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# Various approaches to products of residue currents 

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#### Abstract

We describe various approaches to Coleff-Herrera products of residue currents $R^{j}$ (of Cauchy-Fantappiè-Leray type) associated to holomorphic mappings $f_{j}$. More precisely, we study to which extent (exterior) products of natural regularizations of the individual currents $R^{j}$ yield regularizations of the corresponding Coleff-Herrera products. Our results hold globally on an arbitrary pure-dimensional complex space. © 2012 Elsevier Inc. All rights reserved.


Keywords: Residue currents; Regularization of currents; Coleff-Herrera products; Cauchy-Fantappiè-Leray type currents

## 1. Introduction

Let $f$ be a holomorphic function defined on the unit ball $\mathbb{B} \subset \mathbb{C}^{n}$. If $f$ is a monomial it is elementary to show, e.g., by integrations by parts or by a Taylor expansion, that the principal value current $\varphi \mapsto \lim _{\epsilon \rightarrow 0} \int_{|f|^{2}>\epsilon} \varphi / f, \varphi \in \mathscr{D}_{n, n}(\mathbb{B})$, exists and defines a $(0,0)$-current $1 / f$ that we also denote by $U^{f}$. From Hironaka's theorem it then follows that such limits exist for general $f$ and also that $\mathbb{B}$ may be replaced by a complex space [20]. The $\bar{\partial}$-image, $R^{f}:=\bar{\partial}(1 / f)$, is the residue current of $f$ and by Stokes' theorem it is given by $\varphi \mapsto \lim _{\epsilon \rightarrow 0} \int_{|f|^{2}=\epsilon} \varphi / f, \varphi \in$ $\mathscr{D}_{n, n-1}(\mathbb{B})$. It has the useful property that its annihilator ideal is equal to the principal ideal $\langle f\rangle$

[^0]and, moreover, it gives a factorization of Lelong's integration current; $2 \pi i[f=0]=\bar{\partial}(1 / f) \wedge$ $d f$.

There are (at least) two natural ways of regularizing $U^{f}$ and $R^{f}$. If $\lambda \in \mathbb{C}$ and $\mathfrak{R e} \lambda \gg 0$, then $\lambda \mapsto \int \varphi|f|^{2 \lambda} / f$ is holomorphic for any test form $\varphi$. It is well known (cf., Lemma 6) that the current-valued map $\lambda \mapsto|f|^{2 \lambda} / f=: U^{f, \lambda}$ has a meromorphic extension to $\mathbb{C}$ with poles contained in the set of negative rational numbers and that the value at $\lambda=0$ is $U^{f}$. It follows that $\lambda \mapsto \bar{\partial}|f|^{2 \lambda} / f=: R^{f, \lambda}$ is meromorphic in $\mathbb{C}$, analytic in a half-space containing the origin, and that the value at the origin is $R^{f}$. The technique of using analytic continuation in residue current theory has its roots in the work of Atiyah [8], and Bernstein and Gel'fand [14]. In the context of residue currents it has been developed by several authors, e.g., Barlet and Maire [9], Yger [33], Berenstein, Gay and Yger [11], Passare and Tsikh [26], and recently by the second author in [30]. The second regularization method, inspired by Passare [24], is more explicit and concrete; $U^{f}$ and $R^{f}$ are obtained as weak limits of explicit smooth forms. Let $\chi$ be a smooth regularization of the characteristic function $\mathbf{1}_{[1, \infty)}$ and let $U^{f, \epsilon}:=\chi\left(|f|^{2} / \epsilon\right) / f$ and $R^{f, \epsilon}:=$ $\bar{\partial} \chi\left(|f|^{2} / \epsilon\right) / f$. Then (see, e.g., [24]) $U^{f}=\lim _{\epsilon \rightarrow 0^{+}} U^{f, \epsilon}$ and $R^{f}=\lim _{\epsilon \rightarrow 0^{+}} R^{f, \epsilon}$ in the sense of currents. Notice that the original definition mentioned above corresponds to $\chi=\mathbf{1}_{[1, \infty)}$.

If $f$ is a tuple of functions or a section of a vector bundle there are natural analogues of the currents $1 / f$ and $\bar{\partial}(1 / f)$ introduced in [28] and [1]. The construction of these more general currents, still denoted $U^{f}$ and $R^{f}$, is based on Bochner-Martinelli and Cauchy-Fantappiè-Leray type formulas; see Section 2 for details. In this paper we consider products of regularized currents of this kind and we investigate their limit behavior. It turns out that both the $\lambda$-approach and the $\epsilon$-approach yield the same current as the classical Coleff-Herrera approach.

Let $Z$ be a reduced complex space of pure dimension $n$, let $E_{1}, \ldots, E_{p}$ be hermitian holomorphic vector bundles over $Z$, and let $f_{j}$ be a holomorphic section of $E_{j}^{*}$. Then $U^{f_{j}}=: U^{j}$ and $R^{f_{j}}=: R^{j}$ become currents with values in $\bigwedge E_{j}$; if $\operatorname{rank} E_{j}=1$ then $U^{j}$ is the principal value current associated with the meromorphic section $1 / f_{j}$ of $E_{j}$ and $R^{j}=\bar{\partial} U^{j}$. In complete analogy with the regularization methods discussed above we have

$$
U^{j}=\left.U^{j, \lambda}\right|_{\lambda=0}=\lim _{\epsilon \rightarrow 0^{+}} U^{j, \epsilon} \quad \text { and } \quad R^{j}=\left.R^{j, \lambda}\right|_{\lambda=0}=\lim _{\epsilon \rightarrow 0^{+}} R^{j, \epsilon}
$$

see Section 2. We define products of the $R^{j}$ (for simplicity we restrict attention to such products in this section) recursively as follows: Having defined $R^{k-1} \wedge \cdots \wedge R^{1}$ it turns out (see [7] or Section 2) that

$$
\lambda \mapsto R^{k, \lambda} \wedge R^{k-1} \wedge \cdots \wedge R^{1}
$$

has an analytic continuation to a neighborhood of $\lambda=0$ and we define $R^{k} \wedge \cdots \wedge R^{1}$ as the value at $\lambda=0$. From the proof of Proposition 5.4 in [6] it follows that one can compute the product in the following way: If $a_{1}>\cdots>a_{p}>0$ are integers then

$$
R^{p} \wedge \cdots \wedge R^{1}=\left.R^{p, \lambda^{a_{p}}} \wedge \cdots \wedge R^{1, \lambda^{a_{1}}}\right|_{\lambda=0}
$$

That is, the recursive definition can be replaced by the evaluation of a one-variable analytic (current-valued) function at the origin; we just have to make sure that $\lambda^{a_{1}}$ tends to zero much faster than $\lambda^{a_{2}}$ and so on.

We now consider the smooth form $R^{p, \epsilon_{p}} \wedge \cdots \wedge R^{1, \epsilon_{1}}$ and limits of it of the following kind:
Definition 1. Let $\vartheta$ be a function defined on $(0, \infty)^{p}$. We let

$$
\lim _{\epsilon_{1}<\cdots \ll \epsilon_{p} \rightarrow 0} \vartheta\left(\epsilon_{1}, \ldots, \epsilon_{p}\right)
$$

denote the limit (if it exists and is well defined) of $\vartheta$ along any path $\delta \mapsto \epsilon(\delta)$ towards the origin such that for all $\ell \in \mathbb{N}$ and $j=2, \ldots, p$ there are positive constants $C_{j \ell}$ such that $\epsilon_{j-1}(\delta) \leqslant$ $C_{j \ell} \epsilon_{j}^{\ell}(\delta)$. Here, we extend the domain of definition of $\vartheta$ to points $\left(0, \ldots, 0, \epsilon_{m+1}, \ldots, \epsilon_{p}\right)$, where $\epsilon_{m+1}, \ldots, \epsilon_{p}>0$, by defining

$$
\vartheta\left(0, \ldots, 0, \epsilon_{m+1}, \ldots, \epsilon_{p}\right)=\lim _{\epsilon_{m} \rightarrow 0} \ldots \lim _{\epsilon_{1} \rightarrow 0} \vartheta\left(\epsilon_{1}, \ldots, \epsilon_{m}, \epsilon_{m+1}, \ldots, \epsilon_{p}\right),
$$

if the limits exist.
Recall that $\left(\epsilon_{1}, \ldots, \epsilon_{p}\right)$ tends to zero along an admissible paths in the sense of Coleff and Herrera [17], if it tends to zero along a path inside $(0, \infty)^{p}$ such that $\epsilon_{j-1} / \epsilon_{j}^{\ell} \rightarrow 0$ for all $\ell \in \mathbb{N}$ and $j=2, \ldots, p$. The limits in Definition 1 are (slightly) more general since, e.g., $\epsilon_{1}$ is allowed to attain the value 0 before the other $\epsilon_{j}$ go to zero. In particular, it thus includes the iterated limit letting $\epsilon_{k} \rightarrow 0$ one at a time. The following theorem is a special case of Theorem 11 below. The proof shares many similarities with the proof of [24, Proposition 1] (even though the statements differ). However, in our case, extra technical difficulties arise since the bundles $E_{j}$ may have non-trivial metrics.

Theorem 2. In the sense of currents we have

$$
R^{p} \wedge \cdots \wedge R^{1}=\lim _{\epsilon_{1}<\cdots \ll \epsilon_{p} \rightarrow 0} R^{p, \epsilon_{p}} \wedge \cdots \wedge R^{1, \epsilon_{1}}
$$

To connect with the classical Coleff-Herrera approach, assume temporarily that rank $E_{j}=1$, $j=1, \ldots, p$, so that $R^{j}=\bar{\partial}\left(1 / f_{j}\right)$. Then Theorem 2 says that for any test form $\varphi$ of bidegree ( $n, n-p$ )

$$
\bar{\partial} \frac{1}{f_{p}} \wedge \cdots \wedge \bar{\partial} \frac{1}{f_{1}} \cdot \varphi=\lim _{\epsilon_{1}<\cdots \ll \epsilon_{p} \rightarrow 0} \int_{Z} \frac{\bar{\partial} \chi^{\epsilon_{p}}}{f_{p}} \wedge \cdots \wedge \frac{\bar{\partial} \chi^{\epsilon_{1}}}{f_{1}} \wedge \varphi,
$$

where $\chi^{\epsilon_{j}}=\chi\left(\left|f_{j}\right|^{2} / \epsilon_{j}\right)$. We will refer to the integral on the right-hand side as the residue integral and denote it by $\mathcal{I}_{f}^{\varphi}(\epsilon)$. If the $\chi$-functions tend to $\mathbf{1}_{[1, \infty)}$ (for a fixed generic $\epsilon \in(0, \infty)^{p}$ ) then $\mathcal{I}_{f}^{\varphi}(\epsilon)$ tends to Coleff-Herrera's original residue integral

$$
\begin{equation*}
I_{f}^{\varphi}(\epsilon)=\int_{T(\epsilon)} \varphi /\left(f_{1} \cdots f_{p}\right) \tag{1}
\end{equation*}
$$

where $T(\epsilon)=\bigcap_{1}^{p}\left\{\left|f_{j}\right|^{2}=\epsilon_{j}\right\}$ is oriented as the distinguished boundary of the corresponding polyhedron. In [17] Coleff and Herrera prove that the limit of $I_{f}^{\varphi}(\epsilon)$ along an admissible path
exists and defines a current, the nowadays called Coleff-Herrera product. We show (see Theorem 11) that the Coleff-Herrera product equals the product $\bar{\partial}\left(1 / f_{p}\right) \wedge \cdots \wedge \bar{\partial}\left(1 / f_{1}\right)$; this is folklore but to our knowledge not completely proved before (except in the case of complete intersection when it follows from [24] and [23] together with [30]).

A result much in the same spirit was proven by Passare in [23], where he relates the original Coleff-Herrera product to residue currents defined by $\lambda$-regularizations. Passare considers the regularization

$$
\begin{equation*}
\left.\frac{\bar{\partial}\left|f_{p}\right|^{2 \lambda}}{f_{p}} \wedge \cdots \wedge \frac{\bar{\partial}\left|f_{1}\right|^{2 \lambda}}{f_{1}}\right|_{\lambda=0} \tag{2}
\end{equation*}
$$

i.e., instead of letting the $\lambda_{i}$ go to zero successively, all the $\lambda_{i}$ are equal to a single $\lambda$ that tends to 0 . In that case, Passare proves that this current coincides with an average of limits along parabolic paths of the residue integral, as considered in [24], irrespectively of whether $f$ defines a complete intersection or not.

The product $R^{k} \wedge \cdots \wedge R^{1}$ does in general not have any natural commutation properties. For instance, $\bar{\partial}(1 /(z w)) \wedge \bar{\partial}(1 / z)=0$ while $\bar{\partial}(1 / z) \wedge \bar{\partial}(1 /(z w))=\bar{\partial}\left(1 / z^{2}\right) \wedge \bar{\partial}(1 / w)$, where the last product simply is the tensor product. However, if the $f_{j}$ define a complete intersection, i.e., $\operatorname{codim}\left\{f_{1}=\cdots=f_{p}=0\right\}=\sum_{j} \operatorname{rank} E_{j}$, then it is known (see, e.g., [3]) that the product is commutative; the case when all the $E_{j}$ have rank 1 is proved in [17].

Remark 3. Recall that the currents $R^{j}$ take values in $\bigwedge E_{j}$. The sum of the degree of $R^{j}$ in $\bigwedge E_{j}$ and its form-degree is even. Therefore the product is naturally commutative. If the $E_{j}$ are trivial line bundles that we do not make any distinction between, then the product is anti-commutative; this is the classical Coleff-Herrera setting.

Theorem 4. Assume that the $f_{j}$ define a complete intersection. Then for every test form $\varphi$

$$
\left(\lambda_{1}, \ldots, \lambda_{p}\right) \mapsto \int_{Z} R^{p, \lambda_{p}} \wedge \cdots \wedge R^{1, \lambda_{1}} \wedge \varphi
$$

has an analytic continuation to a neighborhood of the origin in $\mathbb{C}^{p}$.
This result is a special case of our Theorem 14, which generalizes [30, Theorem 1]. The case when $p=2$ and rank $E_{j}=1$ was proved by Berenstein-Yger (see, e.g., [10]). The following result is a special case of Theorem 13, which generalizes [16, Theorem 1].

Theorem 5. Assume that the $f_{j}$ define a complete intersection. Then for every test form $\varphi$

$$
\left(\epsilon_{1}, \ldots, \epsilon_{p}\right) \mapsto \int_{Z} R^{p, \epsilon_{p}} \wedge \cdots \wedge R^{1, \epsilon_{1}} \wedge \varphi
$$

is Hölder continuous on $[0, \infty)^{p}$.
For this result it is crucial that the $\chi$-functions used to regularize the $R^{j}$ are smooth. In fact, Passare and Tsikh [27], found a quite simple tuple ( $f_{1}, f_{2}$ ) defining a complete intersection
in $\mathbb{C}^{2}$ and a test form $\varphi$ such that the classical Coleff-Herrera residue integral $I_{\left(f_{1}, f_{2}\right)}^{\varphi}(\epsilon)$ is discontinuous at $\epsilon=0$. Soon after Björk found generic families of such examples, see, e.g., [15].

Let us give some background and motivation for the kind of products considered here. Products of Cauchy-Fantappiè-Leray type currents were first studied by Wulcan [32]. Wulcan defines the product as the value at $\lambda=0$ of the analytic continuation of $\lambda \mapsto R^{p, \lambda} \wedge \cdots \wedge R^{1, \lambda}$. In the non-complete intersection case Wulcan's product is different from our; in the case that all $E_{j}$ have rank $1,\left.R^{p, \lambda} \wedge \cdots \wedge R^{1, \lambda}\right|_{\lambda=0}$ coincides with Passare's product, (2). Passare-Wulcan products satisfy several natural computation rules and are quite useful but it has turned out that the recursive definition discussed above often is more natural. In particular, the Stückrad-Vogel intersection algorithm in non-proper intersection theory is conveniently expressed using recursively defined products, see [6].

In the complete intersection case there is no ambiguity, the Coleff-Herrera product is commutative and if $f=\left(f_{1}, \ldots, f_{p}\right)$ then $R^{f}$ equals $\bigwedge_{j} \bar{\partial}\left(1 / f_{j}\right)$, see [28] and [1]. This indicates that the Coleff-Herrera product is the "correct" current to associated to a complete intersection. The Coleff-Herrera product is the minimal current extension of Grothendieck's cohomological residue (see, e.g., [25] for definitions) in the sense that it annihilated by anti-holomorphic functions vanishing on its support. Moreover, if $f$ defines a complete intersection then the annihilator ideal of $R^{f}$ equals the ideal generated by $f$, see [25] and [18]. This property is very useful and lies behind many applications, e.g., explicit division-interpolation formulas and Briançon-Skoda type results [2,10], explicit versions of the fundamental principle [13], the $\bar{\partial}$-equation on complex spaces [4,5,19], and explicit Green currents in arithmetic intersection theory [12].

In Section 2, we give the necessary background and the general formulations of our results. Section 3 contains the proof of Theorems 2 and 11. The proof of Theorems 4, 5, 13 and 14 is the content of Section 4; the crucial part is Lemma 19 which enables us to effectively use the assumption about complete intersection.

## 2. Formulation of the general results

Let $Z$ be a reduced complex space of pure dimension $n$. We say that $\varphi$ is a smooth $(p, q)$-form on $Z$ if $\varphi$ is smooth on $Z_{\text {reg }}$, and in a neighborhood of any $p \in Z$, there is a smooth $(p, q)$-form $\tilde{\varphi}$ in an ambient complex manifold such that the pull-back of $\tilde{\varphi}$ to $Z_{\text {reg }}$ coincides with $\left.\varphi\right|_{Z_{\text {reg }}}$ close to $p$. The $(p, q)$-test forms on $Z, \mathscr{D}_{p, q}(Z)$, are defined as the smooth compactly supported ( $p, q$ )-forms (with a suitable topology) and the space of $(p, q)$-currents on $Z, \mathscr{D}_{p, q}^{\prime}(Z)$, is the dual of $\mathscr{D}_{n-p, n-q}(Z)$. More concretely, if $i: Z \rightarrow \Omega \subset \mathbb{C}^{N}$ is an embedding and $\mu$ is a $(p, q)$ current on $Z$ then $i_{*} \mu$ is an $(N-n+p, N-n+q)$-current in $\Omega$ that vanishes on test forms $\xi$ such that $i^{*} \xi=0$ on $Z_{\text {reg }}$. Conversely, such a current in $\Omega$ defines a current on Z. See, e.g., [22] for a more thorough discussion.

Let $x$ be a complex coordinate on $\mathbb{C}$. Recall that the principal value current $1 / x^{m}$ can be computed as the value at $\lambda=0$ of the analytic continuation of $|x|^{2 \lambda} / x^{m}$; the residue current $\bar{\partial}\left(1 / x^{m}\right)$ then is the value at $\lambda=0$ of $\bar{\partial}|x|^{2 \lambda} / x^{m}$. Since one can take tensor products of onevariable currents it follows that

$$
\begin{equation*}
T=\frac{1}{x_{1}^{\alpha_{1}}} \wedge \cdots \wedge \frac{1}{x_{p}^{\alpha_{p}}} \wedge \frac{\vartheta(x)}{x_{p+1}^{\alpha_{p+1}} \cdots x_{n}^{\alpha_{n}}} \tag{3}
\end{equation*}
$$

is a well-defined current in $\mathbb{C}^{n}$; here $\alpha_{1}, \ldots, \alpha_{p}$ are positive integers, $\alpha_{p+1}, \ldots, \alpha_{n}$ are nonnegative integers, and $\vartheta$ is a smooth compactly supported form. Such a current $T$ is called an
elementary pseudomeromorphic current. Following [7] we say that a current $\mu$ on $Z$ is pseudomeromorphic, $\mu \in \mathcal{P} \mathcal{M}(Z)$, if $\mu$ locally is a finite sum of push-forwards $\pi_{*}^{1} \cdots \pi_{*}^{m} \tau$ under maps

$$
X^{m} \xrightarrow{\pi^{m}} \cdots \xrightarrow{\pi^{2}} X^{1} \xrightarrow{\pi^{1}} Z,
$$

where each $\pi^{j}$ is either a modification or an open inclusion and $\tau$ is an elementary pseudomeromorphic current on $X^{m}$. It follows that the class of pseudomeromorphic currents is closed under $\bar{\partial}$ and multiplication with smooth forms, and that the push-forward of a pseudomeromorphic current by a modification is pseudomeromorphic.

Lemma 6. Let $f$ be a holomorphic function, and let $T \in \mathcal{P} \mathcal{M}(Z)$. If $\tilde{f}$ is a holomorphic function such that $\{\tilde{f}=0\}=\{f=0\}$ and $v$ is a smooth non-zero function, then $\left(|\tilde{f} v|^{2 \lambda} / f\right) T$ and $\left(\bar{\partial}|\tilde{f} v|^{2 \lambda} / f\right) \wedge T$ have current-valued analytic continuations to $\lambda=0$ and the values at $\lambda=0$ are pseudomeromorphic and independent of the choices of $\tilde{f}$ and $v$. Moreover, if $\chi=\mathbf{1}_{[1, \infty)}$, or a smooth approximation thereof, then

$$
\begin{equation*}
\left.\frac{|\tilde{f} v|^{2 \lambda}}{f} T\right|_{\lambda=0}=\lim _{\epsilon \rightarrow 0^{+}} \frac{\chi^{\epsilon}}{f} T \quad \text { and }\left.\quad \frac{\bar{\partial}|\tilde{f} v|^{2 \lambda}}{f} \wedge T\right|_{\lambda=0}=\lim _{\epsilon \rightarrow 0^{+}} \frac{\bar{\partial} \chi^{\epsilon}}{f} \wedge T \tag{4}
\end{equation*}
$$

where $\chi^{\epsilon}=\chi\left(|\tilde{f} v|^{2} / \epsilon\right)$.

Proof. The first part is essentially Proposition 2.1 in [7], except that there, $Z$ is a complex manifold, $\tilde{f}=f$ and $v \equiv 1$. However, with suitable resolutions of singularities, the proof in [7] goes through in the same way in our situation, as long as we observe that in $\mathbb{C}$

$$
\frac{\left|x^{\alpha^{\prime}} v\right|^{2 \lambda}}{x^{\alpha}} \frac{1}{x^{\beta}} \quad \text { and } \quad \frac{\left|x^{\alpha^{\prime}} v\right|^{2 \lambda}}{x^{\alpha}} \bar{\partial} \frac{1}{x^{\beta}}
$$

have analytic continuations to $\lambda=0$, and the values at $\lambda=0$ are $1 / x^{\alpha+\beta}$ and 0 respectively, independently of $\alpha^{\prime}$ and $v$, as long as $\alpha^{\prime}>0$ and $v \neq 0$ (and similarly with $\bar{\partial}\left|x^{\alpha^{\prime}} v\right|^{2 \lambda} / x^{\alpha}$ ).

By Leibniz rule, it is enough to consider the first equality in (4), since if we have proved the first equality, then

$$
\begin{aligned}
\lim _{\epsilon \rightarrow 0} \frac{\bar{\partial} \chi^{\epsilon}}{f} \wedge T & =\lim _{\epsilon \rightarrow 0} \bar{\partial}\left(\frac{\chi^{\epsilon}}{f} T\right)-\frac{\chi^{\epsilon}}{f} \bar{\partial} T \\
& =\left.\left(\bar{\partial}\left(\frac{|\tilde{f} v|^{2 \lambda}}{f} T\right)-\frac{|\tilde{f} v|^{2 \lambda}}{f} \bar{\partial} T\right)\right|_{\lambda=0}=\left.\frac{\bar{\partial}|\tilde{f} v|^{2 \lambda}}{f} \wedge T\right|_{\lambda=0}
\end{aligned}
$$

To prove the first equality in (4), we observe first that in the same way as in the first part, we can assume that $f=x^{\gamma} u$ and $\tilde{f}=x^{\tilde{\gamma}} \tilde{u}$, where $u$ and $\tilde{u}$ are non-zero holomorphic functions. Since $T$ is a sum of push-forwards of elementary currents, we can assume that $T$ is of the form (3). Note that if supp $\gamma \cap \operatorname{supp} \beta \neq \emptyset$, then $\left(|\tilde{f} v|^{2 \lambda} / f\right) T=0$ for $\mathfrak{R e} \lambda \gg 0$ and $\left(\chi\left(|\tilde{f} v|^{2} / \epsilon\right) / f\right) T=0$ for $\epsilon>0$, since $\operatorname{supp} T \subseteq\left\{x_{i}=0, i \in \operatorname{supp} \beta\right\}$. Thus, we can assume that supp $\gamma \cap \operatorname{supp} \beta=\emptyset$. By a smooth (but non-holomorphic) change of variables, as in Section 3 (Eqs. (13)), we can assume
that $|\tilde{u} v|^{2} \equiv 1$. Thus, since $\left(\left|x^{\tilde{\gamma}}\right|^{2 \lambda} / x^{\gamma}\right)\left(1 / x^{\alpha}\right),\left(\chi\left(\left|x^{\tilde{\gamma}}\right|^{2} / \epsilon\right) / x^{\gamma}\right)\left(1 / x^{\alpha}\right)$ depend on variables disjoint from the ones that $\bigwedge_{\beta_{i} \neq 0} \bar{\partial}\left(1 / x_{i}^{\beta_{i}}\right)$ depends on, it is enough to prove that

$$
\left.\frac{\left|x^{\tilde{\gamma}}\right|^{2 \lambda}}{x^{\gamma}} \frac{1}{x^{\alpha}}\right|_{\lambda=0}=\lim _{\epsilon \rightarrow 0} \frac{\chi\left(\left|x^{\tilde{\gamma}}\right|^{2} / \epsilon\right)}{x^{\gamma}} \frac{1}{x^{\alpha}},
$$

which is Lemma 2 in [16].
Let $E_{1}, \ldots, E_{q}$ be holomorphic hermitian vector bundles over $Z$, let $f_{j}$ be a holomorphic section of $E_{j}^{*}, j=1, \ldots, q$, and let $s_{j}$ be the section of $E_{j}$ with pointwise minimal norm such that $f_{j} \cdot s_{j}=\left|f_{j}\right|^{2}$. Outside $\left\{f_{j}=0\right\}$, define

$$
u_{k}^{j}=\frac{s_{j} \wedge\left(\bar{\partial} s_{j}\right)^{k-1}}{\left|f_{j}\right|^{2 k}}
$$

It is easily seen that if $f_{j}=f_{j}^{0} f_{j}^{\prime}$, where $f_{j}^{0}$ is a holomorphic function and $f_{j}^{\prime}$ is a non-vanishing section, then $u_{k}^{j}=\left(1 / f_{j}^{0}\right)^{k}\left(u^{\prime}\right)_{k}^{j}$, where $\left(u^{\prime}\right)_{k}^{j}$ is smooth across $\left\{f_{j}=0\right\}$. We let

$$
\begin{equation*}
U^{j}=\left.\sum_{k=1}^{\infty}\left|\tilde{f}_{j}\right|^{2 \lambda} u_{k}^{j}\right|_{\lambda=0} \tag{5}
\end{equation*}
$$

where $\tilde{f}_{j}$ is any holomorphic section of $E_{j}^{*}$ such that $\left\{\tilde{f}_{j}=0\right\}=\left\{f_{j}=0\right\}$. The existence of the analytic continuation is a local statement, so we can assume that $f_{j}=\sum f_{j, k} \mathfrak{e}_{j, k}^{*}$, where $\mathfrak{e}_{j, k}^{*}$ is a local holomorphic frame for $E_{j}^{*}$. After principalization we can assume that the ideal $\left\langle f_{j, 1}, \ldots, f_{j, k_{j}}\right\rangle$ is generated by, e.g., $f_{j, 0}$. By the representation $u_{k}^{j}=\left(1 / f_{j, 0}\right)^{k}\left(u^{\prime}\right)_{k}^{j}$, the existence of the analytic continuation of $U^{j}$ in (5) then follows from Lemma 6. Let $U_{k}^{j}$ denote the term of $U^{j}$ that takes values in $\bigwedge^{k} E_{j} ; U_{k}^{j}$ is thus a $(0, k-1)$-current with values in $\bigwedge^{k} E_{j}$. Let $\delta_{f_{j}}$ denote interior multiplication with $f_{j}$ and put $\nabla_{f_{j}}=\delta_{f_{j}}-\bar{\partial}$; it is not hard to verify that $\nabla_{f_{j}} U=1$ outside $f_{j}=0$. We define the Cauchy-Fantappiè-Leray type residue current, $R^{j}$, of $f_{j}$ by $R^{j}=1-\nabla_{f_{j}} U^{j}$. One readily checks that

$$
\begin{align*}
R^{j} & =R_{0}^{j}+\sum_{k=1}^{\infty} R_{k}^{j} \\
& =\left.\left(1-\left|\tilde{f}_{j}\right|^{2 \lambda}\right)\right|_{\lambda=0}+\left.\sum_{k=1}^{\infty} \bar{\partial}\left|\tilde{f}_{j}\right|^{2 \lambda} \wedge \frac{s_{j} \wedge\left(\bar{\partial} s_{j}\right)^{k-1}}{\left|f_{j}\right|^{2 k}}\right|_{\lambda=0} \tag{6}
\end{align*}
$$

where, as above, $\tilde{f}_{j}$ is a holomorphic section such that $\left\{\tilde{f}_{j}=0\right\}=\left\{f_{j}=0\right\}$.
Remark 7. Notice that if $E_{j}$ has rank 1, then $U_{j}$ simply equals $1 / f_{j}$ and $R^{j}=1-\nabla_{f_{j}}\left(1 / f_{j}\right)=$ $1-f_{j} \cdot\left(1 / f_{j}\right)+\bar{\partial}\left(1 / f_{j}\right)=\bar{\partial}\left(1 / f_{j}\right)$.

We now define a non-commutative calculus for the currents $U_{k}^{i}$ and $R_{\ell}^{j}$ recursively as follows.
Definition 8. If $T$ is a product of some $U_{k}^{i}$ and $R_{\ell}^{j}$, then we define
-

$$
\begin{gathered}
U_{k}^{j} \wedge T=\left.\left|\tilde{f}_{j}\right|^{2 \lambda} \frac{s_{j} \wedge\left(\bar{\partial} s_{j}\right)^{k-1}}{\left|f_{j}\right|^{2 k}} \wedge T\right|_{\lambda=0} \\
R_{0}^{j} \wedge T=\left.\left(1-\left|\tilde{f}_{j}\right|^{2 \lambda}\right) T\right|_{\lambda=0} \\
R_{k}^{j} \wedge T=\left.\bar{\partial}\left|\tilde{f}_{j}\right|^{2 \lambda} \wedge \frac{s_{j} \wedge\left(\bar{\partial} s_{j}\right)^{k-1}}{\left|f_{j}\right|^{2 k}} \wedge T\right|_{\lambda=0}
\end{gathered}
$$

where $\tilde{f}_{j}$ is any holomorphic section of $E_{j}^{*}$ with $\left\{\tilde{f}_{j}=0\right\}=\left\{f_{j}=0\right\}$.
Notice that after principalization the pull-back of $u_{k}^{j}$ is semi-meromorphic; in particular $U^{j}$ and $R^{j}$ are pseudomeromorphic. Thus, by Lemma 6, the analytic continuations of Definition 8 exist and the values at $\lambda=0$ are pseudomeromorphic as well.

Remark 9. Under assumptions about complete intersection, these products have the suggestive commutation properties, e.g., if $\operatorname{codim}\left\{f_{i}=f_{j}=0\right\}=\operatorname{rank} E_{i}+\operatorname{rank} E_{j}$, then $R_{k}^{i} \wedge R_{\ell}^{j}=R_{\ell}^{j} \wedge$ $R_{k}^{i}, R_{k}^{i} \wedge U_{\ell}^{j}=U_{\ell}^{j} \wedge R_{k}^{i}$, and $U_{k}^{i} \wedge U_{\ell}^{j}=-U_{\ell}^{j} \wedge U_{k}^{i}$ (see, e.g., [3]). In general, there are no simple relations. However, products involving only $U$ :s are always anti-commutative.

Now, consider collections $R=\left\{R_{k_{1}}^{1}, \ldots, R_{k_{p}}^{p}\right\}$ and $U=\left\{U_{k_{p+1}}^{p+1}, \ldots, U_{k_{q}}^{q}\right\}$ and put $\left(P_{1}, \ldots\right.$, $\left.P_{q}\right)=\left(R_{k_{1}}^{1}, \ldots, R_{k_{p}}^{p}, U_{k_{p+1}}^{p+1}, \ldots, U_{k_{q}}^{q}\right)$. For a permutation $v$ of $\{1, \ldots, q\}$ we define

$$
\begin{equation*}
(U R)^{\nu}=P_{\nu(q)} \wedge \cdots \wedge P_{\nu(1)} \tag{7}
\end{equation*}
$$

From (5) and (6) we get natural $\lambda$-regularizations, $P_{j}^{\lambda}$, of $P_{j}$ and from Definition 8 we have $(U R)^{\nu}=P_{\nu(q)}^{\lambda_{q}} \wedge \cdots \wedge P_{v(1)}^{\lambda_{1}}\left|\lambda_{1}=0 \cdots\right|_{\lambda_{q}=0}$, i.e., we set successively $\lambda_{1}=0$, then $\lambda_{2}=0$ and so on. The following result is proved in [6].

Theorem 10. Let $a_{1}>\cdots>a_{q}>0$ be integers and $\lambda$ a complex variable. Then

$$
\lambda \mapsto P_{\nu(q)}^{\lambda_{q}^{a_{q}}} \wedge \cdots \wedge P_{\nu(1)}^{\lambda^{a_{1}}}
$$

has a current-valued analytic continuation to a neighborhood of the half-axis $[0, \infty) \subset \mathbb{C}$ and the value at $\lambda=0$ equals $(U R)^{\nu}$.

The recursively defined product $(U R)^{v}$ can thus be obtained as the value at zero of a onevariable $\zeta$-type function. From an algebraic point of view, this is desirable since one can derive functional equations and use Bernstein-Sato theory to study $(U R)^{\nu}$.

There are also more concrete and explicit regularizations of the currents $U_{k}^{i}$ and $R_{\ell}^{j}$ inspired by [17] and [24]. Let $\chi=\mathbf{1}_{[1, \infty)}$, or a smooth approximation thereof that is 0 close to 0 and 1 close to $\infty$. It follows from [29], or after principalization from Lemma 6, that

$$
\begin{gather*}
U_{k}^{j}=\lim _{\epsilon \rightarrow 0^{+}} \chi\left(\left|\tilde{f}_{j}\right|^{2} / \epsilon\right) \frac{s_{j} \wedge\left(\bar{\partial} s_{j}\right)^{k-1}}{\left|f_{j}\right|^{2 k}},  \tag{8}\\
R_{k}^{j}=\lim _{\epsilon \rightarrow 0^{+}} \bar{\partial} \chi\left(\left|\tilde{f}_{j}\right|^{2} / \epsilon\right) \wedge \frac{s_{j} \wedge\left(\bar{\partial} s_{j}\right)^{k-1}}{\left|f_{j}\right|^{2 k}}, \quad k>0, \tag{9}
\end{gather*}
$$

and similarly for $k=0$; as usual, $\left\{\tilde{f}_{j}=0\right\}=\left\{f_{j}=0\right\}$. Of course, the limits are in the current sense and if $\chi=\mathbf{1}_{[1, \infty)}$, then $\epsilon$ is supposed to be a regular value for $\left|f_{j}\right|^{2}$ and $\bar{\partial} \chi\left(\left|f_{j}\right|^{2} / \epsilon\right)$ is to be interpreted as integration over the manifold $\left|f_{j}\right|^{2}=\epsilon$. We denote the regularizations given by (8) and (9) by $P_{j}^{\epsilon}$.

Theorem 11. Let $R=\left\{R_{k_{1}}^{1}, \ldots, R_{k_{p}}^{p}\right\}$ and $U=\left\{U_{k_{p+1}}^{p+1}, \ldots, U_{k_{q}}^{q}\right\}$ be collections of currents defined in (5) and (6). Let $v$ be a permutation of $\{1, \ldots, q\}$ and let $(U R)^{v}$ be the product defined in (7). Then

$$
(U R)^{\nu}=\lim _{\epsilon_{1} \ll \cdots \ll \epsilon_{q} \rightarrow 0} P_{\nu(q)}^{\epsilon_{q}} \wedge \cdots \wedge P_{\nu(1)}^{\epsilon_{1}}
$$

where, as above, $\left(P_{1}, \ldots, P_{q}\right)=\left(R_{k_{1}}^{1}, \ldots, R_{k_{p}}^{p}, U_{k_{p+1}}^{p+1}, \ldots, U_{k_{q}}^{q}\right)$; see Definition 1 for the meaning of the limit. If $\chi=\mathbf{1}_{[1, \infty)}$, we require that $\epsilon \rightarrow 0$ along an admissible path in the sense of Coleff-Herrera.

Thus $(U R)^{v}$ can be computed as the weak limit of an explicit smooth form and moreover, Definition 8 give the Coleff-Herrera product (in case the bundles $E_{j}$ have rank 1).

Remark 12. It might be more natural to consider products of whole Cauchy-Fantappiè-Leray type currents, $U^{j}$ and $R^{j}$, as in (5) and (6), and not just products of their components $U_{k}^{j}$ and $R_{k}^{j}$, cf., for example [6]. However, since such a product is a sum of products of their components, it follows readily that Theorem 11 holds also for products of whole Cauchy-Fantappiè-Leray type currents.

### 2.1. The complete intersection case

Assume that $f_{1}, \ldots, f_{q}$ define a complete intersection, i.e., that $\operatorname{codim}\left\{f_{1}=\cdots=f_{q}=0\right\}=$ $\operatorname{rank} E_{1}+\cdots+\operatorname{rank} E_{q}$. Then we know that the calculus defined in Definition 8 satisfies the suggestive commutation properties, but we have in fact the following much stronger results.

Theorem 13. Assume that $f_{1}, \ldots, f_{q}$ define a complete intersection on $Z$, let $\left(P_{1}, \ldots, P_{q}\right)=$ $\left(R_{k_{1}}^{1}, \ldots, R_{k_{p}}^{p}, U_{k_{p+1}}^{p+1}, \ldots, U_{k_{q}}^{q}\right)$, and let $P_{j}^{\epsilon_{j}}$ be an $\epsilon$-regularization of $P_{j}$ defined by (8) and (9) with smooth $\chi$-functions. Then we have

$$
\left|\int_{Z} P_{1}^{\epsilon_{1}} \wedge \cdots \wedge P_{q}^{\epsilon_{q}} \wedge \varphi-P_{1} \wedge \cdots \wedge P_{q} \cdot \varphi\right| \leqslant C\|\varphi\|_{C^{M}}\left(\epsilon_{1}^{\omega}+\cdots+\epsilon_{q}^{\omega}\right)
$$

where $M$ and $\omega$ only depend on $f_{1}, \ldots, f_{q}, Z$, and $\operatorname{supp} \varphi$ while $C$ also depends on the $C^{M}$-norm of the $\chi$-functions.

Theorem 14. Assume that $f_{1}, \ldots, f_{q}$ define a complete intersection on $Z$, let $\left(P_{1}, \ldots, P_{q}\right)=$ $\left(R_{k_{1}}^{1}, \ldots, R_{k_{p}}^{p}, U_{k_{p+1}}^{p+1}, \ldots, U_{k_{q}}^{q}\right)$, and let $P_{j}^{\lambda_{j}}$ be the $\lambda$-regularization of $P_{j}$ given by (5) and (6). Then the current-valued function

$$
\lambda \mapsto P_{1}^{\lambda_{1}} \wedge \cdots \wedge P_{q}^{\lambda_{q}}
$$

a priori defined for $\mathfrak{R e} \lambda_{j} \gg 0$, has an analytic continuation to a neighborhood of the half-space $\bigcap_{1}^{q}\left\{\mathfrak{R e} \lambda_{j} \geqslant 0\right\}$.

Remark 15. In case the $E_{j}$ are trivial with trivial metrics, Theorems 13 and 14 follow quite easily from, respectively, [16, Theorem 1] and [30, Theorem 1] by taking averages. As an illustration, let $\varepsilon_{1}, \ldots, \varepsilon_{r}$ be a nonsense basis and let $f_{1}, \ldots, f_{r}$ be holomorphic functions. Then we can write $s=\bar{f} \cdot \varepsilon$ and so $u_{k}=(\bar{f} \cdot \varepsilon) \wedge(d \bar{f} \cdot \varepsilon)^{k-1} /|f|^{2 k}$. A standard computation shows that

$$
\int_{\alpha \in \mathbb{C P}^{r-1}} \frac{|\alpha \cdot f|^{2 \lambda} \alpha \cdot \varepsilon}{(\alpha \cdot f)|\alpha|^{2 \lambda}} d V=A(\lambda)|f|^{2 \lambda} \frac{\bar{f} \cdot \varepsilon}{|f|^{2}},
$$

where $d V$ is the (normalized) Fubini-Study volume form and $A$ is holomorphic with $A(0)=1$. It follows that

$$
\int_{\alpha_{1}, \ldots, \alpha_{k} \in \mathbb{C P}^{r-1}} \bigwedge_{1}^{k} \frac{\bar{\partial}\left|\alpha_{j} \cdot f\right|^{2 \lambda}}{\alpha_{j} \cdot f} \wedge \frac{\alpha_{j} \cdot \varepsilon}{\left|\alpha_{j}\right|^{2 \lambda}} d V\left(\alpha_{j}\right)=A(\lambda)^{k} \bar{\partial}\left(|f|^{2 k \lambda} u_{k}\right) .
$$

Elaborating this formula and using [30, Theorem 1] one can show Theorem 14 in the case of trivial $E_{j}$ with trivial metrics. The general case can probably also be handled in a similar manner but the computations become more involved and we prefer to give direct proofs.

## 3. Proof of Theorem 11

The structure of this proof is rather similar to the structure of the proof of Proposition 5.4 in [6].

We start by making a Hironaka resolution of singularities [21], of $Z$ such that the pre-image of $\bigcup_{j}\left\{f_{j}=0\right\}$ has normal crossings. We then make further toric resolutions (e.g., as in [28]) such that, in local charts, the pull-back of each $f_{i}$ is a monomial, $x^{\alpha_{i}}$, times a non-vanishing holomorphic tuple. One checks that the pull-back of $P_{j}^{\epsilon}$ is of one of the following forms:

$$
\frac{\chi\left(\left|x^{\tilde{\alpha}}\right|^{2} \xi / \epsilon\right)}{x^{\alpha}} \vartheta, \quad 1-\chi\left(\left|x^{\tilde{\alpha}}\right|^{2} \xi / \epsilon\right), \quad \frac{\bar{\partial} \chi\left(\left|x^{\tilde{\alpha}}\right|^{2} \xi / \epsilon\right)}{x^{\alpha}} \wedge \vartheta
$$

where $\xi$ is smooth and positive, $\operatorname{supp} \tilde{\alpha}=\operatorname{supp} \alpha$, and $\vartheta$ is a smooth bundle valued form; by localizing on the blow-up we may also suppose that $\vartheta$ has as small support as we wish. If the $\chi$-functions are smooth, the following special case of Theorem 11 now immediately follows from Lemma 6:

$$
\begin{equation*}
(U R)^{\nu}=\lim _{\epsilon_{q} \rightarrow 0} \cdots \lim _{\epsilon_{1} \rightarrow 0} P_{\nu(q)}^{\epsilon_{q}} \wedge \cdots \wedge P_{\nu(1)}^{\epsilon_{1}} \tag{10}
\end{equation*}
$$

For smooth $\chi$-functions we put

$$
\mathcal{I}(\epsilon)=\int \frac{\bar{\partial} \chi_{1}^{\epsilon} \wedge \cdots \wedge \bar{\partial} \chi_{p}^{\epsilon} \chi_{p+1}^{\epsilon} \cdots \chi_{q}^{\epsilon}}{x^{\alpha_{1}+\cdots+\alpha_{p}+\cdots+\alpha_{q^{\prime}}}} \wedge \varphi
$$

where $q^{\prime} \leqslant q, \varphi$ is a smooth $(n, n-p)$-form with support close to the origin, and $\chi_{j}^{\epsilon}=$ $\chi\left(\left|x^{\tilde{\alpha}_{j}}\right|^{2} \xi_{j} / \epsilon_{j}\right)$ for smooth positive $\xi_{j}$. We note that we may replace the $\bar{\partial}$ in $\mathcal{I}(\epsilon)$ by $d$ for bidegree reasons. In case $\chi=\mathbf{1}_{[1, \infty)}$ we denote the corresponding integral by $I(\epsilon)$. We also put $\mathcal{I}^{\nu}\left(\epsilon_{1}, \ldots, \epsilon_{q}\right)=\mathcal{I}\left(\epsilon_{\nu(1)}, \ldots, \epsilon_{\nu(q)}\right)$ and similarly for $I^{\nu}$. In view of (10), the special case of Theorem 11 when the $\chi$-functions are smooth will be proved if we can show that

$$
\begin{equation*}
\lim _{\epsilon_{1}<\cdots \ll \epsilon_{q} \rightarrow 0} \mathcal{I}^{\nu}(\epsilon) \tag{11}
\end{equation*}
$$

exists. The case with $\chi=\mathbf{1}_{[1, \infty)}$ will then follow if we can show

$$
\begin{equation*}
\lim _{\delta \rightarrow 0}\left(\mathcal{I}^{\nu}(\epsilon(\delta))-I^{\nu}(\epsilon(\delta))\right)=0 \tag{12}
\end{equation*}
$$

where $\delta \mapsto \epsilon(\delta)$ is any admissible path.
For notational convenience, we will consider $\mathcal{I}^{\nu}(\epsilon)$ (unless otherwise stated), but our arguments apply just as well to $I^{\nu}(\epsilon)$ until we arrive at the integral (16).

Denote by $\tilde{A}$ the $q \times n$-matrix with rows $\tilde{\alpha}_{i}$. We will first show that we can assume that $\tilde{A}$ has full rank. The idea is the same as in [17] and [24], however because of the paths along which our limits are taken, we have to modify the argument slightly. The following lemma follows from the proof of Lemma III.12.1 in [31].

Lemma 16. Assume that $\alpha$ is a $q \times n$-matrix with rows $\alpha_{i}$ such that there exists $\left(v_{1}, \ldots, v_{q}\right) \neq 0$ with $\sum v_{i} \alpha_{i}=0$. Let $j=\min \left\{i ; v_{i} \neq 0\right\}$. Then there exist constants $C, c>0$ such that if $\epsilon_{j}<$ $C\left(\epsilon_{j+1} \ldots \epsilon_{q}\right)^{c}$, then $\chi\left(\left|x^{\alpha_{j}}\right|^{2} \xi_{j} / \epsilon_{j}\right) \equiv 1$ and $\bar{\partial} \chi\left(\left|x^{\alpha_{j}}\right|^{2} \xi_{j} / \epsilon_{j}\right) \equiv 0$ for all $x \in \Delta \cap\left\{\left|x^{\alpha_{i}}\right|^{2} \geqslant\right.$ $\left.C_{i} \epsilon_{i}, i=j+1, \ldots, q\right\}$, where $\Delta$ is the unit polydisc.

Assume that $\tilde{A}$ does not have full rank, and let $v$ be a column vector such that $v^{t} \tilde{A}=0$. Since $\left(\epsilon_{1}, \ldots, \epsilon_{q}\right)$ is replaced by $\left(\epsilon_{\nu(1)}, \ldots, \epsilon_{\nu(q)}\right)$ in $\mathcal{I}^{v}(\epsilon)$, we choose instead $j_{0}$ such that $v\left(j_{0}\right) \leqslant \nu(i)$ for all $i$ such that $v_{i} \neq 0$. If $j_{0} \leqslant p$, we let $\widetilde{\mathcal{I}}^{\nu}(\epsilon)=0$, and if $j_{0} \geqslant p+1$, we let $\widetilde{\mathcal{I}}^{\nu}(\epsilon)$ be $\mathcal{I}^{\nu}(\epsilon)$ but with $\chi_{j_{0}}^{\epsilon}$ replaced by 1 . If $\epsilon=\epsilon(\delta)$ is such that $\epsilon_{\nu\left(j_{0}\right)}>0$, then $\mathcal{I}^{\nu}(\epsilon)$ is a current acting on a test form with support on a set of the form

$$
\Delta \cap\left\{\left|x^{\alpha_{i}}\right|^{2} \geqslant C_{i} \epsilon_{\nu(i)} ; \text { for all } i \text { such that } v(i) \geqslant v\left(j_{0}\right)\right\} .
$$

In particular, if $\epsilon_{\nu\left(j_{0}\right)}(\delta)$ is sufficiently small compared to $\left(\epsilon_{\nu\left(j_{0}\right)+1}(\delta), \ldots, \epsilon_{q}(\delta)\right)$, then by Lemma 16, if $j_{0} \leqslant p$, the factor $\bar{\partial} \chi_{j_{0}}^{\epsilon}$ is identically 0 , and if $j_{0} \geqslant p+1$, the factor $\chi_{j_{0}}^{\epsilon}$ is identically 1 and thus is equal to $\widetilde{\mathcal{I}}^{\nu}(\epsilon)$ for such $\epsilon$. Similarly, if $\epsilon_{\nu\left(j_{0}\right)}=0$, we have that $\mathcal{I}^{\nu}(\epsilon)$ is defined as a limit along $\epsilon_{\nu\left(j_{0}\right)} \rightarrow 0$, with $\epsilon_{\nu\left(j_{0}\right)+1}, \ldots, \epsilon_{q}$ fixed and in the limit we get again that for sufficiently small $\epsilon_{\nu\left(j_{0}\right)}$, we can replace $\mathcal{I}^{\nu}(\epsilon)$ by $\widetilde{\mathcal{I}}^{\nu}(\epsilon)$. Thus we have

$$
\lim _{\epsilon_{1}<\cdots \ll \epsilon_{q} \rightarrow 0} \mathcal{I}^{\nu}(\epsilon)=\lim _{\epsilon_{1} \ll \cdots<\epsilon_{q} \rightarrow 0} \widetilde{\mathcal{I}}^{v}(\epsilon),
$$

and we have reduced to the case that $\tilde{A}$ is a $(q-1) \times n$-matrix of the same rank. We continue this procedure until $\tilde{A}$ has full rank.

By re-numbering the coordinates, we may suppose that the minor $A=\left(\tilde{\alpha}_{i j}\right)_{1 \leqslant i, j \leqslant q}$ of $\tilde{A}$ is invertible and we put $A^{-1}=B=\left(b_{i j}\right)$. We now use complex notation to make a non-holomorphic, but smooth change of variables:

$$
\begin{align*}
& y_{1}=x_{1} \xi^{b_{1} / 2}, \ldots, y_{q}=x_{q} \xi^{b_{q} / 2}, y_{q+1}=x_{q+1}, \ldots, y_{n}=x_{n} \\
& \bar{y}_{1}=\bar{x}_{1} \xi^{b_{1} / 2}, \ldots, \bar{y}_{q}=\bar{x}_{q} \xi^{b_{q} / 2}, \bar{y}_{q+1}=\bar{x}_{q+1}, \ldots, \bar{y}_{n}=\bar{x}_{n} \tag{13}
\end{align*}
$$

where $\xi^{b_{i} / 2}=\xi_{1}^{b_{i 1} / 2} \cdots \xi_{q}^{b_{i q} / 2}$. One easily checks that $d y \wedge d \bar{y}=\xi^{b_{1}} \cdots \xi^{b_{q}} d x \wedge d \bar{x}+O(|x|)$, so (13) defines a smooth change of variables between neighborhoods of the origin. A simple linear algebra computation then shows that $\left|x^{\tilde{\alpha}_{i}}\right|^{2} \xi_{i}=\left|y^{\tilde{\alpha}_{i}}\right|^{2}$. Of course, this change of variables does not preserve bidegrees so $\varphi(y)$ is merely a smooth compactly supported $(2 n-p)$-form. We thus have

$$
\begin{equation*}
\mathcal{I}^{v}(\epsilon)=\int_{\Delta} \frac{d \chi_{1}^{\epsilon} \wedge \cdots \wedge d \chi_{p}^{\epsilon} \chi_{p+1}^{\epsilon} \cdots \chi_{q}^{\epsilon}}{y^{\alpha_{1}+\cdots+\alpha_{p}+\cdots+\alpha_{q^{\prime}}}} \wedge \varphi^{\prime}(y) \tag{14}
\end{equation*}
$$

where $\chi_{j}^{\epsilon}=\chi\left(\left|y^{\tilde{\alpha}_{j}}\right|^{2} / \epsilon_{\nu(j)}\right)$ and $\varphi^{\prime}(y)=\sum_{|I|+|J|=2 n-p} \psi_{I J} d y_{I} \wedge d \bar{y}_{J}$. By linearity we may assume that the sum only consists of one term $\varphi^{\prime}(y)=\psi d y_{K} \wedge d \bar{y}_{L}$, and by scaling, we may assume that $\operatorname{supp} \psi \subseteq \Delta, \Delta$ being the unit polydisc. By Lemma 2.4 in [17], we can write the function $\psi$ as

$$
\begin{equation*}
\psi(y)=\sum_{I+J<\sum_{1}^{q^{\prime}} \alpha_{j}-1} \psi_{I J} y^{I} \bar{y}^{J}+\sum_{I+J=\sum_{1}^{q^{\prime}} \alpha_{j}-\mathbf{1}} \psi_{I J} y^{I} \bar{y}^{J}, \tag{15}
\end{equation*}
$$

where $a<b$ for tuples $a$ and $b$ means that $a_{i}<b_{i}$ for all $i$. In the decomposition (15) each of the smooth functions $\psi_{I J}$ in the first sum on the left-hand side is independent of some variable. We now show that this implies that the first sum on the left-hand side of (15) does not contribute to the integral (14). In case $\varphi^{\prime}(y)$ has bidegree $(n, n-p)$ this is a well-known fact but we must show it for an arbitrary $(2 n-p)$-form.

We change to polar coordinates:

$$
d y_{K} \wedge d \bar{y}_{L}=d\left(r_{K_{1}} e^{i \theta_{K_{1}}}\right) \wedge \cdots \wedge d\left(r_{L_{1}} e^{-i \theta_{L_{1}}}\right) \wedge \cdots
$$

Since $\chi_{j}^{\epsilon}$ in (14) is independent of $\theta$, it follows that we must have full degree $=n$ in $d \theta$. The only terms in the expansion of $d y_{K} \wedge d \bar{y}_{L}$ above that will contribute to (14) are therefore of the form

$$
c r_{1} \cdots r_{n} e^{i \theta \cdot \gamma} d r_{M} \wedge d \theta
$$

where $|M|=n-p, c$ is a constant, and $\gamma$ is a multiindex with entries equal to $1,-1$, or 0 . Substituting this and a term $\psi_{I J} y^{I} \bar{y}^{J}=\psi_{I J} r^{I+J} e^{i \theta \cdot(I-J)}$ from (15) into (14) gives rise to an "inner" $\theta$-integral (by Fubini's theorem):

$$
\mathscr{J}_{I J}(r)=\int_{\theta \in[0,2 \pi)^{n}} \psi_{I J}(r, \theta) e^{i \theta \cdot\left(I-J-\sum_{1}^{q^{\prime}} \alpha_{j}+\gamma\right)} d \theta
$$

If $I+J<\sum_{1}^{q^{\prime}} \alpha_{j}-\mathbf{1}$, then $I-J-\sum_{1}^{q^{\prime}} \alpha_{j}+\gamma<0$ and $\psi_{I J}$ is independent of some $y_{j}=r_{j} e^{i \theta_{j}}$. Integrating over $\theta_{j} \in[0,2 \pi)$ thus yields $\mathscr{J}_{I J}=0$ if $I+J<\sum_{1}^{q^{\prime}} \alpha_{j}-\mathbf{1}$. If instead $I+J=$ $\sum_{1}^{q^{\prime}} \alpha_{j}-1$, then $\mathscr{J}_{I J}(r)$ is smooth on $[0, \infty)^{n}$.

Summing up, we see that we can write (14) as

$$
\begin{equation*}
\mathcal{I}^{\nu}(\epsilon)=\int_{r \in(0,1)^{n}} d \chi_{1}^{\epsilon} \wedge \cdots \wedge d \chi_{p}^{\epsilon} \chi_{p+1}^{\epsilon} \cdots \chi_{q}^{\epsilon} \mathscr{J}(r) d r_{M} \tag{16}
\end{equation*}
$$

where $\chi_{j}^{\epsilon}=\chi\left(r^{2 \alpha_{j}} / \epsilon_{\nu(j)}\right), \mathscr{J}$ is smooth, and $|M|=n-p$.
After these reductions, the integral (16) we arrive at is the same as Eq. (16) in [24], and we will use the fact proven there, that $\lim _{\delta \rightarrow 0} \mathcal{I}^{\nu}(\epsilon(\delta))$ exists along any admissible path $\epsilon(\delta)$, and is well defined independently of the choice of admissible path. (This is not exactly what is proven there, but the fact that if $b \in \mathbb{Q}^{p}$, then $\lim _{\delta \rightarrow 0} \epsilon(\delta)^{b}$ is either 0 or $\infty$ independently of the admissible path chosen is the only addition we need to make for the argument to go through in our case.) Using this, if we let $\epsilon(\delta)$ be any admissible path, we will show by induction over $q$ that

$$
\lim _{\epsilon_{1} \lll<\epsilon_{q} \rightarrow 0} \mathcal{I}^{\nu}(\epsilon)=\lim _{\delta \rightarrow 0} \mathcal{I}^{\nu}(\epsilon(\delta)) .
$$

For $q=1$ this is trivially true, so we assume $q>1$. Let $\epsilon^{k}$ be any sequence satisfying the conditions in Definition 1. Consider a fixed $k$, and let $m$ be such that $\epsilon^{k}=\left(0, \ldots, 0, \epsilon_{m+1}^{k}, \ldots, \epsilon_{q}^{k}\right)$ with $\epsilon_{m+1}^{k}>0$. Let $I_{1}=v^{-1}(\{1, \ldots, m\}) \cap\{1, \ldots, p\}$ and $I_{2}=v^{-1}(\{1, \ldots, m\}) \cap\{p+1, \ldots, q\}$. We consider $\epsilon_{m+1}^{k}, \ldots, \epsilon_{q}^{k}$ fixed in $\mathcal{I}^{v}(\epsilon)$, and define

$$
\mathcal{I}_{k}\left(\epsilon_{1}, \ldots, \epsilon_{m}\right)=\int_{[0,1]^{n}} \bigwedge_{i \in I_{1}} d \chi\left(r^{\alpha_{i}} / \epsilon_{\nu(i)}\right) \prod_{i \in I_{2}} \chi\left(r^{\alpha_{i}} / \epsilon_{\nu(i)}\right) \mathscr{J}_{k}(r) d r_{M}
$$

originally defined on $(0, \infty)^{p}$, but extended according to Definition 1, where

$$
\mathscr{J}_{k}(r)= \pm \bigwedge_{i \in\{1, \ldots, p\} \backslash I_{1}} d \chi\left(r^{\alpha_{i}} / \epsilon_{\nu(i)}^{k}\right) \prod_{i \in\{p+1, \ldots, q\} \backslash I_{2}} \chi\left(r^{\alpha_{i}} / \epsilon_{\nu(i)}^{k}\right) \mathscr{J}(r)
$$

(where the sign is chosen such that $\mathcal{I}_{k}(0)=\mathcal{I}^{\nu}\left(\epsilon^{k}\right)$ ). Since $m<q$ and $\mathscr{J}_{k}$ is smooth, we have by induction that

$$
\mathcal{I}_{k}(0)=\lim _{\epsilon_{m} \rightarrow 0} \ldots \lim _{\epsilon_{1} \rightarrow 0} \mathcal{I}_{k}\left(\epsilon_{1}, \ldots, \epsilon_{m}\right)=\lim _{\delta \rightarrow 0} \mathcal{I}_{k}\left(\epsilon^{\prime}(\delta)\right)
$$

where $\epsilon^{\prime}(\delta)$ is any admissible path, and the first equality follows by definition of $\mathcal{I}_{k}(0)$. We fix an admissible path $\epsilon^{\prime}(\delta)$. For each $k$ we can choose $\delta_{k}$ such that if $\epsilon^{k^{\prime}}=\left(\epsilon_{1}^{\prime}\left(\delta_{k}\right), \ldots, \epsilon_{m}^{\prime}\left(\delta_{k}\right)\right)$, then $\lim _{k \rightarrow \infty}\left(\mathcal{I}_{k}\left(\epsilon^{k^{\prime}}\right)-\mathcal{I}_{k}(0)\right)=0$ and if $\tilde{\epsilon}^{k}=\left(\epsilon^{k^{\prime}}, \epsilon_{m+1}^{k}, \ldots, \epsilon_{q}^{k}\right)$, then $\tilde{\epsilon}^{k}$ forms a subsequence of an admissible path. Since $\mathcal{I}_{k}(0)=\mathcal{I}^{\nu}\left(\epsilon^{k}\right)$, and $\mathcal{I}_{k}\left(\epsilon^{k^{\prime}}\right)=\mathcal{I}^{\nu}\left(\tilde{\epsilon}^{k}\right)$, we thus have

$$
\lim _{k \rightarrow \infty} \mathcal{I}^{\nu}\left(\epsilon^{k}\right)=\lim _{k \rightarrow \infty} \mathcal{I}^{\nu}\left(\tilde{\epsilon}^{k}\right)=\lim _{\delta \rightarrow 0} \mathcal{I}^{\nu}(\epsilon(\delta))
$$

where the second equality follows from the existence and uniqueness of $\mathcal{I}^{\nu}(\epsilon(\delta))$ along any admissible path. Hence we have shown that the limit in (11) exists and is well defined.

Finally, if we start from (16), as (23) in [24] shows, either

$$
\lim _{\epsilon_{1} \ll \cdots \ll \epsilon_{q} \rightarrow 0} \mathcal{I}^{\nu}(\epsilon)= \pm \int_{r_{M} \in(0,1)^{n-p}} \mathscr{J}\left(0, r_{M}\right) d r_{M}
$$

or the limit is 0 , depending only on $\alpha$. If we consider $I^{\nu}(\epsilon)$ instead, we get the same limit, see [31, pp. 79-80], and (12) follows.

## 4. Proof of Theorems 13 and 14

As in [30] and [16] the key-step of the proof is a Whitney type division lemma, Lemma 19 below. Recall that

$$
\left(P_{1}, \ldots, P_{q}\right)=\left(R_{k_{1}}^{1}, \ldots, R_{k_{p}}^{p}, U_{k_{p+1}}^{p+1}, \ldots, U_{k_{q}}^{q}\right)
$$

and that $P_{j}^{\epsilon_{j}}$ and $P_{j}^{\lambda_{j}}$ are the $\epsilon$-regularizations with smooth $\chi$ (given by (8), (9)) and the $\lambda$-regularizations (cf., (5), (6)) respectively of $P_{j}$. We will consider the following two integrals:

$$
\begin{aligned}
& \mathcal{I}(\epsilon)=\int_{Z} P_{1}^{\epsilon_{1}} \wedge \cdots \wedge P_{q}^{\epsilon_{q}} \wedge \varphi \\
& \Gamma(\lambda)=\int_{Z} P_{1}^{\lambda_{1}} \wedge \cdots \wedge P_{q}^{\lambda_{q}} \wedge \varphi
\end{aligned}
$$

where $\varphi$ is a test form on $Z$, supported close to a point in $\left\{f_{1}=\cdots=f_{q}=0\right\}$, of bidegree $\left(n, n-k_{1}-\cdots-k_{q}+q-p\right)$ with values in $\bigwedge\left(E_{1}^{*} \oplus \cdots \oplus E_{q}^{*}\right)$. In the arguments below, we will assume for notational convenience that $\tilde{f}_{j}=f_{j}$ (cf., e.g., (5)); the modifications to the general case are straightforward.

The main parts of the proofs of Theorems 13 and 14 are contained in the following propositions.

Proposition 17. Assume that $f_{1}, \ldots, f_{q}$ define a complete intersection. For $p<s \leqslant q$ we have

$$
\left|\mathcal{I}(\epsilon)-\mathcal{I}\left(\epsilon_{1}, \ldots, \epsilon_{s-1}, 0, \ldots, 0\right)\right| \leqslant C\|\varphi\|_{M}\left(\epsilon_{s}^{\omega}+\cdots+\epsilon_{q}^{\omega}\right)
$$

Note that $\mathcal{I}\left(\epsilon_{1}, \ldots, \epsilon_{s-1}, 0, \ldots, 0\right)$ is well defined; it is the action of $U_{k_{s}}^{s} \wedge \cdots \wedge U_{k_{q}}^{q}$ on a smooth form.

Proposition 18. Assume that $f_{1}, \ldots, f_{q}$ define a complete intersection. Then $\Gamma(\lambda)$ has a meromorphic continuation to all of $\mathbb{C}^{q}$ and its only possible poles in a neighborhood of $\bigcap_{1}^{q}\left\{\mathfrak{R e} \lambda_{j} \geqslant\right.$ $0\}$ are along hyperplanes of the form $\sum_{j=1}^{p} \lambda_{j} \alpha_{j}=0$, where $\alpha_{j} \in \mathbb{N}$ and at least two $\alpha_{j}$ are positive. In particular, for $p=1, \Gamma(\lambda)$ is analytic in a neighborhood of $\bigcap_{1}^{q}\left\{\mathfrak{R e} \lambda_{j} \geqslant 0\right\}$.

Using that

$$
\begin{equation*}
\bar{\partial}\left|f_{j}\right|^{2 \lambda} \wedge u_{k}^{j}=\bar{\partial}\left(\left|f_{j}\right|^{2 \lambda} u_{k}^{j}\right)-f_{j} \cdot\left(\left|f_{j}\right|^{2 \lambda} u_{k+1}^{j}\right) \tag{17}
\end{equation*}
$$

the proof of Theorem 14 follows from Proposition 18 in a similar way as Theorem 1 in [30] follows from Proposition 4 in [30].

We indicate one way Proposition 17 can be used to prove Theorem 13. To simplify notation somewhat, we let $R^{j}$ denote any $R_{k}^{j}$ and $R_{\epsilon}^{j}$ denotes a smooth $\epsilon$-regularization of $R^{j} ; U^{j}$ and $U_{\epsilon}^{j}$ are defined similarly. The uniformity in the estimate of Proposition 17 implies that we have estimates of the form

$$
\begin{equation*}
\left|\bigwedge_{1}^{m} R_{\epsilon}^{j} \wedge \bigwedge_{m+1}^{p} R^{j} \wedge \bigwedge_{p+1}^{q} U_{\epsilon}^{j}-\bigwedge_{1}^{m} R_{\epsilon}^{j} \wedge \bigwedge_{m+1}^{p} R^{j} \wedge \bigwedge_{p+1}^{q} U^{j}\right| \lesssim\left(\epsilon_{p+1}^{\omega}+\cdots+\epsilon_{q}^{\omega}\right) \tag{18}
\end{equation*}
$$

where, e.g., $R^{m+1} \wedge \cdots \wedge R^{p}$ a priori is defined as a Coleff-Herrera product. We prove (a slightly stronger result than) Theorem 13 by induction over $p$. Let $R^{*}$ denote the Coleff-Herrera product of some $R^{j}$ :s with $j>p$ and let $U^{*}$ and $U_{\epsilon}^{*}$ denote the product of some $U^{j}$ :s and $U_{\epsilon}^{j}$ :s respectively, also with $j>p$ but only $j$ :s not occurring in $R^{*}$. We prove

$$
\left|R_{\epsilon}^{1} \wedge \cdots \wedge R_{\epsilon}^{p} \wedge R^{*} \wedge U_{\epsilon}^{*}-R^{1} \wedge \cdots \wedge R^{p} \wedge R^{*} \wedge U^{*}\right| \lesssim \epsilon^{\omega}
$$

i.e., we prove Theorem 13 on the current $R^{*}$. The induction start, $p=0$, follows immediately from (18). If we add and subtract $R_{\epsilon}^{1} \wedge \cdots \wedge R_{\epsilon}^{p} \wedge R^{*} \wedge U^{*}$, the induction step follows easily from (17) (construed in setting of $\epsilon$-regularizations) and estimates like (18).

Proof of Propositions $\mathbf{1 7}$ and 18. We may assume that $\varphi$ has arbitrarily small support. Hence, we may assume that $Z$ is an analytic subset of a domain $\Omega \subseteq \mathbb{C}^{N}$ and that all bundles are trivial, and thus make the identification $f_{j}=\left(f_{j 1}, \ldots, f_{j e_{j}}\right)$, where $f_{j i}$ are holomorphic in $\Omega$. We choose a Hironaka resolution $\hat{Z} \rightarrow Z$ such that the pulled-back ideals $\left\langle\hat{f}_{j}\right\rangle$ are all principal, and moreover, so that in a fixed chart with coordinates $x$ on $\hat{Z}$ (and after a possible re-numbering), $\left\langle\hat{f}_{j}\right\rangle$ is generated by $\hat{f}_{j 1}$ and $\hat{f}_{j 1}=x^{\alpha_{j}} h_{j}$, where $h_{j}$ is holomorphic and non-zero. We then have

$$
\left|\hat{f}_{j}\right|^{2}=\left|\hat{f}_{j 1}\right|^{2} \xi_{j}, \quad \hat{u}_{k_{j}}^{j}=v^{j} / \hat{f}_{j 1}^{k_{j}}
$$

where $\xi_{j}$ is smooth and positive and $v^{j}$ is a smooth (bundle valued) form. We thus get

$$
\bar{\partial} \chi_{j}\left(\left|\hat{f}_{j}\right|^{2} / \epsilon_{j}\right)=\tilde{\chi}_{j}\left(\left|\hat{f}_{j}\right|^{2} / \epsilon_{j}\right)\left(\frac{d \overline{\hat{f}}_{j 1}}{\overline{\hat{f}}_{j 1}}+\frac{\bar{\partial} \xi_{j}}{\xi_{j}}\right),
$$

where $\tilde{\chi}_{j}(t)=t \chi_{j}^{\prime}(t)$, and

$$
\bar{\partial}\left|\hat{f}_{j}\right|^{2 \lambda_{j}}=\lambda_{j}\left|\hat{f}_{j}\right|^{2 \lambda_{j}}\left(\frac{d \hat{\hat{f}}_{j 1}}{\hat{f}_{j 1}}+\frac{\bar{\partial} \xi_{j}}{\xi_{j}}\right) .
$$

It follows that $\mathcal{I}(\epsilon)$ and $\Gamma(\lambda)$ are finite sums of integrals which we without loss of generality can assume to be of the form

$$
\begin{align*}
& \pm \int_{\mathbb{C}_{x}^{n}} \prod_{1}^{p} \tilde{\chi}_{j}^{\epsilon} \prod_{p+1}^{q} \chi_{j}^{\epsilon} \bigwedge_{1}^{m} \frac{d \overline{\hat{f}}_{j 1}}{\hat{\hat{f}}_{j 1}} \wedge \bigwedge_{m+1}^{p} \frac{\bar{\partial} \xi_{j}}{\xi_{j}} \wedge \bigwedge_{1}^{q} \frac{v^{j}}{\hat{f}_{j 1}^{k_{j}}} \wedge \varphi \rho,  \tag{19}\\
\pm & \lambda_{1} \cdots \lambda_{p} \int_{\mathbb{C}_{x}^{n}} \prod_{1}^{q}\left|\hat{f}_{j}\right|^{2 \lambda_{j}} \bigwedge_{1}^{m} \frac{d \overline{\hat{f}}_{j 1}}{\hat{\hat{f}}_{j 1}} \wedge \bigwedge_{m+1}^{p} \frac{\bar{\partial} \xi_{j}}{\xi_{j}} \wedge \bigwedge_{1}^{q} \frac{v^{j}}{\hat{f}_{j 1}^{k_{j}}} \wedge \varphi \rho, \tag{20}
\end{align*}
$$

where $\rho$ is a cutoff function.
Recall that $\hat{f}_{j 1}=x^{\alpha_{j}} h_{j}$ and let $\mu$ be the number of vectors in a maximal linearly independent subset of $\left\{\alpha_{1}, \ldots, \alpha_{m}\right\}$; say that $\alpha_{1}, \ldots, \alpha_{\mu}$ are linearly independent. We then can define new holomorphic coordinates (still denoted by $x$ ) so that $\hat{f}_{j 1}=x^{\alpha_{j}}, j=1, \ldots, \mu$, see [24, p. 46] for details. Then we get

$$
\begin{align*}
\bigwedge_{1}^{m} d \hat{f}_{j 1} & =\bigwedge_{1}^{\mu} d x^{\alpha_{j}} \wedge \bigwedge_{\mu+1}^{m}\left(x^{\alpha_{j}} d h_{j}+h_{j} d x^{\alpha_{j}}\right) \\
& =x^{\sum_{\mu+1}^{m} \alpha_{j}} \bigwedge_{1}^{\mu} d x^{\alpha_{j}} \wedge \bigwedge_{\mu+1}^{m} d h_{j} \tag{21}
\end{align*}
$$

where the last equality follows because $d x^{\alpha_{1}} \wedge \cdots \wedge d x^{\alpha_{\mu}} \wedge d x^{\alpha_{j}}=0, \mu+1 \leqslant j \leqslant m$, since $\alpha_{1}, \ldots, \alpha_{\mu}, \alpha_{j}$ are linearly dependent. From the beginning we could also have assumed that $\varphi=\varphi_{1} \wedge \varphi_{2}$, where $\varphi_{1}$ is an anti-holomorphic $\left(n-\sum_{1}^{q} k_{j}+q-p\right)$-form and $\varphi_{2}$ is a (bundle valued) $(n, 0)$-test form on $Z$. We now define

$$
\Phi=\bigwedge_{\mu+1}^{m} \frac{d \bar{h}_{j}}{\bar{h}_{j}} \wedge \bigwedge_{m+1}^{p} \frac{\bar{\partial} \xi_{j}}{\xi_{j}} \wedge \bigwedge_{1}^{q} v^{j} \wedge \hat{\varphi}_{1}
$$

Using (21) we can now write (19) and (20) as

$$
\begin{align*}
& \pm \int_{\mathbb{C}_{x}^{n}} \frac{\prod_{1}^{p} \tilde{x}_{j}^{\epsilon} \prod_{p+1}^{q} \chi_{j}^{\epsilon}}{\prod_{1}^{q} \hat{f}_{j 1}^{k_{j}}} \frac{d \bar{x}^{\alpha_{1}}}{\bar{x}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d \bar{x}^{\alpha_{\mu}}}{\bar{x}^{\alpha_{\mu}}} \wedge \Phi \wedge \hat{\varphi}_{2} \rho,  \tag{22}\\
& \pm \lambda_{1} \cdots \lambda_{p} \int_{\mathbb{C}_{x}^{n}} \frac{\prod_{1}^{q}\left|\hat{f}_{j}\right|^{2 \lambda_{j}}}{\prod_{1}^{q} \hat{f}_{j 1}^{k_{j}}} \frac{d \bar{x}^{\alpha_{1}}}{\bar{x}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d \bar{x}^{\alpha_{\mu}}}{\bar{x}^{\alpha_{\mu}}} \wedge \Phi \wedge \hat{\varphi}_{2} \rho . \tag{23}
\end{align*}
$$

Lemma 19. Let $\mathcal{K}=\left\{i ; x_{i} \mid x^{\alpha_{j}}\right.$, some $\left.p+1 \leqslant j \leqslant q\right\}$. For any fixed $r \in \mathbb{N}$, one can replace $\Phi$ in (22) and (23) by

$$
\Phi^{\prime}:=\Phi-\left.\sum_{J \subseteq \mathcal{K}}(-1)^{|J|} \sum_{k_{1}, \ldots, k_{|J|}=0}^{r+1} \frac{\partial^{|k|} \Phi}{\partial x_{J}^{k}}\right|_{x_{J}=0} \frac{x_{J}^{k}}{k!}
$$

without affecting the integrals. Moreover, for any $I \subseteq \mathcal{K}$, we have that $\Phi^{\prime} \wedge \bigwedge_{i \in I}\left(d \bar{x}_{i} / \bar{x}_{i}\right)$ is $C^{r}$-smooth.

We replace $\Phi$ by $\Phi^{\prime}$ in (22) and (23) and we write $d=d_{\mathcal{K}}+d_{\mathcal{K}^{c}}$, where $d_{\mathcal{K}}$ differentiates with respect to the variables $x_{i}, \bar{x}_{i}$ for $i \in \mathcal{K}$ and $d_{\mathcal{K}^{c}}$ differentiates with respect to the rest. Then we can write $\left(d \bar{x}^{\alpha_{1}} / \bar{x}^{\alpha_{1}}\right) \wedge \cdots \wedge\left(d \bar{x}^{\alpha_{\mu}} / \bar{x}^{\alpha_{\mu}}\right) \wedge \Phi^{\prime}$ as a sum of terms, which we without loss of generality can assume to be of the form

$$
\begin{aligned}
& \frac{d_{\mathcal{K}^{c}} \bar{x}^{\alpha_{1}}}{\bar{x}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d_{\mathcal{K}^{c}} \bar{x}^{\alpha_{\nu}}}{\bar{x}^{\alpha_{v}}} \wedge \frac{d_{\mathcal{K}} \bar{x}^{\alpha_{v+1}}}{\bar{x}^{\alpha_{v+1}}} \wedge \cdots \wedge \frac{d_{\mathcal{K}} \bar{x}^{\alpha_{\mu}}}{\bar{x}^{\alpha_{\mu}}} \wedge \Phi^{\prime} \\
& \quad=\frac{d_{\mathcal{K}^{c}} \bar{x}^{\alpha_{1}}}{\bar{x}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d_{\mathcal{K}^{c}} \bar{x}^{\alpha_{v}}}{\bar{x}^{\alpha_{\nu}}} \wedge \Phi^{\prime \prime} \wedge d \bar{x}_{\mathcal{K}},
\end{aligned}
$$

where $\Phi^{\prime \prime}$ is $C^{r}$-smooth and of bidegree $(0, n-v-|\mathcal{K}|)$ (possibly, $\Phi^{\prime \prime}=0$ ). Thus, (22) and (23) are finite sums of integrals of the following type

$$
\begin{align*}
& \int_{\mathbb{C}_{x}^{n}} \frac{\prod_{1}^{p} \tilde{x}_{j}^{\epsilon} \prod_{p+1}^{q} \chi_{j}^{\epsilon}}{\prod_{1}^{q} \hat{f}_{j 1}^{k_{j}}} \frac{d \bar{x}^{\alpha_{1}}}{\bar{x}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d \bar{x}^{\alpha_{\nu}}}{\bar{x}^{\alpha_{v}}} \wedge \psi \wedge d \bar{x}_{\mathcal{K}} \wedge d x,  \tag{24}\\
& \lambda_{1} \cdots \lambda_{p} \int_{\mathbb{C}_{x}^{n}} \frac{\prod_{1}^{q}\left|\hat{f}_{j}\right|^{2 \lambda_{j}}}{\prod_{1}^{q} \hat{f}_{j 1}^{k_{j}}} \frac{d \bar{x}^{\alpha_{1}}}{\bar{x}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d \bar{x}^{\alpha_{\nu}}}{\bar{x}^{\alpha_{\nu}}} \wedge \psi \wedge d \bar{x}_{\mathcal{K}} \wedge d x, \tag{25}
\end{align*}
$$

where $\psi$ is $C^{r}$-smooth and compactly supported.
We now first finish the proof of Proposition 18. First of all, it is well known that $\Gamma(\lambda)$ has a meromorphic continuation to $\mathbb{C}^{q}$. We have

$$
\frac{d \bar{x}^{\alpha_{1}}}{\bar{x}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d \bar{x}^{\alpha_{\nu}}}{\bar{x}^{\alpha_{\nu}}} \wedge d \bar{x}_{\mathcal{K}}=\sum_{\substack{|I|=v \\ I \subseteq \mathcal{K}^{c}}} C_{I} \frac{d \bar{x}_{I}}{\bar{x}_{I}} \wedge d \bar{x}_{\mathcal{K}}
$$

Let us assume that $I=\{1, \ldots, \nu\} \subseteq \mathcal{K}^{c}$ and consider the contribution to (25) corresponding to this subset. This contribution equals

$$
\begin{align*}
& C_{I} \lambda_{1} \cdots \lambda_{p} \int_{\mathbb{C}_{x}^{n}} \frac{\left|x^{\sum_{1}^{q} \lambda_{j} \alpha_{j}}\right|^{2}}{x^{\sum_{1}^{q} k_{j} \alpha_{j}}} \bigwedge_{1}^{v} \frac{d \bar{x}_{j}}{\bar{x}_{j}} \wedge \Psi(\lambda, x) \wedge d \bar{x}_{\mathcal{K}} \wedge d x \\
& =\frac{C_{I} \prod_{1}^{p} \lambda_{j}}{\prod_{i=1}^{v}\left(\sum_{1}^{q} \lambda_{j} \alpha_{j i}\right)} \int_{\mathbb{C}_{x}^{n}} \frac{\bigwedge_{i=1}^{v} \bar{\partial}\left|x_{i}\right|^{2 \sum_{1}^{q} \lambda_{j} \alpha_{j i}} \prod_{i=v+1}^{n}\left|x_{i}\right|^{2 \sum_{1}^{q} \lambda_{j} \alpha_{j i}}}{x^{\sum_{1}^{q} k_{j} \alpha_{j}}} \\
& \quad \wedge \Psi(\lambda, x) \wedge d \bar{x}_{\mathcal{K}} \wedge d x, \tag{26}
\end{align*}
$$

where $\Psi(\lambda, x)=\psi(x) \prod_{1}^{q}\left(\xi_{j}^{\lambda_{j}} / h_{j}^{k_{j}}\right)$. It is well known (and not hard to prove, e.g., by integrations by parts as in [1, Lemma 2.1]) that the integral on the right-hand side of (26) has an analytic continuation in $\lambda$ to a neighborhood of $\bigcap_{1}^{q}\left\{\mathfrak{R e} \lambda_{j} \geqslant 0\right\}$. (We thus choose $r$ in Lemma 19 large enough so that we can integrate by parts.) If $p=0$, then the coefficient in front of the integral is to be interpreted as 1 and Proposition 18 follows in this case. For $p>0$, we see that the poles of (26), and consequently of $\Gamma(\lambda)$, in a neighborhood of $\bigcap_{1}^{q}\left\{\mathfrak{R e} \lambda_{j} \geqslant 0\right\}$ are along hyperplanes of the form $0=\sum_{1}^{q} \lambda_{j} \alpha_{j i}, 1 \leqslant i \leqslant \nu$. But if $j>p$ and $i \leqslant \nu$, then $\alpha_{j i}=0$ since $\{1, \ldots, v\} \subseteq \mathcal{K}^{c}=\left\{i ; x_{i} \nmid x^{\alpha_{j}}, \forall j=p+1, \ldots, q\right\}$. Thus, the hyperplanes are of the form $0=\sum_{1}^{p} \lambda_{j} \alpha_{j i}$ and Proposition 18 is proved except for the statement that at least for two $j$ :s, the $\alpha_{j i}$ are non-zero. However, we see from (26) that if for some $i$ we have $\alpha_{j i}=0$ for all $j$ but one, then the appearing $\lambda_{j}$ in the denominator will be canceled by the numerator. Moreover, we may assume that the constant $C_{I}=\operatorname{det}\left(\alpha_{j i}\right)_{1 \leqslant i, j \leqslant \nu}$ is non-zero which implies that we cannot have any $\lambda_{j}^{2}$ in the denominator.

We now prove Proposition 17. Consider (24). We have that $\alpha_{1}, \ldots, \alpha_{\nu}$ are linearly independent so we may assume that $A=\left(\alpha_{i j}\right)_{1 \leqslant i, j \leqslant \nu}$ is invertible with inverse $B=\left(b_{i j}\right)$. We make the non-holomorphic change of variables (13), where the " $q$ " of (13) now should be understood as $\nu$. Then we get $x^{\alpha_{j}}=y^{\alpha_{j}} \eta_{j}$, where $\eta_{j}>0$ and smooth and $\eta_{j}^{2}=1 / \xi_{j}, j=1, \ldots, \nu$. Hence, $\left|\hat{f}_{j}\right|^{2}=\left|y^{\alpha_{j}}\right|^{2}, j=1, \ldots, \nu$. Expressed in the $y$-coordinates we get that $\bigwedge_{1}^{\nu}\left(d \bar{x}^{\alpha_{j}} / \bar{x}^{\alpha_{j}}\right) \wedge \psi \wedge$ $d \bar{x}_{\mathcal{K}} \wedge d x$ is a finite sum of terms of the form

$$
\begin{equation*}
\frac{d \bar{y}^{\alpha_{1}}}{\bar{y}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d \bar{y}^{\alpha_{v^{\prime}}}}{\bar{y}^{\alpha_{v^{\prime}}}} \wedge \bar{y}_{\mathcal{K}^{\prime}} d \overline{\mathcal{K}}_{\mathcal{K}^{\prime \prime}} \wedge \psi_{1} \tag{27}
\end{equation*}
$$

where $\nu^{\prime} \leqslant \nu, \psi_{1}$ is a $C^{r}$-smooth compactly supported form, and $\mathcal{K}^{\prime}$ and $\mathcal{K}^{\prime \prime}$ are disjoint sets such that $\mathcal{K}^{\prime} \cup \mathcal{K}^{\prime \prime}=\mathcal{K}$. In order to give a contribution to (24) we see that $\psi_{1}$ must contain $d y$. In (27) we write $d=d_{\mathcal{K}}+d_{\mathcal{K}^{c}}$, and arguing as we did immediately after Lemma 19 , (27) is a finite sum of terms of the form

$$
\frac{d \bar{y}^{\alpha_{1}}}{\bar{y}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d \bar{y}^{\alpha_{v^{\prime \prime}}}}{\bar{y}^{\alpha_{\nu^{\prime \prime}}}} \wedge \psi_{2} \wedge d \bar{y}_{\mathcal{K}} \wedge d y
$$

where $\nu^{\prime \prime} \leqslant v$ and $\psi_{2}$ is $C^{r}$-smooth and compactly supported. With abuse of notation we thus have that (24) is a finite sum of integrals of the form

$$
\begin{align*}
\int_{\mathbb{C}_{x}^{n}} & \frac{\prod_{1}^{p} \tilde{\chi}_{j}^{\epsilon} \prod_{p+1}^{q} \chi_{j}^{\epsilon}}{\prod_{1}^{q} \hat{f}_{j 1}^{k_{j}}} \frac{d \bar{y}^{\alpha_{1}}}{\bar{y}^{\alpha_{1}}} \wedge \cdots \wedge \frac{d \bar{y}^{\alpha_{v}}}{\bar{y}^{\alpha_{v}}} \wedge \psi \wedge d \bar{y}_{\mathcal{K}} \wedge d y \\
& =\int_{\mathbb{C}_{x}^{n}} \frac{\bigwedge_{1}^{v} d \chi_{j}^{\epsilon} \prod_{v+1}^{p} \tilde{\chi}_{j}^{\epsilon} \prod_{p+1}^{q} \chi_{j}^{\epsilon}}{y^{\sum_{1}^{q} k_{j} \alpha_{j}}} \wedge \Psi \wedge d \bar{y}_{\mathcal{K}} \wedge d y \tag{28}
\end{align*}
$$

where $\Psi$ is a $C^{r}$-smooth compactly supported ( $n-|\mathcal{K}|-v$ )-form; the equality follows since $\chi_{j}^{\epsilon}=\chi_{j}\left(\left|y^{\alpha_{j}}\right|^{2} / \epsilon_{j}\right), j=1, \ldots, v$. Now, (28) is essentially equal to Eq. (24) of [16] and the proof of Proposition 17 is concluded as in the proof of Proposition 8 in [16].

Proof of Lemma 19. The proof is similar to the proof of Lemma 9 in [16] but some modifications have to be done. First, it is easy to check by induction over $|\mathcal{K}|$ that $\Phi^{\prime} \wedge \bigwedge i \in I\left(d \bar{x}_{i} / \bar{x}_{i}\right)$ is $C^{r}$-smooth for any $I \subseteq \mathcal{K}$; for $|\mathcal{K}|=1$ this is just Taylor's formula for forms. It thus suffices to show that

$$
\left.d \bar{x}^{\alpha_{1}} \wedge \cdots \wedge d \bar{x}^{\alpha_{\mu}} \wedge \frac{\partial^{|k|} \Phi}{\partial x_{I}^{k}}\right|_{x_{I}=0}=0, \quad \forall I \subseteq \mathcal{K}, k=\left(k_{i_{1}}, \ldots, k_{i_{|I|}}\right) .
$$

To show this, fix an $I \subseteq \mathcal{K}$ and let $L=\left\{j ; x_{i} \nmid x^{\alpha_{j}} \forall i \in I\right\}$. Say for simplicity that

$$
L=\left\{1, \ldots, \mu^{\prime}, \mu+1, \ldots, m^{\prime}, m+1, \ldots, p^{\prime}, p+1, \ldots, q^{\prime}\right\}
$$

where $\mu^{\prime} \leqslant \mu, m^{\prime} \leqslant m, p^{\prime} \leqslant p$, and $q^{\prime}<q$. The fact that $q^{\prime}<q$ follows from the definitions of $\mathcal{K}, I$, and $L$.

Consider, on the base variety $Z$, the smooth form

$$
F=\bigwedge_{1}^{\mu^{\prime}} d \bar{f}_{j 1} \bigwedge_{\mu+1}^{m^{\prime}} d \bar{f}_{j 1} \bigwedge_{m+1}^{p^{\prime}}\left(\left|f_{j 1}\right|^{2} \bar{\partial}\left|f_{j}\right|^{2}-\bar{\partial}\left|f_{j 1}\right|^{2}\left|f_{j}\right|^{2}\right) \bigwedge_{j \in L}\left|f_{j}\right|^{2 k_{j}} u_{k_{j}}^{j} \wedge \varphi_{1}
$$

It has bidegree $\left(0, n-\sum_{j \in L^{c}} k_{j}+q-q^{\prime}\right)$ so $F$ has a vanishing pull-back to $\bigcap_{j \in L^{c}}\left\{f_{j}=0\right\}$ since this set has dimension $n-\sum_{j \in L^{c}} e_{j}<n-\sum_{j \in L^{c}} k_{j}+q-q^{\prime}$ by our assumption about complete intersection. Thus, $\hat{F}$ has a vanishing pull-back to $\left\{x_{I}=0\right\} \subseteq \bigcap_{j \in L^{c}}\left\{\hat{f}_{j}=0\right\}$. In fact, this argument shows that

$$
\begin{equation*}
\hat{F}=\sum \phi_{j} \tag{29}
\end{equation*}
$$

where the $\phi_{j}$ are smooth linearly independent forms such that each $\phi_{j}$ is divisible by $\bar{x}_{i}$ or $d \bar{x}_{i}$ for some $i \in I$. (It is the pull-back to $\left\{x_{I}=0\right\}$ of the anti-holomorphic differentials of $\hat{F}$ that vanishes.) For the rest of the proof we let $\sum \phi_{j}$ denote such expressions and we note that they are invariant under holomorphic differential operators. Computing $\hat{F}$ we get

$$
\hat{F}=\prod_{m+1}^{p^{\prime}}\left|\hat{f}_{j 1}\right|^{4} \prod_{j \in L} \frac{\left|\hat{f}_{j}\right|^{2 k_{j}}}{\hat{f}_{j 1}^{k_{j}}} \bigwedge_{1}^{\mu^{\prime}} d \bar{x}^{\alpha_{j}} \bigwedge_{\mu+1}^{m^{\prime}} d\left(\bar{x}^{\alpha_{j}} \bar{h}_{j}\right) \bigwedge_{m+1}^{p^{\prime}} \bar{\partial} \xi_{j} \bigwedge_{j \in L} v^{j} \wedge \hat{\varphi}_{1}
$$

The "coefficient" $\prod_{m+1}^{p^{\prime}}\left|\hat{f}_{j 1}\right|^{4} \prod_{j \in L}\left(\left|\hat{f}_{j}\right|^{2 k_{j}} / \hat{f}_{j 1}^{k_{j}}\right)$ does not contain any $\bar{x}_{i}$ with $i \in I$ so we may divide (29) by it (recall that the $\phi_{j}$ are linearly independent) and we obtain

$$
\begin{aligned}
\sum \phi_{j}= & \bigwedge_{1}^{\mu^{\prime}} d \bar{x}^{\alpha_{j}} \bigwedge_{\mu+1}^{m^{\prime}} d\left(\bar{x}^{\alpha_{j}} \bar{h}_{j}\right) \bigwedge_{m+1}^{p^{\prime}} \bar{\partial} \xi_{j} \bigwedge_{j \in L} v^{j} \wedge \hat{\varphi}_{1} \\
= & \prod_{\mu+1}^{m^{\prime}} \bar{x}^{\alpha_{j}} \bigwedge_{1}^{\mu^{\prime}} d \bar{x}^{\alpha_{j}} \bigwedge_{\mu+1}^{m^{\prime}} d \bar{h}_{j} \bigwedge_{m+1}^{p^{\prime}} \bar{\partial} \xi_{j} \bigwedge_{j \in L} v^{j} \wedge \hat{\varphi}_{1} \\
& +\bigwedge_{1}^{\mu^{\prime}} d \bar{x}^{\alpha_{j}} \wedge \sum_{\mu+1}^{m^{\prime}} d \bar{x}^{\alpha_{j}} \wedge \tau_{j}
\end{aligned}
$$

for some $\tau_{j}$. We multiply this equality with

$$
\bigwedge_{m^{\prime}+1}^{m} d \bar{h}_{j} \bigwedge_{p^{\prime}+1}^{p} \bar{\partial} \xi_{j} \bigwedge_{j \in L^{c}} v^{j} /\left(\prod_{\mu+1}^{m} \bar{h}_{j} \prod_{m+1}^{p} \xi_{j}\right)
$$

and get

$$
\prod_{\mu+1}^{m^{\prime}} \bar{x}^{\alpha_{j}} \bigwedge_{1}^{\mu^{\prime}} d \bar{x}^{\alpha_{j}} \wedge \Phi+\bigwedge_{1}^{\mu^{\prime}} d \bar{x}^{\alpha_{j}} \wedge \sum_{\mu+1}^{m^{\prime}} d \bar{x}^{\alpha_{j}} \wedge \tau_{j}=\sum \phi_{j}
$$

for some new $\tau_{j}$. We apply the operator $\partial^{|k|} / \partial x_{I}^{k}$ to this equality and then we pull-back to $\left\{x_{I}=0\right\}$, which makes the right-hand side vanish; (we construe however the result in $\mathbb{C}_{x}^{n}$ ). Finally, taking the exterior product with $\bigwedge_{\mu^{\prime}+1}^{\mu} d \bar{x}^{\alpha_{j}}$, which will make each term in under the summation sign on the left-hand side vanish, we arrive at

$$
\left.\prod_{\mu+1}^{m^{\prime}} \bar{x}^{\alpha_{j}} \bigwedge_{1}^{\mu} d \bar{x}^{\alpha_{j}} \wedge \frac{\partial^{|k|} \Phi}{\partial x_{I}^{k}}\right|_{x_{I}=0}=0
$$

and we are done.

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