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Neighbour-dependent point shifts and random exchange models: Invariance and attractors

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Consider a partition of the real line into intervals by the points of a stationary renewal point process. Subdivide the intervals in proportions given by i.i.d. random variables with distribution G supported by $[0, 1]$. We ask ourselves for what interval length distribution F and what division distribution G, the subdivision points themselves form a renewal process with the same F? An evident case is that of degenerate F and G. As we show, the only other possibility is when F is Gamma and G is Beta with related parameters. In particular, the process of division points of a Poisson process is again Poisson, if the division distribution is Beta: $\mathrm{B}(r, 1-r)$ for some $0 < r < 1$.

We show a similar behaviour of random exchange models when a countable number of "agents" exchange randomly distributed parts of their "masses" with neighbours. More generally, a Dirichlet distribution arises in these models as a fixed point distribution preserving independence of the masses at each step. We also show that for each G there is a unique attractor, a distribution of the infinite sequence of masses, which is a fixed point of the random exchange and to which iterations of a non-equilibrium configuration of masses converge weakly. In particular, iteratively applying $\mathrm{B}(r, 1-r)$-divisions to a realisation of any renewal process with finite second moment of F yields a Poisson process of the same intensity in the limit.

Keywords: adjustment process; attractor; Dirichlet distribution; Gamma distribution; neighbour-dependent shifts; Poisson process; random exchange; random operator; renewal process

1. Introduction

The Poisson point process is a fundamental object in Probability. Despite being a very well studied model, recent developments in stochastic calculus, stochastic geometry, differential geometry of configuration spaces, variational analysis on measures, continue to bring new insights and deepen our understanding of this seemingly elementary concept, see, for example, [11] and the references therein. It is hard to overestimate the usefulness of the Poisson process in applications which are due to its appealing properties often enabling to produce mathematically tractable models. One of such fundamental properties of the Poisson process on $\mathbb{R}^d$ is that whenever we apply independent random shifts to its points, the resulting process is again Poisson. In particular, if the shifts are i.i.d., the result of such a transformation of a homogeneous Poisson process is again a homogeneous Poisson process of the same intensity.

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The fact that random independent shifts preserve the Poisson process distribution reflects "independence" of its points. However, transformations which depend on two or more neighbouring points destroy this independence and hence, as one may naturally think, the Poisson process. For instance, the mid-points of the consecutive segments in a homogeneous Poisson process on the line do *not* form a Poisson process, as it can be easily checked. If one considers again the mid-points of this process, this second iteration of the initial process actually corresponds to the transformation by which the points move to the centres of their one-dimensional Voronoi cells (which are also their centres of gravity). Such a transformation, known as the *adjustment process*, can be defined in any dimension with respect to any metric and it forms the basis of a popular Lloyd's algorithm in computational geometry. The adjustment process is used to model behaviour of repulsing particles or animals, see, for example, [10], Chapter 7.3.2, Lloyd's algorithm and its variations is widely used in image compression and optimisation, see, for example, [4] and the references therein. It can be shown that the variance of the inter-point distances in one-dimensional adjustment model on any compact set vanishes, so iterations of the adjustment procedure converge (in a suitable sense) to a regular array of points, see [5]. A similar phenomenon of convergence to a lattice is observed in a multi-dimensional case, although questions of the uniqueness of the limiting configuration still remain, see [3].

If dividing the consecutive segments in the Poisson process in half (as well as in any other given non-random proportion) produces a non-Poisson process, an interesting question arises: is there a way to divide each of the segments independently of the others in a *random* proportion so that the division points still form a Poisson process?

Somewhat surprisingly, the answer is *yes*: the segments must be divided in Beta-distributed proportions with parameters $(r, 1-r)$ for some $r \in (0, 1)$, and this is *the only* class of division distributions that preserves the Poisson process!

More generally, in Section 2 we consider a stationary renewal point process, the segments between its consecutive points having lengths drawn independently from a distribution F. Divide its every segment in a random proportion independently drawn from a given distribution G on $[0, 1]$. One may think of the processes of the division points as a transformation of the original point process by which its every point moves to the closest division point on its right, for instance. Since the distribution of the division points depends on the distance to the neighbouring point to the right only, we call such transformation *neighbour-dependent shifts*. We show in Section 2 that Beta-distributed shifts are the only non-trivial ones which preserve a renewal process and that a renewal process preserved by neighbour-dependent shifts is necessarily a process with Gamma-distributed segments. Poisson process with exponential segment lengths provides an example.

Since the resulting process has inter-point segments each composed of pieces of two original segments, one may naturally consider general compositions involving more than two of these. To this end, in Section 3 we establish correspondence of the neighbour-dependent shifts model to the so-called random exchange process which allows for such a composition interpretation. In this process a set of "agents" simultaneously exchange their "masses" with the neighbours in randomly drawn proportions. We show that independent Gamma-distributed initial masses are preserved by Dirichlet-distributed proportions. Moreover, we prove that the iterations of the exchange process starting from any stationary sequence of masses with a finite second moment weakly converge to a limit which is a sequence of independent Gamma-distributed masses in the case of Dirichlet-distributed proportions. To our knowledge, so far only the exchange models

with a finite number of agents were studied in the literature, the main tool here is analysis of convergence of the product of random matrices, see [9] and the references therein. The rôle of the agents and masses in our models play the segments and their lengths, respectively, so we have to deal with the product of infinitely-dimensional linear operators instead. Although the results of Section 3 are somewhat reminiscent of the finite case, the behaviour in this non-compact framework is rather different.

Throughout this paper, $\Gamma(a, \gamma)$ denotes the Gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\gamma > 0$, its density is given by

$$\frac{\gamma^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\gamma x}, \qquad x > 0,$$

and $\mathrm{B}(\alpha, \beta)$ is the Beta distribution, $\alpha > 0$, $\beta > 0$ with density

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad x \in (0, 1).$$

More generally, a random vector $X = (X_1, \dots, X_r)$ with support on a $(r-1)$-dimensional simplex $\{(x_1, x_2, \dots, x_r) : x_1 + x_2 + \cdots + x_r = 1\}$, $r \geq 2$ has a Dirichlet distribution with positive real parameters $(\alpha_1, \alpha_2, \dots, \alpha_r)$ if its multidimensional density is given by

$$\frac{\Gamma(\sum_{i=1}^r \alpha_i)}{\prod_{i=1}^r \Gamma(\alpha_i)} \prod_{i=1}^r x_i^{\alpha_i - 1}.$$

We write $X \sim \mathrm{Dirichlet}(\alpha_1, \alpha_2, \dots, \alpha_r)$. Note that for a random variable b, $b \sim \mathrm{B}(\alpha, \beta)$ is equivalent to $(b, 1-b) \sim \mathrm{Dirichlet}(\alpha, \beta)$. Furthermore, if Y_i, $i = 1, \dots, r$ are independent random variables $Y_i \sim \Gamma(\alpha_i, \gamma)$ with a common $\gamma > 0$, then the vector $(Y_1/Y, \dots, Y_r/Y)$, where $Y = Y_1 + \cdots + Y_r$, has $\mathrm{Dirichlet}(\alpha_1, \dots, \alpha_r)$ distribution.

2. Invariance under neighbour-dependent shifts

The main object of our study in this section is a stationary point process without multiple points T on the real line. Each realisation of T can be associated with a countable set of the intervals $\mathcal{I}(T) = \{I_k\}_{k\in\mathbb{Z}}$ between its consecutive points. Given a realisation, we are going to introduce its transformation which involves dividing the segments between its consecutive points in a random proportion drawn independently from a given distribution and then studying the distribution of the resulting process of the division points. We fix a *division distribution* G supported by $[0, 1]$ and define the following (random) operator Ψ_G acting on the set of countable subsets of $\mathbb{R}$ without accumulation points:

$$\Psi_G T = T' = \big\{c(I) : I \in \mathcal{I}(T)\big\},$$

where $c(I) = x + b_I(y - x)$ for an interval $I = (x, y) \in \mathcal{I}(T)$ and b_I is a random variable taken from the distribution G independently of anything. Geometrically, one may think that every point

x of T is shifted to a new location $c(I)$, where I is the interval to the right from x, this is why we call Ψ_G the *neighbour-dependent shift operator*. Alternatively, the shift operator can be thought of as a non-stochastic operator $\overline{\Psi}_G$ acting on the distributions of stationary processes since the stationarity is preserved. Obviously, when G is concentrated on 0 or on 1, the corresponding operator Ψ_G preserves the distribution of any stationary point process.

We do not specify how exactly the indexing of the intervals is done. A common way to define I_0 as the zero-interval, that is, the one containing the origin, introduces a size bias. In addition, the zero-interval I'_0 of T' may be composed either of pieces of I_0 and I_1 or of I_{-1} and I_0. To avoid unnecessary technicalities involving either reindexing of T' or a point-stationary indexing of T, we choose to work with the Palm version of the underlying point process instead.

It is well known that there is one-to-one correspondence between the distributions of a stationary point process with finite intensity and of a stationary sequence $\mathcal{T} = (\tau_k)_{k\in\mathbb{Z}}$ of positive random variables with finite mean. This sequence is related to the Palm version of the point process: under the Palm distribution there is almost surely a point $T_0 = 0$ of the process at the origin, and if T_n denotes the $|n|$th closest process point to the origin on the positive semi-axis for $n \geq 1$ and on the negative semi-axis for $n < 0$, then $\tau_k = T_k - T_{k-1}$, $k \in \mathbb{Z}$, represents the lengths of the kth inter-point interval I_k. In the case of a renewal process, the sequence $\mathcal{T}$ is i.i.d. drawn from a distribution F with finite mean. Relation between the distributions of the point process and the corresponding stationary sequence is given in general by the Ryll–Nardzewski exchange formula, see, for example, [2], Section 13.3, for details.

This allows us to define a stochastic *exchange operator* Φ_G acting realisationwise on $\mathcal{T}$ as follows: take an i.i.d. sequence $\{b_k\}_{k\in\mathbb{Z}}$ of G-distributed random variables and set

$$\Phi_G\mathcal{T} = \mathcal{T}' = \big(\tau'_k\big)_{k\in\mathbb{Z}} \qquad \text{where } \tau'_k = (1-b_k)\tau_k + b_{k+1}\tau_{k+1}. \tag{1}$$

The main result of this section is a characterisation of the class of fixed points of the exchange operator Φ_G, and thus of the random shift operator Ψ_G (more exactly, of the operator $\overline{\Psi}_G$). The degenerate distribution concentrated at point x is denoted by δ_x.

Theorem 1. *Let $\mathcal{T}$ be an i.i.d. sequence of positive integrable random variables with the distribution F corresponding to a stationary renewal process T and Φ_G be the exchange operator* (1). *Then $\Phi_G(\mathcal{T}) \stackrel{D}{=} \mathcal{T}$, and thus $\Psi_G(T) \stackrel{D}{=} T$, if and only if one of the following alternatives is true*:

(i) $F = \Gamma(\alpha, \gamma)$ *and* $G = \mathrm{B}(r\alpha, (1-r)\alpha)$ *for some constants* $\alpha > 0$, $\gamma > 0$ *and* $r \in (0, 1)$,
(ii) $F = \delta_s$ *for some* $s \in (0, \infty)$ *and* $G = \delta_b$ *for some* $b \in [0, 1]$.

Proof. *Necessity.* Considering three consecutive elements X, Y, Z in $\mathcal{T}$, we note that $\Phi_G(\mathcal{T}) = \mathcal{T}' \stackrel{D}{=} \mathcal{T}$ implies, in particular, that the two consecutive elements in $\mathcal{T}'$ they contribute to, should also be independent, identically distributed with F, which in turn implies the following condition:

> If X, Y, Z are independent F-distributed random variables and a, b, c are independent G-distributed, then the random variables
>
> $$(1-a)X + bY, \qquad (1-b)Y + cZ$$
>
> are also independent and F-distributed.

In terms of the Laplace transforms, for $x_1, x_2 < 0$ we obtain:

$$\begin{aligned}\phi_{(1-a)X+bY,(1-b)Y+cZ}(x_1,x_2) &= \mathbf{E}\exp\{x_1((1-a)X+bY)+x_2((1-b)Y+cZ)\}\\ &= \mathbf{E}\exp\{x_1(1-a)X\}\mathbf{E}\exp\{x_2cZ\}\mathbf{E}\exp\{x_1bY+x_2(1-b)Y\}\\ &= \phi_{(1-a)X}(x_1)\phi_{cZ}(x_2)\phi_{bY,(1-b)Y}(x_1,x_2).\end{aligned}$$

On the other hand, the independence of $(1-a)X+bY$ and $(1-b)Y+cZ$ implies

$$\begin{aligned}\phi_{(1-a)X+bY,(1-b)Y+cZ}(x_1,x_2) &= \phi_{(1-a)X+bY}(x_1)\phi_{(1-b)Y+cZ}(x_2)\\ &= \phi_{(1-a)X}(x_1)\phi_{bY}(x_1)\phi_{(1-b)Y}(x_2)\phi_{cZ}(x_2),\end{aligned}$$

and hence

$$\phi_{(1-b)Y,bY}(x_1,x_2) = \phi_{(1-b)Y}(x_1)\phi_{bY}(x_2),$$

so the random variables $\eta_1 = bY$ and $\eta_2 = (1-b)Y$ are independent.

Let us first suppose $Y \sim F$ is degenerate. Then the only case when Y is a sum of the two independent random variables bY and $(1-b)Y$ is when both of them are degenerate, too. That means, the random variable $b \sim G$ must be degenerate, leading us to alternative (ii).

Suppose now that $Y \sim F$ is non-degenerate. Then so are η_1 and η_2. In that case, if we put $b = \eta_1/(\eta_1+\eta_2)$ and $Y = \eta_1+\eta_2$, then $(b, 1-b)$ and Y can be understood as a shape vector and a size variable of the random vector (η_1, η_2). Note that b and Y are independent by construction, that is, the shape is independent of the size. Therefore, the only possibility for the joint distributions of η_1, η_2 is for them to be independent, Gamma-distributed with some positive shape parameters a_1, a_2 and a common rate γ, as given by the Lukacs theorem [8]. Put $r = a_1/(a_1+a_2)$ and $\alpha = a_1 + a_2$. Then b becomes $\mathrm{B}(r\alpha, (1-r)\alpha)$-distributed and Y conforms to $\Gamma(\alpha, \gamma)$, proving the alternative (i).

Sufficiency. The alternative (ii) trivially leads to the invariance.

For the sufficiency in case (i) it is enough to notice that due to the Lukacs theorem [8], for every $k \in \mathbb{Z}$ the random variables $\tau_k b_k$ and $\tau_k(1-b_k)$ are independent, distributed as $\Gamma(r\alpha, \gamma)$ and $\Gamma((1-r)\alpha, \gamma)$, respectively. Hence, the random variable $\tau'_k = (1-b_k)\tau_k + b_{k+1}\tau_{k+1}$ is again Gamma-distributed, and moreover, it is independent of $\tau'_{k+1} = (1-b_{k+1})\tau_{k+1} + b_{k+2}\tau_{k+2}$, since all their summands are mutually independent. Hence, the sequence $\{\tau'_k\}_{k\in\mathbb{Z}}$ is again i.i.d., $\tau'_0 \sim \Gamma(\alpha, \gamma)$, thus finishing the proof. □

Remark 1. Since a homogeneous Poisson process with rate γ has exponential $\Gamma(1, \gamma)$ distributed inter-point distances, the Beta division point distribution $G = B(r, 1-r)$ for some $0 < r < 1$ is the only non-degenerate distribution preserving the Poisson process.

Remark 2. The process formed by every second point in a homogeneous Poisson process is a renewal process with $\Gamma(2, \gamma)$-distributed inter-point distances. Thus a uniform division distribution which is also $G = B(1, 1)$ preserves it. This also follows from a known elementary fact that if X, Y are independent Exponentially-distributed random variables and U is a uniform variable independent of them, then $U(X+Y)$ and $(1-U)(X+Y)$ are independent Exponentially-distributed random variables.

3. Random exchange model: Fixed points and convergence

As we have shown in the previous section, Beta division distribution G defines a neighbour-dependent shift operator which preserves a renewal process with gamma-distributed inter-point distances. Two immediate questions arise. Suppose we start from a renewal process realisation $\mathcal{T}$ which is *not* Gamma and apply the operator Φ_G to it iteratively. Will the iterations $\Phi_G^{(n)}(\mathcal{T}) = \Phi_G(\Phi_G^{(n-1)}(\mathcal{T}))$ converge to a Gamma renewal process? The answer is *yes*, provided the inter-point distances have a finite second moment. Another question: if the "new" inter-point intervals are composed of more than two "old" ones, how much of the previous results still hold? We show below that Gamma-distributed renewal process appears again, but the rôle of the Beta distribution is now played by the Dirichlet distribution.

Recall that at each iteration of Φ_G each inter-point segment, independently of everything else, cuts a G-distributed proportion of its length and passes it to the segment to the right of it, while at the same time receiving a portion of length from the interval to the left of it.

If we regard the intervals as "agents", and their lengths as "masses", then the shift procedure defined by (1) can be interpreted as a simultaneous random exchange, where at each application of Φ_G every agent i splits its current mass τ_i into two random pieces in proportion $b_i : 1 - b_i$ and shares it between itself and its neighbour to the right, while at the same time receiving a piece of length of its neighbour to the left. More generally, we define a *random mass exchange* model in discrete time as follows.

Consider a countable collection of *agents*, labelled by a sequence of integers $i \in \mathbb{Z}$. Each agent is supplied with a non-negative entity called its *mass*. Assume that at the beginning of step n the ith agent has mass τ_i^n, starting from some initial mass (row-)vector $\tau^0 = (\tau_i^0)_{i\in\mathbb{Z}}$ at $n = 0$. Then each agent i samples a new vector of proportions $(\pi_{i,j}(n+1))_{j\in\mathbb{Z}}$, $\sum_j \pi_{i,j}(n+1) = 1$, and distributes all of its mass between itself and other agents accordingly, so that agent j gets a portion $\tau_i^n \pi_{i,j}(n+1)$ of its mass from i, or, in the vector form,

$$\tau^{n+1} = \tau^n \Pi(n+1), \qquad n = 0, 1, 2, \ldots. \tag{2}$$

Here

$$\Pi(n) = \big(\pi_{ij}(n)\big)_{i,j\in\mathbb{Z}}, \qquad n = 1, 2, \ldots$$

is a mass exchange (two-side infinite) matrix, with proportion vectors $(\pi_{i,j}(n))_{j\in\mathbb{Z}}$ as its rows. Obviously for every n, $\Pi(n)$ is row-stochastic:

- $\pi_{ij}(n) \geq 0, i, j \in \mathbb{Z}, n = 1, 2, \ldots$.
- $\sum_{j\in\mathbb{Z}} \pi_{ij}(n) = 1, i \in \mathbb{Z}, n = 1, 2, \ldots$.

The mass exchange model can be regarded as a discrete version of a randomised Potlatch process first defined in [6,7], where instead of having a Poisson clock at each site, all of the sites' transitions are synchronised.

For the sequel we assume two conditions on the initial mass configuration τ^0:

(A1) τ_i^0 are non-negative i.i.d. random variables for different $i \in \mathbb{Z}$ with $\mu = \mathbf{E}\tau_i^0 < \infty$;

(A2) The second moments of initial masses are finite: $\sigma^2 = \operatorname{var} \tau_i^0 < \infty$.

To stay in the stationary framework, we will also assume that the random exchange model is translation invariant on $\mathbb{Z}$:

(B) There exists a random probability *sharing distribution* $(\pi_j)_{j\in\mathbb{Z}}$ on $\mathbb{Z}$ such that the vectors $(\pi_{i,i+j}(n))_{j\in\mathbb{Z}}$ for different $i \in \mathbb{Z}$, and $n = 1, 2, \ldots$ are i.i.d. copies of $(\pi_j)_{j\in\mathbb{Z}}$.

Introduce $p_{ij} = \mathbf{E}\pi_{ij}(n)$ and $p_i = \mathbf{E}\pi_i, i, j \in \mathbb{Z}, n = 1, 2, \ldots$. Condition (B) implies that the matrix of proportions' expected values

$$P = (p_{ij})_{i,j\in\mathbb{Z}}$$

does not depend on n and the following local balance condition is satisfied for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$:

$$\mathbf{E}\sum_i \pi_{ij}(n) = \sum_i p_{0,j-i} = \sum_k p_k = 1. \tag{3}$$

Here and below the indices in the sums run over all integers, unless specified otherwise.

We will also use the following notation for matrix products:

$$\Pi(1:n) = \Pi(1)\cdots\Pi(n), \qquad \Pi(n:1) = \Pi(n)\cdots\Pi(1).$$

Note that the neighbour-dependent shifts in the previous section can be regarded as a random exchange model with the two-diagonal exchange matrix $\Pi(n) = (\pi_{i,j}(n))_{i,j\in\mathbb{Z}}$ given by

$$\pi_{i,j}(n) = \begin{cases} 1 - b_i(n), & i = j, \\ b_i(n), & i = j+1, \\ 0, & \text{otherwise}, \end{cases}$$

where $b_i(n), i \in \mathbb{Z}, n = 1, 2, \ldots$ are i.i.d. with distribution G.

Random walk in random environment (RWRE). There is a correspondence between the random exchange (2) and a certain version of RWRE which we define here.

Assume the translation invariance condition (B). Introduce $\{W^n\}_{n\geq 0}$, a random walk in a random environment (RWRE) on $\mathbb{Z}$, governed by the random transition probabilities $\pi_{i,i+j}(n)$, conditional on the environment $\mathcal{E} = \sigma(\{\Pi(n)\}_{n\geq 1})$:

$$\mathbf{P}\big(W^{n+1} = i + j | W^n = i, \mathcal{E}\big) = \pi_{i,i+j}(n+1) \stackrel{D}{=} \pi_j.$$

Then its n-step transition matrix is given by $\Pi(1:n)$.

In this interpretation of RWRE, the environment is resampled on every step $n = 1, 2, \ldots$. Note that if we integrate out the environment, then due to independence of $\Pi(n)$ we can regard a single random walker's trajectory as a usual time-homogeneous random walk on $\mathbb{Z}$ with transition probabilities given by

$$\mathbf{P}\big(W^{n+1} = i + j | W^n = i\big) = \mathbf{E}\pi_{i,i+j}(n+1) = p_j, \qquad i, j \in Z, n = 1, 2, \ldots.$$

However if there is more than one random walker, the joint dynamics are more involved. Below we will be interested in running several copies of W^n together in the same realisation

of the environment $\{\pi_{i,i+j}(n)\}$. In particular, we are going to consider the process Z_n of the difference between two conditionally on $\mathcal{E}$ independent copies W^n, $\widetilde{W}^n$ of such a random walk:

$$Z^n = W^n - \widetilde{W}^n, \qquad n = 1, 2, \ldots.$$

Note that given $\mathcal{E}$, Z^n is not a random walk, and not even a Markov chain, since we have to know the positions of both walkers to determine the conditional probabilities for the next step of Z_n. However, if we integrate out the environment, Z^n becomes a Markov chain on $\mathbb{Z}$ with the transition probabilities

$$\mathbf{P}\big(Z^{n+1} = i + j | Z^n = i\big) = \begin{cases} \displaystyle\sum_{j_1 - j_2 = j} \mathbf{E}\pi_{j_1}\pi_{j_2}, & i = 0, \\ \displaystyle\sum_{j_1 - j_2 = j} p_{j_1} p_{j_2}, & i \neq 0. \end{cases} \tag{4}$$

Having this machinery at hand, we are ready to prove the sufficient conditions for existence and uniqueness of an equilibrium point of a random exchange model. Below we use notation "$\Rightarrow$" to denote the weak convergence of a sequence of infinite random sequences, meaning the weak convergence of all of their finite-dimensional sub-vectors.

Theorem 2. *Under condition* (B), *there exists a unique* (*up to a distributional copy and scaling by a constant factor*) *fixed point for dynamics* (2), *i.e. a sequence* $\tau^\infty = (\tau_i^\infty)_{i\in\mathbb{Z}}$ *of* (*not necessarily independent*) *random variables*, *independent of* $\{\Pi(n)\}_{n\geq 1}$ *such that*:

(i) $\tau^\infty \Pi(1) \stackrel{D}{=} \tau^\infty$,
(ii) *for any* τ^0 *satisfying* (A1)–(A2), *independent of* $\{\Pi(n)\}_{n\geq 1}$,

$$\tau^0 \Pi(1:n) \Rightarrow \tau^\infty \qquad \text{as } n \to \infty. \tag{5}$$

Proof. Introduce a dual version of the process:

$$\widetilde{\tau}^n = \tau^0 \Pi(n:1).$$

The proof is based on the distributional equality:

$$\widetilde{\tau}^n \stackrel{D}{=} \tau^n,$$

that holds for any fixed $n = 1, 2, \ldots$.

First, prove the theorem for the constant initial conditions $\theta^0 = \mathbf{1}$, where $\mathbf{1}$ is the sequence of ones: $\theta_j^0 = 1$ for all $j \in \mathbb{Z}$. Denote by $\widetilde{\pi}_{ij}^n$ the elements of the matrix $\Pi(n:1)$ and by $\mathcal{B}^n$ the σ-algebra generated by $\{\Pi(k), k = 1, \ldots, n\}$. Denoting $\theta^n = \theta^0 \Pi(n:1)$, we have that

$$\begin{aligned} \mathbf{E}\big[\theta_j^{n+1} | \mathcal{B}^n\big] &= \mathbf{E}\left[\sum_i \widetilde{\pi}_{ij}^{n+1} \Big| \mathcal{B}^n\right] = \mathbf{E}\left[\sum_i \sum_k \pi_{ik}(n+1)\widetilde{\pi}_{kj}^n \Big| \mathcal{B}^n\right] \\ &= \sum_k \widetilde{\pi}_{kj}^n \mathbf{E} \sum_i \pi_{ik}(n+1). \end{aligned} \tag{6}$$

Due to (3), $\mathbf{E}\sum_i \pi_{ik}(n+1) = 1$ for any $k \in \mathbb{Z}$, so we continue (6) with

$$\sum_k \widetilde{\pi}^n_{kj} \mathbf{E} \sum_i \pi_{ik}(n+1) = \sum_k \widetilde{\pi}^n_{kj} = \theta^n_j.$$

Thus, for every j, the sequence $\{\theta^n_j\}_{n\geq 0}$ is a non-negative martingale, therefore it has an almost sure limit, call it θ^∞_j. The sequence $\theta^\infty = (\theta^\infty_j)_{j\in\mathbb{Z}}$ obviously satisfies (i) as well as (ii) for the initial condition $\theta^0 = \mathbf{1}$. Moreover,

$$\begin{aligned}\operatorname{var}\theta^n_j &= \operatorname{var}\sum_i \widetilde{\pi}^n_{ij} \leq \mathbf{E}\Big(\sum_i \widetilde{\pi}^n_{ij}\Big)^2 = \mathbf{E}\sum_i \big(\widetilde{\pi}^n_{ij}\big)^2 + \mathbf{E}\sum_{k,l,k\neq l} \widetilde{\pi}^n_{kj}\widetilde{\pi}^n_{lj} \\ &\leq \sum_i \mathbf{E}\big(\widetilde{\pi}^n_{ij}\big)^2 + \sum_{k\neq l} p_k p_l \leq \sum_i \mathbf{E}\big(\widetilde{\pi}^n_{ij}\big)^2 + 1.\end{aligned}$$

Below in (10) we show that $\sum_i \mathbf{E}(\widetilde{\pi}^n_{ij})^2$ vanishes as $n\to\infty$, so for any j the martingale $\{\theta^n_j\}_{n\geq 0}$ is bounded in $\mathcal{L}^2$ and hence it converges in $\mathcal{L}^2$. Therefore any finite subvector of θ^n converges in $\mathcal{L}^2$ as well.

Moving to the general case, assume τ^0 is arbitary, satisfying (A1), (A2). Define

$$\Delta^n = \Delta^0 \Pi(n:1) \qquad \text{where } \Delta^0 = \tau^0 - \mu\mathbf{1}.$$

Now we can rewrite $\widetilde{\tau}^n$ as the sum:

$$\widetilde{\tau}^n = \Delta^n + \mu\theta^n. \tag{7}$$

The second term on the right-hand side converges to the limit $\mu\theta^\infty := \tau^\infty$ coordinate-wise almost surely and in $\mathcal{L}^2$ as $n\to\infty$. Now show that each coordinate of the first term vanishes in $\mathcal{L}^2$ as $n\to\infty$, this will imply the desired weak convergence in (ii).

The variance of Δ^n_j is given by the following expression:

$$\begin{aligned}\operatorname{var}\Delta^n_j &= \operatorname{var}\sum_i \Delta^0_i \widetilde{\pi}^n_{ij} = \sum_i \operatorname{var}\Delta^0_i \widetilde{\pi}^n_{ij} + \sum_{i\neq k} \operatorname{cov}\big(\Delta^0_i\widetilde{\pi}^n_{ij}, \Delta^0_k \widetilde{\pi}^n_{kj}\big) \\ &= \sum_i \mathbf{E}\big(\widetilde{\pi}^n_{ij}\big)^2 \mathbf{E}\big(\Delta^0_i\big)^2 = \sigma^2 \sum_i \mathbf{E}\big(\pi^n_{ij}\big)^2.\end{aligned} \tag{8}$$

Here the third equality is based on the following fact that can be checked directly: if ξ, η, X, Y are random variables such that (X,Y) is independent of (ξ,η) and $\operatorname{cov}(X,Y) = 0$, then

$$\operatorname{cov}(\xi X, \eta Y) = \mathbf{E}X\mathbf{E}Y \operatorname{cov}(\xi,\eta),$$

implying all the covariance terms in (8) are 0.

We now prove that $\mathbf{E}\sum_i (\pi^n_{ij})^2 \to 0$, $n\to\infty$ by making use of the RWRE construction introduced earlier in this section. Let $W^n, \widetilde{W}^n, n = 0,1,\ldots$ be the two copies of conditionally independent (given $\mathcal{E}$) RWRE with transition probabilities $\Pi(n)$, and let $Z^n = W^n - \widetilde{W}^n$, $n = 0,1,\ldots$ with $W^0 = \widetilde{W}^0 = j$. As already noted, Z^n is a Markov chain, with transition probabilities (4).

The key observation is the following:

$$\mathbf{P}\big(Z^n = 0\big) = \mathbf{E}\sum_i \mathbf{P}\big(W^n = \widetilde{W}^n = i + j|\mathcal{E}\big) = \mathbf{E}\sum_i \big(\pi^n_{j,i+j}\big)^2, \tag{9}$$

where the last equality follows from the conditional independence of $W^n, \widetilde{W}^n$. Now, use the translation invariance (B) to continue:

$$\mathbf{E}\sum_i \big(\pi^n_{j,i+j}\big)^2 = \sum_i \mathbf{E}\big(\pi^n_{j-i,j}\big)^2 = \sum_i \mathbf{E}\big(\widetilde{\pi}^n_{ij}\big)^2.$$

It is easy to see that 0 is a null state of the Markov chain Z^n. Indeed, starting from 0, Z^n leaves 0 after a geometrically distributed with parameter $(1 - \sum_i \mathbf{E}\pi_i^2)$ number of steps, and while out of 0, Z^n behaves as a symmetrical random walk on integers, therefore

$$\mathbf{P}\big(Z^n = 0\big) = \sum_i \mathbf{E}\big(\widetilde{\pi}^n_{ij}\big)^2 \to 0, \qquad n \to \infty, \tag{10}$$

hence by (8), Δ^n_j tends to 0 in $\mathcal{L}^2$ for any j.

We have shown that both terms in the decomposition (7) converge in $\mathcal{L}^2$ coordinate-wise: $\widetilde{\tau}^n_j \xrightarrow{\mathcal{L}^2} \tau^\infty_j$ for all $j \in \mathbb{Z}$, which in turn implies the weak convergence $\tau^n \Rightarrow \tau^\infty$, as well as the uniqueness of the distribution of τ^∞, thus finishing the proof. □

The finite second moment condition (A2) is essential for the convergence (5) even in much more restrictive settings than ours as the following example shows.

Example 1. Assume that all exchange proportions are non-random $\pi_{i,i+j}(n) = p_j$ with $p_{-1} = 1 - p_0 = p \in (0, 1)$ and $p_j = 0$, $j \in \mathbb{Z} \setminus \{0, -1\}$. Then

$$\tau^n_0 = \sum_{j=0}^n \tau^0_j p^j (1-p)^{n-j} \binom{n}{j} \tag{11}$$

and the (almost sure) limit of the latter expression when n goes to infinity, if it exists, is called the *Euler sum* of the sequence $\tau^0 = (\tau^0_j)_{j\in\mathbb{Z}}$ with parameter p. If τ^0 is a sequence of i.i.d. random variables, the almost sure limit of (11) exists if and only if the second moment of τ^0_0 is finite, see [1], Theorem 1, and the references therein. In that case the limit is equal to $\mu = \mathbf{E}\tau^0_0$.

Corollary 1. *If the vector of proportions π is distributed as $\pi \stackrel{D}{=} (\eta(i))_{i\in\mathbb{Z}}$, where η is a Dirichlet process on $\mathbb{Z}$ with a parameter non-negative measure α, $\alpha(\mathbb{Z}) = a < \infty$, then the fixed point of a random exchange process is a vector τ^∞ of independent $\Gamma(a, \gamma)$-distributed random variables.*

In particular, if τ^0 *satisfies the conditions* (A1), (A2), *then*

$$\tau^0 \Pi(1:n) \Rightarrow \tau^\infty,$$

where the components of τ^∞ *are independent, with distribution* $\Gamma(a,\gamma)$ *where* $\gamma = a/\mu = a/\mathbf{E}\tau_0^0$.

When $\alpha_0 = (1-r)\alpha$ and $\alpha_1 = r\alpha$ for some $\alpha > 0$ and $r \in (0,1)$, the statement of the last corollary is the sufficiency part of Theorem 1(i). It is still an open question, whether the necessity statement holds for the sharing proportion distribution π with the support on more than two neighbours.

A partial answer is provided by the next theorem: in the case when π is exchangeable on some finite subset of $\mathbb{Z}$, the only scenario for which the masses of different agents in the equilibrium are independent is indeed when π has a Dirichlet distribution.

Theorem 3. *Assume the conditions* (A1), (A2), (B) *are satisfied. Assume additionally, that the support of* π *is almost surely in a compact set*: $|K| = |\{i \in \mathbb{Z} : \mathbf{P}(\pi_i > 0) > 0\}| < \infty$, *and that every non-zero subvector of* π *is exchangeable. Then*

$$\tau' = \tau\Pi \stackrel{D}{=} \tau \tag{12}$$

if and only if the components of τ *are Gamma-distributed*: $\tau_j \sim \Gamma(a,\gamma)$ *for some* $a,\gamma > 0$ *and* π *is Dirichlet-distributed*: $(\pi_i)_{i\in K} \sim \mathrm{Dirichlet}((a/|K|)_{i\in K})$.

Proof. For the simplicity of the presentation, take $K = \{0,1,2,\dots,m\}$, $m \geq 1$.

The "if" part follows directly from the shape vs. size independence property of Gamma random vectors, as in the proof of Theorem 1. Now prove the "only if" part.

First, consider the joint distribution of the two components of τ' which are at distance m: (τ'_0, τ'_m). By the invariance assumption they are independent, that is, in terms of Laplace transforms we have:

$$\begin{aligned}\phi_{\tau'_0,\tau'_m}(x_1,x_2) &= \phi_{\tau'_0}(x_1)\phi_{\tau'_m}(x_2)\\ &= \prod_{i=-m}^{0} \phi_{\tau_i\pi_{i0}}(x_1) \times \prod_{j=0}^{m} \phi_{\tau_j\pi_{jm}}(x_2).\end{aligned} \tag{13}$$

Alternatively, we can express the Laplace transform of the pair (τ'_0, τ'_m) directly, taking into account the independence of $\tau_i\pi_{ij}$ for different i:

$$\begin{aligned}\phi_{\tau'_0,\tau'_m}(x_1,x_2) &= \mathbf{E}\exp\{x_1\tau'_0 + x_2\tau'_m\}\\ &= \mathbf{E}\exp\left\{x_1\sum_{i=-m}^{0}\tau_i\pi_{i0} + x_2\sum_{j=0}^{m}\tau_j\pi_{jm}\right\}\end{aligned} \tag{14}$$

$$= \prod_{i=-m}^{-1} \mathbf{E}\exp\{x_1\tau_i\pi_{i0}\} \times \mathbf{E}\exp\{x_1\tau_0\pi_{00} + x_2\tau_0\pi_{0m}\} \times \prod_{j=1}^{m} \mathbf{E}\exp\{x_2\tau_j\pi_{jm}\}$$

$$= \prod_{i=-m}^{-1} \phi_{\tau_i\pi_{i0}}(x_1) \times \phi_{\tau_0\pi_{00},\tau_0\pi_{0m}}(x_1, x_2) \times \prod_{j=1}^{m} \phi_{\tau_j\pi_{jm}}(x_2).$$

Comparing (13) and (14), we conclude that the two quantities $\tau_0\pi_{00}, \tau_0\pi_{0m}$ are independent. Since the distribution of the vector $(\pi_{00}, \pi_{01}, \ldots, \pi_{0m})$ is exchangeable, the random variables $\tau_0\pi_{00}, \tau_0\pi_{01}, \ldots, \tau_0\pi_{0m}$ are pairwise independent.

Next, consider the joint Laplace transform of the three components in τ': say, τ'_0, τ'_1 and τ'_m. We can repeat the previous argument to arrive at the conclusion that the joint Laplace transform of the three quantities $\tau_0\pi_{00}, \tau_0\pi_{01}, \tau_0\pi_{0m}$ factorises into the product of their marginal Laplace transforms, making them mutually independent. Using the exchangeability assumption, we conclude that the random variables $\tau_0\pi_{00}, \tau_0\pi_{01}, \ldots, \tau_0\pi_{0m}$ are 3-independent.

Repeating this argument m times yields the joint independence of all the components of the random vector $(\tau_0\pi_{00}, \tau_0\pi_{01}, \ldots, \tau_0\pi_{0m})$. Notice that $(\pi_{00}, \pi_{01}, \ldots, \pi_{0m})$ is its shape vector and τ_0 is its size variable. Moreover, τ_0 is independent of $(\pi_{00}, \pi_{01}, \ldots, \pi_{0m})$ by construction, so the application of the Lukacs theorem [8] finishes the proof. □

4. Open problems and generalisations

The models we have considered here admit a variety of generalisations and raise many intriguing questions. First of all, the already mentioned extension of Theorem 3 to a non-exchangeable sharing distribution would give a generalisation of Theorem 1. We conjecture that a non-degenerate i.i.d. limiting sequence τ^∞ is possible only for a Dirichlet sharing distribution π, but a counterexample may well exist. If, however, the conjecture *is* true, there are arguments that a more general statement may hold without assuming shift-invariance of $\Pi = (\pi_{ij})$. For instance, the agents may be indexed by another countable group such as $\mathbb{Z}^d$ with $d \geq 2$ rather than by $\mathbb{Z}$. Assume there exists a sequence $q = (q_k)_{k\in\mathbb{Z}}$ satisfying the balance equation $qP = q$ for the matrix P of the expectations $p_{ij} = \mathbf{E}\pi_{ij}$. Using the same relations between Gamma and Dirichlet distributions, one can show that the distribution of the vector of masses τ^0 with independent components τ^0_i distributed as $\Gamma(aq_i, \gamma)$ with some $a, \gamma > 0$ is left invariant by the following sharing distributions now depending on the node k: $(\pi_{ki})_{i\in\mathbb{Z}} \sim \text{Dirichlet}((aq_k p_{ki})_{i\in\mathbb{Z}})$, $k \in \mathbb{Z}$. In the shift-invariant case $\pi_{ij} = \pi_{j-i}$ for all $i, j \in \mathbb{Z}$, the unit sequence $\mathbf{1}$ satisfies the balance equation and we obtain Corollary 1. It is interesting if this is the only possibility to have non-trivial independent masses as a fixed point.

Finally, an intriguing question is, if there exists a fixed point for multi-dimensional analog of the neighbour-dependent shifts models. For a point process in $\mathbb{R}^d, d \geq 2$ neighbouring relation can be defined in many different ways, for instance, neighbours can be declared the nodes connected by en edge in any stationary graph having the process points as vertices. We already mentioned in the Introduction the adjustment process where the nodes move to the centre of mass of their Voronoi cells. This provides an example of a shift depending on the Delaunay

graph neighbours with a hexagon lattice vertices in $\mathbb{R}^2$ left intact. If there are any non-degenerate point processes preserved by these neighbouring shifts is an open question, as well as if there are neighbour-dependent shifts of any kind preserving a multi-dimensional Poisson process. Note in this respect that even the balance equation is hard to satisfy on stationary graphs in $\mathbb{R}^d$, the reason being typically unbounded degree of their vertices.

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