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V.S. Shulman [a], I.G. Todorov [b], L. Turowska [c,*,1]

[a] *Department of Mathematics, Vologda State Technical University, Vologda, Russia*
[b] *Pure Mathematics Research Centre, Queen's University Belfast, Belfast BT7 1NN, United Kingdom*
[c] *Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, Gothenburg SE-412 96, Sweden*

A R T I C L E I N F O

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To the memory of William Arveson, with gratitude and admiration

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A B S T R A C T

We undertake a detailed study of the sets of multiplicity in a second countable locally compact group G and their operator versions. We establish a symbolic calculus for normal completely bounded maps from the space $\mathcal{B}(L^2(G))$ of bounded linear operators on $L^2(G)$ into the von Neumann algebra $\mathrm{VN}(G)$ of G and use it to show that a closed subset $E \subseteq G$ is a set of multiplicity if and only if the set $E^* = \{(s,t) \in G \times G : ts^{-1} \in E\}$ is a set of operator multiplicity. Analogous results are established for M_1-sets and M_0-sets. We show that the property of being a set of multiplicity is preserved under various operations, including taking direct products, and establish an Inverse Image Theorem for such sets. We characterise the sets of finite width that are also sets of operator multiplicity, and show that every compact operator supported on a set of finite width can be approximated by sums of rank one operators supported on the same set. We show that, if G satisfies a mild approximation condition, pointwise multiplication by a given measurable function $\psi : G \to \mathbb{C}$ defines a closable multiplier on the reduced C^*-algebra $C_r^*(G)$ of G if and only if Schur multiplication by the function $N(\psi) : G \times G \to \mathbb{C}$, given by $N(\psi)(s,t) = \psi(ts^{-1})$, is a closable operator when

* Corresponding author.
E-mail addresses: shulman.victor80@gmail.com (V.S. Shulman), i.todorov@qub.ac.uk (I.G. Todorov), turowska@chalmers.se (L. Turowska).
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viewed as a densely defined linear map on the space of compact operators on $L^2(G)$. Similar results are obtained for multipliers on $\mathrm{VN}(G)$.

Contents

1. Introduction

The connections between Harmonic Analysis and the Theory of Operator Algebras have a long and illustrious history. With his pivotal paper [2], W.B. Arveson opened up a new avenue in that direction by introducing the notion of operator synthesis. The relation between operator synthesis and spectral synthesis for locally compact groups was explored in detail in [14,25,39,9,10], among others. In this connection, J. Froelich [14] studied the question of when the operator algebra associated with a commutative subspace lattice contains a non-zero compact operator. For any compact abelian group G and a closed subset $E \subseteq G$, he constructed a commutative subspace lattice $\mathcal{L}_E$, such that the corresponding operator algebra contains a non-zero compact operator if and only if E is a set of multiplicity in the sense of (commutative) Harmonic Analysis.

Recently, we observed in [34] a connection between sets of multiplicity and the closability of linear transformations that are a natural unbounded analogue of Schur multipliers. Motivated originally by Schur multiplication of matrices, Schur multipliers have played an important role in a number of contexts in Operator Theory, see *e.g.* [18] and [30]. In the context of Harmonic Analysis, they provide the basis for a useful characterisation of completely bounded multipliers of the Fourier algebra $A(G)$ of a locally compact group G introduced in [7]. Namely, a measurable function $\psi : G \to \mathbb{C}$ is a completely bounded multiplier of $A(G)$ precisely when the function $N(\psi)$, given by $N(\psi)(s,t) = \psi(ts^{-1})$, is a Schur multiplier on $G \times G$ [6] (see also [19] and [38]). The functions ψ satisfying the latter condition are known as Herz–Schur multipliers. Any multiplier ψ on $A(G)$ determines a bounded transformation on the corresponding reduced group C^*-algebra defined by the pointwise multiplication of $L^1(G)$ by ψ.

Unbounded transformations of Schur type, acting on group C^*-algebras, have been considered in the literature in connection with problems arising in Non-commutative Geometry (see [1] and the references therein). However, unbounded versions of transformations on group C^*-algebras corresponding to multipliers of Fourier algebras and their connection with (unbounded) operators of Schur type have not been explored until the present work.

These considerations gave the motivation for our present study of sets of multiplicity in the general setting of locally compact groups and their connection with closable multipliers on group algebras.

Sets of multiplicity for the group of the circle initially arose in connection with the problem of uniqueness of trigonometric series and have been extensively studied (see [15]). In a general locally compact group G, sets of uniqueness (or, equivalently, of non-multiplicity) were introduced by M. Bożejko in [4] as those closed subsets $E \subseteq G$ which do not support non-zero elements of the reduced C^*-algebra $C_r^*(G)$ of G.

An operator counterpart of sets of multiplicity was introduced in [34]. On the operator level, as well as on the level of locally compact groups, two classes of sets of multiplicity have been mostly examined: (operator) M-sets and (operator) M_1-sets. Here we introduce the class of operator M_0-sets and show, in Section 4, that a closed subset E of a second countable locally compact group G is an M-set (resp. M_1-set, M_0-set) if and only if the set $E^* = \{(s,t) : ts^{-1} \in E\} \subseteq G \times G$ is an operator M-set (resp. operator M_1-set, operator M_0-set). These results should be compared to the result established in [14,25,39] stating that E is a set of local spectral synthesis if and only if E^* is a set of operator synthesis. They permit the use of operator theoretic methods in the study of concepts pertinent purely to Harmonic Analysis.

An important role in our approach plays the technique of pseudo-integral operators introduced in [2]. Some results on these operators, which are used in the sequel, are collected in Section 3 of the paper. En route, we give an affirmative answer of a question of J. Froelich [14] concerning the validity of a tensor product formula for masa-bimodules (see Theorem 3.8).

The main technical tool we develop and use is a symbolic calculus for weak* continuous completely bounded maps from the algebra $\mathcal{B}(L^2(G))$ of bounded operators on $L^2(G)$ into the von Neumann algebra $\mathrm{VN}(G)$ of G (see Theorem 4.6). A significant role in our approach is played by a locally compact version of the uniform Roe algebra which was introduced for discrete groups in [32] and has been studied in various contexts.

In Section 5, we show that the property of being a set of (operator) multiplicity is preserved under some natural operations. These include direct products and a certain type of generalised union. As a corollary of a more general operator algebraic statement, we recover M. Bożejko's result [5,4] that every countable closed set in a non-discrete locally compact group is a set of uniqueness. We also establish an Inverse Image Theorem for sets of operator multiplicity (see Theorem 5.5).

In Section 6, we examine sets of finite width. This class of sets has played a fundamental role in the field since their introduction in [2] (see [9,10,35] and the references therein). We characterise the sets of finite width that are also sets of operator multiplicity, and show that, in general, every compact operator supported on a set of finite width is the norm limit of sums of rank one operators supported on this set.

Sections 7 and 8 are devoted to the main applications of the previously described results. Namely, in Section 7, we establish a "closable" version of the aforementioned characterisation of completely bounded multipliers, showing that for groups G satisfying a certain approximation property (more general than weak amenability), ψ is a closable multiplier on $C_r^*(G)$, in the sense that the pointwise multiplication of $L^1(G)$ by ψ is a closable map on $C_r^*(G)$, if and only if $N(\psi)$ is a closable multiplier in the sense of [34]. We present various examples of closable and non-closable multipliers.

In Section 8, we discuss similar multiplier maps on the group von Neumann algebra $\mathrm{VN}(G)$. We introduce the notion of a weak* closable operator, which is suitable for the setting of dual Banach spaces, such as $\mathrm{VN}(G)$. We show that a continuous function ψ is a weak* closable multiplier if and only if $N(\psi)$ is a local Schur multiplier [34], which occurs precisely when ψ belongs locally to the Fourier algebra $A(G)$. Weak** closable multipliers on $C_r^*(G)$ [34] (see Section 2.1) are shown to form a proper subset of the class of weak* closable multipliers, which in turn form a proper subset of the class of closable multipliers.

Finally, in Section 2, we collect the necessary preliminary material and set notation for the subsequent sections.

2. Preliminaries

In this section, we collect some definitions and results that will be needed in the sequel.

2.1. Closable operators

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T : D(T) \to \mathcal{Y}$ be a linear operator, where the domain $D(T)$ of T is a dense linear subspace of $\mathcal{X}$. The operator T is called *closable* if the closure $\overline{\mathrm{Gr}\, T}$ of its graph

$$\mathrm{Gr}\, T = \big\{(x, Tx) : x \in D(T)\big\} \subseteq \mathcal{X} \oplus \mathcal{Y}$$

is the graph of a linear operator. Equivalently, T is closable if $(x_k)_{k\in\mathbb{N}} \subseteq D(T)$, $y \in \mathcal{Y}$, $\|x_k\| \to_{k\to\infty} 0$ and $\|T(x_k) - y\| \to_{k\to\infty} 0$ imply that $y = 0$. The operator T is called *weak** closable* [34] if the weak* closure $\overline{\operatorname{Gr} T}^{w^*}$ of $\operatorname{Gr} T$ in $\mathcal{X}^{**} \oplus \mathcal{Y}^{**}$ is the graph of a linear operator. Equivalently, T is weak** closable if whenever $(x_j)_{j\in J} \subseteq D(T)$ is a net, $y \in \mathcal{Y}^{**}$, $x_j \xrightarrow{w^*}_{j\in J} 0$ and $T(x_j) \xrightarrow{w^*}_{j\in J} y$, we have that $y = 0$. We note that in [34] weak** closable operators were called weak* closable. We have chosen to alter our terminology since we feel that the term "weak* closable" is better suited for the notion introduced and studied in Section 8 of the present paper.

The domain of the adjoint operator of T is the subspace

$$D(T^*) = \{g \in \mathcal{Y}^* : \exists f \in \mathcal{X}^* \text{ such that } g(Tx) = f(x) \text{ for all } x \in D(T)\}$$

and the adjoint of T is the operator $T^* : D(T^*) \to \mathcal{X}^*$ defined by letting $T^*(g) = f$, where f is the functional associated with g in the definition of $D(T^*)$.

In the following proposition, which was stated in [34], the equivalence (iii) $\Leftrightarrow$ (iv) is well-known (see, for example, [22, Chapter III, Section 5]), while the other implications can be proved easily.

Proposition 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $D(T) \subseteq \mathcal{X}$, $T : D(T) \to \mathcal{Y}$ be a densely defined linear operator and set $\mathcal{D} = D(T^*)$. Consider the following conditions:*

(i) *T is weak** closable;*
(ii) $\overline{\mathcal{D}}^{\|\cdot\|} = \mathcal{Y}^*$;
(iii) $\overline{\mathcal{D}}^{w^*} = \mathcal{Y}^*$;
(iv) *T is closable.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

2.2. Locally compact groups

If H, H_1 and H_2 are Hilbert spaces, we denote by $\mathcal{B}(H_1, H_2)$ the space of all bounded linear operators from H_1 to H_2, and set $\mathcal{B}(H) = \mathcal{B}(H, H)$. Let G be a locally compact group. Left Haar measure on G will be denoted by m_G or m and integration with respect to m_G along the variable s will be denoted by ds. We denote by $L^p(G)$, $p = 1, 2, \infty$, the corresponding Lebesgue spaces associated with m_G. For a function $\xi : G \to \mathbb{C}$, we set as customary $\check{\xi}(s) = \xi(s^{-1})$, $s \in G$. Let $\lambda : G \to \mathcal{B}(L^2(G))$ be the left regular representation of G, that is, $\lambda_s f(t) = f(s^{-1}t)$, $f \in L^2(G)$, $s, t \in G$, and $M(G)$ be the *measure algebra* of G, consisting by definition of all bounded complex Borel measures on G. We denote the variation of $\theta \in M(G)$ by $|\theta|$ and let $\|\theta\| = |\theta|(G)$. The support of a measure $\theta \in M(G)$ is the (closed) subset

$$\operatorname{supp} \theta = \Big(\bigcup \{U \subseteq G : U \text{ open}, \ |\theta|(U) = 0\}\Big)^c;$$

it is the smallest closed subset E of G with the property that if $U \subseteq E^c$ is a Borel set then $\theta(U) = 0$. For a closed set $E \subseteq G$, let $M(E)$ be the set of all measures θ in

$M(G)$ with $\operatorname{supp}\theta \subseteq E$. If $\theta \in M(G)$ then the operator $\lambda(\theta)$ of convolution by θ is given by $\lambda(\theta)(f)(t) = \int_G f(s^{-1}t)d\theta(s)$; the map $\lambda : M(G) \to \mathcal{B}(L^2(G))$ is a representation of $M(G)$ of $L^2(G)$. Since $L^1(G)$ is a Banach subalgebra of $M(G)$, the restriction of λ to $L^1(G)$ is a representation of $L^1(G)$; we have

$$\lambda(f)g(t) = f * g(t) = \int f(s)g\big(s^{-1}t\big)ds, \quad f \in L^1(G),\ g \in L^2(G),\ t \in G.$$

The *Fourier algebra* $A(G)$ of G [12] is the algebra of coefficients of λ, that is, the algebra of functions of the form $s \to (\lambda_s\xi, \eta)$, for $\xi, \eta \in L^2(G)$. The *Fourier–Stieltjes algebra* $B(G)$ of G [12] is, on the other hand, the algebra of coefficients of all continuous unitary representations of G acting on some Hilbert space, that is, the algebra of all functions of the form $s \to (\pi(s)\xi, \eta)$, where $\pi : G \to \mathcal{B}(H)$ is a continuous unitary representation, and $\xi, \eta \in H$. We denote by $C_r^*(G)$ the *reduced C^*-algebra* of G, that is, the closure of $\lambda(L^1(G))$ in the operator norm. We let $\mathrm{VN}(G) = \overline{C_r^*(G)}^{w^*}$ be the *von Neumann algebra* of G, and $C^*(G)$ be the *full C^*-algebra* of G. It is known [12] that $A(G)$ is a semisimple, regular, commutative Banach algebra with spectrum G, which can be identified with the predual $\mathrm{VN}(G)_*$ of $\mathrm{VN}(G)$ via the pairing $\langle u, T\rangle = (T\xi, \eta)$, where $u \in A(G)$ is given by $u(s) = (\lambda_s\xi, \eta)$. If $T \in \mathrm{VN}(G)$ and $u \in A(G)$, the operator $u \cdot T \in \mathrm{VN}(G)$ is given by the relations $\langle u \cdot T, v\rangle = \langle T, uv\rangle$, $v \in A(G)$. The map $(u, T) \mapsto u \cdot T$ turns $\mathrm{VN}(G)$ into a Banach $A(G)$-module.

Let

$$MA(G) = \big\{v : G \to \mathbb{C} : vu \in A(G),\ \text{for all } u \in A(G)\big\}$$

be the multiplier algebra of $A(G)$. For each $v \in MA(G)$, the map $u \mapsto vu$ on $A(G)$ is bounded; its norm will be denoted by $\|v\|_{MA(G)}$. As usual, let $M^{\mathrm{cb}}A(G)$ be the subalgebra of $MA(G)$ consisting of those v for which the map $u \mapsto vu$ on $A(G)$ is completely bounded [7]. We refer the reader to [27] and [31] for the basic of Operator Space Theory and completely bounded maps.

We denote by $C_0(G)$ the space of all continuous functions on G vanishing at infinity. The dual of $C_0(G)$ can be canonically identified with $M(G)$; the duality between the two spaces will be denoted by $\langle \cdot, \cdot \rangle$. Note that $A(G) \subseteq C_0(G)$ and that the adjoint of this inclusion gives rise to the inclusion $\lambda(M(G)) \subseteq \mathrm{VN}(G)$. We refer the reader to [12] for more details about the notions discussed above.

If $J \subseteq A(G)$ is an ideal, let

$$\operatorname{null} J = \big\{s \in G : u(s) = 0 \text{ for all } u \in J\big\}.$$

On the other hand, for a closed set $E \subseteq G$, let

$$I(E) = \big\{f \in A(G) : f(s) = 0,\ s \in E\big\},$$
$$J_0(E) = \big\{f \in A(G) : f \text{ has compact support disjoint from } E\big\}$$

and $J(E) = \overline{J_0(E)}$. We have that $\operatorname{null} J(E) = \operatorname{null} I(E) = E$ and that if $J \subseteq A(G)$ is a closed ideal with $\operatorname{null} J = E$, then $J(E) \subseteq J \subseteq I(E)$. The support $\operatorname{supp}(T)$ of an operator $T \in \mathrm{VN}(G)$ is given by

$$\operatorname{supp}(T) = \{t \in G : u \cdot T \neq 0 \text{ whenever } u \in A(G) \text{ and } u(t) \neq 0\}.$$

It is known (see [12]) that the annihilator $J(E)^\perp$ of $J(E)$ in $\mathrm{VN}(G)$ coincides with the space of all operators $T \in \mathrm{VN}(G)$ with $\operatorname{supp}(T) \subseteq E$.

2.3. Masa-bimodules

We fix, throughout the paper, standard measure spaces (X, μ) and (Y, ν); this means that μ and ν are Radon measures with respect to some complete metrisable separable locally compact topologies (henceforth called admissible topologies) on X and Y, respectively. A subset of $X \times Y$ will be called a *rectangle* if it is of the form $\alpha \times \beta$, where $\alpha \subseteq X$ and $\beta \subseteq Y$ are measurable. We equip $X \times Y$ with the σ-algebra generated by all rectangles and denote by $\mu \times \nu$ the product measure. A subset $E \subseteq X \times Y$ is called *marginally null* if $E \subseteq (X_0 \times Y) \cup (X \times Y_0)$, where $\mu(X_0) = \nu(Y_0) = 0$. We call two subsets $E, F \subseteq X \times Y$ *marginally equivalent* (and write $E \simeq F$) if their symmetric difference is marginally null.

A subset E of $X \times Y$ is called *ω-open* if it is marginally equivalent to the union of a countable set of rectangles. The complements of ω-open sets are called *ω-closed*. It is clear that the class of all ω-open (resp. ω-closed) sets is closed under countable unions (resp. intersections) and finite intersections (resp. unions). Let $\mathfrak{B}(X \times Y)$ be the space of all measurable complex valued functions defined on the measure space $(X \times Y, \mu \times \nu)$. We say that two functions $\varphi, \psi \in \mathfrak{B}(X \times Y)$ are *equivalent*, and write $\varphi \sim \psi$, if the set $D = \{(x, y) \in X \times Y : \psi(x, y) \neq \varphi(x, y)\}$ is null with respect to $\mu \times \nu$. If D is marginally null then we say that φ and ψ coincide marginally almost everywhere or that they are *marginally equivalent*, and write $\varphi \simeq \psi$.

The following lemma was proved in [11].

Lemma 2.2. *Suppose that compact admissible topologies can be chosen on X and Y and that μ and ν are finite. Let $E \subseteq \bigcup_{n=1}^\infty \gamma_n$ where E is ω-closed and γ_n is ω-open, $n \in \mathbb{N}$. Then for each $\varepsilon > 0$ there are subsets $X_\varepsilon \subseteq X$, $Y_\varepsilon \subseteq Y$ such that $\mu(X \setminus X_\varepsilon) < \varepsilon$, $\nu(Y \setminus Y_\varepsilon) < \varepsilon$ and $E \cap (X_\varepsilon \times Y_\varepsilon)$ is contained in the union of finitely many of the subsets γ_n, $n \in \mathbb{N}$.*

For Hilbert spaces H_1 and H_2, we denote by $\mathcal{K}(H_1, H_2)$ (resp. $\mathcal{C}_1(H_1, H_2)$, $\mathcal{C}_2(H_1, H_2)$) the space of compact (resp. nuclear, Hilbert–Schmidt) operators in $\mathcal{B}(H_1, H_2)$. We often write $\mathcal{K} = \mathcal{K}(H_1, H_2)$. Throughout the paper, we let $H_1 = L^2(X, \mu)$ and $H_2 = L^2(Y, \nu)$. The operator norm of $T \in \mathcal{B}(H_1, H_2)$ is denoted by $\|T\|$. The space $\mathcal{C}_1(H_2, H_1)$ (resp. $\mathcal{B}(H_1, H_2)$) can be naturally identified with the Banach space dual of $\mathcal{K}(H_1, H_2)$ (resp.

$\mathcal{C}_1(H_2,H_1)$), the duality being given by the map $(T,S) \mapsto \langle T,S\rangle \stackrel{def}{=} \operatorname{tr}(TS)$. Here $\operatorname{tr} A$ denotes the trace of a nuclear operator A.

The space $L^2(Y\times X)$ will be identified with $\mathcal{C}_2(H_1,H_2)$ via the map sending an element $k\in L^2(Y\times X)$ to the integral operator T_k given by $T_k\xi(y)=\int_X k(y,x)\xi(x)d\mu(x)$, $\xi\in H_1$, $y\in Y$. In a similar fashion, $\mathcal{C}_1(H_2,H_1)$ will be identified with the space $\Gamma(X,Y)$ of all (marginal equivalence classes of) functions $h: X\times Y\to\mathbb{C}$ which admit a representation

$$h(x,y)=\sum_{i=1}^{\infty} f_i(x)g_i(y),$$

where $f_i\in H_1$, $g_i\in H_2$, $i\in\mathbb{N}$, $\sum_{i=1}^{\infty}\|f_i\|_2^2<\infty$ and $\sum_{i=1}^{\infty}\|g_i\|_2^2<\infty$. Equivalently, $\Gamma(X,Y)$ can be defined as the projective tensor product $H_1\hat{\otimes} H_2$; we write $\|h\|_\Gamma$ for the projective norm of $h\in\Gamma(X,Y)$. The duality between $\mathcal{B}(H_1,H_2)$ and $\Gamma(X,Y)$ is given by

$$\langle T, f\otimes g\rangle = (Tf,\bar{g}),$$

for $T\in\mathcal{B}(H_1,H_2)$, $f\in L^2(X,\mu)$ and $g\in L^2(Y,\nu)$.

If $f\in L^\infty(X,\mu)$, let $M_f\in\mathcal{B}(H_1)$ be the operator on H_1 of multiplication by f. The collection $\{M_f: f\in L^\infty(X,\mu)\}$ is a maximal abelian selfadjoint algebra (for short, masa) on H_1. If $\alpha\subseteq X$ is measurable, we write $P(\alpha)=M_{\chi_\alpha}$ for the multiplication by the characteristic function of the set α. The same notation will be used for H_2. A subspace $\mathcal{W}\subseteq\mathcal{B}(H_1,H_2)$ will be called a *masa-bimodule* if $M_\psi TM_\varphi\in\mathcal{W}$ for all $T\in\mathcal{W}$, $\varphi\in L^\infty(X,\mu)$ and $\psi\in L^\infty(Y,\nu)$.

We say that an ω-closed subset $\kappa\subseteq X\times Y$ *supports* an operator $T\in\mathcal{B}(H_1,H_2)$ (or that T *is supported on* κ) if $P(\beta)TP(\alpha)=0$ whenever $(\alpha\times\beta)\cap\kappa\simeq\emptyset$. For any subset $\mathcal{M}\subseteq\mathcal{B}(H_1,H_2)$, there exists a smallest (up to marginal equivalence) ω-closed set $\operatorname{supp}\mathcal{M}$ which supports every operator $T\in\mathcal{M}$ [11]. By [2] and [35], for any ω-closed set κ there exists a smallest (resp. largest) weak* closed masa-bimodule $\mathfrak{M}_{\min}(\kappa)$ (resp. $\mathfrak{M}_{\max}(\kappa)$) with support κ, in the sense that if $\mathfrak{M}\subseteq\mathcal{B}(H_1,H_2)$ is a weak* closed masa-bimodule with $\operatorname{supp}\mathfrak{M}=\kappa$ then $\mathfrak{M}_{\min}(\kappa)\subseteq\mathfrak{M}\subseteq\mathfrak{M}_{\max}(\kappa)$.

Let

$$\Phi(\kappa)=\{h\in\Gamma(X,Y): h\chi_\kappa\simeq 0\}$$

and

$$\Psi(\kappa)=\overline{\{h\in\Gamma(X,Y): h \text{ vanishes on an } \omega\text{-open nbhd of } \kappa\}}^{\|\cdot\|_\Gamma}.$$

By [35, Theorems 4.3, 4.4], $\mathfrak{M}_{\min}(\kappa)=\Phi(\kappa)^\perp$ and $\mathfrak{M}_{\max}(\kappa)=\Psi(\kappa)^\perp$.

2.4. Schur multipliers

If φ is a function defined on a measure space (Z,θ), and $\mathcal{E}$ is a space of measurable functions on Z, we write $\varphi \in^{\theta} \mathcal{E}$ when there exists a function $\psi \in \mathcal{E}$ such that φ and ψ differ on a θ-null set. Let

$$J_{\varphi}^{\mathcal{E}} = \{h \in \mathcal{E} : \varphi h \in^{\theta} \mathcal{E}\}.$$

For $\varphi \in \mathfrak{B}(X \times Y)$, the function $\hat{\varphi} : Y \times X \to \mathbb{C}$ is given by $\hat{\varphi}(y,x) = \varphi(x,y)$, $x \in X$, $y \in Y$. We set $D(S_\varphi) = J_{\hat{\varphi}}^{L^2(Y\times X)}$. Identifying $L^2(Y \times X)$ with $\mathcal{C}_2(H_1,H_2) \subseteq \mathcal{K}(H_1,H_2)$, define $S_\varphi : D(S_\varphi) \to \mathcal{K}(H_1,H_2)$ to be the mapping given by $S_\varphi(T_k) = T_{\hat{\varphi}k}$. We say that $\varphi \in \mathfrak{B}(X \times Y)$ is a *closable multiplier* (resp. *weak** closable multiplier*) [34] if the map S_φ is closable (resp. weak** closable) when viewed as a densely defined linear operator on $\mathcal{K}(H_1,H_2)$. If S_φ is moreover bounded in the operator norm, φ is called a *Schur multiplier*. If φ is a Schur multiplier then the mapping S_φ extends by continuity to a (bounded) mapping on $\mathcal{K}(H_1,H_2)$. After taking its second dual, one obtains a bounded weak* continuous linear transformation on $\mathcal{B}(H_1,H_2)$ which will also be denoted by S_φ. We set $\|\varphi\|_{\mathfrak{S}} = \|S_\varphi\|$. The map S_φ is automatically completely bounded and its completely bounded norm is still equal to $\|\varphi\|_{\mathfrak{S}}$ (the reader is referred to [27] and [31] for the basics of Operator Space Theory, which will be used throughout the paper). By a result of V.V. Peller [29] (see also [21] and [38]), a function $\varphi \in \mathfrak{B}(X \times Y)$ is a Schur multiplier if and only if there exist sequences $(a_k)_{k\in\mathbb{N}} \subseteq L^\infty(X,\mu)$ and $(b_k)_{k\in\mathbb{N}} \subseteq L^\infty(Y,\nu)$ with $\operatorname{ess\,sup}_{x\in X} \sum_{k=1}^{\infty} |a_k(x)|^2 < \infty$ and $\operatorname{ess\,sup}_{y\in Y} \sum_{k=1}^{\infty} |b_k(y)|^2 < \infty$ such that

$$\varphi(x,y) = \sum_{k=1}^{\infty} a_k(x) b_k(y), \quad \text{a.e. } (x,y) \in X \times Y.$$

In this case, $S_\varphi(T) = \sum_{k=1}^{\infty} M_{b_k} T M_{a_k}$, $T \in \mathcal{B}(H_1,H_2)$.

Let $\mathfrak{S}(X,Y)$ be the set of all Schur multipliers (we will also write $\mathfrak{S}(X \times Y)$ in the place of $\mathfrak{S}(X,Y)$ if there is no risk of confusion). By [29],

$$\mathfrak{S}(X,Y) = \{\varphi \in L^\infty(X \times Y) : \varphi h \in^{\mu\times\nu} \Gamma(X,Y),\ \forall h \in \Gamma(X,Y)\}.$$

If $\varphi \in \mathfrak{S}(X,Y)$, let $m_\varphi : \Gamma(X,Y) \to \Gamma(X,Y)$ be the mapping given by $m_\varphi(h) = \varphi h$, $h \in \Gamma(X,Y)$; then the adjoint of m_φ coincides with S_φ.

Let G be a locally compact group. The map $P : \Gamma(G,G) \to A(G)$ given by

$$P(f \otimes g)(t) = \langle \lambda_t, f \otimes g\rangle = (\lambda_t f, \bar{g}) = \int_G f(t^{-1}s) g(s) ds = g * \check{f}(t) \tag{1}$$

is a contractive surjection. The next lemma will be used repeatedly.

Lemma 2.3. *If $h \in \Gamma(G,G)$ then*

$$P(h)(t) = \int_G h\left(t^{-1}s, s\right) ds, \quad t \in G. \tag{2}$$

Proof. Identity (2) is a direct consequence of (1) if h is a finite sum of elementary tensors. Let $h = \sum_{i=1}^{\infty} f_i \otimes g_i \in \Gamma(G,G)$, where $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$, and let h_n be the nth partial sum of this series. By the continuity of P, $\|P(h_n) - P(h)\| \to 0$ in $A(G)$; since $\|\cdot\|_\infty$ is dominated by the norm of $A(G)$, we conclude that $P(h_n)(t) \to P(h)(t)$ for every $t \in G$.

By [35, Lemma 2.1], there exists a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ of $(h_n)_{n\in\mathbb{N}}$ such that $h_{n_k} \to h$ marginally almost everywhere. It follows that, for every $t \in G$, one has $h_{n_k}(t^{-1}s, s) \to h(t^{-1}s, s)$ for almost all $s \in G$. By [25, (4.3)], the function $s \to \sum_{i=1}^{\infty} |f_i(t^{-1}s)||g_i(s)|$ is integrable, and hence an application of the Lebesgue Dominated Convergence Theorem shows that $\int_G h_{n_k}(t^{-1}s, s) ds \to_{k\to\infty} \int_G h(t^{-1}s, s) ds$, for every $t \in G$. The proof is complete. $\square$

For a function $f : G \to \mathbb{C}$, let $N(f) : G \times G \to \mathbb{C}$ be the function given by

$$N(f)(s,t) = f\left(ts^{-1}\right), \quad s,t \in G. \tag{3}$$

Note that in [25] and [39], the map N' given by $N'(f)(s,t) = f(st^{-1})$ was used instead of N, but the results established in these papers remain valid with the current definition as well. It follows from [6] (see also [19] and [38]) that N maps $M^{\mathrm{cb}}A(G)$ isometrically into $\mathfrak{S}(G,G)$. Note that, if G is compact, then $\Gamma(G,G)$ contains the constant functions and hence $\mathfrak{S}(G,G) \subseteq \Gamma(G,G)$; thus, in this case N maps $A(G)$ into $\Gamma(G,G)$.

3. Arveson measures and pseudo-integral operators

3.1. Measures

Let σ be a complex measure of finite total variation, defined on the product σ-algebra $\mathcal{F}$ of $X \times Y$. We let $|\sigma|$ denote the variation of σ; thus, for a subset $E \in \mathcal{F}$, the quantity $|\sigma|(E)$ equals the total variation of σ on the set E. We let $|\sigma|_X$ be the X-marginal measure of $|\sigma|$, that is, the measure on X given by $|\sigma|_X(\alpha) = |\sigma|(\alpha \times Y)$. We define $|\sigma|_Y$ similarly by setting $|\sigma|_Y(\beta) = |\sigma|(X \times \beta)$. A complex measure σ on $\mathcal{F}$ will be called an *Arveson measure* if σ has finite total variation and there exists a constant $c > 0$ such that

$$|\sigma|_X \le c\mu \quad \text{and} \quad |\sigma|_Y \le c\nu. \tag{4}$$

We denote by $\mathbb{A}(X,Y)$ the set of all Arveson measures on $X \times Y$ and let $\|\sigma\|_{\mathbb{A}}$ be the smallest constant c which satisfies the inequalities (4). We note that if $\sigma \in \mathbb{A}(X,Y)$ then $|\sigma| \in \mathbb{A}(X,Y)$ as well.

It was shown in [34], that given a family $\mathcal{E}$ of ω-open sets, there exists a minimal (with respect to inclusion up to a marginally null set) ω-open set E which marginally contains every element from $\mathcal{E}$. The set E is called the *ω-union* of E and denoted by $\bigcup_\omega \mathcal{E}$.

Recall that (X,μ) and (Y,ν) are standard measure spaces and let σ be an Arveson measure on $Y\times X$. Denote by $\operatorname{supp}\sigma$ the ω-closed subset of $Y\times X$ defined by

$$(\operatorname{supp}\sigma)^c = \bigcup_\omega \{R \subseteq Y\times X : R \text{ is a rectangle such that } \\ \sigma(R') = 0 \text{ for each rectangle } R' \subseteq R\}.$$

Proposition 3.1. *Let $\sigma \in \mathbb{A}(Y,X)$.*

(i) *The set $\operatorname{supp}\sigma$ is the smallest (up to marginal equivalence) ω-closed subset E of $Y\times X$ such that $\sigma(R)=0$ for every rectangle $R\subseteq E^c$.*
(ii) *If $E\subseteq Y\times X$ is an ω-closed set then $\operatorname{supp}\sigma\subseteq E$ if and only if $|\sigma|(E^c)=0$.*

Proof. (i) Let $\mathcal{R}$ be the set of all rectangles $R\subseteq Y\times X$ such that $\sigma(R')=0$ for every rectangle R' contained in R. By [34, Lemma 2.1], $(\operatorname{supp}\sigma)^c \simeq \bigcup_{i=1}^\infty R_i$ for some family $\{R_i\}_{i\in\mathbb{N}}\subseteq\mathcal{R}$. Let $R\subseteq(\operatorname{supp}\sigma)^c$ be a rectangle. We will show that $\sigma(R)=0$; without loss of generality, we may assume that the measures μ and ν are finite. By Lemma 2.2, for every $n\in\mathbb{N}$ there exist measurable subsets $X_n\subseteq X$ and $Y_n\subseteq Y$ such that $\mu(X\setminus X_n)<1/n$, $\nu(Y\setminus Y_n)<1/n$ and $R\cap(Y_n\times X_n)$ is contained in the union of a finite subfamily of $\{R_i\}_{i\in\mathbb{N}}$. It follows that $\sigma(R\cap(Y_n\times X_n))=0$ for every n and, since $\bigcup_{n=1}^\infty X_n$ and $\bigcup_{n=1}^\infty Y_n$ have full measure, $\sigma(R)=0$.

Suppose that E is an ω-closed set with the property that $\sigma(R)=0$ for every rectangle $R\subseteq E^c$. By the definition of $\operatorname{supp}\sigma$, the set E^c is marginally contained in $(\operatorname{supp}\sigma)^c$, and hence $\operatorname{supp}\sigma\subseteq E$ up to marginal equivalence.

(ii) Suppose that $E^c\simeq\Omega=\bigcup_{i=1}^\infty R_i$, where $R_i\subseteq Y\times X$ is a rectangle, $i\in\mathbb{N}$. Assume, without loss of generality, that $R_i\cap R_j=\emptyset$ if $i\neq j$. Fix $i\in\mathbb{N}$. By (i), if $R\subseteq R_i$ is a rectangle, then $\sigma(R)=0$. Since the product σ-algebra on R_i is generated by the rectangles contained in R_i, it follows that $\sigma(F)=0$ for every measurable (with respect to the product σ-algebra) subset $F\subseteq R_i$. Thus, if $F\subseteq\Omega$ is an arbitrary measurable subset then $\sigma(F\cap R_i)=0$ for each i; therefore, $\sigma(F)=0$.

Now suppose that $F\subseteq E^c$ is a measurable subset. Then $F\subseteq F'\cup F''$ as a disjoint union, where $F'\subseteq\Omega$ and F'' is marginally null. By the previous paragraph, $\sigma(F')=0$, while, since σ is an Arveson measure, $\sigma(F'')=0$. It follows that $\sigma(F)=0$. Thus, $|\sigma|(E^c)=0$.

Conversely, if $|\sigma|(E^c)=0$ then $\sigma(R)=0$ for every measurable rectangle contained in E^c. By (i), $\operatorname{supp}\sigma\subseteq E$. $\square$

For an ω-closed set $F\subseteq Y\times X$, we denote by $\mathbb{A}(F)$ the set of all measures σ in $\mathbb{A}(Y,X)$ such that $\operatorname{supp}\sigma\subseteq F$.

3.2. Operators

The importance of Arveson measures is explained by the fact that they define special operators called *pseudointegral* in [2], where they were introduced. This class of operators will be essential for our considerations.

The first part of the following result was established in [2, Theorem 1.5.1]; we include its full proof for completeness.

Theorem 3.2. *Let $\sigma \in \mathbb{A}(Y,X)$. There exists a unique operator $T_\sigma : H_1 \to H_2$ such that*

$$(T_\sigma f, g) = \int\limits_{Y\times X} f(x)\overline{g(y)}d\sigma(y,x), \quad f \in H_1,\ g \in H_2.$$

Moreover, $\|T_\sigma\| \le \|\sigma\|_{\mathbb{A}}$ and, for a given ω-closed subset $\kappa \subseteq X \times Y$, the operator T_σ is supported on κ if and only if $\operatorname{supp}\sigma \subseteq \hat{\kappa} \stackrel{def}{=} \{(y,x) : (x,y) \in \kappa\}$. If $h \in \Gamma(X,Y)$ and $\sigma \in \mathbb{A}(Y,X)$ then $\langle T_\sigma, h\rangle = \int_{Y\times X} \hat{h} d\sigma$.

Proof. Fix $\sigma \in \mathbb{A}(Y,X)$ and consider the sesqui-linear form $\phi : H_1 \times H_2 \to \mathbb{C}$ given by

$$\phi(f,g) = \int\limits_{Y\times X} f(x)\overline{g(y)}d\sigma(y,x).$$

Note that ϕ is well-defined:

$$\begin{aligned}\Big|\int\limits_{Y\times X} f(x)\overline{g(y)}d\sigma(y,x)\Big|^2 &\le \Big(\int\limits_{Y\times X} |f(x)||g(y)|d|\sigma|(y,x)\Big)^2 \\ &\le \int\limits_{Y\times X} |f(x)|^2 d|\sigma|(y,x) \int\limits_{Y\times X} |g(y)|^2 d|\sigma|(y,x) \\ &= \int\limits_X |f(x)|^2 d|\sigma|_X(x) \int\limits_Y |g(y)|^2 d|\sigma|_Y(y) \le \|\sigma\|_{\mathbb{A}}^2 \|f\|_2^2 \|g\|_2^2.\end{aligned}$$

By the Riesz Representation Theorem, there exists a unique operator $T_\sigma : H_1 \to H_2$ such that $(T_\sigma f, g) = \phi(f,g)$; moreover, $\|T_\sigma\| \le \|\sigma\|_{\mathbb{A}}$.

Let $\kappa \subseteq X \times Y$ and suppose that $\operatorname{supp}\sigma \subseteq \hat{\kappa}$. Let $\alpha \subseteq X$ and $\beta \subseteq Y$ be measurable subsets with $(\alpha \times \beta) \cap \kappa \simeq \emptyset$. By deleting null sets from α and β we may assume that, in fact, $(\alpha \times \beta) \cap \kappa = \emptyset$. If $f \in H_1$ (resp. $g \in H_2$) is supported on α (resp. β) then, by Proposition 3.1,

$$(T_\sigma f, g) = \int\limits_{(\beta\times\alpha)\cap\hat{\kappa}} f(x)\overline{g(y)}d\sigma(y,x) = 0;$$

thus, T_σ is supported on κ.

Conversely, suppose that T_σ is supported on κ and let $\beta \times \alpha \subseteq Y \times X$ be a rectangle of finite measure, marginally disjoint from $\hat{\kappa}$. Then

$$\sigma(\beta \times \alpha) = (T_\sigma \chi_\alpha, \chi_\beta) = 0,$$

and Proposition 3.1 implies that $\operatorname{supp} \sigma \subseteq \hat{\kappa}$, up to a marginally null set.

Finally, suppose that $h \in \Gamma(X, Y)$ and $\sigma \in \mathbb{A}(Y, X)$. Write $h = \sum_{i=1}^{\infty} f_i \otimes g_i$, where $(f_i)_{i\in\mathbb{N}} \subseteq H_1$ and $(g_i)_{i\in\mathbb{N}} \subseteq H_2$ are sequences of functions with $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$. The estimate in the first paragraph of the proof shows that $\int_{Y\times X} \sum_{i=1}^{\infty} |f_i(x)||g_i(y)| d|\sigma|(y, x) < \infty$.

Let $h_n = \sum_{i=1}^{n} f_i \otimes g_i$; by the Lebesgue Dominated Convergence Theorem, $\int_{Y\times X} \hat{h}_n d\sigma \to_{n\to\infty} \int_{Y\times X} \hat{h} d\sigma$. Thus,

$$\begin{aligned}
\langle T_\sigma, h\rangle &= \lim_{n\to\infty} \langle T_\sigma, h_n\rangle = \lim_{n\to\infty} \sum_{i=1}^{n} \langle T_\sigma, f_i \otimes g_i\rangle \\
&= \lim_{n\to\infty} \int\limits_{Y\times X} \sum_{i=1}^{n} f_i(x) g_i(y) d\sigma(y, x) = \int\limits_{Y\times X} h(x, y) d\sigma(y, x). \qquad \square
\end{aligned}$$

We recall some facts from [35] that will be needed subsequently. If $\kappa \subseteq X \times Y$ is ω-closed, a *κ-pair* is an element

$$(P, Q) \in \big(\mathcal{B}(\ell^2) \bar{\otimes} L^\infty(X, \mu)\big) \times \big(\mathcal{B}(\ell^2) \bar{\otimes} L^\infty(Y, \nu)\big)$$

such that, after the identification of P and Q with operator-valued weakly measurable functions, defined on X and Y, respectively, P and Q take values that are projections and $P(x)Q(y) = 0$ marginally almost everywhere on κ. A κ-pair is called simple if P and Q take finitely many values. The following was established in [35].

Theorem 3.3. *Let $\kappa \subseteq X \times Y$ be an ω-closed set. Then*

$$\mathfrak{M}_{\min}(\kappa) = \big\{T \in \mathcal{B}(H_1, H_2) : Q(I \otimes T)P = 0,\ \forall \kappa\text{-pair } (P, Q)\big\}$$

and

$$\mathfrak{M}_{\max}(\kappa) = \big\{T \in \mathcal{B}(H_1, H_2) : Q(I \otimes T)P = 0,\ \forall \text{simple } \kappa\text{-pair } (P, Q)\big\}.$$

A version of the next lemma for $\mathfrak{M}_{\min}(\kappa)$ was proved in [36, Proposition 5.3].

Lemma 3.4. *If $\kappa \subseteq X \times Y$ is an ω-closed set then*

$$\begin{aligned}
\mathfrak{M}_{\max}(\kappa) = \big\{T \in \mathcal{B}(H_1, H_2) : S_\varphi(T) = 0,\ &\text{for all } \varphi \in \mathfrak{S}(X, Y), \\
&\text{vanishing on an } \omega\text{-open neighbourhood of } \kappa\big\}.
\end{aligned}$$

Proof. Suppose that $T \in \mathcal{B}(H_1, H_2)$ belongs to the set on the right hand side of the above equality. If $\kappa \cap (\alpha \times \beta) \simeq \emptyset$ then $\chi_{\alpha\times\beta} \in \mathfrak{S}(X,Y)$ vanishes on the ω-open neighbourhood $(\alpha \times \beta)^c$ of κ and hence $M_{\chi_\beta} T M_{\chi_\alpha} = S_{\chi_{\alpha\times\beta}}(T) = 0$; thus, $T \in \mathfrak{M}_{\max}(\kappa)$.

Conversely, suppose that $T \in \mathfrak{M}_{\max}(\kappa)$ and let $\varphi \in \mathfrak{S}(X,Y)$ vanish on an ω-open neighbourhood of κ. If $h \in \Gamma(X,Y)$ then $\varphi h \in \Gamma(X,Y)$ and vanishes on an ω-open neighbourhood of κ. By [35],

$$\langle S_\varphi(T), h\rangle = \langle T, \varphi h\rangle = 0,$$

showing that $S_\varphi(T) = 0$. $\square$

Now we obtain two technical results; apart of applications to the mainstream of the paper they have additional applications which we believe are interesting on their own right. Namely, we will show that a tensor product formula holds for the minimal masa-bimodules, answering in this way affirmatively a question posed by J. Froelich in [14]. Simultaneously, we show that the minimal masa-bimodule $\mathfrak{M}_{\min}(\kappa)$ associated with an ω-closed set κ is the closure of all pseudo-integral operators with symbols supported on κ; this provides an alternative, "synthetic" description of $\mathfrak{M}_{\min}(\kappa)$ in measure-theoretic terms, similar to the topological one given originally by Arveson in [2].

Let (X,μ) and (Y,ν) are standard measure spaces. Recall that $\mathcal{F}$ denotes the product σ-algebra on $Y \times X$.

Lemma 3.5. *If $\sigma \in \mathbb{A}(Y,X)$ and $E \in \mathcal{F}$ then the measure σ_E given by $\sigma_E(F) = \sigma(E\cap F)$, $F \in \mathcal{F}$, belongs to $\mathbb{A}(Y,X)$.*

Proof. Let $\sigma \in \mathbb{A}(Y,X)$ and $E \in \mathcal{F}$. If $\alpha \subseteq X$ is measurable then, denoting by $\dot{\bigcup}$ the union of a family of pairwise disjoint measurable sets, we have

$$\begin{aligned}
|\sigma_E|_X(\alpha) = |\sigma_E|(Y\times\alpha) &= \sup\left\{\sum_{i=1}^k \bigl|\sigma_E(F_i)\bigr| : \dot{\bigcup_{i=1}^k} F_i = Y\times\alpha\right\} \\
&= \sup\left\{\sum_{i=1}^k \bigl|\sigma(E\cap F_i)\bigr| : \dot{\bigcup_{i=1}^k} F_i = Y\times\alpha\right\} \\
&\le \sup\left\{\sum_{i=1}^k |\sigma|(E\cap F_i) : \dot{\bigcup_{i=1}^k} F_i = Y\times\alpha\right\} \\
&\le \sup\left\{\sum_{i=1}^k |\sigma|(F_i) : \dot{\bigcup_{i=1}^k} F_i = Y\times\alpha\right\} = |\sigma|(Y\times\alpha) \\
&= |\sigma|_X(\alpha).
\end{aligned}$$

One shows similarly that $|\sigma_E|_Y \le |\sigma|_Y$; it now follows that $\sigma_E \in \mathbb{A}(Y,X)$. $\square$

Theorem 3.6. *Let $\kappa \subseteq X \times Y$ be an ω-closed set. Then*

$$\mathfrak{M}_{\min}(\kappa) = \overline{\{T_\sigma : \sigma \in \mathbb{A}(Y,X),\ \operatorname{supp}\sigma \subseteq \hat{\kappa}\}}^{w^*}.$$

Proof. Let $\mathfrak{M}_0(\kappa)$ denote the right hand side of the identity. We first show that $\mathfrak{M}_0(\kappa)$ is a weak* closed masa-bimodule. Since $T_\sigma + T_\nu = T_{\sigma+\nu}$, we have that $\mathfrak{M}_0(\kappa)$ is a (weak*) closed subspace of $\mathcal{B}(H_1, H_2)$. It is moreover easy to check that if $\varphi \in L^\infty(X,\mu)$ and $\psi \in L^\infty(Y,\nu)$ then $M_\psi T_\sigma M_\varphi = T_{\sigma'}$, where $\sigma' \in \mathbb{A}(Y,X)$ is given by

$$\sigma'(E) = \int\limits_{Y\times X} \psi(y)\varphi(x)d\sigma(y,x).$$

If σ is supported on $\hat{\kappa}$ then clearly so is σ'; hence, $\mathfrak{M}_0(\kappa)$ is a masa-bimodule.

We next claim that $\operatorname{supp}\mathfrak{M}_0(\kappa) = \kappa$. Suppose that $\alpha\times\beta$ is a rectangle of finite measure such that $P(\beta)T_\sigma P(\alpha) = 0$ for all $\sigma \in \mathbb{A}(Y,X)$ with $\operatorname{supp}\sigma \subseteq \hat{\kappa}$. Let $\tau \in \mathbb{A}(X,Y)$ be arbitrary, and $\tau_{\hat{\kappa}}$ be the measure defined as in Lemma 3.5. Then $\operatorname{supp}\tau_{\hat{\kappa}} \subseteq \hat{\kappa}$ and hence

$$\tau\big((\beta\times\alpha)\cap\hat{\kappa}\big) = \tau_{\hat{\kappa}}\big((\beta\times\alpha)\cap\hat{\kappa}\big) = \big(P(\beta)T_{\tau_{\hat{\kappa}}}P(\alpha)\chi_\alpha, \chi_\beta\big) = 0.$$

By Arveson's Null Set Theorem [2, Theorem 1.4.3], $(\beta\times\alpha)\cap\hat{\kappa} \simeq \emptyset$. It follows that κ is contained in the support of $\mathfrak{M}_0(\kappa)$; on the other hand, by Theorem 3.2, $\operatorname{supp}\mathfrak{M}_0(\kappa) \subseteq \kappa$, up to a marginally null set. It follows that $\kappa \simeq \operatorname{supp}\mathfrak{M}_0(\kappa)$.

Thus $\mathfrak{M}_{\min}(\kappa) \subseteq \mathfrak{M}_0(\kappa)$. To show the converse inclusion, it suffices, by [35, Theorem 4.4], to show that if a function $h \in \Gamma(X,Y)$ vanishes on κ then $\langle T_\sigma, h\rangle = 0$ for each $\sigma \in \mathbb{A}(Y,X)$ supported by $\hat{\kappa}$. But this follows from the equality $\langle T_\sigma, h\rangle = \int_{Y\times X}\hat{h}d\sigma$ (see Theorem 3.2). $\square$

Corollary 3.7. *Let $\kappa \subseteq X\times X$ be an ω-closed set such that $\mathfrak{M}_{\max}(\kappa)$ is a unital algebra. Then $\mathfrak{M}_{\min}(\kappa)$ is a (unital) algebra.*

Proof. It was shown in [2] that the set of all pseudo-integral operators is an algebra. Since $\mathfrak{M}_{\max}(\kappa)$ is an algebra, the set $\mathfrak{M}_0(\kappa)$ of all pseudo-integral operators in $\mathfrak{M}_{\max}(\kappa)$ is also an algebra. Hence its weak* closure $\overline{\mathfrak{M}_0(\kappa)}^{w^*}$ is also an algebra. By Theorems 3.2 and 3.6, $\mathfrak{M}_{\min}(\kappa) = \overline{\mathfrak{M}_0(\kappa)}^{w^*}$ and the proof is complete. $\square$

The next theorem establishes a tensor product formula for the minimal masa-bimodules. Let (X_i,μ_i) and (Y_i,ν_i) be standard measure spaces, $i = 1, 2$, and consider the flip

$$\rho : (X_1\times Y_1)\times(X_2\times Y_2) \to (X_1\times X_2)\times(Y_1\times Y_2)$$

given by

$$\rho\big((x_1,y_1),(x_2,y_2)\big) = \big((x_1,x_2),(y_1,y_2)\big).$$

Below, for two weak* closed subspaces $\mathcal{U}$ and $\mathcal{V}$ of operators, we denote by $\mathcal{U}\bar{\otimes}\mathcal{V}$ the weak* closed subspace generated by the elementary tensors $A\otimes B$ where $A\in\mathcal{U}$ and $B\in\mathcal{V}$.

Theorem 3.8. *Let (X_i,μ_i) and (Y_i,ν_i) be standard measure spaces and $\kappa_i\subseteq X_i\times Y_i$ be ω-closed sets, $i=1,2$. Then*

$$\mathfrak{M}_{\min}(\kappa_1)\bar{\otimes}\mathfrak{M}_{\min}(\kappa_2)=\mathfrak{M}_{\min}\big(\rho(\kappa_1\times\kappa_2)\big). \tag{5}$$

Proof. We first note that, by [26],

$$\operatorname{supp}\big(\mathfrak{M}_{\min}(\kappa_1)\bar{\otimes}\mathfrak{M}_{\min}(\kappa_2)\big)\simeq\rho(\kappa_1\times\kappa_2). \tag{6}$$

By the minimality property of $\mathfrak{M}_{\min}(\rho(\kappa_1\times\kappa_2))$ we have that

$$\mathfrak{M}_{\min}\big(\rho(\kappa_1\times\kappa_2)\big)\subseteq\mathfrak{M}_{\min}(\kappa_1)\bar{\otimes}\mathfrak{M}_{\min}(\kappa_2).$$

To see the reverse inclusion, it is enough prove that if $m\in\mathbb{A}(Y_1,X_1)$ and $n\in\mathbb{A}(Y_2,X_2)$ then $T_m\otimes T_n=T_\sigma$ for some measure $\sigma\in\mathbb{A}(Y_1\times Y_2,X_1\times X_2)$. Indeed, by (6), $\operatorname{supp}T_\sigma\subseteq\rho(\kappa_1\times\kappa_2)$ and hence Theorem 3.2 implies that $\operatorname{supp}\sigma\subseteq\rho(\widehat{\kappa_1\times\kappa_2})$. By Theorem 3.6, $T_\sigma\in\mathfrak{M}_{\min}(\rho(\kappa_1\times\kappa_2))$.

Let

$$\sigma(E)=\int\limits_{Y_2\times X_2}\int\limits_{Y_1\times X_1}\chi_E(y,x)dm(y_1,x_1)dn(y_2,x_2)$$

for every measurable $E\subseteq(Y_1\times Y_2)\times(X_1\times X_2)$. If $\beta_i\subseteq Y_i$, $i=1,2$, are measurable then

$$\begin{aligned}
&|\sigma|\big((\beta_1\times\beta_2)\times(X_1\times X_2)\big)\\
&\quad\le\int\limits_{Y_2\times X_2}\int\limits_{Y_1\times X_1}\chi_{(\beta_1\times\beta_2)\times(X_1\times X_2)}(y,x)d|m|(y_1,x_1)d|n|(y_2,x_2)\\
&\quad=|m|(\beta_1\times X_1)|n|(\beta_2\times X_2)\\
&\quad\le\|m\|_{\mathbb{A}}\|n\|_{\mathbb{A}}\nu_1(\beta_1)\nu_2(\beta_2)\\
&\quad=\|m\|_{\mathbb{A}}\|n\|_{\mathbb{A}}(\nu_1\times\nu_2)(\beta_1\times\beta_2).
\end{aligned}$$

It now easily follows that $|\sigma|(F\times(X_1\times X_2))\le\|m\|_{\mathbb{A}}\|n\|_{\mathbb{A}}(\nu_1\times\nu_2)(F)$, for any element F in the product σ-algebra on $Y_1\times Y_2$. Similar arguments show that $|\sigma|((Y_1\times Y_2)\times E)\le\|m\|_{\mathbb{A}}\|n\|_{\mathbb{A}}(\mu_1\times\mu_2)(E)$, for every measurable $E\subseteq X_1\times X_2$. Hence σ is an Arveson measure and T_σ is a bounded operator from $L^2(X_1\times X_2)$ to $L^2(Y_1\times Y_2)$.

If $f_i \in L^2(X_i, \mu_i)$, $g_i \in L^2(Y_i, \nu_i)$, $i = 1, 2$, we have

$$\begin{aligned}
&\big((T_m \otimes T_n)f_1 \otimes f_2, g_1 \otimes g_2\big) \\
&\quad = \int\limits_{Y_2 \times X_2} \int\limits_{Y_1 \times X_1} f_1(x_1)f_2(x_2)\overline{g_1(y_1)g_2(y_2)}dm(y_1, x_1)dn(y_2, x_2) \\
&\quad = \int\limits_{(Y_1 \times Y_2) \times (X_1 \times X_2)} (f_1 \otimes f_2)(x)\overline{(g_1 \otimes g_2)(y)}d\sigma(y, x),
\end{aligned}$$

and hence $T_m \otimes T_n = T_\sigma$, proving the statement. $\square$

4. Sets of multiplicity and their operator versions

In this section, we study sets of multiplicity and their operator versions, and examine the relations between them.

4.1. Sets of multiplicity in arbitrary locally compact groups

Let us recall the classical notion of a set of multiplicity, where $G = \mathbb{T}$ is the group of the circle; in this case, $A(\mathbb{T}) = \{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}}|c_n| < \infty\} \simeq \ell^1(\mathbb{Z})$. The space of *pseudo-measures* $PM(\mathbb{T}) = A(\mathbb{T})^*$ can be identified with $\ell^\infty(\mathbb{Z})$ via Fourier transform $F \mapsto (\hat{F}(n))_{n\in\mathbb{Z}}$, and the space of *pseudo-functions* $PF(\mathbb{T}) = \{F \in PM(\mathbb{T}) : \hat{F}(n) \to 0$, as $n \to \infty\}$ is $*$-isomorphic to $C^*(\mathbb{T}) = C_r^*(\mathbb{T})$. Note that there is a canonical embedding $M(\mathbb{T}) \subseteq PM(\mathbb{T})$ arising from the inclusion $A(\mathbb{T}) \subseteq C(\mathbb{T})$.

If E is a closed subset of $\mathbb{T}$, let $PM(E)$ denote the space of all pseudo-measures supported on E, $M(E)$ the space of measures $\mu \in M(G)$ with $\operatorname{supp}\mu \subseteq E$, and $N(E)$ the weak* closure of $M(E)$. For an ideal $J \subseteq A(G)$, let $J^\perp$ denote the annihilator of J in $PM(\mathbb{T})$; then $PM(E) = J(E)^\perp$ and $N(E) = I(E)^\perp$ (see, *e.g.*, [15]).

A closed set $E \subseteq \mathbb{T}$ is called an M-set if $PM(E) \cap PF(\mathbb{T}) \neq \{0\}$, an M_1-set if $N(E) \cap PF(\mathbb{T}) \neq \{0\}$, and an M_0-set if $M(E) \cap PF(\mathbb{T}) \neq \{0\}$. The closed sets that are not M-sets are called sets of uniqueness.

A definition of sets of multiplicity for locally compact abelian groups was proposed by I. Piatetski-Shapiro (see [16, p. 190]). In [4], M. Bożejko introduced sets of uniqueness in general locally compact groups. Here we extend his definition to include versions of M_1-sets and of M_0-sets.

Definition 4.1. A closed subset $E \subseteq G$ will be called

(i) an *M-set* if $J(E)^\perp \cap C_r^*(G) \neq \{0\}$;
(ii) an *M_1-set* if $I(E)^\perp \cap C_r^*(G) \neq \{0\}$;
(iii) an *M_0-set* if $\lambda(M(E)) \cap C_r^*(G) \neq \{0\}$.

The set E will be called a *U-set* (resp. a *U_1-set*, a *U_0-set*) if it is not an M-set (resp. an M_1-set, an M_0-set).

Remark 4.2. (i) Since $\lambda(M(E)) \subseteq I(E)^{\perp} \subseteq J(E)^{\perp}$, every M_0-set is an M_1-set, and every M_1-set is an M-set. It is known that these three classes of sets are distinct, see [15].

(ii) If G is amenable then $C_r^*(G)$ is $*$-isomorphic to $C^*(G)$ and it is a direct consequence of the definition that a closed set $E \subseteq G$ is an M-set (resp. an M_1-set) if and only if $J(E)$ (resp. $I(E)$) is not weak* dense in $B(G)$.

(iii) Measures $\mu \in M(G)$ satisfying the condition $\lambda(\mu) \in C_r^*(G)$ were studied in [3] where the author characterised them in terms of their values on certain Borel subsets of G. If G is compact or abelian then this class of measures coincides with the *Rajchman measures* on G, that is, the measures whose Fourier–Stieltjes coefficients vanish at infinity (see [3]).

We point out an easy source of examples of sets of multiplicity:

Remark 4.3. Every closed subset of positive Haar measure in a locally compact second countable group is an M_0-set.

Proof. Let $E \subseteq G$ be a measurable subset of positive Haar measure and $E_0 \subseteq E$ be a compact set of positive Haar measure; then $m(E_0) < \infty$. Let θ be the measure given by $d\theta(x) = \chi_{E_0}(x)dm(x)$. Clearly, $\operatorname{supp}\theta \subseteq E$ and $0 \neq \lambda(\theta) = \lambda(\chi_{E_0}) \in C_r^*(G)$. $\square$

4.2. Sets of operator multiplicity

For an ω-closed set $F \subseteq Y \times X$, we denote by $\mathbb{A}(F)$ the set of all measures σ in $\mathbb{A}(Y, X)$ such that $\operatorname{supp}\sigma \subseteq F$.

Operator versions of M-sets and M_1-sets were introduced by the authors in [34] in connection with the study of closable multipliers. We recall the relevant definition now, introducing the additional notion of an M_0-set.

Definition 4.4. Let (X, μ) and (Y, ν) be standard measure spaces. An ω-closed set $\kappa \subseteq X \times Y$ is called

(i) an *operator M-set* if $\mathcal{K}(H_1, H_2) \cap \mathfrak{M}_{\max}(\kappa) \neq \{0\}$;
(ii) an *operator M_1-set* if $\mathcal{K}(H_1, H_2) \cap \mathfrak{M}_{\min}(\kappa) \neq \{0\}$;
(iii) an *operator M_0-set* if there exists a non-zero measure $\sigma \in \mathbb{A}(\hat{\kappa})$ such that $T_\sigma \in \mathcal{K}(H_1, H_2)$.

We call κ an *operator U-set* (resp. an *operator U_1-set*, an *operator U_0-set*) if it is not an operator M-set (resp. an operator M_1-set, an operator M_0-set).

(Operator) M-sets will be referred to as *sets of (operator) multiplicity*, while (operator) U-sets – as *sets of (operator) uniqueness*. It will follow from Theorem 3.6 that if $\sigma \in \mathbb{A}(\hat{\kappa})$ then $T_\sigma \in \mathfrak{M}_{\min}(\kappa)$. Therefore, every operator M_0-set is an operator M_1-set, while every operator M_1-set is trivially an operator M-set.

Remark. Recall that $\mu \in M(G)$ is called a Rajchman measure if $\lambda(\mu) \in C_r^*(G)$. The compact operators of the form T_σ, where $\sigma \in \mathbb{A}(Y,X)$, can be thought of as an operator version of these measures.

4.3. A symbolic calculus

Aiming at applications to multiplicity sets we establish here a kind of symbolic calculus for completely bounded maps from $\mathcal{B}(L^2(G))$ to $\mathrm{VN}(G)$ (Theorem 4.6). We first recall the Stone–von Neumann Theorem in a suitable for our needs form. Let $\mathcal{D} = \{M_a : a \in L^\infty(G)\}$ and $\mathcal{D}_0 = \{M_a : a \in C_0(G)\}$. For each $s \in G$, let $\alpha_s : C_0(G) \to C_0(G)$ be given by $\alpha_s f(t) = f(s^{-1}t)$. The map $s \mapsto \alpha_s$ is a homomorphism from G into the automorphism group of $C_0(G)$, and thus gives rise to the (C^*-algebraic) crossed product $C_0(G) \rtimes_\alpha G$. Denoting for a moment by $\pi : C_0(G) \to \mathcal{B}(L^2(G))$ the representation given by $\pi(g) = M_g$, we have that the pair (π,λ) (where λ is the left regular representation of G on $L^2(G)$) is a covariant representation of the dynamical system $(C_0(G), G, \alpha)$. Thus, (π,λ) gives rise to a representation $\pi \times \lambda$ of $C_0(G) \rtimes_\alpha G$ on $L^2(G)$. By the Stone–von Neumann Theorem (see [41, Theorem 4.23]), this representation is faithful and its image coincides with the algebra $\mathcal{K}$ of all compact operators on $L^2(G)$. In particular, we claim that

$$\mathcal{K} = \overline{\left[AT : A \in \mathcal{D}_0,\ T \in C_r^*(G)\right]}^{\|\cdot\|} = \overline{\left[ATB : A, B \in \mathcal{D}_0,\ T \in C_r^*(G)\right]}^{\|\cdot\|} \tag{7}$$

(here, and in the sequel, $[\mathcal{E}]$ denotes the linear span of $\mathcal{E}$). To see that (7) holds, note that if $f \in L^1(G)$, $T = \lambda(f)$ and $A, B \in \mathcal{D}_0$, then

$$AT = \int_G f(t) A\lambda_t dt \in (\pi \times \lambda)\big(C_0(G) \rtimes_\alpha G\big) = \mathcal{K},$$

and thus $ATB \in \mathcal{K}$ as well. Conversely, it is easy to observe (see, *e.g.*, [28]) that the operators of the form $\sum_{i=1}^k \int_{E_i} A_i\lambda_s ds$, where $E_i \subseteq G$ are measurable sets of finite measure and $A_i \in \mathcal{D}_0$, $i = 1, \dots, k$, form a dense subset of $(\pi \times \lambda)(C_0(G) \rtimes_\alpha G)$; however, $\int_{E_i} A_i\lambda_s ds = A_i\lambda(\chi_{E_i})$, and the first equality in (7) is established. To complete the proof of the second equality, let $(B_i)_{i=1}^\infty \subseteq \mathcal{D}_0$ be a sequence strongly converging to the identity operator on $L^2(G)$, and note that if $A \in \mathcal{D}_0$ and $T \in C_r^*(G)$, then $AT = \lim_i ATB_i$ in norm, by the compactness of AT.

In the sequel, we will use the norm closed $\mathcal{D}$-bimodule generated by $C_r^*(G)$

$$\mathcal{A} = \overline{\left[ATB : A, B \in \mathcal{D},\ T \in C_r^*(G)\right]}^{\|\cdot\|} \tag{8}$$

and the smallest norm closed subspace of $\mathcal{B}(L^2(G))$ containing $C_r^*(G)$ and invariant under Schur multipliers

$$\mathcal{R} = \overline{\left[S_\varphi(T) : T \in C_r^*(G),\ \varphi \in \mathfrak{S}(G,G)\right]}^{\|\cdot\|}. \tag{9}$$

By (7),

$$\mathcal{K} \subseteq \mathcal{A} \subseteq \mathcal{R}. \tag{10}$$

Remark 4.5. (i) Let G be discrete. Then $\mathcal{A} = \mathcal{R}$. Indeed, in this case $C_r^*(G)$ is generated as a closed linear space by the unitaries λ_s, $s \in G$, which are normalisers of the multiplication masa $\mathcal{D}$. However, if $\varphi \in \mathfrak{S}(G,G)$ then $S_\varphi(\lambda_s) = M_f\lambda_s \in \mathcal{A}$ for some $f \in L^\infty(G)$ (see, *e.g.*, [21, Proposition 14]). It follows that $\mathcal{A}$ is invariant under Schur multiplication, and hence $\mathcal{R} = \mathcal{A}$. Note that, in the case G is infinite, $\mathcal{K}$ is strictly contained in $\mathcal{A}$ since λ_s is a unitary operator in $C_r^*(G)$ which is not compact.

In [32], given a discrete group G, J. Roe introduced what is now known as *the uniform Roe algebra* $UC_r^*(G)$ which equals, by definition, to the uniform closure in $\mathcal{B}(\ell^2(G))$ of the space of all matrices indexed by $G \times G$ with uniformly bounded entries supported on sets of the form $\{(s,t) \in G \times G : ts^{-1} \in E\}$, where E is finite. We note that $UC_r^*(G)$ coincides in this case with $\mathcal{R}$. Indeed, the unitary generators λ_s are represented by matrices (indexed by $G \times G$) whose sth diagonal has all entries equal to 1, and all other diagonals are zero. Multiplying by an operator of the form M_a, where $a \in \ell^\infty(G)$, we see that all matrices which, on a given diagonal, have a sequence from $\ell^\infty(G)$, are in $\mathcal{A} = \mathcal{R}$; thus, $UC_r^*(G) \subseteq \mathcal{R}$. Conversely, since $C_r^*(G)$ is generated as a norm closed subspace by the operators of the form λ_s, we have that $\mathcal{A} \subseteq UC_r^*(G)$, and hence $UC_r^*(G) = \mathcal{R}$.

The previous paragraph shows that the space $\mathcal{R}$ can be thought of as a locally compact version of the uniform Roe algebra.

(ii) If G is compact then $\mathcal{K} = \mathcal{A} = \mathcal{R}$. Indeed, in this case $C_r^*(G) \subseteq \mathcal{K}$ and since the compact operators are invariant under Schur multipliers, we have that $\mathcal{R} \subseteq \mathcal{K}$, and the equalities follow from (10).

In view of Remark 4.5, it is natural to ask whether $\mathcal{A} = \mathcal{R}$ for every locally compact group G; we do not know whether this equality always holds.

If G is compact then $N(A(G)) \subseteq \Gamma(G,G)$ and hence the formula

$$\langle E(T), u\rangle = \langle T, N(u)\rangle, \quad T \in \mathcal{B}(L^2(G)),\ u \in A(G),$$

defines a canonical expectation E from $\mathcal{B}(L^2(G))$ onto $\mathrm{VN}(G)$. This is the motivation behind the next theorem, where we exhibit a symbolic calculus for completely bounded maps from $\mathcal{B}(L^2(G))$ into $\mathrm{VN}(G)$ (that are not necessarily projections). Let us denote by $CB^{w^*}(\mathcal{B}(L^2(G)), \mathrm{VN}(G))$ the space of weak* continuous completely bounded maps from $\mathcal{B}(L^2(G))$ into $\mathrm{VN}(G)$. It has a natural structure of a right Banach module over

$\mathfrak{S}(G,G)$, the action being given by $\Phi \cdot \varphi = \Phi \circ S_\varphi$. Note that $\Gamma(G,G)$ is also a right Banach module over $\mathfrak{S}(G,G)$ under the action $\psi \cdot \varphi = \psi\varphi$.

Theorem 4.6. *For every $\varphi \in \Gamma(G,G)$ and every $T \in \mathcal{B}(L^2(G))$, there exists a unique operator $E_\varphi(T) \in \mathrm{VN}(G)$ such that*

$$\langle E_\varphi(T), u\rangle = \langle T, \varphi N(u)\rangle, \quad u \in A(G).$$

The transformation $\varphi \to E_\varphi$ is a contractive $\mathfrak{S}(G,G)$-module map from $\Gamma(G,G)$ into $CB^{w^*}(\mathcal{B}(L^2(G)), \mathrm{VN}(G))$. Moreover, if $\varphi \in \Gamma(G,G)$ then $E_\varphi(\lambda_s) = P(\varphi)(s)\lambda_s$, $s \in G$, and $E_\varphi(T) \in C_r^*(G)$, for all $T \in \mathcal{R}$.

Proof. Fix $\varphi \in \Gamma(G,G)$ and consider the mapping $e_\varphi : A(G) \to \Gamma(G,G)$ given by $e_\varphi(u) = \varphi N(u)$, $u \in A(G)$. The mapping $N : A(G) \to \mathfrak{S}(G,G)$ is completely isometric (see, *e.g.*, [38]). On the other hand, the mapping $\psi \to \varphi\psi$ from $\mathfrak{S}(G,G)$ into $\Gamma(G,G)$ is completely bounded with completely bounded norm not exceeding $\|\varphi\|_\Gamma$. Indeed, let $\psi_{i,j} \in \mathfrak{S}(G,G)$, $i,j = 1,\dots,n$; then, denoting by F_φ the functional on $\mathcal{B}(L^2(G))$ given by $F_\varphi(T) = \langle \varphi, T\rangle$, we have

$$\begin{aligned}
\big\|(\varphi\psi_{i,j})_{i,j}\big\|_{M_n(\Gamma(G,G))} &= \big\|(\varphi\psi_{i,j})_{i,j}\big\|_{CB(\mathcal{B}(L^2(G)),M_n(\mathbb{C}))} \\
&= \sup_{\|(T_{p,q})_{p,q}\|\le 1} \big\|\big(\langle \varphi\psi_{i,j}, T_{p,q}\rangle\big)_{(i,p),(j,q)}\big\| \\
&= \sup_{\|(T_{p,q})_{p,q}\|\le 1} \big\|\big(\langle \varphi, S_{\psi_{i,j}}(T_{p,q})\rangle\big)_{(i,p),(j,q)}\big\| \\
&\le \sup_{\|(T_{p,q})_{p,q}\|\le 1} \|F_\varphi\|\big\|\big(S_{\psi_{i,j}}(T_{p,q})\big)_{(i,p),(j,q)}\big\| \\
&\le \|\varphi\|_\Gamma \big\|(S_{\psi_{i,j}})_{i,j}\big\|_{\mathrm{cb}} \\
&= \|\varphi\|_\Gamma \big\|(\psi_{i,j})_{i,j}\big\|_{M_n(\mathfrak{S}(G,G))}.
\end{aligned}$$

Thus, e_φ is completely bounded and $\|e_\varphi\|_{\mathrm{cb}} \le \|\varphi\|_\Gamma$. It follows that the map $E_\varphi = e_\varphi^*$ is a normal completely bounded map from $\mathcal{B}(L^2(G))$ into $\mathrm{VN}(G)$ and $\|E_\varphi\|_{\mathrm{cb}} \le \|\varphi\|_\Gamma$. The identity

$$\langle E_\varphi(T), u\rangle = \langle T, \varphi N(u)\rangle, \quad u \in A(G),\ T \in \mathcal{B}\big(L^2(G)\big),$$

holds by the definition of E_φ.

It is obvious that the map $E : \varphi \to E_\varphi$ is linear and, by the previous paragraph, it is contractive. Moreover, if $\varphi \in \Gamma(G,G)$, $\psi \in \mathfrak{S}(G,G)$ and $u \in A(G)$, then

$$\langle E_{\varphi\psi}(T), u\rangle = \langle T, \psi\varphi N(u)\rangle = \langle S_\psi(T), \varphi N(u)\rangle = \langle (E_\varphi \circ S_\psi)(T), u\rangle,$$

which shows that E is an $\mathfrak{S}(G,G)$-module map.

Using (2), for every $u \in A(G)$ we have

$$\begin{aligned}\langle E_\varphi(\lambda_s), u\rangle &= \langle \lambda_s, \varphi N(u)\rangle = P\big(N(u)\varphi\big)(s)\\ &= u(s)P(\varphi)(s) = \langle P(\varphi)(s)\lambda_s, u\rangle,\end{aligned}$$

which shows that $E_\varphi(\lambda_s) = P(\varphi)(s)\lambda_s$.

Now suppose that $f \in L^1(G)$ and let $a, b \in L^2(G)$. Then

$$S_{N(u)}\big(\lambda(f)\big) = \lambda(uf), \quad u \in A(G). \tag{11}$$

Indeed, write $N(u) = \sum_{i=1}^\infty f_i \otimes g_i$, where $\|\sum_{i=1}^\infty |f_i|^2\|_\infty \le C < \infty$ and $\|\sum_{i=1}^\infty |g_i|^2\|_\infty \le C < \infty$. Then

$$\big(S_{N(u)}\big(\lambda(f)\big)a, b\big) = \sum_{i=1}^\infty \iint g_i(t)f(s)f_i\big(s^{-1}t\big)a\big(s^{-1}t\big)\overline{b(t)}dsdt.$$

Applying Fubini's arguments we obtain

$$\begin{aligned}\big(S_{N(u)}\big(\lambda(f)\big)a, b\big) &= \iint \sum_{i=1}^\infty g_i(t)f(s)f_i\big(s^{-1}t\big)a\big(s^{-1}t\big)\overline{b(t)}dsdt\\ &= \iint f(s)N(u)\big(s^{-1}t, t\big)a\big(s^{-1}t\big)\overline{b(t)}dtds\\ &= \iint u(s)f(s)a\big(s^{-1}t\big)\overline{b(t)}dtds = \big(\lambda(uf)a, b\big).\end{aligned}$$

Thus, (11) is established.

The mapping $u \mapsto N(u)$ from $A(G)$ into $\mathfrak{S}(G, G)$ is an isometry (see, *e.g.*, [38]); hence $\|S_{N(u)}(\lambda(f))\| \le \|N(u)\|_{\mathfrak{S}}\|\lambda(f)\| = \|u\|_{A(G)}\|\lambda(f)\|$ and therefore the mapping $u \mapsto \lambda(uf)$, $A(G) \to C_r^*(G)$, is continuous. We also have

$$\begin{aligned}\langle E_{a\otimes b}\big(\lambda(f)\big), u\rangle &= \langle \lambda(f), (a\otimes b)N(u)\rangle = \big(S_{N(u)}\big(\lambda(f)\big)a, \bar{b}\big)\\ &= \big(\lambda(uf)a, \bar{b}\big) = \iint u(s)f(s)a\big(s^{-1}t\big)b(t)dsdt\\ &= \int u(s)f(s)\bigg(\int a\big(s^{-1}t\big)b(t)dt\bigg)ds = \int u(s)f(s)(b * \check{a})(s)ds.\end{aligned}$$

Using (1), we conclude that

$$\langle E_{a\otimes b}\big(\lambda(f)\big), u\rangle = \int u(s)f(s)P(a\otimes b)(s)ds.$$

Note that, since $P(a\otimes b) \in A(G)$, the function $P(a\otimes b)f$ belongs to $L^1(G)$ and hence

$$\langle E_\varphi\big(\lambda(f)\big), u\rangle = \langle \lambda\big(P(\varphi)f\big), u\rangle \tag{12}$$

for $\varphi = a \otimes b$. By linearity, (12) holds whenever φ is a finite sum of elementary tensors. By the continuity of the transformations $\varphi \to E_\varphi$, $\varphi \to P(\varphi)$ and $g \to \lambda(gf)$ (the last one mapping $A(G)$ into $C_r^*(G)$), we conclude that (12) holds for all $\varphi \in \Gamma(G,G)$.

Relation (12) implies that $E_\varphi(\lambda(f)) = \lambda(P(\varphi)f) \in C_r^*(G)$, for all $f \in L^1(G)$ and all $\varphi \in \Gamma(G,G)$. Since E_φ is norm continuous and $\lambda(L^1(G))$ is dense in $C_r^*(G)$, we have that $E_\varphi(C_r^*(G)) \subseteq C_r^*(G)$. If $\psi \in \mathfrak{S}(G,G)$ and $T \in C_r^*(G)$ then

$$E_\varphi\big(S_\psi(T)\big) = E_{\varphi\psi}(T) \in C_r^*(G).$$

It follows that $E_\varphi(\mathcal{R}) \subseteq C_r^*(G)$, for every $\varphi \in \Gamma(G,G)$. $\square$

We will assume, for the rest of the paper, that G is second countable. The following lemma will be needed in the proof of Theorem 4.9.

Lemma 4.7. *Suppose that $T \in \mathcal{B}(L^2(G))$ is non-zero. Then there exist $a, b \in L^2(G)$ such that $E_{a\otimes b}(T) \neq 0$.*

Proof. Let $T \in \mathcal{B}(L^2(G))$ be a non-zero operator, and suppose, by way of contradiction, that $E_{a\otimes b}(T) = 0$ for all $a, b \in L^2(G)$. We may assume that $T = M_{\chi_K} T M_{\chi_K}$ for some compact set $K \subseteq G$. By Theorem 4.6, $E_\varphi(T) = 0$ for every $\varphi \in \Gamma(G,G)$. Since

$$\big\langle E_\varphi(T), u\big\rangle = \big\langle T, \varphi N(u)\big\rangle = \big\langle S_{N(u)}(T), \varphi\big\rangle, \quad u \in A(G),\ \varphi \in \Gamma(G,G),$$

we have that $S_{N(u)}(T) = 0$ for every $u \in A(G)$. Let

$$\mathcal{W} = \operatorname{span}\big\{N(u)\psi : \psi \in \Gamma(G,G),\ u \in A(G)\big\}.$$

Then $\mathcal{W} \subseteq \Gamma(G,G)$ is a subspace, invariant under $\mathfrak{S}(G)$, and $T \in \mathcal{W}^\perp$. Denoting by $\operatorname{null}(\mathcal{W})$ the complement of the ω-union [34] of the family $\{h^{-1}(\mathbb{C} \setminus \{0\}) : h \in \mathcal{W}\}$, we have $\operatorname{null}(\mathcal{W}) \simeq \emptyset$. In fact, since G is second countable and locally compact, there exists an increasing sequence of compact sets $\{K_n\}$ such that $\bigcup_{n=1}^\infty K_n = G$. For each $n \in \mathbb{N}$, choose a function $u_n \in A(G)$ that takes the value 1 on K_n. Then, up to a marginally null set,

$$\begin{aligned}
\operatorname{null}(\mathcal{W}) &\subseteq \bigcap_{n,m=1}^{\infty} \operatorname{null}\big(N(u_n)\chi_{K_m} \times \chi_{K_m}\big) \subseteq \bigcap_{n,m=1}^{\infty} \big(K_n^* \cap (K_m \times K_m)\big)^c \\
&= \bigcap_{n=1}^{\infty} \left(\big(K_n^c\big)^* \cup \left(\bigcup_{m=1}^{\infty} K_m \times K_m\right)^c\right) = \bigcap_{n=1}^{\infty} \big(K_n^c\big)^* \\
&= \left(\left(\bigcup_{n=1}^{\infty} K_n\right)^c\right)^* = \emptyset.
\end{aligned}$$

By [35, Corollary 4.3], $\mathcal{W}$ is dense in $\Gamma(G,G)$ and hence $T = 0$, a contradiction. $\square$

If $E \subseteq G$, we let

$$E^* = \{(s,t) \in G \times G : ts^{-1} \in E\}.$$

If E is closed then E^* is closed and hence, if G is second countable, it is ω-closed.

4.4. Multiplicity versus operator multiplicity

In the case of compact abelian groups, a connection between M-sets (resp. M_1-sets) and operator M-sets (resp. operator M_1-sets) was established in [14] (resp. [34]). Our aim now is to extend these results to arbitrary locally compact groups; a corresponding statement for M_0-sets will be proved in the next subsection. We will need the following lemma.

Lemma 4.8. *Let $E \subseteq G$ be a closed set and $T \in \mathrm{VN}(G)$. If $\operatorname{supp} T \subseteq E$ then T is supported on E^*.*

Proof. Let K and L be compact sets such that $(K \times L) \cap E^* = \emptyset$. Then $(LK^{-1}) \cap E = \emptyset$. As the mapping $(s,t) \in G \times G \mapsto ts^{-1} \in G$ is continuous, the set LK^{-1} is compact. If now f and $g \in L^2(G)$ are such that $f\chi_K = 0$ and $g\chi_L = 0$ then the function $u \in A(G)$ given by $u(s) = (\lambda_s(f), g)$, $s \in G$, has support in LK^{-1} and hence $u \in J(E)$. Therefore $(Tf, g) = \langle T, u\rangle = 0$. This implies that $P_L T P_K = 0$. By the regularity of the Haar measure, $P_L T P_K = 0$ whenever K and L are Borels sets with $(K \times L) \cap E^* = \emptyset$. Hence T is supported on E^*. $\square$

Theorem 4.9. *Let G be a locally compact second countable group and let $E \subseteq G$ be a closed subset.*

(a) *The following are equivalent:*
  - (i) *E is an M-set;*
  - (ii) *E^* is an operator M-set;*
  - (iii) *$\mathcal{A} \cap \mathfrak{M}_{\max}(E^*) \neq \{0\}$;*
  - (iv) *$\mathcal{R} \cap \mathfrak{M}_{\max}(E^*) \neq \{0\}$.*

(b) *The following are equivalent:*
  - (i′) *E is an M_1-set;*
  - (ii′) *E^* is an operator M_1-set;*
  - (iii′) *$\mathcal{A} \cap \mathfrak{M}_{\min}(E^*) \neq \{0\}$;*
  - (iv′) *$\mathcal{R} \cap \mathfrak{M}_{\min}(E^*) \neq \{0\}$.*

Proof. (a) (i) $\Rightarrow$ (ii) Let E be an M-set; then there exists a non-zero operator $T \in J(E)^{\perp} \cap C_r^*(G)$. Suppose that $AT = 0$ for all $A \in \mathcal{D}_0$. Since $\mathcal{D}_0$ is weak* dense in $\mathcal{D}$, there exists a net $(A_j)_{j \in \mathbb{J}} \subseteq \mathcal{D}_0$ such that $\lim_{j \in \mathbb{J}} A_j = I$ in the weak* topology. After passing to a limit, we obtain that $T = 0$, a contradiction. Thus, there exists $A \in \mathcal{D}_0$

such that $AT \neq 0$; in view of (7), $AT \in \mathcal{K}$. By Lemma 4.8, $T \in \mathfrak{M}_{\max}(E^*)$ and hence $AT \in \mathfrak{M}_{\max}(E^*)$; thus, E^* is an operator M-set.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow from the inclusions (10).

(iv) $\Rightarrow$ (i) Suppose that $T \in \mathcal{R} \cap \mathfrak{M}_{\max}(E^*)$ is non-zero. By Lemma 4.7, there exist $a, b \in L^2(G)$ such that $E_{a\otimes b}(T) \neq 0$. By Theorem 4.6, $E_{a\otimes b}(T) \in C_r^*(G)$; we claim that, moreover, $E_{a\otimes b}(T) \in J(E)^\perp$. To see this, let $u \in A(G)$ vanish on an open neighbourhood of E and have compact support. Then $N(u) \in \mathfrak{S}(G,G)$ vanishes on an ω-open neighbourhood of E^*, and hence the function $(a \otimes b)N(u) \in \Gamma(G,G)$ vanishes on an ω-open neighbourhood of E^*. On the other hand, by [35, Theorem 4.3], we have

$$\big(S_{N(u)}(T)a, \bar{b}\big) = \big\langle T, (a \otimes b)N(u)\big\rangle = 0,$$

giving $\langle E_{a\otimes b}(T), u\rangle = 0$. Thus, $0 \neq E_{a\otimes b}(T) \in J(E)^\perp$ and hence E is an M-set.

(b) (i′) $\Rightarrow$ (ii′) We claim that $\lambda_s \in \mathfrak{M}_{\min}(E^*)$ for every $s \in E$. To see this, suppose that $w \in \Gamma(G,G)$ vanishes on the set E^*, that is, $w\chi_{E^*} = 0$ marginally almost everywhere. For every $r \in G$ and $s \in E$, we have that $(s^{-1}r, r) \in E^*$ and hence $w(s^{-1}r, r) = 0$ for every $s \in E$ and almost every $r \in G$. By (2), $P(w)(s) = 0$ for every $s \in E$ and hence, by (1), $\langle \lambda_s, w\rangle = 0$ for every $s \in E$; the claim is thus proved.

Suppose that E is an M_1-set, and let $0 \neq T \in I(E)^\perp \cap C_r^*(G)$. A direct verification shows that $I(E)^\perp = \overline{[\lambda_s : s \in E]}^{w^*}$. It follows from the previous paragraph that $T \in \mathfrak{M}_{\min}(E^*)$. As in the proof of the implication (i) $\Rightarrow$ (ii), we conclude that there exists $A \in \mathcal{D}_0$ such that $0 \neq AT \in \mathcal{K} \cap \mathfrak{M}_{\min}(E^*)$, that is, E^* is an M_1-set.

(ii′) $\Rightarrow$ (iii′) $\Rightarrow$ (iv′) follow from the inclusions (10).

(iv′) $\Rightarrow$ (i′) Suppose that $0 \neq T \in \mathcal{R} \cap \mathfrak{M}_{\min}(E^*)$. As in the proof of (a), we can show that there exist $a, b \in L^2(G)$ such that $E_{a\otimes b}(T)$ is a non-zero element of $C_r^*(G)$ annihilating $I(E)$. $\square$

4.5. The case of M_0-sets

In order to establish a statement for M_0-sets, analogous to the ones from Theorem 4.9, we need a couple of auxiliary lemmas.

Lemma 4.10. *If σ is an Arveson measure on $G \times G$ then for every $\varphi \in \Gamma(G,G)$ there exists a unique measure $\sigma_\varphi \in M(G)$ such that $E_\varphi(T_\sigma) = \lambda(\sigma_\varphi)$. Moreover, if* $\operatorname{supp}\sigma \subseteq \widehat{E^*}$ *then* $\operatorname{supp}\sigma_\varphi \subseteq E$.

Proof. Let $\varphi = \sum_{i=1}^\infty f_i \otimes g_i \in \Gamma(G,G)$ (here $\sum_{i=1}^\infty \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^\infty \|g_i\|_2^2 < \infty$); note that

$$\left\| \sum_{i=1}^\infty |f_i| \otimes |g_i| \right\|_\Gamma \leq \|\varphi\|_\Gamma.$$

If $u \in C_0(G)$ then

$$
\begin{aligned}
\left| \int\limits_{G\times G} \varphi(s,t)u\big(ts^{-1}\big)d\sigma(t,s)\right| &\le \int\limits_{G\times G} \big|\varphi(s,t)\big|\big|u\big(ts^{-1}\big)\big|d|\sigma|(t,s) \\
&\le \|u\|_\infty \int\limits_{G\times G} \sum_{i=1}^{\infty}\big|f_i(s)\big|\big|g_i(t)\big|d|\sigma|(t,s) \\
&= \|u\|_\infty \sum_{i=1}^{\infty}\big(T_{|\sigma|}|f_i|,|g_i|\big) \\
&= \|u\|_\infty \left\langle T_{|\sigma|}, \sum_{i=1}^{\infty}|f_i|\otimes|g_i|\right\rangle \\
&\le \|u\|_\infty \|T_{|\sigma|}\| \left\|\sum_{i=1}^{\infty}|f_i|\otimes|g_i|\right\|_{\Gamma} \\
&\le \|u\|_\infty \|\sigma\|_{\mathbb{A}} \|\varphi\|_{\Gamma}.
\end{aligned}
$$

It follows that the functional $R : C_0(G) \to \mathbb{C}$ given by

$$
R(u) = \int\limits_{G\times G} \varphi(s,t)u\big(ts^{-1}\big)d\sigma(t,s), \quad u \in C_0(G),
$$

is well-defined and bounded. Hence, there exists $\sigma_\varphi \in M(G)$ such that

$$
\int\limits_{G\times G} \varphi(s,t)u\big(ts^{-1}\big)d\sigma(t,s) = \int\limits_G u(x)d\sigma_\varphi(x), \quad u \in C_0(G). \tag{13}
$$

On the other hand,

$$
\int\limits_G u(x)d\sigma_\varphi(x) = \big\langle \lambda(\sigma_\varphi), u\big\rangle, \quad u \in A(G).
$$

By (13) and Theorem 3.2,

$$
\big\langle \lambda(\sigma_\varphi), u\big\rangle = \big\langle T_\sigma, \varphi N(u)\big\rangle = \big\langle E_\varphi(T_\sigma), u\big\rangle, \quad u \in A(G);
$$

thus, $E_\varphi(T_\sigma) = \lambda(\sigma_\varphi)$.

Now suppose that $\operatorname{supp}\sigma \subseteq \widehat{E^*}$ and that $U \subseteq G$ is an open set, disjoint from E. For any function $u \in C_0(G)$ with $\operatorname{supp} u \subseteq U$, we have that $\operatorname{supp} N(u) \subseteq \widehat{U^*}$. On the other hand, $\widehat{U^*}$ is disjoint from $\widehat{E^*}$ and hence Proposition 3.1 implies that $|\sigma|(\widehat{U^*}) = 0$. Now (13) shows that $\int_G u(x)d\sigma_\varphi(x) = 0$. It follows that $\sigma_\varphi(U) = 0$; thus, $\operatorname{supp}\sigma_\varphi \subseteq E$. □

We will need the following fact, which was discussed in [37, p. 347] in the case of a finite measure (here we need a σ-finite version of this as the Haar measure on a locally compact non-compact group is such).

Lemma 4.11. *Let (X,μ) and (Y,ν) be σ-finite standard measure spaces and $(\sigma^x)_{x\in X}$ be a family of complex Borel measures on Y such that, for every measurable $F \subseteq Y$, the function $x \mapsto \sigma^x(F)$ is measurable. Suppose that the function $x \mapsto \|\sigma^x\|$ is integrable and essentially bounded (with respect to the measure μ). Then there exists a Borel measure σ on $Y \times X$ such that $\sigma(E) = \int_X \int_Y \chi_E(y,x)d\sigma^x(y)d\mu(x)$, for every measurable set $E \subseteq Y \times X$, and a constant $c > 0$ such that $|\sigma|(Y \times \alpha) \le c\mu(\alpha)$ for every measurable set $\alpha \subseteq X$.*

Proof. First of all, notice that the quantity

$$\sigma(E) = \int\limits_X \int\limits_Y \chi_E(y,x)d\sigma^x(y)d\mu(x)$$

is finite. Indeed,

$$\begin{aligned}\left|\int\limits_X \int\limits_Y \chi_E(y,x)d\sigma^x(y)d\mu(x)\right| &\le \int\limits_X \int\limits_Y \chi_E(y,x)d\left|\sigma^x\right|(y)d\mu(x) \\ &\le \int\limits_X \left|\sigma^x\right|(Y)d\mu(x) < \infty.\end{aligned}$$

A direct verification now shows that σ is a measure. Moreover, the above estimate yields

$$|\sigma|(E) \le \int\limits_X \int\limits_Y \chi_E(y,x)d\left|\sigma^x\right|(y)d\mu(x),$$

for every measurable set $E \subseteq Y \times X$. Letting $c = \operatorname{ess\,sup}_{x\in X}\|\sigma^x\|$, for every measurable $\alpha \subseteq X$, we have

$$|\sigma|(Y \times \alpha) \le \int\limits_\alpha \left\|\sigma^x\right\|d\mu(x) \le c\mu(\alpha). \qquad \square$$

In the next theorem, we let

$$\mathfrak{P}(\kappa) = \left\{T_\mu : \mu \in \mathbb{A}(\hat{\kappa})\right\}.$$

Theorem 4.12. *Let $E \subseteq G$ be a closed set. The following are equivalent:*

(i) *E is an M_0-set;*
(ii) *E^* is an operator M_0-set;*
(iii) *$\mathcal{A} \cap \mathfrak{P}(E^*) \ne \{0\}$;*
(iv) *$\mathcal{R} \cap \mathfrak{P}(E^*) \ne \{0\}$.*

Proof. (i) $\Rightarrow$ (ii) Let $\theta \in M(G)$ be such that $\operatorname{supp}\theta \subseteq E$ and $\lambda(\theta) \in C_r^*(G)$. Then $M_g\lambda(\theta)M_f$ is a compact operator for all $f, g \in C_0(G)$ (see (7)).

For each $x \in G$, let $\theta^x \in M(G)$ be given by $\theta^x(\alpha) = \theta(x\alpha^{-1})$ and θ_x be given by $\theta_x(\alpha) = \theta(\alpha x^{-1})$, for any measurable $\alpha \subseteq G$ (here $\alpha^{-1} = \{s^{-1} : s \in \alpha\}$). Let $\theta^* \in M(G)$ be defined by $d\theta^*(s) = \overline{d\theta(s^{-1})}$; then $\lambda(\theta^*) = \lambda(\theta)^*$. First observe that $\|\theta^x\| = \|\theta\|$ for each $x \in G$. Indeed, if $\{\alpha_j\}_{j=1}^N$ is a measurable partition of G then $\{x\alpha_j^{-1}\}_{j=1}^N$ is also such, and hence

$$\sum_{j=1}^N \left|\theta^x(\alpha_j)\right| = \sum_{j=1}^N \left|\theta\left(x\alpha_j^{-1}\right)\right| \le \|\theta\|.$$

On the other hand, for every $\epsilon > 0$, letting $\{\beta_k\}_{k=1}^K$ be a measurable partition of G such that $\sum_{k=1}^K |\theta(\beta_k)| > \|\theta\| - \epsilon$, we see that $\{\beta_k^{-1}x\}_{k=1}^K$ is a measurable partition of G with $\sum_{k=1}^K |\theta^x(\beta_k^{-1}x)| > \|\theta\| - \epsilon$, and so $\|\theta^x\| \ge \|\theta\|$. Similarly, $\|\theta_x^*\| = \|\theta^*\|$ for all $x \in G$.

If $f, g \in C_0(G)$ then

$$\begin{aligned}\left(M_g\lambda(\theta)M_f\xi, \eta\right) &= \iint f\left(y^{-1}x\right)\xi\left(y^{-1}x\right)g(x)\overline{\eta(x)}d\theta(y)dx \\ &= \iint f(z)\xi(z)g(x)\overline{\eta(x)}d\theta^x(z)dx \end{aligned} \tag{14}$$

and, also,

$$\begin{aligned}\left(M_g\lambda(\theta)M_f\xi, \eta\right) &= \left(M_f\xi, \lambda\left(\theta^*\right)M_{\bar{g}}\eta\right) \\ &= \iint f(z)\xi(z)g\left(x^{-1}z\right)\overline{\eta\left(x^{-1}z\right)}\overline{d\theta^*(x)}dz \\ &= \iint f(z)\xi(z)g(x)\overline{\eta(x)}\overline{d\left(\theta^*\right)^z(x)}dz.\end{aligned}$$

If, moreover, $f, g \in C_0(G) \cap L^1(G)$ and $x \in G$, the total variation of the measure $g(x)f(\cdot)d\theta^x(\cdot)$ equals $\int_G |f(z)|d|g(x)\theta^x|$ which does not exceed $\|f\|_\infty\|g(x)\theta^x\|$. Hence, $\|g(x)f(\cdot)d\theta^x(\cdot)\| \le \|g\|_\infty\|f\|_\infty\|\theta\|$ for all $x \in G$. Furthermore, the function $x \mapsto \|f\|_\infty\|g(x)\theta^x\|$ is integrable since $x \mapsto \|\theta^x\|$ is a constant function.

Similarly, the total variation of the measure $f(z)g(\cdot)d(\theta^*)^z(\cdot)$ does not exceed $\|g\|_\infty\|f\|_\infty\|\theta^*\|$, and the function $z \to \|g\|_\infty\|f(z)d(\theta^*)^z\|$ is integrable. Lemma 4.11 now shows that, if $f, g \in C_0(G) \cap L^1(G)$, then $M_g\lambda(\theta)M_f$ is the pseudo-integral operator of the Arveson measure $\sigma_{f,g,\theta}$ given by $d\sigma_{f,g,\theta}(x,z) = g(x)f(z)\overline{d(\theta^*)^z(x)}dz = g(x)f(z)d\theta^x(z)dx$. On the other hand, since $\lambda(\theta) \in C_r^*(G)$, the operator $M_g\lambda(\theta)M_f$ is compact whenever $f, g \in C_0(G) \cap L^1(G)$. It is now clear that, since $\theta \ne 0$, we can find functions $f, g \in C_0(G) \cap L^1(G)$ such that $M_g\lambda(\theta)M_f$ is non-zero.

Suppose that $\alpha \times \beta$ is a measurable rectangle with $(\alpha \times \beta) \cap E^* = \emptyset$ and $\xi \in L^2(G)$ (resp. $\eta \in L^2(G)$) vanishes everywhere on α^c (resp. β^c). For each $x \in G$, the function

$y \mapsto \xi(y^{-1}x)\overline{\eta(x)}$ vanishes on E and hence, by (14), $(M_g\lambda(\theta)M_f\xi, \eta) = 0$. Thus, $M_g\lambda(\theta)M_f$ is supported on E^*.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (i) Suppose that σ is an Arveson measure supported on $\widehat{E^*}$ such that $0 \neq T_\sigma \in \mathcal{R}$. By Lemma 4.7, there exists $\varphi \in \Gamma(G,G)$ such that $E_\varphi(T_\sigma) \neq 0$. By Lemma 4.10, $E_\varphi(T_\sigma) = \lambda(\sigma_\varphi)$, where σ_φ is supported on E and, by Theorem 4.6, $\lambda(\sigma_\varphi)$ belongs to $C_r^*(G)$. $\square$

4.6. An application: unions of sets of uniqueness

It was shown in [34, Proposition 5.3] that the union of two operator U-sets (resp. operator U_1-sets) is an operator U-set (resp. an operator U_1-set). A similar statement holds for operator U_0-sets.

Proposition 4.13. *Let $E_1, E_2 \subset X \times Y$ be ω-closed operator U_0-sets. Then $E_1 \cup E_2$ is an operator U_0-set.*

Proof. Let T_σ be a pseudo-integral compact operator supported on $E_1 \cup E_2$; we may assume that the total variation of σ is 1. Let $\theta_i \in \Phi(E_i) \cap \mathfrak{S}(X,Y)$, $i = 1,2$, and write $\theta_1(x,y) = \sum_{i=1}^\infty f_i(x)g_i(y)$, where $\|\sum_{i=1}^\infty |f_i|^2\|_\infty \leq C$ and $\|\sum_{i=1}^\infty |g_i|^2\|_\infty \leq C$. We have that $\theta_1\theta_2 \in \Phi(E_1 \cup E_2)$ and hence

$$0 = \langle T_\sigma, \theta_1\theta_2\rangle = \big\langle S_{\theta_1}(T_\sigma), \theta_2\big\rangle. \tag{15}$$

Let ρ be the measure on $Y \times X$ given by

$$\rho(E) = \int_E \theta_1(x,y)d\sigma(y,x).$$

Denoting by $\dot{\bigcup}$ the union of a pairwise disjoint family of measurable sets, we have

$$\begin{aligned}
|\rho|_X(\alpha) &= |\rho|(Y \times \alpha) \\
&= \sup\left\{\sum_{j=1}^r \big|\rho(E_j)\big| : Y \times \alpha = \dot{\bigcup_{j=1}^r} E_j\right\} \\
&= \sup\left\{\sum_{j=1}^r \left|\int_{E_j} \theta_1(x,y)d\sigma(y,x)\right| : Y \times \alpha = \dot{\bigcup_{j=1}^r} E_j\right\} \\
&\leq \sup\left\{\sum_{j=1}^r \int_{E_j} \big|\theta_1(x,y)\big|d|\sigma|(y,x) : Y \times \alpha = \dot{\bigcup_{j=1}^r} E_j\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \int\limits_{Y\times\alpha} \sum_{i=1}^{\infty} \big|f_i(x)\big|\big|g_i(y)\big| d|\sigma|(y,x) \\
&\leq \sum_{i=1}^{\infty} \Big(\int\limits_{Y\times\alpha} \big|f_i(x)\big|^2 d|\sigma|(y,x) \Big)^{1/2} \Big(\int\limits_{Y\times\alpha} \big|g_i(y)\big|^2 d|\sigma|(y,x) \Big)^{1/2} \\
&\leq \Bigg(\int\limits_{Y\times\alpha} \sum_{i=1}^{\infty} \big|f_i(x)\big|^2 d|\sigma|(y,x) \Bigg)^{1/2} \Bigg(\int\limits_{Y\times\alpha} \sum_{i=1}^{\infty} \big|g_i(y)\big|^2 d|\sigma|(y,x) \Bigg)^{1/2} \\
&\leq C^2 |\sigma|_X(\alpha).
\end{aligned}
$$

Similarly, $|\rho|_Y(\beta) \leq C^2|\sigma|_Y(\beta)$ showing that ρ is an Arveson measure. Now the identity

$$
\big(S_{\theta_1}(T_\sigma)\xi, \eta\big) = \int\limits_{Y\times X} \theta_1(x,y)\xi(x)\overline{\eta(y)} d\sigma(y,x), \quad \xi \in H_1, \ \eta \in H_2,
$$

shows that $S_{\theta_1}(T_\sigma) = T_\rho$.

Let $h \in \Phi(E_2)$ and write $h = \sum_{i=1}^{\infty} f_i \otimes g_i$, where $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|_2^2 < \infty$. Let $X_N = \{x \in X : \sum_{i=1}^{\infty} |f_i(x)|^2 \leq N\}$ and $Y_N = \{y \in Y : \sum_{i=1}^{\infty} |g_i(y)|^2 \leq N\}$. Then $\chi_{X_N\times Y_N} h \in \mathfrak{S}(X,Y)$ and $\|\chi_{X_N\times Y_N} h - h\|_\Gamma \to_{N\to\infty} 0$. Thus, $\Phi(E_2) \cap \mathfrak{S}(X,Y)$ is dense in $\Phi(E_2)$, and, by (15), $T_\rho \in \Phi(E_2)^\perp = \mathfrak{M}_{\min}(E_2)$. As E_2 is an operator U_0-set, $T_\rho = 0$ and therefore $\rho = 0$. By Theorem 3.2, $\langle T_\sigma, \theta_1\rangle = \rho(Y\times X) = 0$. Since this holds for any $\theta_1 \in \Phi(E_1) \cap \mathfrak{S}(X,Y)$, the operator T_σ is supported on E_1. Since E_1 is an operator U_0-set, $T_\sigma = 0$. $\square$

Proposition 4.13, [34, Proposition 5.3], Theorem 4.9 and Theorem 4.12 have the following immediate corollary.

Corollary 4.14. *Let G be a locally compact second countable group. Suppose that $E_1, E_2 \subseteq G$ are U-sets (resp., U_1-sets, U_0-sets). Then $E_1 \cup E_2$ is a U-set (resp. a U_1-set, a U_0-set).*

5. Preservation properties

The aim of this section is to show that the property of being a set of multiplicity, or a set of uniqueness, is preserved under some natural operations. The section is divided into three subsections.

5.1. Sets possessing an m-resolution

Here we consider a certain type of a countable union of operator U-sets. Theorem 5.2 should be compared to the classical result of N.K. Barry that a countable union of U-sets is a U-set [24].

Definition 5.1. Let (X,μ) and (Y,ν) be standard measure spaces.

(i) A pair (κ_1,κ_2) of ω-closed subsets of the direct product $X\times Y$ will be called *m-separable* if there exist a function $\varphi_1 \in \mathfrak{S}(X,Y)$ and ω-open neighbourhoods E_1 and E_2 of κ_1 and κ_2, respectively, such that $\varphi|_{E_1}=1$ and $\varphi|_{E_2}=0$.

(ii) Let $\kappa\subseteq X\times Y$ be an ω-closed set and α be a countable ordinal. We call a family $(\kappa_\beta)_{\beta\le\alpha}$ of ω-closed sets an *m-resolution* of κ if

- $\kappa_1=\kappa$;
- $\kappa_{\beta+1}\subseteq\kappa_\beta$, the set $\kappa_\beta\setminus\kappa_{\beta+1}$ is ω-closed and the pair $\kappa_{\beta+1},\kappa_\beta\setminus\kappa_{\beta+1}$ is an m-separable, for every ordinal $\beta<\alpha$;
- $\kappa_\beta=\bigcap_{\gamma<\beta}\kappa_\gamma$, for every limit ordinal $\beta\le\alpha$.

Theorem 5.2. *Let (X,μ) and (Y,ν) be standard measure spaces and $\kappa\subseteq X\times Y$ be an ω-closed set which possesses an m-resolution $(\kappa_\beta)_{\beta\le\alpha}$ such that $\kappa_\beta\setminus\kappa_{\beta+1}$ is an operator U-set, for each $\beta<\alpha$, and κ_α is an operator U-set. Then κ is an operator U-set.*

Proof. Let $(\kappa_\beta)_{\beta\le\alpha}$ be an m-resolution of κ such that $\kappa_\beta\setminus\kappa_{\beta+1}$ is an operator U-set for each $\beta<\alpha$.

We first observe that if $T\in\mathfrak{M}_{\max}(\kappa_\beta)\cap\mathcal{K}$ for some ordinal $\beta<\alpha$, then $T\in\mathfrak{M}_{\max}(\kappa_{\beta+1})\cap\mathcal{K}$. In fact, let $\kappa'_\beta=\kappa_{\beta-1}\setminus\kappa_\beta$. By our assumptions, κ'_β is an ω-closed set and there exists a function $\varphi\in\mathfrak{S}(X,Y)$ such that $\varphi=1$ on an ω-open neighbourhood of $\kappa_{\beta+1}$ and $\varphi=0$ on an ω-open neighbourhood of κ'_β. Clearly, $1-\varphi\in\mathfrak{S}(X,Y)$, $1-\varphi=0$ on an ω-open neighbourhood of $\kappa_{\beta+1}$ and $1-\varphi=1$ on an ω-open neighbourhood of κ'_β. Moreover, $T=S_\varphi(T)+S_{1-\varphi}(T)$.

For each $\psi\in\mathfrak{S}(X,Y)$ vanishing on an ω-open neighbourhood of $\kappa_{\beta+1}$, the function $\psi\varphi\in\mathfrak{S}(X,Y)$ vanishes on an ω-open neighbourhood of κ_β and, since $T\in\mathfrak{M}_{\max}(\kappa_\beta)$, Lemma 3.4 implies that $S_\psi(S_\varphi(T))=S_{\psi\varphi}(T)=0$. By Lemma 3.4 again, $S_\varphi(T)\in\mathfrak{M}_{\max}(\kappa_{\beta+1})$. Similarly, $S_{1-\varphi}(T)\in\mathfrak{M}_{\max}(\kappa'_\beta)$. Since $\mathcal{K}$ is invariant under Schur multipliers, we conclude that $S_{1-\varphi}(T)\in\mathfrak{M}_{\max}(\kappa'_\beta)\cap\mathcal{K}$. However, κ'_β is an operator U-set by assumption. It follows that $S_{1-\varphi}(T)=0$ and hence $T=S_\varphi(T)\in\mathfrak{M}_{\max}(\kappa_{\beta+1})\cap\mathcal{K}$.

Let now $T\in\mathfrak{M}_{\max}(\kappa)\cap\mathcal{K}$. It follows by transfinite induction that $T\in\mathfrak{M}_{\max}(\kappa_\beta)\cap\mathcal{K}$ for all $\beta\le\alpha$. In fact, assuming that the statement holds for all $\gamma<\beta$ we get by the previous paragraph that $T\in\mathfrak{M}_{\max}(\kappa_\beta)\cap\mathcal{K}$ if β has a predecessor while, if β is a limit ordinal, the inclusion follows from the assumption that $\kappa_\beta=\bigcap_{\gamma<\beta}\kappa_\gamma$.

Since κ_α is an operator U-set, we have now $T=0$ and hence κ is an operator U-set. $\square$

The following corollary should be compared to M. Bożejko's result [5,4] that every compact countable set in a non-discrete locally compact group is a U-set.

Corollary 5.3. *Let G be a non-discrete locally compact second countable group and $E\subseteq G$ be a closed countable set. Then E is a U-set.*

Proof. Recall that the successive Cantor–Bendixson derivatives of the set E are defined as follows: let $E_0 = E$ and for an ordinal β, let E_β be equal to the set of all limit points of $E_{\beta-1}$ if β has a predecessor, and to $\bigcap_{\gamma<\beta} E_\gamma$ if β is a limit ordinal. Since E is countable, there exists a countable ordinal α such that $E_\alpha = \emptyset$. Moreover, $E_\beta \setminus E_{\beta+1}$ is a countable set consisting of isolated points of E. By the regularity of $A(G)$, a pair of the form $(\{s\}^*, F^*)$, where F is a closed set and $s \notin F$, is m-separable. One hence easily obtains an m-resolution for E^*. On the other hand, if G is not discrete then $\mathfrak{M}_{\max}(\{s\}^*) = \lambda_s \mathcal{D}$ does not contain non-zero compact operators. It follows from Theorem 5.2 and Theorem 4.9 that E is a U-set. $\square$

5.2. *Inverse images*

In this subsection, we establish an Inverse Image Theorem for sets of multiplicity. Our result, Theorem 5.5, should be compared to [35, Theorem 4.7], an inverse image result for operator synthesis.

Let (X, μ), (X_1, μ_1), (Y, ν) and (Y_1, ν_1) be standard measure spaces. We fix, for the remainder of this section, measurable mappings $\varphi : X \to X_1$ and $\psi : Y \to Y_1$ such that $\varphi(X)$ and $\psi(Y)$ are measurable, the measure $\varphi_*\mu$ on X_1 given by $\varphi_*\mu(\alpha_1) = \mu(\varphi^{-1}(\alpha_1))$ is absolutely continuous with respect to μ_1, and the measure $\psi_*\nu$, defined similarly, is absolutely continuous with respect to ν_1.

Let $r : X_1 \to \mathbb{R}^+$ be the Radon–Nikodym derivative of $\varphi_*\mu$ with respect to μ_1, that is, the μ_1-measurable function such that $\mu(\varphi^{-1}(\alpha_1)) = \int_{\alpha_1} r(x_1) d\mu_1(x_1)$ for every measurable set $\alpha_1 \subseteq X_1$. Similarly, let $s : Y_1 \to \mathbb{R}^+$ be the Radon–Nikodym derivative of $\psi_*\nu$ with respect to ν_1. Let $M_1 = \{x_1 \in X_1 : r(x_1) = 0\}$ and $N_1 = \{y_1 \in Y_1 : s(y_1) = 0\}$. Note that $\mu(\varphi^{-1}(M_1)) = \int_{M_1} r(x_1) d\mu_1(x_1) = 0$. Similarly, $\nu(\psi^{-1}(N_1)) = 0$. Observe that, up to a μ_1-null set, $M_1^c \subseteq \varphi(X)$. Indeed, letting $M_2 = M_1^c \cap \varphi(X)^c \subseteq X_1$, we see that $\varphi^{-1}(M_2) = \emptyset$ and hence $0 = \mu(\varphi^{-1}(M_2)) = \int_{M_2} r_1(x) d\mu_1(x)$. Since $r_1(x) > 0$ for every $x \in M_2$, we have that $\mu_1(M_2) = 0$. Similarly, $N_1^c \subseteq \varphi(Y)$, up to a null set.

We will say that $\varphi : X \to X_1$ is injective up to a null set if there exists a subset $\Lambda \subseteq X$ with $\mu(\Lambda) = 0$, such that $\varphi : \Lambda^c \to X_1$ is injective. By [35, Lemma 4.2], there exists a null set $N \subseteq X_1$ such that $\varphi(\Lambda^c) \setminus N$ is measurable and the inverse of φ^{-1} on $\varphi(\Lambda^c) \setminus N$ is measurable. We moreover have that $\varphi^{-1}(N)$ is null; thus, the function φ, restricted to $\Lambda^c \cap \varphi^{-1}(N)^c$ is a bijection onto $\varphi(\Lambda^c) \setminus N$ and has measurable inverse.

If the set Λ above can moreover be chosen so that $\mu_1(\varphi(\Lambda^c)^c) = 0$, then we say that φ is bijective up to a null set.

The following result must be known but we could not find a precise reference.

Lemma 5.4. *The operator $V_\varphi : L^2(X_1, \mu_1) \to L^2(X, \mu)$ given by*

$$V_\varphi \xi(x) = \begin{cases} \frac{\xi(\varphi(x))}{\sqrt{r(\varphi(x))}} & \textit{if } x \notin \varphi^{-1}(M_1), \\ 0 & \textit{if } x \in \varphi^{-1}(M_1) \end{cases}$$

is a partial isometry with initial space $L^2(M_1^c, \mu_1|_{M_1^c})$*. Moreover, if* φ *is injective up to a null set then* V_φ *is surjective.*

Proof. Note that, if $\xi \in L^2(X_1, \mu_1)$ then

$$\|V_\varphi \xi\|^2 = \int\limits_{\varphi^{-1}(M_1)^c} \left| \frac{\xi(\varphi(x))}{\sqrt{r(\varphi(x))}} \right|^2 d\mu(x) = \int\limits_{M_1^c} \left| \frac{\xi(x_1)}{\sqrt{r(x_1)}} \right|^2 d\varphi_* \mu(x_1)$$
$$= \int\limits_{M_1^c} r(x_1) \left| \frac{\xi(x_1)}{\sqrt{r(x_1)}} \right|^2 d\mu_1(x_1) = \int\limits_{M_1^c} \left|\xi(x_1)\right|^2 d\mu_1(x_1).$$

It follows that V_φ is a partial isometry with initial space $L^2(M_1^c, \mu_1|_{M_1^c})$.

Suppose that φ is injective up to a null set. By the remarks preceding the formulation of the lemma, we may assume that there exists a set $M \subseteq X$ such that $\mu(M) = 0$, $\varphi(M^c)$ is measurable, $\varphi|_{M^c}$ is one-to-one, and $\varphi^{-1} : \varphi(M^c) \to M^c$ is measurable. Let $\eta \in L^2(X, \mu)$ and define $\xi : X_1 \to \mathbb{C}$ by setting $\xi(x_1) = \sqrt{r(x_1)}\eta(\varphi^{-1}(x_1))$ if $x_1 \in \varphi(M^c)$ and $\xi(x_1) = 0$ if $x_1 \notin \varphi(M^c)$. We claim that $\xi \in L^2(X_1, \mu_1)$. To see this, note that

$$\mu\big(\varphi^{-1}(\alpha_1)\big) = \int\limits_{\alpha_1} r(x_1) d\mu_1(x_1),$$

for all μ_1-measurable sets $\alpha_1 \subseteq \varphi(M^c)$. Setting $\tilde{\mu}$ to be the measure on M^c given by $\tilde{\mu}(\alpha) = \mu_1(\varphi(\alpha))$ for μ-measurable subset $\alpha \subseteq M^c$ we have

$$\mu(\alpha) = \int\limits_{\alpha} r\big(\varphi(x)\big) d\tilde{\mu}(x).$$

It follows that

$$\|\xi\|_{L^2(X_1,\mu_1)} = \int\limits_{\varphi(M^c)} r(x_1)\big|\eta\big(\varphi^{-1}(x_1)\big)\big|^2 d\mu_1(x_1)$$
$$= \int\limits_{M^c} r\big(\varphi(x)\big)\big|\eta(x)\big|^2 d\tilde{\mu}(x)$$
$$= \int\limits_{M^c} \big|\eta(x)\big|^2 d\mu(x) = \|\eta\|_{L^2(X,\mu)}$$

since M is μ-null. On the other hand,

$$\Lambda \stackrel{def}{=} \varphi^{-1}\big(M_1^c \cap \varphi\big(M^c\big)^c\big) \subseteq \varphi^{-1}\big(\varphi\big(M^c\big)^c\big) \subseteq M,$$

and hence Λ is μ-null. It follows as in the third paragraph of the present subsection that $M_1^c \cap \varphi(M^c)^c$ is μ_1-null. Thus, $V_\varphi \xi = \eta$, and the proof is complete. □

We now formulate and prove the main result of this subsection.

Theorem 5.5. *Let $\varphi : X \to X_1$ and $\psi : Y \to Y_1$ be measurable functions. Let $\kappa_1 \subseteq X_1 \times Y_1$ and $\kappa = \{(x, y) \in X \times Y : (\varphi(x), \psi(y)) \in \kappa_1\}$.*

(i) *Suppose that φ and ψ are injective up to a null set. If κ_1 is an operator U-set (resp. an operator U_1-set) then κ is an operator U-set (resp. an operator U_1-set).*
(ii) *Suppose that $\kappa_1 \subseteq M_1^c \times N_1^c$. If κ_1 is an operator M-set (resp. an operator M_1-set) then κ is an operator M-set (resp. an operator M_1-set).*
(iii) *Suppose that μ_1 (resp. ν_1) is equivalent to $\varphi_*\mu$ (resp. $\psi_*\nu$) and that φ and ψ are injective up to a null set. Then κ_1 is an operator M-set (resp. an operator M_1-set) if and only if κ is an operator M-set (resp. an operator M_1-set).*

Proof. (i) Let Θ be the linear map from the algebraic tensor product $L^2(X_1, \mu_1) \otimes L^2(Y_1, \nu_1)$ of $L^2(X_1, \mu_1)$ and $L^2(Y_1, \nu_1)$ sending $f \otimes g$ to $V_\varphi f \otimes V_\psi g$. Since V_φ and V_ψ are partial isometries, Θ is contractive in the norm of $\Gamma(X_1, Y_1)$, and hence extends to a contractive linear map $\Theta : \Gamma(X_1, Y_1) \to \Gamma(X, Y)$. By Lemma 5.4, V_φ and V_ψ are surjective, and hence Θ has dense range. Moreover, if $h \in \Gamma(X_1, Y_1)$ then

$$\Theta(h)(x, y) = \frac{h(\varphi(x), \psi(y))}{\sqrt{r(\varphi(x))s(\psi(y))}}, \quad \text{for m.a.e. } (x, y) \in \varphi^{-1}(M_1) \times \psi^{-1}(N_1). \tag{16}$$

To show (16), write $h = \sum_{i=1}^\infty f_i \otimes g_i$, where $\sum_{i=1}^\infty \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^\infty \|g_i\|_2^2 < \infty$, and set $h_n = \sum_{i=1}^n f_i \otimes g_i$, $n \in \mathbb{N}$. By the definition of Θ, identity (16) holds for all h_n, $n \in \mathbb{N}$, if (x, y) belongs to the set $\varphi^{-1}(M_1) \times \psi^{-1}(N_1)$. By [35, Lemma 2.1], there exists a subsequence (h_{n_k}) of (h_n) which converges to h marginally almost everywhere. By passing to a further subsequence, we may assume that $\Theta(h_{n_k})$ converges to $\Theta(h)$ marginally almost everywhere. Identity (16) now follows from the fact that if $E \subseteq X_1 \times Y_1$ is marginally null then $\{(x, y) \in X \times Y : (\varphi(x), \psi(y)) \in E\}$ is marginally null.

The map Θ is the adjoint to the map $\mathcal{K}(L^2(X_1, \mu_1), L^2(Y_1, \nu_1)) \ni K \mapsto V_\psi^* K V_\varphi \in K(L^2(X, \mu), L^2(Y, \nu))$; indeed, if $f \in L^2(X, \mu)$ and $g \in L^2(Y, \nu)$ then

$$\begin{aligned}\langle f \otimes g, V_\psi^* K V_\varphi \rangle &= \left(V_\psi^* K V_\varphi f, \bar{g}\right) = (K V_\varphi f, V_\psi \bar{g}) = (K V_\varphi f, \overline{V_\psi g}) \\ &= \langle V_\varphi f \otimes V_\psi g, K \rangle = \langle \Theta(f \otimes g), K \rangle.\end{aligned}$$

It follows that Θ is weak* continuous and thus, if $\mathcal{M}_1 \subseteq \Gamma(X_1, Y_1)$ then

$$\Theta\left(\overline{\mathcal{M}_1}^{w^*}\right) \subseteq \overline{\Theta(\mathcal{M}_1)}^{w^*}. \tag{17}$$

Suppose that $h|_{\kappa_1} = 0$. If $(x, y) \in \kappa \setminus ((\varphi^{-1}(M_1) \times Y) \cup (X \times \psi^{-1}(N_1)))$ then, by (16),

$$\Theta(h)(x, y) = \frac{h(\varphi(x), \psi(y))}{\sqrt{r(\varphi(x))s(\psi(y))}} = 0;$$

thus, $\Theta(\Phi(\kappa_1)) \subseteq \Phi(\kappa)$. On the other hand, if E_1 is an ω-open neighbourhood of κ_1 then $(\varphi \times \psi)^{-1}(E_1)$ is an ω-open neighbourhood of κ. Applying the same reasoning as above, and using the continuity of Θ with respect to $\|\cdot\|_\Gamma$, we conclude that $\Theta(\Psi(\kappa_1)) \subseteq \Psi(\kappa)$.

Now suppose that κ_1 is an operator U-set, that is, $\overline{\Phi(\kappa_1)}^{w^*} = \Gamma(X_1, Y_1)$. Using (17), we have

$$\Gamma(X,Y) = \overline{\Theta\big(\Gamma(X_1,Y_1)\big)}^{\|\cdot\|} = \overline{\Theta\big(\overline{\Phi(\kappa_1)}^{w^*}\big)}^{\|\cdot\|} \subseteq \overline{\Theta\big(\Phi(\kappa_1)\big)}^{w^*} \subseteq \overline{\Phi(\kappa)}^{w^*}.$$

Thus, $\overline{\Phi(\kappa)}^{w^*} = \Gamma(X,Y)$ and hence κ is an operator U-set. It follows similarly that if κ_1 is an operator U_1-set then κ is an operator U_1-set.

(ii) Suppose that κ_1 is an operator M_1-set and let K_1 be a non-zero compact operator in $\mathfrak{M}_{\min}(\kappa_1)$. Let $K = V_\psi K_1 V_\varphi^*$. As $\kappa_1 \subseteq M_1^c \times N_1^c$, $V_\varphi^* V_\varphi = P(M_1^c)$ and $V_\psi^* V_\psi = P(N_1^c)$, we have that $K_1 = V_\psi^* K V_\varphi$ and hence K is a non-zero compact operator.

Let

$$(P,Q) \in \big(\mathcal{B}\big(\ell^2\big) \mathbin{\bar{\otimes}} L^\infty(X,\mu)\big) \times \big(\mathcal{B}\big(\ell^2\big) \mathbin{\bar{\otimes}} L^\infty(Y,\nu)\big)$$

be a κ-pair [35]; this means that, after the identification of P and Q with operator-valued weakly measurable functions, defined on X and Y, respectively, P and Q are projection-valued and $P(x)Q(y) = 0$ marginally almost everywhere on κ. It follows from the proof of [35, Theorem 4.7] that there exists a κ_1-pair

$$(\hat{P},\hat{Q}) \in \big(\mathcal{B}\big(\ell^2\big) \mathbin{\bar{\otimes}} L^\infty(X_1,\mu)\big) \times \big(\mathcal{B}\big(\ell^2\big) \mathbin{\bar{\otimes}} L^\infty(Y_1,\nu)\big),$$

such that $P(x) \leq \hat{P}(\varphi(x))$ and $Q(y) \leq \hat{Q}(\psi(x))$ for almost all $x \in X$ and almost all $y \in Y$. By Theorem 3.3,

$$\hat{Q}(I \otimes K_1)\hat{P} = 0. \tag{18}$$

We claim that

$$\big(I \otimes V_\varphi^*\big)(R \circ \varphi) = R\big(I \otimes V_\varphi^*\big) \quad \text{and} \quad (S \circ \psi)(I \otimes V_\psi) = (I \otimes V_\psi)S, \tag{19}$$

whenever R and S are bounded operator-valued weakly measurable functions on X_1 and Y_1, respectively. It clearly suffices to show only the first of these identities. Start by observing that $P(\varphi^{-1}(\alpha))V_\varphi = V_\varphi P(\alpha)$, for all measurable subsets $\alpha \subseteq X_1$. It follows that (19) holds when $R = \sum_{j=1}^k a_j \otimes \chi_{E_j}$, where $(E_j)_{j=1}^k$ is a family of pairwise disjoint measurable subsets of X_1 and $(a_i)_{i=1}^k$ is a family of bounded operators on ℓ^2. If R is arbitrary then, by Kaplansky's Density Theorem, it is the strong limit of a sequence $(R_n)_{n\in\mathbb{N}}$, where R_n is of the latter form and $\|R_n\| \leq \|R\|$ for each n. By the proof of [35, Theorem 4.6], there exists $S_1 \subseteq X_1$ with $\mu_1(S_1) = 0$ such that

$$R(x_1) = \operatorname*{s-lim}_{n\to\infty} R_{n_k}(x_1) \quad \text{if } x_1 \notin S_1.$$

Let $S = \varphi^{-1}(S_1)$; then $R(\varphi(x)) = \text{s-lim}_{k\to\infty} R_{n_k}(\varphi(x))$ if $x \notin S$. As $\mu(S) = \varphi_*\mu(S_1)$ and $\varphi_*\mu$ is absolutely continuous with respect to μ_1, we have that $\mu(S) = 0$ and hence $(R_{n_k} \circ \varphi)_{k\in\mathbb{N}}$ converges almost everywhere to $R \circ \varphi$.

Since $\|R_n\| = \operatorname{ess\,sup}_{x_1\in X_1} \|R_n(x)\|_{\mathcal{B}(\ell^2)}$ and $(R_n)_{n\in\mathbb{N}}$ is bounded by $\|R\|$, there exists a μ_1-null set $M \subseteq X_1$ such that $\|R_n(x_1)\|_{\mathcal{B}(\ell^2)} \leq \|R\|$ for all $x_1 \notin M$ and all $n \in \mathbb{N}$. Therefore $\|R_n(\varphi(x))\|_{\mathcal{B}(\ell^2)} \leq \|R\|$ for all $x \notin \varphi^{-1}(M)$ and all $n \in \mathbb{N}$. As $\mu(\varphi^{-1}(M)) = 0$, we have that $\|R_n \circ \varphi\| = \operatorname{ess\,sup}_{x\in X} \|R_n(\varphi(x))\|_{\mathcal{B}(\ell^2)} \leq \|R\|$, for all $n \in \mathbb{N}$. By a straightforward application of the Lebesgue Dominated Convergence Theorem, $(R_{n_k} \circ \varphi)_{k\in\mathbb{N}}$ converges strongly to $R \circ \varphi$. As $(I \otimes V_\varphi^*)(R_n \circ \varphi) = R_n(I \otimes V_\varphi^*)$ holds for every n and $(I \otimes V_\varphi^*)(R_n \circ \varphi) \to (I \otimes V_\varphi^*)(R \circ \varphi)$, $R_n(I \otimes V_\varphi^*) \to R(I \otimes V_\varphi^*)$ in the strong operator topology, (19) is proved. Using (18) and (19), we now obtain

$$\begin{aligned} Q(I \otimes K)P &= Q(\hat{Q} \circ \psi)\big(I \otimes V_\psi K_1 V_\varphi^*\big)(\hat{P} \circ \varphi)P \\ &= Q(I \otimes V_\psi)\hat{Q}(I \otimes K_1)\hat{P}\big(I \otimes V_\psi^*\big)P = 0. \end{aligned}$$

By Theorem 3.3, $K \in \mathfrak{M}_{\min}(E)$; hence, κ is an operator M_1-set.

Now suppose that κ_1 is an operator M-set and let $K_1 \in \mathfrak{M}_{\max}(\kappa_1)$ be a non-zero compact operator. Let (P, Q) be a simple κ-pair [35], that is, a κ-pair (P, Q) for which each of the projection valued functions P and Q takes finitely many values. We recall the construction of the pair $(\hat{P}, \hat{Q})$ from [35]. Let $(\xi_j)_{j\in\mathbb{N}}$ be a dense sequence in ℓ^2. It was shown on [35, p. 311] that there are null sets $M_0^1 \subseteq X_1$ and $M_0 \subseteq X$ and, for each $j \in \mathbb{N}$, a measurable function $g_j : \varphi(X) \setminus M_0^1 \to X$ with $\varphi(g_j(x_1)) = x_1$ for all $x_1 \in \varphi(X) \setminus M_0^1$, and $(P(g_j(\varphi(x)))\xi_j, \xi_j) > (P(x)\xi_j, \xi_j) - \frac{1}{j}$, $x \in X \setminus M_0$. Let, similarly, $(\eta_j)_{j\in\mathbb{N}}$ be a dense sequence in ℓ^2 and for each $j \in \mathbb{N}$, let $h_j : \varphi(Y) \setminus N_0^1 \to Y$ be a measurable function with $\psi(h_j(y_1)) = y_1$ for all $y_1 \in \psi(Y) \setminus N_0^1$, and $(Q(h_j(\psi(y)))\eta_j, \eta_j) > (Q(y)\eta_j, \eta_j) - \frac{1}{j}$, $y \in Y \setminus N_0$, where $N_0^1 \subseteq Y_1$ and $N_0 \subseteq Y$ are null sets. Set

$$\begin{aligned} \hat{P}_n(x_1) &= \bigvee_{j=1}^{n} P\big(g_j(x_1)\big), \qquad \hat{P}(x_1) = \bigvee_{j=1}^{\infty} P\big(g_j(x_1)\big), \quad x_1 \in \varphi(X) \setminus M_0^1, \\ \hat{Q}_n(y_1) &= \bigvee_{j=1}^{n} Q\big(h_j(y_1)\big), \qquad \hat{Q}(y_1) = \bigvee_{j=1}^{\infty} Q\big(h_j(y_1)\big), \quad y_1 \in \psi(Y) \setminus N_0^1. \end{aligned}$$

We have that $\hat{P}_n \to_{n\to\infty} \hat{P}$ and $\hat{Q}_n \to_{n\to\infty} \hat{Q}$ in the strong operator topology. Furthermore, since P (resp. Q) takes only finitely many values, the same is true for $\hat{P}_n$ (resp. $\hat{Q}_n$), $n \in \mathbb{N}$. If

$$(x_1, y_1) \in \kappa_1 \cap \big(\big(\varphi(X) \setminus M_0^1\big) \times \big(\psi(Y) \setminus N_0^1\big)\big)$$

then $(g_j(x_1), h_j(y_1)) \in \kappa$. However, $\kappa_1 \subseteq M_1^c \times N_1^c$, while $M_1^c \times N_1^c$ is marginally contained in $\varphi(X) \times \psi(Y)$. It follows that $\hat{P}_n(x_1)\hat{Q}_n(y_1) = 0$ for marginally almost all $(x_1, y_1) \in \kappa_1$ and every $n \in \mathbb{N}$. Thus, $(\hat{P}_n, \hat{Q}_n)$ is a simple κ_1-pair, $n \in \mathbb{N}$, and hence, by Theorem 3.3,

$\hat{Q}_n(I \otimes K_1)\hat{P}_n = 0$ for every n. Since $P \le \hat{P} \circ \varphi$ and $Q \le \hat{Q} \circ \psi$, it follows from (19) and the first part of the proof that

$$P = P(\hat{P} \circ \varphi) = \operatorname*{s\text{-}lim}_{n\to\infty} P(\hat{P}_n \circ \varphi) \quad \text{and} \quad Q = Q(\hat{Q} \circ \psi) = \operatorname*{s\text{-}lim}_{n\to\infty} Q(\hat{Q}_n \circ \psi).$$

As in the previous paragraph, we conclude that

$$Q(I \otimes K)P = \operatorname*{w\text{-}lim}_{n\to\infty} Q(I \otimes V_\psi)\hat{Q}_n(I \otimes K_1)\hat{P}_n\big(I \otimes V_\psi^*\big)P = 0.$$

By Theorem 3.3, $K \in \mathfrak{M}_{\max}(\kappa)$ and since K is a non-zero compact operator, κ is an operator M-set.

(iii) In this case $\mu_1(M_1) = 0$ and $\nu_1(N_1) = 0$; thus, (iii) is immediate from (i) and (ii). $\square$

Remark 5.6. (i) The statement in Theorem 5.5 (ii) does not hold without the assumption $\kappa_1 \subseteq M_1^c \times N_1^c$; indeed, assuming that M_1 and N_1 are non-null and letting $\kappa_1 = M_1 \times N_1$, we see that κ_1 is an operator M-set; but κ is marginally equivalent to the empty set and hence is an operator U-set.

(ii) G.K. Eleftherakis has recently proved part (i) of Theorem 5.5 without the injectivity assumption on the mappings φ and ψ, see [8].

Corollary 5.7. *Let G and H be locally compact second countable groups with Haar measures m_G and m_H, respectively, $\varphi : G \to H$ be a continuous homomorphism and E be a closed subset of H. Assume that $\varphi_* m_G$ is absolutely continuous with respect to m_H.*

(i) *Suppose that φ is injective and has a continuous inverse on $\varphi(G)$. If E is a U-set (resp. a U_1-set) then $\varphi^{-1}(E)$ is a U-set (resp. a U_1-set).*
(ii) *Suppose that $\varphi_* m_G$ is equivalent to m_H. If E is an M-set (resp. an M_1-set) then $\varphi^{-1}(E)$ is an M-set (resp. an M_1-set).*
(iii) *If φ is an isomorphism then E is an M-set (resp. an M_1-set) if and only if $\varphi^{-1}(E)$ is an M-set (resp. an M_1-set).*

Proof. First observe that, since φ is a homomorphism, $\varphi^{-1}(E)^* = (\varphi \times \varphi)^{-1}(E^*)$. If φ is an isomorphism then $\varphi_* m_G$ is equivalent to m_H, see Remark 5.8. The corollary now follows from Theorems 4.9 and 5.5. $\square$

Remark 5.8. We note that if $m_H(\varphi(G)) \neq 0$ then the condition that $\varphi_* m_G$ is absolutely continuous with respect to m_H can be dropped. In fact, in this case, by Steinhaus's Theorem, $\varphi(G)$ is an open subgroup of H and if φ is injective and has a continuous inverse on $\varphi(G)$, φ is a homeomorphism between G and $\varphi(G)$. One can easily see that the measures $\varphi_* m_G$ and m_H restricted to $\varphi(G)$ satisfy the conditions of (left) Haar measures of $\varphi(G)$. Hence, since $m_H(\varphi(G)) \neq 0$ there exists $c > 0$ such that $\varphi_* m_G|_{\varphi(G)} = c m_H|_{\varphi(G)}$. Let $W \subseteq H$ be any Borel subset of H. Then

$$\varphi_* m_G(W) = \varphi_* m_G\big(W \cap \varphi(G)\big) = c m_H\big(W \cap \varphi(G)\big) = c \int\limits_W \chi_{\varphi(G)}(x) dm_H(x),$$

giving the claim. It follows that the measures $\varphi_* m_G$ and m_H are equivalent in the case φ is an isomorphism.

We next include a characterisation of the closed subgroups that are also sets of multiplicity answering a question posed by M. Bożejko in [5]. We will need the following lemma.

Lemma 5.9. *Let G be a locally compact second countable group with a left Haar measure m. Let H be a closed subgroup of G. Let $q : G \to G/H$ be the quotient map. Then there exists a finite Borel measure μ on G/H, equivalent to $q_* m$.*

Proof. If G is compact then the measure is finite itself and we are done. Suppose that G is non-compact. As G is second countable it is σ-compact, *i.e.*, there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G such that $G = \bigcup_n K_n$. By deleting some of the terms of this sequence, we may assume that $m(K_{n+1} \setminus K_n) > 0$ for each $n \in \mathbb{N}$. Define a measure μ_n on G/H by letting $\mu_n(\alpha) = m(q^{-1}(\alpha) \cap K_n)$. Then μ_n is finite; indeed, $\mu_n(G/H) = m(K_n) < \infty$.

As $K_n \subseteq K_{n+1}$, we have that $\mu_n(\alpha) \le \mu_{n+1}(\alpha)$ for any Borel set α. Now define

$$\mu(\alpha) = \sum_{n=0}^{\infty} \frac{\mu_{n+1}(\alpha) - \mu_n(\alpha)}{2^n m(K_{n+1} \setminus K_n)}$$

where we have $\mu_0(\alpha) = 0$. Clearly, μ is a finite measure. Since $\mu(\alpha) = 0 \Leftrightarrow \mu_{n+1}(\alpha) - \mu_n(\alpha) = 0$ for all $n \ge 0 \Leftrightarrow \mu_n(\alpha) = 0$ for all $n \ge 0 \Leftrightarrow m(q^{-1}(\alpha)) = q_* m(\alpha) = 0$, $q_* m$ is absolutely continuous with respect to μ. $\square$

Corollary 5.10. *Let H be a closed subgroup of a locally compact second countable group G. Then H is an M-set if and only if H is open.*

Proof. We note first that by Steinhaus's Theorem, H is open if and only if $m(H) > 0$. If $m(H) > 0$ then H is an M-set by Remark 4.3. Assume now that $m(H) = 0$. By Theorem 4.9, it suffices to see that

$$H^* = \big\{(s,t) : ts^{-1} \in H\big\} = \big\{(s,t) : Ht = Hs\big\}$$

is an operator U-set. Let $q : G \to G/H$ be the quotient map given by $q(s) = Hs$. By [17, 5.22, 8.14] G/H is a locally compact metrisable separable space. By Lemma 5.9, there exists a finite measure μ on G/H, equivalent to $q_* m$. Thus, μ is non-atomic. Since any finite measure on a locally compact second countable space is regular [13, Theorem 7.8], the measure space $(G/H, \mu)$ is standard.

Let now $D = \{(z,z) : z \in G/H\}$. Since every bounded operator on $L^2(G/H, \mu)$ supported on D is a multiplication operator and μ is non-atomic, the only compact operator on $L^2(G/H,\mu)$ supported on D is the zero operator. Therefore D is an operator U-set. By Remark 5.6 (ii), $H^* = (q^{-1} \times q^{-1})(D)$ is a set of uniqueness. $\square$

5.3. Direct products

In this subsection, we show that direct products preserve the property of being an operator M-set (resp. an operator M_1-set, an operator M_0-set).

Theorem 5.11. *Let (X_i,μ_i) and (Y_i,ν_i) be standard measure spaces and $\kappa_i \subseteq X_i \times Y_i$ be ω-closed sets, $i=1,2$. The set $\rho(\kappa_1 \times \kappa_2)$ is an operator M-set (resp. operator M_1-set) if and only if both κ_1 and κ_2 are operator M-sets (resp. operator M_1-sets).*

Proof. By [26], the support of $\mathfrak{M}_{\max}(\kappa_1) \bar{\otimes} \mathfrak{M}_{\max}(\kappa_2)$ is $\rho(\kappa_1 \times \kappa_2)$. It follows that

$$\mathfrak{M}_{\max}(\kappa_1) \bar{\otimes} \mathfrak{M}_{\max}(\kappa_2) \subseteq \mathfrak{M}_{\max}\big(\rho(\kappa_1 \times \kappa_2)\big). \tag{20}$$

Assume first that κ_1 and κ_2 are operator M_1-sets (resp. operator M sets). Suppose that T_i is a non-zero compact operator in $\mathfrak{M}_{\min}(\kappa_i)$ (resp. $\mathfrak{M}_{\max}(\kappa_i)$), $i=1,2$. By Theorem 3.8 (resp. by (20)), $T_1 \otimes T_2$ is a non-zero compact operator in $\mathfrak{M}_{\min}(\rho(\kappa_1 \times \kappa_2))$ (resp. $\mathfrak{M}_{\max}(\rho(\kappa_1 \times \kappa_2))$). Hence $\rho(\kappa_1 \times \kappa_2)$ is an operator M_1-set (resp. an operator M-set).

We next show that if either κ_1 or κ_2 is an operator U-set then so is $\rho(\kappa_1 \times \kappa_2)$. Suppose that $T \in \mathcal{K}(H_1 \otimes H_2, K_1 \otimes K_2)$ is supported on $\rho(\kappa_1 \times \kappa_2)$. Let $\omega \in (\mathcal{K}(H_2,K_2))^* = \mathcal{C}_1(K_2,H_2)$ and let L_ω be the slice map from $\mathcal{K}(H_1 \otimes H_2, K_1 \otimes K_2)$ to $\mathcal{K}(H_1,K_1)$ defined on elementary tensors by $L_\omega(A \otimes B) = \omega(B)A$. Then $\operatorname{supp} L_\omega(T) \subseteq \kappa_1$. In fact, if $\alpha \times \beta$ is a measurable rectangle marginally disjoint from κ_1, then $((\alpha \times X_2) \times (\beta \times Y_2)) \cap \rho(\kappa_1 \times \kappa_2) \simeq \emptyset$ and

$$P(\beta)L_\omega(T)P(\alpha) = L_\omega\big(\big(P(\beta) \otimes I\big)T\big(P(\alpha) \otimes I\big)\big) = 0.$$

If κ_1 is an operator U-set, $L_\omega(T)=0$ for all ω and hence $T=0$.

If $T \in \mathcal{K}(H_1 \otimes H_2, K_1 \otimes K_2) \cap \mathfrak{M}_{\min}(\rho(\kappa_1 \times \kappa_2))$ and (P,Q) is a κ_1-pair, then $(P \otimes I, Q \otimes I)$ is a $\rho(\kappa_1 \times \kappa_2)$-pair and hence

$$Q\big(I_{\ell^2} \otimes L_\omega(T)\big)P = (\mathrm{id} \otimes L_\omega)\big((Q \otimes I)(I_{\ell^2} \otimes T)(P \otimes I)\big) = 0;$$

by Theorem 3.3, $L_\omega(T) \in \mathfrak{M}_{\min}(\kappa_1)$. If κ_1 is an operator U_1-set, arguments similar to the ones above show that $T=0$ and hence $\rho(\kappa_1 \times \kappa_2)$ is an operator U_1-set. $\square$

Corollary 5.12. *Let G_1 and G_2 be locally compact second countable groups and $E_1 \subseteq G_1$, $E_2 \subseteq G_2$ be closed sets. If E_1 and E_2 are M-sets (resp. M_1-sets) then $E_1 \times E_2$ is an M-set (resp. an M_1-set).*

Proof. Suppose that $E_1 \subseteq G_1$ and $E_2 \subseteq G_2$ are M-sets. By Theorem 4.9, E_1^* and E_2^* are operator M-sets, and by Theorem 5.11, $\rho(E_1^* \times E_2^*) = (E_1 \times E_2)^*$ is an operator M-set. By Theorem 4.9 again, $E_1 \times E_2$ is an M-set. A similar argument applies to M_1-sets. □

6. Sets of finite width

Let (X, μ) and (Y, ν) be standard measure spaces. A subset $E \subseteq X \times Y$ is called a *set of finite width* if there exist measurable functions $f_i : X \to \mathbb{R}$, $g_i : Y \to \mathbb{R}$, $i = 1, \dots, n$, such that

$$E = \{(x, y) \in X \times Y : f_i(x) \leq g_i(y),\ i = 1, \dots, n\}; \tag{21}$$

the *width* of E is the smallest n for which E can be represented in the form (21). By [35, Theorem 4.8] and [40, Theorem 2.1], any such set is operator synthetic. In this section we identify those sets of finite width which are operator M_1-sets, and hence operator M-sets.

We first assume that the measures μ and ν are finite and the standard measure spaces X and Y arise from compact topologies. A *system* is a finite set D of disjoint rectangles $\Pi = \alpha \times \beta$, where $\alpha \subseteq X$ and $\beta \subseteq Y$ are measurable. Set $r(\alpha \times \beta) = \min\{\mu(\alpha), \nu(\beta)\}$. The *volume* of a system $D = \{\Pi_j : 1 \leq j \leq J\}$ is the number $r(D) \stackrel{def}{=} \max_{1 \leq j \leq J} r(\Pi_j)$. Let $U_D = \bigcup_{j=1}^J \Pi_j$ and call the systems D_1 and D_2 disjoint if $U_{D_1} \cap U_{D_2} = \emptyset$; in this case, denote by $D_1 \vee D_2$ their union.

With each system $D = \{\alpha_j \times \beta_j : 1 \leq j \leq J\}$, we associate the projection π_D on $\mathcal{B}(H_1, H_2)$ by setting

$$\pi_D(T) = \sum_{j=1}^J P(\beta_j) T P(\alpha_j), \quad T \in \mathcal{B}(H_1, H_2).$$

It is easy to see that π_D depends only on U_D and that $\pi_{D_1 \vee D_2} = \pi_{D_1} + \pi_{D_2}$; thus, the mapping $U \to \pi_U$ is a projection-valued measure on the algebra of sets generated by all rectangles. Note that the range of π_D coincides with $\mathfrak{M}_{\max}(U_D)$.

A system $D = \{\alpha_j \times \beta_j : 1 \leq j \leq J\}$ will be called *diagonal* if $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$ whenever $i \neq j$. The system D will be called *n-diagonal*, if $D = D_1 \vee D_2 \vee \dots \vee D_n$ where $D_1, \dots, D_n$ are diagonal systems. It is easy to see that $\|\pi_D\| = 1$ if D is diagonal. Hence, $\|\pi_D\| \leq n$ if D is n-diagonal.

Lemma 6.1. *Let $(D^k)_{k \in \mathbb{N}}$ be a sequence of n-diagonal systems such that $r(D^k) \to_{k \to \infty} 0$. Then $\|\pi_{D^k}(T)\| \to_{k \to \infty} 0$ for each compact operator T.*

Proof. It suffices to prove the statement for rank one operators $T = u \otimes v$ where u, v are bounded functions on X and Y, because the set of all linear combinations of such operators is dense in $\mathcal{K}(H_1, H_2)$ and the sequence $(\pi_{D^k})_{k \in \mathbb{N}}$ is uniformly bounded.

If $D = \{\alpha_j \times \beta_j\}_{j=1}^J$ is a diagonal system, then for $T = u \otimes v$, we have

$$\begin{aligned}\|\pi_D(T)\| \le \|\pi_D(T)\|_2 &= \left\|\sum_{j=1}^J (\chi_{\alpha_j} \otimes \chi_{\beta_j})(u \otimes v)\right\|_{L^2(X\times Y,\mu\times\nu)}\\ &\le \|u\|_\infty\|v\|_\infty\left(\sum_{j=1}^J \mu(\alpha_j)\nu(\beta_j)\right)^{1/2}\\ &\le \|u\|_\infty\|v\|_\infty\left(\sum_{j=1}^J r(\Pi_j)\big(\mu(\alpha_j)+\nu(\beta_j)\big)\right)^{1/2}\\ &\le \|u\|_\infty\|v\|_\infty r(D)^{1/2}\big(\mu(X)+\nu(Y)\big)^{1/2}.\end{aligned}$$

It follows that if D is an n-diagonal system then

$$\|\pi_D(T)\| \le n\|u\|_\infty\|v\|_\infty r(D)^{1/2}\big(\mu(X)+\nu(Y)\big)^{1/2}.$$

Hence $\|\pi_{D^k}(T)\| \to_{k\to\infty} 0$ whenever $r(D^k) \to_{k\to\infty} 0$. $\square$

Let us call a set E *n-quasi-diagonal* if for each $\varepsilon > 0$ there is an n-diagonal system D with $E \subseteq U_D$ and $r(D) < \varepsilon$.

We say that a (measurable) function defined on a measure space is non-atomic if it is not constant on any set of positive measure.

Lemma 6.2. *Let $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ be Borel maps and assume that f is non-atomic. Then the set*

$$E_{f,g} = \big\{(x,y) \in X \times Y : f(x) = g(y)\big\}$$

is 1-quasi-diagonal.

Proof. Let μ_f be the measure on the Borel σ-algebra of $\mathbb{R}$ given by $\mu_f(C) = \mu(f^{-1}(C))$. By our assumption, μ_f is non-atomic and finite. Hence, for every $\varepsilon > 0$, there exists a partition $\mathbb{R} = \bigcup_{j=1}^N C_j$ with $\mu_f(C_j) < \varepsilon/\nu(Y)$ for all j. In fact, letting $h(x) = \mu_f((-\infty, x])$ we have that h is a bounded increasing function such that $h(\mathbb{R}) \subseteq [0, C]$, where $C = \mu_f(\mathbb{R})$. As μ_f is non-atomic, h is continuous and $(0, C) \subseteq h(\mathbb{R})$. Let $0 = a_0 < a_1 < \ldots < a_{N+1} = C$ be a partition of $[0, C]$ such that $a_{i+1} - a_i < \varepsilon/\nu(Y)$, $0 \le i \le N$, and $h(x_i) = a_i$, $1 \le i \le N$. Set $C_0 = (0, x_1]$, $C_i = (x_i, x_{i+1}]$ if $0 < i < N$, and $C_N = (x_N, \infty)$. Then $\mathbb{R} = \bigcup_{i=1}^N C_i$ and $\mu_f(C_i) < \varepsilon/\nu(Y)$, $1 \le i \le N$.

Setting $\alpha_j = f^{-1}(C_j)$, $\beta_j = g^{-1}(C_j)$ and $D = \{\alpha_j \times \beta_j : 1 \le j \le N\}$, we now see that D is diagonal, $E \subseteq U_D$ and $r(D) < \varepsilon$. $\square$

Fix $T \in \mathcal{B}(H_1, H_2)$, $F \in \mathcal{C}_1(H_2, H_1)$ and set

$$\varphi(\Pi) = \langle \pi_\Pi(T), F \rangle,$$

for each rectangle $\Pi \subseteq X \times Y$. We say that Π is φ-*null*, if $\varphi(\Pi') = 0$ for all rectangles $\Pi' \subseteq \Pi$.

Lemma 6.3. *If $\Pi = \bigcup_{j=1}^{\infty} \Pi_j$ and each Π_j is φ-null then Π is φ-null.*

Proof. It suffices to show that $\varphi(\Pi) = 0$. Without loss of generality we may assume that all Π_j are mutually disjoint.

By Lemma 2.2, for each ε, there are $X_\varepsilon \subseteq X$ and $Y_\varepsilon \subseteq Y$ such that $\mu(X \setminus X_\varepsilon) < \varepsilon$, $\nu(Y \setminus Y_\varepsilon) < \varepsilon$ and the rectangle $\Pi^\varepsilon = \Pi \cap (X_\varepsilon \times Y_\varepsilon)$ is covered by a finite number of rectangles Π_j, say, $\Pi^\varepsilon \subseteq \bigcup_{j=1}^{m} \Pi_j$. Set $\Pi_j^\epsilon = \Pi_j \cap (X_\epsilon \times Y_\epsilon)$; we have

$$\varphi(\Pi^\varepsilon) = \sum_{j=1}^{m} \varphi(\Pi_j^\varepsilon) = 0.$$

On the other hand, if $\Pi = \alpha \times \beta$ then $\varphi(\Pi^\varepsilon) = \langle P(Y_\varepsilon)P(\beta)TP(\alpha)P(X_\varepsilon), F\rangle$ and, since $P(X_\varepsilon) \to I$, $P(Y_\varepsilon) \to I$ in the strong operator topology, we conclude that $\lim_{\varepsilon\to 0} \varphi(\Pi^\varepsilon) = \varphi(\Pi)$. Thus, $\varphi(\Pi) = 0$ and the proof is complete. $\square$

Theorem 6.4. *If E is a set of finite width then $\mathfrak{M}_{\max}(E) \cap \mathcal{K}$ coincides with the norm-closure $\mathfrak{M}_0(E)$ of the subspace of $\mathfrak{M}_{\max}(E)$ generated by its rank one operators.*

Proof. We may assume that the measures μ and ν are finite and the standard spaces X and Y arise from compact topologies. Indeed, if this is not the case, write $X = \bigcup_{n=1}^{\infty} X_n$ and $Y = \bigcup_{n=1}^{\infty} Y_n$ as increasing unions, where X_n and Y_n are compact, $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$. Then $P(X_n) \to_{n\to\infty} I$ and $P(Y_n) \to_{n\to\infty} I$ in the strong operator topology. If $T \in \mathfrak{M}_{\max}(E) \cap \mathcal{K}$ then $P(Y_n)TP(X_n) \to_{n\to\infty} T$ in norm, and hence we may restrict our attention to each of $E \cap (X_n \times Y_n)$, which is a set of finite width when considered as a subset of $X_n \times Y_n$.

We use induction on the width n of E. With the convention that all measurable rectangles are sets of width zero, the statement clearly holds for $n = 0$. Suppose that the assertion of the theorem is true for sets of width smaller than n, and let

$$E = \{(x, y) \in X \times Y : f_j(x) \le g_j(y),\ j = 1, \dots, n\},$$

where $f_j : X \to \mathbb{R}$ and $g_j : Y \to \mathbb{R}$ are measurable functions, $j = 1, \dots, n$. Let $F \in \Gamma(X, Y)$ be in the annihilator of $\mathfrak{M}_0(E)$. We need to show that $\langle T, F\rangle = 0$ for each compact operator $T \in \mathfrak{M}_{\max}(E)$. Assume first that all f_j, $j = 1, \dots, n$, are non-atomic. By Lemma 6.2, the sets

$$E_j = \{(x, y) : f_j(x) = g_j(y)\}, \quad j = 1, \dots, n,$$

are 1-quasi-diagonal and hence their union $\bigcup_{j=1}^{n} E_j$ is n-quasi-diagonal. Let $E' = E \cap (\bigcup_{j=1}^{n} E_j)$; then E' is n-quasi-diagonal and

$$E'' \stackrel{def}{=} E \setminus E' = \{(x, y) : f_j(x) < g_j(y),\ j = 1, \dots, n\}$$

is ω-open.

Let D be an n-diagonal system with $E' \subseteq U_D$. If Π is a rectangle, disjoint from U_D, then $\Pi \subseteq E'' \cup E^c$; since both E'' and E^c are ω-open, $\Pi = \bigcup_{i=1}^{\infty} \Pi_i$ where each of Π_i is either a subset of E'' or of E^c.

Let, as above, $\varphi(\alpha \times \beta) = \langle P(\beta)TP(\alpha), F\rangle$, where $\alpha \subseteq X$ and $\beta \subseteq Y$ are measurable. If $\Pi_i \subseteq E^c$ and $\Pi_i' \subseteq \Pi$ then $\varphi(\Pi_i') = 0$ by the fact that T is supported on E.

On the other hand, if $\Pi_i = \alpha_i \times \beta_i \subseteq E''$ then, clearly, $\Pi_i \subseteq E$ whence $P(\beta_i)TP(\alpha_i) \in \mathfrak{M}_0(\Pi_i) \subseteq \mathfrak{M}_0(E)$. It follows that $\varphi(\Pi_i) = 0$. The same argument shows that $\varphi(\Pi_i') = 0$ whenever Π_i' is a rectangle with $\Pi_i' \subseteq \Pi_i$, and hence Π_i is φ-null. By Lemma 6.3, Π is φ-null. We thus showed that every rectangle disjoint from U_D is φ-null.

Let $\widetilde{D} = \{\Pi_k' : 1 \le k \le m\}$ be a system such that $(U_D)^c = U_{\widetilde{D}}$. It follows from the previous paragraphs that

$$\langle \pi_{\widetilde{D}}(T), F\rangle = \sum_{k=1}^{m} \varphi(\Pi_k') = 0.$$

Hence

$$\langle T, F\rangle = \langle \pi_D(T), F\rangle + \langle \pi_{\widetilde{D}}(T), F\rangle = \langle \pi_D(T), F\rangle$$

and $|\langle T, F\rangle| \le \|F\| \|\pi_D(T)\|$. Since E' is n-quasi-diagonal, there exists a sequence $(D^k)_{k\in\mathbb{N}}$ of n-diagonal systems such that $E' \subseteq U_{D^k}$ for each k and $r(D^k) \to_{k\to\infty} 0$. By Lemma 6.1, $\|\pi_{D^k}(T)\| \to_{k\to\infty} 0$ and thus $\langle T, F\rangle = 0$.

Now let f_j be arbitrary. Then we can write X as a disjoint union $\bigcup_{k=0}^{\omega} X_k$, $\omega \le \infty$, where X_0 is a subset of X such that all f_j are non-atomic on X_0 and for each $k > 0$ at least one of the functions f_j is constant on X_k.

Set $P_k = P(X_k)$, $F_k(x, y) = \chi_{X_k}(x)F(x, y)$ and $T_k = TP_k$; then $\langle T, F\rangle = \sum_{k=0}^{\omega}\langle T_k, F_k\rangle$ and it hence suffices to show that $\langle T_k, F_k\rangle = 0$ for each k. It is clear that T_k is supported on $E_k \stackrel{def}{=} E \cap (X_k \times Y)$ and F_k annihilates $\mathfrak{M}_0(E_k)$.

The equality $\langle T_0, F_0\rangle = 0$ follows from the first part of the proof. Let $k > 0$, and suppose, for example, that the function f_1 is constant on X_k: $f_1(x) = a$, for $x \in X_k$. Set $Y_k = \{y \in Y : g_1(y) \ge a\}$. Then

$$E_k = \{(x, y) \in X_k \times Y_k : f_j(x) \le g_j(y),\ j = 2, \dots, n\}.$$

Thus E_k is a set of width at most $n - 1$, when considered as a subset of $X_k \times Y_k$. Since T is supported on E_k, we have $T_k = P(Y_k)T_k$. Moreover, $\chi_{X_k\times Y_k}F_k$ annihilates $\mathfrak{M}_0(E_k)$ and hence

$$\langle T_k, F_k\rangle = \langle P(Y_k)T_k, \chi_{X_k\times Y_k}F_k\rangle = 0$$

by the inductive assumption. $\square$

Corollary 6.5. *Let E be a set of finite width. The following conditions are equivalent:*

(i) *E is an operator U-set;*
(ii) *E does not support a non-zero Hilbert–Schmidt operator;*
(iii) *$\mu \times \nu(E) = 0$;*
(iv) *E does not support a non-zero nuclear operator;*
(v) *E does not contain a rectangle of non-zero measure.*

Proof. We may assume that μ and ν are finite, for if $X = \bigcup_{k=1}^{\infty} X_k$, $Y = \bigcup_{k=1}^{\infty} Y_k$, where $(X_k)_{k=1}^{\infty}$ and $(Y_k)_{k=1}^{\infty}$ are increasing sequences of subsets of finite measure and $T \in \mathcal{B}(H_1, H_2)$ is a non-zero compact operator supported in E, then so is $P(Y_k)TP(X_k)$ for some k.

(i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) If $\mu \times \nu(E)$ were non-zero, then T_k, where $k(x, y) = \chi_E(x, y)$, would be a non-zero Hilbert–Schmidt operator supported in E.

(iii) $\Rightarrow$ (iv) If E supports a non-zero nuclear operator then by [11, Theorem 6.7], E supports a non-zero rank one operator $u \otimes v$, $u \in L^2(X, \mu)$, $v \in L^2(Y, \nu)$. As $u \otimes v$ is supported on $\operatorname{supp} u \times \operatorname{supp} v$, we have $\mu \times \nu(E) \neq 0$, a contradiction.

(iv) $\Rightarrow$ (v) If E contains a non-zero rectangle $\alpha \times \beta$ then $\chi_\alpha \otimes \chi_\beta$ is a non-zero nuclear operator supported in E, a contradiction.

(v) $\Rightarrow$ (i) If E supports a non-zero compact operator then it follows from Theorem 6.4 that there exists a non-zero rank one operator $u \otimes v$ supported in E. But then $\operatorname{supp} u \times \operatorname{supp} v$ is a non-zero rectangle contained in E, a contradiction. □

Remark. We note that the conditions from Corollary 6.5 are also equivalent to the set E being a U_1-set, as well as to E being a U_0-set.

We have the following immediate corollary.

Corollary 6.6. *A non-zero bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^m)$ cannot be compact if it is supported on a manifold of the form $y_j = \phi(x_1, \dots, x_n)$, for some measurable function $\phi : \mathbb{R}^n \to \mathbb{R}$ and some $j = 1, \dots, m$, or on a set that can be partitioned into finitely many such sets.*

In particular, the support of a non-zero compact operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^1)$ is not contained in a smooth manifold of dimension strictly less than $n + 1$.

Proof. Assume, without loss of generality, that $j = 1$. Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be given by $\psi(y_1, \dots, y_m) = y_1$ and $E = \{(x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}^n \times \mathbb{R}^m : y_1 = \phi(x_1, \dots, x_n)\} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \psi(y) = \phi(x)\}$. As ψ is non-atomic, Lemma 6.2 implies that E is 1-diagonal. By Lemma 6.1, E does not support a non-zero compact operator. By [34, Proposition 5.3] there is no non-zero compact operator supported on a set that can be partitioned into finitely many sets of the form $\{(x_1, \dots, x_n, y_1, \dots, y_m) : y_j = \phi(x_1, \dots, x_n)\}$. □

If $\omega : G \to \mathbb{R}^+$ is a continuous homomorphism, let $E_{\omega,t} = \{s \in G : \omega(s) \leq t\}$; we call the subsets of G of this form *level sets* (see [9]).

Corollary 6.7. *Let $E_j \subseteq G$, $j = 1, \ldots, n$, be level sets. The set $E \stackrel{def}{=} \bigcap_{j=1}^n E_j$ is an M-set if and only if $m(E) > 0$.*

Proof. It is straightforward to check that if $F \subseteq G$ is a level set then F^* is a set of width one. It follows that E^* is a set of finite width. By Theorem 4.9 and Corollary 6.5, E is an M-set if and only if $(m \times m)(E^*) > 0$. This condition is equivalent to $m(E) > 0$ by the identity

$$(m \times m)\big(E^*\big) = \int_G m(Et)dt = \int_G \Delta(t)m(E)dt,$$

where Δ is the modular function. $\square$

7. Closable multipliers on group C^*-algebras

Let G be a locally compact group equipped with left Haar measure m and $\psi : G \to \mathbb{C}$ be a measurable function. It is well-known [6,19] (see also [31, Theorem 8.3]) that pointwise multiplication on $L^1(G)$ by the function ψ defines a completely bounded map on $C_r^*(G)$ if and only if the function $N(\psi)$ is a Schur multiplier. In this section, we prove a version of this result for closable maps (see Theorem 7.4).

Let

$$D(\psi) = \big\{f \in L^1(G) : \psi f \in L^1(G)\big\};$$

it is easy to see that the operator $f \to \psi f$, $f \in D(\psi)$, viewed as a densely defined operator on $L^1(G)$, is closable. Since $\lambda(L^1(G))$ is dense in $C_r^*(G)$ and $\|\lambda(f)\| \leq \|f\|_1$, $f \in L^1(G)$, the space $\lambda(D(\psi))$ is dense in $C_r^*(G)$ in the operator norm. Thus, the operator $S_\psi : \lambda(D(\psi)) \to C_r^*(G)$ given by $S_\psi(\lambda(f)) = \lambda(\psi f)$ is a densely defined operator on $C_r^*(G)$.

We wish to study the question of when S_ψ is closable. To this end, we recall [12] that the Banach space dual of $C_r^*(G)$ can be canonically identified with the weak* closure $B_\lambda(G)$ of $A(G)$ within the Fourier–Stieltjes algebra $B(G)$. A direct verification shows that the domain of S_ψ^* is equal to

$$J_\psi^\lambda \stackrel{def}{=} J_\psi^{B_\lambda(G)} = \big\{g \in B_\lambda(G) : \psi g \in^m B_\lambda(G)\big\}$$

and that $S_\psi^*(g)$ is equivalent to ψg for every $g \in J_\psi^\lambda$. By Proposition 2.1, S_ψ is closable (resp. weak* closable) if and only if J_ψ^λ is weak* dense (resp. norm dense) in $B_\lambda(G)$. We denote by $\mathrm{Clos}(G)$ the set of all measurable functions ψ for which S_ψ is closable and call the elements of $\mathrm{Clos}(G)$ closable multipliers on $C_r^*(G)$. This notion of multipliers should not be confused with the notion of multipliers in the C^*-algebra sense.

A function f on G is said to *belong to* $A(G)$ (resp. *almost belong to* $A(G)$) *at the point* $t \in G$ if there exists a neighbourhood U of t and a function $u \in A(G)$ such that $f(s) = u(s)$ for all (resp. m-almost all) points $s \in U$. Set [34]

$$E_f = \{t \in G : f \text{ does not almost belong to } A(G) \text{ at } t\}.$$

We say that f (almost) belongs locally to $A(G)$ if f (almost) belongs to $A(G)$ at every point and let $A(G)^{\mathrm{loc}}$ be the set of functions which belong to $A(G)$ at every point. If f almost belongs to $A(G)$ at every point then f is equivalent to a function from $A(G)^{\mathrm{loc}}$. To see this, we first show that, given a compact set $K \subseteq G$ and a function f that almost belong to $A(G)$ at each point of G, there exists $g \in A(G)$ such that f is equivalent to g on K. In fact, for each $t \in G$ there exists a neighbourhood V_t of t and $g_t \in A(G)$ such that $f \sim g_t$ on V_t. Then $K \subseteq \bigcup_{t\in F} V_t$ for some finite $F \subseteq K$. By the regularity of $A(G)$, there exist $h_t \in A(G)$, $t \in F$, such that $\sum_{t\in F} h_t(x) = 1$ if $x \in K$ and $h_t(x) = 0$ if $x \notin V_t$, $t \in F$. Hence, for almost all $x \in K$, we have $f(x) = \sum_{t\in F} f(x)h_t(x) = \sum_{t\in F} g_t(x)h_t(x)$, while $\sum_{t\in F} g_t h_t \in A(G)$. As the group G is σ-compact we can find compact subsets $K_n \subseteq G$, $K_n \subseteq K_{n+1}$ such that $G = \bigcup_{n=1}^{\infty} K_n$, and a sequence of functions $g_n \in A(G)$ such that $f \sim g_n$ on K_n for any n. As g_n are continuous, we obtain $g_{n+1} = g_n$ on K_n. Define a function g by letting $g(x) = g_n(x)$ if $x \in K_n$. Then g is well-defined, continuous and $f \sim g$. Clearly, g belongs to $A(G)$ at every point of G.

The following fact was established in [34] in the case G is abelian; its proof, however, does not use the commutativity of G.

Lemma 7.1. *(See [34].) For every measurable function $\psi : G \to \mathbb{C}$, let*

$$J_\psi \stackrel{def}{=} J_\psi^{A(G)} = \{f \in A(G) : \psi f \in^m A(G)\}.$$

Then $E_\psi = \operatorname{null} J_\psi$.

We say that a locally compact group G has *property* (A) if there exists a net $(u_i) \subseteq A(G)$ such that for each $g \in B_\lambda(G)$, $u_i g \to g$ in the weak*-topology of $B_\lambda(G)$. Note that if $(u_i) \subseteq A(G)$ is a net such that $u_i \to 1$ uniformly on compact sets and $\sup\|u_i\|_{MA(G)} < \infty$ (in particular, if G is weakly amenable) then G has property (A). In fact, for $g \in B_\lambda(G)$ and $f \in C_c(G)$, we have

$$\langle \lambda(f), gu_i - g\rangle = \int_G f(t)g(t)\bigl(u_i(t) - 1\bigr)dt \to 0.$$

It follows from [7, Proposition 1.2] that, if $u \in A(G)$ then $\|u\|_{MA(G)}$ coincides with the norm of u as a multiplier of $B_\lambda(G)$. Thus, $\|gu_i - g\|_{B(G)} \leq \|u_i\|_{MA(G)}\|g\|_{B(G)} + \|g\|_{B(G)}$. The statement now follows from the fact that the set of all $\lambda(f)$, $f \in C_c(G)$, is dense in $C_r^*(G)$.

Since $C_r^*(G)^* = B_\lambda(G)$ and $A(G) \subseteq B_\lambda(G)$, the elements of $C_r^*(G)$ can be identified with functionals on $A(G)$ continuous with respect to the restriction to $A(G)$ of the weak* topology of $B_\lambda(G)$; this identification is made in the next proposition.

Proposition 7.2. *Let G be a locally compact group with property* (A) *and $\psi : G \to \mathbb{C}$ be a measurable function. The operator S_ψ is closable if and only if there is no non-zero operator $T \in C_r^*(G)$ which annihilates J_ψ. In particular,*

(i) *if E_ψ is a U-set then S_ψ is closable;*
(ii) *if E_ψ is an M_1-set then S_ψ is not closable.*

Proof. Since $A(G)$ is an ideal in $B(G)$, property (A) implies that the weak* closures of J_ψ and J_ψ^λ in $B_\lambda(G)$ coincide. The first statement now follows from Proposition 2.1.

By Lemma 7.1,

$$J(E_\psi) \subseteq \overline{J_\psi} \subseteq I(E_\psi).$$

Parts (i) and (ii) follow from these inclusions and the definitions of a U-set and an M_1-set. $\square$

Corollary 7.3. *Let G be a locally compact group with property* (A) *and $\psi : G \to \mathbb{C}$ be a measurable function. If $m(E_\psi) > 0$ then S_ψ is not closable.*

Proof. By Remark 4.3, E_ψ is an M_1-set. Now the claim follows from Proposition 7.2 (ii). $\square$

Recall from Section 2 that, for a measurable function $\varphi : G \times G \to \mathbb{C}$, we let S_φ be the operator, densely defined on $\mathcal{K}(L^2(G))$, with domain

$$D(S_\varphi) = \{T_k \in \mathcal{C}_2(H_1, H_2) : \hat{\varphi}k \in L^2(G \times G)\} \subseteq \mathcal{K}\big(L^2(G)\big).$$

It was shown in [34] that the domain $D(S_\varphi^*) \subseteq \Gamma(G,G)$ of its adjoint is given by

$$D\big(S_\varphi^*\big) = \big\{h \in \Gamma(G,G) : \varphi h \in^{m\times m} \Gamma(G,G)\big\}.$$

Theorem 7.4. *Let G be a second countable locally compact group with property* (A)*, $\psi : G \to \mathbb{C}$ be a measurable function and $\varphi = N(\psi)$. The following are equivalent:*

(i) *the operator S_ψ is closable;*
(ii) *the operator S_φ is closable;*
(iii) *$\mathcal{A} \cap D(S_\varphi^*)^\perp = \{0\}$;*
(iv) *$\mathcal{R} \cap D(S_\varphi^*)^\perp = \{0\}$.*

Proof. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) follows from the fact that $\mathcal{K} \subseteq \mathcal{A} \subseteq \mathcal{R}$ and the fact that S_φ is closable if and only if $\mathcal{K} \cap D(S_\varphi^*)^\perp = \{0\}$.

(ii) $\Rightarrow$ (i) If S_ψ is not closable then, by Proposition 7.2, there exists a non-zero $T \in C_r^*(G)$ which annihilates J_ψ. Let $A \in \mathcal{D}_0$ be such that $AT \neq 0$. In view of (7), it suffices to show that AT annihilates $D(S_\varphi^*)$. Since $D(S_\varphi^*)$ is invariant under $\mathfrak{S}(G,G)$, it suffices to show that T annihilates $D(S_\varphi^*)$.

Let $h \in D(S_\varphi^*)$. Writing $h = \sum_{i=1}^\infty f_i \otimes g_i$, where $\sum_{i=1}^\infty \|f_i\|_2^2 < \infty$ and $\sum_{i=1}^\infty \|g_i\|_2^2 < \infty$, and using Lemma 2.3, for every $f \in L^1(G)$, we have

$$\langle \lambda(f), h \rangle = \left\langle \lambda(f), \sum_{i=1}^\infty f_i \otimes g_i \right\rangle = \sum_{i=1}^\infty \big(\lambda(f)(f_i), \bar{g}_i\big)$$

$$= \iint f(s) \sum_{i=1}^\infty g_i(t) f_i\big(s^{-1}t\big) dt ds = \big\langle \lambda(f), P(h) \big\rangle.$$

It follows that $\langle T, h \rangle = \langle T, P(h) \rangle$. Since $\varphi h \in^{m \times m} \Gamma(G,G)$, identity (2) implies that $\psi P(h) = P(\varphi h) \in^m P(\Gamma(G,G)) = A(G)$, and hence $P(h) \in J_\psi$. Thus, $\langle T, P(h) \rangle = 0$ and hence $\langle T, h \rangle = 0$.

(i) $\Rightarrow$ (iv) Let S_ψ be closable and suppose that $0 \neq T \in \mathcal{R} \cap D(S_\varphi^*)^\perp$. By Lemma 4.7, there exist $a, b \in L^2(G)$ such that $E_{a \otimes b}(T) \neq 0$. Suppose that $u \in J_\psi^\lambda$; then

$$\varphi(a \otimes b) N(u) = (a \otimes b) N(\psi u) \in \Gamma(G,G)$$

and hence $(a \otimes b)N(u) \in D(S_\varphi^*)$. Thus

$$\big\langle E_{a \otimes b}(T), u \big\rangle = \big\langle T, (a \otimes b) N(u) \big\rangle = 0.$$

By Theorem 4.6, $E_{a \otimes b}(T)$ is a (non-zero) element of $C_r^*(G)$; in view of Proposition 7.2, this contradicts the closability of S_ψ. $\square$

Corollary 7.5. *The set* $\mathrm{Clos}(G)$ *is an algebra under pointwise addition and multiplication.*

Proof. Let $\psi_i \in \mathrm{Clos}(G)$, $i = 1, 2$. Then $N\psi_1 + N\psi_2 = N(\psi_1 + \psi_2)$ and $N(\psi_1 \psi_2) = (N\psi_1)(N\psi_2)$. By [34, Theorem 5.2], the closable multipliers on $\mathcal{K}(L^2(G))$ form an algebra under pointwise addition and multiplication. The claim now follows from Theorem 7.4. $\square$

We now give some examples of closable and non-closable multipliers.

Example 7.6 *(A non-closable multiplier on $C_r^*(\mathbb{T})$).* Using the arguments in [33, 7.8.3–7.8.6] (see also [36, Proposition 9.9]), one can show that there exist $c = (c_n)_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$, $p > 2$, and $d = (d_n)_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$ with $\bar{d}_n \stackrel{def}{=} \bar{d}_{-n}$, $n \in \mathbb{Z}$, such that $c * d = 0$ and $c * \bar{d} \neq 0$. Let $f \in A(\mathbb{T}) \subseteq L^1(\mathbb{T})$ be the function whose sequence of Fourier coefficients is d and F be the pseudo-function (that is, the bounded linear functional on $A(\mathbb{T})$) whose sequence of Fourier coefficients is c. We have $f \cdot F = 0$ while $\bar{f} \cdot F \neq 0$. After identifying the dual of $A(\mathbb{T})$ with $\mathrm{VN}(\mathbb{T})$, we view F as the operator on $L^2(\mathbb{T})$ determined by the identities $\widehat{F\xi} = c\hat{\xi}$, $\xi \in L^2(\mathbb{T})$ (where $\hat{\eta}$ denotes the Fourier transform of a function $\eta \in L^2(\mathbb{T})$). Moreover (see the start of Section 3), $F \in C_r^*(\mathbb{T})$. Let $h_n \in L^1(\mathbb{T})$ be such that $\lambda(h_n) \to_{n\to\infty} F$ in the operator norm. Then

$$\|\lambda(h_n) - F\| = \sup\{|\langle\lambda(h_n) - F, u\rangle| : u \in A(\mathbb{T}),\ \|u\| = 1\} \to_{n\to\infty} 0.$$

It follows that

$$\sup\{|\langle\lambda(fh_n) - f \cdot F, u\rangle| : u \in A(\mathbb{T}),\ \|u\| = 1\} \to_{n\to\infty} 0$$

which in turn implies that $\lambda(fh_n) \to f \cdot F$ in the operator norm. Similarly, $\lambda(\bar{f}h_n) \to_{n\to\infty} \bar{f} \cdot F$.

Let $\psi : \mathbb{T} \to \mathbb{C}$ be given by $\psi(t) = \bar{f}(t)/f(t)$ if $f(t) \neq 0$ and $\psi(t) = 0$ otherwise. Then

$$S_\psi(\lambda(fh_n)) = \lambda(\psi f h_n) = \lambda(\bar{f}h_n) \nrightarrow_{n\to\infty} 0$$

while $\lambda(fh_n) \to 0$. Hence ψ is a non-closable multiplier.

Example 7.7 *(A continuous non-closable multiplier on $C_r^*(\mathbb{T})$).* The following example was given in [34]. We recall the construction for completeness. Let $X \subseteq \mathbb{T}$ be a closed set of positive Lebesgue measure and $S \subseteq X$ be a dense subset of Lebesgue measure zero. By [23, Chapter II, Theorem 3.4], there exists $h \in C(\mathbb{T})$ whose Fourier series diverges at every point of S. By the Riemann Localisation Principle, any function which belongs locally to $A(\mathbb{T})$ at $t \in \mathbb{T}$ has a convergent Fourier series at t; hence, $S \subseteq E_h$ and since E_h is closed, $X \subseteq E_h$. Therefore $m(E_h) > 0$ and S_h is not closable by Corollary 7.3.

Example 7.8 *(A class of idempotent closable multipliers on $C_r^*(\mathbb{R})$).* Let $F \subseteq \mathbb{R}$ be a closed set which is the union of countably many intervals. We claim that $\chi_F \in \mathrm{Clos}(\mathbb{R})$. Let $\psi = \chi_F$; then E_ψ is the set of boundary points of F. Thus E_ψ is contained in the set of endpoints of the intervals whose unions are F, and hence E_ψ is countable. The claim now follows from Proposition 7.2 and Corollary 5.3.

This example should be compared with the well-known fact that there are no bounded non-trivial idempotent multipliers on $C_r^*(\mathbb{R})$.

We next discuss the weak** closability of the operator S_ψ (in the sense of Section 2.1). We have the following necessary condition.

Proposition 7.9. *If S_ψ is weak** closable then $\psi \in A(G)^{\mathrm{loc}}$.*

Proof. Suppose that S_ψ is weak** closable. By Proposition 2.1, J_ψ^λ is dense in $B_\lambda(G)$. Thus, $A(G)J_\psi^\lambda$ is dense in $A(G)B_\lambda(G) = A(G)$. However, $A(G)J_\psi^\lambda \subseteq J_\psi$ and hence J_ψ is dense in $A(G)$. By Lemma 7.1, $\psi \in A(G)^{\mathrm{loc}}$. $\square$

We point out that the converse of Proposition 7.9 does not hold for non-compact groups. In fact, let G be a non-discrete locally compact abelian group with dual group Γ. Then $B_\lambda(\Gamma) = B(\Gamma) \neq A(\Gamma)$. By [15, Corollary 8.2.6], there exists $f \in B(\Gamma)$, $|f(x)| > 1$, $x \in \Gamma$, such that $\psi \overset{def}{=} \frac{1}{f} \notin B(\Gamma)$; on the other hand, $\psi \in A(\Gamma)^{\mathrm{loc}}$ (see the arguments in [34, Remark 7.11]). We have that $J_\psi^\lambda \subseteq (f)$, where (f) is the ideal in $B(\Gamma)$ generated by f. As f is not invertible in $B(\Gamma)$, (f) is contained in a maximal ideal, and hence cannot be dense in $B(\Gamma)$.

It follows that a version of Theorem 7.4, with weak** closability in the place of closability, does not hold. Indeed, by [34, Theorem 7.8], for abelian groups, $N(\psi)$ is a weak** closable multiplier if and only if $\psi \in A(G)^{\mathrm{loc}}$. In view of these remarks, the following question arises.

Question. Is S_ψ weak** closable only when S_ψ is bounded?

Note that if G is compact then S_ψ is weak** closable if and only if S_ψ is bounded, that is, if and only if $\psi \in A(G)$; this follows from Proposition 7.9 and the fact that in this case $A(G) = A(G)^{\mathrm{loc}}$.

8. Closable multipliers on group von Neumann algebras

In this section we turn our attention to multipliers acting on $\mathrm{VN}(G)$. We will need an appropriate version of closability suited for working with dual spaces, which we now introduce. Let $\mathcal{X}$ and $\mathcal{Y}$ be dual Banach spaces, with specified preduals $\mathcal{X}_*$ and $\mathcal{Y}_*$, respectively, and $D(\Phi) \subseteq \mathcal{X}$ be a weak* dense subspace. We say that an operator $\Phi : D(\Phi) \to \mathcal{Y}$ is *weak* closable* if the conditions $x_i \in \mathcal{X}$, $y \in \mathcal{Y}$, $x_i \to_{w^*} 0$, $\Phi(x_i) \to_{w^*} y$ imply that $y = 0$. Here, the weak* convergence is in the designated weak* topologies of $\mathcal{X}$ and $\mathcal{Y}$.

Note that, since the *-weak closure of the graph of Φ contains its norm-closure, each weak* closable operator is closable.

We have the following characterisation of weak* closability.

Proposition 8.1. *Let $D(\Phi) \subseteq \mathcal{X}$ be a weak* dense subspace and $\Phi : D(\Phi) \to \mathcal{Y}$ be a linear operator. The following are equivalent:*

(i) *the operator Φ is weak* closable;*
(ii) *the space $D_*(\Phi) = \{g \in \mathcal{Y}_* : x \to \langle \Phi(x), g\rangle$ is w^*-cont. on $D(\Phi)\}$ is dense in $\mathcal{Y}_*$.*

Proof. (ii) $\Rightarrow$ (i) Suppose that $x_i \to 0$ and $\Phi(x_i) \to y$ in the corresponding weak* topologies. If $g \in D_*(\Phi)$ then the map $x \to \langle \Phi(x), g\rangle$ is weak* continuous on $D(\Phi)$. Since

$D(\Phi)$ is weak* dense in $\mathcal{X}$, it extends to a weak* continuous functional on the whole of $\mathcal{X}$ and hence there exists $f \in \mathcal{X}_*$ such that $\langle \Phi(x), g\rangle = \langle x, f\rangle$, $x \in D(\Phi)$. In particular, $\langle \Phi(x_i), g\rangle = \langle x_i, f\rangle \to 0$. On the other hand, $\langle \Phi(x_i), g\rangle \to \langle y, g\rangle$. Thus, $\langle y, g\rangle = 0$ for all $g \in D_*(\Phi)$. Since $D_*(\Phi)$ is (norm) dense in $\mathcal{Y}_*$, we conclude that $y = 0$.

(i) ⇒ (ii) For an operator T with domain D, let $\mathrm{Gr}' T = \{(T\xi, \xi) : \xi \in D\}$. Let $\Phi_* : D_*(\Phi) \to \mathcal{X}_*$ be defined by letting $\Phi_*(g) = f$, where, for $g \in D_*(\Phi)$, the element $f \in \mathcal{X}_*$ is the (unique) weak* continuous functional on $\mathcal{X}$ such that $\langle \Phi(x), g\rangle = \langle x, f\rangle$, $x \in D_*(\Phi)$. We claim that

$$(\mathrm{Gr}\,\Phi)_\perp \subseteq \mathrm{Gr}'(-\Phi_*). \tag{22}$$

To see this, let $(f, g) \in (\mathrm{Gr}\,\Phi)_\perp$; then $\langle f, x\rangle = -\langle g, \Phi(x)\rangle$, for all $x \in D(\Phi)$. It follows that $g \in D(\Phi_*)$ and $\Phi_*(g) = -f$; thus, (22) is proved.

Now suppose that $y \in \mathcal{Y}$ annihilates $D_*(\Phi)$. Then $(0, y)$ annihilates $\mathrm{Gr}'(-\Phi_*)$ and (22) implies that

$$(0, y) \in \big((\mathrm{Gr}\,\Phi)_\perp\big)^\perp = \overline{\mathrm{Gr}\,\Phi}^{w^*}.$$

Since Φ is weak* closable, $y = 0$ and so $D_*(\Phi)$ is norm dense in $\mathcal{Y}_*$. □

The von Neumann algebra $\mathrm{VN}(G)$ possesses two natural and, in the case G is non-discrete, genuinely different, weak* dense selfadjoint subalgebras, one of them being $\lambda(L^1(G))$, and the other being the (non-closed) linear span of the left translation operators

$$\mathrm{VN}_0(G) = [\lambda_s : s \in G].$$

Given a continuous function $\psi : G \to \mathbb{C}$, we can now consider, along with the operator S_ψ with domain $D(\psi)$, a linear operator $S'_\psi : \mathrm{VN}_0(G) \to \mathrm{VN}_0(G)$ given by $S'_\psi(\lambda_s) = \psi(s)\lambda_s$, $s \in G$. Our aim in the next theorem is to characterise the weak* closability of S_ψ and S'_ψ.

Theorem 8.2. *Let $\psi : G \to \mathbb{C}$ be a continuous function and $\varphi = N(\psi)$. The following are equivalent:*

(i) *the operator S_ψ is weak** closable;*
(ii) *the operator S'_ψ is weak* closable;*
(iii) *the function ψ belongs locally to $A(G)$ at every point;*
(iv) *the function φ is a local Schur multiplier on $\mathcal{K}(L^2(G))$;*
(v) *the operator S_φ is weak** closable;*
(vi) $\overline{D(S^*_\varphi)}^{\|\cdot\|_\Gamma} = \Gamma(G, G)$, $\overline{D(S^{**}_\varphi)}^{w^*} = \mathcal{B}(L^2(G))$, $\mathrm{VN}_0(G) \subseteq D(S^{**}_\varphi)$ *and the operator* $S^{**}_\varphi : D(S^{**}_\varphi) \to \mathcal{B}(L^2(G))$ *is weak* closable;*
(vii) $\overline{D(S^*_\varphi)}^{\|\cdot\|_\Gamma} = \Gamma(G, G)$, $\mathrm{VN}_0(G) \subseteq D(S^{**}_\varphi)$ *and the operator* $S^{**}_\varphi : D(S^{**}_\varphi) \to \mathcal{B}(L^2(G))$ *is weak* closable.*

Proof. We have that

$$\begin{aligned}
D_*\big(S'_\psi\big) &= \big\{f \in A(G) : T \to \big\langle S'_\psi(T), f\big\rangle \text{ is w*-continuous on } \mathrm{VN}_0(G)\big\} \\
&= \big\{f \in A(G) : \exists u \in A(G) : \big\langle S'_\psi(T), f\big\rangle = \langle T, u\rangle,\ T \in \mathrm{VN}_0(G)\big\} \\
&= \big\{f \in A(G) : \exists u \in A(G) \text{ with } \big\langle S'_\psi(\lambda_s), f\big\rangle = \langle \lambda_s, u\rangle,\ s \in G\big\} \\
&= \big\{f \in A(G) : \exists u \in A(G) \text{ with } \psi(s)f(s) = u(s),\ s \in G\big\} \\
&= \big\{f \in A(G) : \psi f \in A(G)\big\} \\
&= J_\psi,
\end{aligned}$$

where the last equality follows from the fact that ψ is continuous. The equivalence (ii) $\Leftrightarrow$ (iii) now follows from Lemma 7.1 and Proposition 8.1.

Similarly,

$$\begin{aligned}
D_*(S_\psi) &= \big\{f \in A(G) : g \to \big\langle S_\psi\big(\lambda(g)\big), f\big\rangle \text{ is w*-continuous on } D(\psi)\big\} \\
&= \big\{f \in A(G) : \exists u \in A(G) : \big\langle \lambda(\psi g), f\big\rangle = \big\langle \lambda(g), u\big\rangle,\ g \in D(\psi)\big\} \\
&= \Big\{f \in A(G) : \exists u \in A(G) : \int_G \psi f g = \int_G ug,\ g \in D(\psi)\Big\} \\
&= \big\{f \in A(G) : \exists u \in A(G) \text{ such that } \psi f \sim u\big\} \\
&= J_\psi
\end{aligned}$$

(recall that by $u \sim v$ we mean that $u = v$ almost everywhere on G). The fourth equality in the latter chain can be seen as follows. Let $K \subseteq G$ be a compact set; then $\psi|_K$ is bounded and hence $L^1(K) \subseteq D(\psi)$. It follows that $\int_K \psi f g = \int_K ug$ for all $g \in L^1(K)$. Since $\psi f|_K$ and $u|_K$ belong to $L^\infty(K)$, we conclude that $\psi f|_K = u|_K$ almost everywhere. Since this holds for every compact $K \subseteq G$, we have that $\psi f \sim u$.

The equivalence (i) $\Leftrightarrow$ (iii) follows, as above, from Lemma 7.1 and Proposition 8.1.

(iii) $\Rightarrow$ (iv) We claim that $\psi u \in A(G)$ for every $u \in A(G) \cap C_c(G)$. Indeed, since $\psi \in A(G)^{\mathrm{loc}}$, for every $t \in G$ there exists a neighbourhood V_t of t and a function $g_t \in A(G)$ such that $\psi = g_t$ on V_t. Since $\mathrm{supp}(u)$ is compact there exists a finite set $F \subseteq G$ such that $\mathrm{supp}(u) \subseteq \bigcup_{t\in F} V_t$. It follows from the regularity of $A(G)$ that there exist $h_t \in A(G)$, $t \in F$, such that $\sum_{t\in F} h_t(x) = 1$ if $x \in \mathrm{supp}(u)$ and $h_s(x) = 0$ if $x \notin V_s$ for each $s \in F$ (see the proof of [17, Theorem 39.21]). Then for every $x \in G$ we have

$$\psi(x)u(x) = \sum_{t\in F} \psi(x)h_t(x)u(x) = \sum_{t\in F} g_t(x)h_t(x)u(x),$$

which gives $\psi u \in A(G)$.

Let $(K_n)_{n=1}^\infty$ be an increasing sequence of compact sets such that, up to a null set, $\bigcup_{n=1}^\infty K_n = G$. Choose, for each $n \in \mathbb{N}$, a function $\psi_n \in A(G) \cap C_c(G)$ that takes the value 1 on $K_nK_n^{-1}$. By the previous paragraph, $\psi\psi_n \in A(G)$ and therefore $N(\psi\psi_n)$ is a Schur multiplier. Thus, for each $h \in \Gamma(G,G)$, we have

$$\varphi\chi_{K_n\times K_n}h = N(\psi\psi_n)\chi_{K_n\times K_n}h \in \Gamma(G,G).$$

It follows that $\varphi|_{K_n\times K_n}$ is a Schur multiplier and hence φ is a local Schur multiplier.

(iv) ⇒ (v) follows from the fact that every local Schur multiplier is a weak* closable multiplier [34].

(v) ⇒ (vi) Suppose that S_φ is weak** closable. By Proposition 2.1, the space $D(S_\varphi^*)$ is dense in $\Gamma(G,G)$ in the norm topology. We have that

$$D\big(S_\varphi^{**}\big) = \big\{T \in \mathcal{B}\big(L^2(G)\big) : h \to \big\langle T, S_\varphi^*(h)\big\rangle \text{ is continuous on } D\big(S_\varphi^*\big)\big\}.$$

The space $D(S_\varphi^{**})$ is weak* dense in $\mathcal{B}(L^2(G))$ since it contains the norm dense subspace $D(S_\varphi)$.

Suppose that $h \in D(S_\varphi^*)$; then $S_\varphi^*(h) = \varphi h \in^{m\times m} \Gamma(G,G)$ and hence, if $T \in D(S_\varphi^{**})$ then

$$\langle T, \varphi h\rangle = \big\langle T, S_\varphi^*(h)\big\rangle = \big\langle S_\varphi^{**}(T), h\big\rangle.$$

The mapping

$$T \to \big\langle S_\varphi^{**}(T), h\big\rangle, \quad T \in D\big(S_\varphi^{**}\big),$$

is thus weak* continuous and hence $h \in D_*(S_\varphi^{**})$. In other words, $D(S_\varphi^*) \subseteq D_*(S_\varphi^{**})$; since $D(S_\varphi^*)$ is dense in norm in $\Gamma(G,G)$, the same holds true for $D_*(S_\varphi^{**})$. By Proposition 8.1, S_φ^{**} is weak* closable.

Let $s \in G$. We show that $\lambda_s \in D(S_\varphi^{**})$. Recall that $P : \Gamma(G,G) \to A(G)$ is the canonical contractive surjection satisfying (2); for every $h \in D(S_\varphi^*)$, using Lemma 2.3, we see that

$$\big\langle \lambda_s, S_\varphi^*(h)\big\rangle = \langle \lambda_s, \varphi h\rangle = P(\varphi h)(s) = \psi(s)P(h)(s) = \big\langle \psi(s)\lambda_s, h\big\rangle. \tag{23}$$

Thus, $\lambda_s \in D(S_\varphi^{**})$, $S_\varphi^{**}(\lambda_s) = \psi(s)\lambda_s$, and (vi) is proved.

(vi) ⇒ (vii) is trivial.

(vii) ⇒ (ii) Suppose that $(T_i)_i \subseteq \mathrm{VN}_0(G)$ and $T \in \mathrm{VN}(G)$ are such that $T_i \to^{w^*} 0$ and $S'_\psi(T_i) \to^{w^*} T$. Then (T_i) (resp. $(S'_\psi(T_i))$) converges to zero (resp. T) in the weak* topology of $\mathcal{B}(L^2(G))$. Identity (23) shows that $S_\varphi^{**}(R) = S'_\psi(R)$ for every $R \in \mathrm{VN}_0(G)$. Since S_φ^{**} is weak* closable, $T = 0$. □

Remark. If ψ is not assumed to be continuous, then all conditions in Theorem 8.2 apart from (ii) remain equivalent, provided that in (iii) we require that ψ almost belongs locally to $A(G)$ at every point.

Proposition 7.9 and Theorems 7.4 and 8.2 yield the following implications:

$$S_\psi \text{ is weak}^{**} \text{ closable} \implies S_\psi \text{ is weak}^{*} \text{ closable} \implies S_\psi \text{ is closable.}$$

Theorem 8.2 and the example after Proposition 7.9 show that there exists a continuous function ψ for which S_ψ is weak* closable but not weak** closable. On the other hand, Proposition 7.2 implies that if E_ψ is a non-empty U-set then ψ is closable but $\psi \notin A(G)^{\mathrm{loc}}$; thus, by Theorem 8.2, S_ψ is not weak* closable. For example, for $G = \mathbb{R}$, $\psi = \chi_{[0,+\infty)}$, we have $E_\psi = \{0\}$ which is a non-empty U-set by Corollary 5.3. One can also find a continuous function ψ for which E_ψ is a one-point set of uniqueness. In fact, consider a function $\psi(t)$ on $[0,\pi]$ which is smooth on the open interval $(0,\pi)$ and $\psi(0) = \psi(\pi) = 0$. Assume also that $\psi'(\pi) = 0$ and that the integral $\int_0^1 \psi(t)/t dt$ diverges. Extend ψ to an odd (continuous) function on $[-\pi,\pi]$. By [20, Chapter II.14], $\psi \notin A(\mathbb{T})^{\mathrm{loc}} = A(\mathbb{T})$. As ψ is smooth at any $t \neq 0$, ψ belongs to $A(\mathbb{T})$ at any such point t. Therefore $E_\psi = \{0\}$.

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