Shilov boundary for "holomorphic functions" on a quantum matrix ball

Daniil Proskurin

Kyiv Taras Shevchenko University, Cybernetics Department, Volodymyrska, 64, 01033, Kyiv, Ukraine

prosk@univ.kiev.ua

Lyudmila Turowska

Department of Mathematics, Chalmers University of Technology and University of Gothenburg, 412 96 Göteborg, Sweden

turowska@chalmers.se

Abstract

We describe the Shilov boundary ideal for a q-analog of algebra of holomorphic functions on the unit ball in the space of 2×2 matrices.

1 Introduction

The Shilov boundary of a compact Hausdorff space X relative a uniform algebra $\mathcal{A}$ in $C(X)$ is the smallest closed subset $K \subset X$ such that every function in $\mathcal{A}$ achieves its maximum modulus on K, a notion that is closely related to the maximum modules principle in complex analysis.

One of the most important developments in Analysis in recent years has been "quantisation", starting with the advent of the theory of operator spaces in the 1980's. A quantisation of the Shilov boundary is a Shilov boundary ideal of a C^*-algebra, that was introduced by W. Arveson in his foundational papers [1, 2] and studied intensively by many authors.

In the middle of 1990's within the framework of the quantum group theory L.Vaksman and his coauthors started a "quantisation" of bounded symmetric domains (see [16] and references therein). One of the simplest of such domains is the matrix ball $\mathbb{U} = \{z \in Mat_{m,n} : zz^* \leq I\}$, where $Mat_{m,n}$ is the algebra of complex $m \times n$ matrices. Its q-analog was studied in [11, 14] where the authors defined a non-commutative counterpart of the

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polynomial algebra in the space $Mat_{m,n}$, the $*$-algebra $Pol(Mat_{m,n})_q$. A q-analog of polynomial algebra on the Shilov boundary $S(\mathbb{U})$ of the matrix ball $\mathbb{U}$ and the corresponding Cauchy-Szegö integral representation that recovers holomorphic functions from its values on the Shilov boundary was studied in [14]. The authors used a purely algebraic approach "quantizing" a well known procedure for producing the Shilov boundary in the classical case. In particular, they constructed a $*$-homomorphism $\psi : Pol(Mat_{m,n})_q \to Pol(S(\mathbb{U}))_q$ that corresponds to the restriction of the polynomials onto the Shilov boundary in the classical case. That the kernel of ψ gives rise to Arveson's Shilov boundary ideal for the q-analog of holomorphic functions on the unit ball of $Mat_{n,1} = \mathbb{C}^n$ was shown in [15]. In this paper we prove the statement for the case $m = n = 2$.

Our approach relies on a classification of irreducible representations of $Pol(Mat_{2,2})_q$ obtained in [13] and elaborates methods of quantum groups and the Sz.-Nagy's unitary dilation theory. We note that all irreducible representations of $Pol(Mat_{m,n})_q$ are known only in the cases when either $m = 1$, or $n = 1$ or $m = n = 2$.

In this paper we use the following standard notations: $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{N} = \{1, 2, \ldots\}$. All algebras are assumed to be unital one over the field of complex numbers $\mathbb{C}$ and $q \in (0, 1)$. We write Mat_n for the space of $n \times n$ complex matrices. By $\{e_n : n \in \mathbb{Z}_+\}$ we denote the standard basis in the Hilbert space $\ell^2(\mathbb{Z}_+)$.

2 The $*$-algebra $Pol(\mathbb{C}^n)_q$

The $*$-algebra $Pol(\mathbb{C}^n)_q$ is a $*$-algebra generated by z_j, $j = 1, \ldots, n$, subject to the relations:

$$\begin{aligned} z_j z_k &= q z_k z_j, j < k, \\ z_j^* z_k &= q z_k z_j^*, j \neq k, \\ z_j^* z_j &= q^2 z_j z_j^* + (1 - q^2)(1 - \sum_{k>j} z_k z_k^*). \end{aligned}$$

This $*$-algebra is a q-analog of the $*$-algebra of polynomials on $\mathbb{C}^n$. It was first introduced by Pusz and Woronowicz in [9] but in terms of slightly different generators (see a remark in [15, section 3]). Its irreducible representations are well-known, see [9]. If $n = 1$ we have the following list of representations up to unitary equivalence:

1. the Fock representation ρ_F on $\ell^2(\mathbb{Z}_+)$: $\rho_F(z_1)e_n = \sqrt{1 - q^{2n+2}} e_{n+1}$,
2. one-dimensional representations ρ_φ, $\varphi \in [0, 2\pi)$: $\rho_\varphi(z_1) = e^{i\varphi}$.

The $*$-algebra $\mathbb{C}[SU_2]_q$ of regular functions on the quantum SU_2 is given by its generators t_{ij}, $i, j = 1, 2$, satisfying the relations:

$$\begin{aligned} & t_{11}t_{21} = qt_{21}t_{11},\ t_{11}t_{12} = qt_{12}t_{11},\ t_{12}t_{21} = t_{21}t_{12}, \\ & t_{22}t_{21} = q^{-1}t_{21}t_{11},\ t_{22}t_{12} = q^{-1}t_{12}t_{22}, \\ & t_{11}t_{22} - t_{22}t_{11} = (q - q^{-1})t_{12}t_{21},\ t_{11}t_{22} - qt_{12}t_{21} = 1, \\ & t_{11}^* = t_{22}, t_{12}^* = -qt_{21}. \end{aligned} \tag{1}$$

By [6] any irreducible representation of $\mathbb{C}[SU_2]_q$ is unitarily equivalent to one of the following:

1. one-dimensional representations:
$$\xi_\varphi(t_{11}) = e^{i\varphi}, \xi_\varphi(t_{21}) = 0, \varphi \in [0, 2\pi). \tag{2}$$

2. infinite-dimensional representations π_φ, $\varphi \in [0, 2\pi)$, on $\ell^2(\mathbb{Z}_+)$:
$$\begin{aligned} &\pi_\varphi(t_{11})e_0 = 0, \pi_\varphi(t_{11})e_k = (1-q^{2k})^{1/2}e_{k-1}, k \geq 1, \\ &\pi_\varphi(t_{21})e_k = q^k e^{i\varphi}e_k, \\ &\pi_\varphi(t_{22})e_k = (1-q^{2(k+1)})^{1/2}e_{k+1}, \\ &\pi_\varphi(t_{12})e_k = -q^{k+1}e^{-i\varphi}e_k. \end{aligned} \tag{3}$$

3 The $*$-algebra $Pol(Mat_2)_q$ and $*$-representations

The $*$-algebra $Pol(Mat_2)_q$, a q-analog of polynomials on the space Mat_2 of complex 2×2 matrices, introduced in [11], is given by its generators $\{z_a^\alpha\}_{a=1,2;\alpha=1,2}$ and the following commutation relations:
$$\begin{array}{rclrcl} z_1^1 z_2^1 &=& q z_2^1 z_1^1, & z_2^1 z_1^2 &=& z_1^2 z_2^1, \\ z_1^1 z_1^2 &=& q z_1^2 z_1^1, & z_2^1 z_2^2 &=& q z_2^2 z_2^1, \\ z_1^1 z_2^2 - z_2^2 z_1^1 &=& (q-q^{-1}) z_1^2 z_2^1, & z_1^2 z_2^2 &=& q z_2^2 z_1^2, \end{array} \tag{4}$$

$$\begin{array}{rcl} (z_1^1)^* z_1^1 &=& q^2 z_1^1 (z_1^1)^* - (1-q^2)(z_2^1(z_2^1)^* + z_1^2(z_1^2)^*) + \\ && +q^{-2}(1-q^2)^2 z_2^2 (z_2^2)^* + 1 - q^2, \\ (z_2^1)^* z_2^1 &=& q^2 z_2^1 (z_2^1)^* - (1-q^2) z_2^2 (z_2^2)^* + 1 - q^2, \\ (z_1^2)^* z_1^2 &=& q^2 z_1^2 (z_1^2)^* - (1-q^2) z_2^2 (z_2^2)^* + 1 - q^2, \\ (z_2^2)^* z_2^2 &=& q^2 z_2^2 (z_2^2)^* + 1 - q^2, \end{array} \tag{5}$$

$$\begin{array}{rclrcl} (z_1^1)^* z_2^1 - q z_2^1 (z_1^1)^* &=& (q-q^{-1}) z_2^2 (z_1^2)^*, & (z_2^2)^* z_2^1 &=& q z_2^1 (z_2^2)^*, \\ (z_1^1)^* z_1^2 - q z_1^2 (z_1^1)^* &=& (q-q^{-1}) z_2^2 (z_2^1)^*, & (z_2^2)^* z_1^2 &=& q z_1^2 (z_2^2)^*, \\ (z_1^1)^* z_2^2 &=& z_2^2 (z_1^1)^*, & (z_2^1)^* z_1^2 &=& z_1^2 (z_2^1)^*. \end{array} \tag{6}$$

The irreducible representations of $Pol(Mat_2)_q$ were classified in [13]. Next theorem presents them in a different form that is convenient for our purpose.

Let $C,\ S,\ d(q)\colon \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be operators defined as follows
$$Se_n = e_{n+1},\ Ce_n = (1-q^{2n})^{\frac{1}{2}}e_n,\ d(q)e_n = q^n e_n.$$

Theorem 1 *Any irreducible bounded representation of $Pol(Mat_2)_q$ is unitarily equivalent to one of the following non-equivalent representations:*

1. *the Fock representation acting in $\ell^2(\mathbb{Z}_+)^{\otimes 4}$:*
$$\begin{aligned} &\pi_F(z_2^2) = CS \otimes 1 \otimes 1 \otimes 1, \\ &\pi_F(z_2^1) = d(q) \otimes CS \otimes 1 \otimes 1, \\ &\pi_F(z_1^2) = d(q) \otimes 1 \otimes CS \otimes 1, \\ &\pi_F(z_1^1) = 1 \otimes d(q) \otimes d(q) \otimes CS - q^{-1}S^*C \otimes CS \otimes CS \otimes 1; \end{aligned}$$

2. *representations* τ_φ, $\varphi \in [0, 2\pi)$, *acting in* $\ell^2(\mathbb{Z}_+)^{\otimes 3}$*:*

$$\begin{aligned}
\tau_\varphi(z_2^2) &= CS \otimes 1 \otimes 1,\\
\tau_\varphi(z_2^1) &= d(q) \otimes CS \otimes 1,\\
\tau_\varphi(z_1^2) &= d(q) \otimes 1 \otimes CS,\\
\tau_\varphi(z_1^1) &= e^{i\varphi} 1 \otimes d(q) \otimes d(q) - q^{-1} S^* C \otimes CS \otimes CS;
\end{aligned}$$

3. *representations* $\nu_{1,\varphi}$ *and* $\nu_{2,\varphi}$ *acting in* $\ell^2(\mathbb{Z}_+)^{\otimes 2}$*:*

3a)

$$\begin{aligned}
\nu_{1,\varphi}(z_2^2) &= CS \otimes 1,\\
\nu_{1,\varphi}(z_2^1) &= e^{i\varphi} d(q) \otimes 1,\\
\nu_{1,\varphi}(z_1^2) &= d(q) \otimes CS,\\
\nu_{1,\varphi}(z_1^1) &= -e^{i\varphi} q^{-1} S^* C \otimes CS,
\end{aligned}$$

3b)

$$\begin{aligned}
\nu_{2,\varphi}(z_2^2) &= CS \otimes 1,\\
\nu_{2,\varphi}(z_2^1) &= d(q) \otimes CS,\\
\nu_{2,\varphi}(z_1^2) &= e^{i\varphi} d(q) \otimes 1,\\
\nu_{2,\varphi}(z_1^1) &= -e^{i\varphi} q^{-1} S^* C \otimes CS;
\end{aligned}$$

4. *representations* $\rho_{\varphi_1,\varphi_2}$, $\varphi_1, \varphi_2 \in [0, 2\pi)$, *acting in* $\ell^2(\mathbb{Z}_+)$*:*

$$\begin{aligned}
\rho_{\varphi_1,\varphi_2}(z_2^2) &= CS,\\
\rho_{\varphi_1,\varphi_2}(z_2^1) &= e^{i\varphi_1} d(q),\\
\rho_{\varphi_1,\varphi_2}(z_1^2) &= e^{i\varphi_2} d(q),\\
\rho_{\varphi,\varphi_2}(z_1^1) &= -e^{i(\varphi_1+\varphi_2)} q^{-1} S^* C;
\end{aligned}$$

5. *representations* θ_φ, $\varphi \in [0, 2\pi)$, *acting in* $l_2(\mathbb{Z}_+)$*:*

$$\theta_\varphi(z_2^2) = e^{i\varphi},\ \theta_\varphi(z_2^1) = \theta_\varphi(z_1^2) = 0,\ \theta_\varphi(z_1^1) = q^{-1} CS;$$

6. *one-dimensional representations* $\gamma_{\varphi_1,\varphi_2}$, *where* $\varphi_1, \varphi_2 \in [0, 2\pi)$

$$\gamma_{\varphi_1,\varphi_2}(z_2^2) = e^{i\varphi_1},\ \gamma(z_2^1) = \gamma(z_1^2) = 0,\ \gamma_{\varphi_1,\varphi_2}(z_1^1) = e^{i\varphi_2} q^{-1}.$$

It can be easily seen from the above list of representations that the $*$-algebra $Pol(Mat_2)_q$ is $*$-bounded (see [7]), i.e., there exist constants $C(a)$, $a \in Pol(Mat_2)_q$, such that $||\pi(a)|| \le C(a)$ for any bounded $*$-representation π. Let $C(\mathrm{Mat}_2)_q$ denote the universal enveloping C^*-algebra of $Pol(Mat_2)_q$. The following theorem was proved first in [8] and in general case for $Pol(Mat_n)_q$ in [10].

Theorem 2 *Given an irreducible representation, π, of $C(Mat_2)_q$, let $\mathcal{A}_\pi$ be the C^*-algebra generated by operators of the representation π. Then there exists a homomorphism δ_π from the C^*-algebra $\mathcal{A}_{\pi_F}$ to the C^*-algebra $\mathcal{A}_\pi$ such that*

$$\delta_\pi(\pi_F(z_i^j)) = \pi(z_i^j), \quad i,\ j = 1, 2.$$

Consequently, the Fock representation π_F of $C(Mat_2)_q$ is faithful and $C(Mat_2)_q \simeq \mathcal{A}_{\pi_F}$.

In what follows we will use another description of irreducible representations. For this we need the following $*$-homomorphisms whose existence was indicated in [3] without a proof.

Lemma 1 *The map*

$$\mathcal{D} : z_j^i \mapsto \sum_{a,b=1}^{2} z_b^a \otimes t_{bj} \otimes t_{ai}, i, j = 1, 2,$$

is uniquely extendable to a $*$-homomorphism

$$\mathcal{D} : Pol(Mat_2)_q \to Pol(Mat_2)_q \otimes \mathbb{C}[SU_2]_q \otimes \mathbb{C}[SU_2]_q.$$

Proof. Consider the Hopf algebra $\mathbb{C}[SL_4]_q$ generated by $\{t_{ij}\}_{i,j=1}^4$ and the commutation relations

$$\begin{aligned} t_{ij}t_{kl} - qt_{kl}t_{ij} = 0, &\quad i = k \ \&\ j < l \text{ or } i < k \ \&\ j = l, \\ t_{ij}t_{kl} - t_{kl}t_{ij} = 0 &\quad i < k \ \&\ j > l, \\ t_{ij}t_{kl} - t_{kl}t_{ij} = (q - q^{-1})t_{il}t_{kj} &\quad i < k \ \&\ j < l, \\ \det_q T := \sum_{s \in S_4} (-q)^{l(s)} t_{1s(1)}t_{2s(2)}t_{3s(3)}t_{4s(4)} = 1, & \end{aligned}$$

with $l(s) = \mathrm{card}\{(i,j) : i < j \& s(i) > s(j)\}$. The comultiplication Δ is given by

$$\Delta(t_{ij}) = \sum_{k=1}^{2} t_{ik} \otimes t_{kj}.$$

Let $t = t_{\{1,2\}\{3,4\}} := t_{13}t_{24} - qt_{24}t_{23}$. Consider the map

$$\mathcal{I} : z_a^\alpha \mapsto t^{-1} t_{\{1,2\}J_{a\alpha}}.$$

where $J_{a\alpha} = \{3,4\} \setminus \{5-\alpha\} \cup \{a\}$ and $t_{IJ} := t_{i_1j_1}t_{i_2j_2} - qt_{i_1,j_2}t_{i_2j_1}$ with $I = \{i_1, i_2\}$, $J = \{j_1, j_2\}$. By [12, Proposition 6.10], it determines a homomorphism from $Pol(Mat_2)_q$ to a localisation of $Pol(\tilde{X})_q := (\mathbb{C}[SL_4]_q, *)$ (see [12, (6.6)] for the involution $*$ on $\mathbb{C}[SL_4]_q$) with respect to a multiplicative system tt^*, $(tt^*)^2, \ldots$.

Consider now the two-sided ideal $J \subset \mathbb{C}[SL_4]_q$ generated by t_{kl} with $k \le 2$ and $l > 2$ or $k > 2$ and $l \le 2$, and the canonical onto morphism

$$j : \mathbb{C}[SL_4]_q \to \mathbb{C}[SL_4]_q/J.$$

Let $\mathbb{C}[S(U_2 \times U_2)]_q = (\mathbb{C}[SL_4]_q/J, \star)$, an involutive algebra, where

$$t_{ij}^\star = (-q)^{j-i} \det_q T_{ij}$$

and T_{ij} is derived from $(t_{ij})_{i,j=1}^4$ by deleting its i-th row and j-th column and $\det_q T$ is the quantum determinant of T (see [12] for the definition). By [12, Lemma 9.3] the composition $\tilde{\Delta} = (\mathrm{id} \otimes j)\Delta$ is a homomorphsim of $*$-algebras $\tilde{\Delta} : Pol(\tilde{X})_q \to Pol(\tilde{X})_q \otimes \mathbb{C}[S(U_2 \times U_2)]_q$. The $*$-homomorphism can be naturally extended to the localization of $Pol(\tilde{X})_q$ which we shall also denote by $\tilde{\Delta}$.

Let I be the two-sided ideal of $\mathbb{C}[S(U_2 \times U_2)]_q$ generated by $t_{11}t_{22} - qt_{12}t_{21} - 1$ and $t_{33}t_{44} - qt_{34}t_{43} - 1$ (we note that $1 = \det_q(t_{ij}) = (t_{11}t_{22} - qt_{12}t_{21})(t_{33}t_{44} - qt_{34}t_{43})$ in $\mathbb{C}[S(U_2 \times U_2)]_q$). Then $\mathbb{C}[SU_2]_q \otimes \mathbb{C}[SU_2]_q = \mathbb{C}[S(U_2 \times U_2)]_q/I$.

Let

$$i : \mathbb{C}[S(U_2 \times U_2)]_q \mapsto \mathbb{C}[SU_2]_q \otimes \mathbb{C}[SU_2]_q$$

be the canonical onto homomorphism.

Then $(\mathrm{id}\otimes i)\circ\tilde{\Delta}\circ\mathcal{I}$ is a $*$-homomorphism from $Pol((Mat_2)_q$ to $Pol(Mat_2)_q \otimes \mathbb{C}[SU_2]_q \otimes \mathbb{C}[SU_2]_q$. To prove the lemma it is enough to see now that $\mathcal{D} = (\mathrm{id} \otimes i) \circ \tilde{\Delta} \circ \mathcal{I}$.

Using relations in $[SL_4]_q/J$ we obtain

$$\begin{aligned}
\tilde{\Delta}(t) &= \tilde{\Delta}(t_{13}t_{24} - qt_{14}t_{23}) = (\sum_{k=3}^4 t_{1k} \otimes t_{k3})(\sum_{i=3}^4 t_{2i} \otimes t_{i4}) - q(\sum_{k=3}^4 t_{1k} \otimes t_{k4})(\sum_{i=3}^4 t_{2i} \otimes t_{i3}) \\
&= \sum_{k,i=3}^4 t_{1k}t_{2i} \otimes (t_{k3}t_{i4} - qt_{k4}t_{i3}) = t_{13}t_{24} \otimes (t_{33}t_{44} - qt_{34}t_{43}) \\
&+ t_{14}t_{23} \otimes (t_{43}t_{34} - qt_{44}t_{33}) \\
&= t_{13}t_{24} \otimes (t_{33}t_{44} - qt_{34}t_{43}) - q^{-1}t_{14}t_{23} \otimes (t_{44}t_{33} - q^{-1}t_{43}t_{34}) \\
&= (t_{13}t_{24} - q^{-1}t_{14}t_{23}) \otimes (t_{33}t_{44} - qt_{34}t_{43}) = t \otimes (t_{33}t_{44} - qt_{34}t_{43})
\end{aligned}$$

and hence $(\mathrm{id} \otimes i) \circ \tilde{\Delta}(t) = t \otimes 1 \otimes 1$.

Similarly,

$$\begin{aligned}
\tilde{\Delta}(t_{\{12\}J_{11}}) &= \tilde{\Delta}(t_{11}t_{23} - qt_{13}t_{21}) = (\sum_{k=1}^2 t_{1k} \otimes t_{k1})(\sum_{i=3}^4 t_{2i} \otimes t_{i3}) \\
&- q(\sum_{k=3}^4 t_{1k} \otimes t_{k3})(\sum_{i=1}^2 t_{2i} \otimes t_{i1}) \\
&= \sum_{k=1}^2 \sum_{i=3}^4 t_{1k}t_{2i} \otimes t_{k1}t_{i3} - q \sum_{i=3}^4 \sum_{k=1}^2 t_{1i}t_{2k} \otimes t_{i3}t_{k1} \\
&= \sum_{k=1}^2 \sum_{i=3}^4 (t_{1k}t_{2i} - qt_{1i}t_{2k}) \otimes t_{k1}t_{i3}
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{id} \otimes i) \circ \tilde{\Delta}(t^{-1}t_{\{12\}J_{11}}) &= \sum_{k=1}^2 \sum_{i=3}^4 (t^{-1}(t_{1k}t_{2i} - qt_{1i}t_{2k})) \otimes t_{k1} \otimes t_{(i-2)1} \\
&= \sum_{k=1}^2 \sum_{i=3}^4 \mathcal{I}(z_k^{i-2}) \otimes t_{k1} \otimes t_{(i-2)1}
\end{aligned}$$

giving $(\mathrm{id}\otimes i)\circ\tilde{\Delta}\circ\mathcal{I}(z_1^1)=\mathcal{D}(z_1^1)$. Similarly one checks that $(\mathrm{id}\otimes i)\circ\tilde{\Delta}\circ\mathcal{I}(z_i^j)=\mathcal{D}(z_i^j)$ for other generators z_i^j.

■

Consider now a mapping $\Pi_\varphi : Pol(Mat_2)_q \to Pol(\mathbb{C})_q$ given on the generators by

$$\begin{pmatrix} \Pi_\varphi(z_1^1) & \Pi_\varphi(z_1^2) \\ \Pi_\varphi(z_2^1) & \Pi_\varphi(z_2^2) \end{pmatrix} \to \begin{pmatrix} q^{-1}z & 0 \\ 0 & e^{i\varphi} \end{pmatrix}.$$

It is straight forward to check that Π_φ is a $*$-homomorphism.

Clearly, if ρ is a $*$-representation of $Pol(\mathbb{C})_q$, τ is a $*$-representation of $Pol(Mat_2)_q$, and π_1, π_2 are representation of $\mathbb{C}[SU_2]_q$ then $\rho\circ\Pi_\varphi$ and $(\tau\otimes\pi_1\otimes\pi_2)\circ\mathcal{D}$ are $*$-representations of $Pol(Mat_2)_q$.

Let ρ_F be the Fock representation of $Pol(\mathbb{C})_q$, ρ_φ, $\varphi\in[0,2\pi)$, be the one-dimensional representations of $Pol(\mathbb{C})_q$ and π_φ, $\varphi\in[0,2\pi)$ be the infinite-dimensional representation of $\mathbb{C}[SU_2]_q$ given by (3). Consider the following families of $*$-representations of $Pol(Mat_2)_q$:

$$\mathcal{F}_\varphi = \rho_F\circ\Pi_\varphi, \quad \chi_{\varphi_1,\varphi_2} = \rho_{\varphi_1}\circ\Pi_{\varphi_2}$$

and

$$(\mathcal{F}_\varphi\otimes\pi_0\otimes\pi_0)\circ\mathcal{D}, \quad (\chi_{\varphi_1,\varphi_2}\otimes\pi_0\otimes\pi_0)\circ\mathcal{D},$$

where $\varphi,\varphi_1,\varphi_2\in[0,2\pi)$.

We have

$$\begin{aligned}(\mathcal{F}_\varphi\otimes\pi_0\otimes\pi_0)\circ\mathcal{D}(z_1^1) &= (\mathcal{F}_\varphi\otimes\pi_0\otimes\pi_0)(\sum z_b^a\otimes t_{b1}\otimes t_{a1})\\ &= q^{-1}\rho_F(z)\otimes\pi_0(t_{11})\otimes\pi_0(t_{11}) + e^{i\varphi}\otimes\pi_0(t_{21})\otimes\pi_0(t_{21})\\ &= q^{-1}CS\otimes S^*C\otimes S^*C + e^{i\varphi}\otimes d(q)\otimes d(q),\end{aligned}$$

$$\begin{aligned}(\mathcal{F}_\varphi\otimes\pi_0\otimes\pi_0)\circ\mathcal{D}(z_2^2) &= (\mathcal{F}_\varphi\otimes\pi_0\otimes\pi_0)(\sum z_b^a\otimes t_{b2}\otimes t_{a2})\\ &= q^{-1}\rho_F(z)\otimes\pi_0(t_{12})\otimes\pi_0(t_{12}) + e^{i\varphi}\otimes\pi_0(t_{22})\otimes\pi_0(t_{22})\\ &= qCS\otimes d(q)\otimes d(q) + e^{i\varphi}\otimes CS\otimes CS,\end{aligned}$$

$$\begin{aligned}(\mathcal{F}_\varphi\otimes\pi_0\otimes\pi_0)\circ\mathcal{D}(z_1^2) &= (\mathcal{F}_\varphi\otimes\pi_0\otimes\pi_0)(\sum z_b^a\otimes t_{b1}\otimes t_{a2})\\ &= q^{-1}\rho_F(z)\otimes\pi_0(t_{11})\otimes\pi_0(t_{12}) + e^{i\varphi}\otimes\pi_0(t_{21})\otimes\pi_0(t_{22})\\ &= -CS\otimes S^*C\otimes d(q) + e^{i\varphi}\otimes d(q)\otimes CS,\end{aligned}$$

$$\begin{aligned}(\mathcal{F}_\varphi\otimes\pi_0\otimes\pi_0)\circ\mathcal{D}(z_2^1) &= (\mathcal{F}_\varphi\otimes\pi_0\otimes\pi_0)(\sum z_b^a\otimes t_{b2}\otimes t_{a1})\\ &= q^{-1}\rho_F(z)\otimes\pi_0(t_{12})\otimes\pi_0(t_{11}) + e^{i\varphi}\otimes\pi_0(t_{22})\otimes\pi_0(t_{21})\\ &= -CS\otimes d(q)\otimes S^*C + e^{i\varphi}\otimes CS\otimes d(q),\end{aligned}$$

and

$$\begin{aligned}(\chi_{\varphi_1,\varphi_2}\otimes\pi_0\otimes\pi_0)\circ\mathcal{D}(z_1^1) &= q^{-1}e^{i\varphi_1}S^*C\otimes S^*C + e^{i\varphi_2}d(q)\otimes d(q),\\ (\chi_{\varphi_1,\varphi_2}\otimes\pi_0\otimes\pi_0)\circ\mathcal{D}(z_2^2) &= qe^{i\varphi_1}d(q)\otimes d(q) + e^{i\varphi_2}CS\otimes CS,\\ (\chi_{\varphi_1,\varphi_2}\otimes\pi_0\otimes\pi_0)\circ\mathcal{D}(z_1^2) &= -e^{i\varphi_1}S^*C\otimes d(q) + e^{i\varphi_2}d(q)\otimes CS,\\ (\chi_{\varphi_1,\varphi_2}\otimes\pi_0\otimes\pi_0)\circ\mathcal{D}(z_2^1) &= -e^{i\varphi_1}d(q)\otimes S^*C + e^{i\varphi_2}CS\otimes d(q).\end{aligned}$$

Lemma 2 *The $*$-representation $(\mathcal{F}_\varphi \otimes \pi_0 \otimes \pi_0) \circ \mathcal{D}$, $\varphi \in [0, 2\pi)$, is irreducible and unitarily equivalent to τ_φ.*

Proof. Fix $\varphi \in [0, 2\pi)$. Let $Z_i^j = \tau_\varphi(z_i^j)$ and $W_i^j = (\mathcal{F}_\varphi \otimes \pi_0 \otimes \pi_0) \circ \mathcal{D}(z_i^j)$, $i, j = 1, 2$. The operators act on $\ell^2(\mathbb{Z}_+)^{\otimes 3}$. Let $\Omega = e_0 \otimes e_0 \otimes e_0$. It can be easily verified that Ω is cyclic for both of the families $\{Z_i^j,\ (Z_i^j)^*,\ i, j = 1, 2\}$, $\{W_i^j,\ (W_i^j)^*,\ i, j = 1, 2\}$, and

$$(Z_2^2)^*\Omega = (W_2^2)^*\Omega = 0, \quad (Z_2^1)^*\Omega = (W_2^1)^*\Omega = 0,$$
$$(Z_1^2)^*\Omega = (W_1^2)^*\Omega = 0, \quad (Z_1^1)^*\Omega = (W_1^1)^*\Omega = e^{-i\phi}\Omega.$$

Hence both τ_φ and $(\mathcal{F}_\varphi \otimes \pi_0 \otimes \pi_0) \circ \mathcal{D}$ determine so-called **coherent representations** of the Wick algebra corresponding to $Pol(Mat_2)_q$, with the same coherent state (see [4] for definition and properties of coherent representation of $*$-algebra allowing Wick ordering). Since coherent representation of Wick algebra is unique, up to the unitary equivalence, and irreducible (see [4, Proposiiton 1.3.3]), we have the required statement.

■

4 Shilov boundary

Let E_1 and E_2 be subspaces of C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. We denote by $M_n(E_i)$ be the space of all $n \times n$ matrices with entries in E_i. We equip $M_n(E_i)$ with norms induced from the C^*-algebras $M_n(\mathcal{A}_i)$. Note that the norms are independent of the embeddings of E_i into a C^*-algebra. Let $T : E_1 \to E_2$ be a linear operator. Denote by $T^{(n)}$ the mapping from $M_n(E_1)$ to $M_n(E_2)$ defined by

$$T^{(n)}((a_{ij})_{i,j}) = (T(a_{ij})_{i,j}), (a_{ij})_{i,j} \in M_n(E_1).$$

T is called contractive if $||T|| \leq 1$ and completely contractive if $||T^{(n)}|| \leq 1$ for any $n \geq 1$. T is called an isometry if $||T(a)||_{E_2} = ||a||_{E_1}$, and is a complete isometry if $T^{(n)}$ is an isometry for any $n \geq 1$.

Let A be a linear subspace of a C^*-algebra B such that A contains the identity of B and generates B as a C^*-algebra. The following definition was given by Arveson [1].

Definition 1 *A closed two-sided ideal J in B is called a boundary ideal for A if the canonical quotient map $q : B \to B/J$ is completely isometric on A. A boundary ideal is called the Shilov boundary for A if it contains every other boundary ideal.*

Note that the Shilov boundary exists and unique, [1, 5]. Shilov boundary ideal is a non-commutative analog of Shilov boundary of a compact Hausdorff space X relative to a subspace $\mathcal{A}$ of the space $C(X)$ of continuous functions on X which is by definition the smallest closed subset K of X such that every function in $\mathcal{A}$ achieves its maximum modulus on K.

Let us give some examples of Shilov boundary and Shilov boundary ideals.

Example 1
- If $\mathbb{D} = \{z \in \mathbb{C}^n : |z| \leq 1\}$ is the unit disk. It is known that any holomorphic function on $\mathbb{D}$ attains its maximum on the unit disk $\mathbb{U} = \{z \in \mathbb{C}^n :$

$|z| = 1\}$ and moreover it is the smallest closed set with this property and hence $\mathbb{U}$ is the Shilov boundary of $\mathbb{D}$ with respect to the set of holomorphic functions $A(\mathbb{D})$. The ideal $J = \{f \in C(\mathbb{D}) : f|_{\mathbb{U}} = 0\}$ is the Shilov ideal of the C^*-algebra $C(\mathbb{D})$ with respect to $A(\mathbb{D})$.

- A q-analog of $C(\mathbb{D})$ is the universal enveloping C^*-algebra of $Pol(Mat_{n,1})_q$. It was proved by L.Vaksman, [15] that a closed two-sided ideal generated by $\sum_{j=1}^{n} z_j z_j^* - 1$ is the Shilov boundary ideal for the closed unital algebra generated by z_i, $i = 1, \ldots, n$, which is a q-analog of the algebra of holomorphic functions on $\mathbb{D}$.

In $C(Mat_n)_q$ consider a closed two-sided ideal J generated by

$$\sum_{j=1}^{n} q^{2n-\alpha-\beta} z_j^\alpha (z_j^\beta)^* - \delta^{\alpha\beta}, \alpha, \beta = 1, \ldots n,$$

where $\delta^{\alpha\beta}$ is the Kronecker symbol. The ideal J is a $*$-ideal, i.e. $J = J^*$. The quotient algebra $\mathbb{C}(S(\mathbb{D}))_q := C(Mat_n)_q/J$ is a $U_q su_{n,n}$-module $*$-algebra called the algebra of continuous functions on the Shilov boundary of a quantum matrix ball. The canonical homomorphism

$$j_q : C(Mat_n)_q \to C(S(\mathbb{D}))_q$$

is a q-analog of the restriction operator which maps a continuous functions on the disk $\mathbb{D} = \{z \in Mat_n : zz^* \leq 1\}$ to its restriction to the Shilov boundary $S(\mathbb{D}) = \{z \in Mat_n : zz^* = 1\}$.

In this section we show that for $n = 2$, the ideal J is the Shilov boundary ideal for the (non-involutive) closed subalgebra $A(Mat_2)_q$ of $C(Mat_2)_q$ generated by z_i^j, $i, j = 1, 2$. Our approach, similarly to [15], is based on Sz.-Nagy's and Foyas' dilation theory.

Theorem 3 (Sz.-Nagy's dilation theorem). *Let $T \in B(H)$ with $||T|| \leq 1$. Then there exists a Hilbert space K containing H as a subspace and a unitary U on K with the property that*

$$T^n = P_H U^n|_H \text{ for all nonnegative integers } n.$$

Lemma 3 *The only irreducible representations that annihilate the ideal J are $\rho_{\varphi_1,\varphi_2}$ and $\gamma_{\varphi_1,\varphi_2}$, φ_1, $\varphi_2 \in [0, 2\pi)$.*

Proof. A straightforward verification. ∎

Lemma 4 *Given a representation π of $Pol(Mat_2)_q$ that annihilates the ideal J and $a \in Pol(Mat_2)_q$, $||\pi(a)|| \leq \sup_{\psi_1,\psi_2} ||\rho_{\psi_1,\psi_2}(a)||$.*

Proof. We start by noting that the operators C, S, $d(q)$: $\ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ defined in Section 3 satisfy the equalities

$$C^2 = (1 - q^2) \sum_{n=0}^{\infty} q^{2n} S^{n+1} (S^{n+1})^*, \quad d(q) = \sum_{n=0}^{\infty} q^n \left(S^n (S^n)^* - S^{n+1} (S^{n+1})^* \right) \qquad (7)$$

and hence the C^*-algebra, $C^*(S)$, generated by S coincides with the one generated by S, C and $d(q)$), and the mapping $S \mapsto e^{i\varphi}$ can be naturally extended to $*$-homomorphism

$$\Theta_\varphi \colon C^*(S) \to \mathbb{C}, \quad \Theta_\varphi(S) = e^{i\varphi}, \quad \Theta_\varphi(C) = 1, \quad \Theta_\varphi(d(q)) = 0.$$

Recall that

$$\begin{array}{rcl} \rho_{\varphi_1,\varphi_2}(z_2^2) & = & CS, \\ \rho_{\varphi_1,\varphi_2}(z_2^1) & = & e^{i\varphi_1} d(q), \\ \rho_{\varphi_1,\varphi_2}(z_1^2) & = & e^{i\varphi_2} d(q), \\ \rho_{\varphi,\varphi_2}(z_1^1) & = & -e^{i(\varphi_1+\varphi_2)} q^{-1} S^* C, \end{array} \tag{8}$$

and

$$\gamma_{\varphi_1,\varphi_2}(z_2^2) = e^{i\varphi_1},\ \gamma_{\varphi_1,\varphi_2}(z_2^1) = \gamma_{\varphi_1,\varphi_2}(z_1^2) = 0,\ \gamma_{\varphi_1,\varphi_2}(z_1^1) = e^{i\varphi_2} q^{-1}.$$

For a representation π of $Pol(Mat_2)_q$) we let $\mathcal{A}_\pi$ denote the unital C^*-algebra generated by $\pi(z_i^j)$, $i, j = 1, 2$. Then $\mathcal{A}_{\rho_{\varphi_1,\varphi_2}} = C^*(S)$. In fact it follows from (8) and (7) that $\mathcal{A}_{\rho_{\varphi_1,\varphi_2}} \subset C^*(S)$. To see the other inclusion we note that 0 is an isolated point in the spectrum $\sigma(C)$ of C, and hence the function f given by $f(0) = 0$ and $f(t) = t^{-1}$, $t \in \sigma(C)$, $t \neq 0$, is continuous on $\sigma(C)$. Therefore, since $T := \rho_{\varphi_1,\varphi_2}(z_2^2) = CS$ one has $C = ((1 - q^{-2})I + q^{-2}T^*T)^{1/2} \in \mathcal{A}_{\rho_{\varphi_1,\varphi_2}}$ and $S = f(C)T \in \mathcal{A}_{\rho_{\varphi_1,\varphi_2}}$ implying $C^*(S) \subset \mathcal{A}_{\rho_{\varphi_1,\varphi_2}}$.

Evidently, $\mathcal{B}_{\gamma_{\varphi_1,\varphi_2}} = \mathbb{C}$. The homomorphism Θ_{φ_1} gives rise to a homomorphism between $\mathcal{A}_{\rho_{\varphi_1,\pi+\varphi_2}}$ and $\mathcal{B}_{\varphi_1,\varphi_2}$:

$$\Theta_{\varphi_1}(\rho_{\varphi_1,\pi+\varphi_2}(z_i^j)) = \gamma_{\varphi_1,\varphi_2}(z_i^j), \quad i, j = 1, 2$$

proving that

$$|\gamma_{\varphi_1,\varphi_2}(a)| = |\Theta_{\varphi_1}(\rho_{\varphi_1,\pi+\varphi_2}(a))| \leq \|\rho_{\varphi_1,\pi+\varphi_2}(a)\| \leq \sup_{\psi_1,\psi_2} ||\rho_{\psi_1,\psi_2}(a)||, \quad i = 1, 2.$$

■

Lemma 5 *The ideal J is a boundary ideal, i.e. the restriction $j_{A(Mat_2)_q}$ of j_q to $A(Mat_2)_q$ is a complete isometry.*

Proof. Since j_q is a $*$-homomorphism between C^*-algebras, j_q and hence $j_{A(Mat_2)_q}$ is a complete contraction. Therefore it is enough to prove that for $a_{ij} \in A(Mat_2)_q$, we have

$$||(\pi_F(a_{ij}))_{i,j}||_{M_n(C(Mat_2)_q)} \leq ||j_q^{(n)}((\pi_F(a_{ij}))||_{M_n(C(S(\mathbb{D}))_q)}.$$

Since by Lemma 3 the only representations of $C(Mat_2)_q$ that annihilate the ideal J are $\rho_{\varphi_1,\varphi_2}$ and $\gamma_{\varphi_1,\varphi_2}$, $\varphi_i \in [0, 2\pi)$, and

$$|\gamma_{\varphi_1,\varphi_2}(a)| \leq \sup_{\psi_1,\psi_2} ||\rho_{\psi_1,\psi_2}(a)||$$

we have

$$||b + J||_{C(S(\mathbb{D}))_q} = \sup_{\psi_1,\psi_2} ||\rho_{\psi_1,\psi_2}(b)||.$$

Therefore, we must show that

$$||(\pi_F(a_{ij}))_{i,j}||_{M_n(C(Mat_2)_q)} \le \sup_{\psi_1,\psi_2} ||(\rho_{\psi_1,\psi_2}(a_{ij}))||_{M_n(B(\ell^2(\mathbb{Z}_+)\otimes\ell^2(\mathbb{Z}_+)))}$$

for all $(a_{ij}) \in M_n(A(Mat_2)_q)$. We will do this in two steps.

Step 1. It follows from the definition of operators C and S that $T = CS$ is a contraction on $H = \ell^2(\mathbb{Z}_+)$. By Sz.-Nagy dilation theorem there exists a unitary operator U on a Hilbert space K with $K \supset H$ such that $(CS)^n = P_H U^n|_H$ for any $n = 1, 2, \ldots$. Consider a mapping $\Psi : \{z_i^j, i, j = 1, 2\} \to B(H^{\otimes 4})$ given by

$$\Psi(z_i^j) = \pi_F(z_i^j), (i,j) \ne (1,1), \text{ and } \Psi(z_1^1) = 1 \otimes d(q) \otimes d(q) \otimes U - q^{-1} S^* C \otimes CS \otimes CS \otimes 1.$$

Then Ψ extends uniquely to a homomorphism of $A(Mat_2)_q$ and

$$\pi_F(a) = (1_{H^{\otimes 3}} \otimes P_H)\Psi(a)|_{H^{\otimes 3}\otimes H},\ a \in A(Mat_2)_q.$$

Moreover, it is easy to see that Ψ has an extension to a $*$-representation of $Pol(Mat_2)_q$ whose irreducible subrepresentations are unitarily equivalent to τ_φ, $\varphi \in [0, 2\pi)$. Therefore

$$||\pi_F(a)|| \le ||\Psi(a)|| \le \sup_{\varphi\in[0,2\pi)} ||\tau_\varphi(a)||, a \in A(Mat_2)_q.$$

Similarly,

$$||(\pi_F(a_{ij}))||_{M_n(B(H^{\otimes 4}))} \le \sup_{\varphi\in[0,2\pi)} ||(\tau_\varphi(a_{ij}))||_{M_n(B(H^{\otimes 3}))}, (a_{ij}) \in M_n(A(Mat_2)_q).$$

Step 2. Our next goal is to prove that for any $\varphi \in [0, 2\pi)$

$$||\tau_\varphi(a)|| \le \sup_{\varphi_1,\varphi_2} ||\rho_{\varphi_1,\varphi_2}(a)||, a \in A(Mat_2)_q.$$

It is a routine to verify that the representations $(\chi_{\varphi_1,\varphi_2} \otimes \pi_0 \otimes \pi_0) \circ \mathcal{D}$ annihilate the ideal J for any $\varphi_1, \varphi_2 \in [0, 2\pi)$. In particular this fact implies

$$\sup_{\varphi_1,\varphi_2} ||(\chi_{\varphi_1,\varphi_2} \otimes \pi_0 \otimes \pi_0) \circ \mathcal{D}(z_i^j)|| \le \sup_{\varphi_1,\varphi_2} ||\rho_{\varphi_1,\varphi_2}(z_i^j)||,\ i, j = 1, 2.$$

So, it will be enough for us to show that for any $\varphi \in [0, 2\pi)$ and $i, j = 1, 2$

$$||\tau_\varphi(z_i^j)|| \le \sup_{\varphi_1,\varphi_2} ||(\chi_{\varphi_1,\varphi_2} \otimes \pi_0 \otimes \pi_0) \circ \mathcal{D}(z_i^j)||.$$

Indeed, as in the first step let U be a unitary operator on a Hilbert space K with $K \supset H$, $H = \ell^2(\mathbb{Z}_+)$, such that

$$(CS)^n = P_H U^n|_H.$$

Then the mapping Ψ defined on the generators z_i^j, $i, j = 1, 2$, that replaces CS by U in the first component in the expressions for $\tau_\varphi(z_i^j) = \mathcal{F}_\varphi \otimes \pi_0 \otimes \pi_0(\mathcal{D}(z_i^j))$ can be extended to a $*$-representation Ψ of $Pol(Mat_2)_q$ whose all irreducible subrepresentations are unitarily equivalent to $(\chi_{\varphi_1,\varphi} \otimes \pi_0 \otimes \pi_0) \circ \mathcal{D}$, and, moreover,

$$\mathcal{F}_\varphi \otimes \pi_0 \otimes \pi_0(\mathcal{D}(a)) = (P_H \otimes 1_{H^{\otimes 2}})\Psi(a)|_{H\otimes H^{\otimes 2}}, a \in A(Mat_2)_q$$

Hence for $a \in A(Mat_2)_q$

$$\begin{aligned} ||\tau_\varphi(a)|| &= ||(\mathcal{F}_\varphi \otimes \pi_0 \otimes \pi_0)(\mathcal{D})(a))|| \leq ||\Psi(a)|| \\ &\leq \sup_{\varphi_1 \in [0,2\pi)} ||(\chi_{\varphi_1,\varphi} \otimes \pi_0 \otimes \pi_0)(\mathcal{D}(a))|| \\ &\leq \sup_{\varphi_1,\varphi_2} ||\rho_{\varphi_1,\varphi_2}(a)||, \end{aligned}$$

the last inequality is due to Lemma 4. Using similar arguments one gets that

$$||\tau_\varphi^{(n)}((a_{ij}))||_{M_n(B(H^{\otimes 3}))} \leq \sup_{\varphi_1,\varphi_2} ||(\rho_{\varphi_1,\varphi_2}(a_{ij}))||_{M_n(B(H^{\otimes 2}))}, (a_{ij}) \in M_n(A(Mat_2)_q).$$

Combining the results from Step 1 and Step 2 we obtain

$$||\pi_F^{(n)}((a_{ij}))||_{M_n(B(H^{\otimes 4}))} \leq \sup_{\varphi_1,\varphi_2} ||(\rho_{\varphi_1,\varphi_2}(a_{ij}))||_{M_n(B(H^{\otimes 2}))} \text{ for all } (a_{ij}) \in M_n(A(Mat_2)_q),$$

giving the statement of the theorem.

■

Remark 1 We have proved that for any $a \in A(Mat_2)_q$, $\pi_F(a) = P_H\psi(a)|_H$, where ψ is a $*$-representation of $Pol(Mat_2)_q$ that annihilates the ideal J.

Theorem 4 *The ideal J is the Shilov boundary ideal for the subalgebra $A(Mat_2)_q$.*

Proof. Assume that I is a boundary ideal for $A(Mat_2)_q$ with $I \supset J$. We have, in particular, that the quotient maps $j_q : C(Mat_2)_q \to C(Mat_2)_q/J = C(S(\mathbb{D}))_q$ and $i_q : C(Mat_2)_q \to C(Mat_2)_q/I = (C(Mat_2)_q/J)/(I/J) = C(S(\mathbb{D}))_q/(I/J)$ are isometries when restricted to $A(Mat_2)_q$. Therefore for $a \in A(Mat_2)_q$ we have

$$\|a + J\| = \|a\| = \|(a + J) + I/J\|$$

and hence the quotient map $k_q : C(S(\mathbb{D}))_q \to C(S(\mathbb{D}))_q/(I/J)$ is an isometry when restricted to $A(Mat_2)_q + J$. In particular, $0 \neq ||z_i^j|| = ||z_i^j + J|| = ||(z_i^j + J) + I/J||$, $i \neq j$.

If T is an irreducible representation of $C(S(\mathbb{D}))_q/(I/J)$ such that $T((z_i^j + J) + I/J) \neq 0$ then the representation $T \circ k_q$ is an irreducible representation of $C(Mat_2)_q/J$ which does not vanish on $z_i^j + J$, $i \neq j$, and $T \circ k_q(I/J) = 0$. The only irreducible representations of $C(S(\mathbb{D}))_q$ that do not vanish on $z_i^j + J$, $i \neq j$, are $\tilde{\rho}_{\varphi_1,\varphi_2}(a + J) := \rho_{\varphi_1,\varphi_2}(a)$. Therefore, $T \circ k_q$ is unitarily equivalent to one of $\tilde{\rho}_{\varphi_1,\varphi_2}$ and hence $T \circ k_q(I/J) = 0$ implies $I/J \subset \ker \tilde{\rho}_{\varphi_1,\varphi_2}$. Let

$$K = \{(\varphi_1, \varphi_2) \in [0, 2\pi) \times [0, 2\pi) : \rho_{\varphi_1,\varphi_2}(I) = 0\} \text{ and } X_K = \{(e^{i\varphi_1}, e^{i\varphi_2}) : (\varphi_1, \varphi_2) \in K\}.$$

We want to see that K is dense in $[0, 2\pi) \times [0, 2\pi)$.

In $C(S(\mathbb{D}))_q$ consider the subalgebra generated by $z_i^j + J$, $i \neq j$. It is easily seen that the algebra is commutative and that the elements $z_i^j + J$, $i \neq j$ are normal in $C(S(\mathbb{D}))_q$, i.e. they commute with their adjoints. This follows from the fact that the operators $\rho_{\varphi_1,\varphi_2}(z_i^j)$, $i \neq j$ commute and are normal for any $\varphi_1, \varphi_2 \in [0, 2\pi)$. The joint spectrum of $\{\tilde{\rho}_{\varphi_1,\varphi_2}(z_1^2 + J), \tilde{\rho}_{\varphi_1,\varphi_2}(z_2^1 + J)\}$ is $\{(e^{i\varphi_1}q^k, e^{i\varphi_2}q^k) : k = 0, 1, \ldots\} \cup \{(0, 0)\}$. If T is an irreducible representation of $C(S(\mathbb{D}))_q/(I/J)$ then it follows from the description of the representations $\rho_{\varphi_1,\varphi_2}$ that if $(e^{i\varphi_1}, e^{i\varphi_2})$ is in the joint spectrum of $\{T\circ k_q(z_1^2+J), T\circ k_q(z_2^1+J)\}$ then $T\circ k_q$ is unitarily equivalent to $\tilde\rho_{\varphi_1,\varphi_2}$ and hence $(\varphi_1,\varphi_2)\in K$.

Now, given a holomorphic function on $\mathbb{D}^2=\{(\xi_1,\xi_2)\in\mathbb{C}^2 : |\xi_1|<1, |\xi_2|<1\}$ which is also continuous on $\overline{\mathbb{D}^2}$ we have

$$\|f(z_1^2+J, z_2^1+J)\| = \|f((z_1^2+J)+I/J, (z_2^1+J)+I/J\|.$$

As

$$\begin{aligned}\|f(z_1^2+J, z_2^1+J)\| &= \sup_{(\varphi_1,\varphi_2)\in[0,2\pi)^2}\|f(\rho_{\varphi_1,\varphi_2}(z_1^2), \rho_{\varphi_1,\varphi_2}(z_2^1))\| \\ &= \sup\{|f(\xi_1,\xi_2)| : (\xi_1,\xi_2)\in\cup_{k\geq 0}q^k\mathbb{T}^2\} = \sup_{(\xi_1,\xi_2)\in\mathbb{T}^2}|f(\xi_1,\xi_2)| \\ &= \sup_{(\xi_1,\xi_2)\in\mathbb{D}^2}|f(\xi_1,\xi_2)|,\end{aligned}$$

(here $\mathbb{T}^2=\{(\xi_1,\xi_2)\in\mathbb{C}^2 : |\xi_1|=|\xi_2|=1\}$ and the last two equalities follows from the maximum principle), and

$$\begin{aligned}&\|f((z_1^2+J)+I/J, (z_2^1+J)+I/J)\| \\ = \;& \sup\{f(T\circ k_q(z_1^2+J), T\circ k_q(z_2^1+J)) : T\in \mathrm{Irrep}(C(S(\mathbb{D}))_q/(I/J))\} \\ = \;& \max\{\sup_{(\varphi_1,\varphi_2)\in K}\|f(\rho_{\varphi_1,\varphi_2}(z_1^2), \rho_{\varphi_1,\varphi_2}(z_2^1))\|, f(0,0)\} \\ = \;& \sup\{|f(\xi_1,\xi_2)| : (\xi_1,\xi_2)\in\cup_{k\geq 0}q^kX_K\}\end{aligned}$$

(Irrep(A) denote the set of all irreducible representations of A), we obtain

$$\sup\{|f(\xi_1,\xi_2)| : (\xi_1,\xi_2)\in\mathbb{D}^2\} = \sup\{|f(\xi_1,\xi_2)| : (\xi_1,\xi_2)\in\cup_{k\geq 0}q^kX_K\}.$$

Hence $\overline{\cup_{k\geq 0}q^kX_K}$ contains the Shilov boundary of $\mathbb{D}^2$ which is $\mathbb{T}^2$. Therefore $\mathbb{T}^2\supset\overline{X}_K\supset\mathbb{T}^2$ giving that K is dense in $[0,2\pi)\times[0,2\pi)$.

This implies

$$I/J\subset\cap_{(\varphi_1,\varphi_2)\in K}\ker\tilde\rho_{\varphi_1,\varphi_2}=\{0\}$$

and $I=J$. ■

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