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THE 3G INEQUALITY FOR A UNIFORMLY JOHN DOMAIN

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Dedicated to the memory of Professor Nobuyuki Suita

A . Let G be the Green function for a domain $D \subset \mathbb{R}^d$ with $d \geq 3$. The Martin boundary of D and the 3G inequality:

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leq A(|x-y|^{2-d} + |y-z|^{2-d}) \quad \text{for } x,y,z \in D$$

are studied. We give the 3G inequality for a bounded uniformly John domain D, although the Martin boundary of D need not coincide with the Euclidean boundary. On the other hand, we construct a bounded domain such that the Martin boundary coincides with the Euclidean boundary and yet the 3G inequality does not hold.

1. I

For a bounded Lipschitz domain $D \subset \mathbb{R}^d$ with $d \geq 3$, Cranston, Fabes and Zhao [13] proved the following 3G inequality:

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leq A_0(|x-y|^{2-d} + |y-z|^{2-d}) \quad \text{for } x,y,z \in D, \tag{1}$$

where G is the Green function for D and A_0 is a positive constant depending only on D. They used (1) for the conditional gauge theorem and the Schrödinger equation. Their proof is based on the boundary Harnack principle, a comparison principle among positive harmonic functions vanishing on a portion of the boundary ([6, 15, 18]). The boundary Harnack principle also yields the coincidence of the Martin boundary of D and the Euclidean boundary ([16]). So, one might think that there is a relationship between the 3G inequality and the coincidence of the Martin and the Euclidean boundaries. We shall however see that there is no direct connection between them. We shall prove the 3G inequality for a uniformly John domain, whose Martin boundary need not coincide with the Euclidean boundary (Theorem 1). On the other hand, we shall provide an example of a bounded domain in $\mathbb{R}^d$ with $d \geq 3$, whose Martin boundary coincides with the Euclidean boundary and for which the 3G inequality fails to hold (Proposition 2).

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Throughout the paper, let D be a bounded domain in $\mathbb{R}^d$, $d \geq 3$, and let $\delta_D(x) = \mathrm{dist}(x, \partial D)$. For $x, y \in D$, we define the *internal metric* or the *inner diameter distance* $\rho_D(x, y)$ by

$$\rho_D(x, y) = \inf\{\mathrm{diam}(\gamma)\},$$

where the infimum is taken over all curves γ connecting x and y in D and $\mathrm{diam}(\gamma)$ stands for the diameter of γ. The *inner length distance* $\lambda_D(x, y)$ is defined similarly by $\lambda_D(x, y) = \inf\{\ell(\gamma)\}$, where the infimum is taken over all curves γ connecting x and y in D and $\ell(\gamma)$ stands for the length of γ. Obviously $|x - y| \leq \rho_D(x, y) \leq \lambda_D(x, y)$. If $|x - y| \leq \max\{\delta_D(x), \delta_D(y)\}$, then $|x - y| = \rho_D(x, y) = \lambda_D(x, y)$. We say that D is a *uniformly John domain* if there exists a constant $A_1 \geq 1$ such that each pair of points $x, y \in D$ can be connected by a curve $\gamma \subset D$ for which

$$\begin{aligned} &\min\{|x - z|, |z - y|\} \leq A_1 \delta_D(z) \quad \text{for all } z \in \gamma, \\ &\mathrm{diam}(\gamma) \leq A_1 \rho_D(x, y) \end{aligned} \tag{2}$$

(Balogh and Volberg [7, 8]). We say that D is an *inner uniform domain* if there exists a constant $A_2 \geq 1$ such that each pair of points $x, y \in D$ can be connected by a curve $\gamma \subset D$ for which

$$\begin{aligned} &\min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\} \leq A_2 \delta_D(z) \quad \text{for all } z \in \gamma, \\ &\ell(\gamma) \leq A_2 \lambda_D(x, y), \end{aligned}$$

where $\gamma(x, z)$ is the subarc of γ connecting x and z (Bonk, Heinonen and Koskela [11]). In view of Väisälä [17], the family of uniformly John domains and that of inner uniform domains coincide.

Under some additional assumptions, such as the the uniform perfectness of the boundary or the existence of a strong barrier, Balogh-Volberg and Bonk-Heinonen-Koskela studied the boundary Harnack principle and the Martin boundary for these domains. In [4] Mizutani and the authors gave the boundary Harnack principle and identified the Martin boundary of a bounded uniformly John domain without any other additional assumptions. The Martin boundary is the ideal boundary with respect to the internal metric $\rho_D(x, y)$; it need not be homeomorphic to the Euclidean boundary. In this note we show the following.

Theorem 1. *Let D be a bounded uniformly John domain in $\mathbb{R}^d$ with $d \geq 3$. Then the 3G inequality* (1) *holds.*

In Section 3 we shall construct a bounded domain in $\mathbb{R}^d$ with $d \geq 3$, such that the Martin boundary coincides with the Euclidean boundary and yet the 3G inequality does not hold.

Remark 1. There is a significant difference between the planar case and the case $d \geq 3$. For the planar case Bass and Burdzy [10] established the 3G inequality (with suitable modification of the right hand side) for an arbitrary bounded domain.

2. P T 1

We shall use the following notation as in [4]. By the symbol A we denote a positive constant depending only on the dimension d, whose value is unimportant and may change even in the same line. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \geq 1$ such that $A^{-1} f_1 \leq f_2 \leq A f_1$. The constant A will be called the constant of comparison. By $B(x, r)$ we denote the open ball with center at x and radius r.

Let D^* be the completion of D with respect to the internal metric. That is, D^* is the equivalence class of all ρ_D-Cauchy sequences with equivalence relation "$\sim$", where we say $\{x_j\} \sim \{y_j\}$ if $\{x_j\} \cup \{y_j\}$ is a ρ_D-Cauchy sequence. The completion D^* is a compact space. Let $\partial^* D = D^* \setminus D$, the boundary with respect to ρ_D. Take $\xi^* \in D^*$. Suppose ξ^* is represented by a ρ_D-Cauchy sequence $\{x_j\}$. Since $\{x_j\}$ is also a usual Cauchy sequence, it follows that x_j converges to some point $\xi \in \overline{D}$. The point ξ is independent of the representative sequence $\{x_j\}$ and uniquely determined by ξ^*. We say that ξ^* lies over $\xi \in \overline{D}$. If $\xi \in D$, then ξ and ξ^* coincide. Define the projection $\pi : D^* \to \overline{D}$ by $\pi(\xi^*) = \xi$. Let $B_\rho(\xi^*, r)$ be the open connected component of $D \cap B(\pi(\xi^*), r)$ which can be connected to ξ^* in itself, i.e. for every $x \in B_\rho(\xi^*, r)$ there is an arc $\gamma \subset B_\rho(\xi^*, r)$ starting from x and converging to ξ^*. By definition $\rho_D(\xi^*, x) < 2r$ for $x \in B_\rho(\xi^*, r)$; in other words

$$\text{if } \rho_D(\xi^*, x) \geq 2r, \text{ then } x \in D \setminus B_\rho(\xi^*, r). \tag{3}$$

Let $\xi^* \in \partial^* D$. Observe from (2) that

$$\text{if there exists } y \in D \text{ with } \rho_D(\xi^*, y) \geq 2r, \text{ then there exists } x \in B_\rho(\xi^*, r) \text{ with } \delta_D(x) \approx r. \tag{4}$$

In [4] Mizutani and the authors proved the following.

Theorem A. *Let D be a bounded uniformly John domain. Then the Martin compactification of D is homeomorphic to D^* and each boundary point $\xi^* \in \partial^* D$ is minimal.*

This theorem was deduced as a corollary to a uniform boundary Harnack principle, whose proof is based on the following estimate of the Green function (cf. [3, Lemma 3] and [4, Lemma 3.2]).

Lemma A. *Let $\xi^* \in \partial^* D$. Then*

$$\frac{G(x,y)}{G(x',y)} \approx \frac{G(x,y')}{G(x',y')} \quad \text{for } x, x' \in B_\rho(\xi^*, r) \text{ and } y, y' \in D \setminus B_\rho(\xi^*, 6r)$$

with constant comparison depending only on D.

In [4, Lemma 3.2], the above estimate was given actually for the Green function for the intersection of D and $B_\rho(\xi^*, Ar)$ with A large enough. However, for the case $d \geq 3$, we see that the same estimate holds for the Green function for D itself.

We also need the following lemma whose proof is easy and left to the reader.

Lemma 1. *Let $x, y \in D$. Then $G(x,y) \le A\rho_D(x,y)^{2-d}$. Moreover, if $\delta_D(x) \ge A^{-1}\rho_D(x,y)$ and $\delta_D(y) \ge A^{-1}\rho_D(x,y)$, then $G(x,y) \ge A^{-1}\rho_D(x,y)^{2-d}$.*

Proof of Theorem 1. We have observed $|x-y| \le \rho_D(x,y)$. So, let us prove the following slightly stronger form of the 3G inequality.

$$\frac{G(x,y)G(y,z)}{G(x,z)} \le A(\rho_D(x,y)^{2-d} + \rho_D(y,z)^{2-d}) \quad \text{for } x, y, z \in D. \tag{5}$$

We will prove (5) according to the line of Bass' proof of the 3G inequality. See [9, Theorem 3.6] and its correction. Let $c_1 = \frac{1}{39}$ and $c_2 = \frac{1}{13}c_1$. By symmetry we may assume that

$$\rho_D(x,y) \le \rho_D(y,z). \tag{6}$$

Case 1. $\rho_D(x,y) \ge c_1\rho_D(x,z)$. Let $r = \rho_D(x,z)$. If $\delta_D(x) \ge c_2 r$, then we let $x_1 = x$. If $\delta_D(x) < c_2 r$, then we take x_1 as follows: Let $x' \in \partial D$ with $|x - x'| = \delta_D(x)$. Since the line segment $\overline{xx'}$ is included in $D \cap B(x', c_2 r)$, we find $x^* \in \partial^* D$ lying over x' such that $x \in B_\rho(x^*, c_2 r)$. Then

$$\rho_D(x^*, y) \ge \rho_D(x,y) - \rho_D(x,x^*) \ge (c_1 - c_2)r = 12c_2 r,$$
$$\rho_D(x^*, z) \ge \rho_D(x,z) - \rho_D(x,x^*) \ge (1 - c_2)r > 12c_2 r.$$

By (4) we can take $x_1 \in B_\rho(x^*, c_2 r)$ with $\delta_D(x_1) \approx c_2 r$. Then $x, x_1 \in B_\rho(x^*, c_2 r)$ and $y, z \in D \setminus B_\rho(x^*, 6c_2 r)$ by (3), so that by Lemma A yields

$$\frac{G(x,y)}{G(x_1,y)} \approx \frac{G(x,z)}{G(x_1,z)}.$$

Similarly, if $\delta_D(z) \ge c_2 r$, then we let $z_1 = z$. If $\delta_D(z) < c_2 r$, then we take $z' \in \partial D$ with $|z - z'| = \delta_D(z)$ and $z^* \in \partial^* D$ lying over z' such that $z \in B_\rho(z^*, c_2 r)$. By (6)

$$\rho_D(z^*, y) \ge \rho_D(z,y) - \rho_D(z,z^*) \ge (c_1 - c_2)r = 12c_2 r,$$
$$\rho_D(z^*, x_1) \ge \rho_D(x,z) - \rho_D(x,x_1) - \rho_D(z,z^*) \ge (1 - 2c_2)r > 12c_2 r.$$

Hence we find $z_1 \in B_\rho(z^*, c_2 r)$ such that $\delta_D(z_1) \approx c_2 r$ by (4). Then $z, z_1 \in B_\rho(z^*, c_2 r)$ and $y, x_1 \in D \setminus B_\rho(z^*, 6c_2 r)$, so that Lemma A yields

$$\frac{G(y,z)}{G(x_1,z)} \approx \frac{G(y,z_1)}{G(x_1,z_1)}.$$

Hence

$$\frac{G(x,y)G(y,z)}{G(x,z)} \approx \frac{G(x_1,y)G(y,z_1)}{G(x_1,z_1)}.$$

Now observe that $\delta_D(x_1) \approx \delta_D(z_1) \approx \rho_D(x_1,z_1) \approx r$, so that $G(x_1,z_1) \approx r^{2-d}$ by Lemma 1. Also, $\rho_D(x_1,y) \approx \rho_D(x,y) \ge c_1 r$ and $\rho_D(y,z_1) \approx \rho_D(y,z) \ge \rho_D(x,y)$ by (6). Hence Lemma 1 yields

$$\frac{G(x,y)G(y,z)}{G(x,z)} \le A\frac{\rho_D(x,y)^{2(2-d)}}{r^{2-d}} \le A\rho_D(x,y)^{2-d}.$$

Thus (5) holds in this case.

Case 2. $\rho_D(x,y) < c_1\rho_D(x,z)$. Let $s = \rho_D(x,y)$. By connectedness there is $w \in D$ with $\rho_D(x,w) = s/c_1$. Then

$$\rho_D(y,w) \ge \rho_D(x,w) - \rho_D(x,y) = (\frac{1}{c_1} - 1)s > s = \rho_D(x,y) = c_1\rho_D(x,w), \tag{7}$$

so that Case 1 applies to the triplet x, y, w. Hence

$$\frac{G(x,y)G(y,w)}{G(x,w)} \le A\rho_D(x,y)^{2-d}. \tag{8}$$

We are now going to replace w with z.

Subcase 2a. $s = \rho_D(x,y) > \frac{1}{2}\delta_D(y)$. Let $y' \in \partial D$ with $|y - y'| = \delta_D(y)$. Since the line segment $\overline{yy'}$ is included in $D \cap B(y', 2s)$, we find $y^* \in \partial^* D$ lying over y' such that $\rho_D(y, y^*) = \delta_D(y) < 2s$. Then $x, y \in B_\rho(y^*, 3s)$. Observe from (7) that

$$\rho_D(y^*, w) \ge \rho_D(y,w) - \rho_D(y,y^*) \ge (\frac{1}{c_1} - 3)s = 36s,$$
$$\rho_D(y^*, z) \ge \rho_D(x,z) - \rho_D(x,y) - \rho_D(y,y^*) \ge (\frac{1}{c_1} - 3)s = 36s,$$

so that $w, z \in D \setminus B_\rho(y^*, 18s)$ by (3). Lemma A implies

$$\frac{G(y,w)}{G(x,w)} \approx \frac{G(y,z)}{G(x,z)},$$

which together with (8) yields

$$\frac{G(x,y)G(y,z)}{G(x,z)} \approx \frac{G(x,y)G(y,w)}{G(x,w)} \le A\rho_D(x,y)^{2-d}.$$

Subcase 2b. $s = \rho_D(x,y) \le \frac{1}{2}\delta_D(y)$. Then $G(x,y) \approx |x-y|^{2-d} = \rho_D(x,y)^{2-d}$ by Lemma 1. If furthermore $|y - z| > \frac{3}{4}\delta_D(y)$, then $G(\cdot, z)$ is positive and harmonic in $B(y, \frac{3}{4}\delta_D(y))$, so that the Harnack inequality shows $G(x,z) \approx G(y,z)$ and hence

$$\frac{G(x,y)G(y,z)}{G(x,z)} \approx G(x,y) \approx \rho_D(x,y)^{2-d}.$$

If $|y-z| \le \frac{3}{4}\delta_D(y)$, then $|y-z| = \rho_D(y,z) \ge \rho_D(x,y) = |x-y|$ by (6), so that $|x-z| \le |x-y| + |y-z| \le 2|y-z|$. Moreover, $G(y,z) \approx |y-z|^{2-d}$ and $G(x,z) \approx |x-z|^{2-d}$; and hence,

$$\frac{G(x,y)G(y,z)}{G(x,z)} \approx \frac{|x-y|^{2-d}|y-z|^{2-d}}{|x-z|^{2-d}} \le A|x-y|^{2-d} = A\rho_D(x,y)^{2-d}.$$

Thus (5) also holds in Case 2. The proof is complete. □

3. A E

Let us begin with an application of the 3G inequality.

Proposition 1. *Let D be a domain of finite volume in $\mathbb{R}^d$ with $d \ge 3$. Suppose the 3G inequality* (1) *holds. Then the following Cranston-McConnell inequality*

$$\sup_{x \in D} \frac{1}{u(x)} \int_D G(x,y)u(y)dy \le dV_d^{1-2/d}|D|^{2/d}A_0 \tag{9}$$

holds for every nonnegative superharmonic function u in D, where V_d stands for the volume of a unit ball in $\mathbb{R}^d$.

Proof. Let $B(0,R)$ be the open ball with the same volume as D. Suppose u is a Green potential $\int_D G(\cdot,z)d\mu(z)$ of a measure μ in D. Then (1) and Fubini's theorem yield

$$\begin{aligned}\int_D G(x,y)u(y)dy &= \int_D d\mu(z)\int_D G(x,y)G(y,z)dy\\ &\le A_0\int_D G(x,z)d\mu(z)\int_D(|x-y|^{2-d}+|y-z|^{2-d})dy\\ &\le 2A_0u(x)\int_{B(0,R)}|y|^{2-d}dy = A_0dV_dR^2u(x) = A_0dV_d^{1-2/d}|D|^{2/d}u(x).\end{aligned}$$

Thus (9) holds for a Green potential. Since every nonnegative superharmonic function is approximated from below by Green potentials, the monotone convergence theorem completes the proof. □

For an arbitrary planar domain D of finite area, Cranston and McConnell [14] proved (9) with A_0 bounded by the area of D up to an absolute multiplicative constant. See [12] for a simple proof and [1, 5, 2] for an analytic proof and some generalizations. Cranston and McConnell [14] provided also an example of bounded domain in $\mathbb{R}^3$, failing to satisfy (9). We shall modify their example to construct a bounded domain whose Martin boundary coincides with the Euclidean boundary and which fails to satisfy the Cranston-McConnell inequality (9). In view of Proposition 1, this domain also fails to satisfy the 3G inequality (1).

Construction. Let $R_n \downarrow 0$ and $N_n \uparrow \infty$ be a decreasing sequence of positive numbers and an increasing sequence of positive integers such that

(i) $R_{n+1} + \dfrac{R_{n+1}}{N_{n+1}} \le R_n - \dfrac{R_n}{N_n}$,

(ii) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{R_n}{N_n}\right)^2 N_n^{d-1} = \infty$.

For example, $R_n = \dfrac{1}{\sqrt{n}}$ and $N_n = 8n$ satisfy the above condition. In fact,

$$\begin{aligned}R_n - \frac{R_n}{N_n} - (R_{n+1} + \frac{R_{n+1}}{N_{n+1}}) &\ge \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} - \frac{1}{4n\sqrt{n}} = \frac{1}{\sqrt{n}\sqrt{n+1}(\sqrt{n}+\sqrt{n+1})} - \frac{1}{4n\sqrt{n}}\\ &\ge \frac{1}{2(n+1)\sqrt{n}} - \frac{1}{4n\sqrt{n}} = \frac{2n-(n+1)}{4n(n+1)\sqrt{n}} > 0;\end{aligned}$$

and

$$\left(\frac{R_n}{N_n}\right)^2 N_n^{d-1} = \left(\frac{1}{8n\sqrt{n}}\right)^2(8n)^{d-1} = 8^{d-3}n^{d-4} \ge 8^{d-3}n^{-1}.$$

Let $0 < \eta < 1/4$ be a constant depending only on the dimension such that we can place N_n^{d-1} many mutually disjoint open balls of radius $r_n = \eta R_n/N_n$ with centers on the sphere $S_n = \{x \in$

$\mathbb{R}^d : |x| = R_n\}$. Order these balls and call them B_n^k, $k = 1, \dots, N_n^{d-1}$. In view of (i), we may assume that the family of the doubles of B_n^k ($n = 1, \dots, \infty$ and $k = 1, \dots, N_n^{d-1}$) is mutually disjoint. Fix n and connect each ball B_k^n to the next B_{k+1}^n for $k = 1, \dots, N_n^{d-1} - 1$, in order, by a cylindrical tube lying in $B(0, R_n + 2r_n)$. Then connect the last ball B_n^k with $k = N_n^{d-1}$ to the first ball B_{n+1}^1 of the $(n+1)$-th level by a cylindrical tube lying in $B(0, R_n + 2r_n)$. Moreover, each tube intersects its ball in circular caps subtending solid angle $\varepsilon < \pi/6$ and the two caps in each ball (except the first) are antipodal. We may assume that the tubes are mutually disjoint and the connection is so smooth that the resultant domain D is locally Lipschitz apart from the origin. Hence, we observe that the Martin boundary of D is homeomorphic to the Euclidean boundary except for the origin. We shall show that there is a unique minimal function h corresponding to the origin.

Proposition 2. *Let D be as above. Then there is a unique minimal function h corresponding to the origin. Moreover,*

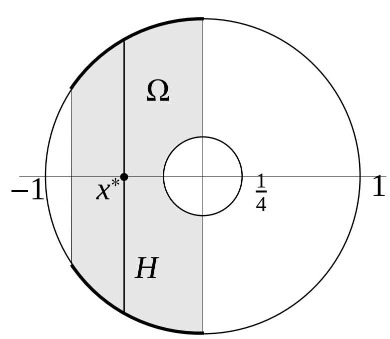
$$\int_D G(x, y)h(y)dy = \infty.$$

Hence, the Martin boundary of D coincides with the Euclidean boundary and yet the Cranston-McConnell inequality (9) *and the 3G inequality* (1) *fail to hold.*

We prepare the proof of Proposition 2 by stating the following boundary Harnack principle for a specific Lipschitz domain. Since we consider near a smooth boundary portion, the boundary Harnack principle can be proved easily. See Figure 1.

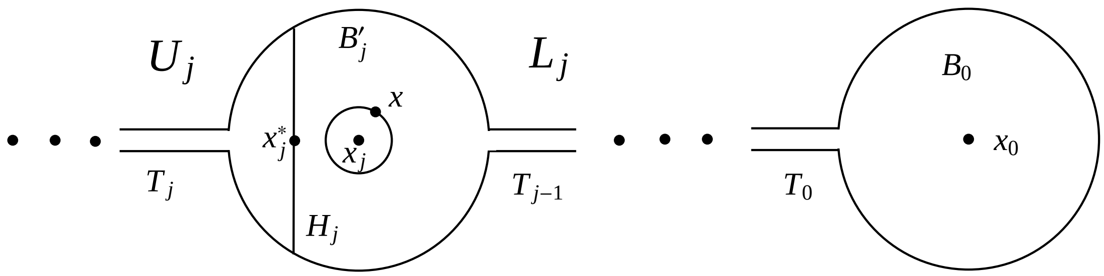
Lemma 2. *Let* $\Omega = \{x = (x_1, \dots, x_d) : \frac{1}{4} < |x| < 1, -\frac{\sqrt{3}}{2} < x_1 < 0\}$, $H = \{x \in \Omega : x_1 = -\frac{1}{2}\}$ *and* $x^* = (-\frac{1}{2}, 0, \dots, 0)$*. If u and v are positive harmonic functions on Ω such that $u = v = 0$ on* $\{x = (x_1, \dots, x_d) : |x| = 1, -\frac{\sqrt{3}}{2} < x_1 < 0\}$*, then*

$$\frac{u(x)}{u(x^*)} \approx \frac{v(x)}{v(x^*)} \approx \frac{\delta_\Omega(x)}{\delta_\Omega(x^*)} \quad \text{for } x \in H.$$



F 1. Boundary Harnack Principle for the shaded domain Ω.

Proof of Proposition 2. Let $B_0 = B(x_0, \rho_0)$, $B_1 = B(x_1, \rho_1)$,... be the enumeration of $\{B_n^k\}_{n,k}$ in order and let T_j be the tube connecting B_j and B_{j+1}. Our domain D looks like a long wiggling string of beads. Take $j \geq 1$. We may assume by rotation that B_j and T_j intersect in a circular cap with center at $(-\rho_j, 0, \dots, 0)+x_j$. Translate and dilate Ω in Lemma 2 so that x_j and $(-\rho_j, 0, \dots, 0)+x_j$ correspond to the origin and $(-1, 0, \dots, 0)$, respectively. Let H_j and x_j^* correspond to H and x^*, respectively. Observe that $B_j \setminus H_j$ consists of two connected components. By B_j' we denote the component containing x_j. Let $L_j = B_0 \cup T_0 \cup \cdots \cup T_{j-1} \cup B_j'$ and let $U_j = D \setminus (L_j \cup H_j)$. See Figure 2



F 2. Counter example to the Cranston-McConnell inequality: a long wiggling string of beads.

Fix x such that $|x - x_j| = \rho_j/4$. Apply Lemma 2 to $u = G(x, \cdot)$ and $v = G(x_0, \cdot)$. Then

$$\frac{G(x,y)}{G(x,x_j^*)} \approx \frac{G(x_0,y)}{G(x_0,x_j^*)} \quad \text{for } y \in H_j \text{ and hence for } y \in U_j$$

by the maximum principle. Since $G(x, x_j^*) \approx \rho_j^{2-d}$, it follows that

$$\frac{G(x,y)}{G(x_0,y)} \approx \frac{\rho_j^{2-d}}{G(x_0,x_j^*)} \quad \text{for } y \in U_j.$$

Let $K(x,y) = G(x,y)/G(x_0,y)$ for $x \in D$ and $y \in D \setminus \{x_0\}$. The Martin kernel is given as the limit of $K(x,y)$ when y tends to a boundary point. Let u and v be Martin kernels at 0 with respect to x_0. Then the above estimate implies

$$u(x) \approx v(x) \approx \frac{\rho_j^{2-d}}{G(x_0,x_j^*)} \quad \text{for } |x - x_j| = \rho_j/4 \text{ and hence for } |x - x_j| \leq \rho_j/4 \tag{10}$$

by the maximum principle. By the Harnack inequality

$$u(x_j^*) \approx v(x_j^*) \approx \frac{\rho_j^{2-d}}{G(x_0,x_j^*)},$$

so that the boundary Harnack principle (Lemma 2) gives a constant $A_3 > 1$ such that

$$A_3^{-1} u(x) \leq v(x) \leq A_3 u(x) \tag{11}$$

for $x \in H_j$ and hence for $x \in L_j$ by the maximum principle. Since j is arbitrary, we have (11) for all $x \in D$.

Now, a standard technique ([3, Theorem 3]) shows that there exists a unique minimal Martin kernel at 0. For the reader's convenience we give a proof. Let $\mathcal{H}_0$ be the family of all positive harmonic functions u on D vanishing on $\partial D \setminus \{0\}$, bounded on $D \setminus B(0, r)$ for each $r > 0$ and taking value $u(x_0) = 1$. Obviously, a Martin kernel at 0 belongs to $\mathcal{H}_0$. Since every $u \in \mathcal{H}_0$ can be represented as an integral over Martin kernels at 0, we see that (11) extends to $u, v \in \mathcal{H}_0$. Let

$$c = \sup_{\substack{u,v\in\mathcal{H}_0\\ x\in D}} \frac{u(x)}{v(x)}. \tag{12}$$

Then $1 \le c \le A_3^2 < \infty$ by (11). Let us show that $c = 1$. Suppose to the contrary $c > 1$. Take arbitrary $u,\ v \in \mathcal{H}_0$. Then $v_1 = (cv - u)/(c - 1) \in \mathcal{H}_0$, so that $u \le cv_1 = c(cv - u)/(c - 1)$ by (12). Hence $(2c - 1)u \le c^2 v$ on D, which would imply

$$c = \sup_{\substack{u,v\in\mathcal{H}_0\\ x\in D}} \frac{u(x)}{v(x)} \le \frac{c^2}{2c-1} < c,$$

a contradiction. Thus $c = 1$ and $\mathcal{H}_0$ is a singleton consisting of the Martin kernel $K(\cdot, 0)$ at 0. Moreover, the Martin kernel $K(\cdot, 0)$ is minimal since there is at least one minimal Martin kernel at 0.

Let $h = K(\cdot, 0)$ be the Martin kernel at 0. Then (10) and the Harnack inequality give

$$\int_{B(x_j,\rho_j/4)} G(x_0, y)h(y)dy \approx G(x_0, x_j^*)\frac{\rho_j^{2-d}}{G(x_0, x_j^*)}\rho_j^d \approx \rho_j^2.$$

In view of Construction (ii), we obtain

$$\int_D G(x_0, y)h(y)dy \ge \sum_{j=1}^{\infty} \int_{B(x_j,\rho_j/4)} G(x_0, y)h(y)dy = \infty.$$

By the Harnack inequality the above integral diverges for every $x \in D$ in place of x_0 as well. The proof is complete. ◻

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