Bit error probability of coherent M-ary PSK over flat Rayleigh fading channels

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Abstract

This paper concerns communication systems using coherent M-ary PSK to communicate over flat Rayleigh fading channels. The aim is to generalize the recently published results on the exact bit error probability (BEP) for AWGN channels to Rayleigh fading channels. We derive a closed-form expression for the exact BEP assuming the binary reflected Gray code is used to map bits to symbols. The result is a general, simple, and correct expression, which has not been previously available in the literature. We also compare the exact BEP obtained from using the average distance spectrum (ADS) in the evaluation with the common approximation of using the Hamming weights of the binary labels associated with the symbols in place of the ADS. We note that the approximation is less accurate for the Rayleigh fading channel than for the Gaussian channel.

1 Introduction

This paper deals with communication systems that uses M-ary phase-shift keying (M-PSK) to transmit information over flat Rayleigh fading communication channels. Communication systems employing M-PSK has been thoroughly studied in the literature for many different types of channels, see, e.g., [2–7]. In a recent publication [1], a method to evaluate the exact bit error probability (BEP) of M-PSK systems was presented for additive, white Gaussian noise (AWGN) channels. Even though the correct expression presented in [1] differs from previously published results on the topic, the difference is only noticeable at very low signal-to-noise ratios (SNR). However, for Rayleigh fading channels the instantaneous SNR is occasionally very small and the discrepancy between the previous and the new expressions may become significant. It is the aim of this paper to establish in what way the results presented in [1] will affect the exact BEP of M-PSK over flat Rayleigh fading channels. In particular, we address the problem of evaluating the bit error probability (BEP) of a coherent M-PSK system using maximum likelihood detection on symbols and present a closed-form expression for the exact BEP of M-PSK for any $M = 2^m$, where m is a positive integer.

2 System Model and Symbol Error Probability

In the discussion that follows, we use the standard complex model of a communication system, see, e.g., [8, ch. 4]. During a signalling interval the M-PSK transmitter can output one out of M signals with quadrature modulation of the form

$$s_k = \sqrt{E_s} e^{j\frac{2\pi}{M}k}, \qquad k = 0, 1, \ldots M-1. \tag{1}$$

In the flat Rayleigh channel model, the channel is assumed to introduce two random variables in the signal observed by the receiver; a random fluctuation of the received signal energy and phase and an additive, white Gaussian noise component, as seen in the system block diagram of Figure 1. Assuming that the receiver is able to perfectly track the channel (amplitude and phase), the detector in the receiver observes the signal $r = |a|^2 s + n$, where a is the complex channel coefficient. The random variable $|a|$ has a Rayleigh probability density function (pdf) with mean $\sqrt{\pi/2}$ and $n = n_I + jn_Q$ is a complex Gaussian noise variable with n_I and n_Q being identically distributed zero-mean Gaussian random variables with variance $N_0/2$, where $N_0/2$ is the double-sided noise power spectral density.

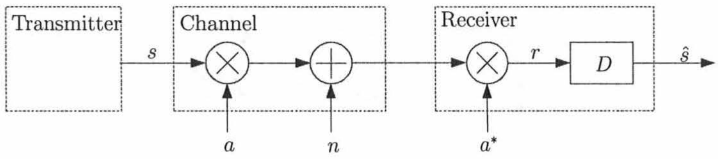

Figure 1: Schematic block diagram of a communication system communicating over a Rayleigh fading channel (a^* denotes the complex conjugate of a).

Each signal alternative is associated with a decision region S_k, and the receiver determines what S_k the received signal r falls within and outputs as its estimate $\hat{s}$ the corresponding signal s_k. A symbol error is said to occur if the estimate $\hat{s}$ differs from the transmitted signal s. We make the reasonable assumption that all signals are equally likely to be selected for transmission, i.e., $\Pr\{s = s_k\} = 1/M$ and we also assume that the receiver uses a maximum likelihood symbol (ML) detector. Under these assumptions, the problem becomes completely symmetrical and the decision regions S_k are wedges of infinite radius, as shown for $M = 8$ in Figure 2. The average probability of symbol error is given by

$$P_\mathrm{e} = \Pr\{r \notin S_k \mid s = s_k\} \tag{2}$$

for any value of k.

For the Gaussian channel (i.e., for a channel for which $|a|$ is constant), P_e is readily available [1]. To be able to handle (2) also for the flat Rayleigh fading channel, we integrate the error probability over the pdf of the *instantaneous* received energy-to-noise ratio per symbol, $\gamma \triangleq |a|^2 E_s/N_0$;

$$P_\mathrm{e} = \int_0^\infty \Pr\{\mathbf{r} \notin S_k \mid \mathbf{s} = \mathbf{s}_k, \gamma\} p(\gamma) d\gamma \tag{3}$$

Since $p(\gamma)$ is given by the assumption of a Rayleigh fading channel model, all we need to evaluate P_e is a tractable expression for $\Pr\{\mathbf{r} \notin S_k \mid \mathbf{s} = \mathbf{s}_k, \gamma\}$. This is the topic of the next section.

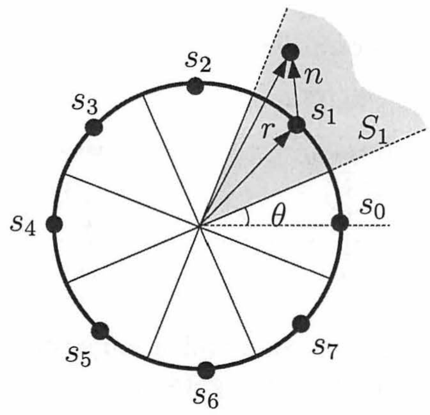

Figure 2: The signal space of a communication system for 8-PSK ($M = 8$). A symbol error occurs when the transmitted signal s_k is displaced by the channel outside the region S_k, i.e., when the received signal r is in the complement of S_k.

3 Symbol Error Probability for AWGN Channels

To evaluate (3), we will make use of an expression proposed by Craig in [9], which will prove particularly useful in section 4. Craig showed that the probability that a transmitted signal point s_0 at a distance $|a|\sqrt{E_s}$ from the origin is displaced outside a surrounding wedge of half-angle θ (see Figure 2) is given by

$$I(\theta \mid \gamma) = \frac{1}{\pi}\int_0^{\pi-\theta} e^{-\frac{|a|^2 E_s}{N_0}\frac{\sin^2\theta}{\sin^2\varphi}} d\varphi = \frac{1}{\pi}\int_0^{\pi-\theta} e^{-\gamma\frac{\sin^2\theta}{\sin^2\varphi}} d\varphi, \qquad 0 \leq \theta \leq \pi \tag{4}$$

As was mentioned in section 2, for M-PSK the (full) angle of the decision regions is $2\pi/M$ and consequently the probability to end up outside such a decision region is

$$I\left(\tfrac{\pi}{M} \mid \gamma\right) = \Pr\{\mathbf{r} \notin S_k \mid \mathbf{s} = \mathbf{s}_k, \gamma\} = \frac{1}{\pi}\int_0^{\pi-\pi/M} e^{-\gamma\frac{\sin^2(\pi/M)}{\sin^2\varphi}} d\varphi \tag{5}$$

If the channel attenuation is constant and known, say, $|a| = 1$, we obtain the average probability of symbol error for M-PSK over a Gaussian channel, see [9] [2, p. 198].

4 Symbol Error Probability for Fading Channels

For the Rayleigh fading channel, the amplitude modulating factor $|a|$ has a Rayleigh pdf, and the pdf of the instantaneous signal-to-noise ratio per symbol, γ, is exponentially distributed with pdf [2, pp. 18]

$$p(\gamma) \triangleq \frac{1}{\bar{\gamma}} e^{-\gamma/\bar{\gamma}}, \qquad \gamma \geq 0 \tag{6}$$

where $\bar{\gamma} \triangleq \mathrm{E}[\gamma] = \mathrm{E}[|a|^2]E_s/N_0 = E_s/N_0$ is the average energy-to-noise ratio per symbol and $\mathrm{E}[x]$ denotes the expected value of the random variable x.

Before we calculate the symbol error probability of an M-PSK system over a Rayleigh fading channel, given by (3), we evaluate the more general expression obtained by averaging (3) over the pdf in (6),

$$\begin{aligned} I(\theta) &= \int_0^\infty p(\gamma) I(\theta \mid \gamma) d\gamma = \frac{1}{\pi} \int_0^\infty p(\gamma) \int_0^{\pi-\theta} e^{-\gamma \frac{\sin^2\theta}{\sin^2\varphi}} d\varphi d\gamma \\ &= \frac{1}{\pi} \int_0^{\pi-\theta} \left[\int_0^\infty p(\gamma) e^{-\gamma \frac{\sin^2\theta}{\sin^2\varphi}} d\gamma \right] d\varphi \end{aligned} \tag{7}$$

The inner integral of (7) is straight-forward to evaluate for the exponential pdf given by (6),

$$\int_0^\infty p(\gamma) e^{-s\gamma} d\gamma = \frac{1}{\bar{\gamma}} \int_0^\infty e^{-(s+\bar{\gamma}^{-1})\gamma} d\gamma = \frac{1}{1+s\bar{\gamma}}, \qquad s > -1/\bar{\gamma} \tag{8}$$

and by defining

$$c(\theta) \triangleq \bar{\gamma} \sin^2\theta \tag{9}$$

we can rewrite (7) as

$$\begin{aligned} I(\theta) &= \frac{1}{\pi} \int_0^{\pi-\theta} \left(1 + \frac{c(\theta)}{\sin^2\varphi}\right)^{-1} d\varphi \\ &= \frac{\pi-\theta}{\pi} - \frac{c(\theta)}{\pi} \int_0^{\pi-\theta} \frac{d\varphi}{\sin^2\varphi + c(\theta)} \end{aligned} \tag{10}$$

Now, a closed form for the integral in the last expression is

$$\sqrt{c(c+1)} \int_0^\phi \frac{d\varphi}{\sin^2\varphi + c} = \begin{cases} \phi, \text{ if } \phi \text{ is an odd multiple of } \frac{\pi}{2} \\ \\ \arctan\left(\sqrt{\frac{c+1}{c}} \tan\phi\right) + \lfloor \frac{\phi}{\pi} \rceil \pi, \text{ otherwise} \end{cases} \tag{11}$$

where $\lfloor x \rceil$ rounds x to the closest integer and c is any positive constant.

By using the relations

$$\arctan x = \begin{cases} \frac{\pi}{2} - \arctan\frac{1}{x}, & x > 0 \\ -\frac{\pi}{2} - \arctan\frac{1}{x}, & x < 0 \end{cases} \tag{12}$$

we may find a single expression for $I(\theta)$ from (11), valid over the interval of primary interest $0 < \theta < \pi$,

$$I(\theta) = 1 - \frac{\theta}{\pi} - \beta(\theta)\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left(\beta(\theta)\cot\theta\right)\right) \tag{13}$$

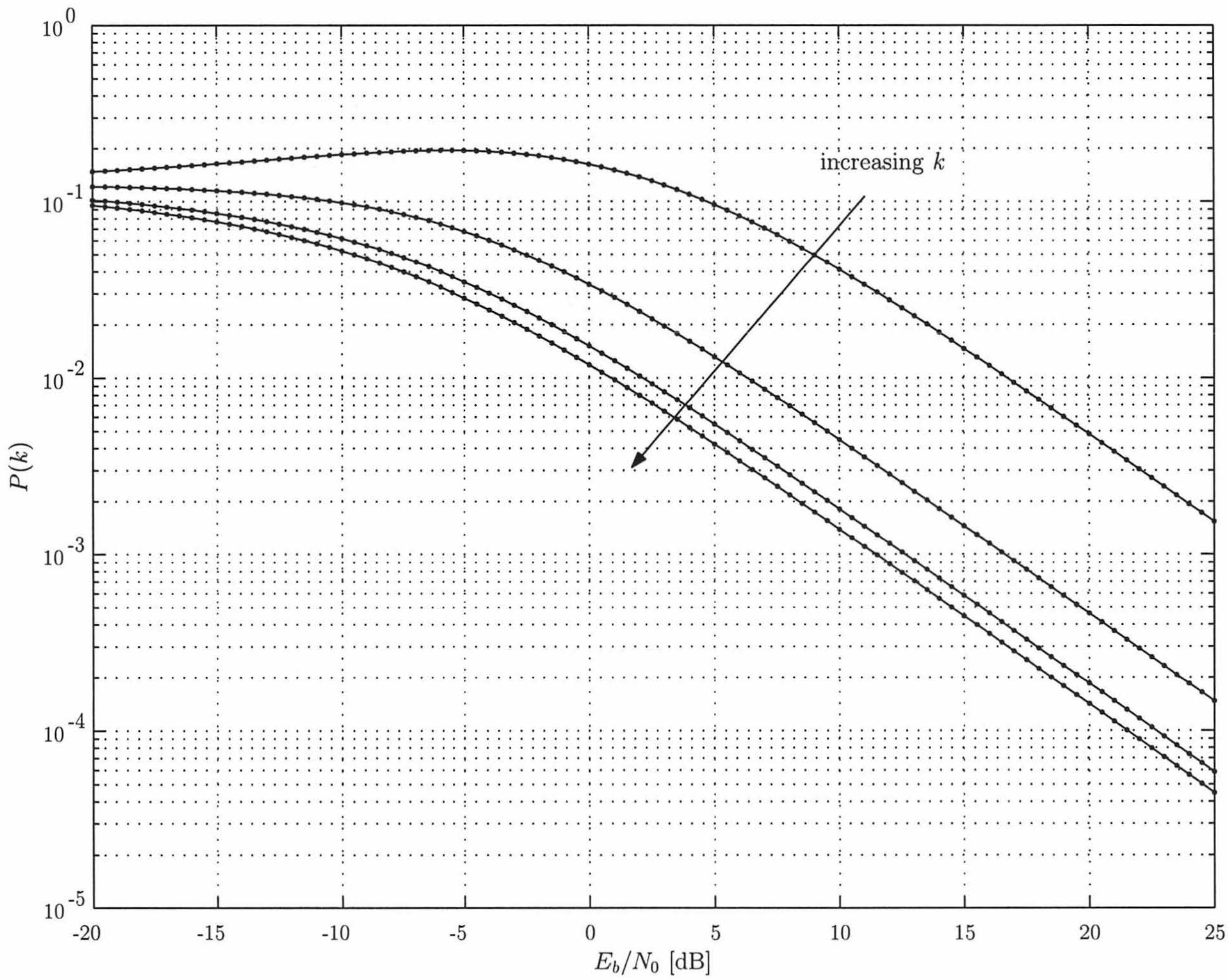

Figure 3: The probabilities $P(k)$, $k = 1, 2, 3, 4$, for an 8-PSK system over a flat Rayleigh fading channel.

where we have defined

$$\beta(\theta) \triangleq \sqrt{\frac{c(\theta)}{c(\theta)+1}} = \sqrt{\frac{\bar{\gamma}\sin^2\theta}{1+\bar{\gamma}\sin^2\theta}} \tag{14}$$

Now, since $P_e = I(\pi/M)$ from (3) and (5), the symbol error probability for the flat Rayleigh fading channel is given by

$$\begin{aligned} P_e = {} & \frac{M-1}{M} \\ & - \frac{1}{\pi}\sqrt{\frac{\bar{\gamma}\sin^2(\pi/M)}{1+\bar{\gamma}\sin^2(\pi/M)}}\left(\frac{\pi}{2} + \arctan\left(\sqrt{\frac{\bar{\gamma}\sin^2(\pi/M)}{1+\bar{\gamma}\sin^2(\pi/M)}}\cot\frac{\pi}{M}\right)\right) \end{aligned} \tag{15}$$

5 Bit Error Probability for Fading Channels

When the average probability of *bit* errors is considered, the mapping of bits onto symbols becomes important. In general, for a system using M-PSK with coherent ML detection, the average bit error probability is given by [1]

$$P_{\mathrm{b}} = \frac{1}{m} \sum_{k=1}^{M-1} \bar{d}(k)\, P(k) \tag{16}$$

where the function $\bar{d}(k)$ depends only on the bit-to-symbol mapping and the function $P(k)$ captures the influence of the channel on the error probability. Note that (16) is valid for any (memoryless) channel and the problem is to find valid and tractable expressions for $\bar{d}(k)$ and $P(k)$.

The probability $P(k)$ is the probability that the detected signal is found within the wedge in the signal space delimited by angles $(2k-1)\pi/M$ and $(2k+1)\pi/M$, assuming that s_0 was transmitted, refer to Figure 2. By defining

$$\theta_k^- \triangleq (2k-1)\pi/M, \tag{17}$$

$$\theta_k^+ \triangleq (2k+1)\pi/M, \tag{18}$$

we can find $P(k)$ directly in terms of (13) above,

$$P(k) = \begin{cases} 1 - I\left(\frac{\pi}{M}\right), & k = 0, \\ \frac{1}{2}\left(I(\theta_k^-) - I(\theta_k^+)\right), & k = 1, 2, \ldots, M/2-1, \\ I\left(\frac{(M-1)\pi}{M}\right), & k = M/2 \end{cases} \tag{19}$$

The values of $P(k)$ for $k = M/2+1, \ldots, M-1$ are obtained from the symmetry of the problem that gives

$$P(M-k) = P(k), \text{ for } k = 1, \ldots, M/2-1.$$

The function $P(k)$ is illustrated for $M = 8$ and $k = 1, 2, 3$, and 4 in Figure 3.

By using (13), we can find closed-form expressions for (19). For $k = 0$, we have

$$P(0) = 1 - \frac{1}{M} + \frac{1}{2}\beta\left(\frac{\pi}{M}\right) + \frac{1}{\pi}\beta\left(\frac{\pi}{M}\right) \arctan\left(\beta\left(\frac{\pi}{M}\right)\cot\frac{\pi}{M}\right), \tag{20}$$

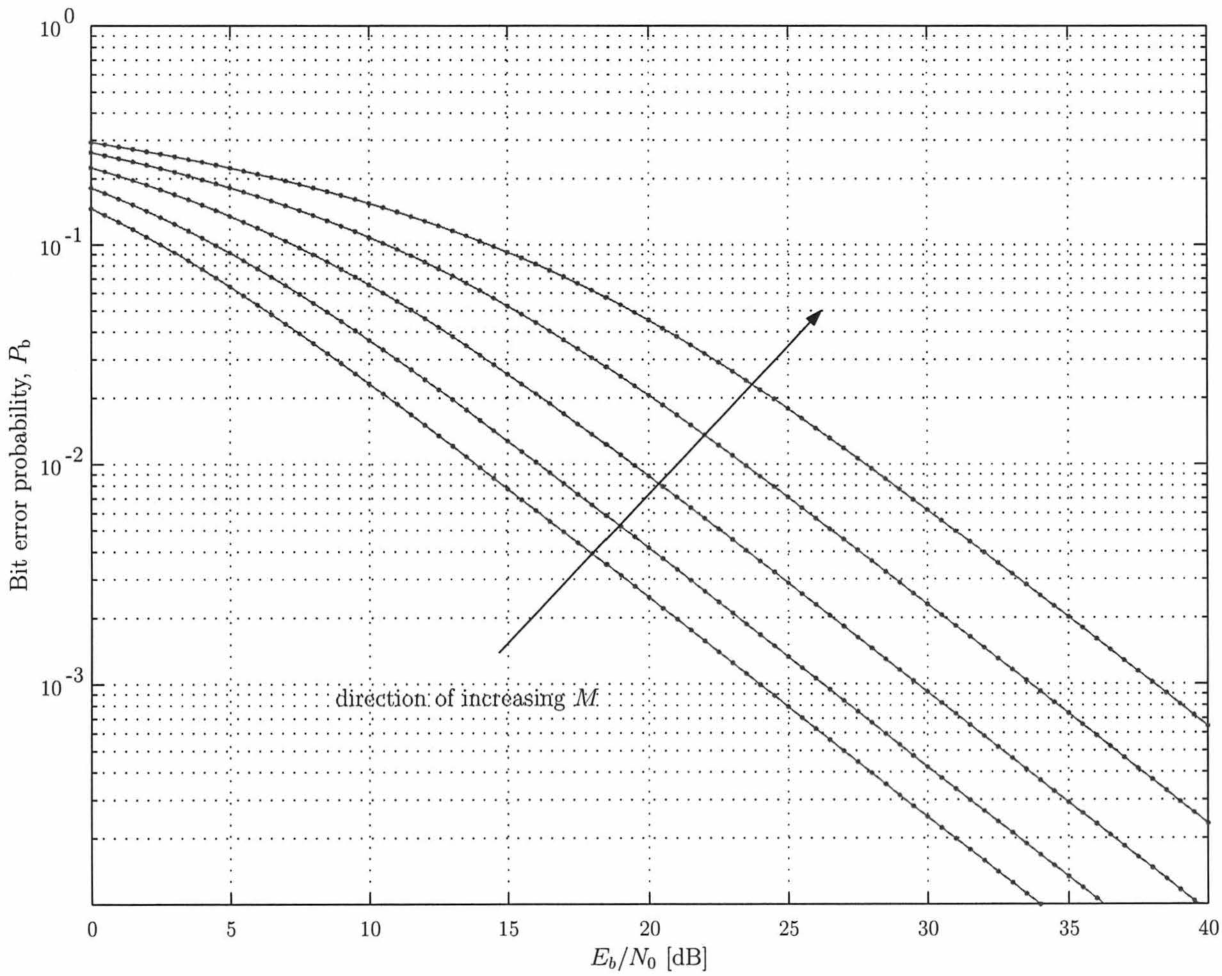

Figure 4: The bit error probability of an M-PSK system using binary reflected Gray mapping from bits to symbols operating over a flat Rayleigh fading channel for $M = 4, 8, 16, 32, 64$.

for $k = 1, 2, \ldots, M/2 - 1$,

$$\begin{aligned} P(k) &= \frac{1}{M} + \frac{\beta(\theta_k^+) - \beta(\theta_k^-)}{4} \\ &+ \frac{1}{2\pi}\left[\beta(\theta_k^+)\arctan\left(\beta(\theta_k^+)\cot\theta_k^+\right) - \beta(\theta_k^-)\arctan\left(\beta(\theta_k^-)\cot\theta_k^-\right)\right] \end{aligned} \tag{21}$$

and for $k = M/2$, we have

$$\begin{aligned} P(k) &= \frac{1}{M} - \frac{1}{2}\beta\left(\frac{(M-1)\pi}{M}\right) \\ &- \frac{1}{\pi}\beta\left(\frac{(M-1)\pi}{M}\right)\arctan\left(\beta\left(\frac{(M-1)\pi}{M}\right)\cot\frac{(M-1)\pi}{M}\right) \end{aligned} \tag{22}$$

The function $\bar{d}(k)$ is called the *average distance spectrum* (ADS) [1], and is defined as the *average* number of bit positions differing between signal alternatives separated by k steps along the M-PSK circle. In order to evaluate the BEP of an M-PSK system, we must specify the particular bit-to-symbol mapping used, since the choice of mapping will affect the BEP. A commonly encountered way of mapping bits to symbols is by means of the binary reflected Gray code (BRGC), which is discussed, e.g., in [1]. In [1, eq. (7)] a closed-form expression for the ADS of the BRGC is given for all integers k,

$$\bar{d}(k) = 2\left|\frac{k}{M} - \left\lfloor\frac{k}{M}\right\rceil\right| + 2\sum_{i=2}^{m}\left|\frac{k}{2^i} - \left\lfloor\frac{k}{2^i}\right\rceil\right| \tag{23}$$

where $\lfloor x \rceil$ was defined in Section 4 (ties are resolved arbitrarily).

By combining (23) with (16) we obtain

$$P_{\mathrm{b}} = \frac{1}{m}\sum_{k=1}^{M-1}\left(2\left|\frac{k}{M} - \left\lfloor\frac{k}{M}\right\rceil\right| + 2\sum_{i=2}^{m}\left|\frac{k}{2^i} - \left\lfloor\frac{k}{2^i}\right\rceil\right|\right) P(k) \tag{24}$$

By symmetry, $P(M-k) = P(k)$ and $\bar{d}(M-k) = \bar{d}(k)$, and we may rewrite (24) as

$$P_{\mathrm{b}} = \frac{2}{m}P\left(\frac{M}{2}\right) + \frac{2}{m}\sum_{k=1}^{M/2-1}\left(2\left|\frac{k}{M} - \left\lfloor\frac{k}{M}\right\rceil\right| + 2\sum_{i=2}^{m}\left|\frac{k}{2^i} - \left\lfloor\frac{k}{2^i}\right\rceil\right|\right) P(k) \tag{25}$$

since $\bar{d}(M/2) = 2$ for all M (for the BRGC). This equation, together with (20)–(22) and the definitions (14), (9), (17), and (18), provides a closed-form expression for the exact BEP of coherent M-PSK over flat Rayleigh fading channels.

6 Comments and Conclusions

In Figure 4, the exact bit error probability of an M-PSK system communicating over a Rayleigh fading channel as dictated by (25) is plotted for $M = 4, 8, 16, 32$, and 64. An approach commonly encountered in the literature when evaluating the BEP of coherent M-PSK systems, see e.g., [3, sec. IV], is to use the Hamming weights of the binary labels assigned to the symbols, rather than the ADS. As was shown in [1], this method only generates approximate results. However, for the Gaussian case, the difference is very small and almost not noticeable. In Figure 5, the ratio between the BEP evaluated using the Hamming weights, $w_H(k)$, and the BEP evaluated using the ADS, $\bar{d}(k)$, is shown for the Gaussian channel and the flat Rayleigh fading channel. We note that the approximation of using $w_H(k)$ instead of $\bar{d}(k)$ is still accurate, although much less accurate than for the AWGN channel. However, since there is a

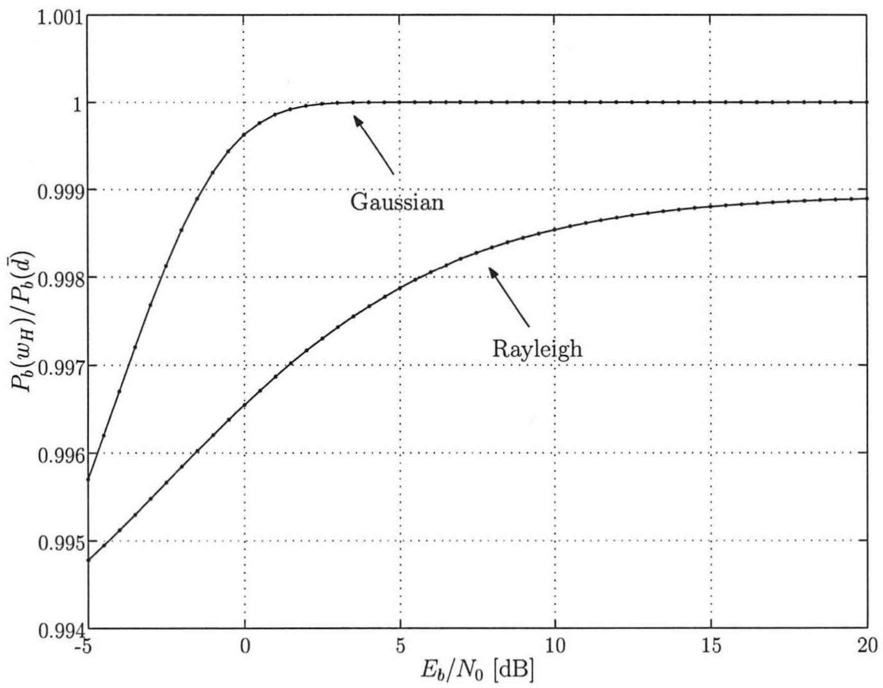

Figure 5: The graph indicates the ratio $P_b(w_H)/P_b(\bar{d})$ for $M = 16$, where $P_b(w_H)$ is (16) evaluated using the Hamming weights of the BRGC labels in place of the ADS, and $P_b(\bar{d}_k)$ is (25). The upper curve gives the ratio for the Gaussian channel and the lower curve is for a flat Rayleigh fading channel.

closed-form expression available for the ADS, there is little motivation for the approximation of using $w_H(k)$. From Figure 5 we also observe that the difference between the approximation and the exact BEP for the fading channel is noticeable over the whole range of practical values of E_b/N_0, which is not the case for the AWGN channel.

An expression similar to (20)–(22) is presented in [2, eq. (8.117)], which is derived from the indefinite form of the integral in (11), for which a closed-form solution can be found in [10, eq. 2.562(1)]. However, when using this integral in its definite form (11), care must be taken when the integration limits enclose one or more of the discontinuity points of the tangent function (i.e., odd multiples of $\pi/2$), so that the resulting solution is not discontinuous at these points. In particular, this is not the case for the form given in [2, eq. (8.117)], and hence it does not give the correct values of $P(k)$ for $k = 0, 1, \ldots, M/4 - 1$ and $k = 3M/4, \ldots, M - 1$. This fact, together with the previously unavailable closed form (23), has until now made it difficult to compute the exact BEP of coherent M-PSK.

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