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Subgap current in superconducting tunnel junctions with diffusive electrodes

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We calculate the subgap current in planar superconducting tunnel junctions with thin-film diffusive leads. It is found that the subharmonic gap structure of the tunnel current scales with an effective tunneling transparency which may exceed the junction transparency by up to two orders of magnitude depending on the junction geometry and the ratio between the coherence length and the elastic scattering length. These results provide an alternative explanation of enhanced values of the subgap current in tunneling experiments often ascribed to imperfection of the insulating layer. We also discuss the effect of finite lifetime of quasiparticles as the possible origin of additional enhancement of multiparticle tunnel currents.

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Subgap quasiparticle current in superconducting junctions at small applied voltages \( eV < 2\Delta \) is the subject of persistent theoretical interest and experimental research. Recently, the problem has attracted new attention, and a number of measurements of the subgap current in high-quality tunnel junctions have been performed,1,2 motivated by the problem of decoherence in Josephson-junction-based superconducting qubits.3 The subgap current at zero temperature is due to decoherence in Josephson-junction-based superconducting electrodes,6 and mesoscopic point contacts with diffusive/elastic reflections have been performed,1,2 to explain by assuming randomly distributed resonant levels within the tunnel barrier,10 transmission junctions could be explained by assuming randomly distributed resonant levels within the tunnel barrier,10 the ballistic scaling6 will be valid.

The enhancement effect can be qualitatively understood by considering a short point contact with the reservoirs located very close to the contact, \( L \ll \xi_0 \) [cf. Fig. 1(a)]. In this case, the current can be calculated within the mesoscopic approach,12 by integrating over contributions of normal conducting eigenmodes with randomly distributed transparencies. The relevant distribution is known to be spread over the interval \( -(L/\ell)D \gg D \). The most transparent modes dominate the subgap current, giving \( D_{\text{eff}} \sim (L/\ell)D \). In our case of junctions with large distance to the reservoirs, the scale of the spatial variation of the Green’s function \( \xi_0 \) plays the role of the effective junction length giving qualitatively our result, \( D_{\text{eff}} \sim (\xi_0/\ell)D \).14 We note that for the long junctions under consideration the statistics of the eigenmode transparencies is not known, and a quantitative result has to be derived from the quasiclassical theory for diffusive superconductors.

Our analysis is based on the diffusive equations for the quasiclassical two-time Keldysh-Green functions \( \hat{G}(r,t_1,t_2) \),

\[
\left[ \hat{H}, \hat{G} \right] = iD \nabla \hat{J}, \quad \hat{G} \ast \hat{G} = 1, \quad \hat{G} = \begin{pmatrix} \hat{g}^R & \hat{G}^\Delta \\ 0 & \hat{g}^A \end{pmatrix}
\]  

(2)

Here \( \hat{H}(t_1,t_2) = [i\sigma_i \partial_i + \Delta \exp(i\sigma_i \phi i\sigma_j) \delta(t_1 - t_2)] \), \( \phi \) is the superconducting phase, the sign \( \ast \) denotes time convolution,

![FIG. 1. One-dimensional (a) and planar (b) models of the tunnel junction with diffusive leads; equilibrium reservoirs are far from the contact, \( L \gg \xi_0 \).](image)
and $\tilde{J} = \tilde{G} \nabla \tilde{G}$. Equation (2) can be decomposed into the Usadel equation for the retarded or advanced Green’s functions $\tilde{g}^R, \tilde{g}^A$ and the equation for the Keldysh function $\tilde{G}^K = \tilde{g}^R \circ \tilde{f} \circ \tilde{g}^A$, where $\tilde{f} + \sigma_3 \tilde{g} f_0$ is the distribution function.

We present detailed calculations for the simpler, 1D geometry of Fig. 1(a). At the left electrode, $x = -L$, the Fourier transformations of the two-time functions $\tilde{g}$ and $\tilde{f}$ with respect to the variable $t_1 - t_2$ are given by the equilibrium expressions

$$\tilde{g}(E) = \sigma_u u(E) + i \sigma_v v(E), \quad \tilde{f}(E) = \tanh(E/2T), \quad (3)$$

$$(u,v) = (E, \Delta / \xi(E), \quad \tilde{g}^R, \tilde{A} = [(E \pm i0)^2 - \Delta^2]^{1/2}. \quad (4)$$

At the right voltage-biased electrode, $x = L$, the function $\tilde{G}$ is defined through the gauge transformation $\tilde{G} = \tilde{g} \circ \tilde{J} \circ \tilde{g}$. The Green’s function matrix structure in Eq. (5) extends to the whole right lead, $\tilde{G}(x)$ for $0 < x < L$. The problem is therefore reduced to the solution of a static equation for the function $\tilde{G}(x, t_1, t_2)$ at $-L < x < 0$ with the time-dependent boundary condition (6) at $x = 0$. The electric current is related to the Keldysh component $\tilde{J}^K$ of the matrix current $\tilde{J}$ as $I(t) = (\pi g/4e) \text{Tr} \sigma_3 \tilde{J}^K(x, t, t)$. Using Eqs. (5) and (6), it can be expressed as

$$I(t) = (\pi i / 8eR) \text{Tr} \sigma_3 \tilde{G} \circ \tilde{G}(t, t). \quad (7)$$

In this and following equations, the functions $\tilde{g}$ and $\tilde{f}$ are taken at the boundary $x = 0$. Expanding all functions over harmonics of the Josephson frequency, $A(E, t) = \sum_{n} A(E, n) e^{-i \nu_0 m t}$, we arrive at the equation for the dc current $I$,

$$I = \frac{1}{16 eR} \int_{-\infty}^{\infty} dE \text{Tr} \sum_m \left[ \tilde{h}(E, m) \tilde{G}^K(E, -m) - \tilde{h}(E, m) \right] \times \tilde{G}^K(E, -m), \quad \tilde{h}(E, m) = \sigma_3 \tilde{g}^R - \tilde{g}^A \sigma_2. \quad (8)$$

In the tunneling limit $W \ll 1$, the amplitude of the nth harmonic is proportional to $W^n$; thus the zero harmonic $m = 0$ of the functions $\tilde{g}$ and $\tilde{G}^K$ in Eq. (8) plays the key role, while the high-order harmonics can be neglected. The effect of these harmonics will be discussed later. Within this approximation, the Green’s function matrix structure in Eq. (3) holds, and the current Eq. (8) exactly transforms to the form

$$I = \frac{1}{eR} \int_{-\infty}^{\infty} dE N(E) N(E - eV) [n(E) - n(E)] \quad (9)$$

Here $N(E) = \text{Re} \, u^R$ is the density of states (DOS) normalized to its value in the normal state, and the distribution function $n = (1/2)(1 - f)$ approaches the Fermi function $n_f$ in equilibrium. Furthermore, we split the integral in Eq. (9) into pieces of length $eV$, denoting $A(E) = A(E + k eV)$,

$$I = \frac{1}{eR} \int_{0}^{eV} dE J(E), \quad J(E) = \sum_{k = \pm 1} j_k(E), \quad (10)$$

$$j_k = (n_{k-1} - n_k) \rho_k^{-1}, \quad \rho_k^{-1} = N_k N_{k-1}. \quad (11)$$

The distribution function $n(E, x = 0)$ satisfies the recurrence relation following from the kinetic equation $\partial_t (D \partial_t n) = 0$, $\Theta(\delta_x E) [\delta_x n(E)] = 0$, $\rho_k^{-1} = N_k N_{k-1}$, we arrive at Eq. (12).

A convenient interpretation of Eqs. (11) and (12) in terms of circuit theory is given by an infinite network in the energy space with the period $eV$, graphically presented in Fig. 2. The electric current spectral density $J(E)$ consists of partial currents $j_k$, which flow through the tunnel “resistors” $\rho_k$ connected to adjacent nodes of the network having “potentials” $n_k$ and $n_{k-1}$. At $E > \Delta$, the nodes are also attached to the distributed “equilibrium source” $n_f(E)$ through equal resistors $r$. Below we impose the equilibrium quasiparticle distribution at $E > \Delta$, $n(E) = n_f(E)$, neglecting the effect of small resistors $r$.

The currents flowing between the nodes outside the gap are related to the thermal current; at $T = 0$, these nodes have equal populations $n_k = 1$ at $E_k < -\Delta$, $n_k = 0$ at $E_k > \Delta$; thus the corresponding partial currents are zero, and the thermal current vanishes. As a result, only the current $j_0$ flowing across the gap through the resistor $\rho_0$ survives at $T = 0$.

Taking the DOS in the BCS form $N = N_f \text{Re} (E / \xi^2)$, we see that if any node falls into the gap, the adjacent resistances turn to infinity, and the current vanishes. For this reason, the network period must exceed the gap, $eV > \Delta$, and the integration in Eq. (10) is confined to the region $\Delta < E < eV - \Delta$.
This recovers the tunneling model result for the single-particle tunnel current: 

\[ I_{\text{exc}} = \frac{W}{2e} \] 

at \( eV < \Delta \), having the threshold value \( I_{\text{exc}}(\Delta) = \pi W^2/4eR \). 

To evaluate the subgap current, the DOS must be calculated to next order in the parameter \( W \), which requires solution of the equations for \( \delta \) following from Eqs. (2) and (6). Using the standard parametrization \( \delta = \sigma e^{\sigma \theta} \), we obtain the equation and the boundary condition for the spectral angle \( \theta \), 

\[ \sinh(\theta - \theta_0) = i \sigma \sinh \theta_0, \quad z = x/\xi_0, \quad (13) \]

\[ \delta, \theta \rightarrow W \sinh \theta \cosh \theta_0 \quad (z = -0). \quad (14) \]

With exponential accuracy, the solution of Eq. (13) for \( z < 0 \) can be approximated by the formula for a semi-infinite wire, 

\[ \tanh[\theta(\xi_0/4)] = \tanh[\theta(0) - \theta_0]/4 \exp(kz), \quad (15) \]

where \( k^{-1}(E) = \sqrt{\sinh \theta_0} \). Equation (15) describes the decay of perturbations of the spectral functions at distances \( \gg \xi_0 \) from the barrier, where the spectral angle approaches its bulk value \( \theta_0 = \arctan(\Delta/E) \). 

The boundary value of \( \theta \) is to be found from the equation following from Eqs. (14) and (15), 

\[ 2k \sin(\theta + \theta_0/2) = W \sinh \theta_0 \cosh \theta + \cosh \theta_0. \quad (16) \]

A direct expansion of \( \theta \) with respect to \( W \) in Eq. (16) leads to the following expression for the DOS within the BCS gap, 

\[ N(E) = W(1 - E^2/\Delta^2)^{-5/4} [N_0(E) + N_0(E - eV)] \quad (17) \]

The DOS divergences at \( |E| \gg \Delta, \Delta - eV \) in Eq. (17) are potentially dangerous (cf. Refs. 4), but they can be eliminated by improving the perturbation procedure by solving a set of recurrences in Eq. (16) in the vicinity of these points.

As follows from Eq. (17), the tunneling processes transfer the DOS in the energy space into the BCS gap at the distances \( \pm eV \) from the regions \( |E| > \Delta \), thus forming an effective spatial potential well of the width \( \approx \xi_0 \) at the tunnel barrier. At \( eV > \Delta \) the BCS gap is entirely filled with the quasiparticle states with a small local DOS \( \sim W \), as shown in Fig. 3(a). The appearance of localized states enables the quasiparticles to overcome the BCS gap at \( eV < \Delta \) via two steps involving intermediate Andreev reflection at energies \( |E| < \Delta \). The population of the intermediate state cannot be taken to be in equilibrium because the subgap quasiparticles cannot access the equilibrium electrodes. In the circuit terms, the node \( k=0 \) is disconnected from the equilibrium source, and the subgap current flows through the two large resistances \( \rho_0, \rho_1 \sim W^{-1} \) (two-particle current) [see Fig. 3(a)]. The corresponding partial currents are equal, \( j_0 = j_1 = [(\rho_0/E_1) - n_{F}(E_{1-1})]/(\rho_0 + \rho_1) \), and their contribution to \( I(V) \) is confined to the energy region \( 0 < eV < \Delta \) (a similar contribution at \( \Delta < eV \) comes from \( j_0 \) and \( j_1 \)). Thus the two-particle current appears above the threshold \( eV = \Delta \), having the threshold value \( I_{12}(2\Delta) = \pi W^2/4eR = 2W_I(2\Delta) \). At \( eV = \Delta \), the two-particle current exhibits a sharp peak with the height \( I_{12}(2\Delta) = 2.3 W^{25}/\Delta/eR \); at larger voltages, it approaches a constant value giving rise to the excess current \( I_{\text{exc}} = 6.2 W^{25}/\Delta/eR \).

At \( eV < \Delta \), a minigap opens in the DOS around the zero energy [see Fig. 3(b)]; however, since the number of subgap resistors increases up to three (three-particle current), the current across the minigap will persist as long as the network period exceeds the minigap size, \( eV > 2(\Delta - eV)) \), i.e., at \( eV > 2\Delta/3 \). The central resistance \( \rho_0 \) is large, \( \rho_0 \sim W^{-2} \), and dominates the net subgap resistance. This leads to a smaller charge current with the threshold value \( I_{12}(2\Delta/3) \approx 2W_I(2\Delta) \). At \( eV < \Delta/3 \) the network period becomes smaller than the minigap, and further correction to the DOS is required.

Similar results were found for the planar junction Fig. 1(b), using the equation for the functions \( \tilde{G}_{\alpha,0} \) at the top (+0) and bottom (−0) sides of the tunnel barrier \( \tilde{I}(\alpha, E + i\sigma, \tilde{G}_{\alpha,0}) = 2\Delta W(\tilde{G}_{\alpha,0}) \tilde{G}_{\alpha,0} \), with the modified tunneling parameter \( W = (3\xi_0^2/4\ell)dD \). This equation is derived by averaging Eq. (2) over the thickness of overlapping leads and using Eq. (6) (cf. Ref. 20). From this equation we obtain a relation for the spectral angle that does not significantly differ from Eq. (16),

\[ k^2 \sinh(\theta_0 - \theta) = W \sinh \theta_0 \cosh \theta_0 + \cosh \theta_0, \quad (18) \]

taking into account the modes which are close to those for the 1D model with the same magnitude of the parameter \( W \).

The presented calculation scheme, combining circuit theory arguments with DOS iteration procedures, suggests an appealingly simple explanation for the diffusive SGS: the decreasing applied voltage results in a shrinking period of the network in Fig. 2; hence a stepwise increase of the number of subgap resistors involved; simultaneously, the number of DOS steps, scaled as \( W^2 \), increases, as shown in Fig. 4(a). This results in the current staircase with the height of the steps given by \( I_{n} \sim (2W)^{-1}I_1, \) at \( 2\Delta/n < eV < 2\Delta/(n-1) \). The quantitative result for the current at arbitrary voltages and temperatures is

\[ I(V) = \int_0^e e(V + eR) \rho_{\Delta} (n_e - n_{e-1}) \int_{eR} e(V + eR) (n_{e-1} - n_{e-2}) \]

\[ \rho_{\Delta} \int_{e(V - eR)} e(V + eR) (n_{e-1} - n_{e-2}) \]

\[ n_{e}(E) = n_{e}(E_{\text{ex}}) \quad (19) \]

In this equation, the second term represents the thermal current, the integers \( \pm n_{e} \) are the indices of the nodes closest to the gap edges outside the gap, \( \int_{x}(x) \) denotes integer part of \( x \), and the quantity \( \rho_{\Delta}(E) = 2\sum_{k=1}^{N_{\Delta}} \rho_{k} \) has the meaning of net subgap resistance. The subgap distribution function reads.
and shown in Fig. 4(b) by a dashed line confirms these considerations.

We conclude our analysis with the estimation of the contribution of higher harmonics of the functions $\hat{g}$ and $\hat{G}^{\theta}$ to the dc current. At $T=0$, the contribution $\delta I$ of the first harmonics $|m|=1$ (the higher harmonics $|m|>1$ are smaller, $\sim W_{m}$) is

$$\delta I = \frac{2W}{eR} \int_0^E dE \Im \left( \frac{1}{p} \cosh^2 \frac{\chi}{2} + \frac{v}{q} \sinh^2 \frac{\chi}{2} \right),$$

where $\chi = \theta + \theta_1$, $v = \theta_1 + \theta_2$, $\theta_1 = (\xi_1 + \xi_{1A})/2$, $\delta_1 = \xi_{1A}$, for this reason, $\delta I$ turns to zero at $E < \Delta$, similar to $I_2$. Numerical calculations show that the contribution of the first harmonics to the IVC does not exceed 30%. From this we conclude that the adopted quasistatic approach gives a rather good approximation to a complete solution.

In our treatment, we have neglected inelastic scattering, which might affect the quasiparticle distribution at subgap energies. Analysis shows that this effect becomes essential under the condition $W_{\tau_r} \Delta < 1$, where $\tau_r$ is the relaxation time. However, this does not affect the estimate of the effective scaling factor and only changes the details of the IVC shape.

In conclusion, we have developed a theory of subgap charge transport and subharmonic gap structure in superconducting tunnel junctions with planar geometry and diffusive thin-film electrodes. We found that the role of scaling factor in the subharmonic gap structure is played by the effective tunneling transparency $D_{eff} = (3\xi_0 / \ell d)D$, which may greatly exceed the bare transparency $D$ of the junction tunnel barrier.

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14 Additional factor $\xi / d$ in Eq. (1) reflects the “anti-point-contact” geometry of the junctions with overlapping leads and the junction cross section exceeding the one of the leads (see Ref. 20).