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## Sets of multiplicity and closable multipliers on group algebras

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# Sets of multiplicity and closable multipliers on group algebras 

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## A B S T R A C T

We undertake a detailed study of the sets of multiplicity in a second countable locally compact group $G$ and their operator versions. We establish a symbolic calculus for normal completely bounded maps from the space $\mathcal{B}\left(L^{2}(G)\right)$ of bounded linear operators on $L^{2}(G)$ into the von Neumann algebra $\operatorname{VN}(G)$ of $G$ and use it to show that a closed subset $E \subseteq G$ is a set of multiplicity if and only if the set $E^{*}=\left\{(s, t) \in G \times G: t s^{-1} \in E\right\}$ is a set of operator multiplicity. Analogous results are established for $M_{1}$-sets and $M_{0}$-sets. We show that the property of being a set of multiplicity is preserved under various operations, including taking direct products, and establish an Inverse Image Theorem for such sets. We characterise the sets of finite width that are also sets of operator multiplicity, and show that every compact operator supported on a set of finite width can be approximated by sums of rank one operators supported on the same set. We show that, if $G$ satisfies a mild approximation condition, pointwise multiplication by a given measurable function $\psi: G \rightarrow \mathbb{C}$ defines a closable multiplier on the reduced $C^{*}$-algebra $C_{r}^{*}(G)$ of $G$ if and only if Schur multiplication by the function $N(\psi): G \times G \rightarrow \mathbb{C}$, given by $N(\psi)(s, t)=\psi\left(t s^{-1}\right)$, is a closable operator when

[^0]viewed as a densely defined linear map on the space of compact operators on $L^{2}(G)$. Similar results are obtained for multipliers on $\mathrm{VN}(G)$.
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## Contents

1. Introduction ..... 1455
2. Preliminaries ..... 1457
2.1. Closable operators ..... 1457
2.2. Locally compact groups ..... 1458
2.3. Masa-bimodules ..... 1460
2.4. Schur multipliers ..... 1462
3. Arveson measures and pseudo-integral operators ..... 1463
3.1. Measures ..... 1463
3.2. Operators ..... 1465
4. Sets of multiplicity and their operator versions ..... 1470
4.1. Sets of multiplicity in arbitrary locally compact groups ..... 1470
4.2. Sets of operator multiplicity ..... 1471
4.3. A symbolic calculus ..... 1472
4.4. Multiplicity versus operator multiplicity ..... 1477
4.5. The case of $M_{0}$-sets ..... 1478
4.6. An application: unions of sets of uniqueness ..... 1482
5. Preservation properties ..... 1483
5.1. Sets possessing an m-resolution ..... 1483
5.2. Inverse images ..... 1485
5.3. Direct products ..... 1492
6. Sets of finite width ..... 1493
7. Closable multipliers on group $C^{*}$-algebras ..... 1498
8. Closable multipliers on group von Neumann algebras ..... 1503
Acknowledgments ..... 1507
References ..... 1507

## 1. Introduction

The connections between Harmonic Analysis and the Theory of Operator Algebras have a long and illustrious history. With his pivotal paper [2], W.B. Arveson opened up a new avenue in that direction by introducing the notion of operator synthesis. The relation between operator synthesis and spectral synthesis for locally compact groups was explored in detail in [14,25,39,9,10], among others. In this connection, J. Froelich [14] studied the question of when the operator algebra associated with a commutative subspace lattice contains a non-zero compact operator. For any compact abelian group $G$ and a closed subset $E \subseteq G$, he constructed a commutative subspace lattice $\mathcal{L}_{E}$, such that the corresponding operator algebra contains a non-zero compact operator if and only if $E$ is a set of multiplicity in the sense of (commutative) Harmonic Analysis.

Recently, we observed in [34] a connection between sets of multiplicity and the closability of linear transformations that are a natural unbounded analogue of Schur multipliers. Motivated originally by Schur multiplication of matrices, Schur multipliers have played an important role in a number of contexts in Operator Theory, see e.g. [18] and [30]. In
the context of Harmonic Analysis, they provide the basis for a useful characterisation of completely bounded multipliers of the Fourier algebra $A(G)$ of a locally compact group $G$ introduced in [7]. Namely, a measurable function $\psi: G \rightarrow \mathbb{C}$ is a completely bounded multiplier of $A(G)$ precisely when the function $N(\psi)$, given by $N(\psi)(s, t)=\psi\left(t s^{-1}\right)$, is a Schur multiplier on $G \times G$ [6] (see also [19] and [38]). The functions $\psi$ satisfying the latter condition are known as Herz-Schur multipliers. Any multiplier $\psi$ on $A(G)$ determines a bounded transformation on the corresponding reduced group $C^{*}$-algebra defined by the pointwise multiplication of $L^{1}(G)$ by $\psi$.

Unbounded transformations of Schur type, acting on group $C^{*}$-algebras, have been considered in the literature in connection with problems arising in Non-commutative Geometry (see [1] and the references therein). However, unbounded versions of transformations on group $C^{*}$-algebras corresponding to multipliers of Fourier algebras and their connection with (unbounded) operators of Schur type have not been explored until the present work.

These considerations gave the motivation for our present study of sets of multiplicity in the general setting of locally compact groups and their connection with closable multipliers on group algebras.

Sets of multiplicity for the group of the circle initially arose in connection with the problem of uniqueness of trigonometric series and have been extensively studied (see [15]). In a general locally compact group $G$, sets of uniqueness (or, equivalently, of non-multiplicity) were introduced by M. Bożejko in [4] as those closed subsets $E \subseteq G$ which do not support non-zero elements of the reduced $C^{*}$-algebra $C_{r}^{*}(G)$ of $G$.

An operator counterpart of sets of multiplicity was introduced in [34]. On the operator level, as well as on the level of locally compact groups, two classes of sets of multiplicity have been mostly examined: (operator) $M$-sets and (operator) $M_{1}$-sets. Here we introduce the class of operator $M_{0}$-sets and show, in Section 4, that a closed subset $E$ of a second countable locally compact group $G$ is an $M$-set (resp. $M_{1}$-set, $M_{0}$-set) if and only if the set $E^{*}=\left\{(s, t): t s^{-1} \in E\right\} \subseteq G \times G$ is an operator $M$-set (resp. operator $M_{1}$-set, operator $M_{0}$-set). These results should be compared to the result established in $[14,25,39]$ stating that $E$ is a set of local spectral synthesis if and only if $E^{*}$ is a set of operator synthesis. They permit the use of operator theoretic methods in the study of concepts pertinent purely to Harmonic Analysis.

An important role in our approach plays the technique of pseudo-integral operators introduced in [2]. Some results on these operators, which are used in the sequel, are collected in Section 3 of the paper. En route, we give an affirmative answer of a question of J. Froelich [14] concerning the validity of a tensor product formula for masa-bimodules (see Theorem 3.8).

The main technical tool we develop and use is a symbolic calculus for weak* continuous completely bounded maps from the algebra $\mathcal{B}\left(L^{2}(G)\right)$ of bounded operators on $L^{2}(G)$ into the von Neumann algebra $\operatorname{VN}(G)$ of $G$ (see Theorem 4.6). A significant role in our approach is played by a locally compact version of the uniform Roe algebra which was introduced for discrete groups in [32] and has been studied in various contexts.

In Section 5, we show that the property of being a set of (operator) multiplicity is preserved under some natural operations. These include direct products and a certain type of generalised union. As a corollary of a more general operator algebraic statement, we recover M. Bożejko's result [5,4] that every countable closed set in a non-discrete locally compact group is a set of uniqueness. We also establish an Inverse Image Theorem for sets of operator multiplicity (see Theorem 5.5).

In Section 6, we examine sets of finite width. This class of sets has played a fundamental role in the field since their introduction in [2] (see [9,10,35] and the references therein). We characterise the sets of finite width that are also sets of operator multiplicity, and show that, in general, every compact operator supported on a set of finite width is the norm limit of sums of rank one operators supported on this set.

Sections 7 and 8 are devoted to the main applications of the previously described results. Namely, in Section 7, we establish a "closable" version of the aforementioned characterisation of completely bounded multipliers, showing that for groups $G$ satisfying a certain approximation property (more general than weak amenability), $\psi$ is a closable multiplier on $C_{r}^{*}(G)$, in the sense that the pointwise multiplication of $L^{1}(G)$ by $\psi$ is a closable map on $C_{r}^{*}(G)$, if and only if $N(\psi)$ is a closable multiplier in the sense of [34]. We present various examples of closable and non-closable multipliers.

In Section 8, we discuss similar multiplier maps on the group von Neumann algebra $\mathrm{VN}(G)$. We introduce the notion of a weak* closable operator, which is suitable for the setting of dual Banach spaces, such as $\operatorname{VN}(G)$. We show that a continuous function $\psi$ is a weak* closable multiplier if and only if $N(\psi)$ is a local Schur multiplier [34], which occurs precisely when $\psi$ belongs locally to the Fourier algebra $A(G)$. Weak** closable multipliers on $C_{r}^{*}(G)[34]$ (see Section 2.1) are shown to form a proper subset of the class of weak* closable multipliers, which in turn form a proper subset of the class of closable multipliers.

Finally, in Section 2, we collect the necessary preliminary material and set notation for the subsequent sections.

## 2. Preliminaries

In this section, we collect some definitions and results that will be needed in the sequel.

### 2.1. Closable operators

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T: D(T) \rightarrow \mathcal{Y}$ be a linear operator, where the domain $D(T)$ of $T$ is a dense linear subspace of $\mathcal{X}$. The operator $T$ is called closable if the closure $\overline{\mathrm{Gr} T}$ of its graph

$$
\operatorname{Gr} T=\{(x, T x): x \in D(T)\} \subseteq \mathcal{X} \oplus \mathcal{Y}
$$

is the graph of a linear operator. Equivalently, $T$ is closable if $\left(x_{k}\right)_{k \in \mathbb{N}} \subseteq D(T), y \in \mathcal{Y}$, $\left\|x_{k}\right\| \rightarrow_{k \rightarrow \infty} 0$ and $\left\|T\left(x_{k}\right)-y\right\| \rightarrow_{k \rightarrow \infty} 0$ imply that $y=0$. The operator $T$ is called
weak ${ }^{* *}$ closable [34] if the weak* closure $\overline{\operatorname{Gr} T}{ }^{w^{*}}$ of $\operatorname{Gr} T$ in $\mathcal{X}^{* *} \oplus \mathcal{Y}^{* *}$ is the graph of a linear operator. Equivalently, $T$ is weak** closable if whenever $\left(x_{j}\right)_{j \in J} \subseteq D(T)$ is a net, $y \in \mathcal{Y}^{* *}, x_{j} \xrightarrow{w^{*}} j \in J 0$ and $T\left(x_{j}\right) \xrightarrow{w^{*}}{ }_{j \in J} y$, we have that $y=0$. We note that in [34] weak $^{* *}$ closable operators were called weak* closable. We have chosen to alter our terminology since we feel that the term "weak* closable" is better suited for the notion introduced and studied in Section 8 of the present paper.

The domain of the adjoint operator of $T$ is the subspace

$$
D\left(T^{*}\right)=\left\{g \in \mathcal{Y}^{*}: \exists f \in \mathcal{X}^{*} \text { such that } g(T x)=f(x) \text { for all } x \in D(T)\right\}
$$

and the adjoint of $T$ is the operator $T^{*}: D\left(T^{*}\right) \rightarrow \mathcal{X}^{*}$ defined by letting $T^{*}(g)=f$, where $f$ is the functional associated with $g$ in the definition of $D\left(T^{*}\right)$.

In the following proposition, which was stated in [34], the equivalence (iii) $\Leftrightarrow$ (iv) is well-known (see, for example, [22, Chapter III, Section 5]), while the other implications can be proved easily.

Proposition 2.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $D(T) \subseteq \mathcal{X}, T: D(T) \rightarrow \mathcal{Y}$ be a densely defined linear operator and set $\mathcal{D}=D\left(T^{*}\right)$. Consider the following conditions:
(i) $T$ is weak ${ }^{* *}$ closable;
(ii) $\overline{\mathcal{D}}^{\|\cdot\|}=\mathcal{Y}^{*}$;
(iii) $\overline{\mathcal{D}}^{w^{*}}=\mathcal{Y}^{*}$;
(iv) $T$ is closable.

Then $(i) \Leftrightarrow(i i) \Rightarrow(i i i) \Leftrightarrow(i v)$.

### 2.2. Locally compact groups

If $H, H_{1}$ and $H_{2}$ are Hilbert spaces, we denote by $\mathcal{B}\left(H_{1}, H_{2}\right)$ the space of all bounded linear operators from $H_{1}$ to $H_{2}$, and set $\mathcal{B}(H)=\mathcal{B}(H, H)$. Let $G$ be a locally compact group. Left Haar measure on $G$ will be denoted by $m_{G}$ or $m$ and integration with respect to $m_{G}$ along the variable $s$ will be denoted by $d s$. We denote by $L^{p}(G), p=1,2, \infty$, the corresponding Lebesgue spaces associated with $m_{G}$. For a function $\xi: G \rightarrow \mathbb{C}$, we set as customary $\check{\xi}(s)=\xi\left(s^{-1}\right), s \in G$. Let $\lambda: G \rightarrow \mathcal{B}\left(L^{2}(G)\right)$ be the left regular representation of $G$, that is, $\lambda_{s} f(t)=f\left(s^{-1} t\right), f \in L^{2}(G), s, t \in G$, and $M(G)$ be the measure algebra of $G$, consisting by definition of all bounded complex Borel measures on $G$. We denote the variation of $\theta \in M(G)$ by $|\theta|$ and let $\|\theta\|=|\theta|(G)$. The support of a measure $\theta \in M(G)$ is the (closed) subset

$$
\operatorname{supp} \theta=(\bigcup\{U \subseteq G: U \text { open, }|\theta|(U)=0\})^{c}
$$

it is the smallest closed subset $E$ of $G$ with the property that if $U \subseteq E^{c}$ is a Borel set then $\theta(U)=0$. For a closed set $E \subseteq G$, let $M(E)$ be the set of all measures $\theta$ in
$M(G)$ with $\operatorname{supp} \theta \subseteq E$. If $\theta \in M(G)$ then the operator $\lambda(\theta)$ of convolution by $\theta$ is given by $\lambda(\theta)(f)(t)=\int_{G} f\left(s^{-1} t\right) d \theta(s)$; the map $\lambda: M(G) \rightarrow \mathcal{B}\left(L^{2}(G)\right)$ is a representation of $M(G)$ of $L^{2}(G)$. Since $L^{1}(G)$ is a Banach subalgebra of $M(G)$, the restriction of $\lambda$ to $L^{1}(G)$ is a representation of $L^{1}(G)$; we have

$$
\lambda(f) g(t)=f * g(t)=\int f(s) g\left(s^{-1} t\right) d s, \quad f \in L^{1}(G), g \in L^{2}(G), t \in G
$$

The Fourier algebra $A(G)$ of $G$ [12] is the algebra of coefficients of $\lambda$, that is, the algebra of functions of the form $s \rightarrow\left(\lambda_{s} \xi, \eta\right)$, for $\xi, \eta \in L^{2}(G)$. The Fourier-Stieltjes algebra $B(G)$ of $G[12]$ is, on the other hand, the algebra of coefficients of all continuous unitary representations of $G$ acting on some Hilbert space, that is, the algebra of all functions of the form $s \rightarrow(\pi(s) \xi, \eta)$, where $\pi: G \rightarrow \mathcal{B}(H)$ is a continuous unitary representation, and $\xi, \eta \in H$. We denote by $C_{r}^{*}(G)$ the reduced $C^{*}$-algebra of $G$, that is, the closure of $\lambda\left(L^{1}(G)\right)$ in the operator norm. We let $\operatorname{VN}(G)={\overline{C_{r}^{*}(G)}}^{w^{*}}$ be the von Neumann algebra of $G$, and $C^{*}(G)$ be the full $C^{*}$-algebra of $G$. It is known [12] that $A(G)$ is a semisimple, regular, commutative Banach algebra with spectrum $G$, which can be identified with the predual $\mathrm{VN}(G)_{*}$ of $\mathrm{VN}(G)$ via the pairing $\langle u, T\rangle=(T \xi, \eta)$, where $u \in A(G)$ is given by $u(s)=\left(\lambda_{s} \xi, \eta\right)$. If $T \in \mathrm{VN}(G)$ and $u \in A(G)$, the operator $u \cdot T \in \mathrm{VN}(G)$ is given by the relations $\langle u \cdot T, v\rangle=\langle T, u v\rangle, v \in A(G)$. The map $(u, T) \mapsto u \cdot T$ turns $\operatorname{VN}(G)$ into a Banach $A(G)$-module.

Let

$$
M A(G)=\{v: G \rightarrow \mathbb{C}: v u \in A(G), \text { for all } u \in A(G)\}
$$

be the multiplier algebra of $A(G)$. For each $v \in M A(G)$, the map $u \mapsto v u$ on $A(G)$ is bounded; its norm will be denoted by $\|v\|_{M A(G)}$. As usual, let $M^{\mathrm{cb}} A(G)$ be the subalgebra of $M A(G)$ consisting of those $v$ for which the map $u \mapsto v u$ on $A(G)$ is completely bounded [7]. We refer the reader to [27] and [31] for the basic of Operator Space Theory and completely bounded maps.

We denote by $C_{0}(G)$ the space of all continuous functions on $G$ vanishing at infinity. The dual of $C_{0}(G)$ can be canonically identified with $M(G)$; the duality between the two spaces will be denoted by $\langle\cdot, \cdot\rangle$. Note that $A(G) \subseteq C_{0}(G)$ and that the adjoint of this inclusion gives rise to the inclusion $\lambda(M(G)) \subseteq \mathrm{VN}(G)$. We refer the reader to [12] for more details about the notions discussed above.

If $J \subseteq A(G)$ is an ideal, let

$$
\text { null } J=\{s \in G: u(s)=0 \text { for all } u \in J\} .
$$

On the other hand, for a closed set $E \subseteq G$, let

$$
I(E)=\{f \in A(G): f(s)=0, s \in E\}
$$

$$
J_{0}(E)=\{f \in A(G): f \text { has compact support disjoint from } E\}
$$

and $J(E)=\overline{J_{0}(E)}$. We have that null $J(E)=\operatorname{null} I(E)=E$ and that if $J \subseteq A(G)$ is a closed ideal with null $J=E$, then $J(E) \subseteq J \subseteq I(E)$. The $\operatorname{support} \operatorname{supp}(T)$ of an operator $T \in \mathrm{VN}(G)$ is given by

$$
\operatorname{supp}(T)=\{t \in G: u \cdot T \neq 0 \text { whenever } u \in A(G) \text { and } u(t) \neq 0\}
$$

It is known (see [12]) that the annihilator $J(E)^{\perp}$ of $J(E)$ in $\mathrm{VN}(G)$ coincides with the space of all operators $T \in \mathrm{VN}(G)$ with $\operatorname{supp}(T) \subseteq E$.

### 2.3. Masa-bimodules

We fix, throughout the paper, standard measure spaces $(X, \mu)$ and $(Y, \nu)$; this means that $\mu$ and $\nu$ are Radon measures with respect to some complete metrisable separable locally compact topologies (henceforth called admissible topologies) on $X$ and $Y$, respectively. A subset of $X \times Y$ will be called a rectangle if it is of the form $\alpha \times \beta$, where $\alpha \subseteq X$ and $\beta \subseteq Y$ are measurable. We equip $X \times Y$ with the $\sigma$-algebra generated by all rectangles and denote by $\mu \times \nu$ the product measure. A subset $E \subseteq X \times Y$ is called marginally null if $E \subseteq\left(X_{0} \times Y\right) \cup\left(X \times Y_{0}\right)$, where $\mu\left(X_{0}\right)=\nu\left(Y_{0}\right)=0$. We call two subsets $E, F \subseteq X \times Y$ marginally equivalent (and write $E \simeq F$ ) if their symmetric difference is marginally null.

A subset $E$ of $X \times Y$ is called $\omega$-open if it is marginally equivalent to the union of a countable set of rectangles. The complements of $\omega$-open sets are called $\omega$-closed. It is clear that the class of all $\omega$-open (resp. $\omega$-closed) sets is closed under countable unions (resp. intersections) and finite intersections (resp. unions). Let $\mathfrak{B}(X \times Y)$ be the space of all measurable complex valued functions defined on the measure space ( $X \times Y, \mu \times \nu$ ). We say that two functions $\varphi, \psi \in \mathfrak{B}(X \times Y)$ are equivalent, and write $\varphi \sim \psi$, if the set $D=\{(x, y) \in X \times Y: \psi(x, y) \neq \varphi(x, y)\}$ is null with respect to $\mu \times \nu$. If $D$ is marginally null then we say that $\varphi$ and $\psi$ coincide marginally almost everywhere or that they are marginally equivalent, and write $\varphi \simeq \psi$.

The following lemma was proved in [11].
Lemma 2.2. Suppose that compact admissible topologies can be chosen on $X$ and $Y$ and that $\mu$ and $\nu$ are finite. Let $E \subseteq \bigcup_{n=1}^{\infty} \gamma_{n}$ where $E$ is $\omega$-closed and $\gamma_{n}$ is $\omega$-open, $n \in \mathbb{N}$. Then for each $\varepsilon>0$ there are subsets $X_{\varepsilon} \subseteq X, Y_{\varepsilon} \subseteq Y$ such that $\mu\left(X \backslash X_{\varepsilon}\right)<\varepsilon$, $\nu\left(Y \backslash Y_{\varepsilon}\right)<\varepsilon$ and $E \cap\left(X_{\varepsilon} \times Y_{\varepsilon}\right)$ is contained in the union of finitely many of the subsets $\gamma_{n}, n \in \mathbb{N}$.

For Hilbert spaces $H_{1}$ and $H_{2}$, we denote by $\mathcal{K}\left(H_{1}, H_{2}\right)$ (resp. $\left.\mathcal{C}_{1}\left(H_{1}, H_{2}\right), \mathcal{C}_{2}\left(H_{1}, H_{2}\right)\right)$ the space of compact (resp. nuclear, Hilbert-Schmidt) operators in $\mathcal{B}\left(H_{1}, H_{2}\right)$. We often write $\mathcal{K}=\mathcal{K}\left(H_{1}, H_{2}\right)$. Throughout the paper, we let $H_{1}=L^{2}(X, \mu)$ and $H_{2}=L^{2}(Y, \nu)$. The operator norm of $T \in \mathcal{B}\left(H_{1}, H_{2}\right)$ is denoted by $\|T\|$. The space $\mathcal{C}_{1}\left(H_{2}, H_{1}\right)$ (resp. $\mathcal{B}\left(H_{1}, H_{2}\right)$ ) can be naturally identified with the Banach space dual of $\mathcal{K}\left(H_{1}, H_{2}\right)$ (resp.
$\mathcal{C}_{1}\left(H_{2}, H_{1}\right)$ ), the duality being given by the $\operatorname{map}(T, S) \mapsto\langle T, S\rangle \stackrel{\text { def }}{=} \operatorname{tr}(T S)$. Here $\operatorname{tr} A$ denotes the trace of a nuclear operator $A$.

The space $L^{2}(Y \times X)$ will be identified with $\mathcal{C}_{2}\left(H_{1}, H_{2}\right)$ via the map sending an element $k \in L^{2}(Y \times X)$ to the integral operator $T_{k}$ given by $T_{k} \xi(y)=\int_{X} k(y, x) \xi(x) d \mu(x)$, $\xi \in H_{1}, y \in Y$. In a similar fashion, $\mathcal{C}_{1}\left(H_{2}, H_{1}\right)$ will be identified with the space $\Gamma(X, Y)$ of all (marginal equivalence classes of) functions $h: X \times Y \rightarrow \mathbb{C}$ which admit a representation

$$
h(x, y)=\sum_{i=1}^{\infty} f_{i}(x) g_{i}(y)
$$

where $f_{i} \in H_{1}, g_{i} \in H_{2}, i \in \mathbb{N}, \sum_{i=1}^{\infty}\left\|f_{i}\right\|_{2}^{2}<\infty$ and $\sum_{i=1}^{\infty}\left\|g_{i}\right\|_{2}^{2}<\infty$. Equivalently, $\Gamma(X, Y)$ can be defined as the projective tensor product $H_{1} \hat{\otimes} H_{2}$; we write $\|h\|_{\Gamma}$ for the projective norm of $h \in \Gamma(X, Y)$. The duality between $\mathcal{B}\left(H_{1}, H_{2}\right)$ and $\Gamma(X, Y)$ is given by

$$
\langle T, f \otimes g\rangle=(T f, \bar{g}),
$$

for $T \in \mathcal{B}\left(H_{1}, H_{2}\right), f \in L^{2}(X, \mu)$ and $g \in L^{2}(Y, \nu)$.
If $f \in L^{\infty}(X, \mu)$, let $M_{f} \in \mathcal{B}\left(H_{1}\right)$ be the operator on $H_{1}$ of multiplication by $f$. The collection $\left\{M_{f}: f \in L^{\infty}(X, \mu)\right\}$ is a maximal abelian selfadjoint algebra (for short, masa) on $H_{1}$. If $\alpha \subseteq X$ is measurable, we write $P(\alpha)=M_{\chi_{\alpha}}$ for the multiplication by the characteristic function of the set $\alpha$. The same notation will be used for $H_{2}$. A subspace $\mathcal{W} \subseteq \mathcal{B}\left(H_{1}, H_{2}\right)$ will be called a masa-bimodule if $M_{\psi} T M_{\varphi} \in \mathcal{W}$ for all $T \in \mathcal{W}, \varphi \in L^{\infty}(X, \mu)$ and $\psi \in L^{\infty}(Y, \nu)$.

We say that an $\omega$-closed subset $\kappa \subseteq X \times Y$ supports an operator $T \in \mathcal{B}\left(H_{1}, H_{2}\right)$ (or that $T$ is supported on $\kappa$ ) if $P(\beta) T P(\alpha)=0$ whenever $(\alpha \times \beta) \cap \kappa \simeq \emptyset$. For any subset $\mathcal{M} \subseteq \mathcal{B}\left(H_{1}, H_{2}\right)$, there exists a smallest (up to marginal equivalence) $\omega$-closed set $\operatorname{supp} \mathcal{M}$ which supports every operator $T \in \mathcal{M}$ [11]. By [2] and [35], for any $\omega$-closed set $\kappa$ there exists a smallest (resp. largest) weak* closed masa-bimodule $\mathfrak{M}_{\min }(\kappa)$ (resp. $\mathfrak{M}_{\max }(\kappa)$ ) with support $\kappa$, in the sense that if $\mathfrak{M} \subseteq \mathcal{B}\left(H_{1}, H_{2}\right)$ is a weak* closed masa-bimodule with supp $\mathfrak{M}=\kappa$ then $\mathfrak{M}_{\text {min }}(\kappa) \subseteq \mathfrak{M} \subseteq \mathfrak{M}_{\text {max }}(\kappa)$.

Let

$$
\Phi(\kappa)=\left\{h \in \Gamma(X, Y): h \chi_{\kappa} \simeq 0\right\}
$$

and

$$
\Psi(\kappa)=\overline{\{h \in \Gamma(X, Y): h \text { vanishes on an } \omega \text {-open nbhd of } \kappa\}}{ }^{\|\cdot\|_{\Gamma}} .
$$

By [35, Theorems 4.3, 4.4], $\mathfrak{M}_{\text {min }}(\kappa)=\Phi(\kappa)^{\perp}$ and $\mathfrak{M}_{\max }(\kappa)=\Psi(\kappa)^{\perp}$.

### 2.4. Schur multipliers

If $\varphi$ is a function defined on a measure space $(Z, \theta)$, and $\mathcal{E}$ is a space of measurable functions on $Z$, we write $\varphi \in^{\theta} \mathcal{E}$ when there exists a function $\psi \in \mathcal{E}$ such that $\varphi$ and $\psi$ differ on a $\theta$-null set. Let

$$
J_{\varphi}^{\mathcal{E}}=\left\{h \in \mathcal{E}: \varphi h \in^{\theta} \mathcal{E}\right\} .
$$

For $\varphi \in \mathfrak{B}(X \times Y)$, the function $\hat{\varphi}: Y \times X \rightarrow \mathbb{C}$ is given by $\hat{\varphi}(y, x)=\varphi(x, y), x \in X$, $y \in Y$. We set $D\left(S_{\varphi}\right)=J_{\hat{\varphi}}^{L^{2}(Y \times X)}$. Identifying $L^{2}(Y \times X)$ with $\mathcal{C}_{2}\left(H_{1}, H_{2}\right) \subseteq \mathcal{K}\left(H_{1}, H_{2}\right)$, define $S_{\varphi}: D\left(S_{\varphi}\right) \rightarrow \mathcal{K}\left(H_{1}, H_{2}\right)$ to be the mapping given by $S_{\varphi}\left(T_{k}\right)=T_{\hat{\varphi} k}$. We say that $\varphi \in \mathfrak{B}(X \times Y)$ is a closable multiplier (resp. weak** closable multiplier) [34] if the map $S_{\varphi}$ is closable (resp. weak ${ }^{* *}$ closable) when viewed as a densely defined linear operator on $\mathcal{K}\left(H_{1}, H_{2}\right)$. If $S_{\varphi}$ is moreover bounded in the operator norm, $\varphi$ is called a Schur multiplier. If $\varphi$ is a Schur multiplier then the mapping $S_{\varphi}$ extends by continuity to a (bounded) mapping on $\mathcal{K}\left(H_{1}, H_{2}\right)$. After taking its second dual, one obtains a bounded weak* continuous linear transformation on $\mathcal{B}\left(H_{1}, H_{2}\right)$ which will also be denoted by $S_{\varphi}$. We set $\|\varphi\|_{\mathfrak{S}}=\left\|S_{\varphi}\right\|$. The map $S_{\varphi}$ is automatically completely bounded and its completely bounded norm is still equal to $\|\varphi\|_{\mathfrak{S}}$ (the reader is referred to [27] and [31] for the basics of Operator Space Theory, which will be used throughout the paper). By a result of V.V. Peller [29] (see also [21] and [38]), a function $\varphi \in \mathfrak{B}(X \times Y)$ is a Schur multiplier if and only if there exist sequences $\left(a_{k}\right)_{k \in \mathbb{N}} \subseteq L^{\infty}(X, \mu)$ and $\left(b_{k}\right)_{k \in \mathbb{N}} \subseteq$ $L^{\infty}(Y, \nu)$ with $\operatorname{ess} \sup _{x \in X} \sum_{k=1}^{\infty}\left|a_{k}(x)\right|^{2}<\infty$ and $\operatorname{ess} \sup _{y \in Y} \sum_{k=1}^{\infty}\left|b_{k}(y)\right|^{2}<\infty$ such that

$$
\varphi(x, y)=\sum_{k=1}^{\infty} a_{k}(x) b_{k}(y), \quad \text { a.e. }(x, y) \in X \times Y
$$

In this case, $S_{\varphi}(T)=\sum_{k=1}^{\infty} M_{b_{k}} T M_{a_{k}}, T \in \mathcal{B}\left(H_{1}, H_{2}\right)$.
Let $\mathfrak{S}(X, Y)$ be the set of all Schur multipliers (we will also write $\mathfrak{S}(X \times Y)$ in the place of $\mathfrak{S}(X, Y)$ if there is no risk of confusion). By [29],

$$
\mathfrak{S}(X, Y)=\left\{\varphi \in L^{\infty}(X \times Y): \varphi h \in^{\mu \times \nu} \Gamma(X, Y), \forall h \in \Gamma(X, Y)\right\}
$$

If $\varphi \in \mathfrak{S}(X, Y)$, let $m_{\varphi}: \Gamma(X, Y) \rightarrow \Gamma(X, Y)$ be the mapping given by $m_{\varphi}(h)=\varphi h$, $h \in \Gamma(X, Y)$; then the adjoint of $m_{\varphi}$ coincides with $S_{\varphi}$.

Let $G$ be a locally compact group. The map $P: \Gamma(G, G) \rightarrow A(G)$ given by

$$
\begin{equation*}
P(f \otimes g)(t)=\left\langle\lambda_{t}, f \otimes g\right\rangle=\left(\lambda_{t} f, \bar{g}\right)=\int_{G} f\left(t^{-1} s\right) g(s) d s=g * \check{f}(t) \tag{1}
\end{equation*}
$$

is a contractive surjection. The next lemma will be used repeatedly.

Lemma 2.3. If $h \in \Gamma(G, G)$ then

$$
\begin{equation*}
P(h)(t)=\int_{G} h\left(t^{-1} s, s\right) d s, \quad t \in G \tag{2}
\end{equation*}
$$

Proof. Identity (2) is a direct consequence of (1) if $h$ is a finite sum of elementary tensors. Let $h=\sum_{i=1}^{\infty} f_{i} \otimes g_{i} \in \Gamma(G, G)$, where $\sum_{i=1}^{\infty}\left\|f_{i}\right\|_{2}^{2}<\infty$ and $\sum_{i=1}^{\infty}\left\|g_{i}\right\|_{2}^{2}<\infty$, and let $h_{n}$ be the $n$th partial sum of this series. By the continuity of $P,\left\|P\left(h_{n}\right)-P(h)\right\| \rightarrow 0$ in $A(G)$; since $\|\cdot\|_{\infty}$ is dominated by the norm of $A(G)$, we conclude that $P\left(h_{n}\right)(t) \rightarrow P(h)(t)$ for every $t \in G$.

By [35, Lemma 2.1], there exists a subsequence $\left(h_{n_{k}}\right)_{k \in \mathbb{N}}$ of $\left(h_{n}\right)_{n \in \mathbb{N}}$ such that $h_{n_{k}} \rightarrow h$ marginally almost everywhere. It follows that, for every $t \in G$, one has $h_{n_{k}}\left(t^{-1} s, s\right) \rightarrow$ $h\left(t^{-1} s, s\right)$ for almost all $s \in G$. By [25, (4.3)], the function $s \rightarrow \sum_{i=1}^{\infty}\left|f_{i}\left(t^{-1} s\right)\right|\left|g_{i}(s)\right|$ is integrable, and hence an application of the Lebesgue Dominated Convergence Theorem shows that $\int_{G} h_{n_{k}}\left(t^{-1} s, s\right) d s \rightarrow_{k \rightarrow \infty} \int_{G} h\left(t^{-1} s, s\right) d s$, for every $t \in G$. The proof is complete.

For a function $f: G \rightarrow \mathbb{C}$, let $N(f): G \times G \rightarrow \mathbb{C}$ be the function given by

$$
\begin{equation*}
N(f)(s, t)=f\left(t s^{-1}\right), \quad s, t \in G \tag{3}
\end{equation*}
$$

Note that in [25] and [39], the map $N^{\prime}$ given by $N^{\prime}(f)(s, t)=f\left(s t^{-1}\right)$ was used instead of $N$, but the results established in these papers remain valid with the current definition as well. It follows from [6] (see also [19] and [38]) that $N$ maps $M^{\text {cb }} A(G)$ isometrically into $\mathfrak{S}(G, G)$. Note that, if $G$ is compact, then $\Gamma(G, G)$ contains the constant functions and hence $\mathfrak{S}(G, G) \subseteq \Gamma(G, G)$; thus, in this case $N$ maps $A(G)$ into $\Gamma(G, G)$.

## 3. Arveson measures and pseudo-integral operators

### 3.1. Measures

Let $\sigma$ be a complex measure of finite total variation, defined on the product $\sigma$-algebra $\mathcal{F}$ of $X \times Y$. We let $|\sigma|$ denote the variation of $\sigma$; thus, for a subset $E \in \mathcal{F}$, the quantity $|\sigma|(E)$ equals the total variation of $\sigma$ on the set $E$. We let $|\sigma|_{X}$ be the $X$-marginal measure of $|\sigma|$, that is, the measure on $X$ given by $|\sigma|_{X}(\alpha)=|\sigma|(\alpha \times Y)$. We define $|\sigma|_{Y}$ similarly by setting $|\sigma|_{Y}(\beta)=|\sigma|(X \times \beta)$. A complex measure $\sigma$ on $\mathcal{F}$ will be called an Arveson measure if $\sigma$ has finite total variation and there exists a constant $c>0$ such that

$$
\begin{equation*}
|\sigma|_{X} \leq c \mu \quad \text { and } \quad|\sigma|_{Y} \leq c \nu \tag{4}
\end{equation*}
$$

We denote by $\mathbb{A}(X, Y)$ the set of all Arveson measures on $X \times Y$ and let $\|\sigma\|_{\mathbb{A}}$ be the smallest constant $c$ which satisfies the inequalities (4). We note that if $\sigma \in \mathbb{A}(X, Y)$ then $|\sigma| \in \mathbb{A}(X, Y)$ as well.

It was shown in [34], that given a family $\mathcal{E}$ of $\omega$-open sets, there exists a minimal (with respect to inclusion up to a marginally null set) $\omega$-open set $E$ which marginally contains every element from $\mathcal{E}$. The set $E$ is called the $\omega$-union of $E$ and denoted by $\bigcup_{\omega} \mathcal{E}$.

Recall that $(X, \mu)$ and $(Y, \nu)$ are standard measure spaces and let $\sigma$ be an Arveson measure on $Y \times X$. Denote by supp $\sigma$ the $\omega$-closed subset of $Y \times X$ defined by

$$
\begin{aligned}
(\operatorname{supp} \sigma)^{c}= & \bigcup_{\omega}\{R \subseteq Y \times X: R \text { is a rectangle such that } \\
& \left.\sigma\left(R^{\prime}\right)=0 \text { for each rectangle } R^{\prime} \subseteq R\right\} .
\end{aligned}
$$

Proposition 3.1. Let $\sigma \in \mathbb{A}(Y, X)$.
(i) The set $\operatorname{supp} \sigma$ is the smallest (up to marginal equivalence) $\omega$-closed subset $E$ of $Y \times X$ such that $\sigma(R)=0$ for every rectangle $R \subseteq E^{c}$.
(ii) If $E \subseteq Y \times X$ is an $\omega$-closed set then supp $\sigma \subseteq E$ if and only if $|\sigma|\left(E^{c}\right)=0$.

Proof. (i) Let $\mathcal{R}$ be the set of all rectangles $R \subseteq Y \times X$ such that $\sigma\left(R^{\prime}\right)=0$ for every rectangle $R^{\prime}$ contained in $R$. By [34, Lemma 2.1], $(\operatorname{supp} \sigma)^{c} \simeq \bigcup_{i=1}^{\infty} R_{i}$ for some family $\left\{R_{i}\right\}_{i \in \mathbb{N}} \subseteq \mathcal{R}$. Let $R \subseteq(\operatorname{supp} \sigma)^{c}$ be a rectangle. We will show that $\sigma(R)=0$; without loss of generality, we may assume that the measures $\mu$ and $\nu$ are finite. By Lemma 2.2, for every $n \in \mathbb{N}$ there exist measurable subsets $X_{n} \subseteq X$ and $Y_{n} \subseteq Y$ such that $\mu\left(X \backslash X_{n}\right)<1 / n, \nu\left(Y \backslash Y_{n}\right)<1 / n$ and $R \cap\left(Y_{n} \times X_{n}\right)$ is contained in the union of a finite subfamily of $\left\{R_{i}\right\}_{i \in \mathbb{N}}$. It follows that $\sigma\left(R \cap\left(Y_{n} \times X_{n}\right)\right)=0$ for every $n$ and, since $\bigcup_{n=1}^{\infty} X_{n}$ and $\bigcup_{n=1}^{\infty} Y_{n}$ have full measure, $\sigma(R)=0$.

Suppose that $E$ is an $\omega$-closed set with the property that $\sigma(R)=0$ for every rectangle $R \subseteq E^{c}$. By the definition of $\operatorname{supp} \sigma$, the set $E^{c}$ is marginally contained in $(\operatorname{supp} \sigma)^{c}$, and hence $\operatorname{supp} \sigma \subseteq E$ up to marginal equivalence.
(ii) Suppose that $E^{c} \simeq \Omega=\bigcup_{i=1}^{\infty} R_{i}$, where $R_{i} \subseteq Y \times X$ is a rectangle, $i \in \mathbb{N}$. Assume, without loss of generality, that $R_{i} \cap R_{j}=\emptyset$ if $i \neq j$. Fix $i \in \mathbb{N}$. By (i), if $R \subseteq R_{i}$ is a rectangle, then $\sigma(R)=0$. Since the product $\sigma$-algebra on $R_{i}$ is generated by the rectangles contained in $R_{i}$, it follows that $\sigma(F)=0$ for every measurable (with respect to the product $\sigma$-algebra) subset $F \subseteq R_{i}$. Thus, if $F \subseteq \Omega$ is an arbitrary measurable subset then $\sigma\left(F \cap R_{i}\right)=0$ for each $i$; therefore, $\sigma(F)=0$.

Now suppose that $F \subseteq E^{c}$ is a measurable subset. Then $F \subseteq F^{\prime} \cup F^{\prime \prime}$ as a disjoint union, where $F^{\prime} \subseteq \Omega$ and $F^{\prime \prime}$ is marginally null. By the previous paragraph, $\sigma\left(F^{\prime}\right)=0$, while, since $\sigma$ is an Arveson measure, $\sigma\left(F^{\prime \prime}\right)=0$. It follows that $\sigma(F)=0$. Thus, $|\sigma|\left(E^{c}\right)=0$.

Conversely, if $|\sigma|\left(E^{c}\right)=0$ then $\sigma(R)=0$ for every measurable rectangle contained in $E^{c}$. By (i), supp $\sigma \subseteq E$.

For an $\omega$-closed set $F \subseteq Y \times X$, we denote by $\mathbb{A}(F)$ the set of all measures $\sigma$ in $\mathbb{A}(Y, X)$ such that supp $\sigma \subseteq F$.

### 3.2. Operators

The importance of Arveson measures is explained by the fact that they define special operators called pseudointegral in [2], where they were introduced. This class of operators will be essential for our considerations.

The first part of the following result was established in [2, Theorem 1.5.1]; we include its full proof for completeness.

Theorem 3.2. Let $\sigma \in \mathbb{A}(Y, X)$. There exists a unique operator $T_{\sigma}: H_{1} \rightarrow H_{2}$ such that

$$
\left(T_{\sigma} f, g\right)=\int_{Y \times X} f(x) \overline{g(y)} d \sigma(y, x), \quad f \in H_{1}, g \in H_{2}
$$

Moreover, $\left\|T_{\sigma}\right\| \leq\|\sigma\|_{\mathbb{A}}$ and, for a given $\omega$-closed subset $\kappa \subseteq X \times Y$, the operator $T_{\sigma}$ is supported on $\kappa$ if and only if $\operatorname{supp} \sigma \subseteq \hat{\kappa} \stackrel{\text { def }}{=}\{(y, x):(x, y) \in \kappa\}$. If $h \in \Gamma(X, Y)$ and $\sigma \in \mathbb{A}(Y, X)$ then $\left\langle T_{\sigma}, h\right\rangle=\int_{Y \times X} \hat{h} d \sigma$.

Proof. Fix $\sigma \in \mathbb{A}(Y, X)$ and consider the sesqui-linear form $\phi: H_{1} \times H_{2} \rightarrow \mathbb{C}$ given by

$$
\phi(f, g)=\int_{Y \times X} f(x) \overline{g(y)} d \sigma(y, x)
$$

Note that $\phi$ is well-defined:

$$
\begin{aligned}
\left.\left.\right|_{Y \times X} f(x) \overline{g(y)} d \sigma(y, x)\right|^{2} & \leq\left(\int_{Y \times X}|f(x)||g(y)| d|\sigma|(y, x)\right)^{2} \\
& \leq \int_{Y \times X}|f(x)|^{2} d|\sigma|(y, x) \int_{Y \times X}|g(y)|^{2} d|\sigma|(y, x) \\
& =\int_{X}|f(x)|^{2} d|\sigma|_{X}(x) \int_{Y}|g(y)|^{2} d|\sigma|_{Y}(y) \leq\|\sigma\|_{\mathbb{A}}^{2}\|f\|_{2}^{2}\|g\|_{2}^{2}
\end{aligned}
$$

By the Riesz Representation Theorem, there exists a unique operator $T_{\sigma}: H_{1} \rightarrow H_{2}$ such that $\left(T_{\sigma} f, g\right)=\phi(f, g)$; moreover, $\left\|T_{\sigma}\right\| \leq\|\sigma\|_{\mathbb{A}}$.

Let $\kappa \subseteq X \times Y$ and suppose that supp $\sigma \subseteq \hat{\kappa}$. Let $\alpha \subseteq X$ and $\beta \subseteq Y$ be measurable subsets with $(\alpha \times \beta) \cap \kappa \simeq \emptyset$. By deleting null sets from $\alpha$ and $\beta$ we may assume that, in fact, $(\alpha \times \beta) \cap \kappa=\emptyset$. If $f \in H_{1}$ (resp. $g \in H_{2}$ ) is supported on $\alpha$ (resp. $\beta$ ) then, by Proposition 3.1,

$$
\left(T_{\sigma} f, g\right)=\int_{(\beta \times \alpha) \cap \hat{\kappa}} f(x) \overline{g(y)} d \sigma(y, x)=0
$$

thus, $T_{\sigma}$ is supported on $\kappa$.

Conversely, suppose that $T_{\sigma}$ is supported on $\kappa$ and let $\beta \times \alpha \subseteq Y \times X$ be a rectangle of finite measure, marginally disjoint from $\hat{\kappa}$. Then

$$
\sigma(\beta \times \alpha)=\left(T_{\sigma} \chi_{\alpha}, \chi_{\beta}\right)=0
$$

and Proposition 3.1 implies that $\operatorname{supp} \sigma \subseteq \hat{\kappa}$, up to a marginally null set.
Finally, suppose that $h \in \Gamma(X, Y)$ and $\sigma \in \mathbb{A}(Y, X)$. Write $h=\sum_{i=1}^{\infty} f_{i} \otimes g_{i}$, where $\left(f_{i}\right)_{i \in \mathbb{N}} \subseteq H_{1}$ and $\left(g_{i}\right)_{i \in \mathbb{N}} \subseteq H_{2}$ are sequences of functions with $\sum_{i=1}^{\infty}\left\|f_{i}\right\|_{2}^{2}<\infty$ and $\sum_{i=1}^{\infty}\left\|g_{i}\right\|_{2}^{2}<\infty$. The estimate in the first paragraph of the proof shows that $\int_{Y \times X} \sum_{i=1}^{\infty}\left|f_{i}(x)\right|\left|g_{i}(y)\right| d|\sigma|(y, x)<\infty$.

Let $h_{n}=\sum_{i=1}^{n} f_{i} \otimes g_{i}$; by the Lebesgue Dominated Convergence Theorem, $\int_{Y \times X} \hat{h}_{n} d \sigma \rightarrow_{n \rightarrow \infty} \int_{Y \times X} \hat{h} d \sigma$. Thus,

$$
\begin{aligned}
\left\langle T_{\sigma}, h\right\rangle & =\lim _{n \rightarrow \infty}\left\langle T_{\sigma}, h_{n}\right\rangle=\lim _{n \rightarrow \infty} \sum_{i=1}^{n}\left\langle T_{\sigma}, f_{i} \otimes g_{i}\right\rangle \\
& =\lim _{n \rightarrow \infty} \int_{Y \times X} \sum_{i=1}^{n} f_{i}(x) g_{i}(y) d \sigma(y, x)=\int_{Y \times X} h(x, y) d \sigma(y, x) .
\end{aligned}
$$

We recall some facts from [35] that will be needed subsequently. If $\kappa \subseteq X \times Y$ is $\omega$-closed, a $\kappa$-pair is an element

$$
(P, Q) \in\left(\mathcal{B}\left(\ell^{2}\right) \bar{\otimes} L^{\infty}(X, \mu)\right) \times\left(\mathcal{B}\left(\ell^{2}\right) \bar{\otimes} L^{\infty}(Y, \nu)\right)
$$

such that, after the identification of $P$ and $Q$ with operator-valued weakly measurable functions, defined on $X$ and $Y$, respectively, $P$ and $Q$ take values that are projections and $P(x) Q(y)=0$ marginally almost everywhere on $\kappa$. A $\kappa$-pair is called simple if $P$ and $Q$ take finitely many values. The following was established in [35].

Theorem 3.3. Let $\kappa \subseteq X \times Y$ be an $\omega$-closed set. Then

$$
\mathfrak{M}_{\min }(\kappa)=\left\{T \in \mathcal{B}\left(H_{1}, H_{2}\right): Q(I \otimes T) P=0, \quad \forall \kappa \text {-pair }(P, Q)\right\}
$$

and

$$
\mathfrak{M}_{\max }(\kappa)=\left\{T \in \mathcal{B}\left(H_{1}, H_{2}\right): Q(I \otimes T) P=0, \forall \text { simple } \kappa \text {-pair }(P, Q)\right\} .
$$

A version of the next lemma for $\mathfrak{M}_{\min }(\kappa)$ was proved in [36, Proposition 5.3].
Lemma 3.4. If $\kappa \subseteq X \times Y$ is an $\omega$-closed set then

$$
\begin{aligned}
\mathfrak{M}_{\max }(\kappa)= & \left\{T \in \mathcal{B}\left(H_{1}, H_{2}\right): S_{\varphi}(T)=0, \text { for all } \varphi \in \mathfrak{S}(X, Y),\right. \\
& \text { vanishing on an } \omega \text {-open neighbourhood of } \kappa\} .
\end{aligned}
$$

Proof. Suppose that $T \in \mathcal{B}\left(H_{1}, H_{2}\right)$ belongs to the set on the right hand side of the above equality. If $\kappa \cap(\alpha \times \beta) \simeq \emptyset$ then $\chi_{\alpha \times \beta} \in \mathfrak{S}(X, Y)$ vanishes on the $\omega$-open neighbourhood $(\alpha \times \beta)^{c}$ of $\kappa$ and hence $M_{\chi_{\beta}} T M_{\chi_{\alpha}}=S_{\chi_{\alpha \times \beta}}(T)=0$; thus, $T \in \mathfrak{M}_{\max }(\kappa)$.

Conversely, suppose that $T \in \mathfrak{M}_{\max }(\kappa)$ and let $\varphi \in \mathfrak{S}(X, Y)$ vanish on an $\omega$-open neighbourhood of $\kappa$. If $h \in \Gamma(X, Y)$ then $\varphi h \in \Gamma(X, Y)$ and vanishes on an $\omega$-open neighbourhood of $\kappa$. By [35],

$$
\left\langle S_{\varphi}(T), h\right\rangle=\langle T, \varphi h\rangle=0
$$

showing that $S_{\varphi}(T)=0$.
Now we obtain two technical results; apart of applications to the mainstream of the paper they have additional applications which we believe are interesting on their own right. Namely, we will show that a tensor product formula holds for the minimal masabimodules, answering in this way affirmatively a question posed by J. Froelich in [14]. Simultaneously, we show that the minimal masa-bimodule $\mathfrak{M}_{\text {min }}(\kappa)$ associated with an $\omega$-closed set $\kappa$ is the closure of all pseudo-integral operators with symbols supported on $\kappa$; this provides an alternative, "synthetic" description of $\mathfrak{M}_{\text {min }}(\kappa)$ in measure-theoretic terms, similar to the topological one given originally by Arveson in [2].

Let $(X, \mu)$ and $(Y, \nu)$ are standard measure spaces. Recall that $\mathcal{F}$ denotes the product $\sigma$-algebra on $Y \times X$.

Lemma 3.5. If $\sigma \in \mathbb{A}(Y, X)$ and $E \in \mathcal{F}$ then the measure $\sigma_{E}$ given by $\sigma_{E}(F)=\sigma(E \cap F)$, $F \in \mathcal{F}$, belongs to $\mathbb{A}(Y, X)$.

Proof. Let $\sigma \in \mathbb{A}(Y, X)$ and $E \in \mathcal{F}$. If $\alpha \subseteq X$ is measurable then, denoting by $\dot{U}$ the union of a family of pairwise disjoint measurable sets, we have

$$
\begin{aligned}
\left|\sigma_{E}\right|_{X}(\alpha) & =\left|\sigma_{E}\right|(Y \times \alpha)=\sup \left\{\sum_{i=1}^{k}\left|\sigma_{E}\left(F_{i}\right)\right|: \bigcup_{i=1}^{k} F_{i}=Y \times \alpha\right\} \\
& =\sup \left\{\sum_{i=1}^{k}\left|\sigma\left(E \cap F_{i}\right)\right|: \bigcup_{i=1}^{k} F_{i}=Y \times \alpha\right\} \\
& \leq \sup \left\{\sum_{i=1}^{k}|\sigma|\left(E \cap F_{i}\right): \bigcup_{i=1}^{k} F_{i}=Y \times \alpha\right\} \\
& \leq \sup \left\{\sum_{i=1}^{k}|\sigma|\left(F_{i}\right): \bigcup_{i=1}^{k} F_{i}=Y \times \alpha\right\}=|\sigma|(Y \times \alpha) \\
& =|\sigma|_{X}(\alpha) .
\end{aligned}
$$

One shows similarly that $\left|\sigma_{E}\right|_{Y} \leq|\sigma|_{Y}$; it now follows that $\sigma_{E} \in \mathbb{A}(Y, X)$.

Theorem 3.6. Let $\kappa \subseteq X \times Y$ be an $\omega$-closed set. Then

$$
\mathfrak{M}_{\min }(\kappa)=\overline{\left\{T_{\sigma}: \sigma \in \mathbb{A}(Y, X), \operatorname{supp} \sigma \subseteq \hat{\kappa}\right\}}{ }^{w^{*}}
$$

Proof. Let $\mathfrak{M}_{0}(\kappa)$ denote the right hand side of the identity. We first show that $\mathfrak{M}_{0}(\kappa)$ is a weak* closed masa-bimodule. Since $T_{\sigma}+T_{\nu}=T_{\sigma+\nu}$, we have that $\mathfrak{M}_{0}(\kappa)$ is a (weak*) closed subspace of $\mathcal{B}\left(H_{1}, H_{2}\right)$. It is moreover easy to check that if $\varphi \in L^{\infty}(X, \mu)$ and $\psi \in L^{\infty}(Y, \nu)$ then $M_{\psi} T_{\sigma} M_{\varphi}=T_{\sigma^{\prime}}$, where $\sigma^{\prime} \in \mathbb{A}(Y, X)$ is given by

$$
\sigma^{\prime}(E)=\int_{Y \times X} \psi(y) \varphi(x) d \sigma(y, x)
$$

If $\sigma$ is supported on $\hat{\kappa}$ then clearly so is $\sigma^{\prime}$; hence, $\mathfrak{M}_{0}(\kappa)$ is a masa-bimodule.
We next claim that supp $\mathfrak{M}_{0}(\kappa)=\kappa$. Suppose that $\alpha \times \beta$ is a rectangle of finite measure such that $P(\beta) T_{\sigma} P(\alpha)=0$ for all $\sigma \in \mathbb{A}(Y, X)$ with $\operatorname{supp} \sigma \subseteq \hat{\kappa}$. Let $\tau \in \mathbb{A}(X, Y)$ be arbitrary, and $\tau_{\hat{\kappa}}$ be the measure defined as in Lemma 3.5. Then $\operatorname{supp} \tau_{\hat{\kappa}} \subseteq \hat{\kappa}$ and hence

$$
\tau((\beta \times \alpha) \cap \hat{\kappa})=\tau_{\hat{\kappa}}((\beta \times \alpha) \cap \hat{\kappa})=\left(P(\beta) T_{\tau_{\hat{\kappa}}} P(\alpha) \chi_{\alpha}, \chi_{\beta}\right)=0
$$

By Arveson's Null Set Theorem [2, Theorem 1.4.3], $(\beta \times \alpha) \cap \hat{\kappa} \simeq \emptyset$. It follows that $\kappa$ is contained in the support of $\mathfrak{M}_{0}(\kappa)$; on the other hand, by Theorem 3.2, $\operatorname{supp} \mathfrak{M}_{0}(\kappa) \subseteq \kappa$, up to a marginally null set. It follows that $\kappa \simeq \operatorname{supp} \mathfrak{M}_{0}(\kappa)$.

Thus $\mathfrak{M}_{\min }(\kappa) \subseteq \mathfrak{M}_{0}(\kappa)$. To show the converse inclusion, it suffices, by [35, Theorem 4.4], to show that if a function $h \in \Gamma(X, Y)$ vanishes on $\kappa$ then $\left\langle T_{\sigma}, h\right\rangle=0$ for each $\sigma \in \mathbb{A}(Y, X)$ supported by $\hat{\kappa}$. But this follows from the equality $\left\langle T_{\sigma}, h\right\rangle=\int_{Y \times X} \hat{h} d \sigma$ (see Theorem 3.2).

Corollary 3.7. Let $\kappa \subseteq X \times X$ be an $\omega$-closed set such that $\mathfrak{M}_{\max }(\kappa)$ is a unital algebra. Then $\mathfrak{M}_{\min }(\kappa)$ is a (unital) algebra.

Proof. It was shown in [2] that the set of all pseudo-integral operators is an algebra. Since $\mathfrak{M}_{\max }(\kappa)$ is an algebra, the set $\mathfrak{M}_{0}(\kappa)$ of all pseudo-integral operators in $\mathfrak{M}_{\text {max }}(\kappa)$ is also an algebra. Hence its weak* closure $\overline{\mathfrak{M}}_{0}(\kappa) w^{*}$ is also an algebra. By Theorems 3.2 and 3.6, $\mathfrak{M}_{\text {min }}(\kappa)=\overline{\mathfrak{M}}_{0}(\kappa)=w^{*}$ and the proof is complete.

The next theorem establishes a tensor product formula for the minimal masabimodules. Let $\left(X_{i}, \mu_{i}\right)$ and $\left(Y_{i}, \nu_{i}\right)$ be standard measure spaces, $i=1,2$, and consider the flip

$$
\rho:\left(X_{1} \times Y_{1}\right) \times\left(X_{2} \times Y_{2}\right) \rightarrow\left(X_{1} \times X_{2}\right) \times\left(Y_{1} \times Y_{2}\right)
$$

given by

$$
\rho\left(\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right)=\left(\left(x_{1}, x_{2}\right),\left(y_{1}, y_{2}\right)\right)
$$

Below, for two weak* closed subspaces $\mathcal{U}$ and $\mathcal{V}$ of operators, we denote by $\mathcal{U} \bar{\otimes} \mathcal{V}$ the weak* closed subspace generated by the elementary tensors $A \otimes B$ where $A \in \mathcal{U}$ and $B \in \mathcal{V}$.

Theorem 3.8. Let $\left(X_{i}, \mu_{i}\right)$ and $\left(Y_{i}, \nu_{i}\right)$ be standard measure spaces and $\kappa_{i} \subseteq X_{i} \times Y_{i}$ be $\omega$-closed sets, $i=1,2$. Then

$$
\begin{equation*}
\mathfrak{M}_{\min }\left(\kappa_{1}\right) \bar{\otimes} \mathfrak{M}_{\min }\left(\kappa_{2}\right)=\mathfrak{M}_{\min }\left(\rho\left(\kappa_{1} \times \kappa_{2}\right)\right) . \tag{5}
\end{equation*}
$$

Proof. We first note that, by [26],

$$
\begin{equation*}
\operatorname{supp}\left(\mathfrak{M}_{\min }\left(\kappa_{1}\right) \bar{\otimes} \mathfrak{M}_{\min }\left(\kappa_{2}\right)\right) \simeq \rho\left(\kappa_{1} \times \kappa_{2}\right) \tag{6}
\end{equation*}
$$

By the minimality property of $\mathfrak{M}_{\min }\left(\rho\left(\kappa_{1} \times \kappa_{2}\right)\right)$ we have that

$$
\mathfrak{M}_{\min }\left(\rho\left(\kappa_{1} \times \kappa_{2}\right)\right) \subseteq \mathfrak{M}_{\min }\left(\kappa_{1}\right) \bar{\otimes} \mathfrak{M}_{\min }\left(\kappa_{2}\right)
$$

To see the reverse inclusion, it is enough prove that if $m \in \mathbb{A}\left(Y_{1}, X_{1}\right)$ and $n \in$ $\mathbb{A}\left(Y_{2}, X_{2}\right)$ then $T_{m} \otimes T_{n}=T_{\sigma}$ for some measure $\sigma \in \mathbb{A}\left(Y_{1} \times Y_{2}, X_{1} \times X_{2}\right)$. Indeed, by (6), $\operatorname{supp} T_{\sigma} \subseteq \rho\left(\kappa_{1} \times \kappa_{2}\right)$ and hence Theorem 3.2 implies that $\operatorname{supp} \sigma \subseteq \rho\left(\widehat{\kappa_{1} \times \kappa_{2}}\right)$. By Theorem 3.6, $T_{\sigma} \in \mathfrak{M}_{\text {min }}\left(\rho\left(\kappa_{1} \times \kappa_{2}\right)\right)$.

Let

$$
\sigma(E)=\int_{Y_{2} \times X_{2}} \int_{Y_{1} \times X_{1}} \chi_{E}(y, x) d m\left(y_{1}, x_{1}\right) d n\left(y_{2}, x_{2}\right)
$$

for every measurable $E \subseteq\left(Y_{1} \times Y_{2}\right) \times\left(X_{1} \times X_{2}\right)$. If $\beta_{i} \subseteq Y_{i}, i=1,2$, are measurable then

$$
\begin{aligned}
& |\sigma|\left(\left(\beta_{1} \times \beta_{2}\right) \times\left(X_{1} \times X_{2}\right)\right) \\
& \quad \leq \int_{Y_{2} \times X_{2}} \int_{Y_{1} \times X_{1}} \chi_{\left(\beta_{1} \times \beta_{2}\right) \times\left(X_{1} \times X_{2}\right)}(y, x) d|m|\left(y_{1}, x_{1}\right) d|n|\left(y_{2}, x_{2}\right) \\
& \quad=|m|\left(\beta_{1} \times X_{1}\right)|n|\left(\beta_{2} \times X_{2}\right) \\
& \quad \leq\|m\|_{\mathbb{A}}\|n\|_{\mathbb{A}} \nu_{1}\left(\beta_{1}\right) \nu_{2}\left(\beta_{2}\right) \\
& \quad=\|m\|_{\mathbb{A}}\|n\|_{\mathbb{A}}\left(\nu_{1} \times \nu_{2}\right)\left(\beta_{1} \times \beta_{2}\right) .
\end{aligned}
$$

It now easily follows that $|\sigma|\left(F \times\left(X_{1} \times X_{2}\right)\right) \leq\|m\|_{\mathbb{A}}\|n\|_{\mathbb{A}}\left(\nu_{1} \times \nu_{2}\right)(F)$, for any element $F$ in the product $\sigma$-algebra on $Y_{1} \times Y_{2}$. Similar arguments show that $|\sigma|\left(\left(Y_{1} \times Y_{2}\right) \times E\right) \leq$ $\|m\|_{\mathbb{A}}\|n\|_{\mathbb{A}}\left(\mu_{1} \times \mu_{2}\right)(E)$, for every measurable $E \subseteq X_{1} \times X_{2}$. Hence $\sigma$ is an Arveson measure and $T_{\sigma}$ is a bounded operator from $L^{2}\left(X_{1} \times X_{2}\right)$ to $L^{2}\left(Y_{1} \times Y_{2}\right)$.

If $f_{i} \in L^{2}\left(X_{i}, \mu_{i}\right), g_{i} \in L^{2}\left(Y_{i}, \nu_{i}\right), i=1,2$, we have

$$
\begin{aligned}
& \left(\left(T_{m} \otimes T_{n}\right) f_{1} \otimes f_{2}, g_{1} \otimes g_{2}\right) \\
& \quad=\int_{Y_{2} \times X_{2}} \int_{Y_{1} \times X_{1}} f_{1}\left(x_{1}\right) f_{2}\left(x_{2}\right) \overline{g_{1}\left(y_{1}\right) g_{2}\left(y_{2}\right)} d m\left(y_{1}, x_{1}\right) d n\left(y_{2}, x_{2}\right) \\
& \quad=\int_{\left(Y_{1} \times Y_{2}\right) \times\left(X_{1} \times X_{2}\right)}\left(f_{1} \otimes f_{2}\right)(x) \overline{\left(g_{1} \otimes g_{2}\right)(y)} d \sigma(y, x),
\end{aligned}
$$

and hence $T_{m} \otimes T_{n}=T_{\sigma}$, proving the statement.

## 4. Sets of multiplicity and their operator versions

In this section, we study sets of multiplicity and their operator versions, and examine the relations between them.

### 4.1. Sets of multiplicity in arbitrary locally compact groups

Let us recall the classical notion of a set of multiplicity, where $G=\mathbb{T}$ is the group of the circle; in this case, $A(\mathbb{T})=\left\{\sum_{n \in \mathbb{Z}} c_{n} e^{i n t}: \sum_{n \in \mathbb{Z}}\left|c_{n}\right|<\infty\right\} \simeq \ell^{1}(\mathbb{Z})$. The space of pseudo-measures $P M(\mathbb{T})=A(\mathbb{T})^{*}$ can be identified with $\ell^{\infty}(\mathbb{Z})$ via Fourier transform $F \mapsto(\hat{F}(n))_{n \in \mathbb{Z}}$, and the space of pseudo-functions $P F(\mathbb{T})=\{F \in P M(\mathbb{T}): \hat{F}(n) \rightarrow 0$, as $n \rightarrow \infty\}$ is $*$-isomorphic to $C^{*}(\mathbb{T})=C_{r}^{*}(\mathbb{T})$. Note that there is a canonical embedding $M(\mathbb{T}) \subseteq P M(\mathbb{T})$ arising from the inclusion $A(\mathbb{T}) \subseteq C(\mathbb{T})$.

If $E$ is a closed subset of $\mathbb{T}$, let $P M(E)$ denote the space of all pseudo-measures supported on $E, M(E)$ the space of measures $\mu \in M(G)$ with $\operatorname{supp} \mu \subseteq E$, and $N(E)$ the weak* closure of $M(E)$. For an ideal $J \subseteq A(G)$, let $J^{\perp}$ denote the annihilator of $J$ in $P M(\mathbb{T})$; then $P M(E)=J(E)^{\perp}$ and $N(E)=I(E)^{\perp}$ (see, e.g., [15]).

A closed set $E \subseteq \mathbb{T}$ is called an $M$-set if $P M(E) \cap P F(\mathbb{T}) \neq\{0\}$, an $M_{1}$-set if $N(E) \cap P F(\mathbb{T}) \neq\{0\}$, and an $M_{0}$-set if $M(E) \cap P F(\mathbb{T}) \neq\{0\}$. The closed sets that are not $M$-sets are called sets of uniqueness.

A definition of sets of multiplicity for locally compact abelian groups was proposed by I. Piatetski-Shapiro (see [16, p. 190]). In [4], M. Bożejko introduced sets of uniqueness in general locally compact groups. Here we extend his definition to include versions of $M_{1}$-sets and of $M_{0}$-sets.

Definition 4.1. A closed subset $E \subseteq G$ will be called
(i) an $M$-set if $J(E)^{\perp} \cap C_{r}^{*}(G) \neq\{0\}$;
(ii) an $M_{1}$-set if $I(E)^{\perp} \cap C_{r}^{*}(G) \neq\{0\}$;
(iii) an $M_{0}$-set if $\lambda(M(E)) \cap C_{r}^{*}(G) \neq\{0\}$.

The set $E$ will be called a $U$-set (resp. a $U_{1}$-set, a $U_{0}$-set) if it is not an $M$-set (resp. an $M_{1}$-set, an $M_{0}$-set).

Remark 4.2. (i) Since $\lambda(M(E)) \subseteq I(E)^{\perp} \subseteq J(E)^{\perp}$, every $M_{0}$-set is an $M_{1}$-set, and every $M_{1}$-set is an $M$-set. It is known that these three classes of sets are distinct, see [15].
(ii) If $G$ is amenable then $C_{r}^{*}(G)$ is $*$-isomorphic to $C^{*}(G)$ and it is a direct consequence of the definition that a closed set $E \subseteq G$ is an $M$-set (resp. an $M_{1}$-set) if and only if $J(E)$ (resp. $I(E)$ ) is not weak* dense in $B(G)$.
(iii) Measures $\mu \in M(G)$ satisfying the condition $\lambda(\mu) \in C_{r}^{*}(G)$ were studied in [3] where the author characterised them in terms of their values on certain Borel subsets of $G$. If $G$ is compact or abelian then this class of measures coincides with the Rajchman measures on $G$, that is, the measures whose Fourier-Stieltjes coefficients vanish at infinity (see [3]).

We point out an easy source of examples of sets of multiplicity:

Remark 4.3. Every closed subset of positive Haar measure in a locally compact second countable group is an $M_{0}$-set.

Proof. Let $E \subseteq G$ be a measurable subset of positive Haar measure and $E_{0} \subseteq E$ be a compact set of positive Haar measure; then $m\left(E_{0}\right)<\infty$. Let $\theta$ be the measure given by $d \theta(x)=\chi_{E_{0}}(x) d m(x)$. Clearly, $\operatorname{supp} \theta \subseteq E$ and $0 \neq \lambda(\theta)=\lambda\left(\chi_{E_{0}}\right) \in C_{r}^{*}(G)$.

### 4.2. Sets of operator multiplicity

For an $\omega$-closed set $F \subseteq Y \times X$, we denote by $\mathbb{A}(F)$ the set of all measures $\sigma$ in $\mathbb{A}(Y, X)$ such that $\operatorname{supp} \sigma \subseteq F$.

Operator versions of $M$-sets and $M_{1}$-sets were introduced by the authors in [34] in connection with the study of closable multipliers. We recall the relevant definition now, introducing the additional notion of an $M_{0}$-set.

Definition 4.4. Let $(X, \mu)$ and $(Y, \nu)$ be standard measure spaces. An $\omega$-closed set $\kappa \subseteq$ $X \times Y$ is called
(i) an operator $M$-set if $\mathcal{K}\left(H_{1}, H_{2}\right) \cap \mathfrak{M}_{\max }(\kappa) \neq\{0\}$;
(ii) an operator $M_{1}$-set if $\mathcal{K}\left(H_{1}, H_{2}\right) \cap \mathfrak{M}_{\text {min }}(\kappa) \neq\{0\}$;
(iii) an operator $M_{0}$-set if there exists a non-zero measure $\sigma \in \mathbb{A}(\hat{\kappa})$ such that $T_{\sigma} \in$ $\mathcal{K}\left(H_{1}, H_{2}\right)$.

We call $\kappa$ an operator $U$-set (resp. an operator $U_{1}$-set, an operator $U_{0}$-set) if it is not an operator $M$-set (resp. an operator $M_{1}$-set, an operator $M_{0}$-set).
(Operator) $M$-sets will be referred to as sets of (operator) multiplicity, while (operator) $U$-sets - as sets of (operator) uniqueness. It will follow from Theorem 3.6 that if $\sigma \in \mathbb{A}(\hat{\kappa})$ then $T_{\sigma} \in \mathfrak{M}_{\min }(\kappa)$. Therefore, every operator $M_{0}$-set is an operator $M_{1}$-set, while every operator $M_{1}$-set is trivially an operator $M$-set.

Remark. Recall that $\mu \in M(G)$ is called a Rajchman measure if $\lambda(\mu) \in C_{r}^{*}(G)$. The compact operators of the form $T_{\sigma}$, where $\sigma \in \mathbb{A}(Y, X)$, can be thought of as an operator version of these measures.

### 4.3. A symbolic calculus

Aiming at applications to multiplicity sets we establish here a kind of symbolic calculus for completely bounded maps from $\mathcal{B}\left(L^{2}(G)\right)$ to $\operatorname{VN}(G)$ (Theorem 4.6). We first recall the Stone-von Neumann Theorem in a suitable for our needs form. Let $\mathcal{D}=\left\{M_{a}: a \in\right.$ $\left.L^{\infty}(G)\right\}$ and $\mathcal{D}_{0}=\left\{M_{a}: a \in C_{0}(G)\right\}$. For each $s \in G$, let $\alpha_{s}: C_{0}(G) \rightarrow C_{0}(G)$ be given by $\alpha_{s} f(t)=f\left(s^{-1} t\right)$. The map $s \mapsto \alpha_{s}$ is a homomorphism from $G$ into the automorphism group of $C_{0}(G)$, and thus gives rise to the ( $C^{*}$-algebraic) crossed product $C_{0}(G) \rtimes_{\alpha} G$. Denoting for a moment by $\pi: C_{0}(G) \rightarrow \mathcal{B}\left(L^{2}(G)\right)$ the representation given by $\pi(g)=M_{g}$, we have that the pair $(\pi, \lambda)$ (where $\lambda$ is the left regular representation of $G$ on $L^{2}(G)$ ) is a covariant representation of the dynamical system $\left(C_{0}(G), G, \alpha\right)$. Thus, $(\pi, \lambda)$ gives rise to a representation $\pi \times \lambda$ of $C_{0}(G) \rtimes_{\alpha} G$ on $L^{2}(G)$. By the Stone-von Neumann Theorem (see [41, Theorem 4.23]), this representation is faithful and its image coincides with the algebra $\mathcal{K}$ of all compact operators on $L^{2}(G)$. In particular, we claim that

$$
\begin{equation*}
\left.\left.\mathcal{K}=\overline{\left[A T: A \in \mathcal{D}_{0}, T \in C_{r}^{*}(G)\right.}\right] \cdot\|\cdot\| \quad \overline{\left[A T B: A, B \in \mathcal{D}_{0}, T \in C_{r}^{*}(G)\right.}\right]\|\cdot\| \tag{7}
\end{equation*}
$$

(here, and in the sequel, $[\mathcal{E}]$ denotes the linear span of $\mathcal{E}$ ). To see that (7) holds, note that if $f \in L^{1}(G), T=\lambda(f)$ and $A, B \in \mathcal{D}_{0}$, then

$$
A T=\int_{G} f(t) A \lambda_{t} d t \in(\pi \times \lambda)\left(C_{0}(G) \rtimes_{\alpha} G\right)=\mathcal{K},
$$

and thus $A T B \in \mathcal{K}$ as well. Conversely, it is easy to observe (see, e.g., [28]) that the operators of the form $\sum_{i=1}^{k} \int_{E_{i}} A_{i} \lambda_{s} d s$, where $E_{i} \subseteq G$ are measurable sets of finite measure and $A_{i} \in \mathcal{D}_{0}, i=1, \ldots, k$, form a dense subset of $(\pi \times \lambda)\left(C_{0}(G) \rtimes_{\alpha} G\right)$; however, $\int_{E_{i}} A_{i} \lambda_{s} d s=A_{i} \lambda\left(\chi_{E_{i}}\right)$, and the first equality in (7) is established. To complete the proof of the second equality, let $\left(B_{i}\right)_{i=1}^{\infty} \subseteq \mathcal{D}_{0}$ be a sequence strongly converging to the identity operator on $L^{2}(G)$, and note that if $A \in \mathcal{D}_{0}$ and $T \in C_{r}^{*}(G)$, then $A T=\lim _{i} A T B_{i}$ in norm, by the compactness of $A T$.

In the sequel, we will use the norm closed $\mathcal{D}$-bimodule generated by $C_{r}^{*}(G)$

$$
\begin{equation*}
\left.\mathcal{A}=\overline{\left[A T B: A, B \in \mathcal{D}, T \in C_{r}^{*}(G)\right]}\right] \cdot \| \tag{8}
\end{equation*}
$$

and the smallest norm closed subspace of $\mathcal{B}\left(L^{2}(G)\right)$ containing $C_{r}^{*}(G)$ and invariant under Schur multipliers

$$
\begin{equation*}
\mathcal{R}={\left.\overline{\left[S_{\varphi}(T)\right.}: T \in C_{r}^{*}(G), \varphi \in \mathfrak{S}(G, G)\right]}^{\|\cdot\|} \tag{9}
\end{equation*}
$$

By (7),

$$
\begin{equation*}
\mathcal{K} \subseteq \mathcal{A} \subseteq \mathcal{R} \tag{10}
\end{equation*}
$$

Remark 4.5. (i) Let $G$ be discrete. Then $\mathcal{A}=\mathcal{R}$. Indeed, in this case $C_{r}^{*}(G)$ is generated as a closed linear space by the unitaries $\lambda_{s}, s \in G$, which are normalisers of the multiplication masa $\mathcal{D}$. However, if $\varphi \in \mathfrak{S}(G, G)$ then $S_{\varphi}\left(\lambda_{s}\right)=M_{f} \lambda_{s} \in \mathcal{A}$ for some $f \in L^{\infty}(G)$ (see, e.g., [21, Proposition 14]). It follows that $\mathcal{A}$ is invariant under Schur multiplication, and hence $\mathcal{R}=\mathcal{A}$. Note that, in the case $G$ is infinite, $\mathcal{K}$ is strictly contained in $\mathcal{A}$ since $\lambda_{s}$ is a unitary operator in $C_{r}^{*}(G)$ which is not compact.

In [32], given a discrete group $G$, J . Roe introduced what is now known as the uniform Roe algebra $U C_{r}^{*}(G)$ which equals, by definition, to the uniform closure in $\mathcal{B}\left(\ell^{2}(G)\right)$ of the space of all matrices indexed by $G \times G$ with uniformly bounded entries supported on sets of the form $\left\{(s, t) \in G \times G: t s^{-1} \in E\right\}$, where $E$ is finite. We note that $U C_{r}^{*}(G)$ coincides in this case with $\mathcal{R}$. Indeed, the unitary generators $\lambda_{s}$ are represented by matrices (indexed by $G \times G$ ) whose $s$ th diagonal has all entries equal to 1 , and all other diagonals are zero. Multiplying by an operator of the form $M_{a}$, where $a \in \ell^{\infty}(G)$, we see that all matrices which, on a given diagonal, have a sequence from $\ell^{\infty}(G)$, are in $\mathcal{A}=\mathcal{R}$; thus, $U C_{r}^{*}(G) \subseteq \mathcal{R}$. Conversely, since $C_{r}^{*}(G)$ is generated as a norm closed subspace by the operators of the form $\lambda_{s}$, we have that $\mathcal{A} \subseteq U C_{r}^{*}(G)$, and hence $U C_{r}^{*}(G)=\mathcal{R}$.

The previous paragraph shows that the space $\mathcal{R}$ can be thought of as a locally compact version of the uniform Roe algebra.
(ii) If $G$ is compact then $\mathcal{K}=\mathcal{A}=\mathcal{R}$. Indeed, in this case $C_{r}^{*}(G) \subseteq \mathcal{K}$ and since the compact operators are invariant under Schur multipliers, we have that $\mathcal{R} \subseteq \mathcal{K}$, and the equalities follow from (10).

In view of Remark 4.5, it is natural to ask whether $\mathcal{A}=\mathcal{R}$ for every locally compact group $G$; we do not know whether this equality always holds.

If $G$ is compact then $N(A(G)) \subseteq \Gamma(G, G)$ and hence the formula

$$
\langle E(T), u\rangle=\langle T, N(u)\rangle, \quad T \in \mathcal{B}\left(L^{2}(G)\right), u \in A(G),
$$

defines a canonical expectation $E$ from $\mathcal{B}\left(L^{2}(G)\right)$ onto $\operatorname{VN}(G)$. This is the motivation behind the next theorem, where we exhibit a symbolic calculus for completely bounded maps from $\mathcal{B}\left(L^{2}(G)\right)$ into $\operatorname{VN}(G)$ (that are not necessarily projections). Let us denote by $C B^{w^{*}}\left(\mathcal{B}\left(L^{2}(G)\right), \mathrm{VN}(G)\right)$ the space of weak* continuous completely bounded maps from $\mathcal{B}\left(L^{2}(G)\right)$ into $\operatorname{VN}(G)$. It has a natural structure of a right Banach module over
$\mathfrak{S}(G, G)$, the action being given by $\Phi \cdot \varphi=\Phi \circ S_{\varphi}$. Note that $\Gamma(G, G)$ is also a right Banach module over $\mathfrak{S}(G, G)$ under the action $\psi \cdot \varphi=\psi \varphi$.

Theorem 4.6. For every $\varphi \in \Gamma(G, G)$ and every $T \in \mathcal{B}\left(L^{2}(G)\right)$, there exists a unique operator $E_{\varphi}(T) \in \mathrm{VN}(G)$ such that

$$
\left\langle E_{\varphi}(T), u\right\rangle=\langle T, \varphi N(u)\rangle, \quad u \in A(G) .
$$

The transformation $\varphi \rightarrow E_{\varphi}$ is a contractive $\mathfrak{S}(G, G)$-module map from $\Gamma(G, G)$ into $C B^{w^{*}}\left(\mathcal{B}\left(L^{2}(G)\right), \mathrm{VN}(G)\right)$. Moreover, if $\varphi \in \Gamma(G, G)$ then $E_{\varphi}\left(\lambda_{s}\right)=P(\varphi)(s) \lambda_{s}, s \in G$, and $E_{\varphi}(T) \in C_{r}^{*}(G)$, for all $T \in \mathcal{R}$.

Proof. Fix $\varphi \in \Gamma(G, G)$ and consider the mapping $e_{\varphi}: A(G) \rightarrow \Gamma(G, G)$ given by $e_{\varphi}(u)=\varphi N(u), u \in A(G)$. The mapping $N: A(G) \rightarrow \mathfrak{S}(G, G)$ is completely isometric (see, e.g., [38]). On the other hand, the mapping $\psi \rightarrow \varphi \psi$ from $\mathfrak{S}(G, G)$ into $\Gamma(G, G)$ is completely bounded with completely bounded norm not exceeding $\|\varphi\|_{\Gamma}$. Indeed, let $\psi_{i, j} \in \mathfrak{S}(G, G), i, j=1, \ldots, n$; then, denoting by $F_{\varphi}$ the functional on $\mathcal{B}\left(L^{2}(G)\right)$ given by $F_{\varphi}(T)=\langle\varphi, T\rangle$, we have

$$
\begin{aligned}
\left\|\left(\varphi \psi_{i, j}\right)_{i, j}\right\|_{M_{n}(\Gamma(G, G))} & =\left\|\left(\varphi \psi_{i, j}\right)_{i, j}\right\|_{C B\left(\mathcal{B}\left(L^{2}(G)\right), M_{n}(\mathbb{C})\right)} \\
& =\sup _{\left\|\left(T_{p, q}\right)_{p, q}\right\| \leq 1}\left\|\left(\left\langle\varphi \psi_{i, j}, T_{p, q}\right\rangle\right)_{(i, p),(j, q)}\right\| \\
& =\sup _{\left\|\left(T_{p, q}\right)_{p, q}\right\| \leq 1}\left\|\left(\left\langle\varphi, S_{\psi_{i, j}}\left(T_{p, q}\right)\right\rangle\right)_{(i, p),(j, q)}\right\| \\
& \leq \sup _{\left\|\left(T_{p, q}\right)_{p, q}\right\| \leq 1}\left\|F_{\varphi}\right\|\left\|\left(S_{\psi_{i, j}}\left(T_{p, q}\right)\right)_{(i, p),(j, q)}\right\| \\
& \leq\|\varphi\|_{\Gamma}\left\|\left(S_{\psi_{i, j}}\right)_{i, j}\right\|_{\mathrm{cb}} \\
& =\|\varphi\|_{\Gamma}\left\|\left(\psi_{i, j}\right)_{i, j}\right\|_{M_{n}(\mathfrak{S}(G, G))} .
\end{aligned}
$$

Thus, $e_{\varphi}$ is completely bounded and $\left\|e_{\varphi}\right\|_{\mathrm{cb}} \leq\|\varphi\|_{\Gamma}$. It follows that the map $E_{\varphi}=e_{\varphi}^{*}$ is a normal completely bounded map from $\mathcal{B}\left(L^{2}(G)\right)$ into $\operatorname{VN}(G)$ and $\left\|E_{\varphi}\right\|_{\mathrm{cb}} \leq\|\varphi\|_{\Gamma}$. The identity

$$
\left\langle E_{\varphi}(T), u\right\rangle=\langle T, \varphi N(u)\rangle, \quad u \in A(G), T \in \mathcal{B}\left(L^{2}(G)\right),
$$

holds by the definition of $E_{\varphi}$.
It is obvious that the map $E: \varphi \rightarrow E_{\varphi}$ is linear and, by the previous paragraph, it is contractive. Moreover, if $\varphi \in \Gamma(G, G), \psi \in \mathfrak{S}(G, G)$ and $u \in A(G)$, then

$$
\left\langle E_{\varphi \psi}(T), u\right\rangle=\langle T, \psi \varphi N(u)\rangle=\left\langle S_{\psi}(T), \varphi N(u)\right\rangle=\left\langle\left(E_{\varphi} \circ S_{\psi}\right)(T), u\right\rangle
$$

which shows that $E$ is an $\mathfrak{S}(G, G)$-module map.

Using (2), for every $u \in A(G)$ we have

$$
\begin{aligned}
\left\langle E_{\varphi}\left(\lambda_{s}\right), u\right\rangle & =\left\langle\lambda_{s}, \varphi N(u)\right\rangle=P(N(u) \varphi)(s) \\
& =u(s) P(\varphi)(s)=\left\langle P(\varphi)(s) \lambda_{s}, u\right\rangle,
\end{aligned}
$$

which shows that $E_{\varphi}\left(\lambda_{s}\right)=P(\varphi)(s) \lambda_{s}$.
Now suppose that $f \in L^{1}(G)$ and let $a, b \in L^{2}(G)$. Then

$$
\begin{equation*}
S_{N(u)}(\lambda(f))=\lambda(u f), \quad u \in A(G) \tag{11}
\end{equation*}
$$

Indeed, write $N(u)=\sum_{i=1}^{\infty} f_{i} \otimes g_{i}$, where $\left\|\sum_{i=1}^{\infty}\left|f_{i}\right|^{2}\right\|_{\infty} \leq C<\infty$ and $\left\|\sum_{i=1}^{\infty}\left|g_{i}\right|^{2}\right\|_{\infty} \leq$ $C<\infty$. Then

$$
\left(S_{N(u)}(\lambda(f)) a, b\right)=\sum_{i=1}^{\infty} \iint g_{i}(t) f(s) f_{i}\left(s^{-1} t\right) a\left(s^{-1} t\right) \overline{b(t)} d s d t
$$

Applying Fubini's arguments we obtain

$$
\begin{aligned}
\left(S_{N(u)}(\lambda(f)) a, b\right) & =\iint \sum_{i=1}^{\infty} g_{i}(t) f(s) f_{i}\left(s^{-1} t\right) a\left(s^{-1} t\right) \overline{b(t)} d s d t \\
& =\iint f(s) N(u)\left(s^{-1} t, t\right) a\left(s^{-1} t\right) \overline{b(t)} d t d s \\
& =\iint u(s) f(s) a\left(s^{-1} t\right) \overline{b(t)} d t d s=(\lambda(u f) a, b)
\end{aligned}
$$

Thus, (11) is established.
The mapping $u \mapsto N(u)$ from $A(G)$ into $\mathfrak{S}(G, G)$ is an isometry (see, e.g., [38]); hence $\left\|S_{N(u)}(\lambda(f))\right\| \leq\|N(u)\|_{\mathfrak{S}}\|\lambda(f)\|=\|u\|_{A(G)}\|\lambda(f)\|$ and therefore the mapping $u \mapsto \lambda(u f), A(G) \rightarrow C_{r}^{*}(G)$, is continuous. We also have

$$
\begin{aligned}
\left\langle E_{a \otimes b}(\lambda(f)), u\right\rangle & =\langle\lambda(f),(a \otimes b) N(u)\rangle=\left(S_{N(u)}(\lambda(f)) a, \bar{b}\right) \\
& =(\lambda(u f) a, \bar{b})=\iint u(s) f(s) a\left(s^{-1} t\right) b(t) d s d t \\
& =\int u(s) f(s)\left(\int a\left(s^{-1} t\right) b(t) d t\right) d s=\int u(s) f(s)(b * \check{a})(s) d s
\end{aligned}
$$

Using (1), we conclude that

$$
\left\langle E_{a \otimes b}(\lambda(f)), u\right\rangle=\int u(s) f(s) P(a \otimes b)(s) d s
$$

Note that, since $P(a \otimes b) \in A(G)$, the function $P(a \otimes b) f$ belongs to $L^{1}(G)$ and hence

$$
\begin{equation*}
\left\langle E_{\varphi}(\lambda(f)), u\right\rangle=\langle\lambda(P(\varphi) f), u\rangle \tag{12}
\end{equation*}
$$

for $\varphi=a \otimes b$. By linearity, (12) holds whenever $\varphi$ is a finite sum of elementary tensors. By the continuity of the transformations $\varphi \rightarrow E_{\varphi}, \varphi \rightarrow P(\varphi)$ and $g \rightarrow \lambda(g f)$ (the last one mapping $A(G)$ into $C_{r}^{*}(G)$ ), we conclude that (12) holds for all $\varphi \in \Gamma(G, G)$.

Relation (12) implies that $E_{\varphi}(\lambda(f))=\lambda(P(\varphi) f) \in C_{r}^{*}(G)$, for all $f \in L^{1}(G)$ and all $\varphi \in \Gamma(G, G)$. Since $E_{\varphi}$ is norm continuous and $\lambda\left(L^{1}(G)\right)$ is dense in $C_{r}^{*}(G)$, we have that $E_{\varphi}\left(C_{r}^{*}(G)\right) \subseteq C_{r}^{*}(G)$. If $\psi \in \mathfrak{S}(G, G)$ and $T \in C_{r}^{*}(G)$ then

$$
E_{\varphi}\left(S_{\psi}(T)\right)=E_{\varphi \psi}(T) \in C_{r}^{*}(G)
$$

It follows that $E_{\varphi}(\mathcal{R}) \subseteq C_{r}^{*}(G)$, for every $\varphi \in \Gamma(G, G)$.

We will assume, for the rest of the paper, that $G$ is second countable. The following lemma will be needed in the proof of Theorem 4.9.

Lemma 4.7. Suppose that $T \in \mathcal{B}\left(L^{2}(G)\right)$ is non-zero. Then there exist $a, b \in L^{2}(G)$ such that $E_{a \otimes b}(T) \neq 0$.

Proof. Let $T \in \mathcal{B}\left(L^{2}(G)\right)$ be a non-zero operator, and suppose, by way of contradiction, that $E_{a \otimes b}(T)=0$ for all $a, b \in L^{2}(G)$. We may assume that $T=M_{\chi_{K}} T M_{\chi_{K}}$ for some compact set $K \subseteq G$. By Theorem 4.6, $E_{\varphi}(T)=0$ for every $\varphi \in \Gamma(G, G)$. Since

$$
\left\langle E_{\varphi}(T), u\right\rangle=\langle T, \varphi N(u)\rangle=\left\langle S_{N(u)}(T), \varphi\right\rangle, \quad u \in A(G), \varphi \in \Gamma(G, G)
$$

we have that $S_{N(u)}(T)=0$ for every $u \in A(G)$. Let

$$
\mathcal{W}=\operatorname{span}\{N(u) \psi: \psi \in \Gamma(G, G), u \in A(G)\}
$$

Then $\mathcal{W} \subseteq \Gamma(G, G)$ is a subspace, invariant under $\mathfrak{S}(G)$, and $T \in \mathcal{W}^{\perp}$. Denoting by $\operatorname{null}(\mathcal{W})$ the complement of the $\omega$-union [34] of the family $\left\{h^{-1}(\mathbb{C} \backslash\{0\}): h \in \mathcal{W}\right\}$, we have $\operatorname{null}(\mathcal{W}) \simeq \emptyset$. In fact, since $G$ is second countable and locally compact, there exists an increasing sequence of compact sets $\left\{K_{n}\right\}$ such that $\bigcup_{n=1}^{\infty} K_{n}=G$. For each $n \in \mathbb{N}$, choose a function $u_{n} \in A(G)$ that takes the value 1 on $K_{n}$. Then, up to a marginally null set,

$$
\begin{aligned}
\operatorname{null}(\mathcal{W}) & \subseteq \bigcap_{n, m=1}^{\infty} \operatorname{null}\left(N\left(u_{n}\right) \chi_{K_{m}} \times \chi_{K_{m}}\right) \subseteq \bigcap_{n, m=1}^{\infty}\left(K_{n}^{*} \cap\left(K_{m} \times K_{m}\right)\right)^{c} \\
& =\bigcap_{n=1}^{\infty}\left(\left(K_{n}^{c}\right)^{*} \cup\left(\bigcup_{m=1}^{\infty} K_{m} \times K_{m}\right)^{c}\right)=\bigcap_{n=1}^{\infty}\left(K_{n}^{c}\right)^{*} \\
& =\left(\left(\bigcup_{n=1}^{\infty} K_{n}\right)^{c}\right)^{*}=\emptyset
\end{aligned}
$$

By [35, Corollary 4.3], $\mathcal{W}$ is dense in $\Gamma(G, G)$ and hence $T=0$, a contradiction.

If $E \subseteq G$, we let

$$
E^{*}=\left\{(s, t) \in G \times G: t s^{-1} \in E\right\} .
$$

If $E$ is closed then $E^{*}$ is closed and hence, if $G$ is second countable, it is $\omega$-closed.

### 4.4. Multiplicity versus operator multiplicity

In the case of compact abelian groups, a connection between $M$-sets (resp. $M_{1}$-sets) and operator $M$-sets (resp. operator $M_{1}$-sets) was established in [14] (resp. [34]). Our aim now is to extend these results to arbitrary locally compact groups; a corresponding statement for $M_{0}$-sets will be proved in the next subsection. We will need the following lemma.

Lemma 4.8. Let $E \subseteq G$ be a closed set and $T \in \mathrm{VN}(G)$. If $\operatorname{supp} T \subseteq E$ then $T$ is supported on $E^{*}$.

Proof. Let $K$ and $L$ be compact sets such that $(K \times L) \cap E^{*}=\emptyset$. Then $\left(L K^{-1}\right) \cap E=\emptyset$. As the mapping $(s, t) \in G \times G \mapsto t s^{-1} \in G$ is continuous, the set $L K^{-1}$ is compact. If now $f$ and $g \in L^{2}(G)$ are such that $f \chi_{K}=0$ and $g \chi_{L}=0$ then the function $u \in A(G)$ given by $u(s)=\left(\lambda_{s}(f), g\right), s \in G$, has support in $L K^{-1}$ and hence $u \in J(E)$. Therefore $(T f, g)=\langle T, u\rangle=0$. This implies that $P_{L} T P_{K}=0$. By the regularity of the Haar measure, $P_{L} T P_{K}=0$ whenever $K$ and $L$ are Borels sets with $(K \times L) \cap E^{*}=\emptyset$. Hence $T$ is supported on $E^{*}$.

Theorem 4.9. Let $G$ be a locally compact second countable group and let $E \subseteq G$ be a closed subset.
(a) The following are equivalent:
(i) $E$ is an $M$-set;
(ii) $E^{*}$ is an operator $M$-set;
(iii) $\mathcal{A} \cap \mathfrak{M}_{\max }\left(E^{*}\right) \neq\{0\}$;
(iv) $\mathcal{R} \cap \mathfrak{M}_{\max }\left(E^{*}\right) \neq\{0\}$.
(b) The following are equivalent:
(i') $E$ is an $M_{1}$-set;
(ii') $E^{*}$ is an operator $M_{1}$-set;
(iii') $\mathcal{A} \cap \mathfrak{M}_{\text {min }}\left(E^{*}\right) \neq\{0\}$;
(iv') $\mathcal{R} \cap \mathfrak{M}_{\min }\left(E^{*}\right) \neq\{0\}$.

Proof. (a) (i) $\Rightarrow$ (ii) Let $E$ be an $M$-set; then there exists a non-zero operator $T \in$ $J(E)^{\perp} \cap C_{r}^{*}(G)$. Suppose that $A T=0$ for all $A \in \mathcal{D}_{0}$. Since $\mathcal{D}_{0}$ is weak* dense in $\mathcal{D}$, there exists a net $\left(A_{j}\right)_{j \in \mathbb{J}} \subseteq \mathcal{D}_{0}$ such that $\lim _{j \in \mathbb{J}} A_{j}=I$ in the weak* topology. After passing to a limit, we obtain that $T=0$, a contradiction. Thus, there exists $A \in \mathcal{D}_{0}$
such that $A T \neq 0$; in view of $(7), A T \in \mathcal{K}$. By Lemma 4.8, $T \in \mathfrak{M}_{\max }\left(E^{*}\right)$ and hence $A T \in \mathfrak{M}_{\max }\left(E^{*}\right)$; thus, $E^{*}$ is an operator $M$-set.
(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow from the inclusions (10).
(iv) $\Rightarrow$ (i) Suppose that $T \in \mathcal{R} \cap \mathfrak{M}_{\max }\left(E^{*}\right)$ is non-zero. By Lemma 4.7, there exist $a, b \in L^{2}(G)$ such that $E_{a \otimes b}(T) \neq 0$. By Theorem 4.6, $E_{a \otimes b}(T) \in C_{r}^{*}(G)$; we claim that, moreover, $E_{a \otimes b}(T) \in J(E)^{\perp}$. To see this, let $u \in A(G)$ vanish on an open neighbourhood of $E$ and have compact support. Then $N(u) \in \mathfrak{S}(G, G)$ vanishes on an $\omega$-open neighbourhood of $E^{*}$, and hence the function $(a \otimes b) N(u) \in \Gamma(G, G)$ vanishes on an $\omega$-open neighbourhood of $E^{*}$. On the other hand, by [35, Theorem 4.3], we have

$$
\left(S_{N(u)}(T) a, \bar{b}\right)=\langle T,(a \otimes b) N(u)\rangle=0
$$

giving $\left\langle E_{a \otimes b}(T), u\right\rangle=0$. Thus, $0 \neq E_{a \otimes b}(T) \in J(E)^{\perp}$ and hence $E$ is an $M$-set.
(b) $\left(\mathrm{i}^{\prime}\right) \Rightarrow(\mathrm{ii})$ We claim that $\lambda_{s} \in \mathfrak{M}_{\min }\left(E^{*}\right)$ for every $s \in E$. To see this, suppose that $w \in \Gamma(G, G)$ vanishes on the set $E^{*}$, that is, $w \chi_{E^{*}}=0$ marginally almost everywhere. For every $r \in G$ and $s \in E$, we have that $\left(s^{-1} r, r\right) \in E^{*}$ and hence $w\left(s^{-1} r, r\right)=0$ for every $s \in E$ and almost every $r \in G$. By (2), $P(w)(s)=0$ for every $s \in E$ and hence, by (1), $\left\langle\lambda_{s}, w\right\rangle=0$ for every $s \in E$; the claim is thus proved.

Suppose that $E$ is an $M_{1}$-set, and let $0 \neq T \in I(E)^{\perp} \cap C_{r}^{*}(G)$. A direct verification shows that $I(E)^{\perp}=\overline{\left[\lambda_{s}: s \in E\right]}{ }^{w^{*}}$. It follows from the previous paragraph that $T \in \mathfrak{M}_{\min }\left(E^{*}\right)$. As in the proof of the implication (i) $\Rightarrow$ (ii), we conclude that there exists $A \in \mathcal{D}_{0}$ such that $0 \neq A T \in \mathcal{K} \cap \mathfrak{M}_{\min }\left(E^{*}\right)$, that is, $E^{*}$ is an $M_{1}$-set.
(ii') $\Rightarrow\left(\mathrm{iii}^{\prime}\right) \Rightarrow\left(\mathrm{iv}^{\prime}\right)$ follow from the inclusions (10).
(iv') $\Rightarrow$ (i') Suppose that $0 \neq T \in \mathcal{R} \cap \mathfrak{M}_{\min }\left(E^{*}\right)$. As in the proof of (a), we can show that there exist $a, b \in L^{2}(G)$ such that $E_{a \otimes b}(T)$ is a non-zero element of $C_{r}^{*}(G)$ annihilating $I(E)$.

### 4.5. The case of $M_{0}$-sets

In order to establish a statement for $M_{0}$-sets, analogous to the ones from Theorem 4.9, we need a couple of auxiliary lemmas.

Lemma 4.10. If $\sigma$ is an Arveson measure on $G \times G$ then for every $\varphi \in \Gamma(G, G)$ there exists a unique measure $\sigma_{\varphi} \in M(G)$ such that $E_{\varphi}\left(T_{\sigma}\right)=\lambda\left(\sigma_{\varphi}\right)$. Moreover, if $\operatorname{supp} \sigma \subseteq \widehat{E^{*}}$ then $\operatorname{supp} \sigma_{\varphi} \subseteq E$.

Proof. Let $\varphi=\sum_{i=1}^{\infty} f_{i} \otimes g_{i} \in \Gamma(G, G)$ (here $\sum_{i=1}^{\infty}\left\|f_{i}\right\|_{2}^{2}<\infty$ and $\left.\sum_{i=1}^{\infty}\left\|g_{i}\right\|_{2}^{2}<\infty\right)$; note that

$$
\left\|\sum_{i=1}^{\infty}\left|f_{i}\right| \otimes\left|g_{i}\right|\right\|_{\Gamma} \leq\|\varphi\|_{\Gamma} .
$$

If $u \in C_{0}(G)$ then

$$
\begin{aligned}
\left|\int_{G \times G} \varphi(s, t) u\left(t s^{-1}\right) d \sigma(t, s)\right| & \leq \int_{G \times G}|\varphi(s, t)| \| u\left(t s^{-1}\right)|d| \sigma \mid(t, s) \\
& \leq\|u\|_{\infty} \int_{G \times G} \sum_{i=1}^{\infty}\left|f_{i}(s)\right|\left|g_{i}(t)\right| d|\sigma|(t, s) \\
& =\|u\|_{\infty} \sum_{i=1}^{\infty}\left(T_{|\sigma|}\left|f_{i}\right|,\left|g_{i}\right|\right) \\
& =\|u\|_{\infty}\left\langle T_{|\sigma|}, \sum_{i=1}^{\infty}\right| f_{i}|\otimes| g_{i}| \rangle \\
& \leq\|u\|_{\infty}\left\|T_{|\sigma|}\right\|\left\|_{i=1}^{\infty}\left|f_{i}\right| \otimes\left|g_{i}\right|\right\|_{\Gamma} \\
& \leq\|u\|_{\infty}\|\sigma\|_{\mathbb{A}}\|\varphi\|_{\Gamma} .
\end{aligned}
$$

It follows that the functional $R: C_{0}(G) \rightarrow \mathbb{C}$ given by

$$
R(u)=\int_{G \times G} \varphi(s, t) u\left(t s^{-1}\right) d \sigma(t, s), \quad u \in C_{0}(G),
$$

is well-defined and bounded. Hence, there exists $\sigma_{\varphi} \in M(G)$ such that

$$
\begin{equation*}
\int_{G \times G} \varphi(s, t) u\left(t s^{-1}\right) d \sigma(t, s)=\int_{G} u(x) d \sigma_{\varphi}(x), \quad u \in C_{0}(G) . \tag{13}
\end{equation*}
$$

On the other hand,

$$
\int_{G} u(x) d \sigma_{\varphi}(x)=\left\langle\lambda\left(\sigma_{\varphi}\right), u\right\rangle, \quad u \in A(G) .
$$

By (13) and Theorem 3.2,

$$
\left\langle\lambda\left(\sigma_{\varphi}\right), u\right\rangle=\left\langle T_{\sigma}, \varphi N(u)\right\rangle=\left\langle E_{\varphi}\left(T_{\sigma}\right), u\right\rangle, \quad u \in A(G) ;
$$

thus, $E_{\varphi}\left(T_{\sigma}\right)=\lambda\left(\sigma_{\varphi}\right)$.
Now suppose that $\operatorname{supp} \sigma \subseteq \widehat{E^{*}}$ and that $U \subseteq G$ is an open set, disjoint from $E$. For any function $u \in C_{0}(G)$ with $\operatorname{supp} u \subseteq U$, we have that $\operatorname{supp} N(u) \subseteq \widehat{U^{*}}$. On the other hand, $\widehat{U^{*}}$ is disjoint from $\widehat{E^{*}}$ and hence Proposition 3.1 implies that $|\sigma|\left(\widehat{U^{*}}\right)=0$. Now (13) shows that $\int_{G} u(x) d \sigma_{\varphi}(x)=0$. It follows that $\sigma_{\varphi}(U)=0$; thus, $\operatorname{supp} \sigma_{\varphi} \subseteq E$.

We will need the following fact, which was discussed in [37, p. 347] in the case of a finite measure (here we need a $\sigma$-finite version of this as the Haar measure on a locally compact non-compact group is such).

Lemma 4.11. Let $(X, \mu)$ and $(Y, \nu)$ be $\sigma$-finite standard measure spaces and $\left(\sigma^{x}\right)_{x \in X}$ be a family of complex Borel measures on $Y$ such that, for every measurable $F \subseteq Y$, the function $x \mapsto \sigma^{x}(F)$ is measurable. Suppose that the function $x \mapsto\left\|\sigma^{x}\right\|$ is integrable and essentially bounded (with respect to the measure $\mu$ ). Then there exists a Borel measure $\sigma$ on $Y \times X$ such that $\sigma(E)=\int_{X} \int_{Y} \chi_{E}(y, x) d \sigma^{x}(y) d \mu(x)$, for every measurable set $E \subseteq$ $Y \times X$, and a constant $c>0$ such that $|\sigma|(Y \times \alpha) \leq c \mu(\alpha)$ for every measurable set $\alpha \subseteq X$.

Proof. First of all, notice that the quantity

$$
\sigma(E)=\int_{X} \int_{Y} \chi_{E}(y, x) d \sigma^{x}(y) d \mu(x)
$$

is finite. Indeed,

$$
\begin{aligned}
\left|\int_{X} \int_{Y} \chi_{E}(y, x) d \sigma^{x}(y) d \mu(x)\right| & \leq \int_{X} \int_{Y} \chi_{E}(y, x) d\left|\sigma^{x}\right|(y) d \mu(x) \\
& \leq \int_{X}\left|\sigma^{x}\right|(Y) d \mu(x)<\infty .
\end{aligned}
$$

A direct verification now shows that $\sigma$ is a measure. Moreover, the above estimate yields

$$
|\sigma|(E) \leq \int_{X} \int_{Y} \chi_{E}(y, x) d\left|\sigma^{x}\right|(y) d \mu(x)
$$

for every measurable set $E \subseteq Y \times X$. Letting $c=\operatorname{ess} \sup _{x \in X}\left\|\sigma^{x}\right\|$, for every measurable $\alpha \subseteq X$, we have

$$
|\sigma|(Y \times \alpha) \leq \int_{\alpha}\left\|\sigma^{x}\right\| d \mu(x) \leq c \mu(\alpha)
$$

In the next theorem, we let

$$
\mathfrak{P}(\kappa)=\left\{T_{\mu}: \mu \in \mathbb{A}(\hat{\kappa})\right\} .
$$

Theorem 4.12. Let $E \subseteq G$ be a closed set. The following are equivalent:
(i) $E$ is an $M_{0}$-set;
(ii) $E^{*}$ is an operator $M_{0}$-set;
(iii) $\mathcal{A} \cap \mathfrak{P}\left(E^{*}\right) \neq\{0\}$;
(iv) $\mathcal{R} \cap \mathfrak{P}\left(E^{*}\right) \neq\{0\}$.

Proof. (i) $\Rightarrow$ (ii) Let $\theta \in M(G)$ be such that $\operatorname{supp} \theta \subseteq E$ and $\lambda(\theta) \in C_{r}^{*}(G)$. Then $M_{g} \lambda(\theta) M_{f}$ is a compact operator for all $f, g \in C_{0}(G)$ (see (7)).

For each $x \in G$, let $\theta^{x} \in M(G)$ be given by $\theta^{x}(\alpha)=\theta\left(x \alpha^{-1}\right)$ and $\theta_{x}$ be given by $\theta_{x}(\alpha)=\theta\left(\alpha x^{-1}\right)$, for any measurable $\alpha \subseteq G$ (here $\alpha^{-1}=\left\{s^{-1}: s \in \alpha\right\}$ ). Let $\theta^{*} \in M(G)$ be defined by $d \theta^{*}(s)=\overline{d \theta\left(s^{-1}\right)}$; then $\lambda\left(\theta^{*}\right)=\lambda(\theta)^{*}$. First observe that $\left\|\theta^{x}\right\|=\|\theta\|$ for each $x \in G$. Indeed, if $\left\{\alpha_{j}\right\}_{j=1}^{N}$ is a measurable partition of $G$ then $\left\{x \alpha_{j}^{-1}\right\}_{j=1}^{N}$ is also such, and hence

$$
\sum_{j=1}^{N}\left|\theta^{x}\left(\alpha_{j}\right)\right|=\sum_{j=1}^{N}\left|\theta\left(x \alpha_{j}^{-1}\right)\right| \leq\|\theta\|
$$

On the other hand, for every $\epsilon>0$, letting $\left\{\beta_{k}\right\}_{k=1}^{K}$ be a measurable partition of $G$ such that $\sum_{k=1}^{K}\left|\theta\left(\beta_{k}\right)\right|>\|\theta\|-\epsilon$, we see that $\left\{\beta_{k}^{-1} x\right\}_{k=1}^{K}$ is a measurable partition of $G$ with $\sum_{k=1}^{K}\left|\theta^{x}\left(\beta_{k}^{-1} x\right)\right|>\|\theta\|-\epsilon$, and so $\left\|\theta^{x}\right\| \geq\|\theta\|$. Similarly, $\left\|\theta_{x}^{*}\right\|=\left\|\theta^{*}\right\|$ for all $x \in G$.

If $f, g \in C_{0}(G)$ then

$$
\begin{align*}
\left(M_{g} \lambda(\theta) M_{f} \xi, \eta\right) & =\iint f\left(y^{-1} x\right) \xi\left(y^{-1} x\right) g(x) \overline{\eta(x)} d \theta(y) d x \\
& =\iint f(z) \xi(z) g(x) \overline{\eta(x)} d \theta^{x}(z) d x \tag{14}
\end{align*}
$$

and, also,

$$
\begin{aligned}
\left(M_{g} \lambda(\theta) M_{f} \xi, \eta\right) & =\left(M_{f} \xi, \lambda\left(\theta^{*}\right) M_{\bar{g}} \eta\right) \\
& =\iint f(z) \xi(z) g\left(x^{-1} z\right) \overline{\eta\left(x^{-1} z\right)} \overline{d \theta^{*}(x)} d z \\
& =\iint f(z) \xi(z) g(x) \overline{\eta(x)} \overline{d\left(\theta^{*}\right)^{z}(x)} d z .
\end{aligned}
$$

If, moreover, $f, g \in C_{0}(G) \cap L^{1}(G)$ and $x \in G$, the total variation of the measure $g(x) f(\cdot) d \theta^{x}(\cdot)$ equals $\int_{G}|f(z)| d\left|g(x) \theta^{x}\right|$ which does not exceed $\|f\|_{\infty}\left\|g(x) \theta^{x}\right\|$. Hence, $\left\|g(x) f(\cdot) d \theta^{x}(\cdot)\right\| \leq\|g\|_{\infty}\|f\|_{\infty}\|\theta\|$ for all $x \in G$. Furthermore, the function $x \mapsto\|f\|_{\infty}\left\|g(x) \theta^{x}\right\|$ is integrable since $x \mapsto\left\|\theta^{x}\right\|$ is a constant function.

Similarly, the total variation of the measure $f(z) g(\cdot) d\left(\theta^{*}\right)^{z}(\cdot)$ does not exceed $\|g\|_{\infty}\|f\|_{\infty}\left\|\theta^{*}\right\|$, and the function $z \rightarrow\|g\|_{\infty}\left\|f(z) d\left(\theta^{*}\right)^{z}\right\|$ is integrable. Lemma 4.11 now shows that, if $f, g \in C_{0}(G) \cap L^{1}(G)$, then $M_{g} \lambda(\theta) M_{f}$ is the pseudo-integral operator of the Arveson measure $\sigma_{f, g, \theta}$ given by $d \sigma_{f, g, \theta}(x, z)=g(x) f(z) \overline{d\left(\theta^{*}\right)^{z}(x)} d z=$ $g(x) f(z) d \theta^{x}(z) d x$. On the other hand, since $\lambda(\theta) \in C_{r}^{*}(G)$, the operator $M_{g} \lambda(\theta) M_{f}$ is compact whenever $f, g \in C_{0}(G) \cap L^{1}(G)$. It is now clear that, since $\theta \neq 0$, we can find functions $f, g \in C_{0}(G) \cap L^{1}(G)$ such that $M_{g} \lambda(\theta) M_{f}$ is non-zero.

Suppose that $\alpha \times \beta$ is a measurable rectangle with $(\alpha \times \beta) \cap E^{*}=\emptyset$ and $\xi \in L^{2}(G)$ (resp. $\eta \in L^{2}(G)$ ) vanishes everywhere on $\alpha^{c}$ (resp. $\beta^{c}$ ). For each $x \in G$, the function
$y \mapsto \xi\left(y^{-1} x\right) \overline{\eta(x)}$ vanishes on $E$ and hence, by (14), $\left(M_{g} \lambda(\theta) M_{f} \xi, \eta\right)=0$. Thus, $M_{g} \lambda(\theta) M_{f}$ is supported on $E^{*}$.
(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.
(iv) $\Rightarrow$ (i) Suppose that $\sigma$ is an Arveson measure supported on $\widehat{E^{*}}$ such that $0 \neq$ $T_{\sigma} \in \mathcal{R}$. By Lemma 4.7, there exists $\varphi \in \Gamma(G, G)$ such that $E_{\varphi}\left(T_{\sigma}\right) \neq 0$. By Lemma 4.10, $E_{\varphi}\left(T_{\sigma}\right)=\lambda\left(\sigma_{\varphi}\right)$, where $\sigma_{\varphi}$ is supported on $E$ and, by Theorem 4.6, $\lambda\left(\sigma_{\varphi}\right)$ belongs to $C_{r}^{*}(G)$.

### 4.6. An application: unions of sets of uniqueness

It was shown in [34, Proposition 5.3] that the union of two operator $U$-sets (resp. operator $U_{1}$-sets) is an operator $U$-set (resp. an operator $U_{1}$-set). A similar statement holds for operator $U_{0}$-sets.

Proposition 4.13. Let $E_{1}, E_{2} \subset X \times Y$ be $\omega$-closed operator $U_{0}$-sets. Then $E_{1} \cup E_{2}$ is an operator $U_{0}$-set.

Proof. Let $T_{\sigma}$ be a pseudo-integral compact operator supported on $E_{1} \cup E_{2}$; we may assume that the total variation of $\sigma$ is 1 . Let $\theta_{i} \in \Phi\left(E_{i}\right) \cap \mathfrak{S}(X, Y), i=1,2$, and write $\theta_{1}(x, y)=\sum_{i=1}^{\infty} f_{i}(x) g_{i}(y)$, where $\left\|\sum_{i=1}^{\infty}\left|f_{i}\right|^{2}\right\|_{\infty} \leq C$ and $\left\|\sum_{i=1}^{\infty}\left|g_{i}\right|^{2}\right\|_{\infty} \leq C$. We have that $\theta_{1} \theta_{2} \in \Phi\left(E_{1} \cup E_{2}\right)$ and hence

$$
\begin{equation*}
0=\left\langle T_{\sigma}, \theta_{1} \theta_{2}\right\rangle=\left\langle S_{\theta_{1}}\left(T_{\sigma}\right), \theta_{2}\right\rangle \tag{15}
\end{equation*}
$$

Let $\rho$ be the measure on $Y \times X$ given by

$$
\rho(E)=\int_{E} \theta_{1}(x, y) d \sigma(y, x) .
$$

Denoting by $\cup \dot{U}$ the union of a pairwise disjoint family of measurable sets, we have

$$
\begin{aligned}
|\rho|_{X}(\alpha) & =|\rho|(Y \times \alpha) \\
& =\sup \left\{\sum_{j=1}^{r}\left|\rho\left(E_{j}\right)\right|: Y \times \alpha=\bigcup_{j=1}^{r} E_{j}\right\} \\
& =\sup \left\{\sum_{j=1}^{r}\left|\int_{E_{j}} \theta_{1}(x, y) d \sigma(y, x)\right|: Y \times \alpha=\bigcup_{j=1}^{r} E_{j}\right\} \\
& \leq \sup \left\{\sum_{j=1}^{r} \int_{E_{j}}\left|\theta_{1}(x, y)\right| d|\sigma|(y, x): Y \times \alpha=\bigcup_{j=1}^{r} E_{j}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \leq \int_{Y \times \alpha} \sum_{i=1}^{\infty}\left|f_{i}(x)\right|\left|g_{i}(y)\right| d|\sigma|(y, x) \\
& \leq \sum_{i=1}^{\infty}\left(\int_{Y \times \alpha}\left|f_{i}(x)\right|^{2} d|\sigma|(y, x)\right)^{1 / 2}\left(\int_{Y \times \alpha}\left|g_{i}(y)\right|^{2} d|\sigma|(y, x)\right)^{1 / 2} \\
& \leq\left(\int_{Y \times \alpha} \sum_{i=1}^{\infty}\left|f_{i}(x)\right|^{2} d|\sigma|(y, x)\right)^{1 / 2}\left(\int_{Y \times \alpha} \sum_{i=1}^{\infty}\left|g_{i}(y)\right|^{2} d|\sigma|(y, x)\right)^{1 / 2} \\
& \leq C^{2}|\sigma|_{X}(\alpha) .
\end{aligned}
$$

Similarly, $|\rho|_{Y}(\beta) \leq C^{2}|\sigma|_{Y}(\beta)$ showing that $\rho$ is an Arveson measure. Now the identity

$$
\left(S_{\theta_{1}}\left(T_{\sigma}\right) \xi, \eta\right)=\int_{Y \times X} \theta_{1}(x, y) \xi(x) \overline{\eta(y)} d \sigma(y, x), \quad \xi \in H_{1}, \eta \in H_{2}
$$

shows that $S_{\theta_{1}}\left(T_{\sigma}\right)=T_{\rho}$.
Let $h \in \Phi\left(E_{2}\right)$ and write $h=\sum_{i=1}^{\infty} f_{i} \otimes g_{i}$, where $\sum_{i=1}^{\infty}\left\|f_{i}\right\|_{2}^{2}<\infty$ and $\sum_{i=1}^{\infty}\left\|g_{i}\right\|_{2}^{2}<\infty$. Let $X_{N}=\left\{x \in X: \sum_{i=1}^{\infty}\left|f_{i}(x)\right|^{2} \leq N\right\}$ and $Y_{N}=\{y \in Y:$ $\left.\sum_{i=1}^{\infty}\left|g_{i}(y)\right|^{2} \leq N\right\}$. Then $\chi_{X_{N} \times Y_{N}} h \in \mathfrak{S}(X, Y)$ and $\left\|\chi_{X_{N} \times Y_{N}} h-h\right\|_{\Gamma} \rightarrow_{N \rightarrow \infty} 0$. Thus, $\Phi\left(E_{2}\right) \cap \mathfrak{S}(X, Y)$ is dense in $\Phi\left(E_{2}\right)$, and, by (15), $T_{\rho} \in \Phi\left(E_{2}\right)^{\perp}=\mathfrak{M}_{\min }\left(E_{2}\right)$. As $E_{2}$ is an operator $U_{0}$-set, $T_{\rho}=0$ and therefore $\rho=0$. By Theorem 3.2, $\left\langle T_{\sigma}, \theta_{1}\right\rangle=\rho(Y \times X)=0$. Since this holds for any $\theta_{1} \in \Phi\left(E_{1}\right) \cap \mathfrak{S}(X, Y)$, the operator $T_{\sigma}$ is supported on $E_{1}$. Since $E_{1}$ is an operator $U_{0}$-set, $T_{\sigma}=0$.

Proposition 4.13, [34, Proposition 5.3], Theorem 4.9 and Theorem 4.12 have the following immediate corollary.

Corollary 4.14. Let $G$ be a locally compact second countable group. Suppose that $E_{1}, E_{2} \subseteq G$ are $U$-sets (resp., $U_{1}$-sets, $U_{0}$-sets). Then $E_{1} \cup E_{2}$ is a $U$-set (resp. a $U_{1}$-set, a $U_{0}$-set).

## 5. Preservation properties

The aim of this section is to show that the property of being a set of multiplicity, or a set of uniqueness, is preserved under some natural operations. The section is divided into three subsections.

### 5.1. Sets possessing an m-resolution

Here we consider a certain type of a countable union of operator $U$-sets. Theorem 5.2 should be compared to the classical result of N.K. Barry that a countable union of $U$-sets is a $U$-set [24].

Definition 5.1. Let $(X, \mu)$ and $(Y, \nu)$ be standard measure spaces.
(i) A pair $\left(\kappa_{1}, \kappa_{2}\right)$ of $\omega$-closed subsets of the direct product $X \times Y$ will be called m-separable if there exist a function $\varphi_{1} \in \mathfrak{S}(X, Y)$ and $\omega$-open neighbourhoods $E_{1}$ and $E_{2}$ of $\kappa_{1}$ and $\kappa_{2}$, respectively, such that $\left.\varphi\right|_{E_{1}}=1$ and $\left.\varphi\right|_{E_{2}}=0$.
(ii) Let $\kappa \subseteq X \times Y$ be an $\omega$-closed set and $\alpha$ be a countable ordinal. We call a family $\left(\kappa_{\beta}\right)_{\beta \leq \alpha}$ of $\omega$-closed sets an m-resolution of $\kappa$ if

- $\kappa_{1}=\kappa$;
- $\kappa_{\beta+1} \subseteq \kappa_{\beta}$, the set $\kappa_{\beta} \backslash \kappa_{\beta+1}$ is $\omega$-closed and the pair $\kappa_{\beta+1}, \kappa_{\beta} \backslash \kappa_{\beta+1}$ is an m-separable, for every ordinal $\beta<\alpha$;
- $\kappa_{\beta}=\bigcap_{\gamma<\beta} \kappa_{\gamma}$, for every limit ordinal $\beta \leq \alpha$.

Theorem 5.2. Let $(X, \mu)$ and $(Y, \nu)$ be standard measure spaces and $\kappa \subseteq X \times Y$ be an $\omega$-closed set which possesses an m-resolution $\left(\kappa_{\beta}\right)_{\beta \leq \alpha}$ such that $\kappa_{\beta} \backslash \kappa_{\beta+1}$ is an operator $U$-set, for each $\beta<\alpha$, and $\kappa_{\alpha}$ is an operator $U$-set. Then $\kappa$ is an operator $U$-set.

Proof. Let $\left(\kappa_{\beta}\right)_{\beta \leq \alpha}$ be an m-resolution of $\kappa$ such that $\kappa_{\beta} \backslash \kappa_{\beta+1}$ is an operator $U$-set for each $\beta<\alpha$.

We first observe that if $T \in \mathfrak{M}_{\max }\left(\kappa_{\beta}\right) \cap \mathcal{K}$ for some ordinal $\beta<\alpha$, then $T \in$ $\mathfrak{M}_{\max }\left(\kappa_{\beta+1}\right) \cap \mathcal{K}$. In fact, let $\kappa_{\beta}^{\prime}=\kappa_{\beta-1} \backslash \kappa_{\beta}$. By our assumptions, $\kappa_{\beta}^{\prime}$ is an $\omega$-closed set and there exists a function $\varphi \in \mathfrak{S}(X, Y)$ such that $\varphi=1$ on an $\omega$-open neighbourhood of $\kappa_{\beta+1}$ and $\varphi=0$ on an $\omega$-open neighbourhood of $\kappa_{\beta}^{\prime}$. Clearly, $1-\varphi \in \mathfrak{S}(X, Y), 1-\varphi=0$ on an $\omega$-open neighbourhood of $\kappa_{\beta+1}$ and $1-\varphi=1$ on an $\omega$-open neighbourhood of $\kappa_{\beta}^{\prime}$. Moreover, $T=S_{\varphi}(T)+S_{1-\varphi}(T)$.

For each $\psi \in \mathfrak{S}(X, Y)$ vanishing on an $\omega$-open neighbourhood of $\kappa_{\beta+1}$, the function $\psi \varphi \in \mathfrak{S}(X, Y)$ vanishes on an $\omega$-open neighbourhood of $\kappa_{\beta}$ and, since $T \in \mathfrak{M}_{\max }\left(\kappa_{\beta}\right)$, Lemma 3.4 implies that $S_{\psi}\left(S_{\varphi}(T)\right)=S_{\psi \varphi}(T)=0$. By Lemma 3.4 again, $S_{\varphi}(T) \in$ $\mathfrak{M}_{\max }\left(\kappa_{\beta+1}\right)$. Similarly, $S_{1-\varphi}(T) \in \mathfrak{M}_{\max }\left(\kappa_{\beta}^{\prime}\right)$. Since $\mathcal{K}$ is invariant under Schur multipliers, we conclude that $S_{1-\varphi}(T) \in \mathfrak{M}_{\max }\left(\kappa_{\beta}^{\prime}\right) \cap \mathcal{K}$. However, $\kappa_{\beta}^{\prime}$ is an operator $U$-set by assumption. It follows that $S_{1-\varphi}(T)=0$ and hence $T=S_{\varphi}(T) \in \mathfrak{M}_{\max }\left(\kappa_{\beta+1}\right) \cap \mathcal{K}$.

Let now $T \in \mathfrak{M}_{\max }(\kappa) \cap \mathcal{K}$. It follows by transfinite induction that $T \in \mathfrak{M}_{\max }\left(\kappa_{\beta}\right) \cap \mathcal{K}$ for all $\beta \leq \alpha$. In fact, assuming that the statement holds for all $\gamma<\beta$ we get by the previous paragraph that $T \in \mathfrak{M}_{\max }\left(\kappa_{\beta}\right) \cap \mathcal{K}$ if $\beta$ has a predecessor while, if $\beta$ is a limit ordinal, the inclusion follows from the assumption that $\kappa_{\beta}=\bigcap_{\gamma<\beta} \kappa_{\gamma}$.

Since $\kappa_{\alpha}$ is an operator $U$-set, we have now $T=0$ and hence $\kappa$ is an operator $U$-set.

The following corollary should be compared to M. Bożejko's result [5,4] that every compact countable set in a non-discrete locally compact group is a $U$-set.

Corollary 5.3. Let $G$ be a non-discrete locally compact second countable group and $E \subseteq G$ be a closed countable set. Then $E$ is a $U$-set.

Proof. Recall that the successive Cantor-Bendixson derivatives of the set $E$ are defined as follows: let $E_{0}=E$ and for an ordinal $\beta$, let $E_{\beta}$ be equal to the set of all limit points of $E_{\beta-1}$ if $\beta$ has a predecessor, and to $\bigcap_{\gamma<\beta} E_{\gamma}$ if $\beta$ is a limit ordinal. Since $E$ is countable, there exists a countable ordinal $\alpha$ such that $E_{\alpha}=\emptyset$. Moreover, $E_{\beta} \backslash E_{\beta+1}$ is a countable set consisting of isolated points of $E$. By the regularity of $A(G)$, a pair of the form $\left(\{s\}^{*}, F^{*}\right)$, where $F$ is a closed set and $s \notin F$, is m-separable. One hence easily obtains an m-resolution for $E^{*}$. On the other hand, if $G$ is not discrete then $\mathfrak{M}_{\max }\left(\{s\}^{*}\right)=\lambda_{s} \mathcal{D}$ does not contain non-zero compact operators. It follows from Theorem 5.2 and Theorem 4.9 that $E$ is a $U$-set.

### 5.2. Inverse images

In this subsection, we establish an Inverse Image Theorem for sets of multiplicity. Our result, Theorem 5.5, should be compared to [35, Theorem 4.7], an inverse image result for operator synthesis.

Let $(X, \mu),\left(X_{1}, \mu_{1}\right),(Y, \nu)$ and $\left(Y_{1}, \nu_{1}\right)$ be standard measure spaces. We fix, for the remainder of this section, measurable mappings $\varphi: X \rightarrow X_{1}$ and $\psi: Y \rightarrow Y_{1}$ such that $\varphi(X)$ and $\psi(Y)$ are measurable, the measure $\varphi_{*} \mu$ on $X_{1}$ given by $\varphi_{*} \mu\left(\alpha_{1}\right)=\mu\left(\varphi^{-1}\left(\alpha_{1}\right)\right)$ is absolutely continuous with respect to $\mu_{1}$, and the measure $\psi_{*} \nu$, defined similarly, is absolutely continuous with respect to $\nu_{1}$.

Let $r: X_{1} \rightarrow \mathbb{R}^{+}$be the Radon-Nikodym derivative of $\varphi_{*} \mu$ with respect to $\mu_{1}$, that is, the $\mu_{1}$-measurable function such that $\mu\left(\varphi^{-1}\left(\alpha_{1}\right)\right)=\int_{\alpha_{1}} r\left(x_{1}\right) d \mu_{1}\left(x_{1}\right)$ for every measurable set $\alpha_{1} \subseteq X_{1}$. Similarly, let $s: Y_{1} \rightarrow \mathbb{R}^{+}$be the Radon-Nikodym derivative of $\psi_{*} \nu$ with respect to $\nu_{1}$. Let $M_{1}=\left\{x_{1} \in X_{1}: r\left(x_{1}\right)=0\right\}$ and $N_{1}=\left\{y_{1} \in Y_{1}: s\left(y_{1}\right)=0\right\}$. Note that $\mu\left(\varphi^{-1}\left(M_{1}\right)\right)=\int_{M_{1}} r\left(x_{1}\right) d \mu_{1}\left(x_{1}\right)=0$. Similarly, $\nu\left(\psi^{-1}\left(N_{1}\right)\right)=0$. Observe that, up to a $\mu_{1}$-null set, $M_{1}^{c} \subseteq \varphi(X)$. Indeed, letting $M_{2}=M_{1}^{c} \cap \varphi(X)^{c} \subseteq X_{1}$, we see that $\varphi^{-1}\left(M_{2}\right)=\emptyset$ and hence $0=\mu\left(\varphi^{-1}\left(M_{2}\right)\right)=\int_{M_{2}} r_{1}(x) d \mu_{1}(x)$. Since $r_{1}(x)>0$ for every $x \in M_{2}$, we have that $\mu_{1}\left(M_{2}\right)=0$. Similarly, $N_{1}^{c} \subseteq \varphi(Y)$, up to a null set.

We will say that $\varphi: X \rightarrow X_{1}$ is injective up to a null set if there exists a subset $\Lambda \subseteq X$ with $\mu(\Lambda)=0$, such that $\varphi: \Lambda^{c} \rightarrow X_{1}$ is injective. By [35, Lemma 4.2], there exists a null set $N \subseteq X_{1}$ such that $\varphi\left(\Lambda^{c}\right) \backslash N$ is measurable and the inverse of $\varphi^{-1}$ on $\varphi\left(\Lambda^{c}\right) \backslash N$ is measurable. We moreover have that $\varphi^{-1}(N)$ is null; thus, the function $\varphi$, restricted to $\Lambda^{c} \cap \varphi^{-1}(N)^{c}$ is a bijection onto $\varphi\left(\Lambda^{c}\right) \backslash N$ and has measurable inverse.

If the set $\Lambda$ above can moreover be chosen so that $\mu_{1}\left(\varphi\left(\Lambda^{c}\right)^{c}\right)=0$, then we say that $\varphi$ is bijective up to a null set.

The following result must be known but we could not find a precise reference.
Lemma 5.4. The operator $V_{\varphi}: L^{2}\left(X_{1}, \mu_{1}\right) \rightarrow L^{2}(X, \mu)$ given by

$$
V_{\varphi} \xi(x)= \begin{cases}\frac{\xi(\varphi(x))}{\sqrt{r(\varphi(x))}} & \text { if } x \notin \varphi^{-1}\left(M_{1}\right) \\ 0 & \text { if } x \in \varphi^{-1}\left(M_{1}\right)\end{cases}
$$

is a partial isometry with initial space $L^{2}\left(M_{1}^{c},\left.\mu_{1}\right|_{M_{1}^{c}}\right)$. Moreover, if $\varphi$ is injective up to a null set then $V_{\varphi}$ is surjective.

Proof. Note that, if $\xi \in L^{2}\left(X_{1}, \mu_{1}\right)$ then

$$
\begin{aligned}
\left\|V_{\varphi} \xi\right\|^{2} & =\int_{\varphi^{-1}\left(M_{1}\right)^{c}}\left|\frac{\xi(\varphi(x))}{\sqrt{r(\varphi(x))}}\right|^{2} d \mu(x)=\int_{M_{1}^{c}}\left|\frac{\xi\left(x_{1}\right)}{\sqrt{r\left(x_{1}\right)}}\right|^{2} d \varphi_{*} \mu\left(x_{1}\right) \\
& =\int_{M_{1}^{c}} r\left(x_{1}\right)\left|\frac{\xi\left(x_{1}\right)}{\sqrt{r\left(x_{1}\right)}}\right|^{2} d \mu_{1}\left(x_{1}\right)=\int_{M_{1}^{c}}\left|\xi\left(x_{1}\right)\right|^{2} d \mu_{1}\left(x_{1}\right)
\end{aligned}
$$

It follows that $V_{\varphi}$ is a partial isometry with initial space $L^{2}\left(M_{1}^{c},\left.\mu_{1}\right|_{M_{1}^{c}}\right)$.
Suppose that $\varphi$ is injective up to a null set. By the remarks preceding the formulation of the lemma, we may assume that there exists a set $M \subseteq X$ such that $\mu(M)=0$, $\varphi\left(M^{c}\right)$ is measurable, $\left.\varphi\right|_{M^{c}}$ is one-to-one, and $\varphi^{-1}: \varphi\left(M^{c}\right) \rightarrow M^{c}$ is measurable. Let $\eta \in L^{2}(X, \mu)$ and define $\xi: X_{1} \rightarrow \mathbb{C}$ by setting $\xi\left(x_{1}\right)=\sqrt{r\left(x_{1}\right)} \eta\left(\varphi^{-1}\left(x_{1}\right)\right)$ if $x_{1} \in \varphi\left(M^{c}\right)$ and $\xi\left(x_{1}\right)=0$ if $x_{1} \notin \varphi\left(M^{c}\right)$. We claim that $\xi \in L^{2}\left(X_{1}, \mu_{1}\right)$. To see this, note that

$$
\mu\left(\varphi^{-1}\left(\alpha_{1}\right)\right)=\int_{\alpha_{1}} r\left(x_{1}\right) d \mu_{1}\left(x_{1}\right)
$$

for all $\mu_{1}$-measurable sets $\alpha_{1} \subseteq \varphi\left(M^{c}\right)$. Setting $\tilde{\mu}$ to be the measure on $M^{c}$ given by $\tilde{\mu}(\alpha)=\mu_{1}(\varphi(\alpha))$ for $\mu$-measurable subset $\alpha \subseteq M^{c}$ we have

$$
\mu(\alpha)=\int_{\alpha} r(\varphi(x)) d \tilde{\mu}(x)
$$

It follows that

$$
\begin{aligned}
\|\xi\|_{L^{2}\left(X_{1}, \mu_{1}\right)} & =\int_{\varphi\left(M^{c}\right)} r\left(x_{1}\right)\left|\eta\left(\varphi^{-1}\left(x_{1}\right)\right)\right|^{2} d \mu_{1}\left(x_{1}\right) \\
& =\int_{M^{c}} r(\varphi(x))|\eta(x)|^{2} d \tilde{\mu}(x) \\
& =\int_{M^{c}}|\eta(x)|^{2} d \mu(x)=\|\eta\|_{L^{2}(X, \mu)}
\end{aligned}
$$

since $M$ is $\mu$-null. On the other hand,

$$
\Lambda \stackrel{\text { def }}{=} \varphi^{-1}\left(M_{1}^{c} \cap \varphi\left(M^{c}\right)^{c}\right) \subseteq \varphi^{-1}\left(\varphi\left(M^{c}\right)^{c}\right) \subseteq M
$$

and hence $\Lambda$ is $\mu$-null. It follows as in the third paragraph of the present subsection that $M_{1}^{c} \cap \varphi\left(M^{c}\right)^{c}$ is $\mu_{1}$-null. Thus, $V_{\varphi} \xi=\eta$, and the proof is complete.

We now formulate and prove the main result of this subsection.
Theorem 5.5. Let $\varphi: X \rightarrow X_{1}$ and $\psi: Y \rightarrow Y_{1}$ be measurable functions. Let $\kappa_{1} \subseteq X_{1} \times Y_{1}$ and $\kappa=\left\{(x, y) \in X \times Y:(\varphi(x), \psi(y)) \in \kappa_{1}\right\}$.
(i) Suppose that $\varphi$ and $\psi$ are injective up to a null set. If $\kappa_{1}$ is an operator $U$-set (resp. an operator $U_{1}$-set) then $\kappa$ is an operator $U$-set (resp. an operator $U_{1}$-set).
(ii) Suppose that $\kappa_{1} \subseteq M_{1}^{c} \times N_{1}^{c}$. If $\kappa_{1}$ is an operator $M$-set (resp. an operator $M_{1}$-set) then $\kappa$ is an operator $M$-set (resp. an operator $M_{1}$-set).
(iii) Suppose that $\mu_{1}\left(\right.$ resp. $\left.\nu_{1}\right)$ is equivalent to $\varphi_{*} \mu$ (resp. $\psi_{*} \nu$ ) and that $\varphi$ and $\psi$ are injective up to a null set. Then $\kappa_{1}$ is an operator $M$-set (resp. an operator $M_{1}$-set) if and only if $\kappa$ is an operator $M$-set (resp. an operator $M_{1}$-set).

Proof. (i) Let $\Theta$ be the linear map from the algebraic tensor product $L^{2}\left(X_{1}, \mu_{1}\right) \otimes$ $L^{2}\left(Y_{1}, \nu_{1}\right)$ of $L^{2}\left(X_{1}, \mu_{1}\right)$ and $L^{2}\left(Y_{1}, \nu_{1}\right)$ sending $f \otimes g$ to $V_{\varphi} f \otimes V_{\psi} g$. Since $V_{\varphi}$ and $V_{\psi}$ are partial isometries, $\Theta$ is contractive in the norm of $\Gamma\left(X_{1}, Y_{1}\right)$, and hence extends to a contractive linear map $\Theta: \Gamma\left(X_{1}, Y_{1}\right) \rightarrow \Gamma(X, Y)$. By Lemma 5.4, $V_{\varphi}$ and $V_{\psi}$ are surjective, and hence $\Theta$ has dense range. Moreover, if $h \in \Gamma\left(X_{1}, Y_{1}\right)$ then

$$
\begin{equation*}
\Theta(h)(x, y)=\frac{h(\varphi(x), \psi(y))}{\sqrt{r(\varphi(x)) s(\psi(y))}}, \quad \text { for m.a.e. }(x, y) \in \varphi^{-1}\left(M_{1}\right) \times \psi^{-1}\left(N_{1}\right) \tag{16}
\end{equation*}
$$

To show (16), write $h=\sum_{i=1}^{\infty} f_{i} \otimes g_{i}$, where $\sum_{i=1}^{\infty}\left\|f_{i}\right\|_{2}^{2}<\infty$ and $\sum_{i=1}^{\infty}\left\|g_{i}\right\|_{2}^{2}<\infty$, and set $h_{n}=\sum_{i=1}^{n} f_{i} \otimes g_{i}, n \in \mathbb{N}$. By the definition of $\Theta$, identity (16) holds for all $h_{n}$, $n \in \mathbb{N}$, if $(x, y)$ belongs to the set $\varphi^{-1}\left(M_{1}\right) \times \psi^{-1}\left(N_{1}\right)$. By [35, Lemma 2.1], there exists a subsequence $\left(h_{n_{k}}\right)$ of $\left(h_{n}\right)$ which converges to $h$ marginally almost everywhere. By passing to a further subsequence, we may assume that $\Theta\left(h_{n_{k}}\right)$ converges to $\Theta(h)$ marginally almost everywhere. Identity (16) now follows from the fact that if $E \subseteq X_{1} \times Y_{1}$ is marginally null then $\{(x, y) \in X \times Y:(\varphi(x), \psi(y)) \in E\}$ is marginally null.

The map $\Theta$ is the adjoint to the map $\mathcal{K}\left(L^{2}\left(X_{1}, \mu_{1}\right), L^{2}\left(Y_{1}, \nu_{1}\right)\right) \ni K \mapsto V_{\psi}^{*} K V_{\varphi} \in$ $K\left(L^{2}(X, \mu), L^{2}(Y, \nu)\right)$; indeed, if $f \in L^{2}(X, \mu)$ and $g \in L^{2}(Y, \nu)$ then

$$
\begin{aligned}
\left\langle f \otimes g, V_{\psi}^{*} K V_{\varphi}\right\rangle & =\left(V_{\psi}^{*} K V_{\varphi} f, \bar{g}\right)=\left(K V_{\varphi} f, V_{\psi} \bar{g}\right)=\left(K V_{\varphi} f, \overline{V_{\psi} g}\right) \\
& =\left\langle V_{\varphi} f \otimes V_{\psi} g, K\right\rangle=\langle\Theta(f \otimes g), K\rangle
\end{aligned}
$$

It follows that $\Theta$ is weak* continuous and thus, if $\mathcal{M}_{1} \subseteq \Gamma\left(X_{1}, Y_{1}\right)$ then

$$
\begin{equation*}
\Theta\left({\overline{\mathcal{M}_{1}}}^{w^{*}}\right) \subseteq{\overline{\Theta\left(\mathcal{M}_{1}\right)}}^{w^{*}} \tag{17}
\end{equation*}
$$

Suppose that $\left.h\right|_{\kappa_{1}}=0$. If $(x, y) \in \kappa \backslash\left(\left(\varphi^{-1}\left(M_{1}\right) \times Y\right) \cup\left(X \times \psi^{-1}\left(N_{1}\right)\right)\right)$ then, by (16),

$$
\Theta(h)(x, y)=\frac{h(\varphi(x), \psi(y))}{\sqrt{r(\varphi(x)) s(\psi(y))}}=0
$$

thus, $\Theta\left(\Phi\left(\kappa_{1}\right)\right) \subseteq \Phi(\kappa)$. On the other hand, if $E_{1}$ is an $\omega$-open neighbourhood of $\kappa_{1}$ then $(\varphi \times \psi)^{-1}\left(E_{1}\right)$ is an $\omega$-open neighbourhood of $\kappa$. Applying the same reasoning as above, and using the continuity of $\Theta$ with respect to $\|\cdot\|_{\Gamma}$, we conclude that $\Theta\left(\Psi\left(\kappa_{1}\right)\right) \subseteq \Psi(\kappa)$.

Now suppose that $\kappa_{1}$ is an operator $U$-set, that is, $\overline{\Phi\left(\kappa_{1}\right)} w^{*}=\Gamma\left(X_{1}, Y_{1}\right)$. Using (17), we have

$$
\Gamma(X, Y)={\overline{\Theta\left(\Gamma\left(X_{1}, Y_{1}\right)\right)}}^{\|\cdot\|}={\overline{\Theta\left({\overline{\Phi\left(\kappa_{1}\right)}}^{w^{*}}\right)}}^{\|\cdot\|} \subseteq{\overline{\Theta\left(\Phi\left(\kappa_{1}\right)\right)}}^{w^{*}} \subseteq \overline{\Phi(\kappa)}^{w^{*}}
$$

Thus, $\overline{\Phi(\kappa)} w^{w^{*}}=\Gamma(X, Y)$ and hence $\kappa$ is an operator $U$-set. It follows similarly that if $\kappa_{1}$ is an operator $U_{1}$-set then $\kappa$ is an operator $U_{1}$-set.
(ii) Suppose that $\kappa_{1}$ is an operator $M_{1}$-set and let $K_{1}$ be a non-zero compact operator in $\mathfrak{M}_{\text {min }}\left(\kappa_{1}\right)$. Let $K=V_{\psi} K_{1} V_{\varphi}^{*}$. As $\kappa_{1} \subseteq M_{1}^{c} \times N_{1}^{c}, V_{\varphi}^{*} V_{\varphi}=P\left(M_{1}^{c}\right)$ and $V_{\psi}^{*} V_{\psi}=P\left(N_{1}^{c}\right)$, we have that $K_{1}=V_{\psi}^{*} K V_{\varphi}$ and hence $K$ is a non-zero compact operator.

Let

$$
(P, Q) \in\left(\mathcal{B}\left(\ell^{2}\right) \bar{\otimes} L^{\infty}(X, \mu)\right) \times\left(\mathcal{B}\left(\ell^{2}\right) \bar{\otimes} L^{\infty}(Y, \nu)\right)
$$

be a $\kappa$-pair [35]; this means that, after the identification of $P$ and $Q$ with operator-valued weakly measurable functions, defined on $X$ and $Y$, respectively, $P$ and $Q$ are projectionvalued and $P(x) Q(y)=0$ marginally almost everywhere on $\kappa$. It follows from the proof of [35, Theorem 4.7] that there exists a $\kappa_{1}$-pair

$$
(\hat{P}, \hat{Q}) \in\left(\mathcal{B}\left(\ell^{2}\right) \bar{\otimes} L^{\infty}\left(X_{1}, \mu\right)\right) \times\left(\mathcal{B}\left(\ell^{2}\right) \bar{\otimes} L^{\infty}\left(Y_{1}, \nu\right)\right),
$$

such that $P(x) \leq \hat{P}(\varphi(x))$ and $Q(y) \leq \hat{Q}(\psi(x))$ for almost all $x \in X$ and almost all $y \in Y$. By Theorem 3.3,

$$
\begin{equation*}
\hat{Q}\left(I \otimes K_{1}\right) \hat{P}=0 . \tag{18}
\end{equation*}
$$

We claim that

$$
\begin{equation*}
\left(I \otimes V_{\varphi}^{*}\right)(R \circ \varphi)=R\left(I \otimes V_{\varphi}^{*}\right) \quad \text { and } \quad(S \circ \psi)\left(I \otimes V_{\psi}\right)=\left(I \otimes V_{\psi}\right) S \tag{19}
\end{equation*}
$$

whenever $R$ and $S$ are bounded operator-valued weakly measurable functions on $X_{1}$ and $Y_{1}$, respectively. It clearly suffices to show only the first of these identities. Start by observing that $P\left(\varphi^{-1}(\alpha)\right) V_{\varphi}=V_{\varphi} P(\alpha)$, for all measurable subsets $\alpha \subseteq X_{1}$. It follows that (19) holds when $R=\sum_{j=1}^{k} a_{j} \otimes \chi_{E_{j}}$, where $\left(E_{j}\right)_{j=1}^{k}$ is a family of pairwise disjoint measurable subsets of $X_{1}$ and $\left(a_{i}\right)_{i=1}^{k}$ is a family of bounded operators on $\ell^{2}$. If $R$ is arbitrary then, by Kaplansky's Density Theorem, it is the strong limit of a sequence $\left(R_{n}\right)_{n \in \mathbb{N}}$, where $R_{n}$ is of the latter form and $\left\|R_{n}\right\| \leq\|R\|$ for each $n$. By the proof of [35, Theorem 4.6], there exists $S_{1} \subseteq X_{1}$ with $\mu_{1}\left(S_{1}\right)=0$ such that

$$
R\left(x_{1}\right)=\mathrm{s}_{n \rightarrow \infty} \lim _{n_{k}}\left(x_{1}\right) \quad \text { if } x_{1} \notin S_{1} .
$$

Let $S=\varphi^{-1}\left(S_{1}\right)$; then $R(\varphi(x))=\operatorname{s-lim}_{k \rightarrow \infty} R_{n_{k}}(\varphi(x))$ if $x \notin S$. As $\mu(S)=\varphi_{*} \mu\left(S_{1}\right)$ and $\varphi_{*} \mu$ is absolutely continuous with respect to $\mu_{1}$, we have that $\mu(S)=0$ and hence $\left(R_{n_{k}} \circ \varphi\right)_{k \in \mathbb{N}}$ converges almost everywhere to $R \circ \varphi$.

Since $\left\|R_{n}\right\|=\operatorname{ess}_{\sup }^{x_{1} \in X_{1}} ⿵ ⺆ R_{n}(x) \|_{\mathcal{B}\left(\ell^{2}\right)}$ and $\left(R_{n}\right)_{n \in \mathbb{N}}$ is bounded by $\|R\|$, there exists a $\mu_{1}$-null set $M \subseteq X_{1}$ such that $\left\|R_{n}\left(x_{1}\right)\right\|_{\mathcal{B}\left(\ell^{2}\right)} \leq\|R\|$ for all $x_{1} \notin M$ and all $n \in \mathbb{N}$. Therefore $\left\|R_{n}(\varphi(x))\right\|_{\mathcal{B}\left(\ell^{2}\right)} \leq\|R\|$ for all $x \notin \varphi^{-1}(M)$ and all $n \in \mathbb{N}$. As $\mu\left(\varphi^{-1}(M)\right)=0$, we have that $\left\|R_{n} \circ \varphi\right\|=\operatorname{ess} \sup _{x \in X}\left\|R_{n}(\varphi(x))\right\|_{\mathcal{B}\left(\ell^{2}\right)} \leq\|R\|$, for all $n \in \mathbb{N}$. By a straightforward application of the Lebesgue Dominated Convergence Theorem, $\left(R_{n_{k}} \circ \varphi\right)_{k \in \mathbb{N}}$ converges strongly to $R \circ \varphi$. As $\left(I \otimes V_{\varphi}^{*}\right)\left(R_{n} \circ \varphi\right)=R_{n}\left(I \otimes V_{\varphi}^{*}\right)$ holds for every $n$ and $\left(I \otimes V_{\varphi}^{*}\right)\left(R_{n} \circ \varphi\right) \rightarrow\left(I \otimes V_{\varphi}^{*}\right)(R \circ \varphi), R_{n}\left(I \otimes V_{\varphi}^{*}\right) \rightarrow R\left(I \otimes V_{\varphi}^{*}\right)$ in the strong operator topology, (19) is proved. Using (18) and (19), we now obtain

$$
\begin{aligned}
Q(I \otimes K) P & =Q(\hat{Q} \circ \psi)\left(I \otimes V_{\psi} K_{1} V_{\varphi}^{*}\right)(\hat{P} \circ \varphi) P \\
& =Q\left(I \otimes V_{\psi}\right) \hat{Q}\left(I \otimes K_{1}\right) \hat{P}\left(I \otimes V_{\psi}^{*}\right) P=0 .
\end{aligned}
$$

By Theorem 3.3, $K \in \mathfrak{M}_{\text {min }}(E)$; hence, $\kappa$ is an operator $M_{1}$-set.
Now suppose that $\kappa_{1}$ is an operator $M$-set and let $K_{1} \in \mathfrak{M}_{\max }\left(\kappa_{1}\right)$ be a non-zero compact operator. Let $(P, Q)$ be a simple $\kappa$-pair [35], that is, a $\kappa$-pair $(P, Q)$ for which each of the projection valued functions $P$ and $Q$ takes finitely many values. We recall the construction of the pair $(\hat{P}, \hat{Q})$ from [35]. Let $\left(\xi_{j}\right)_{j \in \mathbb{N}}$ be a dense sequence in $\ell^{2}$. It was shown on [35, p. 311] that there are null sets $M_{0}^{1} \subseteq X_{1}$ and $M_{0} \subseteq X$ and, for each $j \in \mathbb{N}$, a measurable function $g_{j}: \varphi(X) \backslash M_{0}^{1} \rightarrow X$ with $\varphi\left(g_{j}\left(x_{1}\right)\right)=x_{1}$ for all $x_{1} \in \varphi(X) \backslash M_{0}^{1}$, and $\left(P\left(g_{j}(\varphi(x))\right) \xi_{j}, \xi_{j}\right)>\left(P(x) \xi_{j}, \xi_{j}\right)-\frac{1}{j}, x \in X \backslash M_{0}$. Let, similarly, $\left(\eta_{j}\right)_{j \in \mathbb{N}}$ be a dense sequence in $\ell^{2}$ and for each $j \in \mathbb{N}$, let $h_{j}: \varphi(Y) \backslash N_{0}^{1} \rightarrow Y$ be a measurable function with $\psi\left(h_{j}\left(y_{1}\right)\right)=y_{1}$ for all $y_{1} \in \psi(Y) \backslash N_{0}^{1}$, and $\left(Q\left(h_{j}(\psi(y))\right) \eta_{j}, \eta_{j}\right)>\left(Q(y) \eta_{j}, \eta_{j}\right)-\frac{1}{j}$, $y \in Y \backslash N_{0}$, where $N_{0}^{1} \subseteq Y_{1}$ and $N_{0} \subseteq Y$ are null sets. Set

$$
\begin{array}{ll}
\hat{P}_{n}\left(x_{1}\right)=\bigvee_{j=1}^{n} P\left(g_{j}\left(x_{1}\right)\right), & \hat{P}\left(x_{1}\right)=\bigvee_{j=1}^{\infty} P\left(g_{j}\left(x_{1}\right)\right), \\
x_{1} \in \varphi(X) \backslash M_{0}^{1} \\
\hat{Q}_{n}\left(y_{1}\right)=\bigvee_{j=1}^{n} Q\left(h_{j}\left(y_{1}\right)\right), & \hat{Q}\left(y_{1}\right)=\bigvee_{j=1}^{\infty} Q\left(h_{j}\left(y_{1}\right)\right),
\end{array} \quad y_{1} \in \psi(Y) \backslash N_{0}^{1} .
$$

We have that $\hat{P}_{n} \rightarrow_{n \rightarrow \infty} \hat{P}$ and $\hat{Q}_{n} \rightarrow_{n \rightarrow \infty} \hat{Q}$ in the strong operator topology. Furthermore, since $P($ resp. $Q)$ takes only finitely many values, the same is true for $\hat{P}_{n}$ $\left(\operatorname{resp} . \hat{Q}_{n}\right), n \in \mathbb{N}$. If

$$
\left(x_{1}, y_{1}\right) \in \kappa_{1} \cap\left(\left(\varphi(X) \backslash M_{0}^{1}\right) \times\left(\psi(Y) \backslash N_{0}^{1}\right)\right)
$$

then $\left(g_{j}\left(x_{1}\right), h_{j}\left(y_{1}\right)\right) \in \kappa$. However, $\kappa_{1} \subseteq M_{1}^{c} \times N_{1}^{c}$, while $M_{1}^{c} \times N_{1}^{c}$ is marginally contained in $\varphi(X) \times \psi(Y)$. It follows that $\hat{P}_{n}\left(x_{1}\right) \hat{Q}_{n}\left(y_{1}\right)=0$ for marginally almost all $\left(x_{1}, y_{1}\right) \in \kappa_{1}$ and every $n \in \mathbb{N}$. Thus, $\left(\hat{P}_{n}, \hat{Q}_{n}\right)$ is a simple $\kappa_{1}$-pair, $n \in \mathbb{N}$, and hence, by Theorem 3.3,
$\hat{Q}_{n}\left(I \otimes K_{1}\right) \hat{P}_{n}=0$ for every $n$. Since $P \leq \hat{P} \circ \varphi$ and $Q \leq \hat{Q} \circ \psi$, it follows from (19) and the first part of the proof that

$$
P=P(\hat{P} \circ \varphi)=\underset{n \rightarrow \infty}{\operatorname{s-lim}} P\left(\hat{P}_{n} \circ \varphi\right) \quad \text { and } \quad Q=Q(\hat{Q} \circ \psi)=\underset{n \rightarrow \infty}{\mathrm{~s}-\lim _{\infty}} Q\left(\hat{Q}_{n} \circ \psi\right) .
$$

As in the previous paragraph, we conclude that

$$
Q(I \otimes K) P=\underset{n \rightarrow \infty}{\mathrm{w}-\lim _{\rightarrow \infty}} Q\left(I \otimes V_{\psi}\right) \hat{Q}_{n}\left(I \otimes K_{1}\right) \hat{P}_{n}\left(I \otimes V_{\psi}^{*}\right) P=0 .
$$

By Theorem 3.3, $K \in \mathfrak{M}_{\max }(\kappa)$ and since $K$ is a non-zero compact operator, $\kappa$ is an operator $M$-set.
(iii) In this case $\mu_{1}\left(M_{1}\right)=0$ and $\nu_{1}\left(N_{1}\right)=0$; thus, (iii) is immediate from (i) and (ii).

Remark 5.6. (i) The statement in Theorem 5.5 (ii) does not hold without the assumption $\kappa_{1} \subseteq M_{1}^{c} \times N_{1}^{c}$; indeed, assuming that $M_{1}$ and $N_{1}$ are non-null and letting $\kappa_{1}=M_{1} \times N_{1}$, we see that $\kappa_{1}$ is an operator $M$-set; but $\kappa$ is marginally equivalent to the empty set and hence is an operator $U$-set.
(ii) G.K. Eleftherakis has recently proved part (i) of Theorem 5.5 without the injectivity assumption on the mappings $\varphi$ and $\psi$, see [8].

Corollary 5.7. Let $G$ and $H$ be locally compact second countable groups with Haar measures $m_{G}$ and $m_{H}$, respectively, $\varphi: G \rightarrow H$ be a continuous homomorphism and $E$ be a closed subset of $H$. Assume that $\varphi_{*} m_{G}$ is absolutely continuous with respect to $m_{H}$.
(i) Suppose that $\varphi$ is injective and has a continuous inverse on $\varphi(G)$. If $E$ is a $U$-set (resp. a $U_{1}$-set) then $\varphi^{-1}(E)$ is a $U$-set (resp. a $U_{1}$-set).
(ii) Suppose that $\varphi_{*} m_{G}$ is equivalent to $m_{H}$. If $E$ is an $M$-set (resp. an $M_{1}$-set) then $\varphi^{-1}(E)$ is an $M$-set (resp. an $M_{1}$-set).
(iii) If $\varphi$ is an isomorphism then $E$ is an $M$-set (resp. an $M_{1}$-set) if and only if $\varphi^{-1}(E)$ is an $M$-set (resp. an $M_{1}$-set).

Proof. First observe that, since $\varphi$ is a homomorphism, $\varphi^{-1}(E)^{*}=(\varphi \times \varphi)^{-1}\left(E^{*}\right)$. If $\varphi$ is an isomorphism then $\varphi_{*} m_{G}$ is equivalent to $m_{H}$, see Remark 5.8. The corollary now follows from Theorems 4.9 and 5.5.

Remark 5.8. We note that if $m_{H}(\varphi(G)) \neq 0$ then the condition that $\varphi_{*} m_{G}$ is absolutely continuous with respect to $m_{H}$ can be dropped. In fact, in this case, by Steinhaus's Theorem, $\varphi(G)$ is an open subgroup of $H$ and if $\varphi$ is injective and has a continuous inverse on $\varphi(G), \varphi$ is a homeomorphism between $G$ and $\varphi(G)$. One can easily see that the measures $\varphi_{*} m_{G}$ and $m_{H}$ restricted to $\varphi(G)$ satisfy the conditions of (left) Haar measures of $\varphi(G)$. Hence, since $m_{H}(\varphi(G)) \neq 0$ there exists $c>0$ such that $\left.\varphi_{*} m_{G}\right|_{\varphi(G)}=\left.c m_{H}\right|_{\varphi(G)}$. Let $W \subseteq H$ be any Borel subset of $H$. Then

$$
\varphi_{*} m_{G}(W)=\varphi_{*} m_{G}(W \cap \varphi(G))=c m_{H}(W \cap \varphi(G))=c \int_{W} \chi_{\varphi(G)}(x) d m_{H}(x)
$$

giving the claim. It follows that the measures $\varphi_{*} m_{G}$ and $m_{H}$ are equivalent in the case $\varphi$ is an isomorphism.

We next include a characterisation of the closed subgroups that are also sets of multiplicity answering a question posed by M. Bożejko in [5]. We will need the following lemma.

Lemma 5.9. Let $G$ be a locally compact second countable group with a left Haar measure $m$. Let $H$ be a closed subgroup of $G$. Let $q: G \rightarrow G / H$ be the quotient map. Then there exists a finite Borel measure $\mu$ on $G / H$, equivalent to $q_{*} m$.

Proof. If $G$ is compact then the measure is finite itself and we are done. Suppose that $G$ is non-compact. As $G$ is second countable it is $\sigma$-compact, i.e., there exists an increasing sequence $\left(K_{n}\right)_{n \in \mathbb{N}}$ of compact subsets of $G$ such that $G=\bigcup_{n} K_{n}$. By deleting some of the terms of this sequence, we may assume that $m\left(K_{n+1} \backslash K_{n}\right)>0$ for each $n \in \mathbb{N}$. Define a measure $\mu_{n}$ on $G / H$ by letting $\mu_{n}(\alpha)=m\left(q^{-1}(\alpha) \cap K_{n}\right)$. Then $\mu_{n}$ is finite; indeed, $\mu_{n}(G / H)=m\left(K_{n}\right)<\infty$.

As $K_{n} \subseteq K_{n+1}$, we have that $\mu_{n}(\alpha) \leq \mu_{n+1}(\alpha)$ for any Borel set $\alpha$. Now define

$$
\mu(\alpha)=\sum_{n=0}^{\infty} \frac{\mu_{n+1}(\alpha)-\mu_{n}(\alpha)}{2^{n} m\left(K_{n+1} \backslash K_{n}\right)}
$$

where we have $\mu_{0}(\alpha)=0$. Clearly, $\mu$ is a finite measure. Since $\mu(\alpha)=0 \Leftrightarrow$ $\mu_{n+1}(\alpha)-\mu_{n}(\alpha)=0$ for all $n \geq 0 \Leftrightarrow \mu_{n}(\alpha)=0$ for all $n \geq 0 \Leftrightarrow m\left(q^{-1}(\alpha)\right)=q_{*} m(\alpha)=0$, $q_{*} m$ is absolutely continuous with respect to $\mu$.

Corollary 5.10. Let $H$ be a closed subgroup of a locally compact second countable group $G$. Then $H$ is an $M$-set if and only if $H$ is open.

Proof. We note first that by Steinhaus's Theorem, $H$ is open if and only if $m(H)>0$. If $m(H)>0$ then $H$ is an $M$-set by Remark 4.3. Assume now that $m(H)=0$. By Theorem 4.9, it suffices to see that

$$
H^{*}=\left\{(s, t): t s^{-1} \in H\right\}=\{(s, t): H t=H s\}
$$

is an operator $U$-set. Let $q: G \rightarrow G / H$ be the quotient map given by $q(s)=H s$. By [17, $5.22,8.14] G / H$ is a locally compact metrisable separable space. By Lemma 5.9, there exists a finite measure $\mu$ on $G / H$, equivalent to $q_{*} m$. Thus, $\mu$ is non-atomic. Since any finite measure on a locally compact second countable space is regular [13, Theorem 7.8], the measure space $(G / H, \mu)$ is standard.

Let now $D=\{(z, z): z \in G / H\}$. Since every bounded operator on $L^{2}(G / H, \mu)$ supported on $D$ is a multiplication operator and $\mu$ is non-atomic, the only compact
operator on $L^{2}(G / H, \mu)$ supported on $D$ is the zero operator. Therefore $D$ is an operator $U$-set. By Remark 5.6 (ii), $H^{*}=\left(q^{-1} \times q^{-1}\right)(D)$ is a set of uniqueness.

### 5.3. Direct products

In this subsection, we show that direct products preserve the property of being an operator $M$-set (resp. an operator $M_{1}$-set, an operator $M_{0}$-set).

Theorem 5.11. Let $\left(X_{i}, \mu_{i}\right)$ and $\left(Y_{i}, \nu_{i}\right)$ be standard measure spaces and $\kappa_{i} \subseteq X_{i} \times Y_{i}$ be $\omega$-closed sets, $i=1,2$. The set $\rho\left(\kappa_{1} \times \kappa_{2}\right)$ is an operator $M$-set (resp. operator $M_{1}$-set) if and only if both $\kappa_{1}$ and $\kappa_{2}$ are operator $M$-sets (resp. operator $M_{1}$-sets).

Proof. By [26], the support of $\mathfrak{M}_{\max }\left(\kappa_{1}\right) \bar{\otimes} \mathfrak{M}_{\max }\left(\kappa_{2}\right)$ is $\rho\left(\kappa_{1} \times \kappa_{2}\right)$. It follows that

$$
\begin{equation*}
\mathfrak{M}_{\max }\left(\kappa_{1}\right) \bar{\otimes} \mathfrak{M}_{\max }\left(\kappa_{2}\right) \subseteq \mathfrak{M}_{\max }\left(\rho\left(\kappa_{1} \times \kappa_{2}\right)\right) \tag{20}
\end{equation*}
$$

Assume first that $\kappa_{1}$ and $\kappa_{2}$ are operator $M_{1}$-sets (resp. operator $M$ sets). Suppose that $T_{i}$ is a non-zero compact operator in $\mathfrak{M}_{\min }\left(\kappa_{i}\right)$ (resp. $\left.\mathfrak{M}_{\max }\left(\kappa_{i}\right)\right), i=1,2$. By Theorem 3.8 (resp. by (20)), $T_{1} \otimes T_{2}$ is a non-zero compact operator in $\mathfrak{M}_{\min }\left(\rho\left(\kappa_{1} \times \kappa_{2}\right)\right)$ (resp. $\left.\mathfrak{M}_{\max }\left(\rho\left(\kappa_{1} \times \kappa_{2}\right)\right)\right)$. Hence $\rho\left(\kappa_{1} \times \kappa_{2}\right)$ is an operator $M_{1}$-set (resp. an operator $M$-set).

We next show that if either $\kappa_{1}$ or $\kappa_{2}$ is an operator $U$-set then so is $\rho\left(\kappa_{1} \times \kappa_{2}\right)$. Suppose that $T \in \mathcal{K}\left(H_{1} \otimes H_{2}, K_{1} \otimes K_{2}\right)$ is supported on $\rho\left(\kappa_{1} \times \kappa_{2}\right)$. Let $\omega \in\left(\mathcal{K}\left(H_{2}, K_{2}\right)\right)^{*}=$ $\mathcal{C}_{1}\left(K_{2}, H_{2}\right)$ and let $L_{\omega}$ be the slice map from $\mathcal{K}\left(H_{1} \otimes H_{2}, K_{1} \otimes K_{2}\right)$ to $\mathcal{K}\left(H_{1}, K_{1}\right)$ defined on elementary tensors by $L_{\omega}(A \otimes B)=\omega(B) A$. Then $\operatorname{supp} L_{\omega}(T) \subseteq \kappa_{1}$. In fact, if $\alpha \times \beta$ is a measurable rectangle marginally disjoint from $\kappa_{1}$, then $\left(\left(\alpha \times X_{2}\right) \times\left(\beta \times Y_{2}\right)\right) \cap \rho\left(\kappa_{1} \times\right.$ $\left.\kappa_{2}\right) \simeq \emptyset$ and

$$
P(\beta) L_{\omega}(T) P(\alpha)=L_{\omega}((P(\beta) \otimes I) T(P(\alpha) \otimes I))=0
$$

If $\kappa_{1}$ is an operator $U$-set, $L_{\omega}(T)=0$ for all $\omega$ and hence $T=0$.
If $T \in \mathcal{K}\left(H_{1} \otimes H_{2}, K_{1} \otimes K_{2}\right) \cap \mathfrak{M}_{\min }\left(\rho\left(\kappa_{1} \times \kappa_{2}\right)\right)$ and $(P, Q)$ is a $\kappa_{1}$-pair, then $(P \otimes I$, $Q \otimes I)$ is a $\rho\left(\kappa_{1} \times \kappa_{2}\right)$-pair and hence

$$
Q\left(I_{\ell^{2}} \otimes L_{\omega}(T)\right) P=\left(\mathrm{id} \otimes L_{\omega}\right)\left((Q \otimes I)\left(I_{\ell^{2}} \otimes T\right)(P \otimes I)\right)=0
$$

by Theorem 3.3, $L_{\omega}(T) \in \mathfrak{M}_{\text {min }}\left(\kappa_{1}\right)$. If $\kappa_{1}$ is an operator $U_{1}$-set, arguments similar to the ones above show that $T=0$ and hence $\rho\left(\kappa_{1} \times \kappa_{2}\right)$ is an operator $U_{1}$-set.

Corollary 5.12. Let $G_{1}$ and $G_{2}$ be locally compact second countable groups and $E_{1} \subseteq G_{1}$, $E_{2} \subseteq G_{2}$ be closed sets. If $E_{1}$ and $E_{2}$ are $M$-sets (resp. $M_{1}$-sets) then $E_{1} \times E_{2}$ is an $M$-set (resp. an $M_{1}$-set).

Proof. Suppose that $E_{1} \subseteq G_{1}$ and $E_{2} \subseteq G_{2}$ are $M$-sets. By Theorem 4.9, $E_{1}^{*}$ and $E_{2}^{*}$ are operator $M$-sets, and by Theorem 5.11, $\rho\left(E_{1}^{*} \times E_{2}^{*}\right)=\left(E_{1} \times E_{2}\right)^{*}$ is an operator $M$-set. By Theorem 4.9 again, $E_{1} \times E_{2}$ is an $M$-set. A similar argument applies to $M_{1}$-sets.

## 6. Sets of finite width

Let $(X, \mu)$ and $(Y, \nu)$ be standard measure spaces. A subset $E \subseteq X \times Y$ is called a set of finite width if there exist measurable functions $f_{i}: X \rightarrow \mathbb{R}, g_{i}: Y \rightarrow \mathbb{R}, i=1, \ldots, n$, such that

$$
\begin{equation*}
E=\left\{(x, y) \in X \times Y: f_{i}(x) \leq g_{i}(y), i=1, \ldots, n\right\} \tag{21}
\end{equation*}
$$

the width of $E$ is the smallest $n$ for which $E$ can be represented in the form (21). By [35, Theorem 4.8] and [40, Theorem 2.1], any such set is operator synthetic. In this section we identify those sets of finite width which are operator $M_{1}$-sets, and hence operator $M$-sets.

We first assume that the measures $\mu$ and $\nu$ are finite and the standard measure spaces $X$ and $Y$ arise from compact topologies. A system is a finite set $D$ of disjoint rectangles $\Pi=\alpha \times \beta$, where $\alpha \subseteq X$ and $\beta \subseteq Y$ are measurable. Set $r(\alpha \times \beta)=\min \{\mu(\alpha), \nu(\beta)\}$. The volume of a system $D=\left\{\Pi_{j}: 1 \leq j \leq J\right\}$ is the number $r(D) \stackrel{\text { def }}{=} \max _{1 \leq j \leq J} r\left(\Pi_{j}\right)$. Let $U_{D}=\bigcup_{j=1}^{J} \Pi_{j}$ and call the systems $D_{1}$ and $D_{2}$ disjoint if $U_{D_{1}} \cap U_{D_{2}}=\emptyset$; in this case, denote by $D_{1} \vee D_{2}$ their union.

With each system $D=\left\{\alpha_{j} \times \beta_{j}: 1 \leq j \leq J\right\}$, we associate the projection $\pi_{D}$ on $\mathcal{B}\left(H_{1}, H_{2}\right)$ by setting

$$
\pi_{D}(T)=\sum_{j=1}^{J} P\left(\beta_{j}\right) T P\left(\alpha_{j}\right), \quad T \in \mathcal{B}\left(H_{1}, H_{2}\right)
$$

It is easy to see that $\pi_{D}$ depends only on $U_{D}$ and that $\pi_{D_{1} \vee D_{2}}=\pi_{D_{1}}+\pi_{D_{2}}$; thus, the mapping $U \rightarrow \pi_{U}$ is a projection-valued measure on the algebra of sets generated by all rectangles. Note that the range of $\pi_{D}$ coincides with $\mathfrak{M}_{\max }\left(U_{D}\right)$.

A system $D=\left\{\alpha_{j} \times \beta_{j}: 1 \leq j \leq J\right\}$ will be called diagonal if $\alpha_{i} \cap \alpha_{j}=\beta_{i} \cap \beta_{j}=\emptyset$ whenever $i \neq j$. The system $D$ will be called $n$-diagonal, if $D=D_{1} \vee D_{2} \vee \cdots \vee D_{n}$ where $D_{1}, \ldots, D_{n}$ are diagonal systems. It is easy to see that $\left\|\pi_{D}\right\|=1$ if $D$ is diagonal. Hence, $\left\|\pi_{D}\right\| \leq n$ if $D$ is $n$-diagonal.

Lemma 6.1. Let $\left(D^{k}\right)_{k \in \mathbb{N}}$ be a sequence of $n$-diagonal systems such that $r\left(D^{k}\right) \rightarrow_{k \rightarrow \infty} 0$. Then $\left\|\pi_{D^{k}}(T)\right\| \rightarrow_{k \rightarrow \infty} 0$ for each compact operator $T$.

Proof. It suffices to prove the statement for rank one operators $T=u \otimes v$ where $u, v$ are bounded functions on $X$ and $Y$, because the set of all linear combinations of such operators is dense in $\mathcal{K}\left(H_{1}, H_{2}\right)$ and the sequence $\left(\pi_{D^{k}}\right)_{k \in \mathbb{N}}$ is uniformly bounded.

If $D=\left\{\alpha_{j} \times \beta_{j}\right\}_{j=1}^{J}$ is a diagonal system, then for $T=u \otimes v$, we have

$$
\begin{aligned}
\left\|\pi_{D}(T)\right\| & \leq\left\|\pi_{D}(T)\right\|_{2}=\left\|\sum_{j=1}^{J}\left(\chi_{\alpha_{j}} \otimes \chi_{\beta_{j}}\right)(u \otimes v)\right\|_{L^{2}(X \times Y, \mu \times \nu)} \\
& \leq\|u\|_{\infty}\|v\|_{\infty}\left(\sum_{j=1}^{J} \mu\left(\alpha_{j}\right) \nu\left(\beta_{j}\right)\right)^{1 / 2} \\
& \leq\|u\|_{\infty}\|v\|_{\infty}\left(\sum_{j=1}^{J} r\left(\Pi_{j}\right)\left(\mu\left(\alpha_{j}\right)+\nu\left(\beta_{j}\right)\right)\right)^{1 / 2} \\
& \leq\|u\|_{\infty}\|v\|_{\infty} r(D)^{1 / 2}(\mu(X)+\nu(Y))^{1 / 2}
\end{aligned}
$$

It follows that if $D$ is an $n$-diagonal system then

$$
\left\|\pi_{D}(T)\right\| \leq n\|u\|_{\infty}\|v\|_{\infty} r(D)^{1 / 2}(\mu(X)+\nu(Y))^{1 / 2}
$$

Hence $\left\|\pi_{D^{k}}(T)\right\| \rightarrow_{k \rightarrow \infty} 0$ whenever $r\left(D^{k}\right) \rightarrow_{k \rightarrow \infty} 0$.
Let us call a set $E n$-quasi-diagonal if for each $\varepsilon>0$ there is an $n$-diagonal system $D$ with $E \subseteq U_{D}$ and $r(D)<\varepsilon$.

We say that a (measurable) function defined on a measure space is non-atomic if it is not constant on any set of positive measure.

Lemma 6.2. Let $f: X \rightarrow \mathbb{R}, g: Y \rightarrow \mathbb{R}$ be Borel maps and assume that $f$ is non-atomic. Then the set

$$
E_{f, g}=\{(x, y) \in X \times Y: f(x)=g(y)\}
$$

is 1-quasi-diagonal.
Proof. Let $\mu_{f}$ be the measure on the Borel $\sigma$-algebra of $\mathbb{R}$ given by $\mu_{f}(C)=\mu\left(f^{-1}(C)\right)$. By our assumption, $\mu_{f}$ is non-atomic and finite. Hence, for every $\varepsilon>0$, there exists a partition $\mathbb{R}=\bigcup_{j=1}^{N} C_{j}$ with $\mu_{f}\left(C_{j}\right)<\varepsilon / \nu(Y)$ for all $j$. In fact, letting $h(x)=$ $\mu_{f}((-\infty, x])$ we have that $h$ is a bounded increasing function such that $h(\mathbb{R}) \subseteq[0, C]$, where $C=\mu_{f}(\mathbb{R})$. As $\mu_{f}$ is non-atomic, $h$ is continuous and $(0, C) \subseteq h(\mathbb{R})$. Let $0=a_{0}<a_{1}<\ldots<a_{N+1}=C$ be a partition of $[0, C]$ such that $a_{i+1}-a_{i}<\varepsilon / \nu(Y)$, $0 \leq i \leq N$, and $h\left(x_{i}\right)=a_{i}, 1 \leq i \leq N$. Set $C_{0}=\left(0, x_{1}\right], C_{i}=\left(x_{i}, x_{i+1}\right]$ if $0<i<N$, and $C_{N}=\left(x_{N}, \infty\right)$. Then $\mathbb{R}=\bigcup_{i=1}^{N} C_{i}$ and $\mu_{f}\left(C_{i}\right)<\varepsilon / \nu(Y), 1 \leq i \leq N$.

Setting $\alpha_{j}=f^{-1}\left(C_{j}\right), \beta_{j}=g^{-1}\left(C_{j}\right)$ and $D=\left\{\alpha_{j} \times \beta_{j}: 1 \leq j \leq N\right\}$, we now see that $D$ is diagonal, $E \subseteq U_{D}$ and $r(D)<\varepsilon$.

Fix $T \in \mathcal{B}\left(H_{1}, H_{2}\right), F \in \mathcal{C}_{1}\left(H_{2}, H_{1}\right)$ and set

$$
\varphi(\Pi)=\left\langle\pi_{\Pi}(T), F\right\rangle
$$

for each rectangle $\Pi \subseteq X \times Y$. We say that $\Pi$ is $\varphi$-null, if $\varphi\left(\Pi^{\prime}\right)=0$ for all rectangles $\Pi^{\prime} \subseteq \Pi$.

Lemma 6.3. If $\Pi=\bigcup_{j=1}^{\infty} \Pi_{j}$ and each $\Pi_{j}$ is $\varphi$-null then $\Pi$ is $\varphi$-null.
Proof. It suffices to show that $\varphi(\Pi)=0$. Without loss of generality we may assume that all $\Pi_{j}$ are mutually disjoint.

By Lemma 2.2, for each $\varepsilon$, there are $X_{\varepsilon} \subseteq X$ and $Y_{\varepsilon} \subseteq Y$ such that $\mu\left(X \backslash X_{\varepsilon}\right)<\varepsilon$, $\nu\left(Y \backslash Y_{\varepsilon}\right)<\varepsilon$ and the rectangle $\Pi^{\varepsilon}=\Pi \cap\left(X_{\varepsilon} \times Y_{\varepsilon}\right)$ is covered by a finite number of rectangles $\Pi_{j}$, say, $\Pi^{\varepsilon} \subseteq \bigcup_{j=1}^{m} \Pi_{j}$. Set $\Pi_{j}^{\epsilon}=\Pi_{j} \cap\left(X_{\epsilon} \times Y_{\epsilon}\right)$; we have

$$
\varphi\left(\Pi^{\varepsilon}\right)=\sum_{j=1}^{m} \varphi\left(\Pi_{j}^{\varepsilon}\right)=0
$$

On the other hand, if $\Pi=\alpha \times \beta$ then $\varphi\left(\Pi^{\varepsilon}\right)=\left\langle P\left(Y_{\varepsilon}\right) P(\beta) T P(\alpha) P\left(X_{\varepsilon}\right), F\right\rangle$ and, since $P\left(X_{\varepsilon}\right) \rightarrow I, P\left(Y_{\varepsilon}\right) \rightarrow I$ in the strong operator topology, we conclude that $\lim _{\varepsilon \rightarrow 0} \varphi\left(\Pi^{\varepsilon}\right)=\varphi(\Pi)$. Thus, $\varphi(\Pi)=0$ and the proof is complete.

Theorem 6.4. If $E$ is a set of finite width then $\mathfrak{M}_{\max }(E) \cap \mathcal{K}$ coincides with the normclosure $\mathfrak{M}_{0}(E)$ of the subspace of $\mathfrak{M}_{\max }(E)$ generated by its rank one operators.

Proof. We may assume that the measures $\mu$ and $\nu$ are finite and the standard spaces $X$ and $Y$ arise from compact topologies. Indeed, if this is not the case, write $X=\bigcup_{n=1}^{\infty} X_{n}$ and $Y=\bigcup_{n=1}^{\infty} Y_{n}$ as increasing unions, where $X_{n}$ and $Y_{n}$ are compact, $\mu\left(X_{n}\right)<\infty$ and $\nu\left(Y_{n}\right)<\infty$. Then $P\left(X_{n}\right) \rightarrow_{n \rightarrow \infty} I$ and $P\left(Y_{n}\right) \rightarrow_{n \rightarrow \infty} I$ in the strong operator topology. If $T \in \mathfrak{M}_{\max }(E) \cap \mathcal{K}$ then $P\left(Y_{n}\right) T P\left(X_{n}\right) \rightarrow_{n \rightarrow \infty} T$ in norm, and hence we may restrict our attention to each of $E \cap\left(X_{n} \times Y_{n}\right)$, which is a set of finite width when considered as a subset of $X_{n} \times Y_{n}$.

We use induction on the width $n$ of $E$. With the convention that all measurable rectangles are sets of width zero, the statement clearly holds for $n=0$. Suppose that the assertion of the theorem is true for sets of width smaller than $n$, and let

$$
E=\left\{(x, y) \in X \times Y: f_{j}(x) \leq g_{j}(y), j=1, \ldots, n\right\}
$$

where $f_{j}: X \rightarrow \mathbb{R}$ and $g_{j}: Y \rightarrow \mathbb{R}$ are measurable functions, $j=1, \ldots, n$. Let $F \in$ $\Gamma(X, Y)$ be in the annihilator of $\mathfrak{M}_{0}(E)$. We need to show that $\langle T, F\rangle=0$ for each compact operator $T \in \mathfrak{M}_{\max }(E)$. Assume first that all $f_{j}, j=1, \ldots, n$, are non-atomic. By Lemma 6.2, the sets

$$
E_{j}=\left\{(x, y): f_{j}(x)=g_{j}(y)\right\}, \quad j=1, \ldots, n
$$

are 1-quasi-diagonal and hence their union $\bigcup_{j=1}^{n} E_{j}$ is $n$-quasi-diagonal. Let $E^{\prime}=E \cap$ $\left(\bigcup_{j=1}^{n} E_{j}\right)$; then $E^{\prime}$ is $n$-quasi-diagonal and

$$
E^{\prime \prime} \stackrel{\text { def }}{=} E \backslash E^{\prime}=\left\{(x, y): f_{j}(x)<g_{j}(y), j=1, \ldots, n\right\}
$$

is $\omega$-open.
Let $D$ be an $n$-diagonal system with $E^{\prime} \subseteq U_{D}$. If $\Pi$ is a rectangle, disjoint from $U_{D}$, then $\Pi \subseteq E^{\prime \prime} \cup E^{c}$; since both $E^{\prime \prime}$ and $E^{c}$ are $\omega$-open, $\Pi=\bigcup_{i=1}^{\infty} \Pi_{i}$ where each of $\Pi_{i}$ is either a subset of $E^{\prime \prime}$ or of $E^{c}$.

Let, as above, $\varphi(\alpha \times \beta)=\langle P(\beta) T P(\alpha), F\rangle$, where $\alpha \subseteq X$ and $\beta \subseteq Y$ are measurable. If $\Pi_{i} \subseteq E^{c}$ and $\Pi_{i}^{\prime} \subseteq \Pi$ then $\varphi\left(\Pi_{i}^{\prime}\right)=0$ by the fact that $T$ is supported on $E$.

On the other hand, if $\Pi_{i}=\alpha_{i} \times \beta_{i} \subseteq E^{\prime \prime}$ then, clearly, $\Pi_{i} \subseteq E$ whence $P\left(\beta_{i}\right) T P\left(\alpha_{i}\right) \in$ $\mathfrak{M}_{0}\left(\Pi_{i}\right) \subseteq \mathfrak{M}_{0}(E)$. It follows that $\varphi\left(\Pi_{i}\right)=0$. The same argument shows that $\varphi\left(\Pi_{i}^{\prime}\right)=0$ whenever $\Pi_{i}^{\prime}$ is a rectangle with $\Pi_{i}^{\prime} \subseteq \Pi_{i}$, and hence $\Pi_{i}$ is $\varphi$-null. By Lemma $6.3, \Pi$ is $\varphi$-null. We thus showed that every rectangle disjoint from $U_{D}$ is $\varphi$-null.

Let $\widetilde{D}=\left\{\Pi_{k}^{\prime}: 1 \leq k \leq m\right\}$ be a system such that $\left(U_{D}\right)^{c}=U_{\widetilde{D}}$. It follows from the previous paragraphs that

$$
\left\langle\pi_{\widetilde{D}}(T), F\right\rangle=\sum_{k=1}^{m} \varphi\left(\Pi_{k}^{\prime}\right)=0 .
$$

Hence

$$
\langle T, F\rangle=\left\langle\pi_{D}(T), F\right\rangle+\left\langle\pi_{\tilde{D}}(T), F\right\rangle=\left\langle\pi_{D}(T), F\right\rangle
$$

and $|\langle T, F\rangle| \leq\|F\|\left\|\pi_{D}(T)\right\|$. Since $E^{\prime}$ is $n$-quasi-diagonal, there exists a sequence $\left(D^{k}\right)_{k \in \mathbb{N}}$ of $n$-diagonal systems such that $E^{\prime} \subseteq U_{D^{k}}$ for each $k$ and $r\left(D^{k}\right) \rightarrow_{k \rightarrow \infty} 0$. By Lemma 6.1, $\left\|\pi_{D^{k}}(T)\right\| \rightarrow_{k \rightarrow \infty} 0$ and thus $\langle T, F\rangle=0$.

Now let $f_{j}$ be arbitrary. Then we can write $X$ as a disjoint union $\bigcup_{k=0}^{\omega} X_{k}, \omega \leq \infty$, where $X_{0}$ is a subset of $X$ such that all $f_{j}$ are non-atomic on $X_{0}$ and for each $k>0$ at least one of the functions $f_{j}$ is constant on $X_{k}$.

Set $P_{k}=P\left(X_{k}\right), F_{k}(x, y)=\chi_{X_{k}}(x) F(x, y)$ and $T_{k}=T P_{k}$; then $\langle T, F\rangle=$ $\sum_{k=0}^{\omega}\left\langle T_{k}, F_{k}\right\rangle$ and it hence suffices to show that $\left\langle T_{k}, F_{k}\right\rangle=0$ for each $k$. It is clear that $T_{k}$ is supported on $E_{k} \stackrel{\text { def }}{=} E \cap\left(X_{k} \times Y\right)$ and $F_{k}$ annihilates $\mathfrak{M}_{0}\left(E_{k}\right)$.

The equality $\left\langle T_{0}, F_{0}\right\rangle=0$ follows from the first part of the proof. Let $k>0$, and suppose, for example, that the function $f_{1}$ is constant on $X_{k}: f_{1}(x)=a$, for $x \in X_{k}$. Set $Y_{k}=\left\{y \in Y: g_{1}(y) \geq a\right\}$. Then

$$
E_{k}=\left\{(x, y) \in X_{k} \times Y_{k}: f_{j}(x) \leq g_{j}(y), j=2, \ldots, n\right\}
$$

Thus $E_{k}$ is a set of width at most $n-1$, when considered as a subset of $X_{k} \times Y_{k}$. Since $T$ is supported on $E_{k}$, we have $T_{k}=P\left(Y_{k}\right) T_{k}$. Moreover, $\chi_{X_{k} \times Y_{k}} F_{k}$ annihilates $\mathfrak{M}_{0}\left(E_{k}\right)$ and hence

$$
\left\langle T_{k}, F_{k}\right\rangle=\left\langle P\left(Y_{k}\right) T_{k}, \chi_{X_{k} \times Y_{k}} F_{k}\right\rangle=0
$$

by the inductive assumption.

Corollary 6.5. Let $E$ be a set of finite width. The following conditions are equivalent:
(i) $E$ is an operator $U$-set;
(ii) E does not support a non-zero Hilbert-Schmidt operator;
(iii) $\mu \times \nu(E)=0$;
(iv) E does not support a non-zero nuclear operator;
(v) $E$ does not contain a rectangle of non-zero measure.

Proof. We may assume that $\mu$ and $\nu$ are finite, for if $X=\bigcup_{k=1}^{\infty} X_{k}, Y=\bigcup_{k=1}^{\infty} Y_{k}$, where $\left(X_{k}\right)_{k=1}^{\infty}$ and $\left(Y_{k}\right)_{k=1}^{\infty}$ are increasing sequences of subsets of finite measure and $T \in \mathcal{B}\left(H_{1}, H_{2}\right)$ is a non-zero compact operator supported in $E$, then so is $P\left(Y_{k}\right) T P\left(X_{k}\right)$ for some $k$.
(i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iii) If $\mu \times \nu(E)$ were non-zero, then $T_{k}$, where $k(x, y)=\chi_{E}(x, y)$, would be a non-zero Hilbert-Schmidt operator supported in $E$.
(iii) $\Rightarrow$ (iv) If $E$ supports a non-zero nuclear operator then by [11, Theorem 6.7], $E$ supports a non-zero rank one operator $u \otimes v, u \in L^{2}(X, \mu), v \in L^{2}(Y, \nu)$. As $u \otimes v$ is supported on supp $u \times \operatorname{supp} v$, we have $\mu \times \nu(E) \neq 0$, a contradiction.
(iv) $\Rightarrow(\mathrm{v})$ If $E$ contains a non-zero rectangle $\alpha \times \beta$ then $\chi_{\alpha} \otimes \chi_{\beta}$ is a non-zero nuclear operator supported in $E$, a contradiction.
$(\mathrm{v}) \Rightarrow$ (i) If $E$ supports a non-zero compact operator then it follows from Theorem 6.4 that there exists a non-zero rank one operator $u \otimes v$ supported in $E$. But then $\operatorname{supp} u \times$ $\operatorname{supp} v$ is a non-zero rectangle contained in $E$, a contradiction.

Remark. We note that the conditions from Corollary 6.5 are also equivalent to the set $E$ being a $U_{1}$-set, as well as to $E$ being a $U_{0}$-set.

We have the following immediate corollary.
Corollary 6.6. A non-zero bounded operator from $L^{2}\left(\mathbb{R}^{n}\right)$ to $L^{2}\left(\mathbb{R}^{m}\right)$ cannot be compact if it is supported on a manifold of the form $y_{j}=\phi\left(x_{1}, \ldots, x_{n}\right)$, for some measurable function $\phi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ and some $j=1, \ldots, m$, or on a set that can be partitioned into finitely many such sets.

In particular, the support of a non-zero compact operator from $L^{2}\left(\mathbb{R}^{n}\right)$ to $L^{2}\left(\mathbb{R}^{1}\right)$ is not contained in a smooth manifold of dimension strictly less than $n+1$.

Proof. Assume, without loss of generality, that $j=1$. Let $\psi: \mathbb{R}^{m} \rightarrow \mathbb{R}$ be given by $\psi\left(y_{1}, \ldots, y_{m}\right)=y_{1}$ and $E=\left\{\left(x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}\right) \in \mathbb{R}^{n} \times \mathbb{R}^{m}: y_{1}=\phi\left(x_{1}, \ldots, x_{n}\right)\right\}=$ $\left\{(x, y) \in \mathbb{R}^{n} \times \mathbb{R}^{m}: \psi(y)=\phi(x)\right\}$. As $\psi$ is non-atomic, Lemma 6.2 implies that $E$ is 1-diagonal. By Lemma 6.1, $E$ does not support a non-zero compact operator. By [34, Proposition 5.3] there is no non-zero compact operator supported on a set that can be partitioned into finitely many sets of the form $\left\{\left(x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}\right): y_{j}=\right.$ $\left.\phi\left(x_{1}, \ldots, x_{n}\right)\right\}$.

If $\omega: G \rightarrow \mathbb{R}^{+}$is a continuous homomorphism, let $E_{\omega, t}=\{s \in G: \omega(s) \leq t\}$; we call the subsets of $G$ of this form level sets (see [9]).

Corollary 6.7. Let $E_{j} \subseteq G, j=1, \ldots, n$, be level sets. The set $E \stackrel{\text { def }}{=} \bigcap_{j=1}^{n} E_{j}$ is an $M$-set if and only if $m(E)>0$.

Proof. It is straightforward to check that if $F \subseteq G$ is a level set then $F^{*}$ is a set of width one. It follows that $E^{*}$ is a set of finite width. By Theorem 4.9 and Corollary $6.5, E$ is an $M$-set if and only if $(m \times m)\left(E^{*}\right)>0$. This condition is equivalent to $m(E)>0$ by the identity

$$
(m \times m)\left(E^{*}\right)=\int_{G} m(E t) d t=\int_{G} \Delta(t) m(E) d t
$$

where $\Delta$ is the modular function.

## 7. Closable multipliers on group $C^{*}$-algebras

Let $G$ be a locally compact group equipped with left Haar measure $m$ and $\psi: G \rightarrow \mathbb{C}$ be a measurable function. It is well-known [6,19] (see also [31, Theorem 8.3]) that pointwise multiplication on $L^{1}(G)$ by the function $\psi$ defines a completely bounded map on $C_{r}^{*}(G)$ if and only if the function $N(\psi)$ is a Schur multiplier. In this section, we prove a version of this result for closable maps (see Theorem 7.4).

Let

$$
D(\psi)=\left\{f \in L^{1}(G): \psi f \in L^{1}(G)\right\}
$$

it is easy to see that the operator $f \rightarrow \psi f, f \in D(\psi)$, viewed as a densely defined operator on $L^{1}(G)$, is closable. Since $\lambda\left(L^{1}(G)\right)$ is dense in $C_{r}^{*}(G)$ and $\|\lambda(f)\| \leq\|f\|_{1}$, $f \in L^{1}(G)$, the space $\lambda(D(\psi))$ is dense in $C_{r}^{*}(G)$ in the operator norm. Thus, the operator $S_{\psi}: \lambda(D(\psi)) \rightarrow C_{r}^{*}(G)$ given by $S_{\psi}(\lambda(f))=\lambda(\psi f)$ is a densely defined operator on $C_{r}^{*}(G)$.

We wish to study the question of when $S_{\psi}$ is closable. To this end, we recall [12] that the Banach space dual of $C_{r}^{*}(G)$ can be canonically identified with the weak* closure $B_{\lambda}(G)$ of $A(G)$ within the Fourier-Stieltjes algebra $B(G)$. A direct verification shows that the domain of $S_{\psi}^{*}$ is equal to

$$
J_{\psi}^{\lambda} \stackrel{\text { def }}{=} J_{\psi}^{B_{\lambda}(G)}=\left\{g \in B_{\lambda}(G): \psi g \in^{m} B_{\lambda}(G)\right\}
$$

and that $S_{\psi}^{*}(g)$ is equivalent to $\psi g$ for every $g \in J_{\psi}^{\lambda}$. By Proposition 2.1, $S_{\psi}$ is closable (resp. weak* closable) if and only if $J_{\psi}^{\lambda}$ is weak* dense (resp. norm dense) in $B_{\lambda}(G)$. We denote by $\operatorname{Clos}(G)$ the set of all measurable functions $\psi$ for which $S_{\psi}$ is closable and call the elements of $\operatorname{Clos}(G)$ closable multipliers on $C_{r}^{*}(G)$. This notion of multipliers should not be confused with the notion of multipliers in the $C^{*}$-algebra sense.

A function $f$ on $G$ is said to belong to $A(G)$ (resp. almost belong to $A(G)$ ) at the point $t \in G$ if there exists a neighbourhood $U$ of $t$ and a function $u \in A(G)$ such that $f(s)=u(s)$ for all (resp. $m$-almost all) points $s \in U$. Set [34]

$$
E_{f}=\{t \in G: f \text { does not almost belong to } A(G) \text { at } t\} .
$$

We say that $f$ (almost) belongs locally to $A(G)$ if $f$ (almost) belongs to $A(G)$ at every point and let $A(G)^{\text {loc }}$ be the set of functions which belong to $A(G)$ at every point. If $f$ almost belongs to $A(G)$ at every point then $f$ is equivalent to a function from $A(G)^{\mathrm{loc}}$. To see this, we first show that, given a compact set $K \subseteq G$ and a function $f$ that almost belong to $A(G)$ at each point of $G$, there exists $g \in A(G)$ such that $f$ is equivalent to $g$ on $K$. In fact, for each $t \in G$ there exists a neighbourhood $V_{t}$ of $t$ and $g_{t} \in A(G)$ such that $f \sim g_{t}$ on $V_{t}$. Then $K \subseteq \bigcup_{t \in F} V_{t}$ for some finite $F \subseteq K$. By the regularity of $A(G)$, there exist $h_{t} \in A(G), t \in F$, such that $\sum_{t \in F} h_{t}(x)=1$ if $x \in K$ and $h_{t}(x)=0$ if $x \notin V_{t}$, $t \in F$. Hence, for almost all $x \in K$, we have $f(x)=\sum_{t \in F} f(x) h_{t}(x)=\sum_{t \in F} g_{t}(x) h_{t}(x)$, while $\sum_{t \in F} g_{t} h_{t} \in A(G)$. As the group $G$ is $\sigma$-compact we can find compact subsets $K_{n} \subseteq G, K_{n} \subseteq K_{n+1}$ such that $G=\bigcup_{n=1}^{\infty} K_{n}$, and a sequence of functions $g_{n} \in A(G)$ such that $f \sim g_{n}$ on $K_{n}$ for any $n$. As $g_{n}$ are continuous, we obtain $g_{n+1}=g_{n}$ on $K_{n}$. Define a function $g$ by letting $g(x)=g_{n}(x)$ if $x \in K_{n}$. Then $g$ is well-defined, continuous and $f \sim g$. Clearly, $g$ belongs to $A(G)$ at every point of $G$.

The following fact was established in [34] in the case $G$ is abelian; its proof, however, does not use the commutativity of $G$.

Lemma 7.1. (See [34].) For every measurable function $\psi: G \rightarrow \mathbb{C}$, let

$$
J_{\psi} \stackrel{\text { def }}{=} J_{\psi}^{A(G)}=\left\{f \in A(G): \psi f \in^{m} A(G)\right\}
$$

Then $E_{\psi}=\operatorname{null} J_{\psi}$.
We say that a locally compact group $G$ has property (A) if there exists a net $\left(u_{i}\right) \subseteq A(G)$ such that for each $g \in B_{\lambda}(G), u_{i} g \rightarrow g$ in the weak*-topology of $B_{\lambda}(G)$. Note that if $\left(u_{i}\right) \subseteq A(G)$ is a net such that $u_{i} \rightarrow 1$ uniformly on compact sets and $\sup \left\|u_{i}\right\|_{M A(G)}<\infty$ (in particular, if $G$ is weakly amenable) then $G$ has property (A). In fact, for $g \in B_{\lambda}(G)$ and $f \in C_{c}(G)$, we have

$$
\left\langle\lambda(f), g u_{i}-g\right\rangle=\int_{G} f(t) g(t)\left(u_{i}(t)-1\right) d t \rightarrow 0
$$

It follows from [7, Proposition 1.2] that, if $u \in A(G)$ then $\|u\|_{M A(G)}$ coincides with the norm of $u$ as a multiplier of $B_{\lambda}(G)$. Thus, $\left\|g u_{i}-g\right\|_{B(G)} \leq\left\|u_{i}\right\|_{M A(G)}\|g\|_{B(G)}+\|g\|_{B(G)}$. The statement now follows from the fact that the set of all $\lambda(f), f \in C_{c}(G)$, is dense in $C_{r}^{*}(G)$.

Since $C_{r}^{*}(G)^{*}=B_{\lambda}(G)$ and $A(G) \subseteq B_{\lambda}(G)$, the elements of $C_{r}^{*}(G)$ can be identified with functionals on $A(G)$ continuous with respect to the restriction to $A(G)$ of the weak* topology of $B_{\lambda}(G)$; this identification is made in the next proposition.

Proposition 7.2. Let $G$ be a locally compact group with property (A) and $\psi: G \rightarrow \mathbb{C}$ be a measurable function. The operator $S_{\psi}$ is closable if and only if there is no non-zero operator $T \in C_{r}^{*}(G)$ which annihilates $J_{\psi}$. In particular,
(i) if $E_{\psi}$ is a $U$-set then $S_{\psi}$ is closable;
(ii) if $E_{\psi}$ is an $M_{1}$-set then $S_{\psi}$ is not closable.

Proof. Since $A(G)$ is an ideal in $B(G)$, property (A) implies that the weak* closures of $J_{\psi}$ and $J_{\psi}^{\lambda}$ in $B_{\lambda}(G)$ coincide. The first statement now follows from Proposition 2.1.

By Lemma 7.1,

$$
J\left(E_{\psi}\right) \subseteq \overline{J_{\psi}} \subseteq I\left(E_{\psi}\right)
$$

Parts (i) and (ii) follow from these inclusions and the definitions of a $U$-set and an $M_{1}$-set.

Corollary 7.3. Let $G$ be a locally compact group with property (A) and $\psi: G \rightarrow \mathbb{C}$ be a measurable function. If $m\left(E_{\psi}\right)>0$ then $S_{\psi}$ is not closable.

Proof. By Remark 4.3, $E_{\psi}$ is an $M_{1}$-set. Now the claim follows from Proposition 7.2 (ii).

Recall from Section 2 that, for a measurable function $\varphi: G \times G \rightarrow \mathbb{C}$, we let $S_{\varphi}$ be the operator, densely defined on $\mathcal{K}\left(L^{2}(G)\right)$, with domain

$$
D\left(S_{\varphi}\right)=\left\{T_{k} \in \mathcal{C}_{2}\left(H_{1}, H_{2}\right): \hat{\varphi} k \in L^{2}(G \times G)\right\} \subseteq \mathcal{K}\left(L^{2}(G)\right)
$$

It was shown in [34] that the domain $D\left(S_{\varphi}^{*}\right) \subseteq \Gamma(G, G)$ of its adjoint is given by

$$
D\left(S_{\varphi}^{*}\right)=\left\{h \in \Gamma(G, G): \varphi h \in^{m \times m} \Gamma(G, G)\right\} .
$$

Theorem 7.4. Let $G$ be a second countable locally compact group with property (A), $\psi: G \rightarrow \mathbb{C}$ be a measurable function and $\varphi=N(\psi)$. The following are equivalent:
(i) the operator $S_{\psi}$ is closable;
(ii) the operator $S_{\varphi}$ is closable;
(iii) $\mathcal{A} \cap D\left(S_{\varphi}^{*}\right)^{\perp}=\{0\}$;
(iv) $\mathcal{R} \cap D\left(S_{\varphi}^{*}\right)^{\perp}=\{0\}$.

Proof. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) follows from the fact that $\mathcal{K} \subseteq \mathcal{A} \subseteq \mathcal{R}$ and the fact that $S_{\varphi}$ is closable if and only if $\mathcal{K} \cap D\left(S_{\varphi}^{*}\right)^{\perp}=\{0\}$.
(ii) $\Rightarrow$ (i) If $S_{\psi}$ is not closable then, by Proposition 7.2 , there exists a non-zero $T \in C_{r}^{*}(G)$ which annihilates $J_{\psi}$. Let $A \in \mathcal{D}_{0}$ be such that $A T \neq 0$. In view of (7), it suffices to show that $A T$ annihilates $D\left(S_{\varphi}^{*}\right)$. Since $D\left(S_{\varphi}^{*}\right)$ is invariant under $\mathfrak{S}(G, G)$, it suffices to show that $T$ annihilates $D\left(S_{\varphi}^{*}\right)$.

Let $h \in D\left(S_{\varphi}^{*}\right)$. Writing $h=\sum_{i=1}^{\infty} f_{i} \otimes g_{i}$, where $\sum_{i=1}^{\infty}\left\|f_{i}\right\|_{2}^{2}<\infty$ and $\sum_{i=1}^{\infty}\left\|g_{i}\right\|_{2}^{2}<\infty$, and using Lemma 2.3, for every $f \in L^{1}(G)$, we have

$$
\begin{aligned}
\langle\lambda(f), h\rangle & =\left\langle\lambda(f), \sum_{i=1}^{\infty} f_{i} \otimes g_{i}\right\rangle=\sum_{i=1}^{\infty}\left(\lambda(f)\left(f_{i}\right), \bar{g}_{i}\right) \\
& =\iint f(s) \sum_{i=1}^{\infty} g_{i}(t) f_{i}\left(s^{-1} t\right) d t d s=\langle\lambda(f), P(h)\rangle .
\end{aligned}
$$

It follows that $\langle T, h\rangle=\langle T, P(h)\rangle$. Since $\varphi h \in^{m \times m} \Gamma(G, G)$, identity (2) implies that $\psi P(h)=P(\varphi h) \in^{m} P(\Gamma(G, G))=A(G)$, and hence $P(h) \in J_{\psi}$. Thus, $\langle T, P(h)\rangle=0$ and hence $\langle T, h\rangle=0$.
(i) $\Rightarrow$ (iv) Let $S_{\psi}$ be closable and suppose that $0 \neq T \in \mathcal{R} \cap D\left(S_{\varphi}^{*}\right)^{\perp}$. By Lemma 4.7, there exist $a, b \in L^{2}(G)$ such that $E_{a \otimes b}(T) \neq 0$. Suppose that $u \in J_{\psi}^{\lambda}$; then

$$
\varphi(a \otimes b) N(u)=(a \otimes b) N(\psi u) \in \Gamma(G, G)
$$

and hence $(a \otimes b) N(u) \in D\left(S_{\varphi}^{*}\right)$. Thus

$$
\left\langle E_{a \otimes b}(T), u\right\rangle=\langle T,(a \otimes b) N(u)\rangle=0 .
$$

By Theorem 4.6, $E_{a \otimes b}(T)$ is a (non-zero) element of $C_{r}^{*}(G)$; in view of Proposition 7.2, this contradicts the closability of $S_{\psi}$.

Corollary 7.5. The set $\operatorname{Clos}(G)$ is an algebra under pointwise addition and multiplication.
Proof. Let $\psi_{i} \in \operatorname{Clos}(G), i=1,2$. Then $N \psi_{1}+N \psi_{2}=N\left(\psi_{1}+\psi_{2}\right)$ and $N\left(\psi_{1} \psi_{2}\right)=$ $\left(N \psi_{1}\right)\left(N \psi_{2}\right)$. By [34, Theorem 5.2], the closable multipliers on $\mathcal{K}\left(L^{2}(G)\right)$ form an algebra under pointwise addition and multiplication. The claim now follows from Theorem 7.4.

We now give some examples of closable and non-closable multipliers.
Example 7.6 (A non-closable multiplier on $C_{r}^{*}(\mathbb{T})$ ). Using the arguments in [33, 7.8.3-7.8.6] (see also [36, Proposition 9.9]), one can show that there exist $c=\left(c_{n}\right)_{n \in \mathbb{Z}} \in$ $\ell^{p}(\mathbb{Z}), p>2$, and $d=\left(d_{n}\right)_{n \in \mathbb{Z}} \in \ell^{1}(\mathbb{Z})$ with $\bar{d}_{n} \stackrel{\text { def }}{=} \bar{d}_{-n}, n \in \mathbb{Z}$, such that $c * d=0$ and $c * \bar{d} \neq 0$. Let $f \in A(\mathbb{T}) \subseteq L^{1}(\mathbb{T})$ be the function whose sequence of Fourier coefficients
is $d$ and $F$ be the pseudo-function (that is, the bounded linear functional on $A(\mathbb{T})$ ) whose sequence of Fourier coefficients is $c$. We have $f \cdot F=0$ while $\bar{f} \cdot F \neq 0$. After identifying the dual of $A(\mathbb{T})$ with $\mathrm{VN}(\mathbb{T})$, we view $F$ as the operator on $L^{2}(\mathbb{T})$ determined by the identities $\widehat{F \xi}=c \hat{\xi}, \xi \in L^{2}(\mathbb{T})$ (where $\hat{\eta}$ denotes the Fourier transform of a function $\eta \in L^{2}(\mathbb{T})$ ). Moreover (see the start of Section 3), $F \in C_{r}^{*}(\mathbb{T})$. Let $h_{n} \in L^{1}(\mathbb{T})$ be such that $\lambda\left(h_{n}\right) \rightarrow_{n \rightarrow \infty} F$ in the operator norm. Then

$$
\left\|\lambda\left(h_{n}\right)-F\right\|=\sup \left\{\left|\left\langle\lambda\left(h_{n}\right)-F, u\right\rangle\right|: u \in A(\mathbb{T}),\|u\|=1\right\} \rightarrow_{n \rightarrow \infty} 0
$$

It follows that

$$
\sup \left\{\left|\left\langle\lambda\left(f h_{n}\right)-f \cdot F, u\right\rangle\right|: u \in A(\mathbb{T}),\|u\|=1\right\} \rightarrow_{n \rightarrow \infty} 0
$$

which in turn implies that $\lambda\left(f h_{n}\right) \rightarrow f \cdot F$ in the operator norm. Similarly, $\lambda\left(\bar{f} h_{n}\right) \rightarrow_{n \rightarrow \infty}$ $\bar{f} \cdot F$.

Let $\psi: \mathbb{T} \rightarrow \mathbb{C}$ be given by $\psi(t)=\bar{f}(t) / f(t)$ if $f(t) \neq 0$ and $\psi(t)=0$ otherwise. Then

$$
S_{\psi}\left(\lambda\left(f h_{n}\right)\right)=\lambda\left(\psi f h_{n}\right)=\lambda\left(\bar{f} h_{n}\right) \oiint_{n \rightarrow \infty} 0
$$

while $\lambda\left(f h_{n}\right) \rightarrow 0$. Hence $\psi$ is a non-closable multiplier.
Example 7.7 ( $A$ continuous non-closable multiplier on $C_{r}^{*}(\mathbb{T})$ ). The following example was given in [34]. We recall the construction for completeness. Let $X \subseteq \mathbb{T}$ be a closed set of positive Lebesgue measure and $S \subseteq X$ be a dense subset of Lebesgue measure zero. By [23, Chapter II, Theorem 3.4], there exists $h \in C(\mathbb{T})$ whose Fourier series diverges at every point of $S$. By the Riemann Localisation Principle, any function which belongs locally to $A(\mathbb{T})$ at $t \in \mathbb{T}$ has a convergent Fourier series at $t$; hence, $S \subseteq E_{h}$ and since $E_{h}$ is closed, $X \subseteq E_{h}$. Therefore $m\left(E_{h}\right)>0$ and $S_{h}$ is not closable by Corollary 7.3.

Example 7.8 ( $A$ class of idempotent closable multipliers on $C_{r}^{*}(\mathbb{R})$ ). Let $F \subseteq \mathbb{R}$ be a closed set which is the union of countably many intervals. We claim that $\chi_{F} \in \operatorname{Clos}(\mathbb{R})$. Let $\psi=\chi_{F}$; then $E_{\psi}$ is the set of boundary points of $F$. Thus $E_{\psi}$ is contained in the set of endpoints of the intervals whose unions are $F$, and hence $E_{\psi}$ is countable. The claim now follows from Proposition 7.2 and Corollary 5.3.

This example should be compared with the well-known fact that there are no bounded non-trivial idempotent multipliers on $C_{r}^{*}(\mathbb{R})$.

We next discuss the weak ${ }^{* *}$ closability of the operator $S_{\psi}$ (in the sense of Section 2.1). We have the following necessary condition.

Proposition 7.9. If $S_{\psi}$ is weak ${ }^{* *}$ closable then $\psi \in A(G)^{\text {loc }}$.

Proof. Suppose that $S_{\psi}$ is weak ${ }^{* *}$ closable. By Proposition 2.1, $J_{\psi}^{\lambda}$ is dense in $B_{\lambda}(G)$. Thus, $A(G) J_{\psi}^{\lambda}$ is dense in $A(G) B_{\lambda}(G)=A(G)$. However, $A(G) J_{\psi}^{\lambda} \subseteq J_{\psi}$ and hence $J_{\psi}$ is dense in $A(G)$. By Lemma 7.1, $\psi \in A(G)^{\text {loc }}$.

We point out that the converse of Proposition 7.9 does not hold for non-compact groups. In fact, let $G$ be a non-discrete locally compact abelian group with dual group $\Gamma$. Then $B_{\lambda}(\Gamma)=B(\Gamma) \neq A(\Gamma)$. By [15, Corollary 8.2.6], there exists $f \in B(\Gamma),|f(x)|>1$, $x \in \Gamma$, such that $\psi \stackrel{\text { def }}{=} \frac{1}{f} \notin B(\Gamma)$; on the other hand, $\psi \in A(\Gamma)^{\mathrm{loc}}$ (see the arguments in [34, Remark 7.11]). We have that $J_{\psi}^{\lambda} \subseteq(f)$, where $(f)$ is the ideal in $B(\Gamma)$ generated by $f$. As $f$ is not invertible in $B(\Gamma),(f)$ is contained in a maximal ideal, and hence cannot be dense in $B(\Gamma)$.

It follows that a version of Theorem 7.4, with weak** closability in the place of closability, does not hold. Indeed, by [34, Theorem 7.8], for abelian groups, $N(\psi)$ is a weak ${ }^{* *}$ closable multiplier if and only if $\psi \in A(G)^{\text {loc }}$. In view of these remarks, the following question arises.

Question. Is $S_{\psi}$ weak ${ }^{* *}$ closable only when $S_{\psi}$ is bounded?
Note that if $G$ is compact then $S_{\psi}$ is weak** closable if and only if $S_{\psi}$ is bounded, that is, if and only if $\psi \in A(G)$; this follows from Proposition 7.9 and the fact that in this case $A(G)=A(G)^{\mathrm{loc}}$.

## 8. Closable multipliers on group von Neumann algebras

In this section we turn our attention to multipliers acting on $\operatorname{VN}(G)$. We will need an appropriate version of closability suited for working with dual spaces, which we now introduce. Let $\mathcal{X}$ and $\mathcal{Y}$ be dual Banach spaces, with specified preduals $\mathcal{X}_{*}$ and $\mathcal{Y}_{*}$, respectively, and $D(\Phi) \subseteq \mathcal{X}$ be a weak* dense subspace. We say that an operator $\Phi: D(\Phi) \rightarrow \mathcal{Y}$ is weak* closable if the conditions $x_{i} \in \mathcal{X}, y \in \mathcal{Y}, x_{i} \rightarrow_{w^{*}} 0, \Phi\left(x_{i}\right) \rightarrow_{w^{*}} y$ imply that $y=0$. Here, the weak ${ }^{*}$ convergence is in the designated weak* topologies of $\mathcal{X}$ and $\mathcal{Y}$.

Note that, since the ${ }^{*}$-weak closure of the graph of $\Phi$ contains its norm-closure, each weak* closable operator is closable.

We have the following characterisation of weak* closability.

Proposition 8.1. Let $D(\Phi) \subseteq \mathcal{X}$ be a weak* dense subspace and $\Phi: D(\Phi) \rightarrow \mathcal{Y}$ be a linear operator. The following are equivalent:
(i) the operator $\Phi$ is weak* closable;
(ii) the space $D_{*}(\Phi)=\left\{g \in \mathcal{Y}_{*}: x \rightarrow\langle\Phi(x), g\rangle\right.$ is $w^{*}$-cont. on $\left.D(\Phi)\right\}$ is dense in $\mathcal{Y}_{*}$.

Proof. (ii) $\Rightarrow$ (i) Suppose that $x_{i} \rightarrow 0$ and $\Phi\left(x_{i}\right) \rightarrow y$ in the corresponding weak* topologies. If $g \in D_{*}(\Phi)$ then the map $x \rightarrow\langle\Phi(x), g\rangle$ is weak* continuous on $D(\Phi)$. Since
$D(\Phi)$ is weak* dense in $\mathcal{X}$, it extends to a weak* continuous functional on the whole of $\mathcal{X}$ and hence there exists $f \in \mathcal{X}_{*}$ such that $\langle\Phi(x), g\rangle=\langle x, f\rangle, x \in D(\Phi)$. In particular, $\left\langle\Phi\left(x_{i}\right), g\right\rangle=\left\langle x_{i}, f\right\rangle \rightarrow 0$. On the other hand, $\left\langle\Phi\left(x_{i}\right), g\right\rangle \rightarrow\langle y, g\rangle$. Thus, $\langle y, g\rangle=0$ for all $g \in D_{*}(\Phi)$. Since $D_{*}(\Phi)$ is (norm) dense in $\mathcal{Y}_{*}$, we conclude that $y=0$.
(i) $\Rightarrow$ (ii) For an operator $T$ with domain $D$, let $\operatorname{Gr}^{\prime} T=\{(T \xi, \xi): \xi \in D\}$. Let $\Phi_{*}: D_{*}(\Phi) \rightarrow \mathcal{X}_{*}$ be defined by letting $\Phi_{*}(g)=f$, where, for $g \in D_{*}(\Phi)$, the element $f \in \mathcal{X}_{*}$ is the (unique) weak* continuous functional on $\mathcal{X}$ such that $\langle\Phi(x), g\rangle=\langle x, f\rangle$, $x \in D_{*}(\Phi)$. We claim that

$$
\begin{equation*}
(\operatorname{Gr} \Phi)_{\perp} \subseteq \operatorname{Gr}^{\prime}\left(-\Phi_{*}\right) \tag{22}
\end{equation*}
$$

To see this, let $(f, g) \in(\operatorname{Gr} \Phi)_{\perp}$; then $\langle f, x\rangle=-\langle g, \Phi(x)\rangle$, for all $x \in D(\Phi)$. It follows that $g \in D\left(\Phi_{*}\right)$ and $\Phi_{*}(g)=-f$; thus, (22) is proved.

Now suppose that $y \in \mathcal{Y}$ annihilates $D_{*}(\Phi)$. Then $(0, y)$ annihilates $\operatorname{Gr}^{\prime}\left(-\Phi_{*}\right)$ and (22) implies that

$$
(0, y) \in\left((\operatorname{Gr} \Phi)_{\perp}\right)^{\perp}=\overline{\operatorname{Gr} \Phi^{w^{*}}}
$$

Since $\Phi$ is weak* closable, $y=0$ and so $D_{*}(\Phi)$ is norm dense in $\mathcal{Y}_{*}$.
The von Neumann algebra $\operatorname{VN}(G)$ possesses two natural and, in the case $G$ is nondiscrete, genuinely different, weak* dense selfadjoint subalgebras, one of them being $\lambda\left(L^{1}(G)\right)$, and the other being the (non-closed) linear span of the left translation operators

$$
\mathrm{VN}_{0}(G)=\left[\lambda_{s}: s \in G\right]
$$

Given a continuous function $\psi: G \rightarrow \mathbb{C}$, we can now consider, along with the operator $S_{\psi}$ with domain $D(\psi)$, a linear operator $S_{\psi}^{\prime}: \mathrm{VN}_{0}(G) \rightarrow \mathrm{VN}_{0}(G)$ given by $S_{\psi}^{\prime}\left(\lambda_{s}\right)=\psi(s) \lambda_{s}$, $s \in G$. Our aim in the next theorem is to characterise the weak ${ }^{*}$ closability of $S_{\psi}$ and $S_{\psi}^{\prime}$.

Theorem 8.2. Let $\psi: G \rightarrow \mathbb{C}$ be a continuous function and $\varphi=N(\psi)$. The following are equivalent:
(i) the operator $S_{\psi}$ is weak** closable;
(ii) the operator $S_{\psi}^{\prime}$ is weak* closable;
(iii) the function $\psi$ belongs locally to $A(G)$ at every point;
(iv) the function $\varphi$ is a local Schur multiplier on $\mathcal{K}\left(L^{2}(G)\right)$;
(v) the operator $S_{\varphi}$ is weak** closable;
(vi) $\overline{D\left(S_{\varphi}^{*}\right)}\left\|^{\cdot}\right\|_{\Gamma}=\Gamma(G, G), \overline{D\left(S_{\varphi}^{* *}\right)}{ }^{w^{*}}=\mathcal{B}\left(L^{2}(G)\right), \mathrm{VN}_{0}(G) \subseteq D\left(S_{\varphi}^{* *}\right)$ and the operator $S_{\varphi}^{* *}: D\left(S_{\varphi}^{* *}\right) \rightarrow \mathcal{B}\left(L^{2}(G)\right)$ is weak* closable;
(vii) $\overline{D\left(S_{\varphi}^{*}\right)} \|^{\prime \cdot}=\Gamma(G, G), \mathrm{VN}_{0}(G) \subseteq D\left(S_{\varphi}^{* *}\right)$ and the operator $S_{\varphi}^{* *}: D\left(S_{\varphi}^{* *}\right) \rightarrow$ $\mathcal{B}\left(L^{2}(G)\right)$ is weak* closable.

Proof. We have that

$$
\begin{aligned}
D_{*}\left(S_{\psi}^{\prime}\right) & =\left\{f \in A(G): T \rightarrow\left\langle S_{\psi}^{\prime}(T), f\right\rangle\right. \text { is w} \\
& =\left\{f \in A(G): \exists u \in A(G):\left\langle S_{\psi}^{\prime}(T), f\right\rangle=\langle T, u\rangle, T \in \mathrm{VN}_{0}(G)\right\} \\
& =\left\{f \in A(G): \exists u \in A(G) \text { with }\left\langle S_{\psi}^{\prime}\left(\lambda_{s}\right), f\right\rangle=\left\langle\lambda_{s}, u\right\rangle, s \in G\right\} \\
& =\{f \in A(G): \exists u \in A(G) \text { with } \psi(s) f(s)=u(s), s \in G\} \\
& =\{f \in A(G): \psi f \in A(G)\} \\
& =J_{\psi},
\end{aligned}
$$

where the last equality follows from the fact that $\psi$ is continuous. The equivalence (ii) $\Leftrightarrow$ (iii) now follows from Lemma 7.1 and Proposition 8.1.

Similarly,

$$
\begin{aligned}
D_{*}\left(S_{\psi}\right) & =\left\{f \in A(G): g \rightarrow\left\langle S_{\psi}(\lambda(g)), f\right\rangle \text { is w*-continuous on } D(\psi)\right\} \\
& =\{f \in A(G): \exists u \in A(G):\langle\lambda(\psi g), f\rangle=\langle\lambda(g), u\rangle, g \in D(\psi)\} \\
& =\left\{f \in A(G): \exists u \in A(G): \int_{G} \psi f g=\int_{G} u g, g \in D(\psi)\right\} \\
& =\{f \in A(G): \exists u \in A(G) \text { such that } \psi f \sim u\} \\
& =J_{\psi}
\end{aligned}
$$

(recall that by $u \sim v$ we mean that $u=v$ almost everywhere on $G$ ). The fourth equality in the latter chain can be seen as follows. Let $K \subseteq G$ be a compact set; then $\left.\psi\right|_{K}$ is bounded and hence $L^{1}(K) \subseteq D(\psi)$. It follows that $\int_{K} \psi f g=\int_{K} u g$ for all $g \in L^{1}(K)$. Since $\left.\psi f\right|_{K}$ and $\left.u\right|_{K}$ belong to $L^{\infty}(K)$, we conclude that $\left.\psi f\right|_{K}=\left.u\right|_{K}$ almost everywhere. Since this holds for every compact $K \subseteq G$, we have that $\psi f \sim u$.

The equivalence (i) $\Leftrightarrow$ (iii) follows, as above, from Lemma 7.1 and Proposition 8.1.
(iii) $\Rightarrow$ (iv) We claim that $\psi u \in A(G)$ for every $u \in A(G) \cap C_{c}(G)$. Indeed, since $\psi \in A(G)^{\text {loc }}$, for every $t \in G$ there exists a neighbourhood $V_{t}$ of $t$ and a function $g_{t} \in A(G)$ such that $\psi=g_{t}$ on $V_{t}$. Since $\operatorname{supp}(u)$ is compact there exists a finite set $F \subseteq G$ such that $\operatorname{supp}(u) \subseteq \bigcup_{t \in F} V_{t}$. It follows from the regularity of $A(G)$ that there exist $h_{t} \in A(G), t \in F$, such that $\sum_{t \in F} h_{t}(x)=1$ if $x \in \operatorname{supp}(u)$ and $h_{s}(x)=0$ if $x \notin V_{s}$ for each $s \in F$ (see the proof of [17, Theorem 39.21]). Then for every $x \in G$ we have

$$
\psi(x) u(x)=\sum_{t \in F} \psi(x) h_{t}(x) u(x)=\sum_{t \in F} g_{t}(x) h_{t}(x) u(x),
$$

which gives $\psi u \in A(G)$.
Let $\left(K_{n}\right)_{n=1}^{\infty}$ be an increasing sequence of compact sets such that, up to a null set, $\bigcup_{n=1}^{\infty} K_{n}=G$. Choose, for each $n \in \mathbb{N}$, a function $\psi_{n} \in A(G) \cap C_{c}(G)$ that takes the
value 1 on $K_{n} K_{n}^{-1}$. By the previous paragraph, $\psi \psi_{n} \in A(G)$ and therefore $N\left(\psi \psi_{n}\right)$ is a Schur multiplier. Thus, for each $h \in \Gamma(G, G)$, we have

$$
\varphi \chi_{K_{n} \times K_{n}} h=N\left(\psi \psi_{n}\right) \chi_{K_{n} \times K_{n}} h \in \Gamma(G, G) .
$$

It follows that $\left.\varphi\right|_{K_{n} \times K_{n}}$ is a Schur multiplier and hence $\varphi$ is a local Schur multiplier.
(iv) $\Rightarrow$ (v) follows from the fact that every local Schur multiplier is a weak* closable multiplier [34].
(v) $\Rightarrow$ (vi) Suppose that $S_{\varphi}$ is weak ${ }^{* *}$ closable. By Proposition 2.1, the space $D\left(S_{\varphi}^{*}\right)$ is dense in $\Gamma(G, G)$ in the norm topology. We have that

$$
D\left(S_{\varphi}^{* *}\right)=\left\{T \in \mathcal{B}\left(L^{2}(G)\right): h \rightarrow\left\langle T, S_{\varphi}^{*}(h)\right\rangle \text { is continuous on } D\left(S_{\varphi}^{*}\right)\right\}
$$

The space $D\left(S_{\varphi}^{* *}\right)$ is weak* dense in $\mathcal{B}\left(L^{2}(G)\right)$ since it contains the norm dense subspace $D\left(S_{\varphi}\right)$.

Suppose that $h \in D\left(S_{\varphi}^{*}\right)$; then $S_{\varphi}^{*}(h)=\varphi h \in^{m \times m} \Gamma(G, G)$ and hence, if $T \in D\left(S_{\varphi}^{* *}\right)$ then

$$
\langle T, \varphi h\rangle=\left\langle T, S_{\varphi}^{*}(h)\right\rangle=\left\langle S_{\varphi}^{* *}(T), h\right\rangle
$$

The mapping

$$
T \rightarrow\left\langle S_{\varphi}^{* *}(T), h\right\rangle, \quad T \in D\left(S_{\varphi}^{* *}\right)
$$

is thus weak* continuous and hence $h \in D_{*}\left(S_{\varphi}^{* *}\right)$. In other words, $D\left(S_{\varphi}^{*}\right) \subseteq D_{*}\left(S_{\varphi}^{* *}\right)$; since $D\left(S_{\varphi}^{*}\right)$ is dense in norm in $\Gamma(G, G)$, the same holds true for $D_{*}\left(S_{\varphi}^{* *}\right)$. By Proposition 8.1, $S_{\varphi}^{* *}$ is weak ${ }^{*}$ closable.

Let $s \in G$. We show that $\lambda_{s} \in D\left(S_{\varphi}^{* *}\right)$. Recall that $P: \Gamma(G, G) \rightarrow A(G)$ is the canonical contractive surjection satisfying (2); for every $h \in D\left(S_{\varphi}^{*}\right)$, using Lemma 2.3, we see that

$$
\begin{equation*}
\left\langle\lambda_{s}, S_{\varphi}^{*}(h)\right\rangle=\left\langle\lambda_{s}, \varphi h\right\rangle=P(\varphi h)(s)=\psi(s) P(h)(s)=\left\langle\psi(s) \lambda_{s}, h\right\rangle . \tag{23}
\end{equation*}
$$

Thus, $\lambda_{s} \in D\left(S_{\varphi}^{* *}\right), S_{\varphi}^{* *}\left(\lambda_{s}\right)=\psi(s) \lambda_{s}$, and (vi) is proved.
(vi) $\Rightarrow$ (vii) is trivial.
(vii) $\Rightarrow$ (ii) Suppose that $\left(T_{i}\right)_{i} \subseteq \mathrm{VN}_{0}(G)$ and $T \in \mathrm{VN}(G)$ are such that $T_{i} \rightarrow{ }^{w^{*}} 0$ and $S_{\psi}^{\prime}\left(T_{i}\right) \rightarrow^{w^{*}} T$. Then $\left(T_{i}\right)$ (resp. $\left(S_{\psi}^{\prime}\left(T_{i}\right)\right)$ ) converges to zero (resp. $T$ ) in the weak* topology of $\mathcal{B}\left(L^{2}(G)\right)$. Identity (23) shows that $S_{\varphi}^{* *}(R)=S_{\psi}^{\prime}(R)$ for every $R \in \mathrm{VN}_{0}(G)$. Since $S_{\varphi}^{* *}$ is weak* closable, $T=0$.

Remark. If $\psi$ is not assumed to be continuous, then all conditions in Theorem 8.2 apart from (ii) remain equivalent, provided that in (iii) we require that $\psi$ almost belongs locally to $A(G)$ at every point.

Proposition 7.9 and Theorems 7.4 and 8.2 yield the following implications:

$$
S_{\psi} \text { is weak }{ }^{* *} \text { closable } \Longrightarrow S_{\psi} \text { is weak }{ }^{*} \text { closable } \Longrightarrow S_{\psi} \text { is closable. }
$$

Theorem 8.2 and the example after Proposition 7.9 show that there exists a continuous function $\psi$ for which $S_{\psi}$ is weak* closable but not weak** closable. On the other hand, Proposition 7.2 implies that if $E_{\psi}$ is a non-empty $U$-set then $\psi$ is closable but $\psi \notin A(G)^{\mathrm{loc}}$; thus, by Theorem 8.2, $S_{\psi}$ is not weak* closable. For example, for $G=\mathbb{R}$, $\psi=\chi_{[0,+\infty)}$, we have $E_{\psi}=\{0\}$ which is a non-empty $U$-set by Corollary 5.3. One can also find a continuous function $\psi$ for which $E_{\psi}$ is a one-point set of uniqueness. In fact, consider a function $\psi(t)$ on $[0, \pi]$ which is smooth on the open interval $(0, \pi)$ and $\psi(0)=\psi(\pi)=0$. Assume also that $\psi^{\prime}(\pi)=0$ and that the integral $\int_{0}^{1} \psi(t) / t d t$ diverges. Extend $\psi$ to an odd (continuous) function on $[-\pi, \pi]$. By [20, Chapter II.14], $\psi \notin A(\mathbb{T})^{\text {loc }}=A(\mathbb{T})$. As $\psi$ is smooth at any $t \neq 0, \psi$ belongs to $A(\mathbb{T})$ at any such point $t$. Therefore $E_{\psi}=\{0\}$.

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