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Belavin–Drinfeld solutions of the Yang–Baxter equation: Galois cohomology considerations

Arturo Pianzola[1,2] · **Alexander Stolin**[3]

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Abstract We relate the Belavin–Drinfeld cohomologies (twisted and untwisted) that have been introduced in the literature to study certain families of quantum groups and Lie bialgebras over a non algebraically closed field $\mathbb{K}$ of characteristic 0 to the standard non-abelian Galois cohomology $H^1(\mathbb{K}, \mathbf{H})$ for a suitable algebraic $\mathbb{K}$-group $\mathbf{H}$. The approach presented allows us to establish in full generality certain conjectures that were known to hold for the classical types of the split simple Lie algebras.

Keywords Belavin–Drinfeld · Yang–Baxter · Quantum group · Lie bialgebra · Galois cohomology

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✉ Arturo Pianzola
a.pianzola@gmail.com

1 Department of Mathematical and Statistical Science, University of Alberta, Edmonton, AB T6G 2G1, Canada

2 Centro de Altos Estudios en Ciencias Exactas, Avenida de Mayo 866, (1084), Buenos Aires, Argentina

3 Department of Mathematical Sciences, Chalmers University of Technology and The University of Gothenburg, 412 96 Göteborg, Sweden

1 Introduction

The appearance of Galois cohomology in the classification of certain quantum groups is one of the primary goals of this paper. In order to do this we first need to "linearize" quantum groups (in the same spirit that, via the exponential map, complex simply connected simple Lie groups can be studied/classified by looking at their Lie algebras). The linearization problem is an extremely technical construction brought forward as a conjecture in the work of Drinfeld [5] (see also [3] and [4]), and proved in the seminal work of Etingof and Kazhdan (see [6,7]). An outline of this correspondence can be found in the Introductions of [9,11], wherein one can also find an explanation of why the description of which Lie bialgebras structures exists on the Lie algebra $\mathfrak{g} \otimes_k k((t))$, with $\mathfrak{g}$ simple finite dimensional over an algebraically closed field k of characteristic 0, arise naturally in the classification of quantum groups. The approach to the classification of Lie bialgebra structures on $\mathfrak{g} \otimes_k k((t))$ developed in [9–11] and [14] is by the introduction of the so-called "Belavin–Drinfeld cohomologies". The calculation of these cohomologies is mostly done on a case-by-case basis in the classical types using realizations of the relevant objects as matrices. The main thrust of the present paper is to realize Belavin–Drinfeld cohomologies as usual Galois cohomologies. This allows for uniform realization-free proofs in all types of results that were conjectured (and were known to hold on many of the classical types). The methods that we describe also open an avenue for further studies of Lie bialgebra structures over non-algebraically closed fields.

2 Notation

Throughout this paper $\mathbb{K}$ will denote a field of characteristic 0. We fix an algebraic closure of $\mathbb{K}$ which will be denoted by $\overline{\mathbb{K}}$. The (absolute) Galois group of the extension $\overline{\mathbb{K}}/\mathbb{K}$ will be denoted by $\mathcal{G}$.[1]

If V is a $\mathbb{K}$-space (resp. Lie algebra), we will denote the $\overline{\mathbb{K}}$-space (resp. Lie algebra) $V \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ by $\overline{V}$.

If $\mathbf{K}$ is a linear algebraic group over $\mathbb{K}$ the corresponding (non-abelian) Galois cohomology will be denoted by $H^1(\mathbb{K}, \mathbf{K})$. (See [13] for details. See also [2,12,15] for some of the more technical aspects of this theory that will be used in what follows without further reference). We recall that $H^1(\mathbb{K}, \mathbf{K})$ coincides with the usual non-abelian continuous cohomology of the profinite group $\mathcal{G}$ acting (naturally) on $\mathbf{K}(\overline{\mathbb{K}})$.

Let $\mathfrak{g}$ be a split finite dimensional simple Lie algebra over $\mathbb{K}$. In what follows $\mathbf{G}$ will denote a split (connected) reductive algebraic group over $\mathbb{K}$ with the property that the Lie algebra of the corresponding adjoint group $\mathbf{G}_{\mathrm{ad}}$ is isomorphic to $\mathfrak{g}$.[2]

We fix once and for all a Killing couple $(\mathbf{B}, \mathbf{H})$ of $\mathbf{G}$. The induced Killing couple on $\mathbf{G}_{\mathrm{ad}}$, which we denote by $(\mathbf{B}_{\mathrm{ad}}, \mathbf{H}_{\mathrm{ad}})$, leads to a Borel subalgebra and split Cartan

[1] For the "untwisted" Belavin–Drinfeld cohomologies $\mathbb{K}$ will be arbitrary. In the "twisted" case $\mathbb{K} = k((t))$ where k is algebraically closed.

[2] The case which is most of interest to us is when $\mathbf{G} = \mathbf{G}_{\mathrm{ad}}$. That said, peculiar phenomena appear when $\mathbf{G}$ is either $\mathbf{GL}_n$ or $\mathbf{SL}_n$. Of course $\mathbf{G}_{\mathrm{ad}}$ is then $\mathbf{PGL}_n$ and $\mathfrak{g} = \mathfrak{sl}_n$. The case of $\mathbf{G} = \mathbf{SO}_{2n}$ is also interesting. For all of these reasons we try to maintain our set up as general as possible.

subalgebras of $\mathfrak{g}$ which will be denoted by $\mathfrak{b}$ and $\mathfrak{h}$ respectively. Our fixed Killing couple leads, both at the level of $\mathbf{G}_{\mathrm{ad}}$ and $\mathfrak{g}$, to a root system Δ with a fixed set of positive roots Δ_+ and base $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$.[3]

The Lie bialgebra structures that we will be dealing with are defined by r-matrices, which are element of $\mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ satisfying $\mathrm{CYB}(r) = 0$ where CYB is the classical Yang–Baxter equation (see Sect. 3 below and [8] for definitions). For future use we introduce some terminology and notation. Consider the action of $\mathbf{G}$ on $\mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ induced from the adjoint action of $\mathbf{G}$ on $\mathfrak{g}$. Let R be a commutative ring extension of $\mathbb{K}$. If $X \in \mathbf{G}(R)$ and $v \in (\mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g})_a(R) = (\mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}) \otimes_{\mathbb{K}} R \simeq (\mathfrak{g} \otimes_{\mathbb{K}} R) \otimes_R (\mathfrak{g} \otimes_{\mathbb{K}} R)$, then the adjoint action of X in v will be denoted by $\mathrm{Ad}_X(v)$.[4]

Along similar lines if $\sigma \in \mathcal{G}$ we will write $\sigma(r)$ instead of $(\sigma \otimes \sigma)(r)$.

3 The Belavin–Drinfeld classification

We maintain all of the above notation. Consider a Lie bialgebra structure $(\mathfrak{g}, \delta)$ on $\mathfrak{g}$. By Whitehead's Lemma the cocycle $\delta : \mathfrak{g} \to \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ is a coboundary. Thus $\delta = \delta_r$ for some element $r \in \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$, namely

$$\delta(a) = [a \otimes 1 + 1 \otimes a, r]$$

for all $a \in \mathfrak{g}$. It is well-known when an element $r \in \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ determines a Lie bialgebra structure of $\mathfrak{g}$. See [8] for details.

Assume until further notice that $\mathbb{K}$ is algebraically closed. We then have the Belavin–Drinfeld classification [1], which we now recall. Define an equivalence relation on $\mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ by declaring that r is equivalent to r' if there exist an element $X \in \mathbf{G}_{\mathrm{ad}}(\mathbb{K})$ and a scalar $c \in \mathbb{K}^\times$ such that

$$r' = c\,\mathrm{Ad}_X(r) \tag{3.1}$$

If furthermore $c = 1$ the two elements are called gauge equivalent.

Belavin–Drinfeld provides us with a list of elements $r_{\mathrm{BD}} \in \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ (called Beladin-Drinfeld r-matrices) with the following properties:

1. Each r_{BD} is an r-matrix (i.e. a solution of the classical Yang–Baxter equation) satisfying $r + r^{21} = \Omega$ (where Ω is the Casimir operator of $\mathfrak{g}$.)
2. Any non-skewsymetric r-matrix for $\mathfrak{g}$ is equivalent to a unique r_{BD}.

For the reader's convenience we recall the nature of the Belavin–Drinfeld r-matrices. With respect to our fixed $(\mathfrak{b}, \mathfrak{h})$, any r_{BD} depends on a discrete and a continuous parameter. The discrete parameter is an admissible triple $(\Gamma_1, \Gamma_2, \tau)$, i.e. an isometry $\tau : \Gamma_1 \longrightarrow \Gamma_2$ where $\Gamma_1, \Gamma_2 \subset \Gamma$ such that for any $\alpha \in \Gamma_1$ there exists $k \in \mathbb{N}$ satisfying $\tau^k(\alpha) \notin \Gamma_1$. The continuous parameter is a tensor $r_0 \in \mathfrak{h} \otimes_{\mathbb{K}} \mathfrak{h}$ satisfying $r_0 + r_0^{21} = \Omega_0$ and $(\tau(\alpha) \otimes 1 + 1 \otimes \alpha)(r_0) = 0$ for any $\alpha \in \Gamma_1$. Here Ω_0

[3] The elements of Δ are to be thought as characters of $\mathbf{H}_{\mathrm{ad}}$ or elements of $\mathfrak{h}^*$ depending on whether we are working at the group or Lie algebra level. This will always be clear from the context.

[4] In contrast to the notation $(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(v)$ used elsewhere.

denotes the Cartan part of the quadratic Casimir element Ω. Then

$$r_{\mathrm{BD}} = r_0 + \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + \sum_{\alpha\in(Span\Gamma_1)^+} \sum_{k\in\mathbb{N}} e_\alpha \wedge e_{-\tau^k(\alpha)}.$$

We now return to the case of our general $\mathbb{K}$. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra structure on $\mathfrak{g}$. We will assume that $(\mathfrak{g}, \delta)$ is not triangular, i.e. $\delta = \delta_r$ where $r \in \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ is not skew-symmetric. We view r as an element of $\overline{\mathfrak{g}} \otimes_{\overline{\mathbb{K}}} \overline{\mathfrak{g}}$ in the natural way and denote it by $\overline{r}$. The $\overline{\mathbb{K}}$ Lie bialgebra $(\overline{\mathfrak{g}}, \overline{\delta})$ obtained by base change is given by the r-matrix $\overline{r}$. By the Belavin–Drinfeld classification there exists a unique r_{BD} such that

$$\overline{r} = c\, \mathrm{Ad}_X(r_{\mathrm{BD}}) \tag{3.2}$$

for some $X \in \mathbf{G}(\overline{\mathbb{K}})$ and $c \in \overline{\mathbb{K}}^\times$. Since $\overline{r} + \overline{r}^{21} = c\,\Omega$ we can apply [11] Theorem 2.7 to conclude that $c^2 \in \mathbb{K}$.

This leads to two cases, according to whether c is in $\mathbb{K}$ or not. The first case is treated with the untwisted Belavin–Drinfeld cohomologies, while the second one, in the case when $\mathbb{K} = k((t))$ with k algebraically closed of characteristic 0, leads to twisted Belavin–Drinfeld cohomologies. These and their relations to Galois cohomology are the contents of the next two sections.

4 Untwisted Belavin–Drinfeld cohomology

Assume that in (3.2) we have $c \in \mathbb{K}^\times$. Let $s = c^{-1}\overline{r}$. By (3.2) $r_{\mathrm{BD}} = \mathrm{Ad}_{X^{-1}} s$. For any element $\gamma \in \mathcal{G} = \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ we have $\gamma(s) = s$ and therefore $s = \mathrm{Ad}_{\gamma(X)}\gamma(r_{\mathrm{BD}})$. From the foregoing it follows that

$$r_{\mathrm{BD}} = \mathrm{Ad}_{X^{-1}\gamma(X)}\gamma(r_{\mathrm{BD}}) \tag{4.1}$$

We can now appeal to Theorem 3 of [9] to conclude that.

Theorem 4.1 *Assume that* $\overline{r} = c\,\mathrm{Ad}_X(r_{\mathrm{BD}})$ *are as above. Then* r_{BD} *is rational, i.e. it belongs to* $\mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$. *Furthermore* $X^{-1}\gamma(X) \in \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})(\overline{\mathbb{K}})$ *for all* $\gamma \in \mathcal{G}$. □

We now recall (with our notation) the Belavin–Drinfeld cohomology definitions and results developed in [9]. Let $r_{\mathrm{BD}} \in \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ be a Belavin–Drinfeld r-matrix.

Definition 4.2 An element $X \in \mathbf{G}(\overline{\mathbb{K}})$ is called a *Belavin–Drinfeld cocycle* associated to $\mathbf{G}$ and r_{BD} if $X^{-1}\gamma(X) \in \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})(\overline{\mathbb{K}})$, for any $\gamma \in \mathcal{G}$.

The set of Belavin–Drinfeld cocycles associated to r_{BD} will be denoted by $Z_{BD}(\mathbf{G}, r_{\mathrm{BD}})$. Note that this set contains the identity element of $\mathbf{G}(\overline{\mathbb{K}})$.

Definition 4.3 Two cocycles X_1 and X_2 in $Z_{BD}(\mathbf{G}, r_{\mathrm{BD}})$ are called *equivalent* if there exists $Q \in \mathbf{G}(\mathbb{K})$ and $C \in \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})(\overline{\mathbb{K}})$ such that $X_1 = QX_2C$.

It is easy to check that the above defines an equivalence relation in the non-empty set $Z_{BD}(\mathbf{G}, r_{\mathrm{BD}})$

Definition 4.4 Let $H_{BD}(\mathbf{G}, r_{\mathrm{BD}})$ denote the set of equivalence classes of cocycles in $Z_{BD}(\mathbf{G}, r_{\mathrm{BD}})$.

We call this set the *Belavin–Drinfeld cohomology* associated to $(\mathbf{G}, r_{\mathrm{BD}})$. The Belavin–Drinfeld cohomology is said to be *trivial* if all cocycles are equivalent to the identity, and *non-trivial* otherwise.

Remark 4.5 The relevance of this concept, as explained in [9], is that there exists a one-to-one correspondence between $H_{BD}(\mathbf{G}, r_{\mathrm{BD}})$ and Lie bialgebra structures $(\mathfrak{g}, \delta)$ on $\mathfrak{g}$ with classical double isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$ and $\overline{\delta} = \delta_{r_{\mathrm{BD}}}$ up to equivalence.

Our next goal is to realize $H_{BD}(\mathbf{G}, r_{\mathrm{BD}})$ in terms of usual Galois cohomology. This will allow us to establish some open conjectures, as well as "interpret" some peculiarities observed with $H_{BD}(\mathbf{G}, r_{\mathrm{BD}})$ for certain special orthogonal groups.

Proposition 4.6 *There is a natural injection of pointed sets*

$$H_{BD}(\mathbf{G}, r_{BD}) \to H^1(\mathbb{K}, \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}}))$$

Proof Let $X \in \mathbf{G}(\mathbb{K})$ be a Belavin–Drinfeld cocycle. For $\gamma \in \mathcal{G}$ define

$$u_X : \mathcal{G} \to \mathbf{G}(\mathbb{K})$$

by

$$u_X : \gamma \to u_X(\gamma) := X^{-1}\gamma(X).$$

Clearly u_X satisfies the cocycle condition (it is in fact a cohomologically trivial element of $Z^1(\mathbb{K}, \mathbf{G})$). Since by definition $\gamma(X) = XC$ for some element $C \in \mathbf{C}(\mathbf{G}, r_{BD})(\overline{\mathbb{K}})$, the cocycle u_X takes values in $Z^1(\mathbb{K}, \mathbf{C}(\mathbf{G}, r_{BD}))$.[5] By considering its cohomology class we obtain a map

$$Z_{BD}(\mathbf{G}, r_{BD}) \to H^1(\mathbb{K}, \mathbf{C}(\mathbf{G}, r_{BD})).$$

It remains to show that if X and Y are Belavin–Drinfeld cocycles, then u_X is cohomologous u_Y if and only if X is equivalent to Y.

If X and Y are equivalent then $Y = QXC$ with $C \in \mathbf{C}(\mathbf{G}, r_{BD})(\overline{\mathbb{K}})$ and $Q \in \mathbf{G}(\mathbb{K})$. Since $\gamma(Q) = Q$ for any $\gamma \in \mathcal{G}$, it follows that $u_Y(\gamma) = C^{-1}u_X(\gamma)\gamma(C)$, which means that u_X and u_Y are cohomologous. Conversely, if u_X and u_Y are cohomologous as elements of $Z^1(\mathbb{K}, \mathbf{C}(\mathbf{G}, r_{BD}))$ there exists $C \in \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})(\overline{\mathbb{K}})$ such that

$$Y^{-1}\gamma(Y) = C^{-1}X^{-1}\gamma(X)\gamma(C)$$

for all $\gamma \in \mathcal{G}$. It follows that $Q^{-1} = XCY^{-1} \in \mathbf{G}(\mathbb{K})$. This completes the proof of the proposition. □

[5] As the reader has probably guessed, it will not necessarily be true that the class of our cocycle will any longer be trivial when viewed as taking values in the smaller group $\mathbf{C}(\mathbf{G}, r_{BD})$. This subtlety is in fact the reason that allows Galois cohomology to be brought into be picture.

The remarkable fact is that the the algebraic $\mathbb{K}$-group $\mathbf{C}(\mathbf{G}, r_{BD}))$ is diagonalizable. Indeed since $r_{\mathrm{BD}} \in \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ we can reason exactly as in [9] Theorem 1 to conclude that.

Theorem 4.7 *$\mathbf{C}(\mathbf{G}, r_{BD})$ is a closed subgroup of $\mathbf{H}$.* □

Combining this last result with Proposition 4.6 we obtain, with the aid of Hilbert's theorem 90, that

Corollary 4.8 *If the algebraic $\mathbb{K}$-group $\mathbf{C}(\mathbf{G}, r_{BD})$ is connected then $H_{BD}(\mathbf{G}, r_{\mathrm{BD}})$ is trivial.* □

One of the most important r-matrices is the so-called Drinfeld–Jimbo r_{DJ} given by

Definition 4.9 $r_{\mathrm{DJ}} = \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + \frac{1}{2}\,\Omega_0$

where Ω_0, as has already been mentioned, stands for the $\mathfrak{h} \otimes_{\mathbb{K}} \mathfrak{h}$ component of the Casimir operator Ω of $\mathfrak{g}$ written with respect to our choice of $(\mathfrak{b}, \mathfrak{h})$.

In [9] it was conjectured that $H_{BD}(\mathbf{G}, r_{\mathrm{DJ}})$ is trivial under the assumption that $\mathbf{G}$ be simple and $\mathbb{K} = \mathbb{C}((\hbar))$. The conjecture was established by a case-by-case reasoning for most of the classical groups. Further progress on this problem (still for the classical algebras but now with an arbitrary base field of characteristic 0) is given in [11]. The Galois cohomology interpretation we have given provides an affirmative much more general answer to this question.

Theorem 4.10 *$H_{BD}(\mathbf{G}, r_{\mathrm{DJ}})$ is trivial for any split reductive group $\mathbf{G}$ over a field $\mathbb{K}$ of characteristic 0.*

Proof We already know that $\mathbf{C}(\mathbf{G}, r_{DJ})$ is a closed subgroup of our split torus $\mathbf{H}$. It is also clear from Definition 4.9 that all elements of $\mathbf{H}(\overline{\mathbb{K}})$ fix $\overline{r_{\mathrm{DJ}}}$. This yields $\mathbf{C}(\mathbf{G}, r_{DJ}) = \mathbf{H}$. By the last Corollary the Theorem follows.

Remark 4.11 Since $\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ is a closed subgroup of $\mathbf{H}$ it is of the form

$$\mathbf{C}(\mathbf{G}, r_{\mathrm{BD}}) = \mathbf{T} \times \mu_{m_1} \times \cdots \times \mu_{m_n}$$

where $\mathbf{T}$ is a split torus over $\mathbb{K}$ and μ_m is the finite multiplicative $\mathbb{K}$-group of m–roots of unity.

Thus

$$H^1(\mathbb{K}, \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})) = \mathbb{K}^\times/(\mathbb{K}^\times)^{m_1} \times \cdots \times \mathbb{K}^\times/(\mathbb{K}^\times)^{m_n}$$

It is possible to deduce from the results of [9–11,14] that for $\mathbf{G} = \mathbf{GL}(n), \mathbf{SO}(2n+1), \mathbf{Sp}(n)$ that $H^1_{BD}(\mathbf{G}, r_{\mathrm{BD}})$ is trivial. Though the centralizer of Belavin–Drinfeld r-matrices were not explicitly computed in these papers, it is natural to conjecture that that they are always *connected.* If so, then Corollary 4.8 would *show* that the corresponding H_{BD} is trivial. This approach is not only sensible, but likely the only reasonable way of attacking the problem in the exceptional types.

The situation for $\mathbf{G} = \mathbf{SO}(2n)$ is different. Assume that α_n and α_{n-1} are the end vertices of the Dynkin diagram of $\mathfrak{so}(2n)$. Assume also $\alpha_{n-1} = \tau^k(\alpha_n)$ for some integer k, where $\tau : \Gamma_1 \to \Gamma_2$ defines r_{BD}. It was shown in [9] that $\mathbf{C}(\mathbf{G}, r_{\text{BD}}) = \mathbf{T} \times \mathbb{Z}/2\mathbb{Z}$ in this case and $\mathbf{C}(\mathbf{G}, r_{\text{BD}}) = \mathbf{T}$ otherwise.

From our results it follows that $H_{BD}(\mathbf{G}, r_{\text{BD}})$ is trivial in the second case.

Since $H^1(\mathbb{K}, \mathbf{C}(\mathbf{G}r_{\text{BD}})) = \mathbb{K}^\times/(\mathbb{K}^\times)^2$ in the first case, to prove that the corresponding $H^1(\mathbf{SO}(2n), r_{BD})$ is isomorphic to $\mathbb{K}^\times/(\mathbb{K}^\times)^2$, it is sufficient to construct a non-trivial cocycle for any non-square $d \in \mathbb{K}$. It is not difficult to see that such a cocycle can be defined by means of the element

$$diag(d_1, d_2, \dots, d_{2n}) \in \mathbf{SO}(2n)$$

with $d_1 = d_2 = \cdots = d_{n-1} = d_{n+2} = \cdots = 1$ and $d_n = d_{n+1} = d^{1/2}$.

We see again that the Galois cohomology point of view "explains" why certain Belavin–Drinfeld cohomolgies are trivial, and why in the case of $\mathbf{SO}_{2n}$ the appearance of non-trivial classes is natural.

We end this section with a statement, which provides a complete description of non-twisted Belavin–Drinfeld cohomologies in terms of the Galois cohomologies of algebraic groups.

Theorem 4.12 *Let* $\mathbf{G}$ *be a split reductive group over a field* $\mathbb{K}$ *of characteristic* 0. *Assume that the Lie algebra* $\mathfrak{g}$ *of the adjoint group of* $\mathbf{G}$ *is simple. For any Belavin–Drinfeld r-matrix* r_{BD} *in* $\mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g}$ *the sequence*

$$1 \to H_{BD}(\mathbf{G}, r_{\text{BD}}) \to H^1(\mathbb{K}, \mathbf{C}(\mathbf{G}, r_{\text{BD}})) \to H^1(\mathbb{K}, \mathbf{G})$$

is exact.

Proof This is a direct consequence of the various definitions and of Proposition 4.6 (both the statement and the proof). □

From Steinberg's theorem (see [13] Ch III Theorem 3.2.1') we obtain.

Corollary 4.13 *Assume that* $\mathbb{K}$ *is of cohomological dimension 1.*[6] *Then*

$$H_{BD}(\mathbf{G}, r_{BD}) = H^1(\mathbb{K}, \mathbf{C}(\mathbf{G}, r_{BD}))$$

5 Twisted Belavin–Drinfeld cohomologies

In this section we assume that $\mathbb{K} = k((t))$ where k is algebraically closed of characteristic 0. Fix an element $j \in \overline{\mathbb{K}}$ such that $j^2 = t$. We will denote the quadratic extension $\mathbb{K}(j)$ of $\mathbb{K}$ by $\mathbb{L}$. Twisted Belavin–Drinfeld cohomologies where introduced in [9,11] to describe a new class of Lie bialgebras structure on $\mathfrak{g}$ whose Drinfeld double (see [8] for the definition and constriction of this object) is isomorphic to $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{L}$.

[6] For example $\mathbb{K} = \mathbb{C}((t))$. This is the case most relevant to quantum groups.

In this section our reductive group $\mathbf{G}$ will be assumed to be of adjoint type. Within the general framework described in Sect. 2, our analysis corresponds to the case when in (3.2) the constant c does not belong to $\mathbb{K}$. As we have seen, then $c^2 \in \mathbb{K}$.

Before we recall how these Lie bialgebras appear and what the relevant definitions are, we introduce some notation and give an explicit description of $\mathrm{Gal}(\mathbb{K})$ and $\mathrm{Gal}(\mathbb{L})$ that will be used in the proofs.

Fix a compatible set of primitive mth roots of unity ξ_m, namely such that $\xi_{me}^e = \xi_m$ for all $e > 0$. Fix also, with the obvious meaning, a compatible set $t^{\frac{1}{m}}$ of mth roots of t in $\overline{\mathbb{K}}$. There is no loss of generality in assuming that $t^{\frac{1}{2}} = j$.

Let $\mathbb{K}_m = \mathbb{C}((t^{\frac{1}{m}}))$. We can then identify $\mathrm{Gal}(\mathbb{K}_m/\mathbb{K})$ with $\mathbb{Z}/m\mathbb{Z}$ where for each $e \in \mathbb{Z}$ the corresponding element $\overline{e} \in \mathbb{Z}/m\mathbb{Z}$ acts on $\mathbb{K}_m$ via ${}^{\overline{e}}t_i^{\frac{1}{m}} = \xi_m^e t_i^{\frac{1}{m}}$.

We have $\overline{\mathbb{K}} = \varinjlim \mathbb{K}_m$. The absolute Galois group $\mathrm{Gal}(\mathbb{K})$ is the profinite completion $\widehat{\mathbb{Z}}$ thought as the inverse limit of the Galois groups $\mathrm{Gal}(\mathbb{K}_m/\mathbb{K})$ as described above. It will henceforth be denoted by $\mathcal{G}$ as per our convention. If γ_1 denotes the standard profinite generator of $\widehat{\mathbb{Z}}$, then the action of γ on $\overline{\mathbb{K}}$ is given by

$$^{\gamma_1}t^{\frac{1}{m}} = \xi_m t^{\frac{1}{m}}$$

Note for future reference that $\gamma_2 := 2\gamma_1$ is the canonical profinite generator of $\mathrm{Gal}(\mathbb{L})$.

5.1 Definition of the twisted cohomologies

Twisted cohomologies are a tool in the study of Lie bialgebra structures on $\mathfrak{g}$ such that

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r], \quad x \in \mathfrak{g}$$

with an r-matrix r satisfying condition $r + r^{21} = j\Omega$.[7]

The following result is proved in [11].

Proposition 5.1 *Lie bialgebra structures on $\mathfrak{g} = \mathfrak{sl}_n$ such that the corresponding double is isomorphic to $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{L}$ are given by the formula*

$$\delta(a) = [a \otimes 1 + 1 \otimes a, r]$$

where r satisfies $r + r^{21} = j\Omega$ and $CYB(r) = 0$.

Furthermore there exists a (unique) r-matrix r_{BD} from the Belavin–Drinfeld list of $\mathfrak{g}$ and an element $X \in \mathbf{G}(\overline{\mathbb{K}})$ such that

(i) $r = j Ad_X(r_{\mathrm{BD}})$
(iia) $X^{-1}\gamma(X) \in \mathbf{C}(\mathbf{G}, r)$ *for any* $\gamma \in \mathrm{Gal}(\mathbb{L})$
(iib) $\mathrm{Ad}_{X^{-1}\gamma_1(X)}(r_{\mathrm{BD}}) = r_{\mathrm{BD}}^{21}$.

[7] We are in the situation when c in (3.1) is not in $\mathbb{K}$. Strictly speaking we should have $c = aj$ with $a \in \mathbb{K}^\times$. Since we are working on Lie bialgebras up to equivalence we may assume without loss of generality that $a = 1$.

To define twisted Belavin–Drinfeld cohomology we will need the following more general result.

Proposition 5.2 *Let* $r \in \overline{\mathfrak{g}} \otimes_{\overline{\mathbb{K}}} \overline{\mathfrak{g}}$ *be an r-matrix which defines a Lie bialgebra structure on* $\mathfrak{g}$ *and such that* $r + r^{21} = j\Omega$. *Then*

- $\gamma(r) = r$ *for all* $\gamma \in \mathrm{Gal}(\mathbb{L})$
- $\gamma_1(r) = -r^{21}$

Proof Let $\gamma \in \mathcal{G}$. It is proved in [11] that

$$\gamma(r) = r, \quad \text{or} \tag{5.1}$$

$$\gamma(r) = r - j\Omega. \tag{5.2}$$

Let $\mathcal{H} \subset \mathcal{G}$ be the subgroup of elements satisfying (5.1). Clearly, $\mathcal{H}$ is a proper subgroup because $r + r^{21} = j\Omega$.

Let γ and γ' satisfy (5.2). Then $\gamma\gamma' \in \mathcal{H}$. It follows that $\mathcal{H}$ is a subgroup of $\mathcal{G}$ index 2, in fact $\mathcal{H} = \mathrm{Gal}(\mathbb{L})$. For γ_1 we conclude that $\gamma_1(r) = r - j\Omega = -r^{21}$. □

Remark 5.3 It is easy to see that if r satisfies the conclusions of the proposition above, then r induces a Lie bialgebra structure on $\mathfrak{g}$.

Since $r + r^{21} = j\Omega$, it is clear that $r = j\mathrm{Ad}_X(r_{\mathrm{BD}})$ for some $X \in \mathbf{G}(\overline{\mathbb{K}})$. We will henceforth assume that r_{BD} is *rational*, namely $R_{\mathrm{BD}} \in \mathfrak{g} \otimes_k \mathfrak{g}$ or, what is equivalent, that $\gamma(r_{\mathrm{BD}}) = r_{\mathrm{BD}}$ for all $\gamma \in \mathcal{G}$. Then we get the following two equations for X:

- $X^{-1}\gamma(X) \in \mathbf{C}(\mathbf{G}, r)(\overline{\mathbb{K}})$ for any $\gamma \in \mathrm{Gal}(\mathbb{L})$
- $\mathrm{Ad}_{X^{-1}\gamma_1(X)}(r_{\mathrm{BD}}) = r_{\mathrm{BD}}^{21}$.

Definition 5.4 An element $X \in \mathbf{G}(\overline{\mathbb{K}})$ is called a twisted Belavin–Drinfeld cocycle for $\mathbf{G}$ and r_{BD} if $X^{-1}\gamma(X) \in \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})$ for any $\gamma \in \mathrm{Gal}(\mathbb{L})$ and $\mathrm{Ad}_{X^{-1}\gamma_1(X)}(r_{\mathrm{BD}}) = r_{\mathrm{BD}}^{21}$.

The definition of equivalent cocycles is just as in the untwisted case.

Definition 5.5 Two twisted Belavin–Drinfeld cocycles X and Y are said to be equivalent if $Y = QXC$ for some $C \in \mathbf{C}(\mathbf{G}, r_{\mathrm{BD}})(\overline{\mathbb{K}})$and $Q \in \mathbf{G}(\mathbb{K})$.

It is clear that the above defines an equivalence relation on the set $\overline{Z}_{BD}(\mathbf{G}, r_{\mathrm{BD}})$ of twisted Belavin–Drinfeld cocycles.

Definition 5.6 The twisted Belavin–Drinfeld cohomology related to $\mathbf{G}$ and r_{BD} is the set of equivalence classes of the twisted cocycles. We will denote it by $\overline{H}_{BD}(\mathbf{G}, r_{\mathrm{BD}})$.

Note that, unlike the untwisted case, it is not clear that twisted Belavin–Drinfeld cocycles exist.

Remark 5.7 Assume that r_{BD} is rational. Then the twisted Belavin–Drinfeld cohomology $\overline{H}_{BD}(\mathbf{G}, r_{\mathrm{BD}})$ gives a one-to-one correspondence between equivalence of Lie bialgebra structures on $\mathfrak{g}$ such that over $\overline{\mathbb{K}}$ they become gauge equivalent to the Lie bialgebra structure defined by jr_{BD}.

5.2 Twisted cohomology for the Drinfeld–Jimbo r-matrix

The only good understanding of twisted Belavin–Drinfeld cohomologies is for the Drinfeld–Jimbo r-matrix r_{DJ} (which is clearly rational). Our main goal is to establish the following.

Theorem 5.8 *The set $\overline{H}^1_{\mathrm{BD}}(\mathbf{G}, r_{\mathrm{DJ}})$ consists of one element.*

This result was established in [9,11] for the classical Lie algebras. The key to the proof is the existence of special elements $S \in \mathbf{G}(\mathbb{K})$ and $J \in \mathbf{G}(\mathbb{L})$ with the property

$$\mathrm{Ad}_S(r_{DJ}) = r_{DJ}^{21} \quad \text{and} \quad J^{-1}\gamma_1(J) = S.$$

The existence of these elements is established by a laborious case-by-case analysis (realizing the classical algebras/groups as matrices). We shall provide a uniform and calculation-free proof of the existence of these elements using Steinberg's theorem ("Serre Conjecture I"). We will then relate $\overline{H}^1_{\mathrm{BD}}$ to Galois cohomology to establish Theorem 5.8 for all types.

5.2.1 Construction of S and $J \in \mathbf{G}(\mathbb{L})$ such that $\gamma_1(J) = JS$

Let $\mathrm{Out}(\mathfrak{g})$ be the finite group of automorphisms of the Coxeter–Dynkin diagram of our simple Lie algebra $\mathfrak{g}$. If $\mathbf{Out}(\mathfrak{g})$ is the corresponding constant $\mathbb{K}$-group we know [16] that we have a split exact sequence of algebraic $\mathbb{K}$-groups

$$1 \to \mathbf{G} \to \mathbf{Aut}(\mathfrak{g}) \to \mathbf{Out}(\mathfrak{g}) \to 1 \tag{5.3}$$

We fix a section $\mathbf{Out}(\mathfrak{g}) \to \mathbf{Aut}(\mathfrak{g})$ that stabilizes $(\mathbf{B}, \mathbf{H})$. This gives a copy of $\mathrm{Out}(\mathfrak{g}) = \mathbf{Out}(\mathfrak{g})(\mathbb{K})$ inside $\mathrm{Aut}(\mathfrak{g}) := \mathbf{Aut}(\mathfrak{g})(\mathbb{K})$ that permutes the fundamental root spaces $\mathfrak{g}^{\alpha_i}$ around, and which stabilizes both of our chosen Borel and Cartan subalgebras. Of course $\mathrm{Aut}(\mathfrak{g})$ is the semi-direct product of $\mathbf{G}(\mathbb{K})$ and $\mathrm{Out}(\mathfrak{g})$.

Lemma 5.9 *Let w_0 be the longest element of the Weyl group W of the pair $(\mathbf{B}, \mathbf{H})$. Then there exists an element $S \in \mathbf{G}(\mathbb{K})$ such that $S^2 = 1_{\mathbf{G}(\mathbb{K})}$ and $S(\mathfrak{g}^\alpha) = \mathfrak{g}^{w_0(\alpha)}$ for all roots $\alpha \in \Delta$.*

Proof Let $c \in \mathrm{Aut}(\mathfrak{g})$ be the Chevalley involution. Thus $c^2 = \mathrm{Id}$, $c(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$ and c restricted to the Cartan subalgebra $\mathfrak{h}$ is scalar multiplication by -1. If $\mathrm{Out}(\mathfrak{g})$ is trivial, then $w_0(\alpha) = -\alpha$ and we take $S = c$.

In general note that $-w_0 \in \mathrm{Out}(\mathfrak{g})$, so we can view this as an element $d \in \mathrm{Aut}(\mathfrak{g})$ of order 2. Clearly, $cd = dc$ and we set $S = cd$, which is of order 2.

It remains to be shown that $S \in \mathbf{G}(\mathbb{K})$. Since both c and d stabilize $\mathfrak{h}$, so does S. From this it follows that $S(\mathfrak{g}^\alpha) = \mathfrak{g}^{\theta(\alpha)}$ for some $\theta \in \mathrm{Aut}(\Delta)$ (the automorphism group of our root system). It is well-known that $\mathrm{Aut}(\Delta)$ is a semi-direct product of W and $\mathrm{Out}(\mathfrak{g})$. Moreover, $S \in \mathbf{G}(\mathbb{K})$ if and only if the restriction of S to $\mathfrak{h}$ is in W. But by construction this restriction is $\theta = w_0 \in W$. □

It is clear from Definition 4.9 that $Ad_S(r_{DJ}) = r_{DJ}^{21}$. Since $\mathbf{C}(\mathbf{G}, r_{\mathrm{DJ}}) = \mathbf{H}$ we can redefine twisted Belavin–Drinfeld cocycles for r_{DJ} as follows.

Lemma 5.10 *An element $X \in \mathbf{G}(\overline{\mathbb{K}})$ is a twisted Belavin–Drinfeld cocycle for $\mathbf{G}$ and r_{DJ} if and only if*

(i) $X^{-1}\gamma(X) \in \mathbf{H}(\overline{\mathbb{K}})$ *for any* $\gamma \in \mathrm{Gal}(\mathbb{L})$, *and*
(ii) $\mathrm{Ad}_{X^{-1}\gamma_1(X)}(r_{\mathrm{BD}}) = \mathrm{Ad}_S(r_{\mathrm{BD}})$.

As we shall see this definition will allow us to compute the corresponding twisted Belavin–Drinfeld cohomology by means of usual Galois cohomology.

Proposition 5.11 *Let $S \in \mathbf{G}(\mathbb{K})$ be as in the previous lemma. Then there exists $J \in \mathbf{G}(\mathbb{L})$ such that $\gamma_1(J) = JS$*

Proof There exists a unique continuous group homomorphism $u : \mathcal{G} \to G(\overline{\mathbb{K}})$ such that $u(\gamma_1) = S$. Given that $\gamma_1(S) = S$ our u is a cocycle in $Z^1(\mathbb{K}, \mathbf{G})$.

Since $\mathbb{K}$ is of cohomological dimension 1 by Steinberg's theorem $H^1(\mathbb{K}, \mathbf{G}) = 1$. Therefore, there exists $J \in \mathbf{G}(\overline{\mathbb{K}})$ such that $J^{-1}\gamma_1(J) = S$. It remains to be shown that $J \in \mathbf{G}(\mathbb{L})$. For this note that

$$2\gamma_1(J) = \gamma_1(\gamma_1(J)) = \gamma_1(JS) = \gamma_1(J)S = JS^2 = J$$

Since $2\gamma_1$ pro-generates $\mathrm{Gal}(\mathbb{L})$ it follows that $J \in \mathbf{G}(\mathbb{L})$ as desired. □

Note that our element J is a twisted Belavin–Drinfeld cocycle.

5.2.2 Computation of $\overline{H}_{BD}(\mathbf{G}, r_{\mathrm{DJ}})$

The aim of this section to show that $\overline{H}_{BD}(\mathbf{G}, r_{\mathrm{DJ}})$ consists of one element generated by the class of the element J constructed above. This will in particular prove Theorem 5.8.

It is clear that our element S normalizes (in the functorial sense) $\mathbf{H}$. We can therefore consider the $\mathbb{K}$-group

$$\tilde{\mathbf{H}} = \mathbf{H} \rtimes \{1, S\}.$$

Strictly speaking we should be writing the constant $\mathbb{K}$-group corresponding to the finite group $\{1, S\}$. For this reason we shall also write

$$\tilde{\mathbf{H}} = \mathbf{H} \rtimes \mathbb{Z}/2\mathbb{Z}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbf{H}$ by means of S.

Let us begin by explicitly determining $H^1(\mathbb{K}, \tilde{\mathbf{H}})$. Consider the split exact sequence of $\mathbb{K}$ groups

$$1 \to \mathbf{H} \to \tilde{\mathbf{H}} \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

Passing to cohomology we get

$$H^1(\mathbb{K}, \mathbf{H}) \to H^1(\mathbb{K}, \tilde{\mathbf{H}}) \to H^1(\mathbb{K}, \mathbb{Z}/2\mathbb{Z}) \to 1.$$

The surjectivity of the last map follows from the fact that the original sequence of $\mathbb{K}$-groups splits. We have $H^1(\mathbb{K}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{K}^\times/(\mathbb{K}^\times)^2$. This last is the group of order 2 with representatives $\{1, j\}$ where we recall that $j = t^{\frac{1}{2}}$.

The elements of $H^1(\mathbb{K}, \tilde{\mathbf{H}})$ mapping to the class of 1 are given by the image of $H^1(\mathbb{K}, \mathbf{H})$ which is trivial by Hilbert 90. The elements of $H^1(\mathbb{K}, \tilde{\mathbf{H}})$ mapping to the class of j are given by the image of $H^1(\mathbb{K}, \mathbf{H}')$ where the $\mathbb{K}$-group $\mathbf{H}'$ is a twisted form of $\mathbf{H}$. By Steinberg's theorem $H^1(\mathbb{K}, \mathbf{H}')$ vanishes. It follows that $H^1(\mathbb{K}, \tilde{\mathbf{H}})$ has two elements. More precisely.

Theorem 5.12 *The pointed set $H^1(\mathbb{K}, \mathbf{H} \rtimes \{1, S\})$ consists of the two elements*:

1. *The trivial class*,
2. *The class of the cocycle u_J defined by u_J, $u_J(\gamma) = J^{-1}\gamma(J)$. In particular $u_J(\gamma_1) = S$.*

If $X \in \mathbf{G}(\overline{\mathbb{K}})$ is a twisted Belavin–Drinfeld cocycle for r_{DJ} it is clear from Lemma 5.10 that the map $\tilde{u}_X : \mathrm{Gal}(\mathbb{K}) \to \mathbf{G}(\overline{\mathbb{K}})$ given by

$$\tilde{u}_X : \gamma \mapsto X^{-1}\gamma(X)$$

is a Galois cohomology cocycle in $Z^1(\mathbb{K}, \tilde{\mathbf{H}})$.

Theorem 5.13 *The map $X \mapsto \tilde{u}_X$ described above induces an injection $\overline{H}_{BD}(\mathbf{G}, r_{\mathrm{DJ}}) \to H^1(\mathbb{K}, \tilde{\mathbf{H}}) = \{1, j\}$. More precisely the fiber of the trivial class* 1 *is empty and that of j consist of the class of the Belavin–Drinfeld cocycle J.*

Proof If X and Y are equivalent Belavin–Drinfeld cocycle for r_{DJ} then by definition $Y = QXC$ where $Q \in G(\mathbb{K})$ and $C \in \mathbf{H}(\overline{\mathbb{K}})$. Just as in the untwisted case we see that the Galois cocycles $\tilde{u}_X$ and $\tilde{u}_Y$ are cohomologous. We thus have a canonical map

$$i : \overline{H}_{BD}(\mathbf{G}, r_{\mathrm{DJ}}) \to H^1(\mathbb{K}, \mathbf{H} \rtimes \{1, S\})$$

We now look in detail at the two fibers. Let $X \in \mathbf{G}(\overline{\mathbb{K}})$ be a twisted Belavin–Drinfeld cocycle.

1. Suppose that $\tilde{u}_X$ is in the trivial class $1 \in \{1, j\}$. By definition there exists an element $h \in \tilde{\mathbf{H}}(\overline{\mathbb{K}})$ such that $\tilde{u}_X(\gamma) = h^{-1}{}^{\gamma}h$. Let $C \in \mathbf{H}(\overline{\mathbb{K}})$ and $\epsilon \in \{0, 1\}$ be such that $h = S^\epsilon C$. Since S is fixed by the Galois group $h^{-1}{}^{\gamma}h = C^{-1}{}^{\gamma}C$. But this implies, in particular, that $\tilde{u}_X(\gamma_1) \in \mathbf{H}(\overline{\mathbb{K}})$. This last is false since

$$\tilde{u}_X(\gamma_1)(r_{\mathrm{DJ}}) = X^{-1}\gamma_1(X)(r_{\mathrm{DJ}}) = r_{\mathrm{DJ}}^{21} \neq r_{\mathrm{DJ}}.$$

   The fiber of the trivial class 1 under our canonical map is therefore empty.

2. Suppose that the class of X is mapped to $j \in \{1, j\}$. Then $\tilde{u}_X$ is cohomologous to u_J. By definition there exist $h = S^{\epsilon}C$ as above such that

$$X^{-1}\gamma(X) = C^{-1}S^{\epsilon}J^{-1}\gamma(J)S^{\epsilon}\gamma(C) \tag{5.4}$$

for all $\gamma \in \mathcal{G}$. An arbitrary element of our Galois group is of the form $\gamma_n = n\gamma_1$ Recall that $J \in \mathbf{G}(\mathbb{L})$ (hence it is fixed by all γ_n with n even), that $J^{-1}\gamma(J) = S \in \mathbf{G}(\mathbb{K})$ and that $S^2 = 1$. These easily imply that $J^{-1}\gamma_n(J) = S^n$. Taking this into account we get from (5.4) that for all $n \in \mathbb{Z}$

$$X^{-1}\gamma_n(X) = C^{-1}J^{-1}\gamma_n(J)\gamma_n(C) \quad \text{if } n \text{ is odd} \tag{5.5}$$

From these it readily follows that $Q^{-1} := JCX^{-1}$ is invariant under the action of $\mathcal{G}$. Thus $Q \in \mathbf{G}(\mathbb{K})$. Since $X = QJC$ we have that X and J are equivalent Belavin–Drinfeld cocycles. The fiber of j has therefore exactly one element.

This completes the proof. □

This last result shows that Theorem 5.8 holds. More precisely.

Corollary 5.14 *The twisted Belavin–Drinfeld cohomology $\overline{H}_{BD}(\mathbf{G}, r_{\mathrm{DJ}})$ consists of one class only, namely the class of the cocycle J.* □

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