Novel Approach to Switched Controller Design for Linear Continuous-Time Systems

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Abstract

In this paper, we study a novel approach to the design of an output feedback switched controller with an arbitrary switching algorithm for continuous-time invariant systems that is described by a novel plant model as a gain-scheduled plant using the multiple quadratic stability and quadratic stability approaches. In the proposed design procedure, there is no need to use the notion of the "dwell time". The obtained results are in the form of bilinear matrix inequalities (BMI). Numerical examples show that, in the proposed method, the design procedure is less conservative and gives more possibilities than that described in the papers published previously.

Keywords: Switched system, continuous-time system, output feedback, quadratic stability, multiple quadratic stability

1 Introduction

Switched systems have played an important role in the past decade. Motivation for studying switched systems comes from two facts:

- switched systems have numerous applications in the control of real plants, and
- in real control, there are dynamic systems that cannot be stabilized by any continuous static output/state feedback control law, but a stabilizing switching control scheme can be found.

Switched systems constitute an important class. The switched system consists of a continuous-time or discrete-time system and a switching law that specifies the switching between them. It has been pointed out [1, 2, 3, 4] that the stability of switched systems plays an important role in the analysis and design of switched controllers. There are at least two approaches to the stability analysis of switched systems: The quadratic stability approach with a common Lyapunov function gives the stability under an arbitrary switching law and Multiple Lyapunov functions, which is less conservative. A huge number of references can be found on the switched control

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of linear discrete-time invariant systems, but, in the field of linear continuous-time invariant systems, the number of references is rather small. Representatives are the following references [5, 6, 7, 8, 9]. In the first paper, the authors introduce, into the switched controller design procedure, the notion of dwell time T_d (minimal time interval between switching). In the stability analysis condition of switched systems, the dwell time is in the term $e^{A_c T_d}$ (A_c-closed-loop matrix (4)). The proposed design procedure for stability analysis and switching controller design for the real switching time interval $T \geq T_d$ becomes rather conservative. The above "dwell time term" complicates the switched controller design procedure for continuous-time systems. In [6], sufficient conditions are given for the stability of linear systems with a dwell time and with polytopic type parameter uncertainty. A Lyapunov function, in quadratic form for each mode, which is non-increasing at the switching instants, is assigned to each mode. During the dwell time this function varies piecewise linearly in time after switching occurs. The proposed method was applied to stabilization via the state feedback for both nominal and uncertain cases. Since, within the dwell time the Lyapunov function varies piecewise linearly and the real switching time interval $T > T_d$, the switching controller design procedure becomes rather conservative. In [7], the stability analysis problem for a class of switched positive linear systems with average dwell time switching was investigated. [8] investigated the stability of a class of switched linear systems and proposes a number of new results on the stability analysis. A novel analysis method has been developed via the 2-norm technique, and several stability results have been obtained based on the new analysis method. It is shown that the main results obtained in this paper not only guarantee the stability of the systems under arbitrary switching but also provide an algorithm to find the minimum dwell time with which switches make the switched systems stable. Dwell-time switching is a logic for orchestrating the switching between controllers in order to control a process with a highly uncertain model [9]. The idea of dwell-time switching is to use a parameter tuner, which switches controller parameter values rather than continuously adjusting them. Such a switched system consists of two interconnected subsystems, namely a "continuous part" and a "dwell-time switching logic". Together these two subsystems constitute a parameter tuner, which is a bona fide hybrid (switched) system. The above system generates "switching logic" to make estimation error (performance) "small" in some sense [9]. Note that implementation of dwell-time switching logic requires the capability of minimizing some performance over the set of controllers in real time. In this paper, the proposed switched controller design procedure for continuous-time systems has none of the aforementioned drawbacks due to dwell time consideration. An overview of switched systems can be found in [1, 3, 10].
In this paper, new quadratic stability and multi quadratic stability conditions of closed-loop switched systems for arbitrarily switching [3] are given using a new model of switched continuous-time linear systems. In the proposed approach to switching controller design for continuous-time systems, there is no need to use the approach of the "dwell time" [5, 6]. The proposed switched controller design procedure can be expanded to the case of robust switched controller design [12] with multi parameter dependent quadratic stability.

The organization of the paper is as follows. Section 2 includes problem formulation of the switched controller design using the novel proposed model, and some preliminary results are given. In Section 3, sufficient stability conditions in the form of BMI for the case of quadratic and multi quadratic approach are given, and, in Section 4, the obtained results are illustrated on some examples.

Hereafter, the following notational conditions will be adopted. Given a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$, the inequality $P > 0$ denotes matrix positive definiteness. Symbol $*$ denotes a block that is transposed and complex conjugated to the respective symmetrically placed one. I denotes the identity matrix of corresponding dimensions.

2 Preliminaries and problem formulation

Let us consider a class of linear continuous-time invariant switched systems:

$$\begin{aligned}\sum\sigma:\quad \dot{x}(t) &= A(\theta)x(t)+B(\theta)u(t)\\ y(t) &= Cx(t)\\ x(0) &= x_0\end{aligned}\tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the control vector, $y(t)\in\mathbb{R}^l$ is the output vector of the system to be controlled, and $\sigma\in S=\{1,2,\ldots,p\}$ is an arbitrarily switching algorithm. The arbitrary switching algorithm σ is a piecewise constant, right continuous function that specifies at each time the index of the active closed-loop system [3, 11], p is the number of switched modes of linear systems, and

$$A(\theta)=A_0+\sum_{i=1}^{p}A_i\theta_i\ ,\quad B(\theta)=B_0+\sum_{i=1}^{p}B_i\theta_i\tag{2}$$

where $\sum_{i=1}^{p}\theta_i=1,\ \theta_i\in\langle 0,1\rangle\in\Omega_\theta,\ i=1,2,\ldots,p$ are switching parameters. For calculation of all the above matrices in (1) see the example. Note that, for switching systems, the stable steady state points of switching parameters θ_i, $i=1,2,\ldots,p$ are equal to 0 or 1. If the switching parameter $\theta_i,\ i=1,2,\ldots,p$ differs from 0 or 1, it is moving to one of the stable points with the rate of θ_i change $\dot{\theta}_i$, $i=1,2,\ldots,p$. There are two possibilities for switching parameters:

- the rates of change of switching parameters are infinite, that is $\theta_i=1$ holds for the i-th mode and $\theta_j=0$ for $j=1,2,\ldots,p$, $j\neq i$ (quadratic stability approach), or
- the rates of change of switching parameters are finite, assuming that the system switched from i to j mode. For this case one has $\theta_i+\theta_j=1,\ \dot{\theta}_i+\dot{\theta}_j=0$ and $\theta_k=0,\ k\neq i,j$ (multiple quadratic stability approach).

For both cases, $\theta_i, i=1,2,\ldots,p$ are known and the number of system mode or the number of vertices is equal to "p". Note that, number of vertices for a polytopic system is 2^p.
The switched output feedback control law is considered in the form:

$$u(t)=F(\theta)y(t)=F(\theta)Cx(t)\tag{3}$$

where

$$F(\theta)=F_0+\sum_{i=1}^{p}F_i\theta_i$$

The structure of matrices $F_i,\ i=0,1,\ldots,p$ can be prescribed, for the decentralized control structure F_i is a block diagonal matrix and so on. The closed-loop system is obtained from Eqs. (1) and (3):

$$\dot{x}(t)=\left(A(\theta)+B(\theta)F(\theta)C\right)x(t)=A_c(\theta)x(t)\tag{4}$$

To access the system performance, we consider a standard positive definite quadratic cost function with respect to state x and control u:

$$J=\int_0^\infty\left(x(t)^T(t)Qx(t)+u^T(t)Ru(t)\right)dt=\int_0^\infty J(t)dt\tag{5}$$

Let us recall some results on the optimal control of time varying systems.

Lemma 1 *[13] Let there exist a scalar positive definite function $V(x,t)$ such that $\lim_{t\to\infty} V(x,t) = 0$, which satisfies*

$$\min_{u(t)\in\Omega_u} \left\{ \frac{\partial V}{\partial x} A_c(\theta) + \frac{\partial V}{\partial t} + J(t) \right\} = 0 \tag{6}$$

The control algorithm $u(t) = u^*(x,t) \in \Omega_u$ obtained from (6) ensures closed-loop stability and the optimal value of the cost function (5) as $J^* = J(x_0, t_0) = V(x(0), t_0)$.

Equation (6) is known as the Bellman-Lyapunov equation, and function $V(x,t)$, which satisfies (6) is the Lyapunov function. For a particular structure of the Lyapunov function, the optimal control algorithm reduces from "if and only if" to "if", and for switched systems, robust control and so on to a guaranteed cost.

Definition 1 *Consider System (1) and Controller (3). If there exist a control law $u^*(x,t)$ and a positive scalar J^* such that the respective closed-loop system (4) is stable and the value of the closed-loop cost function (5) satisfies $J \leq J^*$, then J^* is said to be the guaranteed cost and $u^*(x,t)$ is said to be the guaranteed cost control law for System (4).*

Definition 2 *[4] The switched linear closed-loop system (4) is said to be quadratically stabilizable via output feedback if there exists a Lyapunov function of the form $V = x(t)^T P x(t)$, $P > 0$, a positive number $\epsilon > 0$, and an arbitrary switching rule σ such that*

$$\frac{dV}{dt} \leq -\epsilon x(t)^T x(t) \tag{7}$$

Definition 3 *The switched linear closed-loop system (4) is said to be multiple quadratically stabilizable via output feedback if there exists a Lyapunov function of the form $V(\theta) = x(t)^T P(\theta) x(t)$, $P(\theta) > 0$, a positive number $\epsilon > 0$, and an arbitrary switching rule σ such that*

$$\frac{dV(\theta)}{dt} \leq -\epsilon x(t)^T x(t) \tag{8}$$

Lemma 2 *[14], [3] Consider a closed-loop system (4) with a control algorithm (3). Control algorithm (3) will be a cost guaranteed algorithm if there exists a positive scalar ε such that, for the time derivative of the positive definite Lyapunov function, the following condition holds*

$$B_e = \max_{u(t)\in\Omega_u} \left\{ \frac{\partial V}{\partial x} A_c(\theta) + \frac{\partial V}{\partial t} + J(t) \right\} \leq -\varepsilon x(t)^T x(t) \tag{9}$$

when $\varepsilon \to 0$.

The following lemma plays an important role in the next development [15].

Lemma 3 *Consider a scalar quadratic function of θ_i*

$$f(\theta) = a_0 + \sum_{i=1}^{p} a_i\theta_i + \sum_{i=1}^{p}\sum_{j=i}^{p} b_{ij}\theta_i\theta_j \tag{10}$$

and assume that $f(\theta)$ is multiconvex, that is

$$\frac{\partial^2 f(\theta)}{\partial \theta_i^2} \geq 0, \quad i = 1, 2, \ldots, p \tag{11}$$

Then $f(\theta)$ is negative in the hyper rectangle $\theta_i \in \langle 0, 1\rangle$, $i = 1, 2, \ldots, p$, if and only if it takes negative values at the corners, that is if and only if $f(\theta) < 0$ for $\theta_i = 0$ or $\theta_i = 1$, $i = 1, 2, \ldots, p$.

3 Switched controller design

In this paragraph, two methods of switched controller design for linear continuous-time invariant systems are presented. The first method is connected with the notion of quadratic stability with respect to θ. We will assume that the rate of θ change is infinite. Most of the literature on switched controller design concentrates on the case where switching can occur immediately, making the rate of change of the switching signal infinite (ideal switching). In some real cases, the rate of change of the switching signal is finite (non ideal switching). This assumption will be used in the second approach to obtain the switched controller design procedure. In the references on the design of switched controllers for continuous-time systems, there are no solutions for the case of a finite rate of change of switching signal. In this paper, we set up the case of the finite rate of change of the switching signal using the multi quadratic stability approach to the closed-loop switched systems. In the proposed approach to the switching controller design for continuous-time systems, there is no need to use the approach of the "dwell time" [5, 6], which complicates the switched controller design procedure considerably.

3.1 Quadratic stability approach

The quadratic stability approach to the design of the switched controller for continuous-time and discrete-time systems is well established. Because of the new model for continuous-time systems (1), in this part of the paper, the proposed method is connected with the notion of quadratic stability with respect to θ. We will assume that the rate of θ change is infinite (ideal switching). From (9), the following lemma is obtained.

Lemma 4 *Closed-loop $A_c(\theta)$ is quadratically stable with guaranteed cost if there exists a positive definite Lyapunov matrix P such that the following inequality holds:*

$$A_c^T(\theta)P + PA_c(\theta) + Q + C^T F(\theta)^T RF(\theta)C \leq 0 \tag{12}$$

Substituting $A_c(\theta)$ (4) to (12) and using *Lemmas 2* and *3*, the following quadratic stability conditions are obtained.

Theorem 1 *Closed-loop system* (4) *is quadratically stable with guaranteed cost if there exists a Lyapunov matrix $P > 0$ and matrices $Q \geq 0$, $R > 0$ such that, for an arbitrarily switching rule σ, the following matrix inequalities hold:*

a)

$$\begin{aligned}(A_0 + B_0F_0C)^T P + P(A_0 + B_0F_0C)&\\ +Q + C^T F_0^T RF_0C = M_0 \leq 0&\end{aligned} \tag{13}$$

b)

$$M_0 + M_i + M_{ii} \leq 0, \quad i = 1, 2, \ldots, p \tag{14}$$

where

$$\begin{aligned}M_i = {}&(A_i + B_0F_iC + B_iF_0C)^T P\\ &+ P(A_i + B_0F_iC + B_iF_0C)\\ &+ C^T F_i^T RF_0C + C^T F_0^T RF_iC\end{aligned} \tag{15}$$

and

$$M_{ii} = (B_iF_iC)^T P + PB_iF_iC + C^T F_i^T RF_iC \tag{16}$$

c)

$$M_{ii} \geq 0, \quad for\, i = 1, 2, \ldots, p \tag{17}$$

provided that $\theta_i = 0$ or $\theta_i = 1$

Proof 1 *The proof is based on* (9)*, Lemma 2, and Lemma 3 and goes along the same line as the proof of Theorem 2.*

Equation (14) implies that, for stability analysis of the switched system, we have obtained linear matrix inequalities (LMI) but, for the switched controller design, we have obtained bilinear matrix inequalities (BMI).

3.2 Multiple Lyapunov function approach

In this subsection, we will assume that, for some realistic cases, the switching signal rate of change is finite (non ideal switching). As we mentioned above, in the references, there is no solution for these cases. The obtained results for a finite rate of change of switching signal open the new possibilities for the designer (practical realization) and for the theory of the switched controller design procedure, *i.e.*, the design of a switched robust controller, design of a switched controller for some type of nonlinear systems and so on. Specifically,

- for switching parameters it holds that $\sum_{i=1}^{p} \theta_i = 1$, $\theta_i \in \langle 0, 1 \rangle \in \Omega_\theta$.
- the rate of switching parameter variation $\dot{\theta}_i$ is clear at all times and satisfies known boundaries

$$\dot{\theta}_i \in \Omega_t = \left\{ \dot{\theta}_i \in \left\langle \underline{\dot{\theta}}_i, \bar{\dot{\theta}}_i \right\rangle, i = 1, 2, \ldots, p \right\} \tag{18}$$

$$\sum_{i=1}^{p} \dot{\theta}_i = 0.$$

The main results for the switched controller design using the multiple quadratic stability approach are given in the next theorem.

Theorem 2 *Closed-loop system* (4) *is multiple quadratically stable with guaranteed cost if there exist $p+1$ symmetric matrices $P_0, P_1, \ldots, P_p$ such that $P_0 + \sum_{i=1}^{p} P_i\theta_i > 0$ is positive definite for all switching parameters $\theta_i \in \Omega_\theta$, $\dot{\theta}_i \in \Omega_t$ switched controller parameters $F(\theta)$ satisfying*

$$\begin{aligned} W(\theta) &\leq 0 \\ W_{ii} &\geq 0, \quad i = 1, 2, \ldots, p \end{aligned} \tag{19}$$

for an arbitrary switching rule σ, where

$$W(\theta) = W_0 + \sum_{i=1}^{p} W_i\theta_i + \sum_{i=1}^{p}\sum_{j=1}^{p} W_{ij}\theta_i\theta_j$$

$$W_0 = \begin{bmatrix} w_{110} & w_{120} \\ w_{120}^T & w_{220} \end{bmatrix}; W_i = \begin{bmatrix} w_{11i} & w_{12i} \\ w_{12i}^T & w_{22i} \end{bmatrix}$$

$$W_{ij} = \begin{bmatrix} w_{11ij} & w_{12ij} \\ w_{12ij}^T & w_{22ij} \end{bmatrix}$$

$$w_{110} = N_1 + N_1^T, \quad w_{11i} = w_{11ij} = 0$$
$$w_{120} = P_0 + N_2 - N_1^T A_{c0}, \quad w_{12i} = P_i + N_2 - N_1^T A_{ci}$$
$$w_{12ij} = N_2 - N_1^T A_{cij}$$
$$w_{220} = \sum_{k=1}^{p} P_k \bar{\dot{\theta}}_k - N_2^T A_{c0} - A_{c0}^T N_2 + Q + C^T F_0^T R F_0 C$$
$$w_{22i} = -N_2^T A_{ci} - A_{ci}^T N_2 + C^T F_0^T R F_i C + C^T F_i^T R F_0 C$$
$$w_{22ij} = -N_2^T A_{cij} - A_{cij}^T N_2 + C^T F_i^T R F_j C + C^T F_j^T R F_i C$$
$$A_{c0} = A_0 + B_0 F_0 C, \quad A_{ci} = A_i + B_0 F_i C + B_i F_0 C$$
$$A_{cij} = B_i F_j C$$

Proof 2 *For the first derivative of the Lyapunov function, $V(\theta) = x(t)^T P(\theta) x(t)$, one obtains:*

$$\frac{dV(\theta)}{dt} = \dot{x}(t)^T P(\theta) x(t) + x(t)^T P(\dot{\theta}) x(t) + x(t)^T P(\theta) \dot{x}(t) =$$
$$[\dot{x}(t)^T \quad x(t)^T] \begin{bmatrix} 0 & P(\theta) \\ P(\theta) & P(\dot{\theta}) \end{bmatrix} [\dot{x}(t)^T \quad x(t)^T]^T$$

where

$$P(\dot{\theta}) = \sum_{i=1}^{p} P_i \dot{\theta}_i \leq \sum_{i=1}^{p} P_i \bar{\dot{\theta}}_i$$

assuming $P_i > 0$, $i = 1, 2 \ldots, p$. For the closed-loop system (4), one can write ($x(t) = x$)

$$[\dot{x}^T N_1^T + x^T N_2^T][\dot{x} - A_c(\theta)x] + ([\dot{x}^T N_1^T + x^T N_2^T][\dot{x} - A_c(\theta)x])^T = 0 \tag{20}$$

where $N_1, N_2 \in \mathbb{R}^{n \times n}$ are auxiliary matrices. Summarizing the above two equations, for the first derivative of the Lyapunov function it holds that

$$\frac{dV(\theta)}{dt} = [\dot{x}^T \quad x^T] \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^T & u_{22} \end{bmatrix} [\dot{x}^T \quad x^T]^T \tag{21}$$

where

$$u_{11} = N_1^T + N_1, \quad u_{12} = P(\theta) + N_2 - N_1^T A_c(\theta),$$
$$u_{22} = P(\dot{\theta}) - N_2^T A_c(\theta) - A_c(\theta)^T N_2$$

Substituting the control algorithm (3) into the system performance (5) for $J(t)$, one obtains:

$$J(t) = x^T (Q + C^T F(\theta)^T R F(\theta) C) x \tag{22}$$

Rewriting the Bellman-Lyapunov equation (9) to the form

$$B_e = \left\{ \frac{dV(\theta)}{dt} + J(t) \right\} \leq 0 \tag{23}$$

or using Equations (21) and (22) for B_e it holds that

$$B_e = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}^T \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^T & u_{22} + Q + C^T F(\theta)^T R F(\theta) C \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \tag{24}$$

Inequalities (23), (9) hold if the inner matrix of (24) is negative definite (semidefinite), that is

$$W(\theta) = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^T & u_{22} + Q + C^T F(\theta)^T RF(\theta)C \end{bmatrix} \leq 0 \tag{25}$$

Due to the quadratic function of $W(\theta)$ with respect to θ, Lemma 2 gives the stability conditions in the form of (19), which proves the sufficient stability conditions of Theorem 2.

For the switched controller design procedure, the last term of $W(\theta)$ needs to be symmetrized as follows:

$$\sum_{i=1}^{p}\sum_{j=1}^{p} W_{ij}\theta_i\theta_j = \sum_{i=1}^{p} W_{ii}\theta_i^2 + \tag{26}$$

$$\sum_{i=1}^{p}\sum_{j>i}^{p}(W_{ij} + W_{ji})\theta_i\theta_j$$

where

$$W_{ii} = \begin{bmatrix} 0 & -N_1^T A_{cii} \\ A_{cii}^T N_1 & -N_2^T A_{cii} - A_{cii}^T N_2 + C^T F_i^T RF_i C \end{bmatrix} \tag{27}$$

Remarks:

1. Note that the dwell time determines the minimal time interval between switching. If the real switching time interval is greater, $T \geq T_d$, the switched controller design procedure for arbitrarily switching proposed in [5, 6] becomes more complicated and conservative. Previous considerations imply that, for an arbitrarily switching algorithm, the number of active switching plant modes generates the value of the switching variable σ to determine which controller will be active. Switched controller parameters calculations are made off-line by minimizing a given performance. Dwell-time switching s another switching algorithm, which in real time (on-line) "t" requires an algorithm capable of minimizing some performance like output estimation errors over the switching controllers. As a results of minimization the dwell-time switching logic generates the switching variable σ to determine which controller will be active in time "t" [9].

2. To obtain a feasible solution, in the switched controller design procedure proposed in this paper, one can use a free and open source BMI solver, PenLab.

4 Examples

The first example is taken from [16]. This plant has been constructed to include the technical challenge of the control of models in practice, such as models like the air induction system of a turbocharged diesel engine. After a small simplification, one obtains a simple linear time-varying plant with parameter varying coefficients:

$$\begin{aligned} \dot{x}(t) &= a(\alpha)x(t) + b(\alpha)u(t) \\ y(t) &= x(t) \end{aligned} \tag{28}$$

where $\alpha(t) \in \mathbb{R}$ is an exogenous signal that changes the parameters of the plant as follows

$$\begin{aligned} a(\alpha) &= -6 - \frac{2}{\pi}\arctan\left(\frac{\alpha}{20}\right) \\ b(\alpha) &= \frac{1}{2} + \frac{5}{\pi}\arctan\left(\frac{\alpha}{20}\right) \end{aligned} \tag{29}$$

Let the problem be to design a switched PI controller that will guarantee the closed-loop stability and guaranteed cost for System (28), where $\alpha(t)$ is changing in steps between 0, 30, and 100. In these working points, the calculated transfer functions are as follows:

$$\begin{array}{rclrcl} G_{s1}|_{\alpha=0} & = & \frac{0.5}{s+6}, & G_{s2}|_{\alpha=30} & = & \frac{2.064}{s+6.626} \\ G_{s3}|_{\alpha=100} & = & \frac{2.686}{s+6.874}, & & & \end{array} \tag{30}$$

We transform the above transfer functions into the time domain to obtain a gain-scheduling model in the form (1). Matrices A_i, B_i, $i = 0, 1, 2, \dots, p$ for the case $i = 0$ are calculated as middle values of all corresponding matrices, and, for $i = 1, 2, \dots, p$, one can use the standard approach. The obtained model is extended by one state variable for PI controller design. The extended model is given as follows:

$$\begin{array}{rclrcl} A_0 & = & \begin{bmatrix} -6.5 & 0 \\ 1 & 0 \end{bmatrix}, & A_1 & = & \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \\ A_2 & = & \begin{bmatrix} -0.126 & 0 \\ 0 & 0 \end{bmatrix}, & A_3 & = & \begin{bmatrix} -0.374 & 0 \\ 0 & 0 \end{bmatrix}, \\ B_0 & = & \begin{bmatrix} 1.75 \\ 0 \end{bmatrix}, & B_1 & = & \begin{bmatrix} -1.25 \\ 0 \end{bmatrix}, \\ B_2 & = & \begin{bmatrix} 0.314 \\ 0 \end{bmatrix}, & B_3 & = & \begin{bmatrix} 0.936 \\ 0 \end{bmatrix}, \\ C & = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & D & = & 0 \end{array}$$

Using *Theorems 1* and *2* with weighting matrices $Q = qI$, $q = 1 \times 10^{-6}$, $R = rI$, $r = 1$ and $\rho_U = 1 \times 10^7$, $\rho_L = 1 \times 10^{-5}$ which are the upper/lower constraint of the Lyapunov matrix ($\rho_L I < P(\theta) < \rho_U I$), $\bar{\dot{\theta}}_i = 1000$ $[1/s]$ is the maximal rate of switched parameter change for multi-quadratic stability approach, and with $\theta \in \langle 0, 1 \rangle$, we obtain switched controllers in the form (3) where, for the case of quadratic stability (*Theorem 1*), one has

$$\begin{array}{rcl} F_0 & = & [\ -0.8213 \quad -5.2172\ ] = [-0.8213 - \frac{-5.2172}{s}] \\ F_1 & = & [\ -0.4868 \quad -3.0799\ ] \\ F_2 & = & [\ -2.6625 \quad -16.8453\ ] \\ F_3 & = & [\ 0.8072 \quad 5.1071\ ] \end{array} \tag{31}$$

and for the case of multi-quadratic stability approach (*Theorem 2*), one has

$$\begin{array}{rcl} F_0 & = & [\ -0.3412 \quad -10.9588\ ] \\ F_1 & = & [\ -0.0475 \quad -0.3355\ ] \times 10^{-8} \\ F_2 & = & [\ 0.0186 \quad 0.1315\ ] \times 10^{-7} \\ F_3 & = & [\ 0.0637 \quad 0.4492\ ] \times 10^{-8} \end{array} \tag{32}$$

In simulations, α will be switched between values 0, 30, and 100, and a particular arbitrarily switching algorithm in simulations is shown in Figs. 3 and 5. The switched parameters are calculated from signal α and switched with a maximal rate of change $\bar{\dot{\theta}}_i$. Simulation results (Figs. 1, 2 and 3) confirm that *Theorems 1* and *2* hold; thus, the closed-loop switched system is stable for a prescribed rate of switching signal change.

Other switched controllers are designed using *Theorems 1* and *2* with weighting matrices $Q = qI$, $q = 10$, $R = rI$, $r = 1$, $\rho_U = 1 \times 10^7$, $\rho_L = 1 \times 10^{-5}$, $\bar{\dot{\theta}}_i = 2000$ $[1/s]$ as the maximal rate of switched parameters change for multi-quadratic stability approach and with $\theta \in \langle 0, 1 \rangle$.

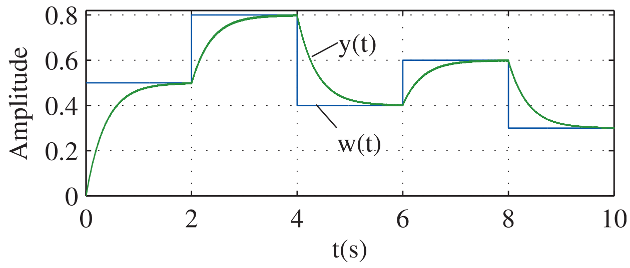

Figure 1: Simulation results $w(t)$, $y(t)$ with switched controller (31) – QS

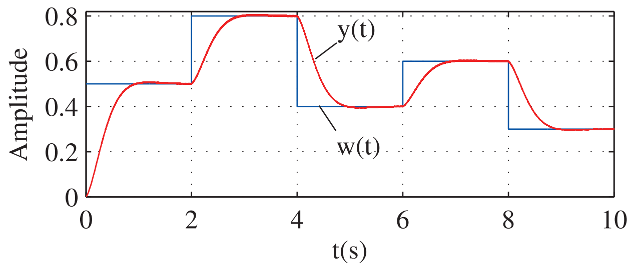

Figure 2: Simulation results $w(t)$, $y(t)$ with switched controller (32) – MPQS

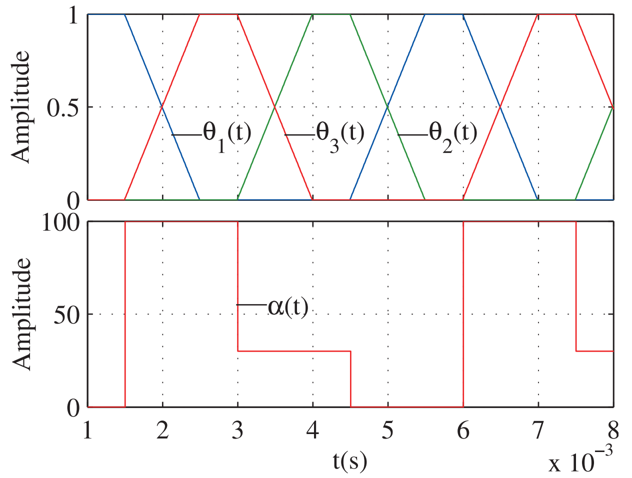

Figure 3: Calculated switched parameters $\theta(t)$ and the switching signal $\alpha(t)$ for the case of $\bar{\dot{\theta}}_i = 1000$ $[1/s]$

The obtained controllers are in the form (3), where, for the case of quadratic stability (*Theorem 1*), one has

$$BMI\ solver\ failed.$$

and for the case of multi-quadratic stability approach (*Theorem 2*), one has

$$\begin{array}{lcl} F_0 & = & [\ \ -540.8653 \quad -1.3937\ \ ] \\ F_1 & = & [\ \ -0.2596 \quad -0.0003\ \ ] \times 10^{-10} \\ F_2 & = & [\ \ 0.1033 \quad 0.0001\ \ ] \times 10^{-9} \\ F_3 & = & [\ \ 0.3467 \quad 0.0005\ \ ] \times 10^{-10} \end{array} \tag{33}$$

For this case, in simulations, α will be switched between the same values 0, 30, and 100, as shown in Fig. 5. The switched parameters are calculated from this signal and switched with maximal rate of change $\bar{\dot{\theta}}_i = 2000$ $[1/s]$. Simulation results (Figs. 4, 5) confirm that the multi-quadratic stability approach is less conservative than the quadratic stability approach. This example implies that, for higher values of weighting matrix $Q = qI$, $q > 0.1$ (higher weight for quality) quadratic stability with BMI solver fails but, with the multi-quadratic stability approach, we can obtain the controller up to $q = 15$.

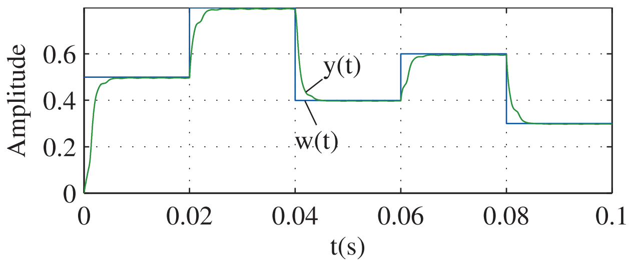

Figure 4: Simulation results $w(t)$, $y(t)$ with switched controller (33) – MPQS

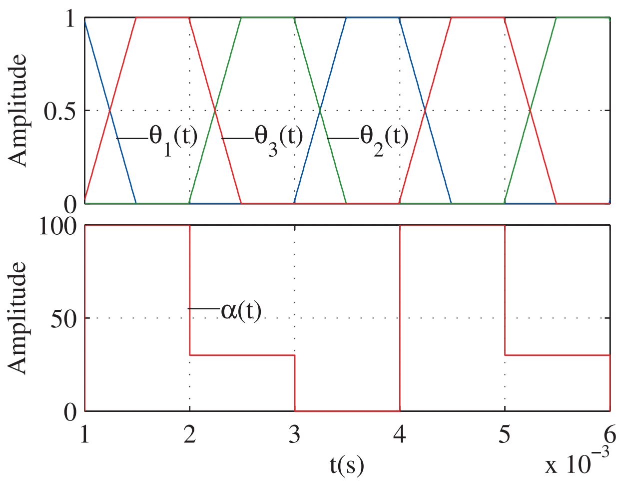

Figure 5: Calculated switching parameters $\theta(t)$ and the switching signal $\alpha(t)$ for the case of $\bar{\dot{\theta}}_i = 2000$ $[1/s]$

The second example is borrowed from [17]. Consider a simplified manual transmission model:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= [-a(x_2)/v + u]/(1 + v) \end{aligned} \tag{34}$$

where x_1 is the ground speed, x_2 is the acceleration, $u \in \langle 0, 1\rangle$ is the throttle position, and $v \in \{1, 2, 3, 4\}$ is the gear shift position. Function $a(.)$ is positive for a positive argument. Model (34) can be transformed into this form:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \frac{-a}{v+v^2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{1+v} \end{bmatrix}}_{B} u \tag{35}$$

Substituting $a = 5$ and $v = [1, 2, 3, 4]$ we can transform (35) into the form (1):

$$\begin{aligned}
A_0 &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, & A_1 &= \begin{bmatrix} 0 & 0 \\ 0 & -1.5 \end{bmatrix} \\
A_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0.1667 \end{bmatrix}, & A_3 &= \begin{bmatrix} 0 & 0 \\ 0 & 0.5833 \end{bmatrix} \\
A_4 &= \begin{bmatrix} 0 & 0 \\ 0 & 0.75 \end{bmatrix}, & B_0 &= \begin{bmatrix} 0 \\ 0.3208 \end{bmatrix} \\
B_1 &= \begin{bmatrix} 0 \\ 0.1792 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0 \\ 0.0125 \end{bmatrix} \\
B_3 &= \begin{bmatrix} 0 \\ -0.0708 \end{bmatrix}, & B_4 &= \begin{bmatrix} 0 \\ -0.1208 \end{bmatrix} \\
C &= \begin{bmatrix} 1 & 0 \\ 0 & 1.75 \end{bmatrix}
\end{aligned}$$

Using *Theorems 1* and *2* with weighting matrices $Q = qI$, $q = 0.1$, $R = rI$, $r = 208$ and $\rho_U = 1 \times 10^5$, $\rho_L = 1 \times 10^{-5}$, $\bar{\dot{\theta}}_i = 1000$ $[1/s]$, we obtain a switched controller in the form (3) where, for the case of quadratic stability (*Theorem 1*), one has

$$BMI\ solver\ failed. \tag{36}$$

and, for the case of multi-quadratic stability approach (*Theorem 2*), one has

$$\begin{aligned}
F_0 &= [\ \ 24.0495 \quad 76.9511\ ] \\
F_1 &= [\ -25.1832 \quad -80.2151\ ] \\
F_2 &= [\ -24.8145 \quad -79.0407\ ] \\
F_3 &= [\ -24.6836 \quad -78.6237\ ] \\
F_4 &= [\ -24.5773 \quad -78.2853\ ]
\end{aligned} \tag{37}$$

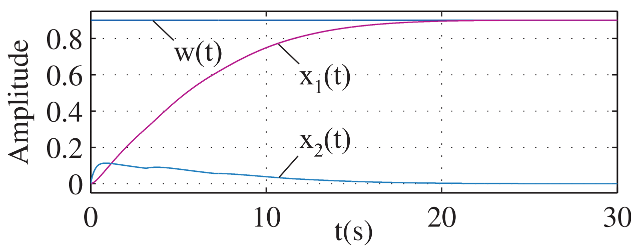

Figure 6: Simulation results $w(t)$, $y(t)$ with switched controller (37) – MPQS

In the simulations, we switched the gear shift as follows $v = 1$ if $x_1 \in \langle 0, 0.3\rangle$, $v = 2$ if $x_1 \in (0.3, 0.6\rangle$, $v = 3$ if $x_1 \in (0.6, 0.8\rangle$, and $v = 4$ if $x_1 \in (0.8, \infty)$, and the switching rate of v is

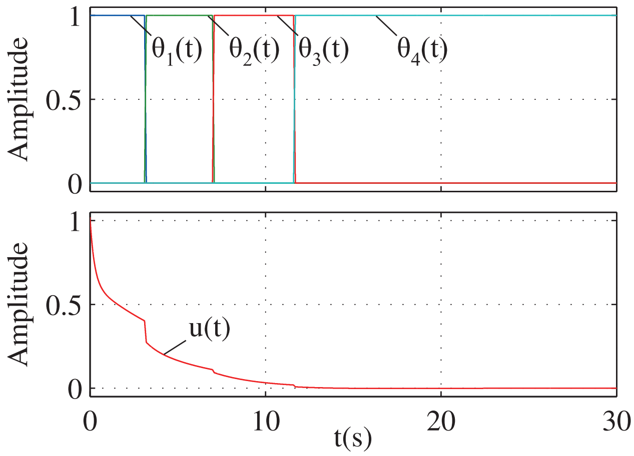

Figure 7: Calculated switching parameters $\theta(t)$ and the controller output with switched controller (37) – MPQS

established with $\bar{\dot{\theta}}_i = 10$ $[1/s]$. From the simulation results (Figs. 6 and 7), it follows that the theorems hold, that the multi-parameter quadratic stability approach is less conservative than the quadratic stability, and that, with the weighting matrices, we can impact the performance quality and tune the system to the desired conditions.

For the third example, we know that control systems over data networks are commonly referred to as networked control systems (NCSs). For the NCSs, the sampled data and controller signals are transmitted through a network. As a result, this leads to a network-induced delay in a networked control closed-loop system. The existence of such a kind of delay in a network-based control loop can induce instability or poor performance of control systems. Assume that a linear system with transfer function $G(s)$ is integrated into NCSs, which inevitably leads to a change of the plant transfer function as $G(s)e^{-T_d s}$, where T_d is a variable plant time delay. The value of T_d depends on the load of the communication network. Assume that, for four communication network loads, one can define four middle values of time delays T_{di}, $i = 1, 2, 3, 4$.

For PI switched controller design and simulation, we will use a laboratory model of a DC-motor, which is one of the real processes built for control education and research at our institute.

We have identified the DC motor system, and the following transfer function has been obtained

$$Sys = \frac{0.0627s + 1.281}{2.081s^2 + 2.506s + 1} \tag{38}$$

For the defined 4 middle values of the induced time delays $T_d = [0.1,\ 0.2,\ 0.3,\ 0.4]\ s$ and using the first order Pade approximation, we computed 4 plant transfer functions, which are transformed to the state space. The obtained 4 plant models are extended with one state for the switched PI controller design. Finally, one obtains the plant models in the form (1):

$$
\begin{aligned}
A_0 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5.0006 & -13.0218 & -11.6208 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
A_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4.6005 & -11.5382 & -9.5832 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
A_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2001 & 0.5018 & 0.4168 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
A_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.8001 & 4.5147 & 3.7498 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
A_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2.6003 & 6.5218 & 5.4168 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
B_0 &= \begin{bmatrix} -0.0308 \\ 0.0498 \\ 4.1652 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.0002 \\ 0.5833 \\ -4.5362 \\ 0 \end{bmatrix} \\
B_2 &= \begin{bmatrix} -0.0002 \\ -0.0261 \\ 2.1004 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.0008 \\ -0.2288 \\ 1.5999 \\ 0 \end{bmatrix} \\
B_4 &= \begin{bmatrix} -0.0003 \\ -0.3284 \\ 0.8360 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

For switched controller design, *Theorem 2* with weighting matrices $Q = qI$, $q = 0.01$, $R = rI$, $r = 2$ and $\rho_U = 10$, $\rho_L = 1 \times 10^{-5}$, $\bar{\dot{\theta}}_i = 10$ $[1/s]$ will be used. The obtained switched PI controller is the form (3)
The case of multi-quadratic stability approach (Theorem 2)

$$
\begin{aligned}
F_0 &= [\ -0.3296 \quad -0.1352\ ] \\
F_1 &= [\ -0.4460 \quad -0.1981\ ] \\
F_2 &= [\ -0.0589 \quad -0.0293\ ] \\
F_3 &= [\ -0.0217 \quad -0.0106\ ] \\
F_4 &= [\ 0.1439 \quad 0.0908\ ] \times 10^{-10}
\end{aligned}
\tag{39}
$$

Simulation results (Figs. 8, 9, and 10) confirm that, *Theorem 2* holds. In the simulation, the switching algorithm (middle time delay) is shown in Fig. 10, from which the scheduled parameters are calculated.

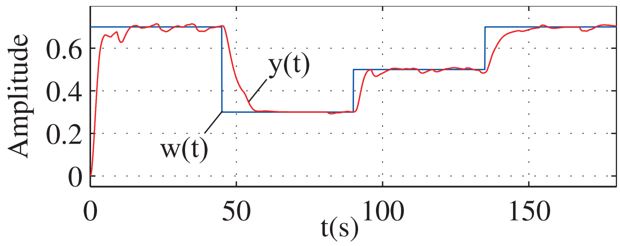

Figure 8: Simulation results $w(t)$, $y(t)$ with switched controller (39)

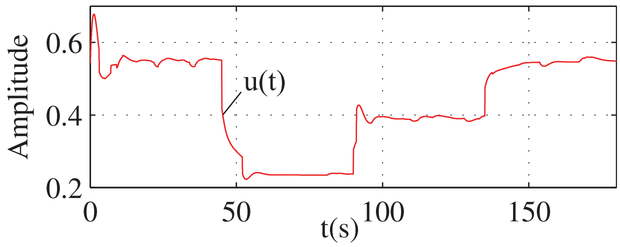

Figure 9: Switched Controller output (39)

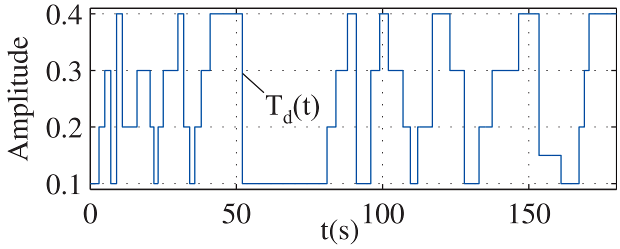

Figure 10: Time delay changes

Conclusion

The paper addresses the problem of the switched controller design with an arbitrarily switching algorithm, which ensures the closed-loop stability and guaranteed cost for a prescribed rate of change of system switching. A novel gain-scheduling plant model is presented for linear continuous-time invariant switched systems. The first proposed method is connected with the notion of quadratic stability with respect to switched parameter θ. In this case, we assume that the rate of θ change is infinite. In some real cases, the rate of change of the switching signal is finite. This assumption was used in the second approach to obtain the switched controller design procedure. The advantages of the proposed method are

- one can obtain less conservative results with respect to using the dwell time approach,
- for the switched controller design, there is no need to use the approach of the "dwell time", which markedly complicates the design procedure,
- the rate of the switching signal change can be prescribed by the designer, which opens new possibilities for practical realizations and development of new theoretical approaches,

- the obtained design procedure for output/state feedback ensures the closed-loop stability of switched systems and guaranteed cost,
- the obtained design procedure can be implemented easily to the standard LMI or BMI approaches,
- the obtained design procedure can be easily transformed to the case of robust switched controller design for continuous-time switched systems with arbitrarily switching.

Numerical examples illustrate the effectiveness of the proposed approach.

Acknowledgment

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