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Citation for the original published paper (version of record):
Lärkäng, R., Samuelsson Kalm, H. (2013). Various approaches to products of residue currents.
Journal of Functional Analysis, 264(1): 118-138. http://dx.doi.org/10.1016/j.jfa.2012.10.004

Available online at www.sciencedirect.com

SciVerse ScienceDirect

Journal of Functional Analysis 264 (2013) 118–138

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Various approaches to products of residue currents

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Received 5 August 2010; accepted 3 October 2012

Available online 24 October 2012

Communicated by D. Voiculescu

Abstract

We describe various approaches to Coleff–Herrera products of residue currents R^j (of Cauchy–Fantappiè–Leray type) associated to holomorphic mappings f_j. More precisely, we study to which extent (exterior) products of natural regularizations of the individual currents R^j yield regularizations of the corresponding Coleff–Herrera products. Our results hold globally on an arbitrary pure-dimensional complex space.

Keywords: Residue currents; Regularization of currents; Coleff–Herrera products; Cauchy–Fantappiè–Leray type currents

1. Introduction

Let f be a holomorphic function defined on the unit ball $\mathbb{B} \subset \mathbb{C}^n$. If f is a monomial it is elementary to show, e.g., by integrations by parts or by a Taylor expansion, that the principal value current $\varphi \mapsto \lim_{\epsilon\to 0}\int_{|f|^2>\epsilon}\varphi/f$, $\varphi \in \mathscr{D}_{n,n}(\mathbb{B})$, exists and defines a $(0,0)$-current $1/f$ that we also denote by U^f. From Hironaka's theorem it then follows that such limits exist for general f and also that $\mathbb{B}$ may be replaced by a complex space [20]. The $\bar\partial$-image, $R^f := \bar\partial(1/f)$, is the residue current of f and by Stokes' theorem it is given by $\varphi \mapsto \lim_{\epsilon\to 0}\int_{|f|^2=\epsilon}\varphi/f$, $\varphi \in \mathscr{D}_{n,n-1}(\mathbb{B})$. It has the useful property that its annihilator ideal is equal to the principal ideal $\langle f\rangle$

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0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2012.10.004

and, moreover, it gives a factorization of Lelong's integration current; $2\pi i[f=0]=\bar\partial(1/f)\wedge df$.

There are (at least) two natural ways of regularizing U^f and R^f. If $\lambda\in\mathbb{C}$ and $\mathfrak{Re}\,\lambda\gg 0$, then $\lambda\mapsto\int\varphi|f|^{2\lambda}/f$ is holomorphic for any test form φ. It is well known (cf., Lemma 6) that the current-valued map $\lambda\mapsto|f|^{2\lambda}/f=:U^{f,\lambda}$ has a meromorphic extension to $\mathbb{C}$ with poles contained in the set of negative rational numbers and that the value at $\lambda=0$ is U^f. It follows that $\lambda\mapsto\bar\partial|f|^{2\lambda}/f=:R^{f,\lambda}$ is meromorphic in $\mathbb{C}$, analytic in a half-space containing the origin, and that the value at the origin is R^f. The technique of using analytic continuation in residue current theory has its roots in the work of Atiyah [8], and Bernstein and Gel'fand [14]. In the context of residue currents it has been developed by several authors, e.g., Barlet and Maire [9], Yger [33], Berenstein, Gay and Yger [11], Passare and Tsikh [26], and recently by the second author in [30]. The second regularization method, inspired by Passare [24], is more explicit and concrete; U^f and R^f are obtained as weak limits of explicit smooth forms. Let χ be a smooth regularization of the characteristic function $\mathbf{1}_{[1,\infty)}$ and let $U^{f,\epsilon}:=\chi(|f|^2/\epsilon)/f$ and $R^{f,\epsilon}:=\bar\partial\chi(|f|^2/\epsilon)/f$. Then (see, e.g., [24]) $U^f=\lim_{\epsilon\to0^+}U^{f,\epsilon}$ and $R^f=\lim_{\epsilon\to0^+}R^{f,\epsilon}$ in the sense of currents. Notice that the original definition mentioned above corresponds to $\chi=\mathbf{1}_{[1,\infty)}$.

If f is a tuple of functions or a section of a vector bundle there are natural analogues of the currents $1/f$ and $\bar\partial(1/f)$ introduced in [28] and [1]. The construction of these more general currents, still denoted U^f and R^f, is based on Bochner–Martinelli and Cauchy–Fantappiè–Leray type formulas; see Section 2 for details. In this paper we consider products of regularized currents of this kind and we investigate their limit behavior. It turns out that both the λ-approach and the ϵ-approach yield the same current as the classical Coleff–Herrera approach.

Let Z be a reduced complex space of pure dimension n, let $E_1,\dots,E_p$ be hermitian holomorphic vector bundles over Z, and let f_j be a holomorphic section of E_j^*. Then $U^{f_j}=:U^j$ and $R^{f_j}=:R^j$ become currents with values in $\bigwedge E_j$; if $\operatorname{rank}E_j=1$ then U^j is the principal value current associated with the meromorphic section $1/f_j$ of E_j and $R^j=\bar\partial U^j$. In complete analogy with the regularization methods discussed above we have

$$U^j=U^{j,\lambda}\big|_{\lambda=0}=\lim_{\epsilon\to0^+}U^{j,\epsilon}\quad\text{and}\quad R^j=R^{j,\lambda}\big|_{\lambda=0}=\lim_{\epsilon\to0^+}R^{j,\epsilon},$$

see Section 2. We define products of the R^j (for simplicity we restrict attention to such products in this section) recursively as follows: Having defined $R^{k-1}\wedge\dots\wedge R^1$ it turns out (see [7] or Section 2) that

$$\lambda\mapsto R^{k,\lambda}\wedge R^{k-1}\wedge\dots\wedge R^1$$

has an analytic continuation to a neighborhood of $\lambda=0$ and we define $R^k\wedge\dots\wedge R^1$ as the value at $\lambda=0$. From the proof of Proposition 5.4 in [6] it follows that one can compute the product in the following way: If $a_1>\dots>a_p>0$ are integers then

$$R^p\wedge\dots\wedge R^1=R^{p,\lambda^{a_p}}\wedge\dots\wedge R^{1,\lambda^{a_1}}\big|_{\lambda=0}.$$

That is, the recursive definition can be replaced by the evaluation of a one-variable analytic (current-valued) function at the origin; we just have to make sure that λ^{a_1} tends to zero much faster than λ^{a_2} and so on.

We now consider the smooth form $R^{p,\epsilon_p} \wedge \cdots \wedge R^{1,\epsilon_1}$ and limits of it of the following kind:

Definition 1. Let ϑ be a function defined on $(0,\infty)^p$. We let

$$\lim_{\epsilon_1 \ll \cdots \ll \epsilon_p \to 0} \vartheta(\epsilon_1, \ldots, \epsilon_p)$$

denote the limit (if it exists and is well defined) of ϑ along any path $\delta \mapsto \epsilon(\delta)$ towards the origin such that for all $\ell \in \mathbb{N}$ and $j = 2, \ldots, p$ there are positive constants $C_{j\ell}$ such that $\epsilon_{j-1}(\delta) \leqslant C_{j\ell}\epsilon_j^{\ell}(\delta)$. Here, we extend the domain of definition of ϑ to points $(0, \ldots, 0, \epsilon_{m+1}, \ldots, \epsilon_p)$, where $\epsilon_{m+1}, \ldots, \epsilon_p > 0$, by defining

$$\vartheta(0, \ldots, 0, \epsilon_{m+1}, \ldots, \epsilon_p) = \lim_{\epsilon_m \to 0} \ldots \lim_{\epsilon_1 \to 0} \vartheta(\epsilon_1, \ldots, \epsilon_m, \epsilon_{m+1}, \ldots, \epsilon_p),$$

if the limits exist.

Recall that $(\epsilon_1, \ldots, \epsilon_p)$ tends to zero along an *admissible paths* in the sense of Coleff and Herrera [17], if it tends to zero along a path inside $(0,\infty)^p$ such that $\epsilon_{j-1}/\epsilon_j^{\ell} \to 0$ for all $\ell \in \mathbb{N}$ and $j = 2, \ldots, p$. The limits in Definition 1 are (slightly) more general since, e.g., ϵ_1 is allowed to attain the value 0 before the other ϵ_j go to zero. In particular, it thus includes the iterated limit letting $\epsilon_k \to 0$ one at a time. The following theorem is a special case of Theorem 11 below. The proof shares many similarities with the proof of [24, Proposition 1] (even though the statements differ). However, in our case, extra technical difficulties arise since the bundles E_j may have non-trivial metrics.

Theorem 2. *In the sense of currents we have*

$$R^p \wedge \cdots \wedge R^1 = \lim_{\epsilon_1 \ll \cdots \ll \epsilon_p \to 0} R^{p,\epsilon_p} \wedge \cdots \wedge R^{1,\epsilon_1}.$$

To connect with the classical Coleff–Herrera approach, assume temporarily that $\operatorname{rank} E_j = 1$, $j = 1, \ldots, p$, so that $R^j = \bar\partial(1/f_j)$. Then Theorem 2 says that for any test form φ of bidegree $(n, n-p)$

$$\bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} . \varphi = \lim_{\epsilon_1 \ll \cdots \ll \epsilon_p \to 0} \int_Z \frac{\bar\partial \chi^{\epsilon_p}}{f_p} \wedge \cdots \wedge \frac{\bar\partial \chi^{\epsilon_1}}{f_1} \wedge \varphi,$$

where $\chi^{\epsilon_j} = \chi(|f_j|^2/\epsilon_j)$. We will refer to the integral on the right-hand side as the residue integral and denote it by $\mathcal{I}_f^{\varphi}(\epsilon)$. If the χ-functions tend to $\mathbf{1}_{[1,\infty)}$ (for a fixed generic $\epsilon \in (0,\infty)^p$) then $\mathcal{I}_f^{\varphi}(\epsilon)$ tends to Coleff–Herrera's original residue integral

$$I_f^{\varphi}(\epsilon) = \int_{T(\epsilon)} \varphi/(f_1 \cdots f_p), \tag{1}$$

where $T(\epsilon) = \bigcap_1^p \{|f_j|^2 = \epsilon_j\}$ is oriented as the distinguished boundary of the corresponding polyhedron. In [17] Coleff and Herrera prove that the limit of $I_f^{\varphi}(\epsilon)$ along an admissible path exists and defines a current, the nowadays called *Coleff–Herrera product*. We show (see Theorem 11) that the Coleff–Herrera product equals the product $\bar\partial(1/f_p) \wedge \cdots \wedge \bar\partial(1/f_1)$; this is folklore but to our knowledge not completely proved before (except in the case of complete intersection when it follows from [24] and [23] together with [30]).

A result much in the same spirit was proven by Passare in [23], where he relates the original Coleff–Herrera product to residue currents defined by λ-regularizations. Passare considers the regularization

$$\frac{\bar\partial|f_p|^{2\lambda}}{f_p} \wedge \cdots \wedge \frac{\bar\partial|f_1|^{2\lambda}}{f_1}\bigg|_{\lambda=0}, \tag{2}$$

i.e., instead of letting the λ_i go to zero successively, all the λ_i are equal to a single λ that tends to 0. In that case, Passare proves that this current coincides with an average of limits along parabolic paths of the residue integral, as considered in [24], irrespectively of whether f defines a complete intersection or not.

The product $R^k \wedge \cdots \wedge R^1$ does in general not have any natural commutation properties. For instance, $\bar\partial(1/(zw)) \wedge \bar\partial(1/z) = 0$ while $\bar\partial(1/z) \wedge \bar\partial(1/(zw)) = \bar\partial(1/z^2) \wedge \bar\partial(1/w)$, where the last product simply is the tensor product. However, if the f_j define a complete intersection, i.e., $\operatorname{codim}\{f_1 = \cdots = f_p = 0\} = \sum_j \operatorname{rank} E_j$, then it is known (see, e.g., [3]) that the product is commutative; the case when all the E_j have rank 1 is proved in [17].

Remark 3. Recall that the currents R^j take values in $\bigwedge E_j$. The sum of the degree of R^j in $\bigwedge E_j$ and its form-degree is even. Therefore the product is naturally commutative. If the E_j are trivial line bundles that we do not make any distinction between, then the product is anti-commutative; this is the classical Coleff–Herrera setting.

Theorem 4. *Assume that the f_j define a complete intersection. Then for every test form φ*

$$(\lambda_1, \dots, \lambda_p) \mapsto \int_Z R^{p,\lambda_p} \wedge \cdots \wedge R^{1,\lambda_1} \wedge \varphi$$

has an analytic continuation to a neighborhood of the origin in $\mathbb{C}^p$.

This result is a special case of our Theorem 14, which generalizes [30, Theorem 1]. The case when $p = 2$ and $\operatorname{rank} E_j = 1$ was proved by Berenstein–Yger (see, e.g., [10]). The following result is a special case of Theorem 13, which generalizes [16, Theorem 1].

Theorem 5. *Assume that the f_j define a complete intersection. Then for every test form φ*

$$(\epsilon_1, \dots, \epsilon_p) \mapsto \int_Z R^{p,\epsilon_p} \wedge \cdots \wedge R^{1,\epsilon_1} \wedge \varphi$$

is Hölder continuous on $[0, \infty)^p$.

For this result it is crucial that the χ-functions used to regularize the R^j are smooth. In fact, Passare and Tsikh [27], found a quite simple tuple (f_1, f_2) defining a complete intersection in $\mathbb{C}^2$ and a test form φ such that the classical Coleff–Herrera residue integral $I^{\varphi}_{(f_1,f_2)}(\epsilon)$ is discontinuous at $\epsilon = 0$. Soon after Björk found generic families of such examples, see, e.g., [15].

Let us give some background and motivation for the kind of products considered here. Products of Cauchy–Fantappiè–Leray type currents were first studied by Wulcan [32]. Wulcan defines the product as the value at $\lambda = 0$ of the analytic continuation of $\lambda \mapsto R^{p,\lambda} \wedge \cdots \wedge R^{1,\lambda}$. In the non-complete intersection case Wulcan's product is different from our; in the case that all E_j have rank 1, $R^{p,\lambda} \wedge \cdots \wedge R^{1,\lambda}|_{\lambda=0}$ coincides with Passare's product, (2). Passare–Wulcan products satisfy several natural computation rules and are quite useful but it has turned out that the recursive definition discussed above often is more natural. In particular, the Stückrad–Vogel intersection algorithm in non-proper intersection theory is conveniently expressed using recursively defined products, see [6].

In the complete intersection case there is no ambiguity, the Coleff–Herrera product is commutative and if $f = (f_1, \dots, f_p)$ then R^f equals $\bigwedge_j \bar\partial(1/f_j)$, see [28] and [1]. This indicates that the Coleff–Herrera product is the "correct" current to associated to a complete intersection. The Coleff–Herrera product is the minimal current extension of Grothendieck's cohomological residue (see, e.g., [25] for definitions) in the sense that it annihilated by anti-holomorphic functions vanishing on its support. Moreover, if f defines a complete intersection then the annihilator ideal of R^f equals the ideal generated by f, see [25] and [18]. This property is very useful and lies behind many applications, e.g., explicit division-interpolation formulas and Briançon–Skoda type results [2,10], explicit versions of the fundamental principle [13], the $\bar\partial$-equation on complex spaces [4,5,19], and explicit Green currents in arithmetic intersection theory [12].

In Section 2, we give the necessary background and the general formulations of our results. Section 3 contains the proof of Theorems 2 and 11. The proof of Theorems 4, 5, 13 and 14 is the content of Section 4; the crucial part is Lemma 19 which enables us to effectively use the assumption about complete intersection.

2. Formulation of the general results

Let Z be a reduced complex space of pure dimension n. We say that φ is a smooth (p,q)-form on Z if φ is smooth on Z_{reg}, and in a neighborhood of any $p \in Z$, there is a smooth (p,q)-form $\tilde\varphi$ in an ambient complex manifold such that the pull-back of $\tilde\varphi$ to Z_{reg} coincides with $\varphi|_{Z_{reg}}$ close to p. The (p,q)-test forms on Z, $\mathscr{D}_{p,q}(Z)$, are defined as the smooth compactly supported (p,q)-forms (with a suitable topology) and the space of (p,q)-currents on Z, $\mathscr{D}'_{p,q}(Z)$, is the dual of $\mathscr{D}_{n-p,n-q}(Z)$. More concretely, if $i : Z \to \Omega \subset \mathbb{C}^N$ is an embedding and μ is a (p,q)-current on Z then $i_*\mu$ is an $(N-n+p, N-n+q)$-current in Ω that vanishes on test forms ξ such that $i^*\xi = 0$ on Z_{reg}. Conversely, such a current in Ω defines a current on Z. See, e.g., [22] for a more thorough discussion.

Let x be a complex coordinate on $\mathbb{C}$. Recall that the principal value current $1/x^m$ can be computed as the value at $\lambda = 0$ of the analytic continuation of $|x|^{2\lambda}/x^m$; the residue current $\bar\partial(1/x^m)$ then is the value at $\lambda = 0$ of $\bar\partial|x|^{2\lambda}/x^m$. Since one can take tensor products of one-variable currents it follows that

$$T = \frac{1}{x_1^{\alpha_1}} \wedge \cdots \wedge \frac{1}{x_p^{\alpha_p}} \wedge \frac{\vartheta(x)}{x_{p+1}^{\alpha_{p+1}} \cdots x_n^{\alpha_n}} \tag{3}$$

is a well-defined current in $\mathbb{C}^n$; here $\alpha_1, \dots, \alpha_p$ are positive integers, $\alpha_{p+1}, \dots, \alpha_n$ are non-negative integers, and ϑ is a smooth compactly supported form. Such a current T is called an *elementary pseudomeromorphic* current. Following [7] we say that a current μ on Z is *pseudomeromorphic*, $\mu \in \mathcal{PM}(Z)$, if μ locally is a finite sum of push-forwards $\pi^1_* \cdots \pi^m_* \tau$ under maps

$$X^m \xrightarrow{\pi^m} \cdots \xrightarrow{\pi^2} X^1 \xrightarrow{\pi^1} Z,$$

where each π^j is either a modification or an open inclusion and τ is an elementary pseudomeromorphic current on X^m. It follows that the class of pseudomeromorphic currents is closed under $\bar\partial$ and multiplication with smooth forms, and that the push-forward of a pseudomeromorphic current by a modification is pseudomeromorphic.

Lemma 6. *Let f be a holomorphic function, and let $T \in \mathcal{PM}(Z)$. If $\tilde f$ is a holomorphic function such that $\{\tilde f = 0\} = \{f = 0\}$ and v is a smooth non-zero function, then $(|\tilde f v|^{2\lambda}/f)T$ and $(\bar\partial|\tilde f v|^{2\lambda}/f) \wedge T$ have current-valued analytic continuations to $\lambda = 0$ and the values at $\lambda = 0$ are pseudomeromorphic and independent of the choices of $\tilde f$ and v. Moreover, if $\chi = \mathbf{1}_{[1,\infty)}$, or a smooth approximation thereof, then*

$$\frac{|\tilde f v|^{2\lambda}}{f} T\bigg|_{\lambda=0} = \lim_{\epsilon\to 0^+} \frac{\chi^\epsilon}{f} T \quad \text{and} \quad \frac{\bar\partial|\tilde f v|^{2\lambda}}{f} \wedge T\bigg|_{\lambda=0} = \lim_{\epsilon\to 0^+} \frac{\bar\partial\chi^\epsilon}{f} \wedge T, \tag{4}$$

where $\chi^\epsilon = \chi(|\tilde f v|^2/\epsilon)$.

Proof. The first part is essentially Proposition 2.1 in [7], except that there, Z is a complex manifold, $\tilde f = f$ and $v \equiv 1$. However, with suitable resolutions of singularities, the proof in [7] goes through in the same way in our situation, as long as we observe that in $\mathbb{C}$

$$\frac{|x^{\alpha'} v|^{2\lambda}}{x^\alpha} \frac{1}{x^\beta} \quad \text{and} \quad \frac{|x^{\alpha'} v|^{2\lambda}}{x^\alpha} \bar\partial \frac{1}{x^\beta}$$

have analytic continuations to $\lambda = 0$, and the values at $\lambda = 0$ are $1/x^{\alpha+\beta}$ and 0 respectively, independently of α' and v, as long as $\alpha' > 0$ and $v \neq 0$ (and similarly with $\bar\partial|x^{\alpha'} v|^{2\lambda}/x^\alpha$).

By Leibniz rule, it is enough to consider the first equality in (4), since if we have proved the first equality, then

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{\bar\partial\chi^\epsilon}{f} \wedge T &= \lim_{\epsilon\to 0} \bar\partial\left(\frac{\chi^\epsilon}{f} T\right) - \frac{\chi^\epsilon}{f} \bar\partial T \\ &= \left(\bar\partial\left(\frac{|\tilde f v|^{2\lambda}}{f} T\right) - \frac{|\tilde f v|^{2\lambda}}{f} \bar\partial T\right)\bigg|_{\lambda=0} = \frac{\bar\partial|\tilde f v|^{2\lambda}}{f} \wedge T\bigg|_{\lambda=0}.\end{aligned}$$

To prove the first equality in (4), we observe first that in the same way as in the first part, we can assume that $f = x^\gamma u$ and $\tilde f = x^{\tilde\gamma} \tilde u$, where u and $\tilde u$ are non-zero holomorphic functions. Since T is a sum of push-forwards of elementary currents, we can assume that T is of the form (3). Note that if $\operatorname{supp}\gamma \cap \operatorname{supp}\beta \neq \emptyset$, then $(|\tilde f v|^{2\lambda}/f)T = 0$ for $\mathfrak{Re}\,\lambda \gg 0$ and $(\chi(|\tilde f v|^2/\epsilon)/f)T = 0$ for $\epsilon > 0$, since $\operatorname{supp} T \subseteq \{x_i = 0,\ i \in \operatorname{supp}\beta\}$. Thus, we can assume that $\operatorname{supp}\gamma \cap \operatorname{supp}\beta = \emptyset$. By a smooth (but non-holomorphic) change of variables, as in Section 3 (Eqs. (13)), we can assume that $|\tilde{u}v|^2 \equiv 1$. Thus, since $(|x^{\tilde{\gamma}}|^{2\lambda}/x^{\gamma})(1/x^{\alpha})$, $(\chi(|x^{\tilde{\gamma}}|^2/\epsilon)/x^{\gamma})(1/x^{\alpha})$ depend on variables disjoint from the ones that $\bigwedge_{\beta_i \neq 0} \bar{\partial}(1/x_i^{\beta_i})$ depends on, it is enough to prove that

$$\frac{|x^{\tilde{\gamma}}|^{2\lambda}}{x^{\gamma}} \frac{1}{x^{\alpha}}\bigg|_{\lambda=0} = \lim_{\epsilon \to 0} \frac{\chi(|x^{\tilde{\gamma}}|^2/\epsilon)}{x^{\gamma}} \frac{1}{x^{\alpha}},$$

which is Lemma 2 in [16]. $\square$

Let $E_1, \dots, E_q$ be holomorphic hermitian vector bundles over Z, let f_j be a holomorphic section of E_j^*, $j = 1, \dots, q$, and let s_j be the section of E_j with pointwise minimal norm such that $f_j \cdot s_j = |f_j|^2$. Outside $\{f_j = 0\}$, define

$$u_k^j = \frac{s_j \wedge (\bar{\partial} s_j)^{k-1}}{|f_j|^{2k}}.$$

It is easily seen that if $f_j = f_j^0 f_j'$, where f_j^0 is a holomorphic function and f_j' is a non-vanishing section, then $u_k^j = (1/f_j^0)^k (u')_k^j$, where $(u')_k^j$ is smooth across $\{f_j = 0\}$. We let

$$U^j = \sum_{k=1}^{\infty} |\tilde{f}_j|^{2\lambda} u_k^j \big|_{\lambda=0}, \tag{5}$$

where $\tilde{f}_j$ is any holomorphic section of E_j^* such that $\{\tilde{f}_j = 0\} = \{f_j = 0\}$. The existence of the analytic continuation is a local statement, so we can assume that $f_j = \sum f_{j,k} \mathfrak{e}_{j,k}^*$, where $\mathfrak{e}_{j,k}^*$ is a local holomorphic frame for E_j^*. After principalization we can assume that the ideal $\langle f_{j,1}, \dots, f_{j,k_j} \rangle$ is generated by, e.g., $f_{j,0}$. By the representation $u_k^j = (1/f_{j,0})^k (u')_k^j$, the existence of the analytic continuation of U^j in (5) then follows from Lemma 6. Let U_k^j denote the term of U^j that takes values in $\bigwedge^k E_j$; U_k^j is thus a $(0, k-1)$-current with values in $\bigwedge^k E_j$. Let δ_{f_j} denote interior multiplication with f_j and put $\nabla_{f_j} = \delta_{f_j} - \bar{\partial}$; it is not hard to verify that $\nabla_{f_j} U = 1$ outside $f_j = 0$. We define the Cauchy–Fantappiè–Leray type residue current, R^j, of f_j by $R^j = 1 - \nabla_{f_j} U^j$. One readily checks that

$$\begin{aligned} R^j &= R_0^j + \sum_{k=1}^{\infty} R_k^j \\ &= \left(1 - |\tilde{f}_j|^{2\lambda}\right)\big|_{\lambda=0} + \sum_{k=1}^{\infty} \bar{\partial} |\tilde{f}_j|^{2\lambda} \wedge \frac{s_j \wedge (\bar{\partial} s_j)^{k-1}}{|f_j|^{2k}} \bigg|_{\lambda=0}, \end{aligned} \tag{6}$$

where, as above, $\tilde{f}_j$ is a holomorphic section such that $\{\tilde{f}_j = 0\} = \{f_j = 0\}$.

Remark 7. Notice that if E_j has rank 1, then U_j simply equals $1/f_j$ and $R^j = 1 - \nabla_{f_j}(1/f_j) = 1 - f_j \cdot (1/f_j) + \bar{\partial}(1/f_j) = \bar{\partial}(1/f_j)$.

We now define a non-commutative calculus for the currents U_k^i and R_ℓ^j recursively as follows.

Definition 8. If T is a product of some U_k^i and R_ℓ^j, then we define

- $$U_k^j \wedge T = |\tilde{f}_j|^{2\lambda} \frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}} \wedge T\bigg|_{\lambda=0},$$
- $$R_0^j \wedge T = \left(1 - |\tilde{f}_j|^{2\lambda}\right) T\big|_{\lambda=0},$$
- $$R_k^j \wedge T = \bar\partial |\tilde{f}_j|^{2\lambda} \wedge \frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}} \wedge T\bigg|_{\lambda=0},$$

where $\tilde{f}_j$ is any holomorphic section of E_j^* with $\{\tilde{f}_j = 0\} = \{f_j = 0\}$.

Notice that after principalization the pull-back of u_k^j is semi-meromorphic; in particular U^j and R^j are pseudomeromorphic. Thus, by Lemma 6, the analytic continuations of Definition 8 exist and the values at $\lambda = 0$ are pseudomeromorphic as well.

Remark 9. Under assumptions about complete intersection, these products have the suggestive commutation properties, e.g., if $\operatorname{codim}\{f_i = f_j = 0\} = \operatorname{rank} E_i + \operatorname{rank} E_j$, then $R_k^i \wedge R_\ell^j = R_\ell^j \wedge R_k^i$, $R_k^i \wedge U_\ell^j = U_\ell^j \wedge R_k^i$, and $U_k^i \wedge U_\ell^j = -U_\ell^j \wedge U_k^i$ (see, e.g., [3]). In general, there are no simple relations. However, products involving only U:s are always anti-commutative.

Now, consider collections $R = \{R_{k_1}^1, \dots, R_{k_p}^p\}$ and $U = \{U_{k_{p+1}}^{p+1}, \dots, U_{k_q}^q\}$ and put $(P_1, \dots, P_q) = (R_{k_1}^1, \dots, R_{k_p}^p, U_{k_{p+1}}^{p+1}, \dots, U_{k_q}^q)$. For a permutation ν of $\{1, \dots, q\}$ we define

$$(UR)^\nu = P_{\nu(q)} \wedge \cdots \wedge P_{\nu(1)}. \tag{7}$$

From (5) and (6) we get natural λ-regularizations, P_j^λ, of P_j and from Definition 8 we have $(UR)^\nu = P_{\nu(q)}^{\lambda_q} \wedge \cdots \wedge P_{\nu(1)}^{\lambda_1}|_{\lambda_1=0} \cdots |_{\lambda_q=0}$, i.e., we set successively $\lambda_1 = 0$, then $\lambda_2 = 0$ and so on. The following result is proved in [6].

Theorem 10. *Let $a_1 > \cdots > a_q > 0$ be integers and λ a complex variable. Then*

$$\lambda \mapsto P_{\nu(q)}^{\lambda^{a_q}} \wedge \cdots \wedge P_{\nu(1)}^{\lambda^{a_1}}$$

has a current-valued analytic continuation to a neighborhood of the half-axis $[0, \infty) \subset \mathbb{C}$ and the value at $\lambda = 0$ equals $(UR)^\nu$.

The recursively defined product $(UR)^\nu$ can thus be obtained as the value at zero of a one-variable ζ-type function. From an algebraic point of view, this is desirable since one can derive functional equations and use Bernstein–Sato theory to study $(UR)^\nu$.

There are also more concrete and explicit regularizations of the currents U_k^i and R_ℓ^j inspired by [17] and [24]. Let $\chi = \mathbf{1}_{[1,\infty)}$, or a smooth approximation thereof that is 0 close to 0 and 1 close to ∞. It follows from [29], or after principalization from Lemma 6, that

$$U_k^j = \lim_{\epsilon\to 0^+} \chi\big(|\tilde{f}_j|^2/\epsilon\big)\frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}}, \tag{8}$$

$$R_k^j = \lim_{\epsilon\to 0^+} \bar\partial\chi\big(|\tilde{f}_j|^2/\epsilon\big) \wedge \frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}}, \quad k>0, \tag{9}$$

and similarly for $k=0$; as usual, $\{\tilde{f}_j = 0\} = \{f_j = 0\}$. Of course, the limits are in the current sense and if $\chi = \mathbf{1}_{[1,\infty)}$, then ϵ is supposed to be a regular value for $|f_j|^2$ and $\bar\partial\chi(|f_j|^2/\epsilon)$ is to be interpreted as integration over the manifold $|f_j|^2 = \epsilon$. We denote the regularizations given by (8) and (9) by P_j^ϵ.

Theorem 11. *Let $R = \{R^1_{k_1}, \dots, R^p_{k_p}\}$ and $U = \{U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q}\}$ be collections of currents defined in (5) and (6). Let ν be a permutation of $\{1,\dots,q\}$ and let $(UR)^\nu$ be the product defined in (7). Then*

$$(UR)^\nu = \lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} P^{\epsilon_q}_{\nu(q)} \wedge \cdots \wedge P^{\epsilon_1}_{\nu(1)},$$

where, as above, $(P_1,\dots,P_q) = (R^1_{k_1},\dots,R^p_{k_p},U^{p+1}_{k_{p+1}},\dots,U^q_{k_q})$; see Definition 1 for the meaning of the limit. If $\chi = \mathbf{1}_{[1,\infty)}$, we require that $\epsilon \to 0$ along an admissible path in the sense of Coleff–Herrera.

Thus $(UR)^\nu$ can be computed as the weak limit of an explicit smooth form and moreover, Definition 8 give the Coleff–Herrera product (in case the bundles E_j have rank 1).

Remark 12. It might be more natural to consider products of whole Cauchy–Fantappiè–Leray type currents, U^j and R^j, as in (5) and (6), and not just products of their components U^j_k and R^j_k, cf., for example [6]. However, since such a product is a sum of products of their components, it follows readily that Theorem 11 holds also for products of whole Cauchy–Fantappiè–Leray type currents.

2.1. The complete intersection case

Assume that $f_1,\dots,f_q$ define a complete intersection, i.e., that $\operatorname{codim}\{f_1 = \cdots = f_q = 0\} = \operatorname{rank} E_1 + \cdots + \operatorname{rank} E_q$. Then we know that the calculus defined in Definition 8 satisfies the suggestive commutation properties, but we have in fact the following much stronger results.

Theorem 13. *Assume that $f_1,\dots,f_q$ define a complete intersection on Z, let $(P_1,\dots,P_q) = (R^1_{k_1},\dots,R^p_{k_p},U^{p+1}_{k_{p+1}},\dots,U^q_{k_q})$, and let $P_j^{\epsilon_j}$ be an ϵ-regularization of P_j defined by (8) and (9) with smooth χ-functions. Then we have*

$$\left|\int_Z P_1^{\epsilon_1} \wedge \cdots \wedge P_q^{\epsilon_q} \wedge \varphi - P_1 \wedge \cdots \wedge P_q.\varphi\right| \leqslant C\|\varphi\|_{C^M}\big(\epsilon_1^\omega + \cdots + \epsilon_q^\omega\big),$$

where M and ω only depend on $f_1,\dots,f_q$, Z, and $\operatorname{supp}\varphi$ while C also depends on the C^M-norm of the χ-functions.

Theorem 14. *Assume that $f_1, \dots, f_q$ define a complete intersection on Z, let $(P_1, \dots, P_q) = (R^1_{k_1}, \dots, R^p_{k_p}, U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q})$, and let $P_j^{\lambda_j}$ be the λ-regularization of P_j given by (5) and (6). Then the current-valued function*

$$\lambda \mapsto P_1^{\lambda_1} \wedge \cdots \wedge P_q^{\lambda_q},$$

a priori defined for $\mathfrak{Re}\,\lambda_j \gg 0$, has an analytic continuation to a neighborhood of the half-space $\bigcap_1^q \{\mathfrak{Re}\,\lambda_j \geqslant 0\}$.

Remark 15. In case the E_j are trivial with trivial metrics, Theorems 13 and 14 follow quite easily from, respectively, [16, Theorem 1] and [30, Theorem 1] by taking averages. As an illustration, let $\varepsilon_1, \dots, \varepsilon_r$ be a nonsense basis and let $f_1, \dots, f_r$ be holomorphic functions. Then we can write $s = \bar{f} \cdot \varepsilon$ and so $u_k = (\bar{f} \cdot \varepsilon) \wedge (d\bar{f} \cdot \varepsilon)^{k-1}/|f|^{2k}$. A standard computation shows that

$$\int\limits_{\alpha \in \mathbb{CP}^{r-1}} \frac{|\alpha \cdot f|^{2\lambda} \alpha \cdot \varepsilon}{(\alpha \cdot f)|\alpha|^{2\lambda}}\, dV = A(\lambda)|f|^{2\lambda} \frac{\bar{f} \cdot \varepsilon}{|f|^2},$$

where dV is the (normalized) Fubini–Study volume form and A is holomorphic with $A(0) = 1$. It follows that

$$\int\limits_{\alpha_1, \dots, \alpha_k \in \mathbb{CP}^{r-1}} \bigwedge_1^k \frac{\bar{\partial}|\alpha_j \cdot f|^{2\lambda}}{\alpha_j \cdot f} \wedge \frac{\alpha_j \cdot \varepsilon}{|\alpha_j|^{2\lambda}}\, dV(\alpha_j) = A(\lambda)^k \bar{\partial}\big(|f|^{2k\lambda} u_k\big).$$

Elaborating this formula and using [30, Theorem 1] one can show Theorem 14 in the case of trivial E_j with trivial metrics. The general case can probably also be handled in a similar manner but the computations become more involved and we prefer to give direct proofs.

3. Proof of Theorem 11

The structure of this proof is rather similar to the structure of the proof of Proposition 5.4 in [6].

We start by making a Hironaka resolution of singularities [21], of Z such that the pre-image of $\bigcup_j \{f_j = 0\}$ has normal crossings. We then make further toric resolutions (e.g., as in [28]) such that, in local charts, the pull-back of each f_i is a monomial, x^{α_i}, times a non-vanishing holomorphic tuple. One checks that the pull-back of P_j^ϵ is of one of the following forms:

$$\frac{\chi(|x^{\tilde{\alpha}}|^2 \xi/\epsilon)}{x^\alpha} \vartheta, \qquad 1 - \chi\big(|x^{\tilde{\alpha}}|^2 \xi/\epsilon\big), \qquad \frac{\bar{\partial}\chi(|x^{\tilde{\alpha}}|^2 \xi/\epsilon)}{x^\alpha} \wedge \vartheta,$$

where ξ is smooth and positive, $\operatorname{supp} \tilde{\alpha} = \operatorname{supp} \alpha$, and ϑ is a smooth bundle valued form; by localizing on the blow-up we may also suppose that ϑ has as small support as we wish. If the χ-functions are smooth, the following special case of Theorem 11 now immediately follows from Lemma 6:

$$(UR)^\nu = \lim_{\epsilon_q \to 0} \cdots \lim_{\epsilon_1 \to 0} P_{\nu(q)}^{\epsilon_q} \wedge \cdots \wedge P_{\nu(1)}^{\epsilon_1}. \tag{10}$$

For smooth χ-functions we put

$$\mathcal{I}(\epsilon) = \int \frac{\bar\partial \chi_1^\epsilon \wedge \cdots \wedge \bar\partial \chi_p^\epsilon \chi_{p+1}^\epsilon \cdots \chi_q^\epsilon}{x^{\alpha_1+\cdots+\alpha_p+\cdots+\alpha_{q'}}} \wedge \varphi,$$

where $q' \leqslant q$, φ is a smooth $(n, n-p)$-form with support close to the origin, and $\chi_j^\epsilon = \chi(|x^{\tilde\alpha_j}|^2 \xi_j/\epsilon_j)$ for smooth positive ξ_j. We note that we may replace the $\bar\partial$ in $\mathcal{I}(\epsilon)$ by d for bidegree reasons. In case $\chi = \mathbf{1}_{[1,\infty)}$ we denote the corresponding integral by $I(\epsilon)$. We also put $\mathcal{I}^\nu(\epsilon_1, \ldots, \epsilon_q) = \mathcal{I}(\epsilon_{\nu(1)}, \ldots, \epsilon_{\nu(q)})$ and similarly for I^ν. In view of (10), the special case of Theorem 11 when the χ-functions are smooth will be proved if we can show that

$$\lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) \tag{11}$$

exists. The case with $\chi = \mathbf{1}_{[1,\infty)}$ will then follow if we can show

$$\lim_{\delta \to 0} \big(\mathcal{I}^\nu\big(\epsilon(\delta)\big) - I^\nu\big(\epsilon(\delta)\big)\big) = 0, \tag{12}$$

where $\delta \mapsto \epsilon(\delta)$ is any admissible path.

For notational convenience, we will consider $\mathcal{I}^\nu(\epsilon)$ (unless otherwise stated), but our arguments apply just as well to $I^\nu(\epsilon)$ until we arrive at the integral (16).

Denote by $\tilde A$ the $q \times n$-matrix with rows $\tilde\alpha_i$. We will first show that we can assume that $\tilde A$ has full rank. The idea is the same as in [17] and [24], however because of the paths along which our limits are taken, we have to modify the argument slightly. The following lemma follows from the proof of Lemma III.12.1 in [31].

Lemma 16. *Assume that α is a $q \times n$-matrix with rows α_i such that there exists $(v_1, \ldots, v_q) \neq 0$ with $\sum v_i \alpha_i = 0$. Let $j = \min\{i;\, v_i \neq 0\}$. Then there exist constants $C, c > 0$ such that if $\epsilon_j < C(\epsilon_{j+1} \ldots \epsilon_q)^c$, then $\chi(|x^{\alpha_j}|^2 \xi_j/\epsilon_j) \equiv 1$ and $\bar\partial \chi(|x^{\alpha_j}|^2 \xi_j/\epsilon_j) \equiv 0$ for all $x \in \Delta \cap \{|x^{\alpha_i}|^2 \geqslant C_i \epsilon_i, i = j+1, \ldots, q\}$, where Δ is the unit polydisc.*

Assume that $\tilde A$ does not have full rank, and let v be a column vector such that $v^t \tilde A = 0$. Since $(\epsilon_1, \ldots, \epsilon_q)$ is replaced by $(\epsilon_{\nu(1)}, \ldots, \epsilon_{\nu(q)})$ in $\mathcal{I}^\nu(\epsilon)$, we choose instead j_0 such that $\nu(j_0) \leqslant \nu(i)$ for all i such that $v_i \neq 0$. If $j_0 \leqslant p$, we let $\widetilde{\mathcal{I}}^\nu(\epsilon) = 0$, and if $j_0 \geqslant p+1$, we let $\widetilde{\mathcal{I}}^\nu(\epsilon)$ be $\mathcal{I}^\nu(\epsilon)$ but with $\chi_{j_0}^\epsilon$ replaced by 1. If $\epsilon = \epsilon(\delta)$ is such that $\epsilon_{\nu(j_0)} > 0$, then $\mathcal{I}^\nu(\epsilon)$ is a current acting on a test form with support on a set of the form

$$\Delta \cap \big\{\big|x^{\alpha_i}\big|^2 \geqslant C_i \epsilon_{\nu(i)};\ \text{for all } i \text{ such that } \nu(i) \geqslant \nu(j_0)\big\}.$$

In particular, if $\epsilon_{\nu(j_0)}(\delta)$ is sufficiently small compared to $(\epsilon_{\nu(j_0)+1}(\delta), \ldots, \epsilon_q(\delta))$, then by Lemma 16, if $j_0 \leqslant p$, the factor $\bar\partial \chi_{j_0}^\epsilon$ is identically 0, and if $j_0 \geqslant p+1$, the factor $\chi_{j_0}^\epsilon$ is identically 1 and thus is equal to $\widetilde{\mathcal{I}}^\nu(\epsilon)$ for such ϵ. Similarly, if $\epsilon_{\nu(j_0)} = 0$, we have that $\mathcal{I}^\nu(\epsilon)$ is defined as a limit along $\epsilon_{\nu(j_0)} \to 0$, with $\epsilon_{\nu(j_0)+1}, \ldots, \epsilon_q$ fixed and in the limit we get again that for sufficiently small $\epsilon_{\nu(j_0)}$, we can replace $\mathcal{I}^\nu(\epsilon)$ by $\widetilde{\mathcal{I}}^\nu(\epsilon)$. Thus we have

$$\lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) = \lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} \widetilde{\mathcal{I}}^\nu(\epsilon),$$

and we have reduced to the case that $\tilde{A}$ is a $(q-1) \times n$-matrix of the same rank. We continue this procedure until $\tilde{A}$ has full rank.

By re-numbering the coordinates, we may suppose that the minor $A = (\tilde{\alpha}_{ij})_{1 \leqslant i,j \leqslant q}$ of $\tilde{A}$ is invertible and we put $A^{-1} = B = (b_{ij})$. We now use complex notation to make a non-holomorphic, but smooth change of variables:

$$\begin{aligned} &y_1 = x_1 \xi^{b_1/2}, \ldots, y_q = x_q \xi^{b_q/2}, y_{q+1} = x_{q+1}, \ldots, y_n = x_n, \\ &\bar{y}_1 = \bar{x}_1 \xi^{b_1/2}, \ldots, \bar{y}_q = \bar{x}_q \xi^{b_q/2}, \bar{y}_{q+1} = \bar{x}_{q+1}, \ldots, \bar{y}_n = \bar{x}_n, \end{aligned} \tag{13}$$

where $\xi^{b_i/2} = \xi_1^{b_{i1}/2} \cdots \xi_q^{b_{iq}/2}$. One easily checks that $dy \wedge d\bar{y} = \xi^{b_1} \cdots \xi^{b_q}\, dx \wedge d\bar{x} + O(|x|)$, so (13) defines a smooth change of variables between neighborhoods of the origin. A simple linear algebra computation then shows that $|x^{\tilde{\alpha}_i}|^2 \xi_i = |y^{\tilde{\alpha}_i}|^2$. Of course, this change of variables does not preserve bidegrees so $\varphi(y)$ is merely a smooth compactly supported $(2n-p)$-form. We thus have

$$\mathcal{I}^\nu(\epsilon) = \int\limits_\Delta \frac{d\chi_1^\epsilon \wedge \cdots \wedge d\chi_p^\epsilon \chi_{p+1}^\epsilon \cdots \chi_q^\epsilon}{y^{\alpha_1 + \cdots + \alpha_p + \cdots + \alpha_{q'}}} \wedge \varphi'(y), \tag{14}$$

where $\chi_j^\epsilon = \chi(|y^{\tilde{\alpha}_j}|^2/\epsilon_{\nu(j)})$ and $\varphi'(y) = \sum_{|I|+|J|=2n-p} \psi_{IJ}\, dy_I \wedge d\bar{y}_J$. By linearity we may assume that the sum only consists of one term $\varphi'(y) = \psi\, dy_K \wedge d\bar{y}_L$, and by scaling, we may assume that $\operatorname{supp} \psi \subseteq \Delta$, Δ being the unit polydisc. By Lemma 2.4 in [17], we can write the function ψ as

$$\psi(y) = \sum_{I+J < \sum_1^{q'} \alpha_j - \mathbf{1}} \psi_{IJ} y^I \bar{y}^J + \sum_{I+J = \sum_1^{q'} \alpha_j - \mathbf{1}} \psi_{IJ} y^I \bar{y}^J, \tag{15}$$

where $a < b$ for tuples a and b means that $a_i < b_i$ for all i. In the decomposition (15) each of the smooth functions ψ_{IJ} in the first sum on the left-hand side is independent of some variable. We now show that this implies that the first sum on the left-hand side of (15) does not contribute to the integral (14). In case $\varphi'(y)$ has bidegree $(n, n-p)$ this is a well-known fact but we must show it for an arbitrary $(2n-p)$-form.

We change to polar coordinates:

$$dy_K \wedge d\bar{y}_L = d\big(r_{K_1} e^{i\theta_{K_1}}\big) \wedge \cdots \wedge d\big(r_{L_1} e^{-i\theta_{L_1}}\big) \wedge \cdots.$$

Since χ_j^ϵ in (14) is independent of θ, it follows that we must have full degree $= n$ in $d\theta$. The only terms in the expansion of $dy_K \wedge d\bar{y}_L$ above that will contribute to (14) are therefore of the form

$$c r_1 \cdots r_n e^{i\theta \cdot \gamma}\, dr_M \wedge d\theta,$$

where $|M| = n - p$, c is a constant, and γ is a multiindex with entries equal to 1, -1, or 0. Substituting this and a term $\psi_{IJ} y^I \bar{y}^J = \psi_{IJ} r^{I+J} e^{i\theta\cdot(I-J)}$ from (15) into (14) gives rise to an "inner" θ-integral (by Fubini's theorem):

$$\mathscr{J}_{IJ}(r) = \int\limits_{\theta\in[0,2\pi)^n} \psi_{IJ}(r,\theta) e^{i\theta\cdot(I-J-\sum_1^{q'}\alpha_j+\gamma)}\, d\theta.$$

If $I + J < \sum_1^{q'} \alpha_j - \mathbf{1}$, then $I - J - \sum_1^{q'} \alpha_j + \gamma < 0$ and ψ_{IJ} is independent of some $y_j = r_j e^{i\theta_j}$. Integrating over $\theta_j \in [0, 2\pi)$ thus yields $\mathscr{J}_{IJ} = 0$ if $I + J < \sum_1^{q'} \alpha_j - \mathbf{1}$. If instead $I + J = \sum_1^{q'} \alpha_j - \mathbf{1}$, then $\mathscr{J}_{IJ}(r)$ is smooth on $[0,\infty)^n$.

Summing up, we see that we can write (14) as

$$\mathcal{I}^\nu(\epsilon) = \int\limits_{r\in(0,1)^n} d\chi_1^\epsilon \wedge \cdots \wedge d\chi_p^\epsilon \chi_{p+1}^\epsilon \cdots \chi_q^\epsilon \mathscr{J}(r)\, dr_M, \tag{16}$$

where $\chi_j^\epsilon = \chi(r^{2\alpha_j}/\epsilon_{\nu(j)})$, $\mathscr{J}$ is smooth, and $|M| = n - p$.

After these reductions, the integral (16) we arrive at is the same as Eq. (16) in [24], and we will use the fact proven there, that $\lim_{\delta\to 0} \mathcal{I}^\nu(\epsilon(\delta))$ exists along any admissible path $\epsilon(\delta)$, and is well defined independently of the choice of admissible path. (This is not exactly what is proven there, but the fact that if $b \in \mathbb{Q}^p$, then $\lim_{\delta\to 0} \epsilon(\delta)^b$ is either 0 or ∞ independently of the admissible path chosen is the only addition we need to make for the argument to go through in our case.) Using this, if we let $\epsilon(\delta)$ be any admissible path, we will show by induction over q that

$$\lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) = \lim_{\delta\to 0} \mathcal{I}^\nu\big(\epsilon(\delta)\big).$$

For $q = 1$ this is trivially true, so we assume $q > 1$. Let ϵ^k be any sequence satisfying the conditions in Definition 1. Consider a fixed k, and let m be such that $\epsilon^k = (0, \ldots, 0, \epsilon_{m+1}^k, \ldots, \epsilon_q^k)$ with $\epsilon_{m+1}^k > 0$. Let $I_1 = \nu^{-1}(\{1, \ldots, m\}) \cap \{1, \ldots, p\}$ and $I_2 = \nu^{-1}(\{1, \ldots, m\}) \cap \{p+1, \ldots, q\}$. We consider $\epsilon_{m+1}^k, \ldots, \epsilon_q^k$ fixed in $\mathcal{I}^\nu(\epsilon)$, and define

$$\mathcal{I}_k(\epsilon_1, \ldots, \epsilon_m) = \int\limits_{[0,1]^n} \bigwedge_{i\in I_1} d\chi\big(r^{\alpha_i}/\epsilon_{\nu(i)}\big) \prod_{i\in I_2} \chi\big(r^{\alpha_i}/\epsilon_{\nu(i)}\big) \mathscr{J}_k(r)\, dr_M,$$

originally defined on $(0,\infty)^p$, but extended according to Definition 1, where

$$\mathscr{J}_k(r) = \pm \bigwedge_{i\in\{1,\ldots,p\}\setminus I_1} d\chi\big(r^{\alpha_i}/\epsilon_{\nu(i)}^k\big) \prod_{i\in\{p+1,\ldots,q\}\setminus I_2} \chi\big(r^{\alpha_i}/\epsilon_{\nu(i)}^k\big) \mathscr{J}(r)$$

(where the sign is chosen such that $\mathcal{I}_k(0) = \mathcal{I}^\nu(\epsilon^k)$). Since $m < q$ and $\mathscr{J}_k$ is smooth, we have by induction that

$$\mathcal{I}_k(0) = \lim_{\epsilon_m\to 0} \ldots \lim_{\epsilon_1\to 0} \mathcal{I}_k(\epsilon_1, \ldots, \epsilon_m) = \lim_{\delta\to 0} \mathcal{I}_k\big(\epsilon'(\delta)\big),$$

where $\epsilon'(\delta)$ is any admissible path, and the first equality follows by definition of $\mathcal{I}_k(0)$. We fix an admissible path $\epsilon'(\delta)$. For each k we can choose δ_k such that if $\epsilon^{k'} = (\epsilon'_1(\delta_k), \dots, \epsilon'_m(\delta_k))$, then $\lim_{k\to\infty}(\mathcal{I}_k(\epsilon^{k'}) - \mathcal{I}_k(0)) = 0$ and if $\tilde{\epsilon}^k = (\epsilon^{k'}, \epsilon^k_{m+1}, \dots, \epsilon^k_q)$, then $\tilde{\epsilon}^k$ forms a subsequence of an admissible path. Since $\mathcal{I}_k(0) = \mathcal{I}^\nu(\epsilon^k)$, and $\mathcal{I}_k(\epsilon^{k'}) = \mathcal{I}^\nu(\tilde{\epsilon}^k)$, we thus have

$$\lim_{k\to\infty} \mathcal{I}^\nu\big(\epsilon^k\big) = \lim_{k\to\infty} \mathcal{I}^\nu\big(\tilde{\epsilon}^k\big) = \lim_{\delta\to 0} \mathcal{I}^\nu\big(\epsilon(\delta)\big)$$

where the second equality follows from the existence and uniqueness of $\mathcal{I}^\nu(\epsilon(\delta))$ along any admissible path. Hence we have shown that the limit in (11) exists and is well defined.

Finally, if we start from (16), as (23) in [24] shows, either

$$\lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) = \pm \int\limits_{r_M \in (0,1)^{n-p}} \mathscr{I}(0, r_M)\, dr_M,$$

or the limit is 0, depending only on α. If we consider $I^\nu(\epsilon)$ instead, we get the same limit, see [31, pp. 79–80], and (12) follows.

4. Proof of Theorems 13 and 14

As in [30] and [16] the key-step of the proof is a Whitney type division lemma, Lemma 19 below. Recall that

$$(P_1, \dots, P_q) = \big(R^1_{k_1}, \dots, R^p_{k_p}, U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q}\big)$$

and that $P_j^{\epsilon_j}$ and $P_j^{\lambda_j}$ are the ϵ-regularizations with smooth χ (given by (8), (9)) and the λ-regularizations (cf., (5), (6)) respectively of P_j. We will consider the following two integrals:

$$\mathcal{I}(\epsilon) = \int\limits_Z P_1^{\epsilon_1} \wedge \cdots \wedge P_q^{\epsilon_q} \wedge \varphi,$$

$$\Gamma(\lambda) = \int\limits_Z P_1^{\lambda_1} \wedge \cdots \wedge P_q^{\lambda_q} \wedge \varphi,$$

where φ is a test form on Z, supported close to a point in $\{f_1 = \cdots = f_q = 0\}$, of bidegree $(n, n - k_1 - \cdots - k_q + q - p)$ with values in $\bigwedge(E_1^* \oplus \cdots \oplus E_q^*)$. In the arguments below, we will assume for notational convenience that $\tilde{f}_j = f_j$ (cf., e.g., (5)); the modifications to the general case are straightforward.

The main parts of the proofs of Theorems 13 and 14 are contained in the following propositions.

Proposition 17. *Assume that $f_1, \dots, f_q$ define a complete intersection. For $p < s \leqslant q$ we have*

$$\big|\mathcal{I}(\epsilon) - \mathcal{I}(\epsilon_1, \dots, \epsilon_{s-1}, 0, \dots, 0)\big| \leqslant C\|\varphi\|_M\big(\epsilon_s^\omega + \cdots + \epsilon_q^\omega\big).$$

Note that $\mathcal{I}(\epsilon_1,\dots,\epsilon_{s-1},0,\dots,0)$ is well defined; it is the action of $U^s_{k_s}\wedge\cdots\wedge U^q_{k_q}$ on a smooth form.

Proposition 18. *Assume that $f_1,\dots,f_q$ define a complete intersection. Then $\Gamma(\lambda)$ has a meromorphic continuation to all of $\mathbb{C}^q$ and its only possible poles in a neighborhood of $\bigcap_1^q\{\mathfrak{Re}\,\lambda_j\geqslant 0\}$ are along hyperplanes of the form $\sum_{j=1}^p\lambda_j\alpha_j=0$, where $\alpha_j\in\mathbb{N}$ and at least two α_j are positive. In particular, for $p=1$, $\Gamma(\lambda)$ is analytic in a neighborhood of $\bigcap_1^q\{\mathfrak{Re}\,\lambda_j\geqslant 0\}$.*

Using that

$$\bar\partial|f_j|^{2\lambda}\wedge u^j_k=\bar\partial\big(|f_j|^{2\lambda}u^j_k\big)-f_j\cdot\big(|f_j|^{2\lambda}u^j_{k+1}\big), \tag{17}$$

the proof of Theorem 14 follows from Proposition 18 in a similar way as Theorem 1 in [30] follows from Proposition 4 in [30].

We indicate one way Proposition 17 can be used to prove Theorem 13. To simplify notation somewhat, we let R^j denote any R^j_k and R^j_ϵ denotes a smooth ϵ-regularization of R^j; U^j and U^j_ϵ are defined similarly. The uniformity in the estimate of Proposition 17 implies that we have estimates of the form

$$\left|\bigwedge_1^m R^j_\epsilon\wedge\bigwedge_{m+1}^p R^j\wedge\bigwedge_{p+1}^q U^j_\epsilon-\bigwedge_1^m R^j_\epsilon\wedge\bigwedge_{m+1}^p R^j\wedge\bigwedge_{p+1}^q U^j\right|\lesssim\big(\epsilon^\omega_{p+1}+\cdots+\epsilon^\omega_q\big), \tag{18}$$

where, e.g., $R^{m+1}\wedge\cdots\wedge R^p$ a priori is defined as a Coleff–Herrera product. We prove (a slightly stronger result than) Theorem 13 by induction over p. Let R^* denote the Coleff–Herrera product of some R^j:s with $j>p$ and let U^* and U^*_ϵ denote the product of some U^j:s and U^j_ϵ:s respectively, also with $j>p$ but only j:s not occurring in R^*. We prove

$$\big|R^1_\epsilon\wedge\cdots\wedge R^p_\epsilon\wedge R^*\wedge U^*_\epsilon-R^1\wedge\cdots\wedge R^p\wedge R^*\wedge U^*\big|\lesssim\epsilon^\omega,$$

i.e., we prove Theorem 13 *on* the current R^*. The induction start, $p=0$, follows immediately from (18). If we add and subtract $R^1_\epsilon\wedge\cdots\wedge R^p_\epsilon\wedge R^*\wedge U^*$, the induction step follows easily from (17) (construed in setting of ϵ-regularizations) and estimates like (18).

Proof of Propositions 17 and 18. We may assume that φ has arbitrarily small support. Hence, we may assume that Z is an analytic subset of a domain $\Omega\subseteq\mathbb{C}^N$ and that all bundles are trivial, and thus make the identification $f_j=(f_{j1},\dots,f_{je_j})$, where f_{ji} are holomorphic in Ω. We choose a Hironaka resolution $\hat Z\to Z$ such that the pulled-back ideals $\langle\hat f_j\rangle$ are all principal, and moreover, so that in a fixed chart with coordinates x on $\hat Z$ (and after a possible re-numbering), $\langle\hat f_j\rangle$ is generated by $\hat f_{j1}$ and $\hat f_{j1}=x^{\alpha_j}h_j$, where h_j is holomorphic and non-zero. We then have

$$|\hat f_j|^2=|\hat f_{j1}|^2\xi_j,\qquad \hat u^j_{k_j}=v^j/\hat f_{j1}^{k_j},$$

where ξ_j is smooth and positive and v^j is a smooth (bundle valued) form. We thus get

$$\bar\partial\chi_j\big(|\hat f_j|^2/\epsilon_j\big)=\tilde\chi_j\big(|\hat f_j|^2/\epsilon_j\big)\bigg(\frac{d\bar{\hat f}_{j1}}{\bar{\hat f}_{j1}}+\frac{\bar\partial\xi_j}{\xi_j}\bigg),$$

where $\tilde\chi_j(t)=t\chi_j'(t)$, and

$$\bar\partial|\hat f_j|^{2\lambda_j}=\lambda_j|\hat f_j|^{2\lambda_j}\bigg(\frac{d\bar{\hat f}_{j1}}{\bar{\hat f}_{j1}}+\frac{\bar\partial\xi_j}{\xi_j}\bigg).$$

It follows that $\mathcal{I}(\epsilon)$ and $\Gamma(\lambda)$ are finite sums of integrals which we without loss of generality can assume to be of the form

$$\pm\int_{\mathbb{C}^n_x}\prod_1^p\tilde\chi_j^\epsilon\prod_{p+1}^q\chi_j^\epsilon\bigwedge_1^m\frac{d\bar{\hat f}_{j1}}{\bar{\hat f}_{j1}}\wedge\bigwedge_{m+1}^p\frac{\bar\partial\xi_j}{\xi_j}\wedge\bigwedge_1^q\frac{v^j}{\hat f_{j1}^{k_j}}\wedge\varphi\rho, \tag{19}$$

$$\pm\lambda_1\cdots\lambda_p\int_{\mathbb{C}^n_x}\prod_1^q|\hat f_j|^{2\lambda_j}\bigwedge_1^m\frac{d\bar{\hat f}_{j1}}{\bar{\hat f}_{j1}}\wedge\bigwedge_{m+1}^p\frac{\bar\partial\xi_j}{\xi_j}\wedge\bigwedge_1^q\frac{v^j}{\hat f_{j1}^{k_j}}\wedge\varphi\rho, \tag{20}$$

where ρ is a cutoff function.

Recall that $\hat f_{j1}=x^{\alpha_j}h_j$ and let μ be the number of vectors in a maximal linearly independent subset of $\{\alpha_1,\dots,\alpha_m\}$; say that $\alpha_1,\dots,\alpha_\mu$ are linearly independent. We then can define new holomorphic coordinates (still denoted by x) so that $\hat f_{j1}=x^{\alpha_j}$, $j=1,\dots,\mu$, see [24, p. 46] for details. Then we get

$$\begin{aligned}\bigwedge_1^m d\hat f_{j1}&=\bigwedge_1^\mu dx^{\alpha_j}\wedge\bigwedge_{\mu+1}^m\big(x^{\alpha_j}\,dh_j+h_j\,dx^{\alpha_j}\big)\\&=x^{\sum_{\mu+1}^m\alpha_j}\bigwedge_1^\mu dx^{\alpha_j}\wedge\bigwedge_{\mu+1}^m dh_j,\end{aligned} \tag{21}$$

where the last equality follows because $dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_\mu}\wedge dx^{\alpha_j}=0$, $\mu+1\leqslant j\leqslant m$, since $\alpha_1,\dots,\alpha_\mu,\alpha_j$ are linearly dependent. From the beginning we could also have assumed that $\varphi=\varphi_1\wedge\varphi_2$, where φ_1 is an anti-holomorphic $(n-\sum_1^q k_j+q-p)$-form and φ_2 is a (bundle valued) $(n,0)$-test form on Z. We now define

$$\varPhi=\bigwedge_{\mu+1}^m\frac{d\bar h_j}{\bar h_j}\wedge\bigwedge_{m+1}^p\frac{\bar\partial\xi_j}{\xi_j}\wedge\bigwedge_1^q v^j\wedge\hat\varphi_1.$$

Using (21) we can now write (19) and (20) as

$$\pm \int_{\mathbb{C}^n_x} \frac{\prod_1^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\mu}}{\bar{x}^{\alpha_\mu}} \wedge \Phi \wedge \hat{\varphi}_2 \rho, \tag{22}$$

$$\pm \lambda_1 \cdots \lambda_p \int_{\mathbb{C}^n_x} \frac{\prod_1^q |\hat{f}_j|^{2\lambda_j}}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\mu}}{\bar{x}^{\alpha_\mu}} \wedge \Phi \wedge \hat{\varphi}_2 \rho. \tag{23}$$

Lemma 19. *Let $\mathcal{K} = \{i; x_i \mid x^{\alpha_j}$, some $p+1 \leqslant j \leqslant q\}$. For any fixed $r \in \mathbb{N}$, one can replace Φ in (22) and (23) by*

$$\Phi' := \Phi - \sum_{J \subseteq \mathcal{K}} (-1)^{|J|} \sum_{k_1, \ldots, k_{|J|}=0}^{r+1} \frac{\partial^{|k|} \Phi}{\partial x_J^k}\bigg|_{x_J=0} \frac{x_J^k}{k!}$$

without affecting the integrals. Moreover, for any $I \subseteq \mathcal{K}$, we have that $\Phi' \wedge \bigwedge_{i \in I}(d\bar{x}_i/\bar{x}_i)$ is C^r-smooth.

We replace Φ by Φ' in (22) and (23) and we write $d = d_{\mathcal{K}} + d_{\mathcal{K}^c}$, where $d_{\mathcal{K}}$ differentiates with respect to the variables $x_i, \bar{x}_i$ for $i \in \mathcal{K}$ and $d_{\mathcal{K}^c}$ differentiates with respect to the rest. Then we can write $(d\bar{x}^{\alpha_1}/\bar{x}^{\alpha_1}) \wedge \cdots \wedge (d\bar{x}^{\alpha_\mu}/\bar{x}^{\alpha_\mu}) \wedge \Phi'$ as a sum of terms, which we without loss of generality can assume to be of the form

$$\begin{aligned} &\frac{d_{\mathcal{K}^c}\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d_{\mathcal{K}^c}\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \frac{d_{\mathcal{K}}\bar{x}^{\alpha_{\nu+1}}}{\bar{x}^{\alpha_{\nu+1}}} \wedge \cdots \wedge \frac{d_{\mathcal{K}}\bar{x}^{\alpha_\mu}}{\bar{x}^{\alpha_\mu}} \wedge \Phi' \\ &\quad = \frac{d_{\mathcal{K}^c}\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d_{\mathcal{K}^c}\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \Phi'' \wedge d\bar{x}_{\mathcal{K}}, \end{aligned}$$

where Φ'' is C^r-smooth and of bidegree $(0, n - \nu - |\mathcal{K}|)$ (possibly, $\Phi'' = 0$). Thus, (22) and (23) are finite sums of integrals of the following type

$$\int_{\mathbb{C}^n_x} \frac{\prod_1^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \psi \wedge d\bar{x}_{\mathcal{K}} \wedge dx, \tag{24}$$

$$\lambda_1 \cdots \lambda_p \int_{\mathbb{C}^n_x} \frac{\prod_1^q |\hat{f}_j|^{2\lambda_j}}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \psi \wedge d\bar{x}_{\mathcal{K}} \wedge dx, \tag{25}$$

where ψ is C^r-smooth and compactly supported.

We now first finish the proof of Proposition 18. First of all, it is well known that $\Gamma(\lambda)$ has a meromorphic continuation to $\mathbb{C}^q$. We have

$$\frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge d\bar{x}_{\mathcal{K}} = \sum_{\substack{|I|=\nu \\ I \subseteq \mathcal{K}^c}} C_I \frac{d\bar{x}_I}{\bar{x}_I} \wedge d\bar{x}_{\mathcal{K}}.$$

Let us assume that $I = \{1, \ldots, \nu\} \subseteq \mathcal{K}^c$ and consider the contribution to (25) corresponding to this subset. This contribution equals

$$
\begin{aligned}
&C_I\lambda_1\cdots\lambda_p\int\limits_{\mathbb{C}^n_x}\frac{|x^{\sum_1^q\lambda_j\alpha_j}|^2}{x^{\sum_1^q k_j\alpha_j}}\bigwedge_1^{\nu}\frac{d\bar{x}_j}{\bar{x}_j}\wedge\Psi(\lambda,x)\wedge d\bar{x}_{\mathcal{K}}\wedge dx\\
&\quad=\frac{C_I\prod_1^p\lambda_j}{\prod_{i=1}^{\nu}(\sum_1^q\lambda_j\alpha_{ji})}\int\limits_{\mathbb{C}^n_x}\frac{\bigwedge_{i=1}^{\nu}\bar\partial|x_i|^{2\sum_1^q\lambda_j\alpha_{ji}}\prod_{i=\nu+1}^n|x_i|^{2\sum_1^q\lambda_j\alpha_{ji}}}{x^{\sum_1^q k_j\alpha_j}}\\
&\qquad\wedge\Psi(\lambda,x)\wedge d\bar{x}_{\mathcal{K}}\wedge dx,
\end{aligned}
\tag{26}
$$

where $\Psi(\lambda,x)=\psi(x)\prod_1^q(\xi_j^{\lambda_j}/h_j^{k_j})$. It is well known (and not hard to prove, e.g., by integrations by parts as in [1, Lemma 2.1]) that the *integral* on the right-hand side of (26) has an analytic continuation in λ to a neighborhood of $\bigcap_1^q\{\mathfrak{Re}\,\lambda_j\geqslant 0\}$. (We thus choose r in Lemma 19 large enough so that we can integrate by parts.) If $p=0$, then the coefficient in front of the integral is to be interpreted as 1 and Proposition 18 follows in this case. For $p>0$, we see that the poles of (26), and consequently of $\Gamma(\lambda)$, in a neighborhood of $\bigcap_1^q\{\mathfrak{Re}\,\lambda_j\geqslant 0\}$ are along hyperplanes of the form $0=\sum_1^q\lambda_j\alpha_{ji}$, $1\leqslant i\leqslant\nu$. But if $j>p$ and $i\leqslant\nu$, then $\alpha_{ji}=0$ since $\{1,\dots,\nu\}\subseteq\mathcal{K}^c=\{i;\,x_i\nmid x^{\alpha_j},\ \forall j=p+1,\dots,q\}$. Thus, the hyperplanes are of the form $0=\sum_1^p\lambda_j\alpha_{ji}$ and Proposition 18 is proved except for the statement that at least for two j:s, the α_{ji} are non-zero. However, we see from (26) that if for some i we have $\alpha_{ji}=0$ for all j but one, then the appearing λ_j in the denominator will be canceled by the numerator. Moreover, we may assume that the constant $C_I=\det(\alpha_{ji})_{1\leqslant i,j\leqslant\nu}$ is non-zero which implies that we cannot have any λ_j^2 in the denominator.

We now prove Proposition 17. Consider (24). We have that $\alpha_1,\dots,\alpha_\nu$ are linearly independent so we may assume that $A=(\alpha_{ij})_{1\leqslant i,j\leqslant\nu}$ is invertible with inverse $B=(b_{ij})$. We make the non-holomorphic change of variables (13), where the "q" of (13) now should be understood as ν. Then we get $x^{\alpha_j}=y^{\alpha_j}\eta_j$, where $\eta_j>0$ and smooth and $\eta_j^2=1/\xi_j$, $j=1,\dots,\nu$. Hence, $|\hat{f}_j|^2=|y^{\alpha_j}|^2$, $j=1,\dots,\nu$. Expressed in the y-coordinates we get that $\bigwedge_1^{\nu}(d\bar{x}^{\alpha_j}/\bar{x}^{\alpha_j})\wedge\psi\wedge d\bar{x}_{\mathcal{K}}\wedge dx$ is a finite sum of terms of the form

$$
\frac{d\bar{y}^{\alpha_1}}{\bar{y}^{\alpha_1}}\wedge\cdots\wedge\frac{d\bar{y}^{\alpha_{\nu'}}}{\bar{y}^{\alpha_{\nu'}}}\wedge\bar{y}_{\mathcal{K}'}\,d\bar{y}_{\mathcal{K}''}\wedge\psi_1,
\tag{27}
$$

where $\nu'\leqslant\nu$, ψ_1 is a C^r-smooth compactly supported form, and $\mathcal{K}'$ and $\mathcal{K}''$ are disjoint sets such that $\mathcal{K}'\cup\mathcal{K}''=\mathcal{K}$. In order to give a contribution to (24) we see that ψ_1 must contain dy. In (27) we write $d=d_{\mathcal{K}}+d_{\mathcal{K}^c}$, and arguing as we did immediately after Lemma 19, (27) is a finite sum of terms of the form

$$
\frac{d\bar{y}^{\alpha_1}}{\bar{y}^{\alpha_1}}\wedge\cdots\wedge\frac{d\bar{y}^{\alpha_{\nu''}}}{\bar{y}^{\alpha_{\nu''}}}\wedge\psi_2\wedge d\bar{y}_{\mathcal{K}}\wedge dy,
$$

where $\nu''\leqslant\nu$ and ψ_2 is C^r-smooth and compactly supported. With abuse of notation we thus have that (24) is a finite sum of integrals of the form

$$\int_{\mathbb{C}^n_x} \frac{\prod_1^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{y}^{\alpha_1}}{\bar{y}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{y}^{\alpha_\nu}}{\bar{y}^{\alpha_\nu}} \wedge \psi \wedge d\bar{y}_{\mathcal{K}} \wedge dy$$
$$= \int_{\mathbb{C}^n_x} \frac{\bigwedge_1^\nu d\chi_j^\epsilon \prod_{\nu+1}^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{y^{\sum_1^q k_j \alpha_j}} \wedge \Psi \wedge d\bar{y}_{\mathcal{K}} \wedge dy, \tag{28}$$

where Ψ is a C^r-smooth compactly supported $(n - |\mathcal{K}| - \nu)$-form; the equality follows since $\chi_j^\epsilon = \chi_j(|y^{\alpha_j}|^2/\epsilon_j)$, $j = 1, \dots, \nu$. Now, (28) is essentially equal to Eq. (24) of [16] and the proof of Proposition 17 is concluded as in the proof of Proposition 8 in [16]. $\square$

Proof of Lemma 19. The proof is similar to the proof of Lemma 9 in [16] but some modifications have to be done. First, it is easy to check by induction over $|\mathcal{K}|$ that $\Phi' \wedge \bigwedge_{i \in I}(d\bar{x}_i/\bar{x}_i)$ is C^r-smooth for any $I \subseteq \mathcal{K}$; for $|\mathcal{K}| = 1$ this is just Taylor's formula for forms. It thus suffices to show that

$$d\bar{x}^{\alpha_1} \wedge \cdots \wedge d\bar{x}^{\alpha_\mu} \wedge \frac{\partial^{|k|}\Phi}{\partial x_I^k}\bigg|_{x_I=0} = 0, \quad \forall I \subseteq \mathcal{K},\ k = (k_{i_1}, \dots, k_{i_{|I|}}).$$

To show this, fix an $I \subseteq \mathcal{K}$ and let $L = \{j;\, x_i \nmid x^{\alpha_j}\ \forall i \in I\}$. Say for simplicity that

$$L = \{1, \dots, \mu', \mu + 1, \dots, m', m + 1, \dots, p', p + 1, \dots, q'\},$$

where $\mu' \leqslant \mu$, $m' \leqslant m$, $p' \leqslant p$, and $q' < q$. The fact that $q' < q$ follows from the definitions of $\mathcal{K}$, I, and L.

Consider, on the base variety Z, the smooth form

$$F = \bigwedge_1^{\mu'} d\bar{f}_{j1} \bigwedge_{\mu+1}^{m'} d\bar{f}_{j1} \bigwedge_{m+1}^{p'} \big(|f_{j1}|^2 \bar\partial |f_j|^2 - \bar\partial |f_{j1}|^2 |f_j|^2\big) \bigwedge_{j \in L} |f_j|^{2k_j} u^j_{k_j} \wedge \varphi_1.$$

It has bidegree $(0, n - \sum_{j \in L^c} k_j + q - q')$ so F has a vanishing pull-back to $\bigcap_{j \in L^c}\{f_j = 0\}$ since this set has dimension $n - \sum_{j \in L^c} e_j < n - \sum_{j \in L^c} k_j + q - q'$ by our assumption about complete intersection. Thus, $\hat{F}$ has a vanishing pull-back to $\{x_I = 0\} \subseteq \bigcap_{j \in L^c}\{\hat{f}_j = 0\}$. In fact, this argument shows that

$$\hat{F} = \sum \phi_j, \tag{29}$$

where the ϕ_j are smooth linearly independent forms such that each ϕ_j is divisible by $\bar{x}_i$ or $d\bar{x}_i$ for some $i \in I$. (It is the pull-back to $\{x_I = 0\}$ of the anti-holomorphic differentials of $\hat{F}$ that vanishes.) For the rest of the proof we let $\sum \phi_j$ denote such expressions and we note that they are invariant under holomorphic differential operators. Computing $\hat{F}$ we get

$$\hat{F} = \prod_{m+1}^{p'} |\hat{f}_{j1}|^4 \prod_{j \in L} \frac{|\hat{f}_j|^{2k_j}}{\hat{f}_{j1}^{k_j}} \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \bigwedge_{\mu+1}^{m'} d\big(\bar{x}^{\alpha_j}\bar{h}_j\big) \bigwedge_{m+1}^{p'} \bar\partial \xi_j \bigwedge_{j \in L} v^j \wedge \hat{\varphi}_1.$$

The "coefficient" $\prod_{m+1}^{p'} |\hat{f}_{j1}|^4 \prod_{j\in L}(|\hat{f}_j|^{2k_j}/\hat{f}_{j1}^{k_j})$ does not contain any $\bar{x}_i$ with $i \in I$ so we may divide (29) by it (recall that the ϕ_j are linearly independent) and we obtain

$$\sum \phi_j = \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \bigwedge_{\mu+1}^{m'} d\big(\bar{x}^{\alpha_j}\bar{h}_j\big) \bigwedge_{m+1}^{p'} \bar{\partial}\xi_j \bigwedge_{j\in L} v^j \wedge \hat{\varphi}_1$$

$$= \prod_{\mu+1}^{m'} \bar{x}^{\alpha_j} \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \bigwedge_{\mu+1}^{m'} d\bar{h}_j \bigwedge_{m+1}^{p'} \bar{\partial}\xi_j \bigwedge_{j\in L} v^j \wedge \hat{\varphi}_1$$

$$+ \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \wedge \sum_{\mu+1}^{m'} d\bar{x}^{\alpha_j} \wedge \tau_j$$

for some τ_j. We multiply this equality with

$$\bigwedge_{m'+1}^{m} d\bar{h}_j \bigwedge_{p'+1}^{p} \bar{\partial}\xi_j \bigwedge_{j\in L^c} v^j \Big/ \left(\prod_{\mu+1}^{m} \bar{h}_j \prod_{m+1}^{p} \xi_j \right)$$

and get

$$\prod_{\mu+1}^{m'} \bar{x}^{\alpha_j} \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \wedge \Phi + \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \wedge \sum_{\mu+1}^{m'} d\bar{x}^{\alpha_j} \wedge \tau_j = \sum \phi_j$$

for some new τ_j. We apply the operator $\partial^{|k|}/\partial x_I^k$ to this equality and then we pull-back to $\{x_I = 0\}$, which makes the right-hand side vanish; (we construe however the result in $\mathbb{C}_x^n$). Finally, taking the exterior product with $\bigwedge_{\mu'+1}^{\mu} d\bar{x}^{\alpha_j}$, which will make each term in under the summation sign on the left-hand side vanish, we arrive at

$$\prod_{\mu+1}^{m'} \bar{x}^{\alpha_j} \bigwedge_1^{\mu} d\bar{x}^{\alpha_j} \wedge \frac{\partial^{|k|}\Phi}{\partial x_I^k}\bigg|_{x_I=0} = 0$$

and we are done. □

Acknowledgment

We would like to thank the anonymous referee for valuable comments regarding the presentation of the article.

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