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# Spectral correspondences for Maass waveforms on quaternion groups

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We prove that in most cases the Jacquet–Langlands correspondence between newforms for Hecke congruence groups and newforms for quaternion orders is a bijection. Our proof covers almost all cases where the Hecke congruence group is of cocompact type, i.e. when a bijection is possible. The proof uses the Selberg trace formula.



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## 1. Introduction

The spectral theory of automorphic Laplacians is approached either from an adelic-representation theoretic perspective or from the classical perspective of the upper half-plane $\mathcal{H}$ [32]. Within the classical context one can view this theory either through a geometric or an arithmetic lens. Seen through the geometric lens, it is natural to begin with a consideration of the spectral resolution of the Laplacian on $L^2(Y)$, where $Y$ is a compact surface endowed with a Riemannian metric of constant negative curvature. From an arithmetic perspective, it is natural to start with the modular group $\mathrm{SL}_2(\mathbb{Z})$ and its automorphic Laplacian. In this case, the corresponding modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ is non-compact and the spectral resolution has an absolutely continuous part in addition to the discrete one that characterizes the compact case [19]. In certain situations it is possible to relate the spectral resolutions of automorphic Laplacians in the compact case to the non-compact case. These correspondences are well-known. They are predicted by the Jacquet–Langlands correspondence [18], which establishes, among other things, that to any nonconstant eigenfunction of the Laplacian on a cocompact arithmetic Fuchsian group there corresponds a nontrivial cuspform with the same eigenvalue on some non-cocompact but cofinite arithmetic Fuchsian group [14]. This fact, first discovered, independently, by M. Eichler and A. Selberg in the 1950s, was first proved in 1970, using the language of representation theory, by Jacquet and Langlands [18] and re-proved in 1972 by Hideo Shimizu [28] using the language of adelic trace formulas.

In this paper, we prove a stronger and more explicit version of this correspondence between the compact case and the non-compact case. The main result is a one-to-one correspondence between Maass newforms on $\Gamma_0(N)\backslash\mathcal{H}$ and Maass newforms on $\mathcal{O}^1\backslash\mathcal{H}$, where $\mathcal{O}^1$ is the unit group of a certain quaternion order with discriminant $N$. We prove this for any $N$ with at least two prime divisors of odd order. The main tool used is the Selberg trace formula.

The history of the method used in this paper goes back to [14], where Hejhal, using an approach based on unpublished work of A. Selberg from the 1950s [27], illustrated how a part of the correspondence between spaces of automorphic forms for cocompact and non-cocompact Fuchsian groups could be established using completely classical techniques. He showed that a certain integral transform, $\Theta$, mapped Maass waveforms on a Fuchsian group of quaternion type to Maass forms of equal eigenvalue on an appropriate congruence subgroup $\Gamma_0(d)$ [31].

Hejhal's work was extended by Bolte and Johansson[2] in [6] and [5]. In [6] they worked out the details of the spectral correspondence when the cocompact arithmetic Fuchsian group is given by a unit group in an arbitrary order in an indefinite rational division quaternion algebra. In so doing they showed that Hejhal's constructions could be extended to arbitrary orders. They also improved the result concerning the level of the

[2] Johansson is the same person as the second author of this paper.

congruence group by illustrating that the natural correspondence is between the discriminant of the order and the level of the congruence group.

To be precise, let $\mathcal{O}$ be an order in an indefinite rational division quaternion algebra. The group of units of norm one, $\mathcal{O}^1$, can be considered as a cocompact Fuchsian group of the first kind. Bolte and Johansson [6] showed that Maass waveforms for $\mathcal{O}^1$, i.e. eigenfunctions of the automorphic Laplacian associated with $\mathcal{O}^1$, can be lifted to Maass cusp forms for the Hecke congruence group $\Gamma_0(d)$, where $d$ is the (reduced) discriminant of the order $\mathcal{O}$ thus establishing that theta-lifts preserve eigenvalues of the hyperbolic Laplacian. It is these theta-lifts that are considered also in this paper. Left unaddressed was the question as to whether or not theta-lifts provided isomorphisms between the respective Laplace eigenspaces.

In [5] they continued to address this question by concentrating on maximal orders in indefinite rational division quaternion algebras. We note that in this case $d$ is necessarily the product of an even number of different primes. By exploiting several versions of the (classical) Selberg trace formula [26] they showed that the Laplace eigenvalues and their multiplicities for the cocompact group $\mathcal{O}^1$ and those for the newforms of level $d$ coincide. This, however, still did not imply that theta-lifts were isomorphisms between Laplace eigenspaces. Strömbergsson [31], in his doctoral thesis, studied this question independently and proved that $\Theta$ was indeed a bijection between the respective eigenspaces.

In [25] Risager investigated the asymptotic behavior of the counting function of Laplace eigenvalues of Maass newforms of a given level $M$. To be precise: denote by $\Delta_\Gamma$ the automorphic Laplacian related to $\Gamma$ and by $N_\Gamma(\lambda)$ the corresponding spectral counting function. We recall that $N_\Gamma(\lambda)$ is defined as follows:

$$N_\Gamma(\lambda) = \#\{\lambda_n \leq \lambda : \lambda_n \in \mathrm{dSpec}(\Delta_\Gamma)\}, \tag{1.1}$$

where $\mathrm{dSpec}(\Delta_\Gamma)$ denotes the discrete spectrum of $\Delta_\Gamma$. Let $A_\Gamma$ be the hyperbolic area of $\Gamma\backslash\mathcal{H}$. Since $\mathcal{O}^1$ is cocompact $N_{\mathcal{O}^1}(\lambda)$ has an asymptotic expansion of the form [12]:

$$N_{\mathcal{O}^1}(\lambda) = \frac{A_{\mathcal{O}^1}}{4\pi}\lambda + O\left(\frac{\sqrt{\lambda}}{\log\lambda}\right) \tag{1.2}$$

and for congruence subgroups $\Gamma_0(d)$, $N_{\Gamma_0(d)}(\lambda)$ has an asymptotic expansion of the form [17]:

$$N_{\Gamma_0(d)}(\lambda) = \frac{A_{\Gamma_0(d)}}{4\pi}\lambda + O\left(\sqrt{\lambda}\log\lambda\right). \tag{1.3}$$

Risager defined a counting function, $N^{new}_{\Gamma_0(d)}(\lambda)$, which counts only the newforms when $d$ is the product of an even number of different primes [25]. He found that

$$N^{new}_{\Gamma_0(d)}(\lambda) = C_d\frac{A_{\Gamma_0(d)}}{4\pi}\lambda + O\left(\frac{\sqrt{\lambda}}{\log\lambda}\right) \tag{1.4}$$

for some constant $0 < C_d \leq 1$. In other words, the asymptotic expansion of $N^{new}_{\Gamma_0(d)}(\lambda)$ is of a form characteristic of that of the cocompact case! We say that counting functions for newforms possessing this type of asymptotic character are of *cocompact type*, i.e. $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type if it is of the form $C_M\lambda + O(\sqrt{\lambda}/\log\lambda)$. Risager then asked: Are there values of $M$ not equal to the product of an even number of different primes for which $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type? In [25] he identified the values of $M$ for which $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type:

**Theorem 1.1** *(Risager). Let $M \in \mathbb{N}$ and let $n, t \in \mathbb{N}$ be the positive integers defined uniquely by the requirements that $n$ should be squarefree and $M = t^2 n$. Then $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type if and only if $n, t$ satisfies one of the following:*

(1) *$n$ contains at least two primes.*
(2) *$n$ is a prime and $4||M$.*

Evidently, there are a number of cases where $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type and $M$ is not a product of an even number of different primes. The following question thus naturally arises: Suppose that $N^{new}_{\Gamma_0(M)}(\lambda)$ is of cocompact type. Does this imply the existence of a cocompact group $\mathcal{O}^1$ such that $N_{\mathcal{O}^1}(\lambda)$ coincides with $N^{new}_{\Gamma_0(M)}(\lambda)$? In other words, are there spectral correspondences responsible for the remaining cases of Theorem 1.1?

In this paper, building upon the work of Bolte and Johansson [5,6], Strömbergsson [31] and Risager [25] we provide a classical description of the correspondences anticipated above. We show that whenever $n$ contains at least two primes there exists a quaternion group $\mathcal{O}^1$ such that the positive Laplace eigenvalues, including multiplicities, for Maass newforms on $\mathcal{O}^1$ coincide with the Laplace spectrum of Maass newforms for the Hecke congruence group $\Gamma_0(M)$. Specifically we prove:

**Theorem 1.2.** *Assume that $r$ is a positive integer that is divisible by an even number of primes, and that every prime dividing $r$ does so to an odd power. Let $u$ be any positive integer relatively prime to $r$. Then the positive Laplace eigenvalues, including multiplicities, for Maass newforms on $\mathcal{O}^1_{r,u}\backslash\mathcal{H}$ and $\Gamma_0(ru)\backslash\mathcal{H}$ coincide.*

The precise definition of the quaternion group $\mathcal{O}^1_{r,u}$ follows in the next section. This result gives a completely affirmative answer to Risager's question for case (1) in Theorem 1.1 filling in gaps that have not been proved before [3]. We will return to case (2) in the near future.

We prove this result by comparing the geometric side of the Selberg trace formulas for the respective groups $\mathcal{O}^1$ and $\Gamma_0(M)$. The key component of this comparison is the local embedding numbers [9,24,23]. It is the agreement of these local factors which is the main ingredient in the proof of the correspondence. The argument can be seen as a non-holomorphic analogue of part of the work by Hijikata, Pizer and Shemanske on the basis problem for holomorphic modular forms [16,24].

Prior to establishing the main result, we present the necessary background on quaternion orders, Maass waveforms, newforms and the Selberg trace formula.

## 2. Quaternion orders

A general reference for this section is [34]. See also [8] for more details on the different classes of orders.

Let $\mathcal{A}$ be a quaternion algebra over $\mathbb{Q}$. We will assume that $\mathcal{A}$ is *indefinite*, i.e. unramified at infinity so that

$$\mathcal{A}_\infty = \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}).$$

The opposite of this is that $\mathcal{A}$ is *definite*. This means that $\mathcal{A}_\infty \cong \mathbb{H}$, the unique real division algebra defined by

$$\mathbb{H} = \langle 1, i, j, k \rangle_{\mathbb{R}}\,, \text{ where } i^2 = j^2 = -1,\ k = ij \text{ and } ji = -ij.$$

We denote the norm and the trace by $N : \mathcal{A} \longrightarrow \mathbb{Q}$ and $Tr : \mathcal{A} \longrightarrow \mathbb{Q}$. For any representation of $\mathcal{A}$ in $M_2(\mathbb{R})$ this is just the determinant and trace of the matrix.

For any prime $p$ let $\mathbb{Q}_p$ be the $p$-adic numbers. For each $p$ we get a completion $\mathcal{A}_p = \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. The situation for the completion at a finite place $p$ is analogous to the infinite case. In the finite case $\mathcal{A}_p$ is either isomorphic to $M_2(\mathbb{Q}_p)$ (unramified) or a unique division algebra $\mathbb{H}_p$ (ramified). This algebra can be represented in matrix form as

$$\mathbb{H}_p \cong \langle I, x_1, x_2, x_3 \rangle_{\mathbb{Q}_p}\,, \text{ with } x_1 = \begin{pmatrix} 1 & \sqrt{\epsilon} \\ -\sqrt{\epsilon} & 0 \end{pmatrix},\ x_2 = \begin{pmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix},\ x_3 = x_1 x_2,$$

where $\epsilon \in {\mathbb{Z}_p}^*$ and $1 - 4\epsilon \in {\mathbb{Z}_p}^* \setminus ({\mathbb{Z}_p}^*)^2$ [8].

The algebra $\mathcal{A}$ is unramified at almost all places and, since it is unramified at infinity, it is ramified at an even number of finite places. Conversely, given an even number ($\geq 0$) of primes, it is always possible to find, up to isomorphism, a unique quaternion algebra that ramifies at exactly these primes. The (reduced) *discriminant*, $d(\mathcal{A})$, of $\mathcal{A}$ is the product of all the ramified primes. In the special case when $\mathcal{A}$ has no ramified primes at all, the discriminant of $\mathcal{A}$ is equal to 1 and $\mathcal{A} \cong M_2(\mathbb{Q})$.

An *order* $\mathcal{O}$ in $\mathcal{A}$ is a $\mathbb{Z}$-module such that $\mathcal{O}$ is a ring and $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathcal{A}$. An order is *maximal* if it is not contained in a larger order. In contrast to the case of number fields, the maximal order is not unique. There is always infinitely many maximal orders. For example for $\mathcal{A} = M_2(\mathbb{Q})$, the maximal orders are all the conjugates $\alpha M_2(\mathbb{Z})\alpha^{-1}$ including for example

$$\left\{ \begin{pmatrix} a & b/r \\ cr & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\},\ \text{for any } r \in \mathbb{Q} \setminus \{0\}.$$

If $\mathcal{A}$ is indefinite over $\mathbb{Q}$, then this is true in general so all maximal orders are conjugated.

For general orders the situation is more complicated, and the best way to analyze them is to look at completions $\mathcal{O}_p = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where $\mathbb{Z}_p$ are the $p$-adic integers. The (reduced) discriminant, $d(\mathcal{O})$, of an order $\mathcal{O}$ is defined as follows: If $\{e_1, e_2, e_3, e_4\}$ is any $\mathbb{Z}$-basis of $\mathcal{O}$ then

$$d(\mathcal{O}) = \sqrt{|det\,[\mathrm{Tr}\,(e_i e_j)]|}. \tag{2.1}$$

The (reduced) discriminant is always an integer whose value is independent of the choices of $e_1, e_2, e_3, e_4$ and $d(\mathcal{A})$ divides $d(\mathcal{O})$. For any maximal order, $d(\mathcal{O}) = d(\mathcal{A})$ and for any suborder $\mathcal{O}_1 \subseteq \mathcal{O}$, $d(\mathcal{O}_1) = d(\mathcal{O})[\mathcal{O} : \mathcal{O}_1]$.

We define for any order $\mathcal{O}$ the unit quotient group $\mathcal{O}^1$ as follows:

$$\mathcal{O}^1 = \{\gamma \in \mathcal{O} : N(\gamma) = 1\} / \{\pm I\}.$$

The group $\mathcal{O}^1$ will embed as a discrete subgroup in $PSL_2(\mathbb{R})$. The group $PSL_2(\mathbb{R})$ acts by Möbius transformations on the complex upper half plane $\mathcal{H}$. With our restrictions on $\mathcal{A}$, the quotient $\mathcal{O}^1 \backslash \mathcal{H}$ is an orbifold, that with suitable handling of possible elliptic points could be given the structure of a Riemann surface. The surface $\mathcal{O}^1 \backslash \mathcal{H}$ is compact if and only if $\mathcal{A} \ncong M_2(\mathbb{Q})$.

In order to define the orders that we will use for the correspondence, we need some information about the local theory of orders. We will use two different classes of local orders. The first is the well known class of Eichler orders. Any Eichler order $E^{(1)}_{p^n}$ satisfy

$$E^{(1)}_{p^n} \cong \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$$

for some $n \geq 1$ and have $d(E^{(1)}_{p^n}) = p^n$. Obviously these only occur in $\mathcal{A}_p \cong M_2(\mathbb{Q}_p)$.

The other class of orders, $E^{(-1)}_{p^n}$, occur both in $M_2(\mathbb{Q}_p)$ and $\mathbb{H}_p$. In $M_2(\mathbb{Q}_p)$ we have

$$E^{(-1)}_{p^{2n}} \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : a \equiv b + d \pmod{p^n},\ c \equiv b\epsilon \pmod{p^n} \right\}, \tag{2.2}$$

where $\epsilon \in {\mathbb{Z}_p}^*$ and $1 + 4\epsilon \in {\mathbb{Z}_p}^* \setminus ({\mathbb{Z}_p}^*)^2$ [8]. (If $1 + 4\epsilon \in ({\mathbb{Z}_p}^*)^2$, then the order is isomorphic to $E^{(1)}_{p^{2n}}$.) We see that $[M_2(\mathbb{Z}_p) : E^{(-1)}_{p^{2n}}] = p^{2n}$ so $d(E^{(-1)}_{p^{2n}}) = p^{2n}$. In $\mathbb{H}_p$ a maximal order is

$$\mathcal{O}_p \cong \langle I, x_1, x_2, x_3 \rangle_{\mathbb{Z}_p}, \text{ with } x_1 = \begin{pmatrix} 1 & \sqrt{\epsilon} \\ -\sqrt{\epsilon} & 0 \end{pmatrix},\ x_2 = \begin{pmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix},\ x_3 = x_1 x_2,$$

where $\epsilon \in {\mathbb{Z}_p}^*$ and $1 - 4\epsilon \in {\mathbb{Z}_p}^* \setminus ({\mathbb{Z}_p}^*)^2$ and in this case

$$E^{(-1)}_{p^{2n+1}} \cong \langle I, x_1, p^n x_2, p^n x_3 \rangle_{\mathbb{Z}_p},$$

so $[\mathcal{O}_p : E_{p^{2n}}^{(-1)}] = p^{2n}$ and hence $d(E_{p^{2n+1}}^{(-1)}) = p^{2n+1}$ [8]. (If $1 - 4\epsilon \in (\mathbb{Z}_p{}^*)^2$, then the algebra is unramified and the order is isomorphic to $E_{p^{2n+1}}^{(1)}$.) Note in particular that $E_p^{(-1)} = \mathcal{O}_p$ is a maximal order in $\mathbb{H}_p$.

Given local orders $\mathcal{O}_p$ for all primes $p$ with $\mathcal{O}_p \cong M_2(\mathbb{Z}_p)$ for almost all $p$, then it is always possible to find a global order $\mathcal{O}$ such that $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_p$ for all primes $p$. We only need the existence, but for the reader unfamiliar with quaternion orders we give some simple examples. For Eichler orders it is obvious that

$$E_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

has $(E_N)_p \cong E_{p^n}^{(1)}$ for all primes $p$ with $p^n \| N$. It is a little more complicated when $\mathcal{O}_p \cong E_{p^n}^{(-1)}$ for at least one $p$, but it is possible to use the explicit descriptions of the local orders to find a global order. For example in the simple case with $d(\mathcal{O}) = 3^{2n}$ and $\mathcal{O}_3 \cong E_{3^{2n}}^{(-1)}$ we can choose $\epsilon = 1$ and get

$$\mathcal{O} \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : a \equiv b + d \pmod{3^n},\ c \equiv b \pmod{3^n} \right\}.$$

In particular, note that $\mathcal{O}^1$ is a congruence subgroup of level $3^n$, i.e. it contains $\Gamma(3^n)$, since the conditions for $\mathcal{O}$ are obviously fulfilled when $a \equiv d \equiv 1$ and $b \equiv c \equiv 0$. It is clear from (2.2), that in fact it will be true in general that the unit group of any order we consider in $M_2(\mathbb{Q})$ will be a congruence subgroup.

We will need two classes of global orders and we describe them as two subclasses of one class of orders. Let $r \in \mathbb{N}$ be such that all primes dividing $r$ do so to an odd power, and let $\mathcal{A}$ be the quaternion algebra ramified at exactly all primes dividing $r$. Since we assume that $\mathcal{A}$ is indefinite, the number of primes dividing $r$ is even. Now let $u \in \mathbb{N}$ be such that $\gcd(r, u) = 1$. We define $\mathcal{O}_{r,u}$ to be an order in $\mathcal{A}$ such that

$$(\mathcal{O}_{r,u})_p \cong \begin{cases} E_{p^{2n+1}}^{(-1)} & \text{if } p^{2n+1} \| r, \\ E_{p^n}^{(1)} & \text{if } p^n \| u, \\ M_2(\mathbb{Z}_p) & \text{if } p \nmid ru. \end{cases}$$

We will consider the two different natural subclasses of these orders for which $r = 1$ or $r > 1$. If $r = 1$, then $\mathcal{A} = M_2(\mathbb{Q})$ and $\mathcal{O}_{1,u} \cong E_u$ are the usual Eichler orders, so in particular $E_u^1 = \Gamma_0(u)/\{\pm I\}$. When $r > 1$, $\mathcal{A}$ is a division algebra with maximal order $\mathcal{O}_{r_1,1}$ where $r_1$ is the product of the distinct primes dividing $r = r_1 r_2^2$.

The main result of this paper relates Maass waveforms corresponding to $\mathcal{O}_{r,u}$ to those corresponding to $\mathcal{O}_{1,ru}$.

## 3. Maass waveforms

The spectral theory of hyperbolic surfaces has its origins in the efforts by Atle Selberg to use the techniques of harmonic analysis in the study of automorphic forms [15]. Its development was influenced, in part, by the work of Hans Maass [21] who studied non-analytic automorphic functions. We outline below the elements of the theory that are necessary for our purposes. A comprehensive reference for what follows is [17].

Let $\mathcal{A}$ be an indefinite rational quaternion algebra and $\mathcal{O}$ be any order in $\mathcal{A}$ such that $d(\mathcal{O}) = d$. Let $\mathcal{O}^1\backslash\mathcal{H}$ and $\Gamma_0(d)\backslash\mathcal{H}$ be the Riemann surfaces related to $\mathcal{O}^1$ and $\Gamma_0(d)$ respectively and let $L^2\left(\mathcal{O}^1\backslash\mathcal{H}\right)$ and $L^2\left(\Gamma_0(d)\backslash\mathcal{H}\right)$ be the corresponding Hilbert spaces of square integrable functions on $\mathcal{O}^1\backslash\mathcal{H}$ and $\Gamma_0(d)\backslash\mathcal{H}$ respectively. Further, let

$$\Delta_\Gamma = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{3.1}$$

be the Laplace–Beltrami operator (hyperbolic Laplacian) on the respective Hilbert spaces.

A (*cuspidal*) *Maass waveform* on $\Gamma\backslash\mathcal{H}$ is a function $f \in L^2\left(\Gamma\backslash\mathcal{H}\right)$ such that:

(i) $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
(ii) $f$ vanishes at the cusps of $\Gamma$, and
(iii) $\Delta_\Gamma f = \lambda f$ for some $\lambda > 0$.

Maass waveforms are real analytic eigenfunctions of the hyperbolic Laplacian and their eigenvalues (together with 0) constitute the discrete spectrum of $\Delta_\Gamma$. These functions give a basis for $L^2\left(\Gamma\backslash\mathcal{H}\right)$ on $\Gamma\backslash\mathcal{H}$. The spectrum of $\Delta_\Gamma$, denoted by $\operatorname{Spec}\left(\Delta_\Gamma\right)$, decomposes into discrete and continuous parts. We will identify these components by $\operatorname{dSpec}\left(\Delta_\Gamma\right)$ and $\operatorname{cSpec}\left(\Delta_\Gamma\right)$ respectively. It should be noted that solutions to the equation $\Delta_\Gamma f = \lambda f$ on hyperbolic surfaces have deep connections to physics. They are used to describe mathematical models of quantum chaos [4] and they also play a role in the study of cosmology [2].

The discrete spectrum, $\operatorname{dSpec}\left(\Delta_\Gamma\right)$, is infinite and spanned by Maass waveforms (and a constant function) both when $\Gamma = \Gamma_0(d)$ and $\Gamma = \mathcal{O}^1$. Each of the eigenvalues occur with finite multiplicity so we have an infinite list of discrete eigenvalues

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots,\ \lambda_n \to \infty. \tag{3.2}$$

If $\Gamma$ is a cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$ such as $\mathcal{O}^1$, then the Laplace operator $\Delta_\Gamma$ has only a discrete spectrum, i.e. $\operatorname{cSpec}\left(\Delta_{\mathcal{O}^1}\right)$ is empty. Hence the complete spectrum of the hyperbolic Laplacian $\Delta_{\mathcal{O}^1}$ is of the form (3.2).

If $\Gamma$ is a non-cocompact but cofinite subgroup of $\mathrm{PSL}_2(\mathbb{R})$ such as $\Gamma_0(d)$, then the Laplace operator $\Delta_\Gamma$ has both a continuous spectrum $\left[\frac{1}{4}, \infty\right)^c$ and a discrete spectrum contained in $[0, \infty)$ [27]. Here $c$ is the number of inequivalent cusps of $\Gamma\backslash\mathcal{H}$. If $\Gamma$ is a congruence group, e.g. $\Gamma = \Gamma_0(d)$, then Selberg proved in the 1950s that the discrete spectrum is infinite and of the form in (3.2) [33, Chapter 8]. For a generic group $\Gamma$ of this type it is conjectured that $\mathrm{dSpec}\,(\Delta_\Gamma) = \{0\}$, i.e. that there are no Maass waveforms at all [22]. However, for symmetry reasons there are non-congruence groups with infinitely many Maass waveforms and numerical computations strongly suggest that there are other groups with a positive finite number of Maass waveforms not explained by symmetry of the surface [10] (see also [7] where some of the eigenvalues in [10] are refined to high precision).

The spectral correspondence that we focus on is the correspondence between $\mathrm{dSpec}\left(\Delta_{\Gamma_0(d)}\right)$ and $\mathrm{dSpec}\,(\Delta_{\mathcal{O}^1})$ for orders $\mathcal{O} = \mathcal{O}_{r,u}$ defined in Section 2 with $d = ru$. As noted in the introduction, the spectral counting function

$$N_\Gamma(\lambda) = \#\{\lambda_n \leq \lambda : \lambda_n \in \mathrm{dSpec}(\Delta_\Gamma)\}, \tag{3.3}$$

for $\Gamma = \mathcal{O}^1$ and $\Gamma = \Gamma_0(d)$ satisfies

$$N_{\mathcal{O}^1}(\lambda) = \frac{A_{\mathcal{O}^1}}{4\pi}\lambda + O\left(\frac{\sqrt{\lambda}}{\log \lambda}\right),$$
$$N_{\Gamma_0(d)}(\lambda) = \frac{A_{\Gamma_0(d)}}{4\pi}\lambda + O\left(\sqrt{\lambda}\log \lambda\right).$$

We note again the difference in the error terms between the cocompact and non-cocompact case. However, we will see that when restricting to the space of newforms the main error terms for $N_{\Gamma_0(d)}(\lambda)$ will cancel and the error terms will be of the same magnitude as for $N_{\mathcal{O}^1}(\lambda)$. We will then prove that for all orders $\mathcal{O}_{r,u}$ there will actually be a bijection between the newforms spectra.

## 4. Newforms

The theory of newforms and oldforms was developed for holomorphic modular forms in [1], and the theory is completely analogous for Maass forms [30]. In this section we briefly recall the part we need, and also describe the corresponding theory for the quaternion orders we consider. We denote the space of Maass forms on $\Gamma\backslash\mathcal{H}$ by $\mathcal{M}_\Gamma$, and the subspace of forms with Laplace-eigenvalue $\lambda$ by $\mathcal{M}_\Gamma(\lambda)$.

### *4.1. Newforms for $\Gamma_0(d)$*

If $\Gamma_1$ is a subgroup of $\Gamma_2$, then obviously $\mathcal{M}_{\Gamma_2}$ is a subspace of $\mathcal{M}_{\Gamma_1}$. In particular $\mathcal{M}_{\Gamma_0(N)}$ is a subspace of $\mathcal{M}_{\Gamma_0(M)}$ when $N \mid M$. However in this case, if $f \in \mathcal{M}_{\Gamma_0(N)}(\lambda)$ then

$$g_a(z) = f(az) \in \mathcal{M}_{\Gamma_0(M)}(\lambda) \text{ for all } a\,|(M/N)\,.$$

In other words, every Maass form on $\Gamma_0(N)$ gives rise to $\tau(M/N)$ Maass forms on $\Gamma_0(M)$, where $\tau(d)$ is the number of divisors of $d$. It is easy to check that the $g_a$ are linearly independent. This construction of course works for all divisors of $M$, and the subspace of $\mathcal{M}_{\Gamma_0(M)}$ generated by these functions for all proper divisors of $M$ is called the space of *oldforms* and will be denoted by $\mathcal{M}^{old}_{\Gamma_0(M)}$. The complement to $\mathcal{M}^{old}_{\Gamma_0(M)}$ in $\mathcal{M}_{\Gamma_0(M)}$ is called the space of *newforms*, $\mathcal{M}^{new}_{\Gamma_0(M)}$.

For any $\lambda$ we get

$$\dim\left(\mathcal{M}_{\Gamma_0(M)}(\lambda)\right) = \sum_{d|M} \tau\left(\frac{M}{d}\right) \dim\left(\mathcal{M}^{new}_{\Gamma_0(d)}(\lambda)\right).$$

Inverting this formula, we get

$$\dim\left(\mathcal{M}^{new}_{\Gamma_0(M)}(\lambda)\right) = \sum_{d|M} \beta\left(\frac{M}{d}\right) \dim\left(\mathcal{M}_{\Gamma_0(d)}(\lambda)\right), \tag{4.1}$$

where $\beta$ is the inverse (with respect to convolution of arithmetic functions) of $\tau$ defined by [1, (6.7)]

$$\beta(n) = \sum_{d|n} \mu(d)\mu\left(\frac{n}{d}\right).$$

The function $\beta$ is multiplicative and for a prime $p$ it satisfies

$$\begin{aligned} \beta(p) &= -2, \\ \beta(p^2) &= 1, \\ \beta(p^k) &= 0, \text{ if } k > 2. \end{aligned}$$

### *4.2. Newforms for $\mathcal{O}^1_{r,u}$*

We will now describe the corresponding situation for $\mathcal{O}_{r,u}$.

**Proposition 1.** *Let $\mathcal{O}_{r,u}$ and $\mathcal{O}_{s,t}$ be orders in the algebra $\mathcal{A}$ with $d(\mathcal{A}) = r_1$, so $r = r_1 r_2^2$ and $s = r_1 s_2^2$ for some integers $r_2$ and $s_2$. Assume that $\mathcal{O}_{r,u} \subseteq \mathcal{O}_{s,t}$, which is equivalent to $s_2 \mid r_2$ and $t \mid u$. If $f \in \mathcal{M}_{\mathcal{O}^1_{s,t}}$, then*

$$g_a(z) = f(az) \in \mathcal{M}_{\mathcal{O}^1_{r,u}} \text{ for all } a \,|(u/t)\,.$$

**Proof.** We show that $g_a \in \mathcal{M}_{\mathcal{O}^1}$ where $\mathcal{O} = \mathcal{O}_{s,ta}$ from which the result is trivial since $\mathcal{O}_{r,u} \subseteq \mathcal{O}$. Eichler orders $E_N$ are characterized by being the intersection of two (conjugated) maximal orders

$$E_N \cong \sigma_N^{-1} M_2(\mathbb{Z}) \sigma_N \cap M_2(\mathbb{Z}), \text{ where } \sigma_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this (locally) we get

$$\mathcal{O} \cong \mathcal{O}_{s,t} \cap \sigma_a^{-1} \mathcal{O}_{s,t} \sigma_a.$$

In particular, if $\gamma \in \mathcal{O}^1$ then $\sigma_a \gamma \in \mathcal{O}_{s,t}\sigma_a$ so $\sigma_a \gamma = \gamma_1 \sigma_a$ for some $\gamma_1 \in \mathcal{O}^1_{s,t}$. We note that $\sigma_a \cdot z = az$, and hence for $f \in \mathcal{M}_{\mathcal{O}^1_{s,t}}$ and $\gamma \in \mathcal{O}^1$

$$g_a(\gamma z) = f(\sigma_a \gamma z) = f(\gamma_1 \sigma_a z) = f(\sigma_a z) = g_a(z)$$

so $g_a \in \mathcal{M}_{\mathcal{O}^1}$. $\square$

Analogous to the case of $\Gamma_0(N)$, we define the oldforms, $\mathcal{M}^{old}_{\mathcal{O}^1_{r,u}}$, in $\mathcal{M}_{\mathcal{O}^1_{r,u}}$ to be the space of all $g_a$ for all proper divisors $a$ of $u$, and the newforms, $\mathcal{M}^{old}_{\mathcal{O}^1_{r,u}}$, to be its complement.

Let $r = r_1 r_2^2$ and $s = r_1 s_2^2$ as in Proposition 1. For any $\lambda$ we get

$$\dim\left(\mathcal{M}_{\mathcal{O}^1_{r,u}}(\lambda)\right) = \sum_{s_2|r_2} \sum_{t|u} \tau\left(\frac{u}{t}\right) \dim\left(\mathcal{M}^{new}_{\mathcal{O}^1_{s,t}}(\lambda)\right).$$

Inverting this formula, we get

$$\dim\left(\mathcal{M}^{new}_{\mathcal{O}^1_{r,u}}(\lambda)\right) = \sum_{t|u} \beta\left(\frac{u}{t}\right) \sum_{s_2|r_2} \mu\left(\frac{r_2}{s_2}\right) \dim\left(\mathcal{M}_{\mathcal{O}^1_{s,t}}(\lambda)\right). \tag{4.2}$$

Remember that $\gcd(r,u) = 1$ so with $M = ru$ (4.1) can be rewritten as

$$\dim\left(\mathcal{M}^{new}_{\Gamma_0(ru)}(\lambda)\right) = \sum_{t|u} \beta\left(\frac{u}{t}\right) \sum_{s|r} \beta\left(\frac{r}{s}\right) \dim\left(\mathcal{M}_{\Gamma_0(st)}(\lambda)\right). \tag{4.3}$$

## 5. Selberg trace formula

In this section we introduce the Selberg trace formulas for the groups $\mathcal{O}^1$ and $\Gamma_0(d)$. These formulas are the main tools with which we will work in order to prove our main result. The Selberg trace formula establishes a quantitative connection between the Laplace spectrum, Spec $(\Delta_\Gamma)$, and the geometry of the Riemann surface $\Gamma\backslash\mathcal{H}$. It is a general identity connecting geometrical and spectral terms of form:

$$\sum \text{spectral terms} = \sum \text{geometric terms}. \tag{5.1}$$

The spectral terms come from the discrete and continuous spectra of the automorphic hyperbolic Laplacian $\Delta_\Gamma$, and the geometric terms contain the area of a fundamental domain, elliptic points and cusps and a sum over all closed geodesics. The closed geodesics are in one-to-one correspondence to the conjugacy classes of hyperbolic elements that show up in the explicit versions of the trace formula used here. The formula was developed by Selberg in order to establish the existence of infinitely many eigenvalues for subgroups of finite index of $\mathrm{SL}_2(\mathbb{Z})$. It is a non-abelian generalization of the Poisson summation formula [19] and one can find a comprehensive classical introduction to the trace formula in [12,13,17]. The trace formulas for both types of arithmetic Fuchsian groups under consideration are well-known. The result for $\mathcal{O}^1$ can be found in [12, ch. V, Thm. 8.1] and for $\Gamma_0(m)$ in [11].

We recall the known results: In what follows $h : \mathbb{C} \to \mathbb{C}$ always denotes a function satisfying,

(1) $h(r) = h(-r)$,
(2) $h(r)$ is holomorphic in the strip $|\Im(r)| \leq \frac{1}{2} + \varepsilon$, for some $\varepsilon > 0$,
(3) $|h(r)| \leq C(1 + \Re(r))^{-2-\delta}$ for some $C > 0$ and $\delta > 0$.

The Fourier transform of $h$ will then be written as

$$\hat{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(r)\, e^{-iru}\, dr.$$

For a group $\Gamma$ we define $E'(t, \Gamma)$ to be the number of conjugacy classes in $\Gamma$ of primitive elements with trace $t$. For $t > 2$ these are in one-to-one correspondence with closed (primitive) geodesics on $\Gamma\backslash\mathcal{H}$, for $t = 2$ they correspond to inequivalent cusps and for $0 \leq t < 2$ they correspond to elliptic points. For elliptic elements $(0 \leq t < 2)$ $m_t$ denotes the order of the primitive element with trace $t$. We also define $A_\Gamma$ to be the hyperbolic area of $\Gamma\backslash\mathcal{H}$. This is everything needed to formulate the trace formulas.

### *5.1. The Selberg trace formula for cocompact groups $\mathcal{O}^1$*

Since the unit group $\mathcal{O}^1$ is a cocompact Fuchsian group, the Selberg trace formula reads as follows with notation defined above:

**Proposition 5.1.** *Let $\lambda_k = r_k^2 + 1/4$ run through all eigenvalues of the hyperbolic Laplacian on $L^2(\mathcal{O}^1\backslash\mathcal{H})$, counted with multiplicities. Then*

$$\sum_{k=0}^{\infty} h(r_k) = \mathcal{I}_{\mathcal{O}^1} + \mathcal{E}_{\mathcal{O}^1} + \mathcal{H}_{\mathcal{O}^1} \tag{5.2}$$

*where*

$$\mathcal{I}_{\mathcal{O}^1} = \frac{A_{\mathcal{O}^1}}{4\pi} \int_{-\infty}^{+\infty} h(r)\, r \tanh(\pi r)\, dr, \tag{5.3}$$

$$\mathcal{E}_{\mathcal{O}^1} = \sum_{t\in\{0,1\}} \frac{E'(t,\mathcal{O}^1)}{2m_t} \sum_{k=1}^{m_t-1} \frac{1}{\sin(\frac{k\pi}{m_t})} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-\frac{2k\pi r}{m_t}}}{1+e^{-2\pi r}}\, dr, \tag{5.4}$$

$$\mathcal{H}_{\mathcal{O}^1} = \sum_{t=3}^{\infty} E'(t,\mathcal{O}^1) \operatorname{arccosh}(\tfrac{t}{2}) \sum_{k=1}^{\infty} \frac{\hat{h}\left(2k \operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k \operatorname{arccosh}(\frac{t}{2})\right)} \tag{5.5}$$

*is the identity, elliptic and hyperbolic contribution respectively.*

### 5.2. The Selberg trace formula for Hecke congruence groups $\Gamma_0(m)$

We recall [11, Thm. 9.9] together with [11, (10.2), (10.4)] and use the same notation as in Proposition 5.1:

**Proposition 5.2.** *Let $\mu_k = r_k^2 + 1/4$ run through all eigenvalues of the hyperbolic Laplacian on $L^2(\Gamma_0(m)\backslash\mathcal{H})$, counted with multiplicities. Then*

$$\sum_{k=0}^{\infty} h(r_k) = \mathcal{I}_{\Gamma_0(m)} + \mathcal{E}_{\Gamma_0(m)} + \mathcal{H}_{\Gamma_0(m)} + \mathcal{P}_{\Gamma_0(m)} \tag{5.6}$$

*with the identity, elliptic and hyperbolic contribution as in Proposition 5.1 and the parabolic contribution is*

$$\mathcal{P}_{\Gamma_0(m)} = \kappa\Bigg\{ \hat{h}(0)\log(\tfrac{\pi}{2}) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r) \left[\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir)\right] dr \tag{5.7}$$

$$+ 2\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n}\,\hat{h}(2\log n) - \sum_{\substack{p\mid m\\ p\ prime}} \sum_{k=0}^{\infty} \frac{\log p}{p^k}\,\hat{h}(2k\log p)\Bigg\}, \tag{5.8}$$

*where $\kappa$ is the number of cusps of $\Gamma_0(m)\backslash\mathcal{H}$.*

### 5.3. Comparing trace formulas

It is clear from Propositions 5.1 and 5.2 that in order to prove results about correspondences between discrete spectra for different groups one needs to compare areas $A_\Gamma$ and numbers $E'(t,\Gamma)$ for the different groups and also check that the parabolic contributions agree. The area terms are straightforward, but for the numbers $E'(t,\Gamma)$ we do a convenient standard reformulation of the sum in terms of so-called optimal embeddings.

Let $K$ be a quadratic field extension of $\mathbb{Q}$ and $S$ an order in $K$. Assume that there is an embedding $\iota : K \longrightarrow \mathcal{A}$. The order $S$ is said to be optimally embedded into $\mathcal{O}$

with respect to $\iota$ if $\iota(S) = \mathcal{O} \cap \iota(K)$. There is a one-to-one correspondence between conjugacy classes of elements in $\mathcal{O}^1$ and optimal embeddings of quadratic orders $S$ into the quaternion order $\mathcal{O}$, see [5] or [30] for details. Hence, one may rewrite the elliptic and hyperbolic contributions in terms of a sum over quadratic orders that can be optimally embedded into $\mathcal{O}$.

The computation of the number of optimal embeddings of $S$ into $\mathcal{O}$, $E'(S,\mathcal{O})$, involves local embedding numbers, $E(S,\mathcal{O})_p$, i.e. the number of optimal embeddings of $S_p$ into $\mathcal{O}_p$, and global factors which are essentially class numbers of $S$ and $\mathcal{O}$. The numbers $E(S,\mathcal{O})_p$ are known for any orders $S$ and $\mathcal{O} = \mathcal{O}_{r,u}$ and any prime $p$, and the global factors in our case only depend on $S$ and not on $\mathcal{O}_{r,u}$:

**Lemma 1.** *The number of optimal embeddings $E'(S,\mathcal{O})$ of a quadratic order $S$ in an order $\mathcal{O} = \mathcal{O}_{r,u}$ modulo conjugation by elements in $\mathcal{O}^1$ satisfy*

$$E'(S,\mathcal{O}) = c(S) \prod_p E(S,\mathcal{O})_p,$$

*where $E(S,\mathcal{O})_p$ is the number of optimal embeddings of $S_p$ into $\mathcal{O}_p$ modulo conjugation by $\mathcal{O}_p^*$ and $c(S)$ only depends on $S$ and not on $(r,u)$.*

**Proof.** This is [20, (3.3)] using the facts that the class number of $\mathcal{O}_{r,u}$ is always 1 and $\mathcal{O}_{r,u}$ always contains elements with norm equal to $-1$ [20]. □

The number $c(S)$ in Lemma 1 is actually the class number of $S$ divided by 2 or the class number itself depending on whether $S$ contains an element with norm equal to $-1$ or not. However, we do not need that but only that it is independent of the quaternion order $\mathcal{O}_{r,u}$.

Summing up we get that in order to show that the elliptic and/or hyperbolic contributions are equal, it is enough to show that the products of the local embedding numbers agree. For convenience we define

$$E(S,\mathcal{O}) = \prod_p E(S,\mathcal{O})_p,$$

so if one shows that $E(S,\mathcal{O}_1) = E(S,\mathcal{O}_2)$ for all quadratic orders $S$, then the elliptic and hyperbolic contributions in the trace formulas for $\mathcal{O}_1^1\backslash\mathcal{H}$ and $\mathcal{O}_2^1\backslash\mathcal{H}$ agree. This is exactly what will be done in the next two sections for the part of the spectrum corresponding to newforms for $\mathcal{O}_1^1 = \Gamma_0(ru)$ and $\mathcal{O}_2^1 = \mathcal{O}_{r,u}^1$.

## 6. Newforms sieve

In this section we describe how to sieve out the contribution from the newforms to the trace formulas and we show how this can be used to prove that the newforms on $\mathcal{O}_{r,u}^1\backslash\mathcal{H}$

and $\Gamma_0(ru)\backslash\mathcal{H}$ coincide. We will use the notation introduced earlier that $r = r_1 r_2^2$, where $r_1$ is the discriminant of the algebra containing $\mathcal{O}_{r,u}$. Also, we will restrict to the case $u = 1$, because, as we will see at the end of this section, it follows for general $u$ from a simple multiplicativity argument. We remind the reader that $\mathcal{O}_{1,r}$ is the Eichler order of level $r$ so that $\mathcal{O}_{1,r}^1 = \Gamma_0(r)$.

Using the formulas (4.2) and (4.3), it is possible to sieve out the contribution of the newforms in the Selberg trace formula. We use the notation $\lambda = r_\lambda^2 + 1/4$. For the Hecke congruence groups we get

$$\sum_{\lambda \in \mathcal{S}_{\Gamma_0(r)}^{new}} h(r_\lambda) = \sum_{d|r} \beta\left(\frac{r}{d}\right) \sum_{\lambda \in \mathcal{S}_{\Gamma_0(d)}} h(r_\lambda), \tag{6.1}$$

and the corresponding formula for $\mathcal{O}_{r,1}^1$ is

$$\sum_{\lambda \in \mathcal{S}_{\mathcal{O}_{r,1}^1}^{new}} h(r_\lambda) = \sum_{d|r_2} \mu\left(\frac{r_2}{d}\right) \sum_{\lambda \in \mathcal{S}_{\mathcal{O}_{r_1 d^2,1}^1}} h(r_\lambda). \tag{6.2}$$

In order to prove that $\mathcal{S}_{\Gamma_0(r)}^{new} = \mathcal{S}_{\mathcal{O}_{r,1}^1}^{new}$, we will compare the corresponding linear combinations of the right hand sides of the trace formulas.

The fact that $N_{\Gamma_0(ru)}^{new}(\lambda)$ is of cocompact type is equivalent to the fact that the linear combination of the parabolic contributions in (5.6) vanish [25]. Hence, we only need to consider the identity, elliptic and hyperbolic contribution.

We simplify the notation of the areas to $A_d = A_{\Gamma_0(d)}$ and $A_{r,u} = A_{\mathcal{O}_{r,u}^1}$. Combining formulas (5.2) and (5.6) with (6.1) and (6.2), it is natural to define

$$A_r^{new} = \sum_{d|r} \beta\left(\frac{r}{d}\right) A_d \text{ and } A_{r,1}^{new} = \sum_{d|r_2} \mu\left(\frac{r_2}{d}\right) A_{r_1 d^2,1} \tag{6.3}$$

and

$$E(S, \mathcal{O}_{1,r})^{new} = \sum_{d|r} \beta\left(\frac{r}{d}\right) E(S, \mathcal{O}_{1,d}) \text{ and}$$

$$E(S, \mathcal{O}_{r,1})^{new} = \sum_{d|r_2} \mu\left(\frac{r_2}{d}\right) E(S, \mathcal{O}_{r_1 d^2,1}). \tag{6.4}$$

We remark that the entities $A_r^{new}$ and $A_{r,1}^{new}$ are not "areas" and that $E(S, \Gamma_0(r))^{new}$ and $E(S, \mathcal{O}_{r,1})^{new}$ are not "embedding numbers" but should just be regarded as functions in $r$. Note that the areas and the embedding numbers are multiplicative functions as well as $\beta$ and $\mu$, and hence these four new functions are also multiplicative in $r$. For example this means that

$$E(S, \mathcal{O}_{r,1})^{new} = \prod_{p|r} E(S, \mathcal{O}_{r,1})_p^{new} \tag{6.5}$$

where

$$E(S,\mathcal{O}_{r,1})_p^{new}=\sum_{d|p^t}\mu\left(\frac{p^t}{d}\right)E(S,\mathcal{O}_{r'd^2,1})_p$$

with $r=r'p^{2t}$ and $p^{2t+1}$ exactly divides $r$.

From (5.2) and (5.6) we see that the right hand sides of the trace formulas (6.1) and (6.2) for the newforms agree if the areas satisfy

$$A_r^{new}=A_{r,1}^{new} \tag{6.6}$$

and the optimal embedding numbers satisfy

$$E(S,\mathcal{O}_{1,r})^{new}=E(S,\mathcal{O}_{r,1})^{new}, \tag{6.7}$$

for all quadratic orders $S$. The comparison of the embedding numbers is done in the next section. The terms corresponding to the areas are simple and straightforward to determine, and we conclude this section with this computation.

Since $A_r^{new}$ and $A_{r,1}^{new}$ are multiplicative in $r$, it is enough to compute them for $r$ a prime power. It is well known that $A_{p^t}=p^{t-1}(p+1)$. In the cocompact case, even though there is not any algebra ramified at just one prime, the entity $A_{p^{2s+1},1}$ makes sense thanks to multiplicativity. The formula $A_{p^{2s+1},1}=(p-1)p^{2s}$ can be extracted from [20, Section 2]. To be precise, $A_{p,1}=(p-1)$ is equation (2.1) in [20]. Moreover, for any allowable (e.g. prime) $q$

$$\frac{A_{p^{2s+1},1}}{A_{p,1}}=[(\mathcal{O}_{pq,1})_p^1:(\mathcal{O}_{p^{2s+1}q,1})_p^1]=p^{2s}$$

by (2.3) in [20], since in the notation there we have $\mathcal{M}=\mathcal{O}_{pq,1}$, $\mathcal{O}=\mathcal{O}_{p^{2s+1}q,1}$, $d(\mathcal{O}_p)=p^{2s+1}$, $e(\mathcal{O}_p)=-1$, $\mathfrak{A}_p\cong\mathbb{H}_p$ and finally $R_p^*=N(\mathcal{O}_p^*)$ by (5.3) in [20]. Plugging these formulas into (6.3), we get

$$A_{p^t}^{new}=\sum_{i=0}^{t}\beta(p^{t-i})A_{p^i}=\begin{cases}p-1 & t=1,\\ p^2-p-1 & t=2,\\ (p+1)(p-1)^2p^{t-3} & t\geq 3,\end{cases} \tag{6.8}$$

and

$$A_{p^{2s+1},1}^{new}=\sum_{i=0}^{s}\mu(p^{s-i})A_{p^i,1}=\begin{cases}p-1 & s=0,\\ (p+1)(p-1)^2p^{2(s-1)} & s\geq 1.\end{cases} \tag{6.9}$$

We see that $A_{p^{2s+1}}^{new}=A_{p^{2s+1},1}^{new}$ for all $s\geq 0$, and by multiplicativity we have established the formula (6.6).

We end this section with noting that formulas (6.6) and (6.7) will actually prove that

$$A_{ru}^{new} = A_{r,u}^{new} \text{ and } E(S, \Gamma_0(ru))^{new} = E(S, \mathcal{O}_{r,u})^{new}. \tag{6.10}$$

The reason is that if $p \mid u$, then $(\mathcal{O}_{1,ru})_p \cong (\mathcal{O}_{r,u})_p$ so all local factors at $p$ will be identical. Hence, the multiplicativity of the entities in $ru$ implies that they are equal also in the more general case.

## 7. Optimal embeddings

In this section $p$ will be a prime and $L$ a quadratic extension of $\mathbb{Q}_p$. Furthermore, $S_0$ will be the maximal order in $L$ and $S_i$ for $i > 0$ will be the suborder $S_i = \mathbb{Z}_p + p^i S_0$ and $\Delta = \Delta(S_i)$ will be the discriminant of $S_i$. Finally $\mathcal{O}$ will be an order with $\mathcal{O}_p \cong E_{p^{2t+1}}^{(-1)}$ and $\mathcal{O}^1$ will be denoted as $\mathcal{O}^1(p^{2t+1})$.

### 7.1. Odd primes

If $p$ is an odd prime, then there are three different classes of quadratic extensions. We will have

$$\Delta(S_i) = \begin{cases} p^{2i}, & \text{if } L \text{ is split,} \\ p^{2i}u, & \text{if } L \text{ is unramified,} \\ p^{2i+1}v, & \text{if } L \text{ is ramified,} \end{cases}$$

where $u$ is a quadratic non-residue and $v$ could be either 1 or $u$. The formulas for the optimal embedding numbers from [24] are in our notation (with the convention that $p^{-1} = 0$):

| $\Delta = p^{2i}$ | $t < i$ | $t = i$ | $t > i$ |
|---|---|---|---|
| $E(S_i, \mathcal{O}^1(p^{2t+1}))_p$ | $0$ | $0$ | $0$ |
| $E(S_i, \Gamma_0(p^{2t+1}))_p$ | $2p^t$ | $2p^i + 2p^{i-1}$ | $2p^i + 2p^{i-1}$ |
| $E(S_i, \Gamma_0(p^{2t}))_p$ | $p^t + p^{t-1}$ | $p^i + 2p^{i-1}$ | $2p^i + 2p^{i-1}$ |

| $\Delta = p^{2i}u$ | $t < i$ | $t = i$ | $t > i$ |
|---|---|---|---|
| $E(S_i, \mathcal{O}^1(p^{2t+1}))_p$ | $0$ | $2p^i - 2p^{i-1}$ | $2p^i - 2p^{i-1}$ |
| $E(S_i, \Gamma_0(p^{2t+1}))_p$ | $2p^t$ | $0$ | $0$ |
| $E(S_i, \Gamma_0(p^{2t}))_p$ | $p^t + p^{t-1}$ | $p^i$ | $0$ |

| $\Delta = p^{2i+1}v$ | $t < i$ | $t = i$ | $t > i$ |
|---|---|---|---|
| $E(S_i, \mathcal{O}^1(p^{2t+1}))_p$ | $0$ | $p^i$ | $0$ |
| $E(S_i, \Gamma_0(p^{2t+1}))_p$ | $2p^t$ | $p^i$ | $0$ |
| $E(S_i, \Gamma_0(p^{2t}))_p$ | $p^t + p^{t-1}$ | $p^i + p^{i-1}$ | $0$ |

**Table 1**
All nonzero local embedding numbers for newforms for odd prime $p$ for $\mathcal{O}^1(p^{2t+1})$ and $\Gamma_0(p^{2t+1})$.

| $\Delta(S_i)$ | Relation $i$ and $t$ | $E(S_i, \mathcal{O}^1(p^{2t+1}))_p^{new}$ | $E(S_i, \Gamma_0(p^{2t+1}))_p^{new}$ |
|---|---|---|---|
| $p^{2i}u$ | $i = t$ | $2p^i - 2p^{i-1}$ | $2p^{i-1} - 2p^i$ |
| $p^{2i+1}v$ | $i = t$ | $p^i$ | $-p^i$ |
| $p^{2i+1}v$ | $i = t - 1$ | $-p^i$ | $p^i$ |

From these we derive the numbers

$$\begin{aligned} E(S_i, \Gamma_0(p^n))_p^{new} &= \sum_{d \mid p^n} \beta\left(\frac{p^n}{d}\right) E(S_i, \Gamma_0(d))_p \\ &= E(S_i, \Gamma_0(p^n))_p - 2E(S_i, \Gamma_0(p^{n-1}))_p + E(S_i, \Gamma_0(p^{n-2}))_p \end{aligned}$$

and

$$\begin{aligned} E(S_i, \mathcal{O}^1(p^{2t+1}))_p^{new} &= \sum_{d \mid p^t} \mu\left(\frac{p^t}{d}\right) E(S_i, \mathcal{O}^1(pd^2))_p \\ &= E(S_i, \mathcal{O}^1(p^{2t+1}))_p - E(S_i, \mathcal{O}^1(p^{2t-1}))_p. \end{aligned}$$

In most cases the value of $E(S_i, \Gamma_0(p^{2t+1}))_p^{new}$ and $E(S_i, \mathcal{O}^1(p^{2t+1}))_p^{new}$ will be zero, and we collect all the non-zero results in Table 1.

### *7.2. The prime $p = 2$*

If $p = 2$, then there are five distinct cases. The formulas for the optimal embedding numbers from [24] are again in our notation (with the convention that $2^{-1} = 0$):

| $\Delta = 1$ | $t = 0$ | $t > 0$ |
|---|---|---|
| $E(S_i, \mathcal{O}^1(2^{2t+1}))_2$ | 0 | 0 |
| $E(S_i, \Gamma_0(2^{2t+1}))_2$ | 2 | 2 |
| $E(S_i, \Gamma_0(2^{2t}))_2$ | 1 | 2 |

| $\Delta = 5$ | $t = 0$ | $t > 0$ |
|---|---|---|
| $E(S_i, \mathcal{O}^1(2^{2t+1}))_2$ | 2 | 2 |
| $E(S_i, \Gamma_0(2^{2t+1}))_2$ | 0 | 0 |
| $E(S_i, \Gamma_0(2^{2t}))_2$ | 1 | 0 |

| $\Delta = 2^{2i+2}$ | $t \leq i$ | $t = i + 1$ | $t > i + 1$ |
|---|---|---|---|
| $E(S_i, \mathcal{O}^1(2^{2t+1}))_2$ | 0 | 0 | 0 |
| $E(S_i, \Gamma_0(2^{2t+1}))_2$ | $2^{t+1}$ | $3 \cdot 2^{i+1}$ | $3 \cdot 2^{i+1}$ |
| $E(S_i, \Gamma_0(2^{2t}))_2$ | $2^t + 2^{t-1}$ | $2^{i+2}$ | $3 \cdot 2^{i+1}$ |

| $\Delta = 5 \cdot 2^{2i+2}$ | $t \leq i$ | $t = i+1$ | $t > i+1$ |
|---|---|---|---|
| $E(S_i, \mathcal{O}^1(2^{2t+1}))_2$ | 0 | $2^{i+1}$ | $2^{i+1}$ |
| $E(S_i, \Gamma_0(2^{2t+1}))_2$ | $2^{t+1}$ | 0 | 0 |
| $E(S_i, \Gamma_0(2^{2t}))_2$ | $2^t + 2^{t-1}$ | $2^{i+1}$ | 0 |

| $\Delta = a \cdot 2^{2i+2}$ | $t < i$ | $t = i$ | $t > i$ |
|---|---|---|---|
| $E(S_i, \mathcal{O}^1(2^{2t+1}))_2$ | 0 | $2^i$ | 0 |
| $E(S_i, \Gamma_0(2^{2t+1}))_2$ | $2^{t+1}$ | $2^i$ | 0 |
| $E(S_i, \Gamma_0(2^{2t}))_2$ | $2^t + 2^{t-1}$ | $2^i + 2^{i-1}$ | 0 |

In the final table $a$ is any of the numbers $3, 7, 6, 10$ or 14. Again, in most cases the values of $E(S_i, \Gamma_0(2^{2t+1}))_2^{new}$ and $E(S_i, \mathcal{O}^1(2^{2t+1}))_2^{new}$ will be zero, and we collect all the non-zero results in Table 2.

### 7.3. Summing up

It is clear from Tables 1 and 2 that

$$E(S_i, \mathcal{O}^1(p^{2t+1}))_p^{new} = -E(S_i, \Gamma_0(p^{2t+1}))_p^{new}$$

for any $i, t$ and prime $p$. Since it is always an even number of primes that divide $r$ for the orders $\mathcal{O}_{r,1}$ the difference in sign will cancel and (6.5) and the corresponding formula for $E(S, \Gamma_0(r))^{new}$ implies that

$$E(S, \Gamma_0(r))^{new} = E(S, \mathcal{O}_{r,1})^{new}$$

for any $S$ and $r$.

## 8. The correspondence

Finally, we present our main result: That is, in most cases, the Jacquet–Langlands correspondence between newforms for Hecke congruence groups and newforms for certain quaternion orders is a bijection.

**Table 2**
All nonzero local embedding numbers for newforms for $\mathcal{O}^1(2^{2t+1})$ and $\Gamma_0(p^{2t+1})$. Here $a$ is any of the numbers $3, 7, 6, 10$ or 14.

| $\Delta(S_i)$ | Relation $i$ and $t$ | $E(S_i, \mathcal{O}^1(2^{2t+1}))_2^{new}$ | $E(S_i, \Gamma_0(2^{2t+1}))_2^{new}$ |
|---|---|---|---|
| 5 | $i = t = 0$ | 2 | $-2$ |
| $5 \cdot 2^{2i+2}$ | $i = t - 1$ | $2^{i+1}$ | $-2^{i+1}$ |
| $a \cdot 2^{2i+2}$ | $i = t$ | $2^i$ | $-2^i$ |
| $a \cdot 2^{2i+2}$ | $i = t - 1$ | $-2^i$ | $2^i$ |

**Theorem 8.1.** *Assume that $r$ is a positive integer that is divisible by an even number of primes, and that every prime dividing $r$ does so to an odd power. Let $u$ be any positive integer relatively prime to $r$. Then the positive Laplace eigenvalues, including multiplicities, for Maass newforms on $\mathcal{O}^1_{r,u}\backslash\mathcal{H}$ and $\Gamma_0(ru)\backslash\mathcal{H}$ coincide, i.e. $\mathcal{S}^{new}_{\Gamma_0(ru)} = \mathcal{S}^{new}_{\mathcal{O}^1_{r,u}}$.*

**Proof.** As has been noted in Section 6, it is enough to consider the case $u = 1$. It now suffices to show that

$$\sum_{\lambda\in\mathcal{S}^{new}_{\Gamma_0(r)}} h(\lambda) = \sum_{\lambda\in\mathcal{S}^{new}_{\mathcal{O}^1_{r,1}}} h(\lambda) \tag{8.1}$$

for an arbitrary test function $h$.

We recall that

$$\sum_{\lambda\in\mathcal{S}^{new}_{\Gamma_0(r)}} h(\lambda) = \sum_{d|r} \beta\left(\frac{r}{d}\right) \sum_{\lambda\in\mathcal{S}_{\Gamma_0(d)}} h(\lambda), \tag{8.2}$$

and that the corresponding formula for $\mathcal{O}^1_{r,1}$ is

$$\sum_{\lambda\in\mathcal{S}^{new}_{\mathcal{O}^1_{r,1}}} h(\lambda) = \sum_{d|r_2} \mu\left(\frac{r_2}{d}\right) \sum_{\lambda\in\mathcal{S}_{\mathcal{O}^1_{r_1 d^2,1}}} h(\lambda). \tag{8.3}$$

In order to prove that $\mathcal{S}^{new}_{\Gamma_0(r)} = \mathcal{S}^{new}_{\mathcal{O}^1_{r,1}}$, we will compare the corresponding linear combinations of the right hand sides of the trace formulas.

The linear combinations of the area terms are given by $A^{new}_r$ and $A^{new}_{r,1}$ defined in (6.3). These were proved to be equal at the end of Section 6.

The terms that contain embedding numbers are given by $E(S,\Gamma_0(r))^{new}$ and $E(S,\mathcal{O}_{r,1})^{new}$ defined in (6.4). That these agree is clear from Table 1 and Table 2 as explained in Section 7.

Finally the terms from the parabolic contribution for $\Gamma_0(r)$ vanish. This is proved in [25] and is by definition equivalent to $\Gamma_0(r)$ being of cocompact type. Hence all the linear combinations of the right hand sides of the trace formulas agree. □

A direct and obvious consequence of Theorem 8.1 is the following correspondence between Maass newforms on orders in different quaternion algebras.

**Corollary 1.** *Assume that $r_1$ and $r_2$ are positive integers each divisible by an even number of primes, and that every prime dividing $r_1$ or $r_2$ does so to an odd power. Let $u_1$ and $u_2$ be any positive integers relatively prime to $r_1$ and $r_2$ respectively such that $r_1u_1 = r_2u_2$. Then the positive Laplace eigenvalues, including multiplicities, for Maass newforms on $\mathcal{O}^1_{r_1,u_1}\backslash\mathcal{H}$ and $\mathcal{O}^1_{r_2,u_2}\backslash\mathcal{H}$ coincide, i.e. $\mathcal{S}^{new}_{\mathcal{O}^1_{r_1,u_1}} = \mathcal{S}^{new}_{\mathcal{O}^1_{r_2,u_2}}$.*

The smallest example matching the corollary is

$$\mathcal{S}^{new}_{\mathcal{O}^1_{6,5}} = \mathcal{S}^{new}_{\mathcal{O}^1_{10,3}} = \mathcal{S}^{new}_{\mathcal{O}^1_{15,2}}.$$

Theorem 8.1 covers all cases where $\Gamma_0(N)$ is of cocompact type except when $N = 4pu^2$ with $p$ a prime and $u$ odd. The obvious idea in this case is to look at orders in the quaternion algebra with discriminant $2p$. It is easy to check that the orders considered in this paper will not work. The order $\mathcal{O}_{2p,u^2}$ which corresponds to $\Gamma_0(2pu^2)$ gives too few Maass forms and the order $\mathcal{O}_{8p,u^2}$ which corresponds to $\Gamma_0(8pu^2)$ gives too many. We also note that numerical calculations comparing the beginning of the newform spectra for $\Gamma_0(12)$ and $\Gamma_0(24)$ reveal no relation, so there is no reason to believe that part of the newform spectrum corresponding to $\Gamma_0(4pu^2)$ will correspond to a subset of the newform spectrum corresponding to $\Gamma_0(8pu^2)$.

The only other class of orders that we find reasonable to consider are those with Eichler invariant equal to 0 [8]. Here we find suborders of $\mathcal{O}_{2p,u^2}$ of index 2, so the discriminant is a promising $4pu^2$. However when we compute the area term in the trace formula we find a leading term that is twice the one we want. Thus, unless there are some non-trivial "oldforms" showing up, our claim is that there is no order in the case $N = 4pu^2$ that gives the same correspondence as in Theorem 8.1, but we have no proof of it. There could for example be a phenomenon analogous to the one for $\Gamma_0(9)$, where all newforms actually are forms, or twists of forms, corresponding to groups strictly containing $\Gamma_0(9)$ [29]. Hence, we definitely do not exclude the possibility of finding a correspondence with an adjusted notion of newforms in the case $N = 4pu^2$. However, without adjustment we believe that:

**Conjecture 1.** *In the case $N = 4pu^2$ with $p$ a prime and $u$ odd there is no quaternion order with a natural correspondence between newforms as the one in Theorem 8.1.*

Of course we would be very happy if someone proves us wrong.

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