284 \\ 24,306 \\ -8,917 \end{bmatrix} = \begin{bmatrix} -0,337 \\ 0,373 \\ -2,769 \end{bmatrix} \frac{1}{EI}$$

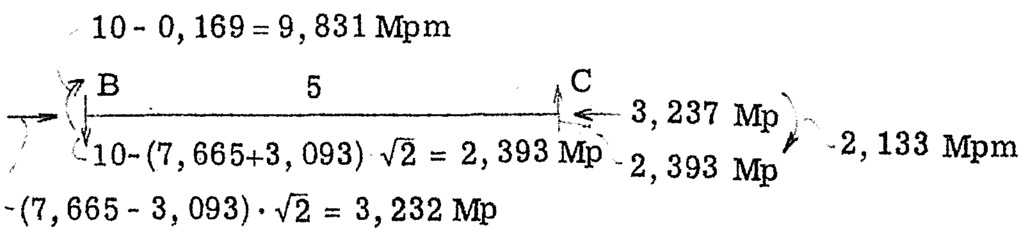
G. The member forces are, in the global coordinate system,

$$(152) \qquad N(0) = N(0)_p + Z(0)R(0) = \begin{bmatrix} N_{4A} \\ N_{5A} \\ N_{6A} \end{bmatrix} = \begin{bmatrix} -7,071 \\ 7,071 \\ -30,000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -0,594 \\ -3,978 \\ 21,083 \\ -0,594 \\ -3,978 \\ 21,083 \\ -0,594 \\ -3,978 \\ 21,083 \end{bmatrix} = \begin{bmatrix} -7,665 \\ 3,093 \\ -8,917 \\ -0,594 \\ -3,978 \\ 21,083 \\ -0,594 \\ -3,978 \\ 21,083 \end{bmatrix}$$

The forces N(0) are transformed to the local coordinate systems of each beam.

$$(169) \qquad N'(0) = \begin{bmatrix} N_4 \\ N_5 \\ N_6 \end{bmatrix} = \begin{bmatrix} T_{4A} & T_{5A} & T_{6A} \end{bmatrix}^D \begin{bmatrix} N_{4A} \\ N_{5A} \\ N_{6A} \end{bmatrix} = \begin{bmatrix} -7,665 \text{ Mp} \\ 3,093 \text{ Mp} \\ -0,169 \text{ Mpm} \\ -3,237 \text{ Mp} \\ -2,393 \text{ Mp} \\ -2,133 \text{ Mpm} \\ -3,978 \text{ Mp} \\ 0,594 \text{ Mp} \\ 0,378 \text{ Mpm} \end{bmatrix}$$

As a check of the calculation of N'(0) we study the equilibrium of beam number 5.



Horisontal force equilibrium gives →: $3,232 - 3,237 = -0,005$ Mp

Moment equilibrium gives D: $9,831 + 2,133 - 2,393 \cdot 5 =$
$= 11,964 - 11,965 = 0,001$ Mpm

Thus the equilibrium is satisfactory.

The member forces N(1) relative to the global coordinate system are

$$(153) \qquad N(1) = F(1)n(1) = F(1)E(1)^*p(1) = F(1)\begin{bmatrix} I \\ I \\ I \end{bmatrix} p(1)$$

$$N(1) = \begin{bmatrix} N_{1A} \\ N_{2A} \\ N_{3A} \end{bmatrix} = \begin{bmatrix} F_{1A} & & \\ & F_{2A} & \\ & & F_{3A} \end{bmatrix} \begin{bmatrix} I \\ I \\ I \end{bmatrix} p(1) = \begin{bmatrix} -11,041 \\ -\ 0,923 \\ -\ 2,486 \\ -16,838 \\ 14,225 \\ -\ 3,692 \\ -\ 0,982 \\ 10,924 \\ -\ 2,736 \end{bmatrix}$$

Transformation to local coordinate systems yields

$$(169) \qquad N'(1) = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} T_{1A} & T_{2A} & T_{3A} \end{bmatrix}^D \begin{bmatrix} N_{1A} \\ N_{2A} \\ N_{3A} \end{bmatrix} = \begin{bmatrix} -11,041 \text{ Mp} \\ -\ 0,923 \text{ Mp} \\ -\ 2,486 \text{ Mpm} \\ -21,965 \text{ Mp} \\ -\ 1,848 \text{ Mp} \\ -\ 3,638 \text{ Mpm} \\ -10,924 \text{ Mp} \\ -\ 0,982 \text{ Mp} \\ -\ 2,699 \text{ Mpm} \end{bmatrix}$$

We now check the equilibrium of the joint A.

Moment equilibrium gives $\widehat{A}$: +2,486 + 3,638 + 2,699 - 8,917 = -0,003 Mpm

Horisontal force equilibrium : 11,276 - 10,924 + 0,923 + 0,982) 0,7071 + + 1,848 - (7,665 - 3,093) 0,7071 = 0,210 Mp

Vertical force equilibrium : 21,965 + (10,924 + 11,276 + 0,982 - 0,923 - - 3,093 - 7,665) 0,7071 - 30,000 = 0,103 Mp

Thus the equilibrium is quite satisfactory.

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