Optimality of the Binary Reflected Gray Code for the Gaussian Channel

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Abstract

The problem of finding the optimal labeling of multilevel coherent PSK and PAM constellations with respect to minimizing the bit error probability (BEP) over a Gaussian channel is addressed. We show that using the binary reflected Gray code (BRGC) to label the signal constellation results in the lowest possible BEP for medium and high signal energy-to-noise ratios. As a consequence, this shows that the BRGC is asymptotically optimal (w.r.t. minimizing the BEP) for the Gaussian channel.

We present simple, closed-form expressions for the BEP of PSK, PAM, and QAM using the BRGC, where the expressions for PAM and QAM are new.

1 Introduction

This report addresses the problem of selecting an optimal labeling with respect to minimizing the bit error probability (BEP) for communication systems using PSK or PAM with coherent symbol detection over an additive white Gaussian noise (AWGN) channel [2]. That labeling signal constellations using Gray codes (in particular, the binary reflected Gray code (BRGC) [3, 7]) is a way to reduce the BEP of the systems considered in this report is established engineering knowledge. On a theoretical level, however, the question whether the BRGC is the best way to label the constellations or not seems to be an open question even in the asymptotic case of infinite signal energy-to-noise ratio (SNR), even though some references indicate that this is the case [10]. That the use of the BRGC (or even Gray codes) is not optimal for all SNRs for at least some M can be demonstrated by explicit evaluation of the BEP for various labelings and modulation forms (see [3, Fig. 8] for a 64-QAM example). The purpose of this report is to establish the optimality of the BRGC for the asymptotic case and also to quantify regions of SNR for which the BRGC is indeed the optimal labeling in the sense of minimizing the BEP.

The BEP of the systems considered herein was shown in [3] to be a function of two quantities; the average distance spectrum (ADS) derived from the constellation labeling and the communication channel. In [3], we established the somewhat artificial result that the BRGC [7] is the optimal labeling for PSK and PAM with respect to certain properties of the ADS. In this report the optimality criterion is extended to the more relevant requirement that the optimal labeling shall minimize the BEP of the communication system.

We address the problem assuming an AWGN channel and show that the minimum achievable BEP is obtained by using the BRGC as long as the signal energy-to-noise ratio is higher than a threshold that depends on the modulation scheme and the size of the constellation.

The report is organized as follows. In Section 2 the preliminaries are presented and a proof outline is given. In Section 3 we derive a particularly convenient partitioning of the set of all possible labelings (competing with the BRGC on being the optimal labeling). This partitioning is used in Section 4 to prove the optimality of the BRGC for PSK, while the corresponding proof for PAM is given in Section 5. An explicit, closed-form expression for the ADS of the

BRGC for PAM is derived in Section 6 resulting in a new exact BEP expression for PAM. Finally, conclusions and comments are given in Section 7.

2 Preliminaries

This section provides the BEP expressions for PSK and PAM used throughout the report. Important definitions and the notation used in the report is introduced. The proof method is outlined and two important analytical results, central to the proofs, are given.

2.1 Bit error probability for PSK

The average BEP of M-PSK, where $M = 2^m$ for any integer $m \geq 1$, over AWGN channels having a two-sided power spectral noise density of $N_0/2$, can be written [1]

$$P_b(\lambda, \gamma) = \frac{1}{m} \sum_{k=1}^{M-1} \bar{d}(k, \lambda) P(k, \gamma) \tag{1}$$

where $\bar{d}(k, \lambda)$ is the ADS of an M-PSK constellation labeling λ, defined for all integers k as

$$\bar{d}(k, \lambda) \triangleq \frac{1}{M} \sum_{l=0}^{M-1} d_H\left(\mathbf{c}_l, \mathbf{c}_{(l+k) \bmod M}\right) \tag{2}$$

where $\mathbf{c}_k$ is the m-bit binary label assigned to the kth signal alternative and the Hamming distance $d_H(\mathbf{c}_j, \mathbf{c}_k)$ is the number of positions in which $\mathbf{c}_j$ and $\mathbf{c}_k$ differ. The ADS denotes the average number of bits that differ between binary labels assigned to signal alternatives separated by k steps in the PSK constellation. If it is clear from the context which labeling λ is concerned, we will simply write $\bar{d}(k)$ for the ADS. The *crossover probability* $P(k, \gamma)$ is the probability that the received signal vector is found in a decision region belonging to a signal point k steps away (clockwise along the PSK circle) from the transmitted signal point. To find an expression for $P(k, \gamma)$ we refer to Figure 1 and consider a rotationally invariant, two-dimensional Gaussian probability density function (pdf) with noise power density $N_0/2$ centered on the point O. In the two-dimensional setting considered herein the noncentral t-distribution gives the probability $\Gamma(a, \gamma)$, which denotes the portion of the Gaussian pdf that is in the region bounded by angles $\pm a$ not containing O for a given symbol energy-to-noise ratio $\gamma \triangleq E_s/N_0$. For $k = 1, \ldots, M/2 - 1$, the probability $P(k, \gamma)$ is related to $\Gamma(a, \gamma)$ through the

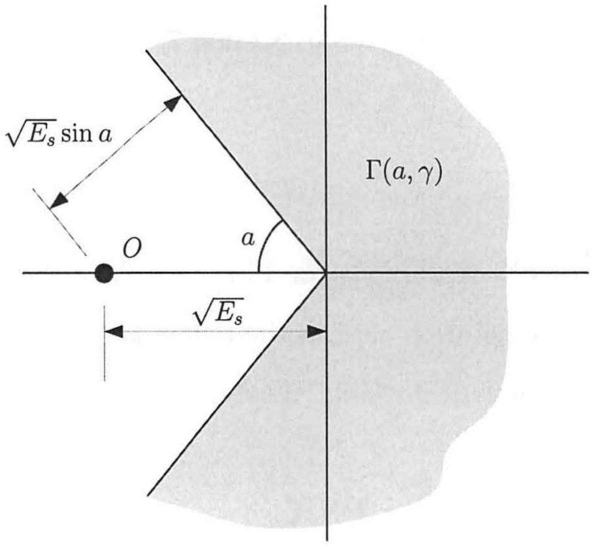

Figure 1: The shaded area represents the probability $\Gamma(a, \gamma)$ that would result from integration of a Gaussian pdf with power spectral density $N_0/2$, centered on O, for a given signal energy-to-noise ratio, $\gamma = E_s/N_0$, over this region.

relation

$$
\begin{aligned}
P(k,\gamma) &= \frac{1}{2}\left[\Gamma\left(\frac{(2k-1)\pi}{M},\gamma\right) - \Gamma\left(\frac{(2k+1)\pi}{M},\gamma\right)\right] \\
&\triangleq \frac{1}{2}\left[\Gamma\left(a_k,\gamma\right) - \Gamma\left(b_k,\gamma\right)\right]
\end{aligned}
\tag{3}
$$

while for $k = 0$,

$$P(0,\gamma) = 1 - \Gamma\left(\frac{\pi}{M},\gamma\right) \tag{4}$$

and for $k = M/2$, we have

$$P(M/2,\gamma) = \Gamma\left(\pi - \frac{\pi}{M},\gamma\right). \tag{5}$$

There exist several expressions for the probability $\Gamma(a, \gamma)$ in the literature, for example [8, p. 198],

$$\Gamma(a,\gamma) = \frac{1}{\pi}\int_0^{\pi-a} e^{-\gamma\frac{\sin^2 a}{\sin^2\varphi}}\,d\varphi. \tag{6}$$

The probability $\Gamma(a, \gamma)$ is closely related to the noncentral t-distribution [11, 12], see Section 6.1.

2.2 Bit error probability for PAM

The BEP expression for M-PAM can be written in a form similar to (1) and is again a function of the labeling λ used to label the constellation and the signal energy-to-noise ratio γ [3]

$$P_{\mathrm{b}}(\lambda, \gamma) = \frac{2}{m} \sum_{k=1}^{\infty} \bar{h}(k, \lambda) \mathcal{P}(k, \gamma) \tag{7}$$

where $\mathcal{P}(k, \gamma)$ is expressed in terms of the Gaussian Q-function

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt, \qquad x \in \mathbb{R} \tag{8}$$

as

$$\mathcal{P}(k, \gamma) = Q\left((2k-1)\sqrt{2\gamma_{\mathrm{eff}}}\right) - Q\left((2k+1)\sqrt{2\gamma_{\mathrm{eff}}}\right) \tag{9}$$

where

$$\gamma_{\mathrm{eff}} \triangleq \frac{3\gamma}{M^2 - 1} \tag{10}$$

The ADS $\bar{h}(k, u)$ of any sequence $u = (\mathbf{c}_0, \ldots, \mathbf{c}_{M-1})$ of M binary vectors is defined for all integers k as

$$\bar{h}(k, u) \triangleq \frac{1}{2M} \sum_{l=0}^{M-1} \left(d_H(\mathbf{c}'_l, \mathbf{c}'_{l+k}) + d_H(\mathbf{c}'_l, \mathbf{c}'_{l-k})\right), \tag{11}$$

with

$$\begin{aligned} \mathbf{c}'_i &= \mathbf{c}_0, && i < 0 \\ \mathbf{c}'_i &= \mathbf{c}_i, && 0 \le i \le M-1 \\ \mathbf{c}'_i &= \mathbf{c}_{M-1}, && i > M-1 \end{aligned} \tag{12}$$

It follows straightforwardly from this definition that for any such sequence, $\bar{h}(0, u) = 0$. More importantly, for the special case when $u = \lambda$ is a labeling, we note that for $k \ge M-1$, (11) counts the average number of ones per label taken over the entire labeling. For any labeling this average is $m/2$, so that $\bar{h}(k, \lambda) = m/2$ for $k \ge M-1$. As for the PSK case, we will write $\bar{h}(k)$ for the ADS if it is obvious from the context what sequence u is concerned.

2.3 Definitions and notation

The presented work deals with binary labelings and in this subsection we introduce the nomenclature and definitions that is used in the discussion.

A *binary labeling* λ of order $m \in \mathbb{Z}^+$ is defined as a sequence of $M = 2^m$ distinct vectors (codewords or labels), $\lambda = (\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{M-1})$, where each $\mathbf{c}_i \in \{0,1\}^m$.

A binary labeling λ' of order m is said to be *optimal* for signal energy-to-noise ratio γ if

$$P_{\mathrm{b}}(\lambda', \gamma) \leq P_{\mathrm{b}}(\lambda, \gamma)$$

for all labelings λ of order m. Here $P_{\mathrm{b}}(\lambda, \gamma)$ is the relevant error probability measure and in this work we use (1) or (7).

Throughout the report, we discuss a particular class of labelings; Gray codes. A *binary Gray code* of order m is a binary labeling with $M = 2^m$ distinct codewords, where adjacent codewords differ in only one of the m positions. If we impose the additional requirement that the first and the last codeword in the labeling also differ in a single position, the labeling is said to be a *cyclic binary Gray code.*

Among the cyclic Gray codes we are particularly interested in the *binary reflected Gray code* (BRGC) [3, 7] and we denote the BRGC of order m by β_m. For reference, we have listed β_m for $m = 1, \ldots, 4$ in Table 1.

Further, letting Λ_m denote the set of all labelings having an ADS that differs from the ADS of β_m, we define the *critical index* $T(\lambda)$ of a labeling $\lambda \in \Lambda_m$ as

$$T(\lambda) \triangleq \min\{k \in \mathbb{Z}^+ : \bar{d}(k, \lambda) \neq \bar{d}(k, \beta_m)\} \tag{13}$$

and similarly for $\bar{h}(k, \lambda)$. From [3, Th. 5] we know that

$$\bar{d}(k, \lambda) > \bar{d}(k, \beta_m) \tag{14}$$

at $k = T(\lambda)$.

The set of all critical indices, the *critical index set,* will be frequently used,

$$\Psi_m = \{T(\lambda) : \lambda \in \Lambda_m\}. \tag{15}$$

In addition, it will be convenient to have a designator for the set of labelings for which $T(\lambda) = i$,

$$\Lambda_m(i) = \{\lambda \in \Lambda_m : T(\lambda) = i\}. \tag{16}$$

Table 1: The binary reflected Gray codes of orders $m = 1, 2, 3$, and 4.

β_1	β_2	β_3	β_4
0	00	000	0000
1	01	001	0001
	11	011	0011
	10	010	0010
		110	0110
		111	0111
		101	0101
		100	0100
			1100
			1101
			1111
			1110
			1010
			1011
			1001
			1000

Although obvious from (15) and (16), we explicitly state that

$$\Lambda_m = \bigcup_{i \in \Psi_m} \Lambda_m(i) \tag{17}$$

and

$$\Lambda_m(i) \cap \Lambda_m(j) = \varnothing \qquad \text{for } i \neq j, \tag{18}$$

since these relations are central to the proof method as described in the next subsection.

2.4 Outline of proof method

Before proceeding to the details, we give an outline of the proof method that will be used. Using the definitions and notation introduced in the previous subsection, the aim of this report is for each m and each modulation form to establish a range of γ for which the following inequality holds:

$$P_\mathrm{b}(\beta_m, \gamma) \leq \min_{\lambda \in \Lambda_m} P_\mathrm{b}(\lambda, \gamma), \tag{19}$$

i.e., for what signal energy-to-noise ratios the labeling β_m will result in the lowest BEP among all possible labelings. We will address (19) by using the equivalent formulation, obvious from (17),

$$P_\mathrm{b}(\beta_m, \gamma) \leq \min_{i \in \Psi_m} \min_{\lambda \in \Lambda_m(i)} P_\mathrm{b}(\lambda, \gamma), \tag{20}$$

meaning that we first find the labeling resulting in lowest BEP among labelings having critical index i for each $i \in \Psi_m$ and then we find the labeling among these that gives the lowest BEP over all i.

Direct evaluation of (20) is cumbersome, so we will make use of an imaginary labeling β'_m which provides an (unattainable) upper bound on the ADS of β_m for upperbounding of the left side of (20). For each $i \in \Psi_m$, we will lowerbound the right-hand side expression using an imaginary labeling $\lambda'_m(i)$ (different for each i) which provides an (unattainable) lower bound on the ADS of any $\lambda \in \Lambda_m(i)$. We then address the inequality

$$P_{\mathrm{b}}(\beta'_m, \gamma) \leq \min_{i \in \Psi_m} P_{\mathrm{b}}(\lambda'_m(i), \gamma), \tag{21}$$

and if we can establish a range of γ for which (21) is satisfied, we know that β_m will provide the lowest BEP among all possible labelings of order m for this range, since (20) will also be satisfied by these γ.

2.5 Lemmas used in the proofs

In order to find the range of γ for which (21) is valid, we make use of two results from calculus, which are derived in this subsection.

Lemma 1 (The strict increase of a convex function) *For three constants x_0, a, and b, such that $0 \leq a < b$, consider the difference*

$$\Delta(x) = f(bx) - f(ax). \tag{22}$$

If

- *$f(x)$ is a continuous function for $x \geq x_0$,*
- *$f'(x) > 0$ for $x > x_0$, and*
- *$f''(x) > 0$ for $x > x_0$,*

then $\Delta(x)$ is a strictly increasing function in x for $x \geq x_0$.

Proof: Since $f''(x) > 0$ for $x > x_0$, we have for $0 \leq a < b$,

$$\int_{ax}^{bx} f''(t)dt > 0, \qquad x > x_0 \tag{23}$$

Rewriting the left side of (23) in terms of the derivative of $f(x)$, we get

$$\int_{ax}^{bx} f''(t)dt = f'(bx) - f'(ax) > 0 \tag{24}$$

for $x > x_0$. Since $f'(x) > 0$ for $x > x_0$, we have

$$f'(bx) - f'(ax) < f'(bx) - \frac{a}{b} f'(ax) = \frac{\Delta'(x)}{b} \tag{25}$$

for $0 \leq a < b$, showing that $\Delta'(x) > 0$, which completes the proof. $\square$

Lemma 2 (The ratio between Q-functions) *For two constants a and b, such that $0 \leq a < b$, the ratio*

$$r(x) = \frac{Q(ax)}{Q(bx)} \tag{26}$$

is a strictly increasing function of x for $x > 0$.

Proof: Let $f(x) = -\log Q(x)$, which is a continuous, twice differentiable, function for all x with first derivative

$$f'(x) = -\frac{Q'(x)}{Q(x)}. \tag{27}$$

For the second derivative we have

$$f''(x) = \frac{Q'(x)^2 - Q''(x)Q(x)}{Q(x)^2} \tag{28}$$

and since

$$Q'(x) = -\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \quad \text{and} \quad Q''(x) = \frac{xe^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \tag{29}$$

we have $f'(x) \geq 0$ for all x and

$$f''(x) = \frac{xe^{-\frac{1}{2}x^2}}{\sqrt{2\pi}Q(x)^2}\left[\frac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}} - Q(x)\right] \tag{30}$$

Now, as $e^{-\frac{1}{2}x^2}/x\sqrt{2\pi}$ is a well known upper bound on $Q(x)$ for $x > 0$ [4, p. 98], we conclude that for $x > 0$, we have $f''(x) > 0$. Applying Lemma 1 to $f(x)$ with $x_0 = 0$, we find that

$$\Delta(x) = -\log Q(bx) + \log Q(ax) = \log \frac{Q(ax)}{Q(bx)} \tag{31}$$

is a strictly increasing function for $x > 0$ and $0 \leq a < b$, which also implies that $r(x) = e^{\Delta(x)}$ is a strictly increasing function of x for $x > 0$. $\square$

Table 2: The number of binary Gray codes and binary cyclic Gray codes that do not have identical ADS as a function of the order m. Note that this table, which was obtained by computer search, does not count the same entities as [3, Tab. I], although the numerical values agree for $m \leq 4$.

m	cyclic Gray	Gray
1	1	1
2	1	1
3	1	3
4	9	131

3 Finding the critical index set

In order to address (21), we need to find the critical set Ψ_m. We will rely on a method called labeling *expansion*, which is a way to construct a labeling λ_m of order m from a labeling λ_{m-1} of order $m-1$. For a labeling λ_{m-1} that is expanded into λ_m the following relations hold for $m \geq 2$ and $k \in \mathbb{Z}$,

$$\bar{d}(4k, \lambda_m) = \bar{d}(2k, \lambda_{m-1}) + f_1 \tag{32}$$

$$\bar{d}(4k+2, \lambda_m) = \bar{d}(2k+1, \lambda_{m-1}) + f_2 \tag{33}$$

$$\bar{d}(2k+1, \lambda_m) = \frac{1}{2}\bar{d}(k, \lambda_{m-1}) + \frac{1}{2}\bar{d}(k+1, \lambda_{m-1}) + f_3 \tag{34}$$

where f_1, f_2 and f_3 are functions of k and m, but independent of λ_{m-1}. The same relations, with different f_1, f_2 and f_3, hold for $\bar{h}(k, \lambda)$ [3]. An important property of labeling expansion is that expanding β_{m-1} gives β_m.

The critical index set depends on the modulation form and the order m, but the method used to find the critical index set for PSK and PAM is the same. We derive the critical index set for PSK in detail and only point out the essential differences in the derivation of the critical index set for PAM.

3.1 The critical index set for PSK

To find the (PSK) critical index set Ψ_m, we start with $\Lambda_m(1)$, i.e., the set of labelings *not* having the cyclic Gray property. It is clear that for all $m \geq 2$ (excluding the trivial case $m = 1$, for which $\Psi_1 = \varnothing$) there is at least one labeling in this set, so definitely $1 \in \Psi_m$.

What about $\Lambda_m(2)$? Labelings in this set are necessarily Gray codes. However, it is easily shown that *all* cyclic Gray codes have $\bar{d}(2) = 2$. Therefore, all cyclic Gray codes have identical ADSs for $k = 1$ and 2 and, hence, $\Lambda_m(2) = \varnothing$.

For $\Lambda_m(3)$ we turn to column 2 of Table 2, where the number of cyclic Gray codes that do not have identical ADS is listed. Since there is only one cyclic Gray code for $m \leq 3$, $\Lambda_m(3) = \varnothing$, for $m \leq 3$. In the Appendix, we assure that $3 \in \Psi_m$ for $m \geq 4$ by giving an explicit construction method, valid for all $m \geq 4$, of a cyclic Gray code that belongs to $\Lambda_m(3)$.

Now, for $T(\lambda) \geq 4$, we are dealing with the class of cyclic Gray codes for which $\bar{d}(k)$ is identical to the ADS of β_m for $k = 1, 2$, and 3. From Lemma 3 in [3], we know that all such labelings of order m can be constructed by expansion of a Gray code of order $m - 1$. This observation results in two important conclusions. First, $\Psi_4 = \{1, 3\}$, since only β_3 exists to start the expansion from, implying that no other labeling has $\bar{d}(k)$ identical to the ADS of β_4 for $k = 1, 2$, and 3. Second, for $m \geq 5$, the ADS of all labelings with $T(\lambda) \geq 4$ can be calculated using (32)–(34). From the recursions we find that the critical index $T(\lambda_{m-1})$ of a labeling of order $m - 1$ will propagate to the expanded labeling λ_m and result in a critical index

$$T(\lambda_m) = 2T(\lambda_{m-1}) - 1. \tag{35}$$

For example, consider a labeling $\lambda_4 \in \Lambda_4(3)$ with $\bar{d}(3, \lambda_4) \neq \bar{d}(3, \beta_4)$. If λ_4 is expanded to λ_5, we see that this difference influences $\bar{d}(k, \lambda_5)$ first at $k = 5$, while for $0 \leq k \leq 4$ we have $\bar{d}(k, \lambda_5) = \bar{d}(k, \beta_5)$.

From the results above we conclude that for $m \geq 5$, Ψ_m is obtained from Ψ_{m-1} by simply adding another element to the set;

$$\Psi_m = \Psi_{m-1} \cup \{2(\max \Psi_{m-1}) - 1\} \tag{36}$$

The PSK critical index set Ψ_m is listed in the second column of Table 3 for $m = 1, \ldots, 8$.

3.2 The critical index set for PAM

For PAM the critical index set is derived in a similar way; the difference is that the ADS is defined by (11) and we exclude the cyclic requirement on the Gray codes.

The change of definition for the ADS results in different values for $\bar{h}(k)$ for $k = 1, 2$ compared

Table 3: The critical index set Ψ_m for PSK and PAM as a function of m.

m	Ψ_m (PSK)	Ψ_m (PAM)
1	$\varnothing$	$\varnothing$
2	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1,3\}$
4	$\{1,3\}$	$\{1,3,5\}$
5	$\{1,3,5\}$	$\{1,3,5,9\}$
6	$\{1,3,5,9\}$	$\{1,3,5,9,17\}$
7	$\{1,3,5,9,17\}$	$\{1,3,5,9,17,33\}$
8	$\{1,3,5,9,17,33\}$	$\{1,3,5,9,17,33,65\}$

to the PSK case, but the conclusion that $\bar{h}(k)$ is identical for all Gray codes of order $m \geq 3$ for $k = 1, 2$ is still valid [3, Lemma 2b].

From column 3 of Table 2, we see that in the class of not necessarily cyclic Gray codes, there are three classes of Gray codes of order $m = 3$ that do not have identical ADSs. This means that, potentially, $3 \in \Psi_m$ for $m \geq 3$ for PAM. As for the PSK case, we show that this is in fact the case, by giving an explicit construction method, valid for $m \geq 3$, of such a Gray code in the Appendix.

Apart from the differences mentioned above, the derivations are identical and the critical index sets are listed in column 3 of Table 3 for $m = 1, \ldots, 8$.

4 Finding the Optimal M-PSK Labelings

At this point we have established the foundation required to address the proof of the optimality of β_m. In this section we use the results in Section 3 to derive sufficient conditions on the signal energy-to-noise ratio for which β_m is optimal for an M-PSK system.

4.1 The bounding ratio

The procedure we use is to compare β_m to all labelings in Λ_m. This is done by finding, for each $i \in \Psi_m$, a signal energy-to-noise ratio γ, such that

$$P_{\mathrm{b}}(\beta_m, \gamma) \leq \min_{\lambda \in \Lambda_m(i)} P_{\mathrm{b}}(\lambda, \gamma) \tag{37}$$

and showing that the highest of these values taken over all $i \in \Psi_m$ provides a γ above which β_m yields the lowest possible BEP over the Gaussian channel.

To find the threshold for given m and i, we use $|\Psi_m|$ (unattainable) lower bounds on the ADS of any $\lambda \in \Lambda_m(i)$

$$\check{d}(k) = \begin{cases} \bar{d}(k, \beta_m), & k = 0, \ldots, i-1 \\ \bar{d}(k, \beta_m) + 2/M, & k = i \\ 1, & k = i+1, \ldots, M/2 \\ \check{d}(M-k) & k = M/2+1, \ldots, M-1 \end{cases} \tag{38}$$

The value at $k = i$ is a lower bound on the difference between the ADS of a labeling $\lambda \in \Lambda_m(i)$ and β_m, which follows from (14) and the fact that the resolution of (2) is $2/M$. Note that the bounds given by (38) are chosen for their simplicity; it is possible to find and use tighter bounds, but the effect on the derived thresholds would be marginal.

For β_m we use $|\Psi_m|$ (similarly unattainable) upper bounds on the ADS

$$\hat{d}(k) = \begin{cases} \bar{d}(k, \beta_m), & k = 0, \ldots, i \\ m, & k = i+1, \ldots, M/2 \\ \hat{d}(M-k) & k = M/2+1, \ldots, M-1 \end{cases} \tag{39}$$

which again is a simple bound which can be improved, but the form given here is convenient for its simplicity.

Using the bounds in (38) and (39) and the the BEP expression in (1), we formulate the inequality

$$\frac{1}{m}\sum_{k=1}^{M-1} \hat{d}(k)P(k,\gamma) \leq \frac{1}{m}\sum_{k=1}^{M-1} \check{d}(k)P(k,\gamma) \tag{40}$$

and observe that if we can find a range of γ for which this inequality is valid, then (37) is also valid for this range. After some manipulation (40) simplifies to

$$\sum_{k=i+1}^{M-i-1} P(k,\gamma) \leq \frac{4P(i,\gamma)}{M(m-1)} \tag{41}$$

for $m \geq 2$ and $1 \leq i \leq M/2 - 1$. This inequality quantifies the relation between the crossover probabilities in terms of the critical index i, which is exactly what we desire. We rewrite (41) and define the *bounding ratio* for PSK as

$$R(i,\gamma) \triangleq \frac{\sum_{k=i+1}^{M-i-1} P(k,\gamma)}{2P(i,\gamma)} \leq \frac{2}{M(m-1)} \tag{42}$$

which will be used in the next subsection to find the γ thresholds.

4.2 BRGC optimality thresholds for M-PSK

We now proceed to derive sufficient conditions of optimality of β_m for M-PSK over the Gaussian channel. We will evaluate (42) for each $i \in \Psi_m$ and find a range of γ for which all these $|\Psi_m|$ inequalities are valid simultaneously.

To do this, we start by rewriting the bounding ratio (42) using (3)–(5) as

$$R(i,\gamma) = \frac{\Gamma(b_i,\gamma)}{\Gamma(a_i,\gamma) - \Gamma(b_i,\gamma)}, \tag{43}$$

valid for $m \geq 2$ and $1 \leq i \leq M/2 - 1$. In general, the bounding ratio is tedious to handle directly, so we derive a more tractable upper bound on $R(i,\gamma)$ using the Q-function defined in (8). Again referring to Figure 1, we have the following upper bound on $\Gamma(a,\gamma)$ for $0 \leq a \leq \pi/2$,

$$\Gamma(a,\gamma) \leq 2Q\left(\sqrt{2\gamma}\sin a\right)$$

and, for $0 \leq a \leq b \leq \pi/2$, the difference $\Gamma(a,\gamma) - \Gamma(b,\gamma)$ is lower bounded by

$$2Q\left(\sqrt{2\gamma}\sin a\right) - 2Q\left(\sqrt{2\gamma}\sin b\right) \leq \Gamma(a,\gamma) - \Gamma(b,\gamma)$$

which directly gives the following upper bound on $R(i,\gamma)$

$$R(i,\gamma) \leq \frac{Q\left(\sqrt{2\gamma}\sin b_i\right)}{Q\left(\sqrt{2\gamma}\sin a_i\right) - Q\left(\sqrt{2\gamma}\sin b_i\right)} \triangleq \hat{R}(i,\gamma) \tag{44}$$

for all $i \geq 1$ such that $0 \leq a_i \leq b_i \leq \pi/2$.

From the upper bound $\hat{R}(i,\gamma)$ and the right-hand side of (42) we formulate the inequality

$$\hat{R}(i,\gamma) \leq \frac{2}{M(m-1)} \tag{45}$$

and, again, it is clear that for all γ for which (45) is satisfied, (42) and (37) are also satisfied. By inverting (45) and inserting (44), we get

$$\frac{Q\left(\sqrt{2\gamma}\sin a_i\right)}{Q\left(\sqrt{2\gamma}\sin b_i\right)} \geq \frac{M(m-1)}{2} + 1 \tag{46}$$

From Lemma 2, we see that for $0 \leq a_i < b_i \leq \pi/2$, i.e., $0 \leq \sin a_i < \sin b_i$, the left side of (46) is strictly increasing with $\sqrt{2\gamma}$ (and therefore also with γ). In addition, invoking well-known bounds on the Q-function [4, p. 98], we have

$$\frac{Q(ax)}{Q(bx)} \geq \frac{b}{a}\left(1 - \frac{1}{a^2x^2}\right) e^{x^2(b^2-a^2)/2} \tag{47}$$

which, for $b > a > 0$, can be made arbitrarily large by increasing x. This means that there exists a *unique* $\gamma = \gamma_m(i)$ which solves the equation

$$\frac{Q\left(\sqrt{2\gamma}\sin a_i\right)}{Q\left(\sqrt{2\gamma}\sin b_i\right)} = \frac{M(m-1)}{2} + 1, \tag{48}$$

for all i such that $0 \leq a_i < b_i \leq \pi/2$.

For each m, we want to find $\gamma_m(i)$ for all $i \in \Psi_m$, and therefore we must establish the values of m for which (48) can be used without violating $a_i < b_i \leq \pi/2$. Since $a_i < b_i$, we focus on b_i. From the implicit definition in (3), we have

$$\max_{i \in \Psi_m} b_i = b_{\max \Psi_m} = \frac{(2\max \Psi_m + 1)\pi}{2^m} \tag{49}$$

and since $\max \Psi_m = 2^{m-3} + 1$ for $m \geq 4$, we have

$$\max_{i \in \Psi_m} b_i = \left(\frac{1}{4} + \frac{3}{2^m}\right)\pi < \frac{\pi}{2}, \qquad \text{for } m \geq 4 \tag{50}$$

For $m = 3$, we have $b_{\max \Psi_3} = 3\pi/8 < \pi/2$, while $b_{\max \Psi_2} = 3\pi/4 > \pi/2$. This means that for $m \geq 3$, we can safely find $\gamma_m(i)$ for $i \in \Psi_m$ using (48), while for $m = 2$ we must use (42) directly. It is straightforward to show, via direct evaluation of (42), that β_2 is optimal for 4-PSK (QPSK) for all γ. For $m \geq 3$, we showed above that $0 \leq a_i < b_i \leq \pi/2$, so that (48) and (47) establish the asymptotic optimality of β_m as γ approaches infinity.

In Figure 2, the function $\gamma_m(i)$ is shown for $i \in \Psi_m$ and $m = 3, 4, \ldots, 10$. The interpretation of $\gamma_m(i)$ is the following. Consider a labeling $\lambda \in \Lambda_m(i)$, i.e., a labeling for which $T(\lambda) = i$. For $\gamma > \gamma_m(i)$, β_m will result in a lower BEP according to (1) over a Gaussian channel, irrespective of the ADS values for $k > i$. If $\gamma < \gamma_m(i)$, there may exist labelings $\lambda \in \Lambda_m(i)$ such that the BEP is lower than for β_m even though $\bar{d}(i, \beta_m) < \bar{d}(i, \lambda)$. For example, we see from Figure 2 that for $m = 4$, any cyclic Gray code will give lower BEP than any non-Gray labeling for $\gamma \geq 9.5$ dB. Compared to the cyclic Gray codes in $\Lambda_4(3)$, β_4 gives a lower BEP than all these

Table 4: The M-PSK optimality threshold $\hat{\gamma}$ obtained from taking the maximum value of $\gamma_m(i)$, $i \in \Psi_m$, for $m = 2, \ldots, 10$. The values for $m = 2$ are obtained through direct evaluation of (42). The bit energy-to-noise ratio is defined as $\gamma_b \triangleq E_s/(mN_0)$.

m	$\hat{\gamma}$ [dB]	$\hat{\gamma}_b$ [dB]	$P_b(\hat{\gamma})$
2	$-\infty$	$-\infty$	0.5
3	3.8	-1.1	0.147
4	10.6	4.6	0.090
5	16.7	9.7	0.070
6	23.6	15.8	0.049
7	30.3	21.9	0.036
8	37.0	28.0	0.027
9	43.6	34.0	0.021
10	50.1	40.1	0.016

labelings for $\gamma \geq 10.6$ dB. For 9.5 dB $\leq \gamma \leq 10.6$ dB, the optimal labeling may be different from β_4, but it must be a cyclic Gray code.

We define the *optimality threshold* of order m as

$$\hat{\gamma}_m = \max_{i \in \Psi_m} \gamma_m(i) \tag{51}$$

and we have shown that we are guaranteed that for $\gamma \geq \hat{\gamma}_m$, β_m is the optimal labeling with respect to minimizing (1). The optimality threshold (as seen in Figure 2) is given in Table 4 for $m = 2, 3, \ldots, 10$, along with the corresponding $P_b(\hat{\gamma}_m)$. We note that for this range of m, it is only for $m = 4$ that $\hat{\gamma} \neq \gamma(1)$.

5 Finding the Optimal M-PAM Labeling

In this section we apply the methods used to prove optimality of β_m for M-PSK to systems using M-PAM.

5.1 The bounding ratio

The proof method for the PAM case is very similar to that of the PSK case; the main difference lies in the evaluation of the crossover probabilities. We again use the inequality (37), this time using (7) for the BEP expression, in order to find a signal energy-to-noise ratio threshold above which β_m gives the lowest possible BEP of all labelings.

We use the following lower bound on the ADS of PAM labelings $\lambda \in \Lambda_m(i)$,

$$\check{h}(k) = \begin{cases} \bar{h}(k, \beta_m), & k = 0, 1, \ldots, i-1 \\ \bar{h}(k, \beta_m) + 1/M, & k = i \\ 1/2, & k = i+1, \ldots, \infty \end{cases} \tag{52}$$

and as an upper bound on the ADS of β_m we use

$$\hat{h}(k) = \begin{cases} \bar{h}(k, \beta_m), & k = 0, 1, \ldots, i \\ m, & k = i+1, \ldots, \infty \end{cases} \tag{53}$$

It is possible to sharpen these bounds in many ways, e.g., by letting $\hat{h}(k) = \check{h}(k) = m/2$ for $k \geq M-1$, but as it can be shown to have marginal effect on the end result, we use the above bounds for simplicity.

Analogously to the PSK case, we use (52), (53) and (7) to formulate an inequality

$$\frac{2}{m} \sum_{k=1}^{\infty} \hat{h}(k) \mathcal{P}(k, \gamma) \leq \frac{2}{m} \sum_{k=1}^{\infty} \check{h}(k) \mathcal{P}(k, \gamma) \tag{54}$$

and for all γ for which (54) is valid, (37) is also satisfied.

After manipulation (54) simplifies to

$$\sum_{k=i+1}^{\infty} \mathcal{P}(k, \gamma) \leq \frac{\mathcal{P}(i, \gamma)}{M\left(m - \frac{1}{2}\right)} \tag{55}$$

for $m \geq 2$ and $0 \leq i \leq M-1$. This is the PAM equivalent of (41) for quantifying the relation between the crossover probabilities in terms of the bounding index i. We rewrite (55) and define

$$\mathcal{R}(i, \gamma) \triangleq \frac{\sum_{k=i+1}^{\infty} \mathcal{P}(k, \gamma)}{\mathcal{P}(i, \gamma)} \leq \frac{1}{M\left(m - \frac{1}{2}\right)}. \tag{56}$$

5.2 BRGC optimality thresholds for M-PAM

For the Gaussian channel, the probability $\mathcal{P}(k, \gamma)$ was given in (9) and using this in (56), we get

$$\frac{Q\left((2i-1)\sqrt{2\gamma_{\text{eff}}}\right)}{Q\left((2i+1)\sqrt{2\gamma_{\text{eff}}}\right)} \geq M\left(m - \frac{1}{2}\right) + 1 \tag{57}$$

for $i \geq 1$. The left side of (57) is strictly increasing in γ for a given i according to Lemma 2 and can be made arbitrarily large, as shown in (47). Therefore, for each m, there exists a unique

Table 5: The M-PAM optimality thresholds $\hat{\gamma}_m$ obtained from taking the maximum value of $\gamma_m(i), i \in \Psi_m$, for $m = 2, \ldots, 9$. The bit energy-to-noise ratio is defined as $\gamma_b \triangleq E_s/(mN_0)$.

m	$\hat{\gamma}$ [dB]	$\hat{\gamma}_b$ [dB]	$P_b(\hat{\gamma})$
2	-0.6	-3.7	0.214
3	8.0	3.2	0.112
4	15.5	9.5	0.072
5	22.6	15.6	0.051
6	29.4	21.6	0.038
7	36.1	27.6	0.029
8	42.7	33.7	0.023
9	49.2	39.7	0.019

$\gamma = \gamma_m(i)$ that satisfies (57) with equality, and the inequality is valid for all $\gamma \geq \gamma_m(i)$. As for the PSK case, this asserts asymptotic optimality of β_m as γ approaches infinity.

The solutions to (57) for $i \in \Psi_m$, are shown in Figure 2 for $m = 2, 3, \ldots, 9$. By taking the maximum of $\gamma_m(i)$, we obtain the optimality threshold $\hat{\gamma}_m$. The optimality thresholds are listed in Table 5 along with the resulting $P_b(\hat{\gamma}_m)$. The significance of the optimality thresholds is, as for the PSK case, that for all $\gamma \geq \hat{\gamma}_m$, β_m is the optimal labeling for systems using M-PAM.

6 Evaluating the BEP of PSK, PAM, and QAM systems

We will conclude the discussion by establishing convenient forms for evaluating the BEP of M-PAM and $(M_1 \times M_2)$-QAM systems. For this, we start by deriving a closed-form expression for the ADS of the BRGC according to (11). For completeness, we also include previously established results for the BEP of M-PSK.

6.1 BEP of M-PSK constellations using β_m

From (1), the BEP of systems using M-PSK is given by

$$P_b(\lambda, \gamma) = \frac{1}{m} \sum_{k=1}^{M-1} \bar{d}(k, \lambda) P(k, \gamma) \qquad (58)$$

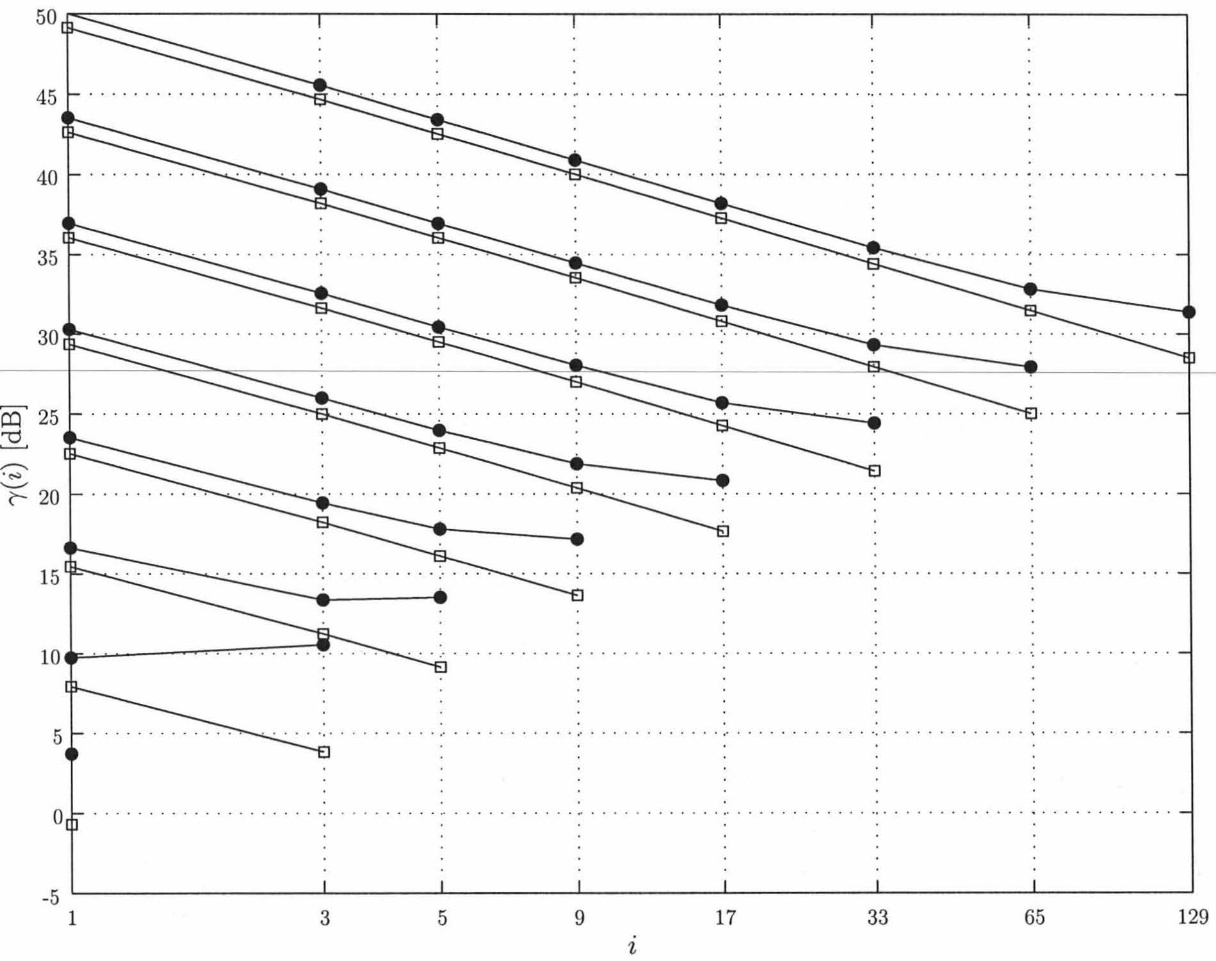

Figure 2: The function $\gamma_m(i)$ for M-PSK given for $m = 3, \ldots, 10$ and $i \in \Psi_m$ and for M-PAM for $m = 2, \ldots, 9$. The curves marked by bullets ($\bullet$) are the M-PSK threshold functions (the bottommost curve (a single point) is for $m = 3$ and the topmost curve is for $m = 10$). The M-PAM threshold functions are marked by squares ($\square$) and the curve for $m = 2$ is the bottommost curve (a single point), while the topmost curve is for $m = 9$.

and in [1] it was shown that if the M-PSK constellation is labeled by β_m, the ADS is given by

$$\bar{d}(k, \beta_m) = \operatorname{tri}(2^m, k) + \sum_{i=2}^{m} \operatorname{tri}(2^i, k) \tag{59}$$

for all integers k. The function $\operatorname{tri}(N, k)$ is a periodic triangular function of period N, defined by

$$\operatorname{tri}(N, k) \triangleq 2\left|\frac{k}{N} - \left\lfloor \frac{k}{N} \right\rceil\right| \tag{60}$$

where the function $\lfloor x \rceil$ rounds x to the closest integer (ties are rounded down).

The crossover probability $P(k, \gamma)$ can be obtained from (3)–(6) or from numerical implementations of the noncentral t-distribution. For example, using MATLAB, we can evaluate $\Gamma(a, x)$ using the command

```
nctcdf(cot(a), 1, sqrt(2x))
```

and obtain $P(k, \gamma)$ from (3). Similarly, using MATHEMATICA with its `ContinuousDistributions` package, the command

```
CDF[NoncentralStudentTDistribution[1, Sqrt[2x]], Cot[a]]
```

evaluates the same expression.

6.2 BEP of M-PAM constellations using β_m

In this section, we start by deriving a closed-form expression, similar to (59), for the ADS $\bar{h}(k, \beta_n)$ used for PAM. We also present a convenient form for evaluating the BEP of M-PAM systems using β_m for constellation labeling.

To find a closed-form expression for $\bar{h}(k)$ of β_m, we use a method similar to the one used in [1] to establish the corresponding form for the ADS of β_m for M-PSK. As we will see, the ADS for M-PAM inherits some of its structure from the M-PSK counterpart.

The expression we are interested in is (11) with $u = \beta_m$, repeated here for convenience,

$$\bar{h}(k, u) = \frac{1}{2M} \sum_{l=0}^{M-1} \left(d_H(\mathbf{c}'_l, \mathbf{c}'_{l+k}) + d_H(\mathbf{c}'_l, \mathbf{c}'_{l-k}) \right), \tag{61}$$

where $\mathbf{c}'_k$ is obtained from $u = \beta_m$ according to (12). As an illustration, the sequence $\mathbf{c}'_k$ used to calculate $\bar{h}(k, \beta_3)$ is given in Table 6.

Now, according to the definition of Hamming distance, we have

$$d_H(\mathbf{c}_l, \mathbf{c}_k) \triangleq \sum_{i=0}^{m-1} [\mathbf{c}_l]_i \oplus [\mathbf{c}_k]_i, \tag{62}$$

where $[\mathbf{c}_k]_i$ denotes the ith bit of the m-bit binary label $\mathbf{c}_k$ and the operator $\oplus$ is defined such that $x \oplus y$ equals the integer 0 if the binary digits x and y are equal and 1 otherwise. Inserting (62) in (61) and exchanging the order of summation, we obtain

$$\bar{h}(k, u) = \sum_{i=0}^{m-1} \bar{h}(k, [u]_i) \tag{63}$$

where $[u]_i$ denotes the sequence of the ith component of all labels in u. Referring to Table 6, $\bar{h}(k, [u]_i)$ is the ADS of column i (from the left) in the label sequence.

Table 6: The sequence of binary labels $\mathbf{c}'_k$ derived from β_3 used to find $\bar{h}(k)$ for 8-PAM.

k	$\mathbf{c}'_k$
$-\infty$	000
...	...
-1	000
0	000
1	001
2	011
3	010
4	110
5	111
6	101
7	100
8	100
...	...
∞	100

For β_m, we see that columns $i = 1, \ldots, m-1$ has a similar structure, while the first column ($i = 0$) must be treated separately. For columns 1 to $m-1$, we start by considering a binary pulse sequence $p(n)$, $n \geq 2$, containing 2^{n-1} ones;

$$p(n) = (\underbrace{0, 0, \ldots, 0}_{2^{n-2}\text{ zeros}}, \underbrace{1, 1, \ldots, 1}_{2^{n-1}\text{ ones}}, \underbrace{0, 0, \ldots, 0}_{2^{n-2}\text{ zeros}}) \tag{64}$$

The total number of elements in $p(n)$ is $N = 2^n$. Let $v(n, m)$ denote the binary vector obtained from $p(n)$ according to

$$v(n, m) = (\underbrace{p(n), \ldots, p(n)}_{2^{m-n}\text{ repetitions}}) \tag{65}$$

for $m \geq n$. If we list $\mathbf{c}'_k$ for β_m as in Table 6, we note that column i, for $i \geq 1$, is $v(m+1-i, m)$, so we must obtain $\bar{h}(k, v(n, m))$.

Now, if v was a periodic sequence v', i.e., consisting of an infinite number of repetitions of the sequence $p(n)$, we would have, for all integers k [1],

$$\bar{h}(k, v'(n, m)) = \text{tri}(N, k) \tag{66}$$

which was defined in (60). The removal of the periodicity introduces a correction term and it is possible to show that for the sequence $v(n, m)$,

$$\bar{h}(k, v(n, m)) = \text{tri}(N, k) - \left(\text{tri}(N, k) - \frac{1}{2}\right) \min\left\{\frac{N}{2M}\left\lceil\frac{2|k|}{N}\right\rceil, 1\right\}, \tag{67}$$

for all integers k. For $1 \leq i \leq m-1$, it is clear from the discussion above that

$$\bar{h}(k, [\beta_m]_i) = \bar{h}(k, v(m+1-i, m)). \tag{68}$$

For the first column ($i = 0$), we note that this is a binary step vector

$$s_m = (\underbrace{0, 0, \ldots, 0}_{2^{m-1} \text{ zeros}}, \underbrace{1, 1, \ldots, 1}_{2^{m-1} \text{ ones}}) \tag{69}$$

For this sequence we have, for all integers k,

$$\bar{h}(k, s_m) = \min\left\{\frac{|k|}{M}, \frac{1}{2}\right\} \tag{70}$$

and since $\bar{h}(k, s_m) = \bar{h}(k, [\beta_m]_0)$, we have established that for β_m the ADS is given by

$$\bar{h}(k, \beta_m) = \bar{h}(k, s_m) + \sum_{i=1}^{m-1} \bar{h}\left(k, v(m+1-i, m)\right) \tag{71}$$

Having established a closed form expression for $\bar{h}(k, \beta_m)$, we again turn our attention to (7). In Section 2.2 we mentioned that, for any labeling, $\bar{h}(k) = m/2$ for $k \geq M-1$, and hence we may rewrite (7) as

$$\begin{aligned} P_b(\beta_m, \gamma) &= \frac{2}{m}\sum_{k=1}^{M-2} \bar{h}(k, \beta_m)\mathcal{P}(k, \gamma) + \frac{2}{m}\sum_{k=M-1}^{\infty} \bar{h}(k, \beta_m)\mathcal{P}(k, \gamma) \\ &= \frac{2}{m}\sum_{k=1}^{M-2} \bar{h}(k, \beta_m)\left[Q\left((2k-1)\sqrt{2\gamma_{\text{eff}}}\right) - Q\left((2k+1)\sqrt{2\gamma_{\text{eff}}}\right)\right] \\ &\quad + Q\left((2M-3)\sqrt{2\gamma_{\text{eff}}}\right) \end{aligned} \tag{72}$$

where $\bar{h}(k, \beta_m)$ is given by (71), (70), and (67).

We believe that this expression, which provides a new means for evaluating the BEP of communication systems using an M-PAM constellation labeled by the BRGC, is the simplest expression for the entity that has been published to date. A double sum is used in [5], while recursive methods are used in [9].

6.3 BEP of $(M_1 \times M_2)$-QAM constellations using $\beta_{m_1} \times \beta_{m_2}$

We may easily extend the results for the BEP of M-PAM systems derived in the previous subsection to include the case of rectangular $(M_1 \times M_2)$-QAM constellations labeled by a two-dimensional $(\beta_{m_1} \times \beta_{m_2})$ Gray code. From [6] we know that the only way to assign a Gray code to a rectangular QAM constellation is by taking the direct product of two M-PAM constellations, each labeled by a Gray code. In addition, the BEP of such direct product constellations is obtained directly from the BEP of the constituent M_1-PAM and M_2-PAM systems [5]. Therefore, the BEP of an $(M_1 \times M_2)$-QAM system labeled by a $(\beta_{m_1} \times \beta_{m_2})$ Gray code is given by

$$P_b(\beta_{m_1} \times \beta_{m_2}, \gamma) = \frac{m_1}{m_1 + m_2} P_b(\beta_{m_1}, \gamma) + \frac{m_2}{m_1 + m_2} P_b(\beta_{m_2}, \gamma) \tag{73}$$

where $m_1 = \log_2(M_1)$, $m_2 = \log_2(M_2)$, and $P_b(\beta_{m_1}, \gamma)$ and $P_b(\beta_{m_2}, \gamma)$ are obtained from (72).

7 Conclusions and comments

We have addressed the problem of finding an optimum signal constellation labeling with respect to minimizing the BEP for M-PSK and M-PAM under the assumption of a Gaussian channel and coherent symbol detection. The report gives the result that for the asymptotic case when the signal energy-to-noise ratio γ approaches infinity, the binary reflected Gray code is the optimal labeling for both modulation types. In addition, the report establishes that the BRGC is in fact the optimal labeling for a significant range of values for γ. In particular, the BRGC is shown to be optimal as long as $\gamma > \hat{\gamma}$, and numerical values of the threshold $\hat{\gamma}$ are given. By evaluating the BEP at the thresholds, the conclusion is drawn that when the BEP is below a few percent, the BRGC is the optimum labeling, at least for constellations of current practical interest ($m \leq 10$).

The discussion in the report concludes to establish a closed-form expression for the ADS used to calculate the BEP of a M-PAM system using the BRGC (and hence also for rectangular QAM systems). Although methods for exactly calculating the BEP is already available in the literature [5,9], the form given in here is simpler and more general (as it easily extends to other labelings).

The various BEP expressions used in the report are all based on the ADS, which is a convenient function to use, as it separates the influence of the labeling on the BEP from that of the crossover probabilities. In addition, the ADS approach allows us to study all labelings, which allows us to compare labelings as is done in the report. Another advantage of basing the calculations on the ADS is the resulting similarity between the PSK and PAM expressions.

A Appendix

In this appendix, we prove that $\Lambda_m(3) \neq \varnothing$ for $m \geq 4$ for PSK and for $m \geq 3$ for PAM by explicit construction of a Gray code having $\bar{d}(3) > 2 = \bar{d}(3, \beta_m)$ for PSK and $\bar{h}(3) > 2-2^{2-m} = \bar{h}(3, \beta_m)$ for PAM. We will demonstrate the construction method using the concept of a hypercube of order m, which we denote by Q_m [3]. We give the proof for the PSK case as it is easy to show that a similar construction establishes the desired result also for PAM.

In Figure 3 we have indicated two different paths on Q_m (the hypercube of order m), for $m \geq 4$, that generates cyclic Gray codes. This means that the construction shown in the lower part of the figure (which obviously is not the BRGC) can be done for all $m \geq 4$. The terms in the sum of (2) for $\bar{d}(3)$ are equally many 1's and 3's for the BRGC (upper) case. For the lower case it is easy to show that there are $M/2 - 2$ terms that are equal to 1 and $M/2 + 2$ terms equal to 3. This means that for any order $m \geq 4$ we can construct a Gray code having $\bar{d}(3) = 2 + 4/M$ by incorporating the sequence of labels in the lower figure into the labeling.

It is straightforward to show that for the PAM case, a similar construction yields a Gray code with $\bar{d}(3) = 7/4$ for $m = 3$ and $\bar{d}(3) = 2$ for all orders $m \geq 4$.

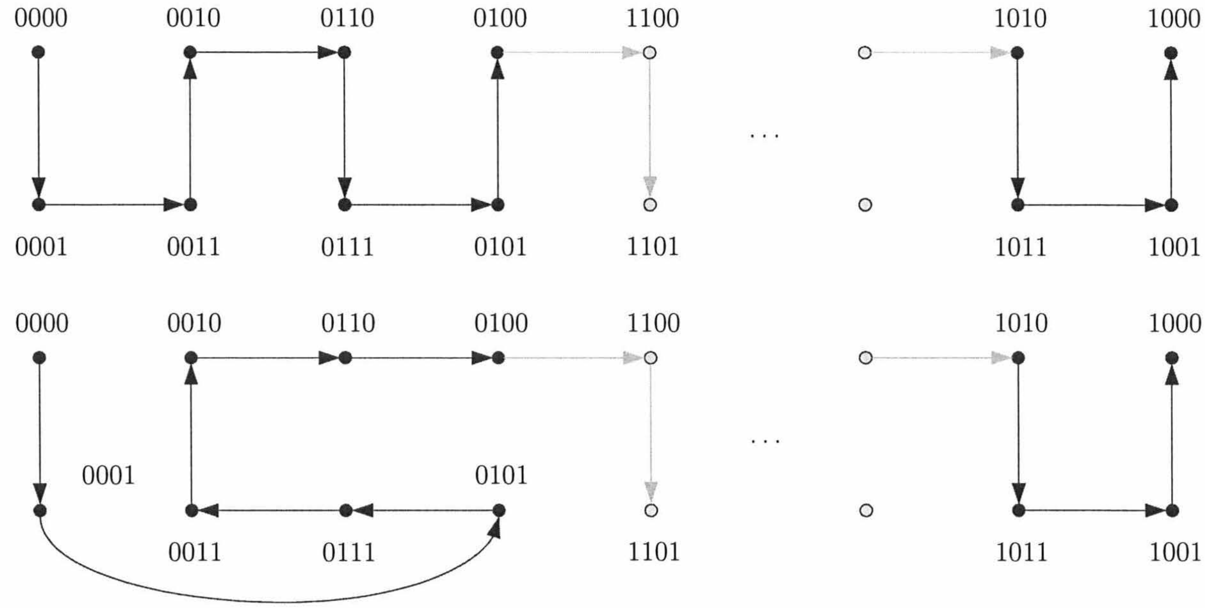

Figure 3: Two paths on Q_m ($m \geq 4$) that forms cyclic Gray codes.

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