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Discrete components in restriction of unitary representations of rank one semisimple Lie groups ☆

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ABSTRACT

We consider spherical principal series representations of the semisimple Lie group of rank one $G = SO(n, 1; \mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. There is a family of unitarizable representations π_ν of G for ν in an interval on $\mathbb{R}$, the so-called complementary series, and subquotients or subrepresentations of G for ν being negative integers. We consider the restriction of (π_ν, G) under the subgroup $H = SO(n-1, 1; \mathbb{K})$. We prove the appearing of discrete components. The corresponding results for the exceptional Lie group $F_{4(-20)}$ and its subgroup $Spin_0(8, 1)$ are also obtained.

1. Introduction

The study of direct components in the restriction to a subgroup $H \subset G$ of a representation (π, G) is one of major subjects in representation theory. Among representations of a semisimple Lie group G there are two somewhat opposite classes, the discrete series and the complementary series; the former appear in the decomposition of $L^2(G)$ and can be treated algebraically, whereas the latter do not contribute to the decomposition and

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their study involves more analytic issues. The study of restriction of discrete series representations has been studied intensively; see e.g. [18,28] and references therein. Motivated by some related questions of [2,3] Speh and Venkataramana [30] studied the restriction of a complementary series representation of $SO(n,1)$ under the subgroup $SO(n-1,1)$. It is proved there, for relatively small parameter ν (in our parametrization), the complementary series π_ν of $SO(n-1,1)$ appears discretely in the complementary series π_ν of $SO(n,1)$ with the same parameter ν. They construct the imbedding of the complementary series of $SO(n-1,1)$ into π_ν of $SO(n,1)$ by using non-compact realizations of the representations as spaces of distributions on Euclidean spaces and by extending distributions on $\mathbb{R}^{n-2}$ to $\mathbb{R}^{n-1}$. Similar results are also obtained for complementary series of differential forms.

In the present paper we shall study the restriction, also called branching, of complementary series of G for all rank one Lie groups G with respect to a symmetric pair (G,H). More precisely we prove the appearance of discrete components for $G = SO(n,1;\mathbb{K})$, $H = SO(n-1,1;\mathbb{K})$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ being the fields of real, complex, quaternion numbers, or for $G = F_{4(-20)}$ and $H = Spin_0(8,1) \subset G$. We shall use the compact realization of the spherical principal series π_ν on the sphere $S = K/M$ in $\mathbb{F}^n$. We prove that for appropriate small parameter ν the natural restriction map of functions on S in π_ν to the lower dimensional sphere $S^\flat$ in $\mathbb{F}^{n-1}$ defines a bounded operator onto a complementary series $\pi^\flat_\nu$ of H. The proof requires rather detailed study of the restriction to $S^\flat \subset S$ of spherical harmonics on S.

The representations π_ν for certain integers ν have also unitarizable subquotients or subrepresentations; some of them are discrete series representations of G. We shall find also irreducible components for these representations under the subgroup H. One easiest case is the subrepresentation $\pi_0^\pm$ (or $\pi_{2n+2}^\pm$ as quotient) of the group $SU(n,1)$. The space $\pi_0^\pm$ consists of holomorphic respectively antiholomorphic polynomials on $\mathbb{C}^n$ modulo constant functions. It can also be treated by using the analytic continuation of scalar holomorphic discrete series at the reducible point [8], and some general decomposition results have been obtained in [19].

The main results in this paper are summarized in the following theorem, the precise statements being given in Theorems 3.6, 3.9 and 4.4; the parametrization of the complementary series (G, π_ν) is done so that the unitary principal series of G appear for $\nu = \rho_G + it$, $t \in \mathbb{R}$, so that the complementary series appear for ν in a symmetric interval around ρ_G.

Theorem 1.1. *Let (G,H) be the pair as above, $G = SO(n,1;\mathbb{K})$, $H = SO(n-1,1;\mathbb{K})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $G = F_{4(-20)}$, and $H = Spin_0(8,1) \subset G$. Let $\rho_G = d - 1 + \frac{d}{2}(n-1)$ and $\rho_H = d - 1 + \frac{d}{2}(n-2)$ be the corresponding half sums of positive roots, where $d = \dim_{\mathbb{R}} \mathbb{F} = 1, 2, 4$. Suppose (π_ν, G) is a complementary series representation of G. We can assume up to Weyl group symmetry that $\nu < \rho_G$.*

(1) *The restriction of (π_ν, G) on H contains a discrete component $(\pi^\flat_\mu, H)$ if $\nu < \rho_H$, and $\mu = \nu$ in our parameterization.*

(2) *Consider unitarizable quotients* $(\mathcal{W}_k, \pi_{\nu(k)}, \mathfrak{g})$ *and* $(\mathcal{V}_k, \pi_{\mu(k)}, \mathfrak{h})$ *for*

$$\nu(k) = \begin{cases} -k \leq 0, & \mathbb{F} = \mathbb{R} \\ -2k \leq -2, & \mathbb{F} = \mathbb{C}, \\ -2k \leq 2, & \mathbb{F} = \mathbb{H}, \end{cases}$$

where k *are integers. Then the restriction of* $(\mathcal{W}_k, \pi_{\nu(k)}, \mathfrak{g})$ *to* $\mathfrak{h}$ *contains the discrete component* $(\mathcal{V}_k, \pi_\mu, \mathfrak{h})$ *with the same parameter* $\mu(k) = \nu(k)$.

We note that our results can be understood heuristically as a kind of boundedness property of the restriction map from certain Sobolev spaces on S to those on $S^\flat$. Indeed for small parameter ν the space π_ν consists of distributions on S whose fractional differentiations are in $L^2(S)$, i.e., they are functions with certain smooth conditions. It is thus expected that their restriction on the sub-sphere $S^\flat$ would make sense in proper Sobolev spaces. (However the precise space of complementary series is not the usual Sobolev space, and only L^2-conditions for the differentiations of functions are required.) A precise formulation can be done and we hope to return to it in future; there has also been a recent development by Kobayashi and Speh [22, Chap. 15]. We remark also that the study of the norm estimates of the restriction of the spherical harmonics on lower dimensional spheres can be put into a general context as the study of growth of L^p-norm of restriction on totally geodesic submanifolds of eigenstates of Laplace–Beltrami operators on Riemannian manifold; see [6]. Our results here give precise estimates of the L^2-norm of the restriction. They might have independent interests on their own right from a view point of harmonic analysis. They might also shed light on the study of $L^p - L^q$ estimate of the above restriction problem for general compact manifolds.

We finish the introduction by briefly mentioning further related works and recent developments. The restriction of spherical representations of $SO(n+1,1)$ to $SO(n,1)$ has been studied in mathematical physics literature in the 1970s [4,27]. Molchanov [25] considered later the case $(SO(n,m), SO(n-1,m))$. The restriction of minimal representations and some induced representations of $O(p,q)$ to the subgroup $O(p',q') \times O(p'',q'')$ is thoroughly studied in [20]; see also [21] for some other groups. The quotient modules $(\mathcal{W}, \pi_\nu)$ treated in Section 3.5 here are of special interests in constructing unitary representations. Their generalizations for the groups $O(p,q)$ are the Zuckerman–Vogan modules, and the restriction of those representations to the subgroup $O(p',q') \times O(p'',q'')$ has been studied in [16,17]. Recently Möllers and Oshima [26] have found a full decomposition for the complementary series of $O(n,1)$ restricted to the symmetric subgroup $O(m,1) \times O(n-m)$. Kobayashi and Speh [22] have found a complete classification of intertwining operators for spherical representations for the pair $(O(n,1), O(n-1,1))$.

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2. Preliminaries

2.1. Classical rank one groups

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ be the real, complex and quaternionic numbers, respectively. Denote by $G := SO_0(n, 1; \mathbb{F}) = SO_0(n, 1), SU(n, 1), Sp(n, 1)$ the connected component of the group $GL(\mathbb{F}^{n+1})$ of $\mathbb{F}$-linear transformations on $\mathbb{F}^{n+1}$ preserving the quadratic form $|x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2$, with $\mathbb{F}$ acting on the right. The group $K := SO_0(n), S(U(n) \times U(1)), Sp(n) \times Sp(1)$ is a maximal compact subgroup of G and G/K is a Riemannian symmetric space of rank one which can further be realized as the unit ball in $\mathbb{F}^n$. Elements in G and $\mathfrak{g}$ will be written as $(n+1) \times (n+1)$ block $\mathbb{F}$-matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are of size $n \times n$, $n \times 1$, $1 \times n$, 1×1, respectively.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. We fix

$$H_0 = \begin{bmatrix} 0 & e_1 \\ e_1^T & 0 \end{bmatrix}$$

in $\mathfrak{p}$, where e_1 is the standard basis vector and e_1^T its transpose, and let $\mathfrak{a} = \mathbb{R}H_0 \subset \mathfrak{p}$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. The root space decomposition of $\mathfrak{g}$ under H_0 is

$$\mathfrak{g} = \mathfrak{g}_{-1} + (\mathfrak{a} + \mathfrak{m}) + \mathfrak{g}_1$$

with roots ± 1, 0 if $\mathbb{F} = \mathbb{R}$, and

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + (\mathfrak{a} + \mathfrak{m}) + \mathfrak{g}_1 + \mathfrak{g}_2$$

with roots ± 2, ± 1, 0, if $\mathbb{F} = \mathbb{C}, \mathbb{H}$. Here $\mathfrak{m} \subset \mathfrak{k}$ is the zero root space in $\mathfrak{k}$. We denote

$$\mathfrak{n} = \mathfrak{g}_1, \quad \mathfrak{n} = \mathfrak{g}_1 + \mathfrak{g}_2$$

the sum of the positive root spaces, in the respective cases. Thus $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is a maximal parabolic subalgebra of $\mathfrak{g}$. Let ρ be the half sum of positive roots. We identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ via $\lambda \to \lambda(H_0)$ and we write $\rho(H_0) = \rho$. We have, denoting $d = \dim_{\mathbb{R}} \mathbb{F}$,

$$\rho = d - 1 + \frac{d}{2}(n-1) = \begin{cases} \frac{n-1}{2}, & \mathbb{F} = \mathbb{R} \\ n, & \mathbb{F} = \mathbb{C} \\ 2n+1, & \mathbb{F} = \mathbb{H}. \end{cases} \tag{2.1}$$

Denote by M the centralizer of $\mathfrak{a}$ in K, A, N the corresponding subgroups with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$. Then MAN is a maximal parabolic subgroup of G.

2.2. Decomposition of $L^2(K/M)$

We identify $\mathfrak{p}$ with $\mathbb{F}^n$ and normalize the K-invariant inner product on $\mathfrak{p}$ so that H_0 is a unit vector. The homogeneous space K/M is then the unit sphere $S := S^{dn-1}$ in $\mathfrak{p} = \mathbb{F}^n$ with M being the isotropic subgroup of the base point $H_0 \in \mathfrak{p}$, with $d = \dim_{\mathbb{R}} F = 1, 2, 4$ as above. We denote dx the area measure on S normalized so that $dx(S) = 1$, and $L^2(S)$ the corresponding L^2-space. For $n = 1$ the decomposition of $L^2(K/M)$ is well-known and elementary, so we assume $n > 1$. Let W^p be the space of spherical harmonics on S. For $\mathbb{F} = \mathbb{C}$ let $W^{p,q}$ be the spherical harmonics of degree $p+q$ on $\mathbb{C}^n$ and holomorphic of degree p and antiholomorphic of degree q. If $\mathbb{F} = \mathbb{H}$, then $K = Sp(n) \times Sp(1)$, and its representations are of the form $\tau_1 \boxtimes \tau_2$, which will be written as (τ_1, τ_2) and further identified with their highest weights. The root system of $Sp(n)$ is of type C and let $\alpha_1, \cdots, \alpha_{n-1}, \alpha_n$ be the simple roots with α_n the longest one. Denote by $\lambda_1, \cdots, \lambda_n$ the corresponding fundamental weights with λ_1 the defining representation on $\mathbb{C}^{2n}$. For $Sp(1) = SU(2)$ the representation on symmetric tensor power $\odot^q(\mathbb{C}^2) = \mathbb{C}^{q+1}$ will be written just as q for simplicity. Denote by $W^{p,q}$ the representation $(q\lambda_1 + \frac{p-q}{2}\lambda_2, q)$ of $K = Sp(n) \times Sp(1)$.

Recall [23,14]

$$L^2(S) = \sum_{\tau}^{\oplus} W^\tau, \quad W^\tau = \begin{cases} W^p,\ p \ge 0, & \mathbb{F} = \mathbb{R} \\ W^{p,q},\ p, q \ge 0, & \mathbb{F} = \mathbb{C} \\ W^{p,q},\ p \ge q \ge 0, p-q\,\text{even}, & \mathbb{F} = \mathbb{H} \end{cases} \tag{2.2}$$

Here and in the following we denote a general representation of K by τ. The subspace $(W^\tau)^M$ of M-fixed vectors is one dimensional

$$(W^\tau)^M = \mathbb{C}\phi_\tau$$

where ϕ_τ is normalized by $\phi_\tau(H_0) = 1$. They depend only on the first variable $x_1 \in \mathbb{H}$ of $x = (x_1, \cdots, x_n)$, and will also be written as $\phi_\tau^n(x_1)$. We recall some explicit formulas for them obtained in [14, Theorem 3.1]. (Note that in the formula for $\psi_{p,q}$ and $e_{p,q}$ in [14, pp. 144–147] the term $\frac{-p-q}{2}$ should be $\frac{-p+q}{2}$.) Those polynomials are obtained as polynomial solutions to differential equations. A variant of these polynomials will be constructed in Lemma 3.3.

Lemma 2.1. *The polynomials ϕ_τ^n are given as follows:*

(1) $\mathbb{F} = \mathbb{R}$, $x_1 = \cos\xi$,

$$\phi_p^n(x_1) := \cos^p \xi F(-\frac{p}{2}, -\frac{p-1}{2}, \frac{n-1}{2}, -\tan^2\xi);$$

(2) $\mathbb{F}=\mathbb{C}$, $x_1=e^{i\theta}\cos\xi$,

$$\phi^n_{p,q}(x_1)=e^{i\theta(p-q)}\cos^{p+q}\xi F(-p,-q,n-1,-\tan^2\xi);$$

(3) $\mathbb{F}=\mathbb{H}$, $x_1=\cos\xi e^{\theta y}=\cos\xi(\cos\theta+y\sin\theta)$ *in quaternionic polar coordinates*, $y\in H$ *being purely imaginary (i.e. in* $\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$*) and* $|y|=1$,

$$\phi^n_{p,q}(x)=\phi^n_{p,q}(x_1):=\frac{\sin(q+1)t}{\sin t}\cos^p\xi\, F(-\frac{p-q}{2},-\frac{p+q+2}{2},2(n-1),-\tan^2\xi).$$

Here $F(a,b,c,x)$ is the Gauss hypergeometric function ${}_2F_1$,

$$F(a,b,c,x)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!}$$

and $(a)_m=\prod_{j=0}^{m-1}(a+j)$ is the Pochammer symbol. Note that all ϕ-functions above are the classical Jacobi polynomials $P^{\alpha,\beta}(t)$ in $t=2|x_1|^2-1$ in the interval $(-1,1)$; see [1, Chap. 6] and [33, Chap. IV].

To indicate the dependence of ϕ_τ on n we write ϕ_τ as ϕ^n_τ.

In particular we have, by Schur's orthogonality relation,

$$\|\phi_\tau\|^2=\frac{1}{\dim(W^\tau)}. \tag{2.3}$$

$\dim(W^\tau)$ can be evaluated by the Weyl's dimension formula: Let $\{\alpha\}$ be the root system of $\mathfrak{k}$ with $\{\alpha>0\}$ the positive roots and $\rho_{\mathfrak{k}}$ the half sum of the positive roots,

$$\dim(W^\tau)=\prod_{\alpha>0}\frac{\langle\tau+\rho_{\mathfrak{k}},\alpha\rangle}{\langle\rho_{\mathfrak{k}},\alpha\rangle};$$

see e.g. [11].

We shall also need a general integral formula: If $f(x)=g(y)h(z)$, $x=(y,z)$ are functions on $\mathbb{R}^m$ with separated variables $y\in\mathbb{R}^k$ and $z\in\mathbb{R}^{m-k}$ with dy the Lebesgue measure then we have

$$\int\limits_{S^{m-1}}f(x)dx=\frac{2\Gamma(\frac{m}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{m-k}{2})\omega_{k-1}}\int\limits_{B^k}g(y)(1-|y|^2)^{\frac12(m-k-2)}$$
$$\times\left(\int\limits_{S^{m-k-1}}h((1-|y|^2)^{\frac12}z)dz\right)dy \tag{2.4}$$

where B^k is the unit ball in $\mathbb{R}^k$, dx and dz are the area measures on the respective spheres normalized with total areas being 1, and $\omega_{k-1}=\frac{2\sqrt{\pi}^k}{\Gamma(\frac{k}{2})}$ is the Lebesgue area of the sphere in $\mathbb{R}^k$ (we shall need $k = 1, 2, 4$ only); see e.g. [29, 1.4.4 (1)] for the case of even m and k. Thus the square norm $\|\phi_\tau\|^2$ can also be proved by using the known integral formulas for Jacobi polynomials. However we shall use mostly the Weyl's dimension formula whenever possible as it is conceptually clearer and as their asymptotics are well-understood.

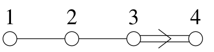
2.3. Exceptional group $F_{4(-20)}$

Let $\mathfrak{g}$ the simple real Lie algebra of split rank 1 of type F, i.e., g is type $F_{4(-20)}$ [10, Chap. X], and let G be the simply connected Lie group of type $F_{4(-20)}$ with Lie algebra $\mathfrak{g}$. This group has been well-studied [13,35]. The maximal compact subgroup K is $Spin(9)$ and the symmetric space G/K can be realized as the unit ball in $\mathbb{O}^2$ with $\mathbb{O}$ being the Cayley division (octonion) algebra. Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition. The space $\mathfrak{p}$ will be identified with $\mathbb{O}^2$ with $\mathfrak{k} = spin(9)$ acting on $\mathbb{O}^2$ via the Spin representation. We fix $H_0 \in \mathbb{O}^2 = \mathfrak{p}$ so that the positive eigenvalues of $\mathrm{ad}(H_0)$ in $\mathfrak{g}$ are 2, 1. The corresponding multiplicities are then 7 and 8. The half sum of positive roots is $\rho = 11$. Let $\mathfrak{m}$ be the zero root space of H_0 in $\mathfrak{k}$, and $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ the maximal parabolic subalgebra.

The algebra $\mathfrak{m} \subset \mathfrak{k}$ is $\mathfrak{spin}(7)$. Let $M = Spin(7)$ be the corresponding simply connected subgroup with Lie algebra $\mathfrak{m}$. Fix the K-invariant inner product on $\mathfrak{p} = \mathbb{O}^2$ with H_0 being unit vector. The homogeneous space K/M is the unit sphere $S = S^{15}$ in $\mathbb{O}^2 = \mathbb{R}^{16}$. To describe the decomposition of $L^2(S)$ under K we observe first that the space $\mathfrak{p} = \mathbb{O}^2$ is decomposed under M as

$$\mathfrak{p} = \mathbb{O} \oplus \mathbb{O} = (\mathbb{R}H_0 \oplus \mathbb{R}^7) \oplus \mathbb{O} \tag{2.5}$$

with $\mathbb{R}^7$ being the defining representation of $SO(7)$ and thus of M via the double covering $M = Spin(7) \to SO(7)$, and $\mathbb{O}$ the Spin representation of M. The Dynkin diagram of $Spin(9)$ is



with the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the corresponding fundamental weights. Let $W^{p,q}$ be the representation of K with highest weight $\frac{p-q}{2}\lambda_1 + q\lambda_4$. Then it follows [23,13] that

$$L^2(S) = \sum_{p \geq q \geq 0,\, p-q \geq 0 \text{ even}}^{\oplus} W^{p,q}, \tag{2.6}$$

and each space $W^{p,q}$ has a unique M-fixed vector $\phi_{p,q}$, $(W^{p,q})^M = \mathbb{C}\phi_{p,q}$, such that $\phi_{p,q}(H_0) = 1$. To describe $\phi_{p,q}$ write elements in $\mathbb{O}^2$ as $x = (x_0, x_1, x_2)$ under the decomposition (2.5), and write their (partial) polar coordinates as $r = |x|$, $\sqrt{x_0^2 + \|x_1\|^2} = r\cos\xi$, $x_0 = r\cos\xi\cos\eta$ with $0 \leq \xi \leq \frac{\pi}{2}$, $0 \leq \eta \leq \pi$. Then

$$\begin{aligned}&\phi_{p,q}(x) = \phi_{p,q}(x_0)\\ &= \cos^q \eta\, F(-\frac{q}{2}, -\frac{q-1}{2}, \frac{7}{2}; -\tan^2\eta) \cos^p \xi\, F(-\frac{p-q}{2}, -\frac{p+q+6}{2}, 4; -\tan^2\xi),\end{aligned}$$

for $x \in S$; see [13].

3. Restriction of $(SO_0(n,1;\mathbb{F}), \pi_\nu)$ to $(SO_0(n-1,1;\mathbb{F})$

3.1. Principal series of G

For $\nu \in \mathbb{C}$ let π_ν be the induced representation of G from MAN consisting of measurable functions f on G (up to sets of measure zero) such that

$$f(gme^{tH_0}n) = e^{-\nu t} f(g), me^{tH_0}n \in MAN \tag{3.1}$$

and $f\big|_K \in L^2(K)$. (Our representation π_ν is $Ind^G_{MAN}(e^{\rho-\nu})$ in the notation in [15]. However the parameter ν has some advantage, it is "stable" under branching; see Theorem 3.6 below.) In particular f in π_ν are invariant under M, and π_ν is further realized on $L^2(K/M) = L^2(S)$. We denote $(X_\nu, \pi_\nu, \mathfrak{g})$ the underlying $(\mathfrak{g}, K)$-module of π_ν of $\mathfrak{g}$, and denote π_ν also the corresponding unitary representation of G when $(X_\nu, \pi_\nu, \mathfrak{g})$ is unitarizable.

The L^2-norm in $L^2(S)$ is not unitary for π_ν except when $\nu = \rho + it$ for $t \in \mathbb{R}$, ρ being given by (2.1). The unitarizable representations $(X_\nu, \pi_\nu, \mathfrak{g})$ for real ν are usually called complementary series. They have been found in [23]. (See also [7] for related results for the real group $SO_0(n,1)$.) Further detailed study of the representations has been done in [14], which we recall now. The constant $\lambda_\nu(\tau)$ below is rewritten in terms of the Pochammer symbol $(a)_m$ and further the Gamma functions.

Theorem 3.1. *There is a positive definite $\mathfrak{g}$-invariant Hermitian form on X_ν given by*

$$\|w\|^2_\nu = \sum_\tau \lambda_\nu(\tau)\|w_\tau\|^2, \quad w = \sum_\tau w_\tau \in X_\nu, \tag{3.2}$$

where $\|w_\tau\|^2$ is the L^2-norm, and its completion forms an unitary irreducible representation of G, if

(1) $\mathbb{F} = \mathbb{R}$, $0 < \nu < n-1$,

$$\lambda_\nu(p) = \frac{(n-1-\nu)_p}{(\nu)_p} = \frac{\Gamma(n-1-\nu+p)}{\Gamma(n-1-\nu)\Gamma(\nu+p)}; \tag{3.3}$$

(2) $\mathbb{F} = \mathbb{C}$, $0 < \nu < 2n$,

$$\lambda_\nu(p,q) = \frac{(n-\frac{\nu}{2})_p}{(\frac{\nu}{2})_p}\frac{(n-\frac{\nu}{2})_q}{(\frac{\nu}{2})_q} = \frac{\Gamma^2(\frac{\nu}{2})\Gamma(n-\frac{\nu}{2}+p)}{\Gamma^2(n-\frac{\nu}{2})\Gamma(\frac{\nu}{2}+p)}\frac{\Gamma(n-\frac{\nu}{2}+q)}{\Gamma(\frac{\nu}{2}+q)}; \tag{3.4}$$

(3) $\mathbb{F} = \mathbb{H}$, $2 < \nu < 4n$,

$$\lambda_\nu(p,q) = \frac{(2n - \frac{\nu}{2})_{\frac{p-q}{2}}}{(\frac{\nu}{2} - 1)_{\frac{p-q}{2}}} \frac{(2n + 1 - \frac{\nu}{2})_{\frac{p+q}{2}}}{(\frac{\nu}{2})_{\frac{p+q}{2}}}$$
$$= \frac{\Gamma(\frac{\nu}{2} - 1)\Gamma(\frac{\nu}{2})}{\Gamma(2n - \frac{\nu}{2})\Gamma(2n + 1 - \frac{\nu}{2})} \frac{\Gamma(2n - \frac{\nu}{2} + \frac{p-q}{2})\Gamma(2n + 1 - \frac{\nu}{2} + \frac{p+q}{2})}{\Gamma(\frac{\nu}{2} - 1 + \frac{p-q}{2})\Gamma(\frac{\nu}{2} + \frac{p+q}{2})}. \tag{3.5}$$

3.2. General criterion of boundedness

We fix n and let $H = SO_0(n-1, 1; \mathbb{F}) \subset G$ be the subgroup of elements of $g \in G$ fixing the n-th coordinate x_n in $\mathbb{F}^{n+1}$. Denote $L := K \cap H$, a maximal subgroup of H. The sub-sphere, or the equator, $S^{d(n-1)-1}$ in $\mathbb{F}^{n-1}$ of the sphere $S = K/M \subset \mathbb{F}^n$ defined by the equation $x_n = 0$ will be written as $S^\flat$, which is homogeneous space of L, $S^b = L/L\cap M$. To avoid confusion we denote by $\pi_\nu^\flat$ the corresponding representations of H and $X_\nu^\flat$ the L-finite vectors, and the corresponding decomposition of $L^2(S^\flat) = L^2(L/L \cap M)$ will be written as

$$L^2(S^\flat) = \sum_\sigma^{\oplus} V^\sigma$$

with σ being specified accordingly.

We shall need a general and elementary criterion for boundedness of intertwining operators. The sufficient part of the following Lemma 3.2 is used in [31] implicitly. Let K temporarily be a compact group and $L \subset K$ a closed subgroup. Let $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be unitary representations of K and respectively L. Consider

$$\mathcal{W}\big|_K = \sum_\tau^{\oplus} W^\tau, \quad \mathcal{V}\big|_L = \sum_\sigma^{\oplus} V^\sigma$$

the irreducible decomposition of $\mathcal{W}$ and $\mathcal{V}$ under K and respectively L counting multiplicities, all assumed being finite. Consider further the branching of $\mathcal{W}^\tau$ under L. Write $\sigma \subset \tau$ if a representation σ appears in τ (counting multiplicities) with $\widetilde{V}^{\tau,\sigma}$ the corresponding isotypic component, and denote $P_{\tau,\sigma}$ the corresponding orthogonal projection, i.e.,

$$W^\tau = \sum_{\sigma\subset\tau}^{\oplus} \widetilde{V}^{\tau,\sigma}, \quad P_{\tau,\sigma} : W^\tau \to \widetilde{V}^{\tau,\sigma}. \tag{3.6}$$

Suppose R is a densely defined L-invariant operator from K-finite elements in $\mathcal{W}$ to L-finite elements in $\mathcal{V}$, and

$$R_{\tau,\sigma} := RP_{\tau,\sigma} : W^\tau \to V^\sigma$$

its components, i.e., $R = \sum_\tau \sum_{\sigma\subset\tau} R_{\tau,\sigma}$ on K-finite functions; we notice that by the assumption of R the operator $RP_{\tau,\sigma}$ maps indeed W^τ into V^σ. We denote $\|R\|_{\mathcal{W},\mathcal{V}}$ its norm whenever it is finite. The following lemma is an easy consequence of the Cauchy–Schwarz inequality whose proof we omit here.

Lemma 3.2. *The restriction operator R extends to a bounded operator from $\mathcal{W}$ to $\mathcal{V}$ if and only if there is a constant C such that for any fixed σ*

$$\sum_{\sigma\subset\tau} \|R_{\tau,\sigma}\|^2_{\mathcal{W},\mathcal{V}} \le C. \tag{3.7}$$

3.3. Restriction of spherical harmonics

We specify the above considerations to the restriction $R : C^\infty(S) \to C^\infty(S^\flat)$, $f(x', x_n) \mapsto f(x')$. The branching of $W^\tau = \sum_\sigma \widetilde{V}^{\tau,\sigma}$ of an irreducible K-component W^τ under L can be read off abstractly from known results. However we need to find all isotypic L-irreducible subspaces $\widetilde{V}^{\tau,\sigma} \subset W^\tau$ with nonzero restriction, i.e. with the restriction

$$R_{\tau,\sigma} : \widetilde{V}^{\tau,\sigma} \to V^\sigma$$

acting as an isomorphism. More precisely we shall study the abstract branching (3.6) along with the concrete restriction

$$W^\tau\big|_{x_n=0} := \{g(x') = f(x', 0),\ x' \in S^\flat; f \in W^\tau\} = \sum_{\sigma\subset\tau} V^\sigma.$$

We shall drop the upper-index τ in $\widetilde{V}^{\tau,\sigma}$ in the lemma below, as it is fixed in the summation. The parameterization of $(\tau;\sigma)$ will be $(p;s)$ for $\mathbb{F} = \mathbb{R}$ and $(p, q; s, t)$ for $\mathbb{F} = \mathbb{C}, \mathbb{H}$. Recall also the notation in Lemma 2.1.

Lemma 3.3.

(1) $\mathbb{F} = \mathbb{R}$. *The branching of W^p under $L = SO(n-1)$ is multiplicity free. The restriction $W^p\big|_{x_n=0}$ under $L = SO(n-1)$ is decomposed as*

$$W^p\big|_{x_n=0} = \sum^{\oplus}_{0\le s\le p,\ p-s\ \text{even}} V^s \tag{3.8}$$

The corresponding unique s-irreducible component in W^p is given by (as functions on S)

$$\widetilde{V^s} = \{h(x')\phi^{n+2s}_{p-s}(x_n); h \in V^s\}$$

(2) $\mathbb{F} = \mathbb{C}$. *The branching of* $W^{p,q}$ *under* $L = U(n-1)$ *is multiplicity free. The space* $W^{p,q}\big|_{x_n=0}$ *under* L *is decomposed as*

$$W^{p,q}\big|_{x_n=0} = \sum_{s\le p, t\le q, p-s=q-t}^{\oplus} V^{s,t}.$$

For each (s,t) *the unique* (s,t)*-irreducible component in* $W^{p,q}$ *is given by*

$$\widetilde{V}^{s,t} = \{h(x')\phi^{n+s+t}_{p-s,q-t}(x_n); h \in V^{s,t}\}.$$

(3) $\mathbb{F} = \mathbb{H}$. *The space* $W^{p,q}\big|_{x_n=0}$ *under* $L = Sp(n-1) \times Sp(1)$ *is decomposed as*

$$W^{p,q}\big|_{x_n=0} = \sum_{0\le p-s \ \ even, t=q}^{\oplus} V^{s,t}.$$

The corresponding (s,t)*-irreducible component is given by*

$$\widetilde{V}^{s,t} = \{h(z')\phi^{n+\frac{s}{2}}_{p-s,0}(x_n); h \in V^{s,t}\}$$

Proof. Let $\mathbb{F} = \mathbb{R}$. The multiplicity free result in this case is well-known, and a proof of it can be found in [36, (9), p. 495]. The proof there relies on explicit computations for the projection into spherical harmonics, which seem not easy to generalize to other cases. We give a slightly different proof which applies also to the other cases and which avoids some redundant computations. Denote $L_n = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, the Laplacian on $\mathbb{R}^n$. Recall that the spherical polynomial $f = r^m C_m^{\frac{n-2}{2}}(\frac{x_n}{r})$ is the unique $SO(n-1)$ invariant polynomial on $\mathbb{R}^n$ of degree m satisfying $L_n f = 0$, where $C_m^{\frac{n-2}{2}}(t)$ is the Gegenbauer polynomial. Let $x = (x', x_n) \in \mathbb{R}^n$, and put $u := |x'|$, $v := x_n$. We have

$$L_n = L_{n-1} + \frac{\partial^2}{\partial v^2} = \frac{\partial^2}{\partial u^2} + \frac{n-2}{u}\frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2},$$

when acting on functions depending only on $|x'|$ and x_n. Rephrasing in terms of u, v we have the unique polynomial solution of the form $f(u,v) = (u^2+v^2)^{\frac{m}{2}} C(\frac{v}{\sqrt{u^2+v^2}})$, of the equation

$$L_n f = \frac{\partial^2 f}{\partial u^2} + \frac{n-2}{u}\frac{\partial f}{\partial u} + \frac{\partial^2 f}{\partial v^2} = 0, \tag{3.9}$$

when $C = C_m^{\frac{n-2}{2}}$. Now for fixed $s \le p$ we search for an isotypic $SO(n-1)$-component in W^p of type V^s consisting of homogeneous polynomials $F(x)$ of degree p of the form $F(x) = h(x')f(u,v) = h(x')f(|x|', x_n)$, where h is a spherical harmonics of degree s on $\mathbb{R}^{n-1}$, i.e. $L_{n-1}h = 0$, and $f(u,v) = (u^2+v^2)^{\frac{p-s}{2}} C(\frac{v}{\sqrt{u^2+v^2}})$. The Laplace equation $L_n F = (L_{n-1} + \frac{\partial^2}{\partial v^2})F = 0$ becomes

$$(L_{n-1}h(x'))f(u,v) + 2\sum_{j=1}^{n-1} x_j \frac{\partial h(x')}{\partial x_j}\frac{1}{u}\frac{\partial h}{\partial u} f(u,x_n) + h(x')L_n f(u,x_n) = 0,$$

with $L_n f$ computed in (3.9). But $L_{n-1}h(x') = 0$ and $\sum_{j=1}^{n-1} x_j \frac{\partial h(x')}{\partial x_j} = sh(x')$ by our assumption. Thus it reduces to

$$2s\frac{1}{u}\frac{\partial h}{\partial u} f(u,v) + L_n f(u,v) = 0, \tag{3.10}$$

i.e.

$$\frac{\partial^2 f}{\partial u^2} + \frac{n-2+2s}{u}\frac{\partial f}{\partial u} + \frac{\partial^2 f}{\partial v^2} = 0. \tag{3.11}$$

This is precisely the equation (3.9) with n replaced by $n+2s$ and m replaced by $p-s$. Thus f is a constant multiple of $r^{p-s}C_{p-s}^{\frac{n-2}{2}+s}(\frac{x_n}{r})$ (which is a posterior polynomial in x). Note that this is non-zero for $x_n = 0$ only if $p-s$ is even. This proves the case for $\mathbb{F} = \mathbb{R}$.

$\mathbb{F} = \mathbb{C}$. The multiplicity free result is also known; see e.g. [24]. The abstract decomposition of $W^{p,q}\big|_{z_n=0}$ follows easily by counting the degrees (s,t). We search an L-isotypic component consisting of polynomials of the form $F(x) = h(x')r^{p+q-s-t}C(\frac{x_n}{r})$ as above. The function $C(\frac{x_n}{r})$ is then exactly the same as ϕ_τ^m as in Lemma 2.1 (2) with $\tau = (p',q')$ and m determined by (p,q) and n.

$\mathbb{F} = \mathbb{H}$. The group $Sp(1)$ acts on the space of polynomials on the right, $h \in Sp(1)$: $f(x) \mapsto f(xh)$, and it acts on the space $W^{p,q}$ as the symmetric tensor $\odot^q(\mathbb{C}^2)$. So does it also on the space $W^{p,q}\big|_{x_n=0}$. Thus any irreducible component must be of type $V^{s,t}$ with $t = q$, again by (2.1). In particular $p-s = (p-q)-(s-t)$ is even since both $p-q$ and $s-t$ are even. This proves the decomposition. The rest of the proof is almost the same as above. (Note that the function $\phi_{p-s,0}^{n+\frac{s}{2}}$ is obtained from $\phi_{p,0}^n$ in Lemma 2.1 (3) by formally replacing n by $n+\frac{s}{2}$, which is not necessarily an integer.) $\square$

We compute now the operator norm of $R_{\tau,\sigma}$. For positive constants $C_{\tau,\sigma}$ and $D_{\tau,\sigma}$ we write $C_{\tau,\sigma} \sim D_{\tau,\sigma}$ if both $\frac{C_{\tau,\sigma}}{D_{\tau,\sigma}}$ and $\frac{D_{\tau,\sigma}}{C_{\tau,\sigma}}$ are dominated by positive constants independent of τ, σ.

Proposition 3.4. *With the notation as above we have the $L^2(S) - L^2(S^\flat)$-norm of $R_{\tau,\sigma}$: $W^\tau \to V^\sigma \subset L^2(S^\flat)$ is given by*

(1) $\mathbb{F} = \mathbb{R}$, $p-s \geq 0$ *even,*

$$\begin{aligned}\|R_{p,s}\|^2 &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}\frac{(2p+n-2)\Gamma(\frac{n+p+s-2}{2})\Gamma(\frac{p-s+1}{2})}{\Gamma(\frac{p-s+2}{2})\Gamma(\frac{n+p+s-1}{2})}\\ &\sim \frac{p+1}{(p+s+1)^{\frac{1}{2}}(p-s+1)^{\frac{1}{2}}};\end{aligned}$$

(2) $\mathbb{F} = \mathbb{C}$, $p \geq s \geq 0$, $q \geq t \geq 0$, $p - q = s - t$,

$$\|R_{(p,q),(s,t)}\|^2 = p + q + n - 1;$$

(3) $\mathbb{F} = \mathbb{H}$, $p - s = 2k \geq 0$ *and* $s - t \geq$ *even*, $q = t$,

$$\begin{aligned}\|R_{(p,q),(s,t)}\|^2 &= \frac{\Gamma(2n-2)}{\Gamma(2n)}(k+1)(2k+2(n-1)+s-1)(k+2(n-1)+s) \\ &\sim (k+1)(k+s+1)^2.\end{aligned}$$

In all other cases of (τ, σ) *we have* $R_{\tau,\sigma} = 0$.

Proof. $\mathbb{F} = \mathbb{R}$. By the previous lemma and Schur lemma we see that $R_{p,s} : W^p \to V^s$ is up to a constant a partial isometry, and $\widetilde{V^s} \to V^s$ is up to a constant an isometry. Thus

$$\|R_{p,s}\|^2 = \frac{\|Rf\|^2}{\|f\|^2}$$

for any $0 \neq f \in \widetilde{V^s}$. Now we take $f = h(x')\phi_{p-s}^{n+2s}(x_n)$, which has a form of variable separation, and we have, by (2.4) and the s-homogeneity of $h(x')$, that

$$\|f\|^2 = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int\limits_{|x_n|<1} (1 - |x_n|^2)^{\frac{n-3}{2}+s} |\phi_{p-s}^{n+2s}(x_n)|^2 \int\limits_{S^\flat} |h(y')|^2 dy' dx_n,$$

and

$$\|Rf\|^2 = |\phi_{p-s}^{n+2s}(0)|^2 \int\limits_{S^\flat} |h(y')|^2 dy'. \tag{3.12}$$

Consequently

$$\|R_{p,s}\|^2 = |\phi_{p-s}^{n+2s}(0)|^2 \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int\limits_{|x_n|<1} (1 - |x_n|^2)^{\frac{n-3}{2}+s} |\phi_{p-s}^{n+2s}(x_n)|^2 dx_n \right)^{-1}.$$

Note that the integral

$$I := \int\limits_{|x_n|<1} (1 - |x_n|^2)^{\frac{n-3}{2}+s} |\phi_{p-s}^{n+2s}(x_n)|^2 dx_n$$

is up to a constant the square norm in $L^2(S^{n+2s-1})$ of the spherical polynomial $\phi_{p-s}^{n+2s}(x_n)$ in dimension $n + 2s$, and can be evaluated by using (2.3) in terms of the dimension

$\dim W^{p-s}_{n+2s}$ of the representation of $SO(n+2s)$. The exact (a rather subtle) constant is computed in (2.4),

$$\int_{|x_n|<1} (1-|x_n|^2)^{\frac{n-3}{2}+s}|\phi^{n+2s}_{p-s}(x_n)|^2 dx_n = \frac{\Gamma(\frac{n+2s-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2s}{2})}\frac{1}{\dim W^{p-s}_{n+2s}}.$$

Thus using the dimension formula that

$$\dim W^j_n = \binom{n+j-1}{j} - \binom{n+j-3}{j-2} = \frac{(n+2j-2)\Gamma(n+j-2)}{\Gamma(j+1)\Gamma(n-1),}$$

we have

$$I^{-1} = \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n+2s-1}{2})\Gamma(\frac{1}{2})}\frac{(2p+n-2)\Gamma(n+p+s-2)}{\Gamma(n+2s-1)\Gamma(p-s+1)}.$$

The evaluation $\phi^{n+2s}_{p-s}(0)$ in (3.12) is zero unless $p-s=2k$ is even, in which case it is

$$(-1)^k\frac{(-k)_k(-\frac{p-s-1}{2})_k}{(\frac{n+2s-1}{2})_k k!}.$$

But $(-k)_k = (-1)^k k!$, $(-\frac{p-s-1}{2})_k = (-1)^k(\frac{1}{2})_k = (-1)^k\frac{\Gamma(\frac{1}{2}+k)}{\Gamma(\frac{1}{2})}$, we find that the evaluation, disregarding the sign $(-1)^k$ and the constant $\Gamma(\frac{1}{2})$, is

$$\frac{\Gamma(\frac{p-s+1}{2})\Gamma(\frac{n+2s-1}{2})}{\Gamma(\frac{n+p+s-1}{2})}.$$

Using the product formula $\Gamma(2x) = \Gamma(\frac{1}{2})^{-1}2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})$ we obtain then the formula for $\|R_{p,s}\|^2$ as stated. The rest follows from the Stirling formula that

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim n^{a-b}, \quad n\to\infty.$$

The case $\mathbb{F}=\mathbb{C}$ is done by similar computations. In the case $\mathbb{F}=\mathbb{H}$ we have

$$\|R_{(p,q),(s,q)}\|^2 = \frac{|\phi^{n+\frac{s}{2}}_{p-s}(0)|^2\int_{S^\flat}|h(x')|^2dx'}{\|\phi^{n+\frac{s}{2}}_{p-s}h\|^2}$$

with $\|\phi^{n+\frac{s}{2}}_{p-s}h\|^2$ being

$$\frac{2\Gamma(n)}{\Gamma(n-1)\omega_3}\int_{x_n\in\mathbb{H},|x_n|<1}|\phi^{n+\frac{s}{2}}_{p-s}(x_n)|^2(1-|x_n|^2)^{\frac{1}{2}(4(n-1)-2+2s)}\int_{S^\flat}|h(x')|^2dx'dx_n$$

by the integral formula above for separated variables. The norm of $\phi^{n+\frac{s}{2}}$ cannot be computed using the dimension formula for s odd as it cannot be interpreted as spherical polynomials on a symmetric space. However we may find it using some known integral formulas [1, Chap. 6] for Jacobi polynomials $P^{(\alpha,\beta)}(t)$. (Alternatively one may also use the theory of Heckman–Opdam [9].) Indeed the function $\phi^n_{k,0}$ in Section 2.2 for any real $n > 1$ can be written as

$$\phi^n_{k,0}(x) = \frac{\Gamma(k+1)\Gamma(2n-2)}{\Gamma(k+2n-2)} P_k^{(2n-3,1)}(2|x|^2 - 1)$$

where $|x|$ is the norm of a quaternionic number $x \in \mathbb{H}$. The norm to be computed is

$$\int\limits_{x\in\mathbb{H},|x|<1} |\phi^m_{m,0}(x)|^2(1-|x|^2)^{\frac{1}{2}(4(n-1)-2}dx = \omega_3 \int\limits_0^1 |\phi^m_{m,0}(x)|^2(1-|x|^2)^{\frac{1}{2}(4(n-1)-2}x^3 dx$$

and which is further [1, (6.4.5)–(6.4.6), pp. 299–301]

$$\omega_3 \frac{\Gamma^2(2n-2)\Gamma(k+1)\Gamma(k+2)}{\Gamma(k+2n-2)\Gamma(k+2n-1)(2k+2n-1)}.$$

The rest is done by a routine computation. $\square$

Note that when $\mathbb{F} = \mathbb{R}$ and $n = 3$ our result coincides with that in [31, Lemma 2.4]. For $\mathbb{F} = \mathbb{C}$, and $W^{p,q} = W^{p,0}$ the space the holomorphic polynomials of degree p, the norm of R can be found directly by computing of the integral $\int_S |x_1^p|^2 dx$ on the sphere S in $\mathbb{C}^n$.

3.4. Discrete components of complementary series

Before stating our first main result we note the following elementary

Lemma 3.5. *Suppose* $0 < \alpha < 1$, $\beta > 0$, $\alpha + \beta > 1$ *and* $\gamma > 1$. *Then*

$$\sum_{j=0}^{\infty} \frac{1}{(j+1)^\alpha (q+j+1)^\beta} \le C \frac{1}{q^{\alpha+\beta-1}}, \quad \sum_{j=0}^{\infty} \frac{1}{(j+q+1)^\gamma} \le C \frac{1}{(q+1)^{\gamma-1}}, \quad \forall q \ge 0.$$

The second estimate is straightforward. The first sum is dominated by the integral

$$\int\limits_0^\infty \frac{1}{x^\alpha (x+q+1)^\beta} dx = \frac{1}{(q+1)^{\alpha+\beta-1}} \int\limits_0^\infty \frac{1}{x^\alpha (x+1)^\beta} dx = \frac{1}{(q+1)^{\alpha+\beta-1}} C$$

since the integral $\int_0^\infty \frac{1}{x^\alpha (x+1)^\beta} dx = C < \infty$ is convergent by our assumption.

Observe also that

$$R : (X_\nu, \pi_\nu, \mathfrak{g}) \to (X_\nu^\flat, \pi_\nu^\flat, \mathfrak{h}),\ f(x) \mapsto f(x', 0)$$

intertwines the respective actions of $\mathfrak{h}$. Thus the boundedness of R implies that $(\pi_\nu^\flat, \mathfrak{h})$ is a discrete component whenever both are unitarizable. In accordance with the notation $\|\cdot\|_\nu$ in Theorem 3.1 we denote $\|T\|_{\nu,\mu}$ the norm of an operator $T : X_\nu \to X_\mu^\flat$. We have then

$$\|R_{\tau,\sigma}\|_{\nu,\mu}^2 = \frac{\lambda_\mu(\sigma)^\flat}{\lambda_\nu(\tau)} \|R_{\tau,\sigma}\|^2,$$

and the criterion (3.7) becomes

$$\sum_{\sigma \subset \tau} \|R_{\tau,\sigma}\|^2 \lambda_\nu(\tau)^{-1} \le \frac{C}{\lambda_\mu^\flat(\sigma)}. \tag{3.13}$$

Theorem 3.6. *The restriction of (π_ν, G) on H contains $(\pi_\nu^\flat, H)$ as a discrete component in the following cases:*

(1) $\mathbb{F} = \mathbb{R}$, $n \ge 3$, $0 < \nu < \frac{n-2}{2}$;
(2) $\mathbb{F} = \mathbb{C}$, $n \ge 3$, $0 < \nu < n-2$;
(3) $\mathbb{F} = \mathbb{H}$, $n \ge 2$, $2 < \nu < 2n-1$.

Proof. $\mathbb{F} = \mathbb{R}$. First note that $\frac{n-2}{2} < n-2 < n-1$, thus both (π_ν, G) and (π_ν, H) are well-defined unitary representations. We use now Lemma 3.2 with $\tau = p$ and $\sigma = s$. The constants $\lambda_\nu(p)$, $\lambda_\nu^\flat(s)$ and the series (3.13) in question are

$$\lambda_\nu(p) \sim (p+1)^{n-1-2\nu}, \quad \lambda_\nu^\flat(s) \sim (s+1)^{n-2-2\nu},$$
$$\sum_{p \ge s, p-s \text{ even}} \frac{p+1}{(p+s+1)^{\frac{1}{2}}(p-s+1)^{\frac{1}{2}}} \frac{1}{(p+1)^{n-1-2\nu}}.$$

Writing $p = s + 2j$ we see the sum is dominated by

$$\sum_{j=0}^\infty \frac{s+2j+1}{(2s+2j+1)^{\frac{1}{2}}(2j+1)^{\frac{1}{2}}} \frac{1}{(s+2j+1)^{n-1-2\nu}} \le C \sum_{j=1}^\infty \frac{1}{j^{\frac{1}{2}}} \frac{1}{(s+j)^{n-1-2\nu-\frac{1}{2}}},$$

and further by $(s+1)^{-(n-2-2\nu)}$ in view of Lemma 3.5, namely by $\frac{1}{\lambda_\nu^\flat(s)}$.

$\mathbb{F} = \mathbb{C}$. $\lambda_\nu(\tau)$, $\tau = (p, q)$, has the asymptotics

$$\lambda_\nu(p, q) \sim (p+1)^{n-\nu}(q+1)^{n-\nu}.$$

For a fixed type $\sigma = (s, t)$ of L the series $\sum_{\sigma \subset \tau} \|R_{\tau,\sigma}\|^2 \lambda(\tau)^{-1}$ is dominated up to a constant by

$$\sum_{p-s=q-t\geq 0}\frac{p+q+2}{(p+1)^{n-\nu}(q+1)^{n-\nu}}$$
$$=\sum_{p-s=q-t\geq 0}\left(\frac{1}{(p+1)^{n-\nu-1}(q+1)^{n-\nu}}+\frac{1}{(p+1)^{n-\nu}(q+1)^{n-\nu-1}}\right)$$

as sum of two, say $I + II$. Now

$$I=\sum_{k=0}^{\infty}\frac{1}{(s+k+1)^{n-\nu-1}(t+k+1)^{n-\nu}},$$

and

$$I\leq\frac{1}{(s+1)^{n-\nu-1}}\sum_{k=0}^{\infty}\frac{1}{(t+k+1)^{n-\nu}}\leq C\frac{1}{(s+1)^{n-\nu-1}(t+1)^{n-\nu-1}}\leq C\frac{1}{\lambda_\nu^\flat(s,t)}$$

by Lemma 3.5. The same holds for II.

$\mathbb{F}=\mathbb{H}$. Writing $p = s + 2k$, $k \geq 0$, we have

$$\begin{aligned}\lambda_\nu(p,q)&\sim(p-q+1)^{2n+1-\nu}(p+q+1)^{2n+1-\nu}\\&\sim(s-q+k+1)^{2n+1-\nu}(s+q+k+1)^{2n+1-\nu}\end{aligned}$$

and

$$\|R_{(p,q),(s,q)}\|^2\sim(k+1)(s+k+1)^2.$$

The sum (3.13) is bounded by

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(k+1)(k+s+1)^2}{(s-q+k+1)^{2n+1-\nu}(s+q+k+1)^{2n+1-\nu}}\\
&\leq\sum_{k=0}^{\infty}\frac{k+1}{(s-q+k+1)^{2n+1-\nu}(s+q+k+1)^{2n-1-\nu}}\\
&\leq\frac{1}{(s+q+1)^{2n-1-\nu}}\sum_{k=0}^{\infty}\frac{k+1}{(s-q+k+1)^{2n+1-\nu}}\\
&\leq\frac{1}{(s+q+1)^{2n-1-\nu}}\sum_{k=0}^{\infty}\frac{1}{(s-q+k+1)^{2n-\nu}}\\
&\leq C\frac{1}{(s+q+1)^{2n-1-\nu}}\frac{1}{(s-q+1)^{2n-\nu-1}}\sim\frac{1}{\lambda_\nu^\flat(s,t)},\end{aligned}$$

finishing the proof. □

Remark 3.7. For $\mathbb{F} = \mathbb{R}$ and $n = 3$ the full decomposition of the complementary series π_ν of $SO_0(3,1)$ under $SO_0(2,1)$ is done in [27]; see further [4] and [25]. If (in terms of our parametrization) $\frac{1}{2} \leq \nu < 1$ then the decomposition is a sum of two direct integrals of spherical principal series, and if $0 < \nu < \frac{1}{2}$ there is one extra discrete component, the complementary series. In [30] a direct proof for the appearance of $(\pi_\nu^\flat, SO_0(n-1,1))$ in $(\pi_\nu, SO_0(n,1))$ is done using the non-compact realization on $\mathbb{R}^{n-1}$. A full decomposition of $(\pi_\nu, O_0(n,1))$ under $O(n-m) \times O(m,1)$ has been found recently in [26].

3.5. The quotients $(\mathcal{W}, \pi_\nu)$ at negative integers ν and their discrete components

The representation π_ν is reducible [14] for ν satisfying certain integral conditions, and there exist unitarizable subrepresentations (or quotients). More precisely we have the following result [34,14], retaining the notation of λ_τ as the Schur proportional constants; here we have rewritten them in similar formulation as in Theorem 3.1. To state the result in a uniform fashion we denote

$$\nu = \nu(k) = \begin{cases} -k, & k \geq 0, \mathbb{F} = \mathbb{R}, \\ -2k, & k \geq 1, \mathbb{F} = \mathbb{C}, \\ -2k, & k \geq -1, \mathbb{F} = \mathbb{H}. \end{cases} \tag{3.14}$$

Here k are integers.

Theorem 3.8. *There is a unitarizable irreducible quotient $(\mathcal{W}_\nu, \pi_\nu)$ at the points $\nu = \nu(k)$ described above whose completion forms a unitary irreducible representation of G in the following cases*

(1) $\mathbb{F} = \mathbb{R}$, $n \geq 3$, $\mathcal{W}_\nu = X_\nu / M_\nu$,

$$M_\nu = \sum_{p=0}^{k} W^p,$$

$$\lambda_\nu(p) = \frac{(n-1-\nu+k+1)_{p-k-1}}{(\nu+k+1)_{p-k-1}} = C_\nu \frac{\Gamma(n-1-\nu+p)}{\Gamma(\nu+p)};$$

(2) $\mathbb{F} = \mathbb{C}$, $n \geq 2$, $\nu = \nu(k) = -2k$, $k > 0$, $\mathcal{W}_\nu = X_\nu / M_\nu$,

$$M_\nu = \sum_{p \leq k, q \geq 0} W^{p,q} + \sum_{q \leq k, p \geq 0} W^{p,q} \quad (k > 0),$$

$$\begin{aligned} \lambda_\nu(p,q) &= \frac{(n - \frac{\nu}{2} + k + 1)_{p-k-1} (n - \frac{\nu}{2} + k + 1)_{q-k-1}}{(\frac{\nu}{2} + k + 1)_{p-k-1} (\frac{\nu}{2} + k + 1)_{q-k-1}} \\ &= C_\nu \frac{\Gamma(n - \frac{\nu}{2} + p) \Gamma(n - \frac{\nu}{2} + q)}{\Gamma(\frac{\nu}{2} + p) \Gamma(\frac{\nu}{2} + q)} \end{aligned}$$

and for $k = 0$ with three quotients $(\mathcal{W}_0^\pm, \pi_0^\pm)$, $(\mathcal{W}_0, \pi_0)$,

$$\mathcal{W}_0^+ = \sum_{p=0}^{\infty} W^{p,0}/\mathbb{C}, \quad \mathcal{W}_0^- = \sum_{q=0}^{\infty} W^{0,q}/\mathbb{C}$$

$$\lambda_\nu^+(p) = \frac{\Gamma(p)}{\Gamma(n+p)}, \lambda_\nu^-(q) = \frac{\Gamma(q)}{\Gamma(n+q)},$$

and

$$\mathcal{W}_0 = X_0/\sum_{p=0}^{\infty}(W^{p,0} + W^{0,p})$$

$$\lambda_0(p,q) = \frac{\Gamma(n+p-1)\Gamma(n+q-1)}{\Gamma(p)\Gamma(q)};$$

(3) $\mathbb{F} = \mathbb{H}$, $n \geq 1$, $\nu = \nu(k) = -2k$, $k \geq -1$, $\mathcal{W}_\nu = X_\nu/M_\nu$,

$$M_\nu = \sum_{p-q \leq 2k+2} W^{p,q}, \quad k \geq 0, \qquad M_\nu = W^{0,0}, \quad k = -1,$$

$$\begin{aligned}\lambda_\nu(p,q) &= \frac{(2n - \frac{\nu}{2} + k + 1)_{\frac{p-q}{2}-k-1}(2n + 1 - \frac{\nu}{2} + k + 1)_{\frac{p+q}{2}-k-1}}{(\frac{\nu}{2} - 1 + k + 1)_{\frac{p-q}{2}-k-1}(\frac{\nu}{2} + k + 1)_{\frac{p+q}{2}-k-1}} \\ &= C_\nu \frac{\Gamma(2n - \frac{\nu}{2} + \frac{p-q}{2})\Gamma(2n + 1 - \frac{\nu}{2} + \frac{p+q}{2})}{\Gamma(\frac{\nu}{2} - 1 + \frac{p-q}{2})\Gamma(\frac{\nu}{2} + \frac{p+q}{2})}.\end{aligned}$$

Note that the same $\nu = \nu(k)$ as above is also a reducible point for $(X_{\nu,\flat}, \pi_\nu^\flat, \mathfrak{h})$. The corresponding quotient representation for $\mathfrak{h}$ will be written as $(\mathcal{V}_k, \pi_{\nu(k)}^\flat, \mathfrak{h})$.

Theorem 3.9. *Let $n \geq 4$ for $\mathbb{F} = \mathbb{R}$, $n \geq 3$ for $\mathbb{F} = \mathbb{C}$, and $n \geq 2$ for $\mathbb{F} = \mathbb{H}$. Let $\nu(k)$ be the integral points in (3.14). The representation $(\mathcal{V}_k, \pi_{\nu(k)}^\flat, \mathfrak{h})$ (and the corresponding completion as representation of H) appears as an irreducible discrete component in $(\mathcal{W}_k, \pi_{\nu(k)}, \mathfrak{g})$ (respectively of G) restricted to $\mathfrak{h}$ (resp. H).*

Proof. Let $Q = Q_\nu$ be the quotient map $Q : X_\nu^\flat \to X_\nu^\flat/M_\nu^\flat := \mathcal{V}_k$ at the reducible point ν as above for the group H. The map $QR : X_\nu \to X_\nu^\flat \to \mathcal{V}_k$ is clearly $(\pi_\nu^\flat, \mathfrak{h})$ intertwining and induces a map

$$QR : \mathcal{W}^k = X_\nu/M_\nu \to X_\nu^\flat/M_\nu^\flat = \mathcal{V}_k.$$

We prove the boundedness of QR by the method above. Notice that the asymptotic exponent of $\lambda(\nu)$ has the same dependence for positive ν, e.g. in the case $\mathbb{F} = \mathbb{R}$ with $\nu = -k$,

$$\lambda_\nu(p) \sim (p+1)^{n-1-2\nu}, \quad p \geq k+1,$$

and $n - 1 - 2\nu \geq 2$. Thus the same proof carries over to all cases, and we omit the details. □

There is some slight difference when $n = 3$ for $\mathbb{F} = \mathbb{R}$, as $H = SO_0(2,1)$ has its maximal compact subgroup being the torus and there is a splitting of the restriction to holomorphic and antiholomorphic discrete series. Note that we have also excluded the case $\mathbb{F} = \mathbb{C}$, $n = 2$, namely $SU(2,1)$, as the restriction map above is zero on the quotient $\mathcal{W}_{-2k}$; actually $\mathcal{W}_{-2k}$ is a discrete series and its branching under $SU(1,1)$ can be studied using some general tools [18,28].

3.6. The representation π_0 and $\pi_0^{\pm}$ for $SU(n,1)$

The representation $\pi^{\pm}$ on the quotient

$$\pi_0^+ = \sum_{p=0}^{\infty} W^{p,0}/\mathbb{C}, \quad \pi_0^- = \sum_{p=0}^{\infty} W^{0,p}/\mathbb{C}$$

is unitarizable representation of $\mathfrak{g}$. π_0^+ can be constructed also by using the analytic continuation of the weighted Bergman space [8, Theorem 5.4], i.e. scalar holomorphic discrete series, on the unit ball G/K in $\mathbb{C}^n$ with reproducing kernel $(1-(z,w))^{-\mu}$ at the reducible point $\mu = 0$; see e.g. [12] where a reproducing kernel and its expansion are found for the space. A full decomposition under $SU(n-1,1)$ of the series and their quotient can be obtained easily. Indeed let $\pi_0^{\pm,\flat}$ be the corresponding representation for H and $\pi_j^{+,\flat}$ the unitary representation of H realized as the space of holomorphic functions on the unit ball $\{z \in \mathbb{C}^{n-1}; |z| < 1\}$ with reproducing kernel $(1-(z,w))^j$, with H acting as

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H, f(z) \mapsto (cz+d)^{-j} f((az+b)(cz+d)^{-1}).$$

$\pi_j^{+,\flat}$ is a discrete series of H only when $j \geq n$. Define analogously $\pi_j^{-,\flat}$ in terms of conjugate holomorphic functions. The following result can be obtained by considering expansion of holomorphic functions $f(x)$ in the last variable x_n. It is also a consequence of the general theory developed in [19].

Proposition 3.10. *The representation $(\pi_0^{\pm}, G)$ is decomposed under H as*

$$\pi_0^{\pm} = \pi_0^{\pm,\flat} \oplus (\sum_{j=1}^{\infty}{}^{\oplus} \pi_j^{\pm,\flat}).$$

4. Restriction of $(F_{4(-20)}, \pi_\nu)$ to $H = Spin_0(8,1)$

4.1. The subgroup $Spin_0(8,1)$

Recall from Section 2.3 that $H_0 \in \mathfrak{p} = \mathbb{O}^2$ has nonzero roots ± 2, ± 1 in $\mathfrak{g}$. Let $\mathfrak{g}_{\pm 2}$ and $\mathfrak{g}_{\pm 1}$ be the respective root spaces. Then the Lie algebras $\mathfrak{g}_{\pm 1}$ generate a subalgebra of $\mathfrak{g}$ of rank one which is easily seen to be $\mathfrak{h} := spin(8,1)$. The Cartan decomposition of $\mathfrak{h}$ is $\mathfrak{h} = spin(8) \oplus \mathbb{O}$ with $spin(8)$ acting on $\mathbb{O}$ by the spin representation. The simply connected subgroup of G with Lie algebra $\mathfrak{h}$ is then $Spin_0(8,1)$ whose maximal compact group is $L = Spin(8)$; see e.g. [5]. The element $H_0 \in \mathfrak{p} = \mathbb{O}^2$ is also in $\mathfrak{h} = \mathfrak{spin}(8,1)$. Notice now that the roots of $\mathfrak{h} = \mathfrak{spin}(8,1) = \mathfrak{so}(8,1)$ under H_0 are $\{\pm 2\}$. Thus there is a discrepancy between the normalization of the H_0 here with that in Section 2.1 for $\mathfrak{so}(8,1)$; the roots of H_0 in Section 2.1 are $\{\pm 1\}$.

It follows from the decomposition (2.5) that the stabilizer of $H_0 \in \mathbb{O} \subset \mathfrak{h}$ in H is also $M = Spin(7)$ and that $L/M = Spin(8)/Spin(7)$ is the sphere S^7. We have thus

$$L^2(S^7) = \sum_{p\geq 0}^{\oplus} V^p$$

where V^p is the space of spherical harmonics of degree p on S^7, defined by the condition $x_2 = 0$ in $S^{15} = \{x = (x_1, x_2) \in \mathbb{O}^2; |x| = 1\}$. The decomposition of $L^2(S^{15}) = L^2(K/M)$ is given in (2.6) with $(W^{p,q})^M = \mathbb{C}\phi_{p,q}$. We consider now the restriction of $W^{p,q}\big|_{x_2=0}$ of the components $W^{p,q}$.

Lemma 4.1. *The decomposition of $W^{p,q}$ under $Spin(8)$ is multiplicity free and $W^{p,q}\big|_{x_2=0} = V^q$; in other words the only irreducible component in the decomposition with non-zero restriction to S^7 is the representation V^q. Moreover the square norm of $R: W^{p,q} \to V^q$ is given by*

$$\begin{aligned}\|R\|^2 &= C\frac{(p+7)\prod_{j=0}^{2}(p+q+8+2j)(p-q+2+2j)(q+4+2j)(q+1+2j)}{(q+3)(q+1)_5} \\ &\sim (p+1)(p+q+1)^3(p-q+1)^3\end{aligned}$$

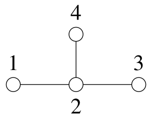
where C is a numerical constant independent of p and q.

Remark 4.2. The representation $W^{p,q}$ of $K = Spin(9)$ is of highest weight $\frac{p-q}{2}\lambda_1 + q\lambda_4$ with λ_1, λ_2, λ_3, λ_4 the fundamental weights dual to the simple roots α_1, α_2, α_3, α_4. In the standard notation they are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$, $\alpha_4 = e_4$ and $\frac{p-q}{2}\lambda_1 + q\lambda_4 = \frac{p}{2}e_1 + \frac{q}{2}e_2 + \frac{q}{2}e_3 + \frac{q}{2}e_4 = \frac{1}{2}(p,q,q,q)$. The four simple roots for $spin(8)$ are $\delta_1 = e_1 - e_2$, $\delta_2 = e_2 - e_3$, $\delta_3 = e_3 - e_4$, $\delta_4 = e_3 + e_4$. The branching rule above can be formulated as

$$W^{\frac{1}{2}(p,q,q,q)}\big|_{x_2=0} = V^{\frac{q}{2}(1,1,1,1)},$$

with $V^{\frac{q}{2}(1,1,1,1)}$ being the space of spherical harmonics of degree q on S^7, which is also of highest weight qe_1 as $SO(8)$ representation. There is a discrepancy of this branching with the usual one for $(SO(9), SO(8))$, which is explained by the triality in $Spin(8)$. The

Dynkin diagram of *Spin*(8) is



There is a symmetry of the permutation group S_3 (as outer automorphisms) acting on the three simple roots δ_1, δ_3, δ_4. The highest weight $\frac{q}{2}(1,1,1,1) = \frac{q}{2}(\delta_1 + 2\delta_2 + \delta_3 + 2\delta_4)$, whereas $qe_1 = \frac{q}{2}(2\delta_1 + 2\delta_2 + \delta_3 + \delta_4)$ and the permutation (134) exchanges the two weights. Also the multiplicity one property of $W^{p,q}$ under $M = Spin(7)$ factors through $Spin(8)$ and we have $(W^{p,q})^M\big|_{x_2=0} = (W^{p,q}\big|_{x_2=0})^M = (V^q)^M = \mathbb{C}\,\phi_q^8$. Note that $M = Spin(7)$ in $Spin(8) \subset Spin(9)$ is not the obvious copy of $Spin(7)$ in $Spin(9)$ defined by the standard inclusion $\mathbb{R}^7 \subset \mathbb{R}^8 \subset \mathbb{R}^9$; in the space $W^{p,q}$ the former copy $M = Spin(7)$ has a unique fixed vector up to scalar, whereas the latter copy $Spin(7)$ has arbitrarily large multiplicities by the construction of Gelfand–Zetlin basis [36].

Proof. The first statement is well-known. Any irreducible representation of $Spin(8)$ in $W^{p,q}\big|_{x_2=0}$ is a constituent in $L^2(S^7)$ and contains thus a unique $M = Spin(7)$-invariant element. But $(W^{p,q}\big|_{x_2=0})^M = (W^{p,q})^M\big|_{x_2=0} = \mathbb{C}\phi^{p,q}\big|_{x_2=0}$, and $\phi^{p,q}\big|_{x_2=0}$ is

$$\phi_{p,q}(\cos\eta, 0) = \cos^q \eta F(-\frac{q}{2}, -\frac{q-1}{2}; \frac{7}{2}; -\tan^2\eta)$$

which is precisely the M-invariant spherical harmonics $\phi_q^8(\cos\eta)$ on S^7, Section 2.2. Thus $W^{p,q}\big|_{x_2=0}$ is nonzero and is just V^q. In particular the element $\phi_{p,q}$ is in the irreducible component $\widetilde{V}^q \subset W^{p,q}$ of V^q. The squared operator norm of R on $W^{p,q}$ is

$$\|R\|^2 = \|R\phi_{p,q}\|^2\|^2\phi_{p,q}\|^{-2}, \quad R\phi_{p,q} = \phi_q^8.$$

Both norms can be evaluated by the dimension formula. Following the notation in the above remark we have $W^{p,q}$ has highest weight $\frac{p}{2}e_1 + \frac{q}{2}e_2 + \frac{q}{2}e_3 + \frac{q}{2}e_4$ with the positive roots being $\{e_i \pm e_j, e_i,\ 1 \le i < j \le 4\}$, and the dimension of $W^{p,q}$ is then

$$\begin{aligned}\dim W^{p,q} &= C_1(p+7)\prod_{j=0}^{2}(p+q+8+2j)(p-q+2+2j)(q+4+2j)(q+1+2j)\\ &\sim (p+1)(p+q+1)^3(p-q+1)^3(q+1)^6\end{aligned}$$

whereas the dimension of V^q is

$$\dim V^q = C_2(q+3)(q+1)_5 \sim (q+1)^6$$

for some constants C_1, C_2 independent of p and q. This completes the proof. $\square$

4.2. Discrete components

Define the principal series representation π_ν of G as in (3.1), realized on $L^2(K/M) = L^2(S^{15})$. We study now the branching of the complementary series under $Spin_0(8,1) \subset G$. Denote by $\pi^\flat_\mu$ the principal series representation of $SO_0(8,1)$, thus also for $Spin_0(8,1)$, as defined and normalized for $SO_0(n,1)$ in (3.1). Recall that the unitary principal series π_ν of G appear on the line $\frac{11}{2} + i\mathbb{R}$ and interval for the complementary series [13] is $(6,16)$, whereas the corresponding line and interval for $SO_o(8,1)$ are $\frac{7}{2} + i\mathbb{R}$, and $(0,7)$.

The restriction map $R : f(x_1,x_2) \mapsto Rf(x_1) = f(x_1,0)$, $(x_1,x_2) \in \mathbb{O}^2$, defines an $\mathfrak{h}$-intertwining operator

$$R : (X_\nu, \pi_\nu, \mathfrak{g}) \to (X^\flat_{\frac{\nu}{2}}, \pi^\flat_{\frac{\nu}{2}}, \mathfrak{h}),$$

the rescaling $\frac{\nu}{2}$ of the parameter ν being due to the discrepancy mentioned above.

We shall need the results in [13] on the unitary norm of each K-type in spherical complementary series of G.

Theorem 4.3. *Let $6 < \nu < 16$. There is a positive definite $(\mathfrak{g}, \pi_\nu)$-invariant form on the $(\mathfrak{g}, K)$-module $\sum_{p,q} W^{p,q}$ defined as in (3.2) with*

$$\|w_{p,q}\|^2_\nu = \lambda_\nu(p,q)\|w_{p,q}\|^2, \quad \lambda_\nu(p,q) = \frac{(8-\frac{\nu}{2})_{\frac{p-q}{2}}}{(\frac{\nu}{2}-3)_{\frac{p-q}{2}}} \frac{(11-\frac{\nu}{2})_{\frac{p+q}{2}}}{(\frac{\nu}{2})_{\frac{p+q}{2}}}.$$

Theorem 4.4. *Let $6 < \nu < 7$. The restriction of (π_ν, G) on H contains $(\pi^\flat_{\frac{\nu}{2}}, H)$ as a discrete component.*

Proof. The λ_ν and $\lambda^\flat_\mu$ in this case are

$$\lambda_\nu(p,q) \sim (p-q+1)^{11-\nu}(p+q+1)^{11-\nu}, \quad \lambda_\mu(q) \sim (q+1)^{7-2\mu}$$

with $p - q = 2k \geq 0$ even. The sum to be treated is

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{q+k+1}{(k+1)^{8-\nu}(2k+q+1)^{8-\nu}} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{(k+1)^{8-\nu}(2k+q+1)^{7-\nu}} \\
&\leq \frac{1}{(q+1)^{7-\nu}} \sum_{k=0}^{\infty} \frac{1}{(k+1)^{8-\nu}} \\
&= C\frac{1}{(q+1)^{7-\nu}} \sim \frac{1}{\lambda^\flat_{\frac{\nu}{2}}(q)},
\end{aligned}$$

completing the proof. □

Remark 4.5. The complementary series for G is parametrized in [5, Example C] as $-5 < \lambda < 5$ using the same parametrization [15]. Our ν is their $\rho + \lambda = 11 + \lambda$, with $\rho = 11$. It is stated there that the point $\lambda = 3$, i.e. $\nu = 8$ is in the automorphic dual $\hat{G}_{\mathrm{aut}}$ of G. Note that this point falls outside the range $6 < \nu < 7$ in our theorem. One can draw some conclusion on the nonexistence of certain intervals in the set $\hat{H}_{\mathrm{Raman}}$ from the Burger–Li–Sarnak conjecture on the Ramanujan dual $\hat{H}_{\mathrm{Raman}}$ for $H = SO(n, 1)$ and our theorem above. In view of [5, Theorem 1] it would be also interesting to study the induction of automorphic representations of H to G.

The representation π_ν has also unitarizable subquotients at integral ν: for $\nu = 6 - 2k$, $k \geq 0$, the quotient

$$\mathcal{W}^\nu = X_\nu / M_\nu, \quad M_\nu = \sum_{p-q \leq k} W^{p,q},$$

is unitarizable; see [13]. However in this case the restriction composed with quotient map is zero. Presumably there is no discrete component under H and it would be interesting to pursue this further.

The main results in the present paper prove the existence of one single discrete component under H of a complementary series of G. There can be more discrete components; see e.g. [4,26,32].

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