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Branching-stable point processes

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Abstract

The notion of stability can be generalised to point processes by defining the scaling operation in a randomised way: scaling a configuration by t corresponds to letting such a configuration evolve according to a Markov branching particle system for -$\log t$ time. We prove that these are the only stochastic operations satisfying basic associativity and distributivity properties and we thus introduce the notion of branching-stable point processes. For scaling operations corresponding to particles that branch but do not diffuse, we characterise stable distributions as thinning-stable point processes with multiplicities given by the quasi-stationary (or Yaglom) distribution of the branching process under consideration. Finally we extend branching-stability to continuous random variables with the help of continuous branching (CB) processes, and we show that, at least in some frameworks, branching-stable integer random variables are exactly Cox (doubly stochastic Poisson) random variables driven by corresponding CB-stable continuous random variables.

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1 Introduction

The concept of stability is central in Probability theory: it inevitably arises in various limit theorems involving scaled sums of random elements. Recall that a random vector ξ (more generally, a random element in a Banach space) is called *strictly α-stable* or $\mathrm{St}\alpha\mathrm{S}$, if

$$t^{1/\alpha}\xi' + (1-t)^{1/\alpha}\xi'' \overset{\mathcal{D}}{=} \xi \qquad \text{for all } t \in [0,1], \tag{1.1}$$

where ξ' and ξ'' are independent copies of ξ and $\overset{\mathcal{D}}{=}$ denotes equality in distribution. When a limiting distribution for the sum of n independent vectors scaled by $n^{1/\alpha}$ exist, it must be $\mathrm{St}\alpha\mathrm{S}$, since one can divide the sum into first tn and the last $(1-t)n$ terms which, in turn, also converge to the same law. This simple observation gives rise to the defining identity (1.1). It is remarkable that the same argument applies to random elements in

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much more general spaces where two abstract operations are defined: a sum and a scaling by positive numbers which should satisfy mild associativity, distributivity and continuity conditions, i.e. in a cone, see [6]. For instance, a random measure ξ on a general complete separable metric space is called strictly α-stable if identity (1.1) is satisfied, where the summation of measures and their multiplication by a number are understood as the corresponding arithmetic operations on the values of these measures on every measurable set. Stable measures are the only class of measures which arise as a weak limit of scaled sums of random measures.

Since the notion of stability relies on multiplication of a random element by a number between 0 and 1, integer valued random variables cannot be $\mathrm{St}\alpha\mathrm{S}$. Therefore Steutel and van Harn in their pioneering work [19] defined a stochastic operation of *discrete multiplication* on positive integer random variables and characterised the corresponding *discrete α-stable* random variables. In a more general context, the discrete multiplication corresponds to the *thinning operation* on point processes when a positive integer random variable is regarded as a trivial point process on a phase space consisting of one point (so it is just the multiplicity of this point). This observation leads to the notion of *thinning stable* or discrete α-stable point processes (notation: $\mathrm{D}\alpha\mathrm{S}$) as the processes Φ which satisfy

$$t^{1/\alpha} \circ \Phi' + (1-t)^{1/\alpha} \circ \Phi'' \overset{\mathcal{D}}{=} \Phi \qquad \text{for all } t \in [0,1], \tag{1.2}$$

when multiplication by a $t \in [0,1]$ is replaced by the operation $t\circ$ of *independent thinning* of their points with the retention probability t. The $\mathrm{D}\alpha\mathrm{S}$ point processes are exactly the processes appearing as a limit in the superposition-thinning schemes (see [14, Ch. 8.3]) and their full characterisation was given in [7].

In its turn, a thinning could be thought of as a particular case of a branching operation where a point either survives with probability t or is removed with the complimentary probability. This observation leads to a new notion of discrete stability for point processes by considering a more general branching operation based on a subcritical Markov branching process $(Y_t)_{t>0}$ with generator semigroup $\mathcal{F} = (F_t)_{t\geq 0}$, satisfying $Y_0 = 1$. Following Steutel and Van Harn [21] who considered the case of integer-valued random variables, we denote this operation by $\circ_{\mathcal{F}}$. In this setting when a point process is “multiplied" by a real number $t \in (0,1]$, every point is replaced by a collection of points located in the same position of their progenitor. The number of points in the collection is a random variable distributed as $Y_{-\log t}$. This operation preserves distributivity and associativity with respect to superposition and generalises the thinning operation. In Section 3 we study stable point processes with respect to this branching operation $\circ_{\mathcal{F}}$ calling them $\mathcal{F}$-stable point processes. We show that $\mathcal{F}$-stable point processes are essentially $\mathrm{D}\alpha\mathrm{S}$ processes with multiplicities which follow the limit distribution Y_∞ of the branching process Y_t conditional on its survival (Yaglom distribution) and we deduce their further properties.

In a broader context, given an abstract associative and distributive stochastic operation $\bullet$ on point processes, a process Φ is stable with respect to $\bullet$ if and only if

$$\forall n \in \mathbb{N}\ \exists c_n \in [0,1] \ : \ \Phi \overset{\mathcal{D}}{=} c_n \bullet (\Phi^{(1)} + \dots + \Phi^{(n)}),$$

where $\Phi^{(1)}, \dots, \Phi^{(n)}$ are independent copies of Φ. In such a context stable point processes arise inevitably in various limiting schemes similar to the central limit theorem involving superposition of point processes. In Section 4 we study and characterise this class of stochastic operations. We prove that a stochastic operation on point processes satisfies associativity and distributivity if and only if it presents a branching structure: “multiplying" a point process by t is equivalent to let the process evolve for time $-\log t$ according to some general Markov branching process which may include diffusion or jumping of the points. We characterise branching-stable (i.e. stable with respect to $\bullet$) point processes for some specific choices of $\bullet$, pointing out possible ways to obtain a characterisation for general branching operations. In order to do so we introduce a stochastic operation for continuous frameworks based on continuous-state branching Markov processes and conjecture that branching stability of point processes and continuous-branching stability of random measures should be related in general: branching-stable point processes are Cox processes driven by branching-stable random measures.

2 Preliminaries

In this section we fix the notation and provide the necessary facts about branching processes and point processes that we will use. We then address the notion of discrete stability for random variables and point processes that we generalise in subsequent sections.

2.1 Branching processes refresher

Here we present some results from [2, Ch.III], [11, Ch.V], [17] and [21] about continuous branching processes that we will need. Let $(Y_s)_{s\geq 0}$ be a $\mathbb{Z}_+$-valued continuous-time Markov branching process with $Y_0 = 1$ almost surely, where $\mathbb{Z}_+$ denotes the set of non-negative integers. Markov branching processes describe the evolution of the total size of a collection of particles undergoing the following dynamic: each particle, independently of the others, lives for an exponential time (with fixed parameter) and then it *branches*, meaning that it is replaced by a random number of offspring particles (according to a fixed probability distribution), which then start to evolve independently. Such a branching process is governed by a family of probability generating functions (p.g.f.'s) $\mathcal{F} = (F_s)_{s\geq 0}$, where F_s is the p.g.f. of the integer-valued random variable Y_s for every $s \geq 0$. It is sufficient for us here to consider the domain of F_s to be $[0, 1]$. It is well known that the family $\mathcal{F}$ is a composition semigroup:

$$F_{s+t}(\cdot) = F_s\big(F_t(\cdot)\big) \quad \forall s, t \geq 0. \tag{C1}$$

Conversely, by writing relation (C1) explicitly in terms of a power series, it is straightforward to see that the system of p.g.f.'s corresponding to an non-negative integer valued random variables describes a system of particles branching independently with the same offspring distribution, so that Y_s is the number of particles at time s, see also [11, Ch.V.5].

We require that the branching process is *subcritical*, i.e. $\mathbb{E}[Y_s] = F_s'(1) < 1$ for $s > 0$. Rescaling, if necessary, the time by a constant factor, we may assume that

$$\mathbb{E}[Y_s] = F_s'(1) = e^{-s}. \tag{C2}$$

Finally we require the following two regularity conditions to hold:

$$\lim_{s\downarrow 0} F_s(z) = F_0(z) = z \qquad 0 \leq z \leq 1, \tag{C3}$$

$$\lim_{s\to\infty} F_s(z) = 1 \qquad 0 \leq z \leq 1. \tag{C4}$$

(C3) implies that the process starts with a single particle $Y_0 = 1$ and (C4) is a consequence of the subcriticality meaning that eventually $Y_s = 0$.

A rationale behind requiring (C2), (C3) and (C4) will become clear later, see Remark 2.5. Identities (C1) and (C3) imply the continuity and differentiability of $F_s(z)$ with respect to s, see, e.g., [2, Sec.III.3], and thus one can define the *generator of the semigroup* $\mathcal{F}$

$$U(z) := \frac{\partial}{\partial s}\Big|_{s=0} F_s(z) \qquad 0 \leq z \leq 1.$$

The function $U(\cdot)$ is continuous and it can be used to define the *A-function* relative to the branching process

$$A(z) := \exp\Big[-\int_0^z \frac{dx}{U(x)}\Big] \qquad 0 \le z \le 1, \tag{2.1}$$

which is a continuous strictly decreasing function with $A(0) = 1$ and $A(1) = 0$, see, e.g., [2, Sec.III.8]. From (C1) it follows that $U(F_s(z)) = U(z)F_s'(z)$ and therefore

$$A\big(F_s(z)\big) = e^{-s}A(z) \qquad s \ge 0,\ 0 \le z \le 1. \tag{2.2}$$

Definition 2.1. *Let $(Y_s)_{s\ge 0}$ and $\mathcal{F} = (F_s)_{s\ge 0}$ be as above. The* limiting conditional distribution *(or Yaglom distribution) of Y_s is the weak limit of the distributions of $(Y_s|Y_s > 0)$ when $s \to +\infty$. We denote by Y_∞ the corresponding random variable and by $B(\cdot)$ its p.g.f., called the B-function of Y_s.*

The *B-function* of Y_s is given by

$$B(z) := 1 - A(z) = \lim_{s\to+\infty} \frac{F_s(z) - F_s(0)}{1 - F_s(0)}, \qquad 0 \le z \le 1. \tag{2.3}$$

From (2.2) and (2.3) it follows that

$$B\big(F_s(z)\big) = 1 - e^{-s} + e^{-s}B(z), \qquad s \ge 0,\ 0 \le z \le 1. \tag{2.4}$$

Both A and B are continuous, strictly monotone, and surjective functions from $[0, 1]$ to $[0, 1]$, thus the inverse functions A^{-1} and B^{-1} exist and have the same properties. Moreover, using (2.2) we obtain

$$\frac{d}{ds}A(F_s(0))\Big|_{s=0} = \frac{d}{ds}\Big|_{s=0} e^{-s} = 1.$$

At the same time

$$\frac{d}{ds}\Big|_{s=0} A(F_s(0)) = A'(0)\frac{d}{ds}\Big|_{s=0} F_s(0) = A'(0)\frac{d}{ds}\Big|_{s=0} \mathbf{P}\{Y(s) = 0\}.$$

Since Y_s is a continuous Markov branching process, every particle branches after exponentially distributed time with a non-null probability to die out and it follows that

$$\frac{d}{ds}\Big|_{s=0} \mathbf{P}\{Y_s = 0\} > 0$$

implying also that

$$A'(0) = \Big[\frac{d}{ds}\Big|_{s=0} \mathbf{P}\{Y_s = 0\}\Big]^{-1} \in (0, +\infty). \tag{2.5}$$

The simplest, but important for the sequel example is provided by a *pure-death process*.

Example 2.2. Let $(Y_s)_{s\ge 0}$ be a continuous-time pure-death process starting with one individual

$$Y_s = \begin{cases} 1 & \text{if } s < \tau, \\ 0 & \text{if } s \ge \tau, \end{cases}$$

where τ is an exponential random variable with parameter 1. The composition semigroup $\mathcal{F} = \big(F_s\big)_{s\ge 0}$ driving such a process is

$$F_s(z) = 1 - e^{-s} + e^{-s}z \qquad 0 \le z \le 1. \tag{2.6}$$

Clearly $\mathcal{F} = \big(F_s\big)_{s\ge 0}$ satisfies (C1)–(C4). The generator $U(z)$ and the A and B-functions defined above are

$$U(z) = A(z) = 1 - z, \qquad B(z) = z, \qquad 0 \le z \le 1. \tag{2.7}$$

Another example is the *birth and death process*.

Example 2.3. Given two positive parameters λ and μ, assume that each particle disappears from the system at rate μ or it is replaced with two particles at rate λ independently of the others. The total number of particles at each time can either grow or diminish by one. In other words, it is a *linear branching process*, as it is sometimes called. Its generator is given by

$$U(z) = \mu - (\lambda + \mu)z + \lambda z^2. \tag{2.8}$$

The process is subcritical whenever $\mu > \lambda$ and in order to satisfy (C2) one needs to scale time so that $\mu = \lambda + 1$. This defines a one-parametric family of semigroups

$$F_s(z) = 1 - \frac{e^{-s}(1-z)}{1 + \lambda(1 - e^{-s})(1-z)},$$

see [2, p. 109]. Conditions (C1),(C3) and (C4) also hold and the functions A and B are given by

$$A(z) = \frac{(\lambda+1)(1-z)}{1+\lambda(1-z)}, \qquad B(z) = \frac{z}{1+\lambda(1-z)},$$

for z in $[0,1]$. Thus B describes the p.g.f. of a (shifted) Geometric distribution with parameter $(1+\lambda)^{-1}$.

2.2 Point processes refresher

We now pass to the necessary definitions related to point processes. The details can be found, for instance, in [4], [5] and [14]. A *random measure* on a *phase space* $\mathcal{X}$ which we assume to be a locally compact second countable Hausdorff space, is a measurable mapping ξ from some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ into the measurable space $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, where $\mathcal{M}$ denote the set of all Radon measures on the Borel σ-algebra $\mathcal{B}(\mathcal{X})$ of subsets of $\mathcal{X}$ and $\mathcal{B}(\mathcal{M})$ is the minimal σ-algebra that makes the mappings $\mu \mapsto \mu(B)$, $\mu \in \mathcal{M}$ measurable for all $B \in \mathcal{B}(\mathcal{X})$.

The distribution of a random measure is characterised by the *Laplace functional* $L_\xi[u]$ which is defined for the class $\mathrm{BM}_+(\mathcal{X})$ of non-negative bounded functions u with bounded support by means of

$$L_\xi[u] = \mathbf{E}\exp\big\{-\langle u, \xi\rangle\big\}, \quad u \in \mathrm{BM}_+(\mathcal{X}). \tag{2.9}$$

Here and below $\langle u, \mu\rangle$ stands for the integral $\int u(x)\,\mu(dx)$ over the the whole $\mathcal{X}$ unless specified otherwise.

A *point process* (p.p.) Φ is a random *counting* measure, i.e. a random measure that with probability one takes values in the set $\mathcal{N}$ of all boundedly finite counting measures on $\mathcal{B}(\mathcal{X})$. The corresponding σ-algebra $\mathcal{B}(\mathcal{N})$ is the restriction of $\mathcal{B}(\mathcal{M})$ onto $\mathcal{N}$. The *support* of $\varphi \in \mathcal{N}$ is the set $\operatorname{supp}\varphi := \{x \in \mathcal{X} :\ \varphi(\{x\}) > 0\}$. A point process is called *simple* (or without multiple points) if $\mathbf{P}\Phi^{-1}\{\varphi \in \mathcal{N} :\ \varphi(\{x\}) \le 1\ \forall x \in \mathcal{X}\} = 1$. The distribution of a point process Φ can be characterised by the probability generating functional (p.g.fl.) $G_\Phi[h]$ defined for functions h such that $0 < h(x) \le 1$ for all $x \in \mathcal{X}$ and such that the set $\{x \in \mathcal{X} :\ h(x) \ne 1\}$ is compact. We denote the class of such functions by $\mathcal{V}(\mathcal{X})$. Then

$$G_\Phi[h] = L_\Phi[-\log h] = \mathbf{E}\exp\big\{\langle \log h, \Phi\rangle\big\}, \quad h \in \mathcal{V}(\mathcal{X}).$$

For a simple p.p. Φ, this expression simplifies to

$$G_\Phi[h] = \mathbf{E}\prod_{x_i \in \operatorname{supp}\Phi} h(x_i), \quad h \in \mathcal{V}(\mathcal{X}).$$

A *Poisson point process* with *intensity measure* Λ is the p.p. Π having the p.g.fl.

$$G_\Pi[h] = \exp\{-\langle 1-h, \Lambda\rangle\}, \quad h \in \mathcal{V}(\mathcal{X}).$$

It is characterised by the following property: given a family of disjoint sets $B_i \in \mathcal{B}(\mathcal{X})$, $i = 1,\dots,n$, the counts $\Pi(B_1),\dots,\Pi(B_n)$ are mutually independent Poisson $\mathsf{Po}(\Lambda(B_i))$ distributed random variables for $i = 1,\dots,n$.

Given a random measure ξ, a *Cox process* with *parameter measure* ξ is the point process with the p.g.fl.

$$G_\Phi[h] = \mathbf{E}\exp\{-\langle 1-h, \xi\rangle\}, \quad h \in \mathcal{V}(\mathcal{X}). \tag{2.10}$$

It is called *doubly-stochastic,* since it can be constructed by first taking a realisation $\xi(\omega)$ of the parameter measure and then taking a realisation of a Poisson p.p. with intensity measure $\xi(\omega)$.

Consider a family of point processes $(\Psi_y)_{y\in\mathcal{Y}}$ on $\mathcal{X}$ indexed by the elements of a locally compact and second countable Hausdorff space $\mathcal{Y}$ which may or may not be $\mathcal{X}$ itself. Such a family is called a *measurable family* if $\mathbf{P}_y(A) := \mathbf{P}(\Psi_y \in A)$ is a $\mathcal{B}(\mathcal{Y})$-measurable function of y for all $A \in \mathcal{B}(\mathcal{N})$.

Given a point process Ξ on $\mathcal{Y}$ and a measurable family of point processes $(\Psi_y)_{y\in\mathcal{Y}}$ on $\mathcal{X}$, the *cluster process* is the following random measure:

$$\Phi(\cdot) = \int_{\mathcal{Y}} \Psi_y(\cdot)\,\Xi(dy) \tag{2.11}$$

The p.p. Ξ is then called the *center* process and $\Psi_y,\ y \in \mathcal{Y}$ are called the *component processes* or *clusters*. The commonest model is when the clusters in (2.11) are independent for different $y_i \in \operatorname{supp}\Xi$ given a realisation of Ξ. In this case, if $G_\Xi[h]$ is the p.g.fl. of the center process and $G_\Psi[h|y]$ are the p.g.fl.'s of Ψ_y, $y \in \mathcal{Y}$, then the p.g.fl. of the corresponding cluster process (2.11) is given by the composition

$$G_\Phi[h] = G_\Xi\big[G_\Psi[h|\cdot]\big]. \tag{2.12}$$

2.3 Stability for discrete random variables

Let X be a $\mathbb{Z}_+$-valued random variable. As in [19], we define an operation of discrete multiplication $\circ$ by a number $t \in [0,1]$

$$t \circ X \overset{\mathcal{D}}{=} \sum_{i=1}^{X} Z^{(i)}, \tag{2.13}$$

where $\{Z^{(i)}\}_{i\in\mathbb{N}}$ are independent and identically distributed (i.i.d.) random variables with Bernoulli distribution $\mathsf{Bin}(1,t)$. A random variable X (or its distribution) is then called *discrete α-stable* (notation: DαS) if

$$t^{1/\alpha} \circ X' + (1-t)^{1/\alpha} \circ X'' \overset{\mathcal{D}}{=} X \qquad \text{for all } t \in [0,1], \tag{2.14}$$

where X', X'' are independent distributional copies of X.

Letting each point i evolve as a pure-death process $Y^{(i)}$ independently of the others (see Example 2.2), after time $-\log t$, $t \in (0,1]$, the number of surviving points will be distributed as (2.13). So alternatively,

$$t \circ X \overset{\mathcal{D}}{=} \sum_{i=1}^{X} Y^{(i)}_{-\log t}\,.$$

Replacing here the pure-death with a general branching process allowed the authors of [21] to define a more general *branching* operation and the corresponding $\mathcal{F}$*-stable* non-negative integer random variables as follows. Let $\{Y^{(i)}\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. continuous-time Markov branching processes driven by a semigroup $\mathcal{F}=(F_s)_{s\geq 0}$ satisfying the conditions (C1)-(C4) in the previous section. Given $t\in(0,1]$ and a $\mathbb{Z}_+$-valued random variable X (independent of $\{Y^{(i)}\}_{i\in\mathbb{N}}$) define

$$t\circ_{\mathcal{F}} X := \sum_{i=1}^{X} Y^{(i)}_{-\log t}\,. \tag{2.15}$$

The notion of $\mathcal{F}$-stability is then defined in an analogous way to discrete stability:

Definition 2.4. *A $\mathbb{Z}_+$-valued random variable X (or its distribution) is called $\mathcal{F}$-*stable *with exponent α if*

$$t^{1/\alpha}\circ_{\mathcal{F}} X' + (1-t)^{1/\alpha}\circ_{\mathcal{F}} X'' \overset{\mathcal{D}}{=} X \qquad \forall t\in[0,1], \tag{2.16}$$

where X' and X'' are independent copies of X.

In terms of the p.g.f. $G_X(z)$ of X, (2.16) is equivalent to

$$G_X(z)=G_X\big(F_{-\alpha^{-1}\log t}(z)\big)\cdot G_X\big(F_{-\alpha^{-1}\log(1-t)}(z)\big) \qquad 0\leq z\leq 1.$$

Let $G_{t\circ_{\mathcal{F}}X}(z)$ denote the p.g.f. of $t\circ_{\mathcal{F}} X$. By independence of $\{Y^{(i)}(\cdot)\}_{i\in\mathbb{N}}$ and X, (2.15) is equivalent to

$$G_{t\circ_{\mathcal{F}}X}(z)=G_X\big(F_{-\log t}(z)\big) \qquad 0\leq z\leq 1. \tag{2.17}$$

It is easy to verify that (C1) and (2.17) make the branching operation $\circ_{\mathcal{F}}$ associative, commutative and distributive with respect to the sum of random variables, i.e. for all $t,t_1,t_2\in[0,1]$ and X independent of X'

$$t_1\circ_{\mathcal{F}}(t_2\circ_{\mathcal{F}} X)\overset{\mathcal{D}}{=}(t_1t_2)\circ_{\mathcal{F}} X\overset{\mathcal{D}}{=} t_2\circ_{\mathcal{F}}(t_1\circ_{\mathcal{F}} X), \tag{2.18}$$

$$t\circ_{\mathcal{F}}(X+X')\overset{\mathcal{D}}{=} t\circ_{\mathcal{F}} X + t\circ_{\mathcal{F}} X'. \tag{2.19}$$

Remark 2.5. As shown in [20, Section V.8, equations (8.6)-(8.8)], conditions (C2), (C3) and (C4) guarantee that $\circ_{\mathcal{F}}$ has some "multiplication-like" properties. In particular (C3) and (C4) imply respectively that $\lim_{t\uparrow 1} t\circ_{\mathcal{F}} X\overset{\mathcal{D}}{=}1\circ_{\mathcal{F}} X\overset{\mathcal{D}}{=}X$ and $\lim_{t\downarrow 0} t\circ_{\mathcal{F}} X\overset{\mathcal{D}}{=}0$. Furthermore, (C2) implies that, in case the expectation of X is finite, $\mathbf{E}[t\circ_{\mathcal{F}} X]=t\,\mathbf{E}\,X$.

The following theorem gives a characterisation of $\mathcal{F}$-stable distributions on $\mathbb{Z}_+$, see [21, Theorem 7.1] and [20, Theorem V.8.6]:

Theorem 2.6. *Let X be a $\mathbb{Z}_+$-valued random variable and $G_X(z)$ its p.g.f., then X is $\mathcal{F}$-stable with exponent α if and only if $0<\alpha\leq 1$ and*

$$G_X(z)=\exp\big\{-cA(z)^\alpha\big\} \qquad 0\leq z\leq 1,$$

where A is the A-function (2.1) *associated to the branching process driven by the semigroup $\mathcal{F}$ and $c>0$. In particular, X is $\mathrm{D}\alpha\mathrm{S}$ if and only if*

$$G_X(z)=\exp\big\{-c(1-z)^\alpha\big\} \qquad 0\leq z\leq 1$$

for some $0<\alpha\leq 1$ and $c>0$, see [19, Theorem 3.2].

2.4 Thinning stable point processes

It was noted in Section 2.3 that the operation of discrete multiplication of an integer random variable $t\circ$ with $t \in [0,1]$ may be thought of as an independent thinning when the random variable is represented as a collection of points and each point is retained with probability t and removed with the complementary probability. Thus the thinning operation generalises the discrete multiplication to general point processes. The corresponding *thinning-stable* or *discrete α-stable* point processes (notation: $\mathrm{D}\alpha\mathrm{S}$) satisfy (1.2) and are exactly the ones which appear as the limit in thinning-superposition schemes, see [14, Ch.8.3]. The full characterisation of these processes is given in [7]. Thinning stable processes exist only for $\alpha \in (0,1]$, and the case $\alpha = 1$ corresponds to the Poisson processes.

To be more specific, we need some further definitions. First we need a way to consistently normalize both finite and infinite measures. Let $B_1, B_2, \ldots$ be a fixed countable base of the topology on $\mathcal{X}$ that consists of relatively compact sets. Append $B_0 = \mathcal{X}$ to this base. For each non-null $\mu \in \mathcal{M}$ consider the sequence of its values $(\mu(B_0), \mu(B_1), \mu(B_2), \ldots)$ possibly starting with infinity, but otherwise finite. Let $i(\mu)$ be the smallest non-negative integer i for which $0 < \mu(B_i) < \infty$, in particular, $i(\mu) = 0$ if μ is a finite measure. Define

$$\mathbb{S} = \{\mu \in \mathcal{M} :\ \mu(B_{i(\mu)}) = 1\}.$$

It can be shown (see [7]) that $\mathbb{S}$ is $\mathcal{B}(\mathcal{M})$-measurable and that $\mathbb{S} \cap \{\mu :\ \mu(\mathcal{X}) < \infty\} = \mathbb{M}_1$ is the family of all probability measures on $\mathcal{X}$. Furthermore, every $\mu \in \mathcal{M} \setminus \{0\}$ can be uniquely associated with the pair $(\hat{\mu}, \mu(B_{i(\mu)})) \in \mathbb{S} \times \mathbb{R}_+$, where $\hat{\mu}$ is defined as $\frac{\mu}{\mu(B_{i(\mu)})}$, and $\mu = \mu(B_{i(\mu)})\hat{\mu}$ is the *polar representation* of μ.

A locally finite random measure ξ is called *strictly stable with exponent* α or $\mathrm{St}\alpha\mathrm{S}$ if it satisfies identity (1.1). It is deterministic in the case $\alpha = 1$ and in the case $\alpha \in (0,1)$ its Laplace functional is given by

$$L_\xi[h] = \exp\Big\{-\int_{\mathcal{M}\setminus\{0\}} (1 - e^{-\langle h,\mu\rangle})\Lambda(d\mu)\Big\}, \quad h \in \mathrm{BM}_+(\mathcal{X}), \tag{2.20}$$

where Λ is a *Lévy measure*, i.e. a Radon measure on $\mathcal{M} \setminus \{0\}$ such that

$$\int_{\mathcal{M}\setminus\{0\}} (1 - e^{-\langle h,\mu\rangle})\Lambda(d\mu) < \infty \tag{2.21}$$

for all $h \in \mathrm{BM}_+(\mathcal{X})$. Such Λ is homogeneous of order $-\alpha$, i.e. $\Lambda(tA) = t^{-\alpha}\Lambda(A)$ for all measurable $A \subset \mathcal{M} \setminus \{0\}$ and $t > 0$, see [7, Th. 2].

Introduce a *spectral measure* σ supported by $\mathbb{S}$ by setting

$$\sigma(A) = \Gamma(1-\alpha)\,\Lambda(\{t\mu :\ \mu \in A,\ t \geq 1\})$$

for all measurable $A \subset \mathbb{S}$, where Γ is the Euler's Gamma-function. Integrating out the radial component in (2.20) leads to the following alternative representation [7, Th. 3]:

$$L_\xi[u] = \exp\Big\{-\int_{\mathbb{S}} \langle u, \mu\rangle^\alpha \sigma(d\mu)\Big\}, \quad u \in \mathrm{BM}_+(\mathcal{X}) \tag{2.22}$$

for some *spectral measure* σ supported by $\mathbb{S}$ which satisfies

$$\int_{\mathbb{S}} \mu(B)^\alpha \sigma(d\mu) < \infty \tag{2.23}$$

for all relatively compact subsets B of $\mathcal{X}$. The latter is a consequence of (2.21) and representation (2.22) is unique.

The importance of $\mathrm{St}\alpha\mathrm{S}$ random measures is explained by the fact that any $\mathrm{D}\alpha\mathrm{S}$ point process Φ is exactly a Cox processes driven by a $\mathrm{St}\alpha\mathrm{S}$ parameter measure ξ: its p.g.fl. has the form

$$G_\Phi[h] = L_\xi[1-h] = \exp\Big\{-\int_{\mathbb{S}} \langle 1-h,\mu\rangle^\alpha \sigma(d\mu)\Big\}, \quad h \in \mathcal{V}(\mathcal{X}) \tag{2.24}$$

for some locally finite spectral measure σ on $\mathbb{S}$ that satisfies (2.23), see [7, Th. 15 and Cor. 16].

In the case when σ charges only probability measures $\mathbb{M}_1$, the corresponding $\mathrm{D}\alpha\mathrm{S}$ p.p.'s are cluster processes. Recall that a positive integer random variable η has *Sibuya* $\mathsf{Sib}(\alpha)$ distribution with parameter α, if its p.g.f. is given by

$$\mathbf{E}\, z^\eta = 1-(1-z)^\alpha\,, \quad z \in (0,1]\,.$$

It corresponds to the number of trials to get the first success in a series of Bernoulli trials with probability of success in the kth trial being α/k.

Definition 2.7. *(See [7, Def.23]) Let μ be a probability measure on $\mathcal{X}$. A point process Υ on $\mathcal{X}$ defined by the p.g.fl.*

$$G_\Upsilon[h] = G_{\Upsilon(\mu)}[h] = 1 - \langle 1-h,\mu\rangle^\alpha, \quad h \in \mathcal{V}(\mathcal{X}), \tag{2.25}$$

is called a Sibuya point process *with* exponent α *and* parameter measure μ. *Its distribution is denoted by* $\mathsf{Sib}(\alpha,\mu)$.

A Sibuya process $\Upsilon \sim \mathsf{Sib}(\alpha,\mu)$ is a.s. finite, the total number of its points $\Upsilon(\mathcal{X})$ follows $\mathsf{Sib}(\alpha)$ distribution and, given the total number of points, these points are independently identically distributed in $\mathcal{X}$ according to μ.

Theorem 2.8 (Th. 24 [7])**.** *A $\mathrm{D}\alpha\mathrm{S}$ point process Φ with a spectral measure σ supported by $\mathbb{M}_1$ can be represented as a cluster process with Poisson centre process on $\mathbb{M}_1$ driven by intensity measure σ and component processes being Sibuya processes $\mathsf{Sib}(\alpha,\mu)$, $\mu \in \mathbb{M}_1$. Its p.g.fl. is given by*

$$G_\Phi[h] = \exp\Big\{\int_{\mathbb{M}_1} (G_{\Upsilon(\mu)}[h]-1)\,\sigma(d\mu)\Big\}, \quad h \in \mathcal{V}(\mathcal{X}),$$

with $G_{\Upsilon(\mu)}[h]$ as in (2.25).

3 $\mathcal{F}$-stability for point processes

We have seen in the previous section that the discrete multiplication operation on integer random variables generalises to the thinning operation on points processes. In a similar fashion, we can extend the branching operation $\circ_\mathcal{F}$ to point processes too.

Let $\{Y_s\}_{s\ge 0}$ be a continuous-time Markov branching process driven by a semigroup $\mathcal{F} = (F_s)_{s\ge 0}$ satisfying conditions (C1)-(C4). Intuitively, given a point process Φ and $t \in (0,1]$, $t \circ_\mathcal{F} \Phi$ is the cluster point process obtained from Φ by replacing every point with $Y_{-\log t}$ points located in the same position (using an independent copy of $Y_{-\log t}$ for each point including the ones in the same position). In this sense, the resulting process is a cluster process. To proceed formally, we first define its component processes.

Definition 3.1. *Given a $\mathbb{Z}_+$-valued random variable Z and $x \in \mathcal{X}$, we denote by Z_x the point process having Z points in x and no points in $\mathcal{X}\backslash\{x\}$. Equivalently Z_x is the point process with p.g.fl. $G_{Z_x}[h] = F\big(h(x)\big)$ for each $h \in \mathcal{V}(\mathcal{X})$, where $F(z)$ is the p.g.f. of Z.*

We can now define the operation $\circ_\mathcal{F}$ for point processes.

Definition 3.2. *Let Φ be a point process and $\{Y_s\}_{s\geq 0}$ be a continuous-time Markov branching process driven by a semigroup $\mathcal{F} = (F_s)_{s\geq 0}$ satisfying conditions* (C1)-(C4)*. For each $t \in (0,1]$, $t \circ_{\mathcal{F}} \Phi$ is the (independent) cluster point process with center process Φ and clusters $\{(Y_{-\log t})_x,\ x \in \mathcal{X}\}$.*

Equivalently $t \circ_{\mathcal{F}} \Phi$ can be defined as the point process with p.g.fl. given by

$$G_{t\circ_{\mathcal{F}}\Phi}[h] = G_\Phi[F_{-\log t}(h)],$$

where G_Φ is the p.g.fl. of Φ. Note that $t \circ_{\mathcal{F}} \Phi$ does not need to be simple (i.e. it can have multiple points), even if Φ is. We define the $\mathcal{F}$-stability for point processes as follows.

Definition 3.3. *A p.p. Φ is $\mathcal{F}$-stable with exponent α (α-stable with respect to $\circ_{\mathcal{F}}$) if*

$$t^{1/\alpha} \circ_{\mathcal{F}} \Phi' + (1-t)^{1/\alpha} \circ_{\mathcal{F}} \Phi'' \overset{\mathcal{D}}{=} \Phi \qquad \forall t \in (0,1], \tag{3.1}$$

where Φ' and Φ'' are independent copies of Φ.

Equivalently, (3.1) can be rewritten in terms of p.g.fl.'s as follows:

$$G_\Phi[h] = G_\Phi\big[F_{-\log t/\alpha}(h)\big]\, G_\Phi\big[F_{-\log(1-t)/\alpha}(h)\big] \qquad \forall t \in (0,1], \forall h \in \mathcal{V}(\mathcal{X}).$$

Remark 3.4. The branching operation $\circ_{\mathcal{F}}$ induced by the pure-death process of Example 2.2 corresponds to the thinning operation. Therefore $\mathrm{D}\alpha\mathrm{S}$ point processes can be seen as a special case of $\mathcal{F}$-stable point processes.

An $\mathcal{F}$-stable point process Φ is necessarily infinitely divisible. Indeed, iterating (3.1) $m-1$ times we obtain

$$m^{-1/\alpha} \circ_{\mathcal{F}} \Phi^{(1)} + \ldots + m^{-1/\alpha} \circ_{\mathcal{F}} \Phi^{(m)} \overset{\mathcal{D}}{=} \Phi, \tag{3.2}$$

where $\Phi^{(1)}, \ldots, \Phi^{(m)}$ are independent copies of Φ.

A characterisation of $\mathcal{F}$-stable point processes is given in the following theorem which generalises [7, Th.15] and which proof we largely follow here.

Theorem 3.5. *A functional $G_\Phi[\cdot]$ is the p.g.fl. of an $\mathcal{F}$-stable point process Φ with exponent of stability α if and only if $0 < \alpha \leq 1$ and there exists a $\mathrm{St}\alpha\mathrm{S}$ random measure ξ such that*

$$G_\Phi[h] = L_\xi\big[A(h)\big] = L_\xi\big[1 - B(h)\big] \qquad \forall h \in \mathcal{V}, \tag{3.3}$$

where $A(z)$ and $B(z)$ are the A-function and B-function of the branching process driven by $\mathcal{F}$.

Proof. Sufficiency: Suppose (3.3) holds. As it was shown in [7], $\mathrm{St}\alpha\mathrm{S}$ random measures exist only for $0 < \alpha \leq 1$, $\alpha = 1$ corresponding to non-random measures. Next, by (2.9) and (2.10), $L_\xi\big[1-h\big]$ as a functional of h is the p.g.fl. of a Cox point process with intensity ξ and $B(z)$ is the p.g.f. of the limiting conditional distribution of the branching process driven by $\mathcal{F}$. Therefore, by (2.12), the functional $G_\Phi[h] = L_\xi\big[1 - B(h)\big]$ is the p.g.fl. of a cluster process, say Φ. We need to prove that Φ is $\mathcal{F}$-stable with exponent α. Given $t \in (0,1]$ and $h \in \mathcal{V}(\mathcal{X})$ it holds that

$$\begin{aligned} &G_\Phi\big[F_{-\log t/\alpha}(h)\big]\, G_\Phi\big[F_{-\log(1-t)/\alpha}(h)\big] = \\ &= L_\xi\Big[A\big(F_{-\log t/\alpha}(h)\big)\Big]\, L_\xi\Big[A\big(F_{-\log(1-t)/\alpha}(h)\big)\Big] \overset{(2.2)}{=} L_\xi\big[t^{1/\alpha}A(h)\big]\, L_\xi\big[(1-t)^{1/\alpha}A(h)\big]. \end{aligned}$$

Since ξ is $\mathrm{St}\alpha\mathrm{S}$, it satisfies (1.1) and thus

$$L_\xi\big[t^{1/\alpha}A(h)\big]\, L_\xi\big[(1-t)^{1/\alpha}A(h)\big] = L_\xi\big[A(h)\big].$$

Therefore

$$G_\Phi\left[F_{-\log t/\alpha}(h)\right] G_\Phi\left[F_{-\log(1-t)/\alpha}(h)\right] = G_\Phi[h]$$

for any h in $\mathcal{V}(\mathcal{X})$, meaning that Φ is $\mathcal{F}$-stable with exponent α.

Necessity: Suppose that Φ is $\mathcal{F}$-stable with exponent α. Writing (3.1) for the values of the measures on a particular compact set $B \in \mathcal{B}(\mathcal{X})$, we see that $\Phi(B)$ is an $\mathcal{F}$-stable random variable with exponent α. Thus by Theorem 2.6 we have $0 < \alpha \le 1$. Now we are going to prove that $G_\Phi[A^{-1}(u)]$, as a functional of u, is the Laplace functional of a $\mathrm{St}\alpha\mathrm{S}$ random measure. While a Laplace functional should be defined on all (bounded) functions with compact support, the expression $G_\Phi[A^{-1}(u)]$ is well defined just for functions with values on $[0,1]$ because $A^{-1} : [0,1] \to [0,1]$. To overcome this difficulty we employ (3.2) which can be written as

$$G_\Phi[h] = \left(G_\Phi[F_{\alpha^{-1}\log m}(h)]\right)^m \qquad \forall h \in \mathcal{V}(\mathcal{X}),$$

and define

$$L[u] = \Big(G_\Phi\left[F_{\alpha^{-1}\log m}(A^{-1}(u))\right]\Big)^m \overset{(2.2)}{=} \Big(G_\Phi[A^{-1}(m^{-1/\alpha}u)]\Big)^m \qquad u \in \mathrm{BM}_+(\mathcal{X}), \quad (3.4)$$

for any $m \ge 1$ such that $m^{-1/\alpha}u < 1$. Note that the right-hand side of (3.4) does not depend on m. Moreover, given $m^{-1/\alpha}u < 1$, the function $A^{-1}(m^{-1/\alpha}u)$ does take values in $[0,1]$ and equals 1 outside of a compact set, implying that $A^{-1}(m^{-1/\alpha}u) \in \mathcal{V}(\mathcal{X})$. Therefore $L[u]$ in (3.4) is well-defined. Since (3.4) holds for all m, it is possible to pass to the limit as $m \to \infty$ to see that

$$L[u] = \exp\Big\{-\lim_{m\to\infty} m(1 - G_\Phi[A^{-1}(m^{-1/\alpha}u)])\Big\} \quad (3.5)$$

We now need the following fact:

$$\lim_{m\to\infty} m(1 - G_\Phi[A^{-1}(m^{-1/\alpha}u)]) = \lim_{m\to\infty} m(1 - G_\Phi[e^{(A^{-1})(0)\, m^{-1/\alpha}u}])\,. \quad (3.6)$$

Since A^{-1} is continuous, strictly decreasing and differentiable in 0 with $A^{-1}(0) = 1$ and $(A^{-1})'(0) < 0$ (see Section 2.1), it follows that for any constant $\varepsilon > 0$ there exists $M(\varepsilon, u) > 0$ such that

$$A^{-1}(m^{-1/\alpha}u(1+\varepsilon)) \le e^{(A^{-1})'(0)m^{-1/\alpha}u} \le A^{-1}(m^{-1/\alpha}u(1-\varepsilon)) \qquad \forall\, m \ge M(\varepsilon, u). \quad (3.7)$$

From (3.5), (3.7) and the monotonicity of G_Φ we can deduce

$$L[(1+\varepsilon)u] \le \exp\Big\{-\lim_{m\to\infty} m(1 - G_\Phi[e^{(A^{-1})'(0)m^{-1/\alpha}u}])\Big\} \le L[(1-\varepsilon)u]\,. \quad (3.8)$$

Note that L is continuous because of (3.4) and the continuity of G_Φ. Therefore taking the limit for ε going to 0 in (3.8) and using (3.5) we obtain (3.6).

From (3.5) and Schoenberg theorem [3, Theorem 3.2.2] it follows that $L[u]$ is positive definite if $\lim_{m\to\infty} m(1 - G_\Phi[1 - B^{-1}(m^{-1/\alpha}u)])$ is negative definite, i.e. by (3.6) if

$$\sum_{i,j=1}^n c_i c_j \lim_{m\to\infty} m(1 - G_\Phi[e^{(A^{-1})(0)\, m^{-1/\alpha}(u_i+u_j)}]) \le 0, \quad (3.9)$$

for all $n \ge 2$, for any $u_1, \dots, u_n \in \mathrm{BM}_+(\mathcal{X})$ and for any $c_1, \dots, c_n$ with $\sum c_i = 0$. If we set $v_i = e^{(A^{-1})(0)\, m^{-1/\alpha}u_i}$, then (3.9) is equivalent to $\sum_{i,j=1}^n c_i c_j \lim_{m\to\infty} G_\Phi[v_i v_j] \ge 0$, which follows from the positive definiteness of G_Φ. Thus, by the Bochner theorem [3, Theorem 4.2.9], the function $L[\sum_{i=1}^k t_i h_i]$ of $t_1, \dots, t_k \ge 0$ is the Laplace transform of a random vector. Moreover $L[\mathbf{0}] = 1$, where $\mathbf{0}$ is the null function on $\mathcal{X}$. Finally from (3.4) and the continuity of the p.g.fl. G_Φ it follows that given $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{BM}_+(\mathcal{X})$, $f_n \uparrow f \in \mathrm{BM}_+(\mathcal{X})$ we have $L[f_n] \to L[f]$ as $n \to \infty$. Therefore we can use Theorem 9.4.II in [5] to obtain that L is the Laplace functional of a random measure ξ.

In order to prove that ξ is $\mathrm{St}\alpha\mathrm{S}$, let $u \in \mathrm{BM}_+(\mathcal{X})$ and take an integer $m \geq (\sup u)^\alpha$ and denote by $\hat{u} = m^{-1/\alpha}u \leq 1$. By (3.4), for any given $t \in (0,1]$ we have

$$L_\xi[u] = G_\Phi^m[A^{-1}(\hat{u})] \overset{(3.1)}{=} G_\Phi^m\big[F_{-\log t/\alpha}(A^{-1}(\hat{u}))\big]\, G_\Phi^m\big[F_{-\log(1-t)/\alpha}(A^{-1}(\hat{u}))\big] \overset{(2.2)}{=}$$
$$G_\Phi^m\big[A^{-1}(t^{1/\alpha}\hat{u})\big]\, G_\Phi^m\big[A^{-1}((1-t)^{1/\alpha}\hat{u})\big] = L_\xi[t^{1/\alpha}u]\, L_\xi[(1-t)^{1/\alpha}u],$$

which implies that ξ is $\mathrm{St}\alpha\mathrm{S}$. □

Corollary 3.6. *A p.p. Φ on $\mathcal{X}$ is $\mathcal{F}$-stable with exponent α if and only if it is a cluster process with $\mathrm{D}\alpha\mathrm{S}$ centre process Ψ on $\mathcal{X}$ and component processes $\big\{(Y_\infty)_x,\ x \in \mathcal{X}\big\}$ (see Definitions 2.1 and 3.1).*

Proof. From Theorem 3.5 and (2.3) it follows that Φ is $\mathcal{F}$-stable if and only if its p.g.fl. satisfies $G_\Phi[h] = L_\xi\big[1 - B(h)\big]$, where $B(\cdot)$ is the p.g.f. of Y_∞, and ξ is a $\mathrm{St}\alpha\mathrm{S}$ random measure. By (2.24) there is a $\mathrm{D}\alpha\mathrm{S}$ point process Ψ with $G_\Psi[h] = L_\xi\big[1-h\big]$. We obtain that $G_\Phi[h] = G_\Psi\big[B(h)\big]$. The result follows from the form (2.12) of the p.g.fl. of a cluster process. □

Corollary 3.6 clarifies the relationship between $\mathcal{F}$-stable and $\mathrm{D}\alpha\mathrm{S}$ point processes: $\mathcal{F}$-stable p.p.'s are an extension of $\mathrm{D}\alpha\mathrm{S}$ p.p.'s, where every point is given an additional multiplicity according to independent copies of Y_∞ (the latter is fixed by $\mathcal{F}$). Note that when the branching operation is thinning, the random variable Y_∞ is identically 1 (that stems from (2.7)) and the $\mathcal{F}$-stable p.p. is the $\mathrm{D}\alpha\mathrm{S}$ centre process itself.

Corollary 3.7. *A p.p. Φ is $\mathcal{F}$-stable with exponent $0 < \alpha \leq 1$ if and only if its p.g.fl. can be written as*

$$G_\Phi[u] = \exp\Big\{ -\int_{\mathbb{S}} \langle 1 - B(u), \mu\rangle^\alpha \sigma(d\mu)\Big\}, \tag{3.10}$$

where σ is a locally finite spectral measure on $\mathbb{S}$ satisfying (2.23).

Proof. If Φ is an $\mathcal{F}$-stable point process with stability exponent α, then by Theorem 3.5 there exist a $\mathrm{St}\alpha\mathrm{S}$ random measure ξ such that

$$G_\Phi[h] = L_\xi\big[A(h)\big] \qquad h \in \mathcal{V}(\mathcal{X}).$$

Thus (3.10) follows from spectral representation (2.22). Conversely, if we have a locally finite spectral measure σ on $\mathbb{S}$ satisfying (2.23) and $\alpha \in (0,1]$, then σ is the spectral measure of a $\mathrm{St}\alpha\mathrm{S}$ random measure ξ, whose Laplace functional is given by (2.22). Therefore (3.10) can be written as

$$G_\Phi[h] = L_\xi\big[1 - B(h)\big],$$

which, by Theorem 3.5 implies the $\mathcal{F}$-stability of Φ. □

We also get the following generalisation of Theorem 2.8.

Theorem 3.8. *An $\mathcal{F}$-stable point process with a spectral measure σ supported only by the set $\mathbb{M}_1$ of probability measures can be represented as a cluster process with centre process being a Poisson process on $\mathbb{M}_1$ driven by the spectral measure σ and daughter processes having p.g.fl. $G_{\Upsilon(\mu)}[B(h)]$, where $\Upsilon(\mu)$ are $\mathsf{Sib}(\alpha,\mu)$ distributed point processes and $B(\cdot)$ is the B-function of the branching process driven by $\mathcal{F}$. The daughter process corresponds to a Sibuya p.p. $\Upsilon(\mu)$ with its every point given a multiplicity according to independent copies of Y_∞.*

Proof. In the case when the spectral measure σ is supported by probability measures, representation (3.10) becomes

$$G_\Phi[h] = \exp\big\{ - \int_{\mathbb{M}_1} \langle 1 - B(h), \mu\rangle^\alpha \sigma(d\mu)\big\} \qquad \forall h \in \mathcal{V}(\mathcal{X}), \tag{3.11}$$

where $\mathbb{M}_1$ is the space of probability measures on $\mathcal{X}$. In terms of the p.g.fl. (2.25) of a Sibuya p.p., this reads

$$G_\Phi[h] = \exp\big\{ - \int_{\mathbb{M}_1} 1 - (1 - \langle 1 - B(h), \mu\rangle^\alpha)\sigma(d\mu)\big\} =$$
$$= \exp\big\{ - \int_{\mathbb{M}_1} \big(1 - G_{\Upsilon(\mu)}[B(h)]\big)\sigma(d\mu)\big\} \qquad h \in \mathcal{V}(\mathcal{X}), \tag{3.12}$$

where $\Upsilon(\mu)$ denotes a point process following the $\mathsf{Sib}(\alpha, \mu)$ distribution. Notice that, since by (2.3), $B(\cdot)$ is the p.g.f. of the distribution Y_∞, $G_{\Upsilon(\mu)}[B(h)]$ is the p.g.fl. of a point process by (2.12). □

As we have seen in (3.2), $\mathcal{F}$-stable processes are infinitely divisible. The latter can be divided into two classes: regular and singular depending on whether their KLM-measure is supported by the set of finite or infinite configurations (see, e.g., [5, Def.10.2.VI]). Similarly to the proof of Theorem 29 in [7] on the decomposition of $\mathrm{D}\alpha\mathrm{S}$ processes, we can extend this result to $\mathcal{F}$-stable processes.

Theorem 3.9. *An $\mathcal{F}$-stable p.p. Φ with a spectral measure σ can be represented as the sum of two independent $\mathcal{F}$-stable point processes:*

$$\Phi = \Phi_r + \Phi_s,$$

where Φ_r is regular and Φ_s singular. Φ_r is an $\mathcal{F}$-stable p.p. with spectral measure being $\sigma\big|_{\mathbb{M}_1} = \sigma(\cdot \cap \mathbb{M}_1)$ and Φ_s is an $\mathcal{F}$-stable p.p. with spectral measure $\sigma\big|_{\mathbb{S}\setminus\mathbb{M}_1}$.

The regular component Φ_r can be represented as a cluster p.p. with p.g.fl. given by

$$G_{\Phi_r}[h] = \exp\big\{ - \int_{\mathbb{M}_1} \big(1 - G_{\Upsilon(\mu)}[B(h)]\big)\sigma|_{\mathbb{M}_1}(d\mu)\big\} \qquad \forall h \in \mathcal{V}(\mathcal{X}).$$

On the contrary, the singular component Φ_s is not a cluster p.p., and its p.g.fl. is given by (3.10) (with σ replaced with $\sigma\big|_{\mathbb{S}\setminus\mathbb{M}_1}$ there).

4 General branching stability for point processes

Stable distributions appear in various limiting schemes because decomposition of a sum into a proportion t and $1 - t$ of the summands inevitably leads to the limiting distribution satisfying (1.1). We have seen that this argument still works for point processes when the multiplication is replaced by a stochastic branching operation, the reason being associativity and distributivity with respect to the sum (superposition). One may ask: to which extent one can generalise this stochastic multiplication operation so that it still satisfies associativity and distributivity? The answer is given in this section: branching operations are, in this sense, the exhaustive generalisation.

4.1 Markov branching processes on $\mathcal{N}$

Markov branching processes on $\mathcal{N}$ (also called branching diffusions or branching particle systems) basically consist of a diffusion component and a branching component: each particle, independently of the others, moves according to a diffusion process and after an exponential time it branches. When a particle branches it is replaced by a random configuration of points (possibly empty, in which case the particle dies) depending on the location of the particle at the branching time (e.g. [1], [9] or [10]).

Alternatively branching particle systems can be defined as Markov processes satisfying the branching property, as follows.

Definition 4.1. A Markov branching process *on $\mathcal{N}$ is a time-homogeneous Markov process $(\Psi_t^\varphi)_{t\geq 0,\,\varphi\in\mathcal{N}}$ on $(\mathcal{N},\mathcal{B}(\mathcal{N}))$, where t denotes time and φ the starting configuration, such that its probability transition kernel $P_t(\varphi,\cdot)$ satisfies the branching property:*

$$P_t(\varphi_1+\varphi_2,\cdot) \;=\; P_t(\varphi_1,\cdot) * P_t(\varphi_2,\cdot), \tag{4.1}$$

for any $t\geq 0$ and φ_1, φ_2 in $\mathcal{N}$.

The branching property (4.1) can also be expressed in terms of p.g.fl.'s as follows:

$$G_t^\varphi[h] \;=\begin{cases} 1, & \text{if } \varphi=\mathbf{0},\\ \prod_{x\in\varphi} G_t^{\delta_x}[h], & \text{if } \varphi\neq\mathbf{0},\end{cases} \qquad h\in\mathcal{V}(\mathcal{X}), \tag{4.2}$$

where G_t^φ and $G_t^{\delta_x}$ are the p.g.fl.'s of Ψ_t^φ and $\Psi_t^{\delta_x}$ respectively (see, e.g., [1, Ch. 5.1] or [8, Ch. 3]). Under some additional regularity assumption, every Markov branching process on $\mathcal{N}$ (defined as above) can be expressed in terms of particles undergoing diffusion and branching see, for example, [12].

In general not every starting configuration $\varphi\in\mathcal{N}$ is allowed. In fact when φ consists of an infinite number of particles, the diffusion component could move an infinite number of particles in a bounded set. Therefore in general one needs to consider only starting configuration φ such that $G_t^\varphi[h]<\infty$ for any $t\geq 0$ (for more details see, e.g., [8, Ch. 5] or [10, Ch. 1.8]).

4.2 General branching operation for point processes

Let $\bullet\,:\;(t,\Phi)\to t\bullet\Phi,\; t\in[0,1]$ be a stochastic operation acting on point processes on $\mathcal{X}$ or, more exactly, on their distributions. We assume $\bullet$ to act independently on each realisation of the point process, meaning that

$$\mathbf{P}(t\bullet\Phi\in A)=\int_{\mathcal{N}}\mathbf{P}(t\bullet\varphi\in A)\mathbf{P}_\Phi(d\varphi), \qquad A\in\mathcal{B}(\mathcal{N}),\; t\in(0,1], \tag{A1}$$

where $\mathbf{P}_\Phi$ is the distribution of Φ. We require $\bullet$ to be associative and distributive with respect to superposition: for any $t,t_1,t_2\in(0,1]$ and Φ,Φ_1,Φ_2 independent p.p.'s on $\mathcal{N}$

$$t_1\bullet(t_2\bullet\Phi) \quad\overset{\mathcal{D}}{=}\quad (t_1t_2)\bullet\Phi \quad\overset{\mathcal{D}}{=}\quad t_2\bullet(t_1\bullet\Phi), \tag{A2}$$

$$t\bullet(\Phi_1+\Phi_2) \quad\overset{\mathcal{D}}{=}\quad t\bullet\Phi_1+t\bullet\Phi_2, \tag{A3}$$

where in (A2) and (A3) the different instances of the $\bullet$ operation are performed independently. Note that (A3) implies that $t\bullet\mathbf{0}\overset{\mathcal{D}}{=}\mathbf{0}$ for all $t\in(0,1]$, where $\mathbf{0}$ is the empty configuration.

Remark 4.2. Because of (A1), $\bullet$ is uniquely defined by its actions on deterministic configurations $\varphi\in\mathcal{N}$. In fact, given (A1), Φ,Φ_1,Φ_2 in (A2) and (A3) can be replaced with deterministic point configurations $\varphi,\varphi_1,\varphi_2\in\mathcal{N}$. Note that, although φ is deterministic, $t\bullet\varphi$ is generally stochastic (as, for example, for the thinning operation).

As we have seen in (2.18) and (2.19), the (local) branching operation $\circ_{\mathcal{F}}$ operation satisfy (A1)-(A3). The following results characterises stochastic operations satisfying (A1)-(A3) in terms of Markov branching processes on $\mathcal{N}$.

Definition 4.3. *We call a stochastic operation* $\bullet$ *acting on p.p.'s on* $\mathcal{X}$ *a* (general) branching operation *if there exist a Markov branching process* $(\Psi_t^{\varphi})_{t\geq 0,\,\varphi\in\mathcal{N}}$ *on* $(\mathcal{N},\mathcal{B}(\mathcal{N}))$*, such that for any p.p.* Φ *on* $\mathcal{X}$

$$\Psi_t^{\Phi} \overset{\mathcal{D}}{=} e^{-t}\bullet\Phi \qquad t\in[0,+\infty). \tag{4.3}$$

Proposition 4.4. *A stochastic operation* $\bullet$ *satisfies* (A1)-(A3) *if and only if it is a general branching operation.*

Proof. Necessity: Let $\bullet$ satisfy (A1)-(A3). Let $P_t(\varphi,\cdot)$ denote the distribution of $e^{-t}\bullet\varphi$. By putting $\Phi = e^{-t_1}\bullet\psi$, $\psi\in\mathcal{N}$, in (A1) we obtain

$$\mathbf{P}\big[e^{-t_2}\bullet(e^{-t_1}\bullet\psi)\in A\big] = \int_{\mathcal{N}} P_{t_2}(\varphi,A)P_{t_1}(\psi,d\varphi), \qquad A\in\mathcal{B}(\mathcal{N}),\ t_1,t_2>0. \tag{4.4}$$

Using the associativity of $\bullet$ (Assumption (A2)) on the left-hand side of (4.4) we obtain the Chapman-Kolmogorov equations

$$P_{t_1+t_2}(\varphi,A) \;=\; \int_{\mathcal{N}} P_{t_1}(\psi,A)P_{t_2}(\varphi,d\psi) \qquad A\in\mathcal{B}(\mathcal{N}),\ t_1,t_2>0. \tag{4.5}$$

Therefore, by the Kolmogorov extension theorem there exists a Markov process Ψ_t^{φ} on $\mathcal{N}$ having transition kernel $\big(P_t(\varphi,\cdot)\big)_{t\geq 0}$. Let $\varphi\in\mathcal{N}\backslash\mathbf{0}$ and $G_t^{\varphi}[\cdot]$ be the p.g.fl. of $P_t(\varphi,\cdot)$ for $t>0$. Using the definition of $P_t(\varphi,\cdot)$ and the distributivity of $\bullet$ (Assumption (A3)) we obtain

$$G_t^{\varphi}[h] = G_{e^{-t}\bullet\varphi}[h] = G_{\sum_{x\in\varphi}e^{-t}\bullet\delta_x}[h] = \prod_{x\in\varphi} G_{e^{-t}\bullet\delta_x}[h] = \prod_{x\in\varphi} G_t^{\delta_x}[h], \qquad h\in\mathcal{V}(\mathcal{X}).$$

From distributivity it also follows that $e^{-t}\bullet\mathbf{0} \overset{\mathcal{D}}{=} \mathbf{0}$ and therefore $G_t^{\mathbf{0}}[h]=1$ for any $h\in\mathcal{V}(\mathcal{X})$. Therefore (4.2) is satisfied and Ψ_t^{φ} is a Markov branching process on $\mathcal{N}$.

Sufficiency: Let $(\Psi_t^{\varphi})_{t\geq 0},\ \varphi\in\mathcal{N}$ be a Markov branching process on $\mathcal{N}$ with transition kernel $P_t(\varphi,\cdot)$. Consider the operation $\bullet$ induced by (4.3), namely $t\bullet\Phi \overset{\mathcal{D}}{=} \Psi_{-\log t}^{\Phi}$. Assumption (A1) follows by the construction:

$$\mathbf{P}(\Psi_t^{\Phi}\in A) = \int_{\mathcal{N}} \mathbf{P}(\Psi_t^{\varphi}\in A)\mathbf{P}_{\Phi}(d\varphi), \qquad A\in\mathcal{B}(\mathcal{N}),\ t>0.$$

Given $\varphi\in\mathcal{N}$ and $t_1,t_2\in(0,1]$, using (A1) and the Chapman-Kolmogorov equations (4.5) we obtain

$$\begin{aligned}\mathbf{P}\left(t_1\bullet(t_2\bullet\varphi)\in A\right) \;&=\; \int_{\mathcal{N}} P_{-\log t_1}(\psi,A)P_{-\log t_2}(\varphi,d\psi) \;=\\ &=\; P_{-\log(t_1t_2)}(\varphi,A) \;=\; \mathbf{P}((t_1t_2)\bullet\varphi\in A) \qquad A\in\mathcal{B}(\mathcal{N}),\end{aligned}$$

i.e. the associativity (A2)) of $\bullet$ holds.

Finally, let $G_t^{\varphi}[\cdot]$ be the p.g.fl. of Ψ_t^{φ} for $t\geq 0$ and $\varphi\in\mathcal{N}$. Given $t\in(0,1]$ and $\varphi_1,\varphi_2\in\mathcal{N}\backslash\mathbf{0}$, using the independent branching property (4.2), it follows that

$$\begin{aligned}G_{t\bullet(\varphi_1+\varphi_2)}[h] = G_{-\log t}^{\varphi_1+\varphi_2}[h] &= \prod_{x\in\varphi_1+\varphi_2} G_t^{\delta_x}[h] = \prod_{x\in\varphi_1} G_t^{\delta_x}[h] \prod_{x\in\varphi_2} G_t^{\delta_x}[h] =\\ &= G_{-\log t}^{\varphi_1}[h]\, G_{-\log t}^{\varphi_2}[h] = G_{t\bullet\varphi_1}[h]G_{t\bullet\varphi_2}[h] \qquad h\in\mathcal{V}(\mathcal{X}).\end{aligned} \tag{4.6}$$

The distributivity (A3) of $\bullet$ follows from (4.6) and Remark 4.2. □

Example 4.5 (Diffusion)**.** Let $(X_t)_{t\geq 0}$ be a strong time-homogeneous Markov process on $\mathcal{X}$, right continuous with left limits. Let $(\Psi_t^\varphi)_{t\geq 0,\,\varphi\in\mathcal{N}}$ be the Markov branching process on $\mathcal{N}$ where every particle moves according to an independent copy of X_t, without branching (see [1, Sec. V.1] for a proof that this is indeed a Markov branching process on $\mathcal{N}$). Denote by $\bullet_d$ the associated branching operation, $t\bullet_d \Phi \overset{D}{=} \Psi^\Phi_{-\log t}$. Let $P_t(x,\cdot)$ be the distribution of X_t^x, where x denotes the starting state, and $P_t h(x) = \mathbf{E}\, h(X_t^x) = \int_\mathcal{X} h(y)P_t(x,dy)$. Then, for any φ in $\mathcal{N}$ we have

$$G_{t\bullet_d\varphi}[h] = G^\varphi_{-\log t}[h] = \prod_{x\in\varphi} G^{\delta_x}_{-\log t}[h] = \\ = \prod_{x\in\varphi} \mathbf{E}\, h(X^x_{-\log t}) = \prod_{x\in\varphi} P_{-\log t} h(x) = G_\varphi[P_{-\log t}h] \qquad h\in\mathcal{V}(\mathcal{X}). \tag{4.7}$$

Example 4.6 (Diffusion with thinning)**.** Let X_t be as in Example 4.6. Let $(\Psi_t^\varphi)_{t\geq 0,\,\varphi\in\mathcal{N}}$ be the Markov Branching process on $\mathcal{N}$, where every particle moves according to an independent copy of X_t and after an exponentially $\mathrm{Exp}(1)$-distributed time it dies (independently of the other particles). We denote by $\bullet_{dt}$ the associated branching operation $t\bullet_{dt}\Phi \overset{D}{=} \Psi^\Phi_{-\log t}$. Similarly to (4.7), it is easy to show that given a p.p. Φ we have

$$G_{t\bullet_{dt}\Phi}[h] = G_\Phi[1-t+t(P_{-\log t}h)], \qquad h\in\mathcal{V}(\mathcal{X}).$$

This operation acts as the composition of the thinning operation $\circ$ and the diffusion operation $\bullet_d$ introduced in Example 4.5, regardless of the order in which these two operations are applied. For any p.p. Φ

$$t\bullet_{dt}\Phi \overset{\mathcal{D}}{=} t\bullet_d (t\circ\Phi) \overset{\mathcal{D}}{=} t\circ(t\bullet_d\Phi).$$

In fact, since $1-t+t(P_{-\log t}h) = P_{-\log t}(1-t+th)$ for any h in $\mathcal{V}(\mathcal{X})$, it holds

$$G_{t\bullet_d(t\circ\Phi)}[h] = G_{t\circ\Phi}[P_{-\log t}h] = G_\Phi[1-t+t(P_{-\log t}h)] = G_{t\bullet_{dt}\Phi}[h] = \\ = G_\Phi[P_{-\log t}(1-t+th)] = G_{t\bullet_d\Phi}[1-t+th] = G_{t\circ(t\bullet_d\Phi)}[h].$$

4.3 Stability for general branching operations

Proposition 4.4 shows that branching operations are the only operations on point processes satisfying assumptions (A1)-(A3). Such assumptions, together with the continuity and subcriticality conditions below, lead to an appropriate definition of stability, as Proposition 4.8 below shows.

Definition 4.7. *We call a branching operation* $\bullet$ continuous *if*

$$t\bullet\varphi \Rightarrow \varphi \quad \text{for } t\uparrow 1 \qquad \text{for every } \varphi\in\mathcal{N}, \tag{A4}$$

where $\Rightarrow$ stands for weak convergence (or equivalently for the convergence in Prokhorov metric). Moreover we say that $\bullet$ is subcritical *if the associated Markov branching process on $\mathcal{N}$, Ψ_t^φ, is subcritical, i.e. if $\mathbf{E}\,\Psi_t^{\delta_x}(\mathcal{X}) < 1$ for every $x\in\mathcal{X}$ and $t>0$.*

Proposition 4.8. *Let Φ be a p.p. on $\mathcal{X}$ with p.g.fl. $G_\Phi[\cdot]$ and $\bullet$ be a subcritical and continuous branching operation on $\mathcal{X}$. Then the following conditions are equivalent:*

1. $\forall\, n\in\mathbb{N}\ \exists\, c_n\in(0,1]$ *such that given* $(\Phi^{(1)},...,\Phi^{(n)})$ *independent copies of* Φ
$$\Phi \overset{\mathcal{D}}{=} c_n\bullet(\Phi^{(1)}+...+\Phi^{(n)}); \tag{4.8}$$
2. $\forall\,\lambda>0\ \exists\, t\in(0,1]$ *such that*
$$G_\Phi[h] = \big(G_{t\bullet\Phi}[h]\big)^\lambda;$$

3. $\exists\, \alpha > 0$ *such that* $\forall\, n \in \mathbb{N}$, *given* $(\Phi^{(1)}, ..., \Phi^{(n)})$ *independent copies of* Φ

$$\Phi \overset{\mathcal{D}}{=} (n^{-\frac{1}{\alpha}}) \bullet (\Phi^{(1)} + ... + \Phi^{(n)}); \tag{4.9}$$

4. $\exists\, \alpha > 0$ *such that* $\forall\, t \in [0,1]$

$$G_\Phi[h] = \big(G_{t\bullet\Phi}[h]\big)^{t^{-\alpha}}; \tag{4.10}$$

5. $\exists\, \alpha > 0$ *such that* $\forall\, t \in [0,1]$, *given* $\Phi^{(1)}$ *and* $\Phi^{(2)}$ *independent copies of* Φ,

$$t^{1/\alpha} \bullet \Phi^{(1)} + (1-t)^{1/\alpha} \bullet \Phi^{(2)} \overset{\mathcal{D}}{=} \Phi. \tag{4.11}$$

Proof. If $\Phi \equiv \mathbf{0}$ then all the conditions are trivially satisfied. So we suppose $\Phi \not\equiv \mathbf{0}$. *4)* $\Rightarrow$ *2)* $\Rightarrow$ *1)* are obvious implications. So if one proves *1)*$\Rightarrow$*4)* then *1)*, *2)* and *4)* are equivalent.

To show *1)*$\Rightarrow$*4)* note that, given $n \in \mathbb{N}$, the coefficient c_n satisfying (4.8) is unique. In fact if c_n and $t\, c_n$ both satisfy (4.8), with $t \in (0,1)$, by associativity it follows

$$\Phi \overset{\mathcal{D}}{=} (tc_n) \bullet (\Phi^{(1)} + ... + \Phi^{(n)}) \overset{\mathcal{D}}{=} t \bullet \Big(c_n \bullet (\Phi^{(1)} + ... + \Phi^{(n)})\Big) \overset{\mathcal{D}}{=} t \bullet \Phi$$

and thus, because of subcriticality, $t = 1$. Using (4.8) and the distributivity and associativity of $\bullet$ we obtain that, given $m, n \in \mathbb{N}$,

$$\begin{aligned}\Phi \ &\overset{\mathcal{D}}{=} c_n \bullet (\Phi^{(1)} + ... + \Phi^{(n)}) \overset{\mathcal{D}}{=} \\ &\overset{\mathcal{D}}{=} c_n \bullet \big(c_m \bullet (\Phi^{(1)} + ... + \Phi^{(m)}) + ... + c_m \bullet (\Phi^{(n-1)m+1} + ... + \Phi^{(nm)})\big) \\ &\overset{\mathcal{D}}{=} (c_n c_m) \bullet (\Phi^{(1)} + ... + \Phi^{(nm)}),\end{aligned}$$

which implies that

$$c_{nm} = c_n c_m. \tag{4.12}$$

Since we are considering the subcritical case, we have

$$n > m \Rightarrow c_n < c_m. \tag{4.13}$$

For every $1 \le m \le n < +\infty$, $m, n \in \mathbb{N}$ define a function $c : [1, +\infty) \cap \mathbb{Q} \to (0, 1]$ by setting

$$c\Big(\frac{n}{m}\Big) := \frac{c_n}{c_m}. \tag{4.14}$$

The function c is well defined because of (4.12) and it takes values in $(0,1]$ because of (4.13). Using associativity, distributivity and (4.8),

$$\begin{aligned}\big(G_{\frac{c_n}{c_m}\bullet\Phi}[h]\big)^{\frac{n}{m}} &= \big(G_{\frac{c_n}{c_m}\bullet\left(c_m\bullet(\Phi^{(1)}+...+\Phi^{(m)})\right)}[h]\big)^{\frac{n}{m}} = \\ &= \big(G_{c_n\bullet(\Phi^{(1)}+...+\Phi^{(m)})}[h]\big)^{\frac{n}{m}} = \Big(\big(G_{c_n\bullet\Phi}[h]\big)^m\Big)^{\frac{n}{m}} = \big(G_{c_n\bullet\Phi}[h]\big)^n = G_\Phi[h].\end{aligned} \tag{4.15}$$

Therefore

$$G_\Phi[h] = \big(G_{c(x)\bullet\Phi}[h]\big)^x \quad \forall x \in [1, +\infty) \cap \mathbb{Q}. \tag{4.16}$$

It follows from (4.13) and (4.14) that c is a strictly decreasing function on $[1, +\infty) \cap \mathbb{Q}$. Therefore we can be extended to the whole $[1, +\infty)$ by putting

$$c(x) := \inf\{c(y) :\ y \in [1, x] \cap \mathbb{Q}\}.$$

From (4.12) and (4.13), taking limits over rational numbers, if follows that $c(xy) = c(x)c(y)$ for every $x, y \in [1, +\infty)$. The only monotone functions c from $[1, +\infty)$ to $(0, 1]$ such that $c(0) = 1$ and $c(xy) = c(x)c(y)$ for every $x, y \in [1, +\infty)$ are $c(x) = x^r$ for some $r \in \mathbb{R}$. Since our function is decreasing then $r < 0$. Let $\alpha > 0$ be such that $r = -1/\alpha$. Fix $x \in [1, +\infty)$ and let $\{x_n\}_{n\in\mathbb{N}} \subset [1, +\infty) \cap \mathbb{Q}$ be such that $x_n \downarrow x$ as $n \to +\infty$, and therefore $x_n^{-1/\alpha} \uparrow x^{-1/\alpha}$ as $n \to +\infty$. Since $\bullet$ is left-continuous in the weak topology (assumption (A4)) it holds that

$$x_n^{-1/\alpha} \bullet \Phi \Rightarrow x^{-1/\alpha} \bullet \Phi \qquad n \to +\infty,$$

which implies

$$G_{x_n^{-1/\alpha} \bullet \Phi}[h] \quad \overset{n\to\infty}{\longrightarrow} \quad G_{x^{-1/\alpha}\bullet\Phi}[h] \qquad \forall\, h \in \mathcal{V}(\mathcal{X}).$$

From (4.16) we have

$$\big(G_\Phi[h]\big)^{1/x} = \lim_{n\to+\infty} G_{c(x_n)\bullet\Phi}[h] = \lim_{n\to+\infty} G_{x_n^{-1/\alpha}\bullet\Phi}[h] \qquad \forall\, h \in \mathcal{V}(\mathcal{X}),$$

and therefore we obtain (4.10) as desired.

4) $\Rightarrow$ *3)* $\Rightarrow$ *1)* are obvious implications and thus *3)* is equivalent to *1)*, *2)* and *4)*.

To show *4)* $\Rightarrow$ *5)* take $x, y \in [1, +\infty)$. Then, because of *4)*,

$$\begin{aligned} G_\Phi[h] &= G_{(x+y)^{-1/\alpha}\bullet\Phi}[h]^{x+y} = G_{x^{-1/\alpha}\left(\frac{x+y}{x}\right)^{-1/\alpha}\bullet\Phi}[h]^x \cdot G_{y^{-1/\alpha}\left(\frac{x+y}{y}\right)^{-1/\alpha}\bullet\Phi}[h]^y = \\ &= G_{\left(\frac{x+y}{x}\right)^{-1/\alpha}\bullet\Phi}[h] \cdot G_{\left(\frac{x+y}{y}\right)^{-1/\alpha}\bullet\Phi}[h] = G_{\left(\frac{x+y}{x}\right)^{-1/\alpha}\bullet\Phi + \left(\frac{x+y}{y}\right)^{-1/\alpha}\bullet\Phi'}[h], \end{aligned} \tag{4.17}$$

where Φ' is an independent copy of Φ. Then *5)* follows since $x, y \in [1, +\infty)$ are arbitrary.

5)$\Rightarrow$*3)*. (4.9) can be obtained iterating (4.11) n-1 times. □

Section 4 shows that branching operations are the most general class of associative and distributive operations that can be used to study stability for point processes. Therefore the following definition generalises all the notions of discrete stability considered so far.

Definition 4.9. *Let* Φ *and* $\bullet$ *be as in Proposition 4.8. If* (4.11) *is satisfied we say that* Φ *is strictly* α*-stable with respect to* $\bullet$ *or, simply,* branching-stable*.*

In this paper we do not provide a characterisation of stable point processes with respect to a general branching operation. Instead, next we consider some specific cases that point towards directions to obtain such a characterisation in full generality. The main idea is that, given a branching operation $\bullet$ acting on point processes, there is a corresponding branching operation $\odot$ acting on random measures such that stable point processes with respect to $\bullet$ are Cox processes driven by stable random measures with respect to $\odot$.

4.4 Stability with respect to thinning and diffusion

Cox characterisation Recall that DαS point processes are Cox processes driven by StαS intensity measures, see Section 2.4. The main reason for this is that the thinned version of a Poisson p.p. with intensity measure μ, Π_μ, is itself a Poisson p.p. with intensity measure $t\mu$, i.e. $t \circ \Pi_\mu \overset{\mathcal{D}}{=} \Pi_{t\mu}$. The same holds for a Cox p.p. Π_ξ driven by a random measure ξ:

$$t \circ \Pi_\xi \overset{\mathcal{D}}{=} \Pi_{t\xi}. \tag{4.18}$$

For the thinning and diffusion operation $\bullet_{dt}$ of Example 4.6,

$$G_{t\bullet_{dt}\Pi_\mu}[h] = G_{\Pi_\mu}[1-t+tP_{-\log t}h] = \exp\{-\langle 1-(1-t+tP_{-\log t}h),\mu\rangle\} = \\ = \exp\{-\langle tP_{-\log t}(1-h),\mu\rangle\} = \exp\{-\langle 1-h, tP^*_{-\log t}\mu\rangle\} = G_{\Pi_{tP^*_{-\log t}\mu}}[h], \quad (4.19)$$

where $P^*_{-\log t}$ is the adjoint to the linear operator $P_{-\log t}$. If we denote by $\odot_{dt}$ the following operation

$$\begin{aligned} \odot_{dt} : (0,1] \times \mathcal{M} &\to \mathcal{M} \\ (t,\mu) &\to t \odot_{dt} \mu := tP^*_{-\log t}\mu, \end{aligned} \quad (4.20)$$

then (4.19) implies $t \bullet_{dt} \Pi_\mu \overset{\mathcal{D}}{=} \Pi_{t\odot_{dt}\mu}$. Similarly for Cox processes,

$$t \bullet_{dt} \Pi_\xi \overset{\mathcal{D}}{=} \Pi_{t\odot_{dt}\xi}, \quad (4.21)$$

where the operation $\odot_{dt}$ acts on each realisation of ξ. The analogy between (4.18) and (4.21) suggests the following result.

Proposition 4.10. *A point process Φ is strictly α-stable with respect to $\bullet_{dt}$ if and only if it is a Cox process Π_ξ with an intensity measure being strictly α-stable with respect to $\odot_{dt}$.*

Proof. Sufficiency. Suppose ξ is strictly α-stable with respect to $\odot_{dt}$. Then using (4.21) and the stability of ξ

$$t^{1/\alpha}\bullet_{dt}\Pi'_\xi + (1-t)^{1/\alpha}\bullet_{dt}\Pi''_\xi \overset{\mathcal{D}}{=} \Pi'_{t^{1/\alpha}\odot_{dt}\xi} + \Pi''_{(1-t)^{1/\alpha}\odot_{dt}\xi} \overset{\mathcal{D}}{=} \Pi_{t^{1/\alpha}\odot_{dt}\xi' + (1-t)^{1/\alpha}\odot_{dt}\xi''} \overset{\mathcal{D}}{=} \Pi_\xi,$$

where the Π'_ξ and Π''_ξ are independent copies of Π_ξ. Therefore Π_ξ is strictly α-stable with respect to $\bullet_{dt}$.

Necessity. Suppose Φ is strictly α-stable with respect to $\bullet_{dt}$. From (4.9) we have that for any positive integer m

$$G_\Phi[h] = \left(G_{m^{-1/\alpha}\bullet_{dt}\Phi}[h]\right)^m = \left(G_\Phi[1-m^{-1/\alpha}+m^{-1/\alpha}P_{\frac{\log m}{\alpha}}h]\right)^m, \qquad h \in \mathcal{V}(\mathcal{X}). \quad (4.22)$$

We need to show that $G_\Phi[1-u]$, as a functional of $u \in BM_+(\mathcal{X})$, is the Laplace functional of a random measure ξ that is strictly α-stable with respect to $\odot_{dt}$. Since $1-u$ may not take values in $[0,1]$, the expression $G_\Phi[1-u]$ may not be well defined. Thus we use (4.22) and define $L[u]$ as

$$L[u] = \left(G_\Phi[1-m^{-1/\alpha}P_{\frac{\log m}{\alpha}}u]\right)^m, \qquad u \in BM_+(\mathcal{X}), \quad (4.23)$$

noting that for some m big enough $1-m^{-1/\alpha}P_{\frac{\log m}{\alpha}}u$ takes values in $[0,1]$ and the right-hand side of (4.23) is well defined. Arguing as in the proof of Theorem 3.5 one can prove that L is the Laplace functional of a random measure ξ, L_ξ. Finally, ξ is strictly α-stable with respect to $\odot_{dt}$ because for any u in $BM_+(\mathcal{X})$

$$L_{t\odot_{dt}\xi}[u] = \mathbf{E}\exp\{-\langle u, t\odot_{dt}\xi\rangle\} = \mathbf{E}\exp\{-\langle u, tP^*_{-\log t}\xi\rangle\} = \\ = \mathbf{E}\exp\{-\langle tP_{-\log t}u, \xi\rangle\} = L_\xi[tP_{-\log t}u],$$

and, supposing $u \le t^{-1}$, and thus $(1-tP_{-\log t}u) \in \mathcal{V}(\mathcal{X})$, we have

$$L_\xi[tP_{-\log t}u] = G_\Phi[1-tP_{-\log t}u] = G_\Phi[1-t+tP_{-\log t}(1-u)] = \\ G_{t\bullet_{dt}\Phi}[1-u] = G_\Phi[1-u]^{t^\alpha} = L_\xi[u]^{t^\alpha}. \quad (4.24)$$

Similar calculations also apply to the general definition of $L[u]$ in (4.23), which includes the case $u > t^{-1}$. The fact that ξ is strictly α-stable with respect to $\odot_{dt}$ follows from (4.24) arguing, for example, as in (4.17). □

Levy characterisation and spectral decomposition Proposition 4.10 characterises stable p.p.'s with respect to $\bullet_{dt}$ as Cox processes driven by stable random measures with respect to $\odot_{dt}$. In this section we describe stable random measures with respect to $\odot_{dt}$ in terms of homogeneous Levy measures (with respect to $\odot_{dt}$) and we show how to decompose such homogeneous Levy measures into a spectral and a radial component.

Given $A \in \mathcal{B}(\mathcal{M})$ and $t \in (0,1]$ we define $t \odot_{dt} A = \{t \odot_{dt} \mu : \ \mu \in A\}$. The idea is to look for homogeneous Levy measures of order α with respect to $\odot_{dt}$, meaning that for any A in $\mathcal{B}(\mathcal{M})$

$$\Lambda(t \odot_{dt} A) = t^{-\alpha}\Lambda(A) \quad \forall t \in (0,1]. \tag{4.25}$$

Let us consider Laplace functionals of the form

$$L[h] = \exp\left\{-\int_{\mathcal{M}\backslash\{0\}} \left(1 - e^{-\langle h,\mu\rangle}\right)\Lambda(d\mu)\right\}, \qquad h \in BM_+(\mathcal{X}), \tag{4.26}$$

where Λ is a Radon measure on $\mathcal{M}\backslash\{0\}$ such that

$$\int_{\mathcal{M}\backslash\{0\}} \left(1 - e^{-\langle h,\mu\rangle}\right)\Lambda(d\mu) < \infty, \tag{4.27}$$

for any h in $BM_+(\mathcal{X})$, and (4.25) holds for any A in $\mathcal{B}(\mathcal{M})$. Arguing as in the proof of Theorem 2 of [7], it can be seen that (4.26) defines the Laplace functional of a random measure, say ξ. Then, defining $L_\xi[h]$ as in (4.26), from (4.25) it follows that

$$L_{t\odot_{dt}\xi}[h] = \exp\left\{-\int_{\mathcal{M}\backslash\{0\}} \left(1 - e^{-\langle h,\mu\rangle}\right) t^{-\alpha}\Lambda(d\mu)\right\} = L_\xi[h]^{t^{-\alpha}}, \qquad h \in BM_+(\mathcal{X}),$$

which means that ξ is α-stable with respect to $\odot_{dt}$ by an argument analogous to the one in (4.17).

We now show how to decompose Levy measures satisfying (4.25) in a radial component (uniquely determined by α) and a spectral component. Such a spectral decomposition depends on the operation $\odot_{dt}$ and thus it is not the one used in Section 2.4 for thinning-stable point processes. For simplicity we restrict ourselves to the case where $\mathcal{M}$ is the space of finite measures on $\mathcal{X} = \mathbb{R}^n$ for some n and the the diffusion process P_t is a Brownian motion, meaning that given μ in $\mathcal{M}$ and t in $(0,1]$, the measure $t \odot_{dt} \mu$ is

$$t \odot_{dt} \mu \ = \ t\,\nu_t * \mu,$$

where $*$ denotes the convolution of measures and ν_t, for t in $(0,1)$, has the following density with respect to the Lebesgue measure

$$\frac{d\nu_t}{d\ell} = f_t(x) = \frac{1}{(2\pi\log t)^{\frac{n}{2}}}\exp\left\{\frac{|x|^2}{-2\log t}\right\},$$

while for $t = 0$, ν_t equals $\delta_{\mathbf{0}}$, with $\mathbf{0}$ being the origin of $\mathbb{R}^n$.

We now show that $\mathcal{M}\backslash\{0\}$ can be decomposed as $\widetilde{\mathbb{S}} \times (0,1]$, for the following $\widetilde{\mathbb{S}}$

$$\widetilde{\mathbb{S}} := \{\mu \in \mathcal{M}\backslash\{0\} : \ \nexists (t,\rho) \in (0,1) \times \mathcal{M}\backslash\{0\} \text{ such that } t \odot_{dt} \rho = \mu\}.$$

Note that $\widetilde{\mathbb{S}} \in \mathcal{B}(\mathcal{M})$ because $\widetilde{\mathbb{S}} = \cup_{t\in(0,1)\cap\mathbb{Q}} t \odot_{dt} (\mathcal{M}\backslash\{0\})$ and $t \odot_{dt} (\mathcal{M}\backslash\{0\}) \in \mathcal{B}(\mathcal{M})$ for any t in $(0,1)\cap\mathbb{Q}$. In fact if $A \in \mathcal{B}(\mathcal{M})$, then also $t \odot_{dt} A \in \mathcal{B}(\mathcal{M})$ because the map $\mu \to t \odot_{dt} \mu$ is injective (see Proposition 4.11 below) and therefore the image of a Borel set is still Borel (see, e.g., Section 15.A of [15]).

Proposition 4.11. *The map*

$$\begin{aligned} (0,1] \times \widetilde{\mathbb{S}} &\mapsto \mathcal{M}\backslash\{0\} \\ (t, \mu_0) \quad &\mapsto t \odot_{dt} \mu_0, \end{aligned} \tag{4.28}$$

is a bijection.

Proof. Injectivity: First note that given $t \in (0,1)$ and $\mu, \rho \in \mathcal{M}$ such that $\nu_t * \mu = \nu_t * \rho$, then $\mu = \rho$. This follows, for example, by noting that, for any t in $(0,1]$, the moment generating function of ν_t is strictly positive on $\mathbb{R}^n$. Then, given $t_1, t_2 \in (0,1]$ and $\mu_0, \rho_0 \in \widetilde{\mathbb{S}}$

$$\begin{aligned} t_1 \odot_{dt} \mu_0 = (t_1 t_2) \odot_{dt} \rho_0 \Leftrightarrow t_1(\nu_{t_1} * \mu_0) = (t_1 t_2)(\nu_{t_1 t_2} * \rho_0) \Leftrightarrow \\ \Leftrightarrow \nu_{t_1} * \mu_0 = \nu_{t_1} * \big(\nu_{t_2} * (t_2 \rho_0)\big) \Leftrightarrow \mu_0 = \nu_{t_2} * (t_2 \rho_0) \Leftrightarrow t_2 = 1 \text{ and } \mu_0 = \rho_0, \end{aligned}$$

which means that the map $(t, \mu_0) \mapsto t \odot_{dt} \mu_0$ is injective.

Surjectivity: Let $\mu \in \mathcal{M} \setminus \{0\}$. Without loss of generality we can consider μ to be a probability measure, otherwise consider $\frac{\mu}{\mu(\mathcal{X})}$. Define

$$I_\mu := \{t \in (0,1] : \ \exists \rho \in \mathcal{M} \setminus \{0\}, t \in (0,1] \text{ such that } t \odot_{dt} \rho = \mu\}.$$

Note that $1 \in I_\mu$ because $1 \odot_{dt} \mu = \mu$ and, thanks to the associativity of $\odot_{dt}$, I_μ is an interval. We define $t_0 := \inf I_\mu$ and we prove $t_0 > 0$. Suppose $t_0 = 0$. Then for any $\varepsilon \in (0,1]$ there is $\rho^{(\varepsilon)} \in \mathcal{M} \setminus \{0\}$ such that $\mu = \varepsilon \nu_\varepsilon * \rho^{(\varepsilon)}$. Setting $\mu^{(\varepsilon)} = \varepsilon \rho^{(\varepsilon)}$ we have $\mu = \nu_\varepsilon * \mu^{(\varepsilon)}$. Since both μ and $\mu^{(\varepsilon)}$ by associativity of $\odot_{dt}$ are obtained by convolution with some ν_t, they both admit bounded density functions, say, g and $g^{(\varepsilon)}$, respectively. Moreover $\|g^{(\varepsilon)}\|_1 = 1$ because

$$1 = \|g\|_1 = \|f_\varepsilon * g^{(\varepsilon)}\|_1 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_\varepsilon(x-y) g^{(\varepsilon)}(y) dy\, dx = \int_{\mathbb{R}^n} g^{(\varepsilon)}(y) dy = \|g^{(\varepsilon)}\|_1 \,,$$

were we used the fact that both f_ε and $g^{(\varepsilon)}$ are positive, Fubini's theorem (which holds for positive functions) and $\|\nu_\varepsilon\|_1 = 1$ and. Then

$$\|g\|_\infty = \|f_\varepsilon * g^{(\varepsilon)}\|_\infty \leq \|f_\varepsilon\|_\infty \|g^{(\varepsilon)}\|_1 = \|f_\varepsilon\|_\infty \to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore, by contradiction, $t_0 > 0$. Given $t \in (t_0, 1]$ let $\mu^{(t)} \in \mathcal{M}\backslash\{0\}$ such that $\nu_t * \mu^{(t)} = \mu$. We now prove that $\{\mu^{(t)}\}$ as $t \downarrow t_0$ is Cauchy in the Prokhorov metric. Let $\varepsilon > 0$ be fixed and $\delta = \delta(\varepsilon) > 0$ to be fixed later. Consider t_1, t_2 such that $t_0 < t_1 < t_2 < t_0 + \delta \leq 1$. We have $\mu^{(t_2)} = \nu_{t_1/t_2} * \mu^{(t_1)}$. As $\delta \to 0$ we have $\nu_{t_1/t_2}(\mathbb{R}^n \setminus B_\varepsilon(0)) \to 0$. Thus we can choose δ such that $\nu_{t_1/t_2}(\mathbb{R}^n \setminus B_\varepsilon(0)) < \varepsilon$. Therefore we have

$$\begin{aligned} \mu^{(t_2)}(A) = \nu_{t_1/t_2} * \mu^{(t_1)}(A) = \int_{\mathbb{R}^n} \nu_{t_1/t_2}(A-y)\, \mu^{(t_1)}(dy) = \\ = \int_{\mathbb{R}^n \setminus A^\varepsilon} \nu_{t_1/t_2}(A-y)\, \mu^{(t_1)}(dy) + \int_{A^\varepsilon} \nu_{t_1/t_2}(A-y)\, \mu^{(t_1)}(dy) < \varepsilon + \mu^{(t_1)}(A^\varepsilon), \end{aligned}$$

where $A - y$ is defined as $\{x \in \mathcal{X} \,:\, x + y \in A\}$ and $A^\varepsilon = \{x \in \mathcal{X} \,:\, B_\varepsilon(x) \cap A \neq \emptyset\}$. Thus there exists a probability measure $\mu^{(t_0)} \in \mathcal{M} \setminus \{0\}$ such that $\mu^{(t)} \Rightarrow \mu^{(t_0)}$ as $t \downarrow t_0$ implying $\nu_t * \mu^{(t)} \Rightarrow \nu_{t_0} * \mu^{(t_0)}$. Therefore $\nu_{t_0} * \mu^{(t_0)} = \mu$. Finally we note that $\mu^{(t_0)} \in \widetilde{\mathbb{S}}$ because of the definition of t_0. □

Thanks to Proposition 4.11, for any μ in $\mathcal{M} \setminus \{0\}$ there is one and only one couple $(t, \mu_0) \in (0,1] \times \widetilde{\mathbb{S}}$ such that $t \odot_{dt} \mu_0 = \mu$, meaning that $\mathcal{M} \setminus \{0\}$ can be decomposed as

$$\mathcal{M} \setminus \{0\} \ = (0,1] \times \widetilde{\mathbb{S}}. \tag{4.29}$$

Any measure Λ satisfying (4.25) can then be represented as $\Lambda = \theta_\alpha \otimes \sigma$, where $\theta_\alpha((a,b]) = (b^{-\alpha} - a^{-\alpha})$ for any $(a,b] \subseteq (0,1]$ and $\sigma(A) = \Lambda((0,1] \times A)$ for any $A \in \mathcal{B}(\widetilde{\mathbb{S}})$. In fact for any $(a,b] \subseteq (0,1]$ and $A \in \mathcal{B}(\widetilde{\mathbb{S}})$

$$\Lambda((a,b] \times A) = \Lambda((0,b] \times A) - \Lambda((0,a] \times A) \overset{(4.25)}{=} \\ = b^{-\alpha}\Lambda((0,1] \times A) - a^{-\alpha}\Lambda((0,1] \times A) = \theta_\alpha((a,b])\,\sigma(A).$$

Since θ_α is fixed by α, there is a one-to-one correspondence between Levy measures satisfying (4.25) and spectral measures σ on $\widetilde{\mathbb{S}}$. Thus a Cox point process driven by the parameter measure ξ is strictly α-stable with respect to the diffusion-thinning operation, if ξ is $\odot_{dt}$-stable, which, in turn, can be obtained by choosing an arbitrary spectral measure σ on $\widetilde{\mathbb{S}}$ satisfying

$$\int_{\widetilde{\mathbb{S}}} \mu(B)^\alpha \sigma(d\mu) < \infty$$

for all any compact subsets $B \subset \mathbb{R}^n$. And then taking ξ with the Laplace functional

$$L_\xi[u] = \exp\Big\{ - \int_{\widetilde{\mathbb{S}}} \langle u, \mu \rangle^\alpha \sigma(d\mu) \Big\}, \quad u \in \mathrm{BM}_+(\mathcal{X}).$$

As an example, consider a finite measure $\hat{\sigma}$ on $\mathbb{R}^n$ and σ its push-forward under the map $x \in \mathcal{X} \mapsto \delta_x \in \mathcal{M}$. As shown above, the homogeneous Levy measure on $\mathcal{M} = (0,1] \times \widetilde{\mathbb{S}}$ having σ as spectral measure is $\Lambda = \theta_\alpha \otimes \sigma$. Therefore Λ is supported by the following subset of $\mathcal{M}$:

$$\mathcal{Y} := \{\nu_t(\cdot - m) : \ t \in (0,1], m \in \mathbb{R}^n\} \subset \mathcal{M}.$$

4.5 Stability on $\mathbb{Z}_+$ and $\mathbb{R}_+$

In all the examples considered so far: thinning stability, $\mathcal{F}$-stability and an example of a general branching stability, the operation $\bullet_{dt}$ in Section 4.4, the corresponding stable point processes were Cox processes driven by a parameter measure which is itself stable with respect to a corresponding operation on measures. Unlike its counterpart operation on point processes, this operation was not stochastic, meaning that its result on a deterministic measure is also deterministic. For the case of thinning stability, this was an operation of ordinary multiplication, for a point-processes branching operation $\bullet_{dt}$, this was the operation $\odot_{dt}$. Nevertheless this does not need to be the case in general: in this section we consider point-processes branching operations $\bullet$ whose corresponding measure branching operation $\odot$ is also stochastic.

We consider the case of a trivial phase space $\mathcal{X}$ consisting of one point, or in other words the case of random variables taking values in $\mathbb{Z}_+$ and $\mathbb{R}_+$. For $\mathbb{Z}_+$, since the phase space consists of one point, the general branching stability corresponds to the $\mathcal{F}$-stability described in Section 2.3. We introduce the notion of branching stability in $\mathbb{R}_+$ using the theory of continuous-state branching processes (CB-processes, see [17]). We show that, at least in the cases we consider, branching stable (or $\mathcal{F}$-stable) discrete random variables are Cox processes driven by a branching stable continuous random variable. Finally we show how to use quasi-stationary distributions to construct branching stable continuous random variables.

We argue that the theory of superprocesses (e.g. [10]) should be relevant to extend the ideas presented in this section to general branching stable point processes.

Continuous-state branching processes Continuous-state branching processes were first considered in [13] and [17], and can be thought of as an analogue of continuous time branching processes on $\mathbb{Z}_+$ on a continuous space $\mathbb{R}_+$.

Definition 4.12. *A continuous-state branching process (CB-process) is a Markov process $(Z_t^x)_{x,t\geq 0}$ on $\mathbb{R}_+$, where t denotes time and x the starting state, with transition probabilities $(P_t^x)_{t,x\geq 0}$ satisfying the following* branching property*:*

$$P_t^{x+y} \;=\; P_t^x * P_t^y, \tag{4.30}$$

for any $t, x, y \geq 0$, where $*$ denotes convolution.

A useful tool to study CB-processes is the spatial Laplace transform V_t, defined by

$$x\, V_t(z) \quad = \quad -\log \int_{\mathbb{R}_+} e^{-z\, y} P_t^x(dy) \qquad z \geq 0, \tag{4.31}$$

for $t \geq 0$ and $x > 0$. The value of x in (4.31) is irrelevant because of the branching property, it could be simply set to 1. Using the Chapman-Kolmogorov equations it follows from (4.31) that $(V_t)_{t\in\mathbb{R}_+}$ is a composition semigroup

$$V_t(V_s(z)) \quad = \quad V_{t+s}(z), \qquad s, t, z \geq 0. \tag{C1$'$}$$

Similarly to the discrete case in Section 2.1, we focus in the subcritical case, $\mathbb{E}[Z_t^1] < 1$, and we assume regularity conditions analogous to (C2)-(C4). More specifically, rescaling the time by a constant factor if necessary, we may assume that

$$\mathbb{E}[Z_t^1] \;=\; e^{-t}\,, \tag{C2$'$}$$

$$\lim_{t\downarrow 0} V_t(z) \;=\; V_0(z) \;=\; z\,, \tag{C3$'$}$$

$$\lim_{t\to\infty} V_t(z) \;=\; 0\,. \tag{C4$'$}$$

Example 4.13. A well known example of CB-process is the diffusion process with Kolmogorov backward equations given by

$$\frac{\partial u}{\partial t} \;=\; ax\frac{\partial u}{\partial x} + \frac{bx}{2}\frac{\partial^2 u}{\partial x^2}.$$

The spatial Laplace transform of the corresponding CB-process is

$$V_t(z) \;=\begin{cases} \frac{z\,\exp(at)}{1-(1-\exp(at))\frac{bz}{2a}}, & \text{if } a \neq 0,\\ \frac{z}{1+t\frac{bz}{2}}, & \text{if } a = 0. \end{cases} \tag{4.32}$$

The sub-critical case corresponds to $a < 0$. Rescaling time to satisfy (C2$'$) corresponds to setting $a = -1$.

The *limiting conditional distribution* (or Yaglom distribution) of $(Z_t^x)_{x,t\geq 0}$ is the weak limit of $(Z_t^x | Z_t^x > 0)$, when $t \to +\infty$. Such limit does not depend on x (e.g. [16, Th. 3.1] or [18, Th. 4.3]) and we denote by Z_∞ the corresponding random variable and by L_{Z_∞} its Laplace transform. The Yaglom distribution is also a quasi-stationary distribution, meaning that $\left(Z_t^{Z_\infty} | Z_t^{Z_\infty} > 0\right) \overset{\mathcal{D}}{=} Z_\infty$ (e.g. [16, Th. 3.1]). Given (C2$'$) and the quasi-stationarity of Z_∞ it follows that

$$L_{Z_\infty}\big(V_t(z)\big) = 1 - e^{-t} + e^{-t} L_{Z_\infty}(z), \qquad s, z \geq 0\,. \tag{4.33}$$

$\mathcal{V}$-stability and Cox characterisation of $\mathcal{F}$-stable random variables Let $(Z_t^x)_{x,t\geq 0}$ be a CB-process with spatial Laplace transform $\mathcal{V} = (V_t)_{t\geq 0}$, satisfying assumptions (C1′)-(C4′) of the previous section. Define a corresponding stochastic operation acting on random variables on $\mathbb{R}_+$ as follows:

$$t \odot_{\mathcal{V}} \xi \ \stackrel{\mathcal{D}}{=}\ Z^{\xi}_{-\log t} \qquad 0 < z \leq 1, \tag{4.34}$$

where ξ is an $\mathbb{R}_+$-valued random variable and Z_t^{ξ} is the CB-process with random starting state ξ. Similarly to Proposition 4.4, from the Markov and branching properties of $(Z_t^x)_{x,t\geq 0}$ it follows that $\odot_{\mathcal{V}}$ is associative and distributive with respect to the usual sum.

The notion of $\mathcal{V}$-stability for continuous random variables is analogous to the notion of $\mathcal{F}$-stability for discrete frameworks:

Definition 4.14. *A $\mathbb{R}_+$-valued random variable X (or its distribution) is $\mathcal{V}$-stable with exponent α if*

$$t^{1/\alpha} \odot_{\mathcal{V}} X' + (1-t)^{1/\alpha} \odot_{\mathcal{V}} X'' \ \stackrel{\mathcal{D}}{=}\ X \qquad 0 < t < 1, \tag{4.35}$$

where X' and X'' are independent copies of X.

In terms of Laplace transform L, the definition of $\odot_{\mathcal{V}}$ in (4.34) can be written as $L_{t\odot_{\mathcal{V}} X}(z) = L_X(V_{-\log t}(z))$. Thus, arguing as in Proposition 4.8, (4.35) is equivalent to

$$L_X(V_{-\log t}(z)) \ =\ L_X(z)^{t^{\alpha}} \qquad 0 < t < 1. \tag{4.36}$$

Suppose we have a continuous-time branching process on $\mathbb{Z}_+$ with p.g.f.'s $\mathcal{F} = (F_t)_{t\geq 0}$ and a CB-process with spatial Laplace transform $\mathcal{V} = (V_t)_{t\geq 0}$ such that

$$F_t(z) \ =\ 1 - V_t(1-z) \qquad 0 \leq z \leq 1. \tag{4.37}$$

The relation between the discrete and continuous stochastic operations, $\circ_{\mathcal{F}}$ and $\odot_{\mathcal{V}}$, is that Cox random variables driven by a $\mathcal{V}$-stable random intensity are $\mathcal{F}$-stable.

Proposition 4.15. *Let $\mathcal{F} = (F_t)_{t\geq 0}$ and $\mathcal{V} = (V_t)_{t\geq 0}$ satisfy* (C1)-(C4) *and* (C1′)-(C4′) *respectively, and let* (4.37) *be satisfied. Let ξ be a $\mathcal{V}$-stable random variable with exponent α and let X be a Cox random variable driven by ξ, meaning that $X|\xi \sim Po(\xi)$. Then X is $\mathcal{F}$-stable with exponent α.*

Proof. The pg.f. of X is given by $G_X(z) = L_\xi(1-z)$, see e.g. (2.10). Therefore

$$G_X(F_{-\log t}(z)) = L_\xi(1-(F_{-\log t}(z))) \stackrel{(4.37)}{=} L_\xi(V_{-\log t}(1-z)) = L_\xi(1-z)^{t^{\alpha}} = G_X(z)^{t^{\alpha}},$$

which implies that X is $\mathcal{F}$-stable with exponent α. □

Examples of discrete and continuous operations, $\circ_{\mathcal{F}}$ and $\odot_{\mathcal{V}}$, with $\mathcal{F}$ and $\mathcal{V}$ coupled by (4.37) are thinning and multiplication, as well as the birth and death process of Example 2.3 and the CB-process of Example 4.13 (with $b = 1$). Also the operations $\bullet_{dt}$ and $\odot_{dt}$ of Section 4.4, in a point process and random measures framework, satisfy (4.37). A natural question is whether for any continuous time branching process on $\mathbb{Z}_+$ there is a CB-process such that (4.37) is satisfied and viceversa. Note that, given (4.37), $(F_t)_{t\geq 0}$ is a composition semigroup if and only if $(V_t)_{t\geq 0}$ is. Indeed,

$$F_t(F_s(z)) \ =\ 1 - V_t(1-(1-V_s(1-z))) \ =\ 1 - V_t(V_s(1-z)) \ =\ 1 - V_{t+s}(1-z) = F_{t+s}(z),$$

and similarly for V_t. Therefore one would only need to prove that if V_t is the spatial Laplace transform of a random variable on $\mathbb{R}_+$ then F_t defined by (4.37) is the p.g.f. of a random variable on $\mathbb{Z}_+$ or viceversa.

Finally, note that Proposition 4.15 suggests that, at least in some cases, $\mathcal{F}$-stable random variables on $\mathbb{Z}_+$ are Cox processes driven by $\mathcal{V}$-stable random variables on $\mathbb{R}_+$. It is therefore natural to ask whether we can characterise $\mathcal{V}$-stable random variables.

Equations (4.33) and (4.34) imply that

$$t \odot_{\mathcal{V}} Z_\infty \overset{\mathcal{D}}{=} \sum_{i=0}^{t\circ 1} Z_\infty \qquad t \in (0,1], \tag{4.38}$$

where $t \circ 1$ is, by the definition of thinning, a binomial $\mathsf{Bin}(1,t)$ random variable (independent of Z_∞). Therefore the Yaglom distribution allows us to pass from $\odot_{\mathcal{V}}$ to thinning and use such a property to construct $\mathcal{V}$-stable random variables from DαS random variables.

Proposition 4.16. *Let X be a* DαS *random variable on $\mathbb{Z}_+$ (see* (2.14)*), and $\xi = \sum_{i=1}^{X} Z_\infty^{(i)}$, where $Z_\infty^{(1)}, Z_\infty^{(2)}, \dots$ are i.i.d. copies of the Yaglom distribution Z_∞. Then ξ is $\mathcal{V}$-stable with exponent α.*

Proof. Given its definition, the Laplace transform of ξ is given by $L_\xi(z) = G_X\left(L_{Z_\infty}(z)\right)$. Therefore

$$L_\xi(V_{-\log t}(z)) \;=\; G_X\left(L_{Z_\infty}(V_{-\log t}(z))\right) \overset{(4.33)}{=} G_X\left(1 - t + tL_{Z_\infty}(z)\right) \;=\; G_{t\circ X}\left(L_{Z_\infty}(z)\right) \overset{(4.10)}{=} G_X\left(L_{Z_\infty}(z)\right)^{t^\alpha} \;=\; L_\xi(z)^{t^\alpha},$$

which implies that ξ if $\mathcal{V}$-stable. □

5 Discussion

In this paper we have studied discrete stability with respect to the most general branching operation on counting measures which unifies all notions considered so far: discrete stable and $\mathcal{F}$-stable integer random variables, thinning-stable and $\mathcal{F}$-stable point processes characterised above. We considered in detail an important example of thinning-diffusion branching stable point processes and established the corresponding spectral representation of their laws. We demonstrate that branching stability of integer random variables may be associated with a stability with respect to a stochastic operation of continuous branching on the positive real line and we conjecture that this association may still be true in general for point processes and its continuous counterpart, random measures. A full characterisation of the branching-stable point processes, as well as of the associated stable random measures is yet to be established.

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