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Citation for the original published paper (version of record):
Kirchner, K., Lang, A., Larsson, S. (2017). Covariance structure of parabolic stochastic partial differential equations with multiplicative
Lévy noise. Journal of Differential Equations, 262(12): 5896-5927.
http://dx.doi.org/10.1016/j.jde.2017.02.021

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 262 (2017) 5896–5927

Journal of Differential Equations

www.elsevier.com/locate/jde

Covariance structure of parabolic stochastic partial differential equations with multiplicative Lévy noise

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Received 14 July 2016
Available online 23 February 2017

Abstract

The characterization of the covariance function of the solution process to a stochastic partial differential equation is considered in the parabolic case with multiplicative Lévy noise of affine type. For the second moment of the mild solution, a well-posed deterministic space–time variational problem posed on projective and injective tensor product spaces is derived, which subsequently leads to a deterministic equation for the covariance function.

MSC: 60H15; 35K90; 35R60

Keywords: Stochastic partial differential equations; Multiplicative Lévy noise; Space–time variational problems on tensor product spaces; Projective and injective tensor product space

1. Introduction

The covariance function of a stochastic process is an interesting quantity for the following reasons: It provides information about the correlation of the process with itself at pairs of time points. In addition, it shows if this relation is stationary, i.e., whether or not it changes when shifted in time, and if it follows a trend. In [6] the covariance of the solution process to a

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http://dx.doi.org/10.1016/j.jde.2017.02.021

parabolic stochastic partial differential equation driven by an additive Q-Wiener process has been described as the solution to a deterministic, tensorized evolution equation. In this case the solution process is also Gaussian with mean zero and therefore completely characterized by its covariance. It is now natural to ask whether it is possible to establish such an equation also for covariance functions of solutions to stochastic partial differential equations driven by multiplicative noise.

In the present paper, we extend the study of the covariance function to solution processes of parabolic stochastic partial differential equations driven by multiplicative Lévy noise in the framework of [8]. In this case the solution process is no longer fully characterized by the covariance, but the covariance function is still of interest as mentioned above. We emphasize that it is the extension to multiplicative noise which is the main motivation and challenge here; the extension to Lévy noise is rather straightforward since the theory of the corresponding Itô integral is more or less parallel to the Wiener case. The multiplicative operator is assumed to be affine. Clearly, under appropriate assumptions on the driving Lévy process, the mean function of the mild solution satisfies the corresponding deterministic, parabolic evolution equation as in the case of additive Wiener noise, since in both cases the stochastic integral has expectation zero. However, the presence of a multiplicative term changes the behavior of the second moment and the covariance. We prove that also in this case the second moment as well as the covariance of the square-integrable mild solution satisfy deterministic space–time variational problems posed on tensor products of Bochner spaces. In contrast to the case of additive Wiener noise considered in [6], the trial and the test space are not Hilbert tensor product spaces. Instead we use different notions of tensor product spaces to obtain well-posed variational problems. These tensor product spaces are non-reflexive Banach spaces. In addition, the resulting bilinear form in the variational problem does not arise from taking the tensor product of the corresponding deterministic parabolic operator with itself, but it involves a non-separable operator mapping to the dual space of the test space. For these reasons, well-posedness of the derived deterministic variational problems is not an immediate consequence, and operator theory on the tensor product spaces is used to derive it. We emphasize that, although the present manuscript is rather abstract, numerical methods based on this variational problem are currently under investigation [1].

The structure of the present paper is as follows: In Section 2 we present the parabolic stochastic differential equation and its mild solution, whose covariance function we aim to describe. The results formulated in Section 3 will be used for proving the main results of this paper in Sections 4–6. More precisely, in Subsections 3.1–3.3 we present different notions of tensor product spaces and several operators arising in the variational problems satisfied by the second moment and the covariance of the mild solution. The weak Itô isometry, which we introduce in Subsection 3.4, is crucial for the derivation of the deterministic variational problems. Theorems 4.2 and 5.5 in Sections 4 and 5 provide the main results of this paper: In Theorem 4.2 we show that the second moment of the mild solution satisfies a deterministic space–time variational problem posed on non-reflexive tensor product spaces. In order to be able to formulate this variational problem, we need some additional regularity of the second moment which we prove first. The aim of Section 5 is to establish well-posedness of the derived variational problem. Since the variational problem is posed on non-reflexive Banach spaces, it is not possible to apply standard inf-sup theory to achieve this goal. Instead, we show that the operator associated with the bilinear form appearing in the variational problem is bounded from below, which implies uniqueness of the solution to the variational problem. Finally, in Section 6 we use the results of the previous sections to obtain a well-posed space–time variational problem satisfied by the covariance function of the mild solution.

2. The stochastic partial differential equation

In this section the investigated stochastic partial differential equation as well as the setting that we impose on it are presented. In addition, we formulate the definition as well as existence, uniqueness, and regularity results of the mild solution to this equation in Definition 2.2 and Theorem 2.3. Finally, in Lemma 2.4 we state a property of the mild solution which will be essential for the derivation of the deterministic equation satisfied by its second moment and its covariance function in Sections 4 and 6, respectively.

For two Banach spaces E_1 and E_2, we denote by $\mathcal{L}(E_1; E_2)$ the space of bounded linear operators mapping from E_1 to E_2. In addition, we write $\mathcal{L}_p(H_1; H_2)$ for the space of Schatten class operators of p-th order mapping from H_1 to H_2, where H_1 and H_2 are separable Hilbert spaces. Here, for $1 \le p < \infty$, an operator $T \in \mathcal{L}(H_1; H_2)$ is called a *Schatten-class operator of p-th order*, if T has a finite p-th Schatten norm, i.e.,

$$\|T\|_{\mathcal{L}_p(H_1;H_2)} := \left(\sum_{n\in\mathbb{N}} s_n(T)^p \right)^{\frac{1}{p}} < \infty,$$

where $s_1(T) \ge s_2(T) \ge \ldots \ge s_n(T) \ge \ldots \ge 0$ are the singular values of T, i.e., the eigenvalues of the operator $(T^*T)^{1/2}$ and $T^* \in \mathcal{L}(H_2; H_1)$ denotes the adjoint of T. If $H_1 = H_2 = H$, we abbreviate $\mathcal{L}_p(H; H)$ by $\mathcal{L}_p(H)$. For the case $p = 1$ and a separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$ and orthonormal basis $(e_n)_{n\in\mathbb{N}}$, we introduce the *trace* of an operator $T \in \mathcal{L}_1(H)$ by

$$\operatorname{tr}(T) := \sum_{n\in\mathbb{N}} \langle T e_n, e_n \rangle_H .$$

The trace $\operatorname{tr}(T)$ is independent of the choice of the orthonormal basis and it satisfies $|\operatorname{tr}(T)| \le \|T\|_{\mathcal{L}_1(H)}$, cf. [2, Proposition C.1]. By $\mathcal{L}_1^+(H)$ we denote the space of all nonnegative, symmetric trace class operators on H, i.e.,

$$\mathcal{L}_1^+(H) := \{T \in \mathcal{L}_1(H) : \langle T\varphi, \varphi \rangle_H \ge 0, \langle T\varphi, \psi \rangle_H = \langle \varphi, T\psi \rangle_H \quad \forall \varphi, \psi \in H\} .$$

For $p = 2$, the norm $\|T\|_{\mathcal{L}_2(H_1;H_2)}$ coincides with the *Hilbert–Schmidt* norm.

In the following, U and H denote separable Hilbert spaces with norms $\|\cdot\|_U$ and $\|\cdot\|_H$ induced by the inner products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_H$, respectively.

Let $L := (L(t), t \ge 0)$ be an adapted, square-integrable, U-valued Lévy process defined on a complete filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$. More precisely, we assume that

(i) L has independent increments, i.e., for all $0 \le t_0 < t_1 < \ldots < t_n$ the U-valued random variables $L(t_1) - L(t_0), L(t_2) - L(t_1), \ldots, L(t_n) - L(t_{n-1})$ are independent;

(ii) L has stationary increments, i.e., the distribution of $L(t) - L(s)$, $s \le t$, depends only on the difference $t - s$;

(iii) $L(0) = 0$ $\mathbb{P}$-almost surely;

(iv) L is stochastically continuous, i.e.,

$$\lim_{\substack{s\to t \\ s\ge 0}} \mathbb{P}(\|L(t) - L(s)\|_U > \epsilon) = 0 \quad \forall \epsilon > 0, \quad \forall t \ge 0;$$

(v) L is adapted, i.e., $L(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$;
(vi) L is square-integrable, i.e., $\mathbb{E}\left[\|L(t)\|_U^2\right] < \infty$ for all $t \geq 0$.

Furthermore, we assume that, for $t > s \geq 0$, the increment $L(t) - L(s)$ is independent of $\mathcal{F}_s$ and that L has zero mean and covariance operator $Q \in \mathcal{L}_1^+(U)$, i.e., for all $s, t \geq 0$ and $x, y \in U$ it holds: $\mathbb{E}\langle L(t), x\rangle_U = 0$ and

$$\mathbb{E}\left[\langle L(s), x\rangle_U \langle L(t), y\rangle_U\right] = \min\{s, t\} \langle Qx, y\rangle_U, \tag{2.1}$$

cf. [8, Theorem 4.44]. Note that under these assumptions, the Lévy process L is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ by [8, Proposition 3.25].

In addition, since $Q \in \mathcal{L}_1^+(U)$ is a nonnegative, symmetric trace class operator, there exists an orthonormal eigenbasis $(e_n)_{n\in\mathbb{N}} \subset U$ of Q with corresponding eigenvalues $(\gamma_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$, i.e., $Qe_n = \gamma_n e_n$ for all $n \in \mathbb{N}$, and for $x \in U$ we may define the fractional operator $Q^{1/2}$ by

$$Q^{\frac{1}{2}}x := \sum_{n\in\mathbb{N}} \gamma_n^{\frac{1}{2}} \langle x, e_n\rangle_U \, e_n$$

as well as its pseudo-inverse $Q^{-1/2}$ by

$$Q^{-\frac{1}{2}}x := \sum_{n\in\mathbb{N}\,:\,\gamma_n\neq 0} \gamma_n^{-\frac{1}{2}} \langle x, e_n\rangle_U \, e_n.$$

We introduce the vector space $\mathcal{H} := Q^{1/2}U$. Then $\mathcal{H}$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}} := \langle Q^{-1/2}\cdot, Q^{-1/2}\cdot\rangle_U$.

Furthermore, let $A\colon \mathcal{D}(A) \subset H \to H$ be a densely defined, self-adjoint, positive definite linear operator, which is not necessarily bounded but which has a compact inverse. In this case $-A$ is the generator of an analytic semigroup of contractions $(S(t), t \geq 0)$ and for $r \geq 0$ the fractional power operator $A^{r/2}$ is well-defined on a domain $\mathcal{D}(A^{r/2}) \subset H$, cf. [7, Chapter 2]. We define the Hilbert space $\dot{H}^r$ as the completion of $\mathcal{D}(A^{r/2})$ equipped with the inner product

$$\langle \varphi, \psi\rangle_{\dot{H}^r} := \langle A^{r/2}\varphi, A^{r/2}\psi\rangle_H$$

and obtain a scale of Hilbert spaces with $\dot{H}^s \subset \dot{H}^r \subset \dot{H}^0 = H$ for $0 \leq r \leq s$. Its role is to measure spatial regularity. We denote the special case when $r = 1$ by $V := \dot{H}^1$. In this way we obtain a Gelfand triple

$$V \hookrightarrow H \cong H^* \hookrightarrow V^*,$$

where we use * to denote the identification of the dual spaces of H and V with respect to the pivot space H. Later on, the notation $'$ will be used when addressing the dual space in its classical sense, i.e., as the space of all linear continuous mappings to $\mathbb{R}$. In addition, although the operator A is assumed to be self-adjoint, we denote by $A^*\colon V \to V^*$ its adjoint for clarification whenever we consider the adjoint instead of the operator itself. With these definitions, the operator A and its adjoint are bounded, i.e., $A, A^* \in \mathcal{L}(V; V^*)$, since for $\varphi, \psi \in V$ it holds

$$ {}_{V^*}\langle A\varphi, \psi\rangle_V = \langle A^{1/2}\varphi, A^{1/2}\psi\rangle_H = \langle \varphi, \psi\rangle_V = {}_V\langle \varphi, A^*\psi\rangle_{V^*}, $$

where ${}_{V^*}\langle\cdot,\cdot\rangle_V$ and ${}_V\langle\cdot,\cdot\rangle_{V^*}$ denote dual pairings between V and V^*.

We consider the stochastic partial differential equation

$$ \begin{aligned} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t &= G(X(t))\,\mathrm{d}L(t), \quad t \in \mathbb{T} := [0, T], \\ X(0) &= X_0, \end{aligned} \tag{2.2} $$

for finite $T > 0$. In order to obtain existence and uniqueness of a solution to this problem as well as additional regularity for its second moment, which will be needed later on, we impose the following assumptions on the initial value X_0 and the operator G.

Assumption 2.1. The initial value X_0 and the operator G in (2.2) satisfy:

(i) X_0 is a square-integrable, H-valued random variable, i.e., $X_0 \in L^2(\Omega; H)$, which is $\mathcal{F}_0$-measurable;
(ii) $G: H \to \mathcal{L}_2(\mathcal{H}; H)$ is an affine operator, i.e., $G(\varphi) = G_1(\varphi) + G_2$ with operators $G_1 \in \mathcal{L}(H, \mathcal{L}_2(\mathcal{H}; H))$ and $G_2 \in \mathcal{L}_2(\mathcal{H}; H)$;
(iii) There exists a regularity exponent $r \in [0, 1]$ such that $X_0 \in L^2(\Omega; \dot{H}^r)$ and $A^{r/2}S(\cdot)G_1 \in L^2(\mathbb{T}; \mathcal{L}(\dot{H}^r; \mathcal{L}_2(\mathcal{H}; H)))$, i.e.,

$$ \int_0^T \|A^{\frac{r}{2}} S(t) G_1\|^2_{\mathcal{L}(\dot{H}^r; \mathcal{L}_2(\mathcal{H}; H))}\,\mathrm{d}t < \infty; $$

(iv) $A^{1/2}S(\cdot)G_1 \in L^2(\mathbb{T}; \mathcal{L}(\dot{H}^r; \mathcal{L}_2(\mathcal{H}; H)))$, i.e.,

$$ \int_0^T \|A^{\frac{1}{2}} S(t) G_1\|^2_{\mathcal{L}(\dot{H}^r; \mathcal{L}_2(\mathcal{H}; H))}\,\mathrm{d}t < \infty, $$

with the same value for $r \in [0, 1]$ as in (iii);
(v) $G_1 \in \mathcal{L}(V, \mathcal{L}(U; H))$ and $G_2 \in \mathcal{L}(U; H)$.

Note that the assumption on G_1 in part (iv) implies the one in part (iii). Conditions (i)–(iii) guarantee $\dot{H}^r$ regularity of the mild solution (cf. Theorem 2.3), but we need all five assumptions for our main results in Sections 4 and 6.

Before we derive the deterministic variational problems satisfied by the second moment and the covariance of the solution X to (2.2) in Sections 4 and 6, we have to specify which kind of solvability we consider. In addition, existence and uniqueness of this solution must be guaranteed.

Definition 2.2. A predictable process $X: \Omega \times \mathbb{T} \to H$ is called a mild solution to (2.2), if $\sup_{t\in\mathbb{T}} \|X(t)\|_{L^2(\Omega;H)} < \infty$ and

$$ X(t) = S(t)X_0 + \int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s), \quad t \in \mathbb{T}. \tag{2.3} $$

It is a well-known result that there exists a unique mild solution to equations driven by affine multiplicative noise as considered above. More precisely, we have the following theorem.

Theorem 2.3. *Under Assumption 2.1 (i)–(ii) there exists (up to modification) a unique mild solution X of (2.2). If additionally Condition (iii) of Assumption 2.1 holds, then the mild solution satisfies*

$$\sup_{t\in\mathbb{T}} \|X(t)\|_{L^2(\Omega;\dot{H}^r)} < \infty,$$

i.e., $X \in L^\infty\big(\mathbb{T}; L^2(\Omega; \dot{H}^r)\big)$.

Proof. The first part of the theorem follows from [8, Theorem 9.29]. Suppose now that Condition (iii) is satisfied. By the dominated convergence theorem, the sequence of integrals

$$\int_0^T \|A^{\frac{r}{2}} S(\tau) G_1\|^2_{\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H))} \mathbb{1}_{(0,T/n)}(\tau)\, \mathrm{d}\tau,$$

where $n \in \mathbb{N}$ and $\mathbb{1}_{(0,T/n)}$ denotes the indicator function on the interval $(0, T/n)$, converges to zero as $n \to \infty$. Therefore, there exists $\widetilde{T} \in (0, T]$ such that

$$\kappa^2 := \int_0^{\widetilde{T}} \|A^{\frac{r}{2}} S(\tau) G_1\|^2_{\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H))} \, \mathrm{d}\tau < 1.$$

Define $\widetilde{\mathbb{T}} := \big[0, \widetilde{T}\big]$, $\mathcal{Z} := L^\infty\big(\widetilde{\mathbb{T}}; L^2(\Omega; \dot{H}^r)\big)$ and

$$\Upsilon \colon \mathcal{Z} \to \mathcal{Z}, \quad \Upsilon(Z)(t) := S(t)X_0 + \int_0^t S(t-s)G(Z(s))\, \mathrm{d}L(s), \quad t \in \widetilde{\mathbb{T}}.$$

Then Υ is a contraction: For every $t \in \widetilde{\mathbb{T}}$ and $Z_1, Z_2 \in \mathcal{Z}$ we have

$$\begin{aligned} \|\Upsilon(Z_1)(t) - \Upsilon(Z_2)(t)\|^2_{L^2(\Omega;\dot{H}^r)} &= \mathbb{E}\Big\| \int_0^t S(t-s)G_1(Z_1(s) - Z_2(s))\, \mathrm{d}L(s) \Big\|^2_{\dot{H}^r} \\ &= \mathbb{E}\Big\| \int_0^t A^{\frac{r}{2}} S(t-s)G_1(Z_1(s) - Z_2(s))\, \mathrm{d}L(s) \Big\|^2_{H}, \end{aligned}$$

since A and, hence, $A^{r/2}$ are closed operators. Now the application of Itô's isometry for the case of a Lévy process, cf. [8, Corollary 8.17], yields

$$\|\Upsilon(Z_1)(t) - \Upsilon(Z_2)(t)\|^2_{L^2(\Omega;\dot{H}^r)} = \mathbb{E}\int_0^t \|A^{\frac{r}{2}}S(t-s)G_1(Z_1(s)-Z_2(s))\|^2_{\mathcal{L}_2(\mathcal{H};H)}\,\mathrm{d}s$$

$$\leq \int_0^t \|A^{\frac{r}{2}}S(t-s)G_1\|^2_{\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H))}\mathbb{E}\Big[\|Z_1(s)-Z_2(s)\|^2_{\dot{H}^r}\Big]\,\mathrm{d}s,$$

where the interchanging of the expectation and the time integral is justified by Tonelli's theorem. Therefore, we obtain the estimate

$$\|\Upsilon(Z_1)(t) - \Upsilon(Z_2)(t)\|^2_{L^2(\Omega;\dot{H}^r)} \leq \kappa^2 \sup_{s\in\widetilde{\mathbb{T}}} \mathbb{E}\|Z_1(s)-Z_2(s)\|^2_{\dot{H}^r}$$

for all $t \in \mathbb{T}$ and $\|\Upsilon(Z_1) - \Upsilon(Z_2)\|_{\mathcal{Z}} \leq \kappa\|Z_1 - Z_2\|_{\mathcal{Z}}$, which shows that Υ is a contraction. By the Banach fixed point theorem, there exists a unique fixed point X_* of Υ in $\mathcal{Z}$. Hence, $X = X_*$ is the unique mild solution to (2.2) on $\widetilde{\mathbb{T}}$ and

$$\|X\|^2_{\mathcal{Z}} = \sup_{t\in\widetilde{\mathbb{T}}} \mathbb{E}\|X(t)\|^2_{\dot{H}^r} < \infty.$$

The claim of the theorem follows from iterating the same argument on the intervals

$$\big[(m-1)\widetilde{T},\ \min\{m\widetilde{T}, T\}\big], \quad m \in \big\{1, 2, \dots, \lceil T/\widetilde{T}\rceil\big\}. \qquad \square$$

Lemma 2.4 relates the concepts of weak and mild solutions of stochastic partial differential equations, cf. [8, Section 9.3], and provides the basis for establishing the connection between the second moment of the mild solution and a space–time variational problem. In order to state it, we first have to define the differential operator ∂_t and the weak stochastic integral. For a vector-valued function $u\colon \mathbb{T} \to H$ taking values in a Hilbert space H we define the distributional derivative $\partial_t u$ as the H-valued distribution satisfying

$$\langle(\partial_t u)(w), \varphi\rangle_H = -\int_0^T \frac{\mathrm{d}w}{\mathrm{d}t}(t)\langle u(t), \varphi\rangle_H\,\mathrm{d}t$$

for all $\varphi \in H$ and $w \in C_0^\infty(\mathbb{T};\mathbb{R})$, cf. [3, Definition 3 in §XVIII.1].

In the following, we consider the spaces $L^2(\Omega\times\mathbb{T};\mathcal{L}_2(\mathcal{H};H))$ as well as $L^2(\Omega\times\mathbb{T};\mathcal{L}(U;H))$ of square-integrable functions taking values in $\mathcal{L}_2(\mathcal{H};H)$ and $\mathcal{L}(U;H)$, respectively, with respect to the measure space $(\Omega\times\mathbb{T}, \mathcal{P}_{\mathbb{T}}, \mathbb{P}\otimes\lambda)$, where $\mathcal{P}_{\mathbb{T}}$ denotes the σ-algebra of predictable subsets of $\Omega\times\mathbb{T}$ and λ the Lebesgue measure on $\mathbb{T}$. For a predictable process $\Phi \in L^2(\Omega\times\mathbb{T};\mathcal{L}_2(\mathcal{H};H))$ and a continuous H-valued function $v \in C^0(\mathbb{T};H)$, we define the stochastic process $\Psi \in L^2(\Omega\times\mathbb{T};\mathcal{L}_2(\mathcal{H};\mathbb{R}))$ by

$$\Psi(t)\colon z \mapsto \langle v(t), \Phi(t)z\rangle_H \quad \forall z \in \mathcal{H},$$

for all $t \in \mathbb{T}$. The predictability of Ψ follows from the continuity of v on $\mathbb{T}$ and the predictability of Φ.

The weak stochastic integral $\int_0^T \langle v(t), \Phi(t)\,\mathrm{d}L(t)\rangle_H$ is then defined as the stochastic integral with respect to the integrand Ψ, i.e.,

$$\int_0^T \langle v(t), \Phi(t)\,\mathrm{d}L(t)\rangle_H := \int_0^T \Psi(t)\,\mathrm{d}L(t) \quad \mathbb{P}\text{-a.s.}, \tag{2.4}$$

cf. [8, p. 151]. Its properties imply by [8, Equation (9.20)] the following lemma.

Lemma 2.4. *Let Assumption 2.1 (i)–(ii) be satisfied and let X be the mild solution to (2.2). Then it holds $\mathbb{P}$-almost surely that*

$$\langle X, (-\partial_t + A^*)v\rangle_{L^2(\mathbb{T};H)} = \langle X_0, v(0)\rangle_H + \int_0^T \langle v(t), G(X(t))\,\mathrm{d}L(t)\rangle_H$$

for all $v \in C^1_{0,\{T\}}(\mathbb{T}; \mathcal{D}(A^*)) := \{w \in C^1(\mathbb{T}, \mathcal{D}(A^*)) : w(T) = 0\}$.

3. Auxiliary results

The aim of this section is to prove some auxiliary results that will be needed later on to derive the main results in Sections 4, 5, and 6.

In Subsection 3.1 we introduce different notions of tensor product spaces and some of their properties. The deterministic equations satisfied by the second moment and the covariance will be posed on these kinds of spaces.

Next, in Subsection 3.2, we use these tensor product spaces to define the covariance kernel associated with the driving Lévy process L and derive some additional results for the interaction of this covariance kernel with the operators G_1 and G_2, see Lemmas 3.4 and 3.5.

In order to formulate our main results in Sections 4–6 in a compact way, we introduce two operators in Subsection 3.3. These operators appear in the deterministic equations in Sections 4 and 6 and the results of this subsection provide the basis for proving their well-posedness in Section 5.

Finally, Subsection 3.4 is devoted to an Itô isometry for the weak stochastic integral driven by a Lévy process L.

3.1. Tensor product spaces

Before we formulate the first result, we have to introduce some definitions and notation: For two Banach spaces E_1 and E_2, we denote the *algebraic tensor product*, i.e., the tensor product of E_1 and E_2 as vector spaces, by $E_1 \otimes E_2$. The algebraic tensor product $E_1 \otimes E_2$ consists of all finite sums of the form

$$\sum_{k=1}^N \varphi_k \otimes \psi_k, \quad \varphi_k \in E_1,\ \psi_k \in E_2,\ k = 1, \ldots, N.$$

There are several ways to define a norm on this space. Here we introduce three of them:

(i) *Projective tensor product:* By taking the completion of the algebraic tensor product $E_1 \otimes E_2$ with respect to the projective norm defined for $x \in E_1 \otimes E_2$ by

$$\|x\|_{E_1 \hat{\otimes}_\pi E_2} := \inf \left\{ \sum_{k=1}^{N} \|\varphi_k\|_{E_1} \|\psi_k\|_{E_2} : x = \sum_{k=1}^{N} \varphi_k \otimes \psi_k \right\},$$

the projective tensor product space $E_1 \hat{\otimes}_\pi E_2$ is obtained. We abbreviate $E^{(\pi)} := E \hat{\otimes}_\pi E$, whenever $E_1 = E_2 = E$.

(ii) *Injective tensor product:* The injective norm of an element x in the algebraic tensor product space $E_1 \otimes E_2$ is defined as

$$\|x\|_{E_1 \hat{\otimes}_\varepsilon E_2} := \sup \left\{ \left| \sum_{k=1}^{N} f(\varphi_k)\, g(\psi_k) \right| : f \in B_{E_1'},\ g \in B_{E_2'} \right\},$$

where $B_{E_1'}$, $B_{E_2'}$ denote the closed unit balls in the dual spaces $E_j' := \mathcal{L}(E_j; \mathbb{R})$, $j = 1, 2$, and $\sum_{k=1}^{N} \varphi_k \otimes \psi_k$ is any representation of $x \in E_1 \otimes E_2$. Note that the value of the supremum is independent of the choice of the representation of x, cf. [9, p. 45]. The completion of $E_1 \otimes E_2$ with respect to this norm is called injective tensor product space and denoted by $E_1 \hat{\otimes}_\varepsilon E_2$. If $E_1 = E_2 = E$, the abbreviation $E^{(\varepsilon)} := E \hat{\otimes}_\varepsilon E$ is used.

(iii) *Hilbert tensor product:* If E_1 and E_2 are Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{E_1}$ and $\langle \cdot, \cdot \rangle_{E_2}$, the tensor product $E_1 \hat{\otimes} E_2$ is defined as the completion of the algebraic tensor product $E_1 \otimes E_2$ with respect to the norm induced by the inner product

$$\langle x, y \rangle_{E_1 \hat{\otimes} E_2} := \sum_{k=1}^{N} \sum_{\ell=1}^{M} \langle \varphi_k, \vartheta_\ell \rangle_{E_1} \langle \psi_k, \chi_\ell \rangle_{E_2},$$

where $x = \sum_{k=1}^{N} \varphi_k \otimes \psi_k$ and $y = \sum_{\ell=1}^{M} \vartheta_\ell \otimes \chi_\ell$ are representations of $x, y \in E_1 \otimes E_2$. For $E_1 = E_2 = E$, set $E^{(2)} := E \hat{\otimes} E$.

In the latter case, we obtain again a Hilbert space, whereas the vector spaces in (i) and (ii) are Banach spaces, which are in general *not reflexive*, cf. [9, Theorem 4.21]. The following lemma is an immediate consequence of the definitions above.

Lemma 3.1. *Let E_1, E_2, F_1, F_2 be Banach spaces and H_1, H_2, U_1, U_2 be Hilbert spaces.*

(i) *For bounded linear operators $S \in \mathcal{L}(E_1; F_1)$ and $T \in \mathcal{L}(E_2; F_2)$, there exists a unique bounded linear operator $S \hat{\otimes}_\pi T : E_1 \hat{\otimes}_\pi E_2 \to F_1 \hat{\otimes}_\pi F_2$ such that $(S \hat{\otimes}_\pi T)(x \otimes y) = (Sx) \otimes (Ty)$ for every $x \in E_1$, $y \in E_2$ and it holds*

$$\|S \hat{\otimes}_\pi T\|_{\mathcal{L}(E_1 \hat{\otimes}_\pi E_2; F_1 \hat{\otimes}_\pi F_2)} = \|S\|_{\mathcal{L}(E_1; F_1)} \|T\|_{\mathcal{L}(E_2; F_2)}.$$

(ii) *For bounded linear operators $S \in \mathcal{L}(H_1; U_1)$ and $T \in \mathcal{L}(H_2; U_2)$, there exists a unique bounded linear operator $S \hat{\otimes} T : H_1 \hat{\otimes} H_2 \to U_1 \hat{\otimes} U_2$ such that $(S \hat{\otimes} T)(x \otimes y) = (Sx) \otimes (Ty)$ for every $x \in H_1$, $y \in H_2$ and it holds*

$$\|S \hat{\otimes} T\|_{\mathcal{L}(H_1 \hat{\otimes} H_2; U_1 \hat{\otimes} U_2)} = \|S\|_{\mathcal{L}(H_1;U_1)} \|T\|_{\mathcal{L}(H_2;U_2)}.$$

(iii) *The following chain of continuous embeddings holds:*

$$H_1 \hat{\otimes}_\pi H_2 \hookrightarrow H_1 \hat{\otimes} H_2 \hookrightarrow H_1 \hat{\otimes}_\varepsilon H_2,$$

where all embedding constants are equal to 1.

Proof. For (i) see [9, Proposition 2.3].

To see that $S \otimes T$ is a bounded mapping with respect to the Hilbert tensor products in (ii), one may proceed as in [5, Section I.2.3] – there for the case $H_1 = U_1$ and $H_2 = U_2$. We may write $S \otimes T$ as $S \otimes T = (I_{U_1} \otimes T)(S \otimes I_{H_2})$ and for $x \in H_1 \otimes H_2$ we can choose a representation $\sum_{k=1}^N \varphi_k \otimes \psi_k$ of x, such that the vectors $\{\psi_k\}_{k=1}^N$ are orthonormal in H_2. Then we obtain

$$\begin{aligned}\|(S \otimes I_{H_2})x\|^2_{U_1 \hat{\otimes} H_2} &= \left\| \sum_{k=1}^N S\varphi_k \otimes \psi_k \right\|^2_{U_1 \hat{\otimes} H_2} = \sum_{k=1}^N \|S\varphi_k\|^2_{U_1} \\ &\le \|S\|^2_{\mathcal{L}(H_1;U_1)} \sum_{k=1}^N \|\varphi_k\|^2_{H_1} = \|S\|^2_{\mathcal{L}(H_1;U_1)} \|x\|^2_{H_1 \hat{\otimes} H_2}\end{aligned}$$

and, thus,

$$\|(S \otimes I_{H_2})x\|_{U_1 \hat{\otimes} H_2} \le \|S\|_{\mathcal{L}(H_1;U_1)} \|x\|_{H_1 \hat{\otimes} H_2}$$

for all $x \in H_1 \otimes H_2$. In the same way, one can prove that

$$\|(I_{U_1} \otimes T)y\|_{U_1 \hat{\otimes} U_2} \le \|T\|_{\mathcal{L}(H_2;U_2)} \|y\|_{U_1 \hat{\otimes} H_2}$$

for every $y \in U_1 \otimes H_2$ and conclude for $x \in H_1 \otimes H_2$

$$\|(S \otimes T)x\|_{U_1 \hat{\otimes} U_2} \le \|T\|_{\mathcal{L}(H_2;U_2)} \|(S \otimes I_{H_2})x\|_{U_1 \hat{\otimes} H_2} \le \|T\|_{\mathcal{L}(H_2;U_2)} \|S\|_{\mathcal{L}(H_1;U_1)} \|x\|_{H_1 \hat{\otimes} H_2}.$$

Therefore, there exists a unique continuous extension $S \hat{\otimes} T \in \mathcal{L}(H_1 \hat{\otimes} H_2; U_1 \hat{\otimes} U_2)$ with $\|S \hat{\otimes} T\|_{\mathcal{L}(H_1 \hat{\otimes} H_2; U_1 \hat{\otimes} U_2)} = \|S\|_{\mathcal{L}(H_1;U_1)} \|T\|_{\mathcal{L}(H_2;U_2)}$.

In order to prove (iii), let $x \in H_1 \otimes H_2$. Then we estimate

$$\|x\|_{H_1 \hat{\otimes} H_2} = \left\| \sum_{k=1}^N \varphi_k \otimes \psi_k \right\|_{H_1 \hat{\otimes} H_2} \le \sum_{k=1}^N \|\varphi_k \otimes \psi_k\|_{H_1 \hat{\otimes} H_2} = \sum_{k=1}^N \|\varphi_k\|_{H_1} \|\psi_k\|_{H_2}$$

for any representation $\sum_{k=1}^N \varphi_k \otimes \psi_k$ of x. This shows that $\|x\|_{H_1 \hat{\otimes} H_2} \le \|x\|_{H_1 \hat{\otimes}_\pi H_2}$ for all $x \in H_1 \otimes H_2$ and, thus, $H_1 \hat{\otimes}_\pi H_2 \hookrightarrow H_1 \hat{\otimes} H_2$ with embedding constant 1.

Furthermore, by the Riesz representation theorem, for $f \in B_{H_1'}$ and $g \in B_{H_2'}$ there exist $\chi_f \in B_{H_1}$ and $\chi_g \in B_{H_2}$ such that $\langle \chi_f, \varphi \rangle_{H_1} = f(\varphi)$, $\langle \chi_g, \psi \rangle_{H_2} = g(\psi)$ for all $\varphi \in H_1$, $\psi \in H_2$. This yields

$$\begin{aligned}\Big|\sum_{k=1}^{N} f(\varphi_k)\, g(\psi_k)\Big|^2 &= \Big|\sum_{k=1}^{N} \langle \chi_f, \varphi_k\rangle_{H_1} \langle \chi_g, \psi_k\rangle_{H_2}\Big|^2 \\ &= \sum_{k=1}^{N}\sum_{\ell=1}^{N} \langle \chi_f, \varphi_k\rangle_{H_1} \langle \chi_g, \psi_k\rangle_{H_2} \langle \chi_f, \varphi_\ell\rangle_{H_1} \langle \chi_g, \psi_\ell\rangle_{H_2} \\ &= \sum_{k=1}^{N}\sum_{\ell=1}^{N} \langle \langle \chi_f, \varphi_\ell\rangle_{H_1} \chi_f, \varphi_k\rangle_{H_1} \langle \langle \chi_g, \psi_\ell\rangle_{H_2} \chi_g, \psi_k\rangle_{H_2} \\ &= \sum_{k=1}^{N}\sum_{\ell=1}^{N} \langle P_{\chi_f}\varphi_\ell, \varphi_k\rangle_{H_1} \|\chi_f\|_{H_1}^2 \langle P_{\chi_g}\psi_\ell, \psi_k\rangle_{H_2} \|\chi_g\|_{H_2}^2,\end{aligned}$$

where P_{χ_f} and P_{χ_g} denote the orthogonal projections on the subspaces $\operatorname{span}\{\chi_f\} := \{\alpha\,\chi_f : \alpha \in \mathbb{R}\} \subset H_1$ and $\operatorname{span}\{\chi_g\} := \{\alpha\,\chi_g : \alpha \in \mathbb{R}\} \subset H_2$, i.e.,

$$P_{\chi_f}\varphi := \frac{\langle \chi_f, \varphi\rangle_{H_1}}{\|\chi_f\|_{H_1}^2}\,\chi_f, \quad \varphi \in H_1, \qquad P_{\chi_g}\psi := \frac{\langle \chi_g, \psi\rangle_{H_2}}{\|\chi_g\|_{H_2}^2}\,\chi_g, \quad \psi \in H_2.$$

By using the properties of orthogonal projections we estimate

$$\begin{aligned}\Big|\sum_{k=1}^{N} f(\varphi_k)\, g(\psi_k)\Big|^2 &= \|\chi_f\|_{H_1}^2 \|\chi_g\|_{H_2}^2 \sum_{k=1}^{N}\sum_{\ell=1}^{N} \langle P_{\chi_f}\varphi_\ell, P_{\chi_f}\varphi_k\rangle_{H_1} \langle P_{\chi_g}\psi_\ell, P_{\chi_g}\psi_k\rangle_{H_2} \\ &= \|\chi_f\|_{H_1}^2 \|\chi_g\|_{H_2}^2 \sum_{k=1}^{N}\sum_{\ell=1}^{N} \langle P_{\chi_f}\varphi_\ell \otimes P_{\chi_g}\psi_\ell, P_{\chi_f}\varphi_k \otimes P_{\chi_g}\psi_k\rangle_{H_1\hat{\otimes} H_2} \\ &= \|\chi_f\|_{H_1}^2 \|\chi_g\|_{H_2}^2 \Big\langle \sum_{\ell=1}^{N} P_{\chi_f}\varphi_\ell \otimes P_{\chi_g}\psi_\ell, \sum_{k=1}^{N} P_{\chi_f}\varphi_k \otimes P_{\chi_g}\psi_k \Big\rangle_{H_1\hat{\otimes} H_2} \\ &= \|\chi_f\|_{H_1}^2 \|\chi_g\|_{H_2}^2 \Big\| \sum_{k=1}^{N} P_{\chi_f}\varphi_k \otimes P_{\chi_g}\psi_k \Big\|_{H_1\hat{\otimes} H_2}^2 \\ &= \|\chi_f\|_{H_1}^2 \|\chi_g\|_{H_2}^2 \Big\| (P_{\chi_f} \hat{\otimes} P_{\chi_g}) \sum_{k=1}^{N} \varphi_k \otimes \psi_k \Big\|_{H_1\hat{\otimes} H_2}^2,\end{aligned}$$

where $P_{\chi_f} \hat{\otimes} P_{\chi_g}$ denotes the extension of $P_{\chi_f} \otimes P_{\chi_g}$ to $H_1 \hat{\otimes} H_2$, which has been introduced in Lemma 3.1 (ii). This lemma and $\|\chi_f\|_{H_1} \le 1$, $\|\chi_g\|_{H_2} \le 1$ yield

$$\begin{aligned}\Big|\sum_{k=1}^{N} f(\varphi_k)\, g(\psi_k)\Big|^2 &\le \|P_{\chi_f} \hat{\otimes} P_{\chi_g}\|_{\mathcal{L}(H_1\hat{\otimes} H_2; H_1\hat{\otimes} H_2)}^2 \Big\| \sum_{k=1}^{N} \varphi_k \otimes \psi_k \Big\|_{H_1\hat{\otimes} H_2}^2 \\ &= \|P_{\chi_f}\|_{\mathcal{L}(H_1;H_1)}^2 \|P_{\chi_g}\|_{\mathcal{L}(H_2;H_2)}^2 \|x\|_{H_1\hat{\otimes} H_2}^2 = \|x\|_{H_1\hat{\otimes} H_2}^2\end{aligned}$$

for any representation $\sum_{k=1}^{N} \varphi_k \otimes \psi_k$ of $x \in H_1 \otimes H_2$. Since $f \in B_{H_1'}$ and $g \in B_{H_2'}$ were arbitrarily chosen we obtain

$$\|x\|_{H_1 \hat{\otimes}_\varepsilon H_2} = \sup \left\{ \left| \sum_{k=1}^{N} f(\varphi_k)\, g(\psi_k) \right| : f \in B_{H_1'},\ g \in B_{H_2'} \right\} \le \|x\|_{H_1 \hat{\otimes} H_2}.$$

This yields $H_1 \hat{\otimes} H_2 \hookrightarrow H_1 \hat{\otimes}_\varepsilon H_2$ with embedding constant 1 and completes the proof. □

For our purpose – formulating variational problems on tensor product spaces for the second moment and the covariance of the mild solution to the stochastic partial differential equation – the following result on the dual space of the injective tensor product of separable Hilbert spaces will be important.

Lemma 3.2. *Let H_1 and H_2 be separable Hilbert spaces. Then the dual space of the injective tensor product space is isometrically isomorphic to the projective tensor product of the dual spaces, i.e., $(H_1 \hat{\otimes}_\varepsilon H_2)' \cong H_1' \hat{\otimes}_\pi H_2'$.*

Proof. The proof can be extracted from [9] as follows: The dual space of the injective tensor product space can be identified with the Banach space of integral bilinear forms on $H_1 \times H_2$ by [9, Proposition 3.14]. In addition, since H_1 and H_2 are separable Hilbert spaces, the dual spaces H_1' and H_2' have the so-called approximation property, which implies that the projective tensor product of them can be identified with the Banach space of nuclear bilinear forms on $H_1 \times H_2$ by [9, Corollary 4.8 (b)]. In general, the space of nuclear bilinear forms is only a subspace of the space of integral bilinear forms. Since we assume that H_1 and H_2 are separable Hilbert spaces, they have monotone shrinking Schauder bases and this fact implies that every integral bilinear form on $H_1 \times H_2$ is nuclear and the integral and nuclear norms coincide, cf. [9, Corollary 4.29]. Hence, the spaces $(H_1 \hat{\otimes}_\varepsilon H_2)'$ and $H_1' \hat{\otimes}_\pi H_2'$ are isometrically isomorphic. □

3.2. The covariance kernel and the multiplicative noise

For a U-valued Lévy process L with covariance operator Q as considered in Section 2, we define the covariance kernel $q \in U^{(2)}$ as the unique element in the tensor space $U^{(2)}$ satisfying

$$\langle q, x \otimes y \rangle_{U^{(2)}} = \langle Qx, y \rangle_U \tag{3.1}$$

for all $x, y \in U$. Note that for an orthonormal eigenbasis $(e_n)_{n\in\mathbb{N}} \subset U$ of Q with corresponding eigenvalues $(\gamma_n)_{n\in\mathbb{N}}$ we may expand

$$q = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \langle q, e_n \otimes e_m \rangle_{U^{(2)}} (e_n \otimes e_m) = \sum_{m\in\mathbb{N}} \gamma_m (e_m \otimes e_m) \tag{3.2}$$

with convergence of the series in $U^{(2)}$, since $(e_n \otimes e_m)_{n,m\in\mathbb{N}}$ is an orthonormal basis of $U^{(2)}$ and $\langle q, e_n \otimes e_m \rangle_{U^{(2)}} = \gamma_m \delta_{nm}$, where δ_{nm} denotes the Kronecker delta. In addition, we obtain convergence of the series also with respect to $U^{(\pi)}$, which is shown in the following lemma.

Lemma 3.3. *The series in (3.2) converges in $U^{(\pi)}$, i.e.,*

$$\lim_{M\to\infty} \Big\| q - \sum_{m=1}^{M} \gamma_m (e_m \otimes e_m) \Big\|_{U^{(\pi)}} = 0.$$

Proof. For $M \in \mathbb{N}$ define

$$q_M := \sum_{m=1}^{M} \gamma_m (e_m \otimes e_m). \tag{3.3}$$

The trace class property of Q implies that $\sum_{m\in\mathbb{N}} \gamma_m < \infty$. Hence, for any $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $\sum_{m=M+1}^{L} \gamma_m < \epsilon$ for all $L > M \geq N_0$ and $(q_M)_{M\in\mathbb{N}}$ is a Cauchy sequence in $U^{(\pi)}$, since for any $L > M \geq N_0$ we obtain

$$\|q_L - q_M\|_{U^{(\pi)}} = \Big\| \sum_{m=M+1}^{L} \gamma_m (e_m \otimes e_m) \Big\|_{U^{(\pi)}} \leq \sum_{m=M+1}^{L} \gamma_m < \epsilon.$$

The completeness of the space $U^{(\pi)}$ implies the existence of $q_* \in U^{(\pi)}$ such that $\lim\limits_{M\to\infty} \|q_M - q_*\|_{U^{(\pi)}} = 0$. The convergence $\lim\limits_{M\to\infty} q_M = q$ in $U^{(2)}$ and the continuous embedding $U^{(\pi)} \hookrightarrow U^{(2)}$, cf. Lemma 3.1 (iii), yield $q = q_* \in U^{(\pi)}$. □

The bilinear form and the right-hand side appearing in the deterministic variational problems in Sections 4 and 6, contain several terms depending on the operators G_1 and G_2 as well as on the kernel q that is associated with the covariance operator Q via (3.1). To verify that they are well-defined we introduce the following Bochner spaces as well as their inner products

$$\mathcal{W} := L^2(\mathbb{T}; H), \qquad \langle u_1, u_2 \rangle_{\mathcal{W}} := \int_0^T \langle u_1(t), u_2(t) \rangle_H \, \mathrm{d}t,$$

$$\mathcal{X} := L^2(\mathbb{T}; V), \qquad \langle v_1, v_2 \rangle_{\mathcal{X}} := \int_0^T \langle v_1(t), v_2(t) \rangle_V \, \mathrm{d}t$$

and derive the results of the two lemmas below.

Lemma 3.4. *For operators G_1 and G_2 satisfying Assumption 2.1 (v) the following hold:*

(i) *The linear operator $G_1 \otimes G_1 \colon U \otimes U \to \mathcal{L}(\mathcal{X}; \mathcal{W}) \otimes \mathcal{L}(\mathcal{X}; \mathcal{W})$,*

$$\sum_{\ell=1}^{M} \varphi_\ell^1 \otimes \varphi_\ell^2 \mapsto \sum_{\ell=1}^{M} G_1(\cdot)\varphi_\ell^1 \otimes G_1(\cdot)\varphi_\ell^2$$

admits a unique extension $G_1 \hat{\otimes}_\pi G_1 \in \mathcal{L}(U^{(\pi)}; \mathcal{L}(\mathcal{X}^{(\pi)}; \mathcal{W}^{(\pi)}))$.

(ii) *The linear operators* $G_1 \otimes G_2 \colon U \otimes U \to \mathcal{L}(\mathcal{X}; \mathcal{W}) \otimes H$ *and* $G_2 \otimes G_1 \colon U \otimes U \to H \otimes \mathcal{L}(\mathcal{X}; \mathcal{W})$,

$$\sum_{\ell=1}^{M} \varphi_\ell^1 \otimes \varphi_\ell^2 \mapsto \sum_{\ell=1}^{M} G_1(\cdot)\varphi_\ell^1 \otimes G_2\varphi_\ell^2, \quad \sum_{\ell=1}^{M} \varphi_\ell^1 \otimes \varphi_\ell^2 \mapsto \sum_{\ell=1}^{M} G_2\varphi_\ell^1 \otimes G_1(\cdot)\varphi_\ell^2$$

admit unique extensions $G_1 \hat{\otimes}_\pi G_2 \in \mathcal{L}(U^{(\pi)}; \mathcal{L}(\mathcal{X}; \mathcal{W} \hat{\otimes}_\pi H))$ *and* $G_2 \hat{\otimes}_\pi G_1 \in \mathcal{L}(U^{(\pi)}; \mathcal{L}(\mathcal{X}; H \hat{\otimes}_\pi \mathcal{W}))$.

(iii) *The linear operator* $G_2 \otimes G_2 \colon U \otimes U \to H \otimes H$,

$$\sum_{\ell=1}^{M} \varphi_\ell^1 \otimes \varphi_\ell^2 \mapsto \sum_{\ell=1}^{M} G_2\varphi_\ell^1 \otimes G_2\varphi_\ell^2$$

admits a unique extension $G_2 \hat{\otimes}_\pi G_2 \in \mathcal{L}(U^{(\pi)}; H^{(\pi)})$.

Proof. We first note that $G_1 \in \mathcal{L}(V; \mathcal{L}(U; H))$ implies that G_1 can be identified with an element in $\mathcal{L}(U; \mathcal{L}(\mathcal{X}; \mathcal{W}))$, because for any $\varphi \in U$ we estimate

$$\|G_1(\cdot)\varphi\|_{\mathcal{L}(\mathcal{X};\mathcal{W})} = \sup_{\substack{u\in\mathcal{X}\\ \|u\|_{\mathcal{X}}=1}} \|G_1(u)\varphi\|_{\mathcal{W}} = \sup_{\substack{u\in\mathcal{X}\\ \|u\|_{\mathcal{X}}=1}} \left(\int_0^T \|G_1(u(t))\varphi\|_H^2 \,\mathrm{d}t \right)^{\frac{1}{2}}$$

$$\leq \|\varphi\|_U \sup_{\substack{u\in\mathcal{X}\\ \|u\|_{\mathcal{X}}=1}} \left(\int_0^T \|G_1(u(t))\|_{\mathcal{L}(U;H)}^2 \,\mathrm{d}t \right)^{\frac{1}{2}}$$

$$\leq \|\varphi\|_U \sup_{\substack{u\in\mathcal{X}\\ \|u\|_{\mathcal{X}}=1}} \left(\int_0^T \|G_1\|_{\mathcal{L}(V;\mathcal{L}(U;H))}^2 \|u(t)\|_V^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \leq \|G_1\|_{\mathcal{L}(V;\mathcal{L}(U;H))} \|\varphi\|_U.$$

This inequality shows that

$$G_1 \in \mathcal{L}(U; \mathcal{L}(\mathcal{X}; \mathcal{W})), \quad \|G_1\|_{\mathcal{L}(U;\mathcal{L}(\mathcal{X};\mathcal{W}))} \leq \|G_1\|_{\mathcal{L}(V;\mathcal{L}(U;H))}.$$

In order to prove (i), note that by Lemma 3.1 (i) for two vectors $\varphi^1, \varphi^2 \in U$ there exists a unique operator $G_1(\cdot)\varphi^1 \hat{\otimes}_\pi G_1(\cdot)\varphi^2 \colon \mathcal{X}^{(\pi)} \to \mathcal{W}^{(\pi)}$ satisfying

$$\left(G_1(\cdot)\varphi^1 \hat{\otimes}_\pi G_1(\cdot)\varphi^2 \right)(u) = \sum_{k=1}^{N} G_1(u_k^1)\varphi^1 \otimes G_1(u_k^2)\varphi^2$$

for any representation $\sum_{k=1}^N u_k^1 \otimes u_k^2$ of $u \in \mathcal{X} \otimes \mathcal{X}$. This operator is bounded because

$$\|G_1(\cdot)\varphi^1 \hat{\otimes}_\pi G_1(\cdot)\varphi^2\|_{\mathcal{L}(\mathcal{X}^{(\pi)};\mathcal{W}^{(\pi)})} = \|G_1(\cdot)\varphi^1\|_{\mathcal{L}(\mathcal{X};\mathcal{W})} \|G_1(\cdot)\varphi^2\|_{\mathcal{L}(\mathcal{X};\mathcal{W})}.$$

In addition, for a representation $\sum_{\ell=1}^{M} \varphi_\ell^1 \otimes \varphi_\ell^2$ of $\varphi \in U \otimes U$ we may extend

$$(G_1(\cdot) \otimes G_1(\cdot))\varphi = \sum_{\ell=1}^{M} G_1(\cdot)\varphi_\ell^1 \otimes G_1(\cdot)\varphi_\ell^2 \colon \mathcal{X} \otimes \mathcal{X} \to \mathcal{W} \otimes \mathcal{W}$$

to a bounded linear operator $(G_1(\cdot) \hat{\otimes}_\pi G_1(\cdot))\varphi \in \mathcal{L}(\mathcal{X}^{(\pi)}; \mathcal{W}^{(\pi)})$, since

$$\begin{aligned}\|(G_1(\cdot) \otimes G_1(\cdot))\varphi\|_{\mathcal{L}(\mathcal{X}^{(\pi)};\mathcal{W}^{(\pi)})} &\le \sum_{\ell=1}^{M} \|G_1(\cdot)\varphi_\ell^1 \otimes G_1(\cdot)\varphi_\ell^2\|_{\mathcal{L}(\mathcal{X}^{(\pi)};\mathcal{W}^{(\pi)})} \\ = \sum_{\ell=1}^{M} \|G_1(\cdot)\varphi_\ell^1\|_{\mathcal{L}(\mathcal{X};\mathcal{W})} \|G_1(\cdot)\varphi_\ell^2\|_{\mathcal{L}(\mathcal{X};\mathcal{W})} &\le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}(U;H))} \sum_{\ell=1}^{M} \|\varphi^1\|_U \|\varphi^2\|_U\end{aligned}$$

by the observations above. Therefore, $(G_1(\cdot) \otimes G_1(\cdot))\varphi \in \mathcal{L}(\mathcal{X}^{(\pi)}; \mathcal{W}^{(\pi)})$ for all $\varphi \in U \otimes U$ with

$$\|(G_1(\cdot) \otimes G_1(\cdot))\varphi\|_{\mathcal{L}(\mathcal{X}^{(\pi)};\mathcal{W}^{(\pi)})} \le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}(U;H))} \|\varphi\|_{U^{(\pi)}}.$$

This estimate shows that $G_1 \otimes G_1 \colon U \otimes U \to \mathcal{L}(\mathcal{X}; \mathcal{W}) \otimes \mathcal{L}(\mathcal{X}; \mathcal{W})$ admits a unique continuous extension to an operator $G_1 \hat{\otimes}_\pi G_1 \in \mathcal{L}(U^{(\pi)}; \mathcal{L}(\mathcal{X}^{(\pi)}; \mathcal{W}^{(\pi)}))$.

For part (ii), let $\sum_{\ell=1}^{M} \varphi_\ell^1 \otimes \varphi_\ell^2$ be again a representation of $\varphi \in U \otimes U$. Then, for $u \in \mathcal{X}$, we calculate

$$\begin{aligned}\Big\|\sum_{\ell=1}^{M} G_1(u)\varphi_\ell^1 \otimes G_2\varphi_\ell^2\Big\|_{\mathcal{W}\hat{\otimes}_\pi H} &\le \sum_{\ell=1}^{M} \|G_1(u)\varphi_\ell^1\|_{\mathcal{W}} \|G_2\varphi_\ell^2\|_H \\ &\le \sum_{\ell=1}^{M} \|G_1(\cdot)\varphi_\ell^1\|_{\mathcal{L}(\mathcal{X};\mathcal{W})} \|u\|_{\mathcal{X}} \|G_2\|_{\mathcal{L}(U;H)} \|\varphi_\ell^2\|_U \\ &\le \|G_1\|_{\mathcal{L}(V;\mathcal{L}(U;H))} \|G_2\|_{\mathcal{L}(U;H)} \|u\|_{\mathcal{X}} \sum_{\ell=1}^{M} \|\varphi_\ell^1\|_U \|\varphi_\ell^2\|_U.\end{aligned}$$

This calculation implies that $(G_1(\cdot) \otimes G_2)\varphi \in \mathcal{L}(\mathcal{X}; \mathcal{W} \hat{\otimes}_\pi H)$ for any $\varphi \in U \otimes U$ with

$$\|(G_1(\cdot) \otimes G_2)\varphi\|_{\mathcal{L}(\mathcal{X};\mathcal{W}\hat{\otimes}_\pi H)} \le \|G_1\|_{\mathcal{L}(V;\mathcal{L}(U;H))} \|G_2\|_{\mathcal{L}(U;H)} \|\varphi\|_{U^{(\pi)}},$$

and that there exists a unique extension $G_1 \hat{\otimes}_\pi G_2 \in \mathcal{L}(U^{(\pi)}; \mathcal{L}(\mathcal{X}; \mathcal{W} \hat{\otimes}_\pi H))$. It is obvious that the same argumentation yields existence and uniqueness of an extension $G_2 \hat{\otimes}_\pi G_1 \in \mathcal{L}(U^{(\pi)}; \mathcal{L}(\mathcal{X}; H \hat{\otimes}_\pi \mathcal{W}))$ of $G_2 \otimes G_1$.

Assertion (iii) follows immediately, since $G_2 \in \mathcal{L}(U; H)$ implies the existence of $G_2 \hat{\otimes}_\pi G_2 \in \mathcal{L}(U^{(\pi)}; H^{(\pi)})$ by Lemma 3.1 (i). □

Lemma 3.5. *Define $q \in U^{(2)}$ as in (3.1) and let G_1 and G_2 satisfy Assumption 2.1 (v).*

(i) $(G_1 \otimes G_1)(\cdot)q \colon \mathcal{X}^{(\pi)} \to \mathcal{W}^{(\pi)}$ *is bounded and*

$$\|(G_1 \otimes G_1)(\cdot)q\|_{\mathcal{L}(\mathcal{X}^{(\pi)};\mathcal{W}^{(\pi)})} \le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))}; \tag{3.4}$$

(ii) $(G_1(\cdot) \otimes G_2)q \in \mathcal{L}(\mathcal{X}; \mathcal{W} \hat{\otimes}_\pi H)$ *and* $(G_2 \otimes G_1(\cdot))q \in \mathcal{L}(\mathcal{X}; H \hat{\otimes}_\pi \mathcal{W})$*;*
(iii) $(G_2 \otimes G_2)q \in H^{(\pi)}$.

Proof. The results $(G_1 \otimes G_1)(\cdot)q \in \mathcal{L}(\mathcal{X}^{(\pi)}; \mathcal{W}^{(\pi)})$, $(G_1(\cdot) \otimes G_2)q \in \mathcal{L}(\mathcal{X}; \mathcal{W} \hat{\otimes}_\pi H)$, $(G_2 \otimes G_1(\cdot))q \in \mathcal{L}(\mathcal{X}; H \hat{\otimes}_\pi \mathcal{W})$ and $(G_2 \otimes G_2)q \in H^{(\pi)}$ are immediate consequences of Lemma 3.4, since $q \in U^{(\pi)}$ by Lemma 3.3.

In order to prove the bound in (3.4), let $M \in \mathbb{N}$ and define $q_M \in U \otimes U$ as in (3.3). Set $f_m := \sqrt{\gamma_m}\, e_m$, $m \in \mathbb{N}$, and let $\sum_{k=1}^N u_k^1 \otimes u_k^2$ be a representation of $u \in \mathcal{X} \otimes \mathcal{X}$. Then we have

$$\begin{aligned}
\|(G_1 \hat{\otimes}_\pi G_1)(u)q_M\|_{\mathcal{W}^{(\pi)}} &\le \sum_{k=1}^N \sum_{m=1}^M \|G_1(u_k^1) f_m\|_{\mathcal{W}} \|G_1(u_k^2) f_m\|_{\mathcal{W}} \\
&\le \sum_{k=1}^N \left(\sum_{m=1}^M \|G_1(u_k^1) f_m\|^2_{\mathcal{W}} \right)^{\frac{1}{2}} \left(\sum_{m=1}^M \|G_1(u_k^2) f_m\|^2_{\mathcal{W}} \right)^{\frac{1}{2}} \\
&\le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))} \sum_{k=1}^N \|u_k^1\|_{\mathcal{X}} \|u_k^2\|_{\mathcal{X}},
\end{aligned}$$

since for $v \in \mathcal{X}$ we obtain

$$\sum_{m=1}^M \|G_1(v) f_m\|^2_{\mathcal{W}} = \int_0^T \sum_{m=1}^M \|G_1(v(t)) f_m\|^2_H \,\mathrm{d}t \le \int_0^T \|G_1(v(t))\|^2_{\mathcal{L}_2(\mathcal{H};H)} \,\mathrm{d}t,$$

where the last inequality follows from the fact that the set $\{f_j : j \in \mathbb{N}, \gamma_j \neq 0\}$ forms an orthonormal basis of $\mathcal{H}$. Therefore,

$$\sum_{m=1}^M \|G_1(v) f_m\|^2_{\mathcal{W}} \le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))} \int_0^T \|v(t)\|^2_V \,\mathrm{d}t = \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))} \|v\|^2_{\mathcal{X}}$$

and, hence, $(G_1 \hat{\otimes}_\pi G_1)(\cdot)q_M \in \mathcal{L}(\mathcal{X}^{(\pi)}; \mathcal{W}^{(\pi)})$ for all $M \in \mathbb{N}$ with

$$\|(G_1 \otimes G_1)(\cdot)q_M\|_{\mathcal{L}(\mathcal{X}^{(\pi)};\mathcal{W}^{(\pi)})} \le \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))}.$$

The bound for $(G_1 \hat{\otimes}_\pi G_1)(\cdot)q$ in (3.4) follows from Lemmas 3.3 and 3.4 (i), since $\lim_{M\to\infty} q_M = q$ in $U^{(\pi)}$ and $G_1 \hat{\otimes}_\pi G_1 \in \mathcal{L}(U^{(\pi)}; \mathcal{L}(\mathcal{X}^{(\pi)}; \mathcal{W}^{(\pi)}))$. $\square$

3.3. The diagonal trace operator

We introduce the spaces $H^1_{0,\{T\}}(\mathbb{T}; V^*) := \{v \in H^1(\mathbb{T}; V^*) : v(T) = 0\}$ as well as $\mathcal{Y} := L^2(\mathbb{T}; V) \cap H^1_{0,\{T\}}(\mathbb{T}; V^*)$, which is a Hilbert space with respect to the inner product

$$\langle v_1, v_2\rangle_{\mathcal{Y}} := \langle v_1, v_2\rangle_{L^2(\mathbb{T};V)} + \langle \partial_t v_1, \partial_t v_2\rangle_{L^2(\mathbb{T};V^*)}, \quad v_1, v_2 \in \mathcal{Y}.$$

Moreover, we obtain the following two continuous embeddings.

Lemma 3.6. *It holds that $\mathcal{Y} \hookrightarrow C^0(\mathbb{T}; H)$ with embedding constant $C \le 1$, i.e., $\sup_{s\in\mathbb{T}} \|v(s)\|_H \le \|v\|_{\mathcal{Y}}$ for every $v \in \mathcal{Y}$.*

Proof. For every $v \in \mathcal{Y} = L^2(\mathbb{T}; V) \cap H^1_{0,\{T\}}(\mathbb{T}; V^*)$ we have the relation

$$\|v(r)\|_H^2 - \|v(s)\|_H^2 = \int_s^r 2\ {}_{V^*}\langle \partial_t v(t), v(t)\rangle_V \,\mathrm{d}t, \quad r, s \in \mathbb{T},\ r > s,$$

cf. [3, §XVIII.1, Theorem 2]. Choosing $r = T$ and observing that $v(T) = 0$ leads to

$$\|v(s)\|_H^2 \le 2\,\|\partial_t v\|_{L^2(\mathbb{T};V^*)}\|v\|_{L^2(\mathbb{T};V)} \le \|\partial_t v\|^2_{L^2(\mathbb{T};V^*)} + \|v\|^2_{L^2(\mathbb{T};V)} = \|v\|^2_{\mathcal{Y}}. \qquad \square$$

Lemma 3.7. *The injective tensor product space satisfies $\mathcal{Y}^{(\varepsilon)} \hookrightarrow C^0(\mathbb{T}; H)^{(\varepsilon)}$ with embedding constant $C \le 1$.*

Proof. The continuous embedding of Lemma 3.6 implies that $\|g\|_{\mathcal{Y}'} \le \|g\|_{C^0(\mathbb{T};H)'}$ for all $g \in C^0(\mathbb{T}; H)'$. Therefore, the unit balls of the dual spaces satisfy $B_{C^0(\mathbb{T};H)'} \subset B_{\mathcal{Y}'}$ and the embedding of the injective tensor product spaces follows, since for $\sum_{k=1}^N v_k^1 \otimes v_k^2 \in \mathcal{Y} \otimes \mathcal{Y}$ we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^N v_k^1 \otimes v_k^2\Big\|_{C^0(\mathbb{T};H)^{(\varepsilon)}} &= \sup\left\{\Big|\sum_{k=1}^N f\big(v_k^1\big)\, g\big(v_k^2\big)\Big| : f, g \in B_{C^0(\mathbb{T};H)'}\right\} \\ &\le \sup\left\{\Big|\sum_{k=1}^N f\big(v_k^1\big)\, g\big(v_k^2\big)\Big| : f, g \in B_{\mathcal{Y}'}\right\} = \Big\|\sum_{k=1}^N v_k^1 \otimes v_k^2\Big\|_{\mathcal{Y}^{(\varepsilon)}}. \qquad \square\end{aligned}$$

In the deterministic equations satisfied by the second moment and the covariance, an operator associated with the *diagonal trace* will play an important role. For $u \in \mathcal{W} \otimes \mathcal{W}$, $v \in \mathcal{Y} \otimes \mathcal{Y}$ and representations $\sum_{k=1}^N u_k^1 \otimes u_k^2$ and $\sum_{\ell=1}^M v_\ell^1 \otimes v_\ell^2$ of u and v, respectively, we define

$$T_\delta(u)v := \sum_{k=1}^N \sum_{\ell=1}^M \int_0^T \langle u_k^1(t), v_\ell^1(t)\rangle_H \langle u_k^2(t), v_\ell^2(t)\rangle_H \,\mathrm{d}t. \tag{3.5}$$

In addition, for $\widetilde{u} \in \mathcal{W} \otimes H$ and $\hat{u} \in H \otimes \mathcal{W}$ with representations $\sum_{k=1}^{N} u_k \otimes \varphi_k$, and $\sum_{k=1}^{N} \varphi_k \otimes u_k$, $u_k \in \mathcal{W}$, $\varphi_k \in H$, respectively, as well as $\varphi \in H \otimes H$ with representation $\sum_{k=1}^{N} \varphi_k^1 \otimes \varphi_k^2$ we define T_δ accordingly,

$$T_\delta(\widetilde{u})v := \sum_{k=1}^{N} \sum_{\ell=1}^{M} \int_0^T \langle u_k(t), v_\ell^1(t) \rangle_H \langle \varphi_k, v_\ell^2(t) \rangle_H \, \mathrm{d}t,$$

$$T_\delta(\hat{u})v := \sum_{k=1}^{N} \sum_{\ell=1}^{M} \int_0^T \langle \varphi_k, v_\ell^1(t) \rangle_H \langle u_k(t), v_\ell^2(t) \rangle_H \, \mathrm{d}t,$$

$$T_\delta(\varphi)v := \sum_{k=1}^{N} \sum_{\ell=1}^{M} \int_0^T \langle \varphi_k^1, v_\ell^1(t) \rangle_H \langle \varphi_k^2, v_\ell^2(t) \rangle_H \, \mathrm{d}t.$$

With these definitions, T_δ admits unique extensions to bounded linear operators mapping from the projective tensor product spaces $\mathcal{W} \hat{\otimes}_\pi \mathcal{W}$, $\mathcal{W} \hat{\otimes}_\pi H$, $H \hat{\otimes}_\pi \mathcal{W}$, and $H \hat{\otimes}_\pi H$, respectively, to the dual space $\mathcal{Y}^{(\varepsilon)\prime} = \mathcal{L}(\mathcal{Y}^{(\varepsilon)}; \mathbb{R})$ of the injective tensor product space $\mathcal{Y} \hat{\otimes}_\varepsilon \mathcal{Y}$ as we prove in the following proposition.

Proposition 3.8. *The operator $T_\delta \colon (\mathcal{W} \otimes \mathcal{W}) \times (\mathcal{Y} \otimes \mathcal{Y}) \to \mathbb{R}$ defined in (3.5) admits a unique extension to a bounded linear operator $T_\delta \in \mathcal{L}(\mathcal{W}^{(\pi)}; \mathcal{Y}^{(\varepsilon)\prime})$ with $\|T_\delta\|_{\mathcal{L}(\mathcal{W}^{(\pi)}; \mathcal{Y}^{(\varepsilon)\prime})} \leq 1$. Furthermore, T_δ as an operator acting on $\mathcal{W} \otimes H$, $H \otimes \mathcal{W}$, and $H \otimes H$ admits unique extensions to $T_\delta \in \mathcal{L}(\mathcal{W} \hat{\otimes}_\pi H; \mathcal{Y}^{(\varepsilon)\prime})$, $T_\delta \in \mathcal{L}(H \hat{\otimes}_\pi \mathcal{W}; \mathcal{Y}^{(\varepsilon)\prime})$, and $T_\delta \in \mathcal{L}(H^{(\pi)}; \mathcal{Y}^{(\varepsilon)\prime})$, respectively.*

Proof. Let $u \in \mathcal{W} \otimes \mathcal{W}$ and $v \in \mathcal{Y} \otimes \mathcal{Y}$ with representations $u = \sum_{k=1}^{N} u_k^1 \otimes u_k^2$ and $v = \sum_{\ell=1}^{M} v_\ell^1 \otimes v_\ell^2$ be given. Then we obtain

$$\begin{aligned} |T_\delta(u)v| &= \Big| \sum_{k=1}^{N} \sum_{\ell=1}^{M} \int_0^T \langle u_k^1(t), v_\ell^1(t) \rangle_H \langle u_k^2(t), v_\ell^2(t) \rangle_H \, \mathrm{d}t \Big| \\ &\leq \sum_{k=1}^{N} \int_0^T \Big| \sum_{\ell=1}^{M} \langle u_k^1(t), v_\ell^1(t) \rangle_H \langle u_k^2(t), v_\ell^2(t) \rangle_H \Big| \, \mathrm{d}t \\ &\leq \sum_{k=1}^{N} \int_0^T \|u_k^1(t)\|_H \|u_k^2(t)\|_H \Big\| \sum_{\ell=1}^{M} v_\ell^1(t) \otimes v_\ell^2(t) \Big\|_{H^{(\varepsilon)}} \, \mathrm{d}t, \end{aligned}$$

since $\langle \varphi, \cdot \rangle_H \in B_{H'}$ for $\varphi \in B_H$. Therefore,

$$|T_\delta(u)v| \leq \sup_{t \in \mathbb{T}} \Big\| \sum_{\ell=1}^{M} v_\ell^1(t) \otimes v_\ell^2(t) \Big\|_{H^{(\varepsilon)}} \sum_{k=1}^{N} \int_0^T \|u_k^1(t)\|_H \|u_k^2(t)\|_H \, \mathrm{d}t$$

$$\leq \sup_{t\in\mathbb{T}} \sup_{f,g\in B_{H'}} \Big|\sum_{\ell=1}^{M} f\big(v_\ell^1(t)\big)\, g\big(v_\ell^2(t)\big)\Big| \sum_{k=1}^{N} \|u_k^1\|_{\mathcal{W}} \|u_k^2\|_{\mathcal{W}}$$

$$\leq \sup_{s,t\in\mathbb{T}} \sup_{f,g\in B_{H'}} \Big|\sum_{\ell=1}^{M} f\big(\delta_s(v_\ell^1)\big)\, g\big(\delta_t(v_\ell^2)\big)\Big| \sum_{k=1}^{N} \|u_k^1\|_{\mathcal{W}} \|u_k^2\|_{\mathcal{W}},$$

where $\delta_t\colon C^0(\mathbb{T}; H) \to H$ denotes the evaluation functional in $t \in \mathbb{T}$, i.e., $\delta_t(v) := v(t)$. We obtain the estimate

$$|T_\delta(u)v| \leq \sup_{\tilde f,\tilde g\in B_{C^0(\mathbb{T};H)'}} \Big|\sum_{\ell=1}^{M} \tilde f\big(v_\ell^1\big)\, \tilde g\big(v_\ell^2\big)\Big| \sum_{k=1}^{N} \|u_k^1\|_{\mathcal{W}} \|u_k^2\|_{\mathcal{W}},$$

because $f \circ \delta_t \in B_{C^0(\mathbb{T};H)'}$ for $f \in B_{H'}$ and $t \in \mathbb{T}$. Hence,

$$|T_\delta(u)v| \leq \|v\|_{C^0(\mathbb{T};H)^{(\varepsilon)}} \|u\|_{\mathcal{W}^{(\pi)}} \leq \|v\|_{\mathcal{Y}^{(\varepsilon)}} \|u\|_{\mathcal{W}^{(\pi)}},$$

since $\mathcal{Y}^{(\varepsilon)} \hookrightarrow C^0(\mathbb{T}; H)^{(\varepsilon)}$ with embedding constant 1 by Lemma 3.7, and T_δ admits a unique extension $T_\delta \in \mathcal{L}(\mathcal{W}^{(\pi)}; \mathcal{Y}^{(\varepsilon)\prime})$.

For $\widetilde{u} \in \mathcal{W} \otimes H$ and $\hat{u} \in H \otimes \mathcal{W}$ with representations $\sum_{k=1}^{N} u_k^1 \otimes \varphi_k$ and $\sum_{k=1}^{N} \varphi_k \otimes u_k^2$, respectively, one can prove in the same way as above that

$$|T_\delta(\widetilde{u})v| \leq \sqrt{T}\, \|v\|_{\mathcal{Y}^{(\varepsilon)}} \|\widetilde{u}\|_{\mathcal{W}\hat{\otimes}_\pi H}, \quad |T_\delta(\hat{u})v| \leq \sqrt{T}\, \|v\|_{\mathcal{Y}^{(\varepsilon)}} \|\hat{u}\|_{H\hat{\otimes}_\pi \mathcal{W}}$$

for all $v \in \mathcal{Y}^{(\varepsilon)}$. Finally, for $\varphi \in H \otimes H$ with representation $\sum_{k=1}^{N} \varphi_k^1 \otimes \varphi_k^2$ we obtain for all $v \in \mathcal{Y}^{(\varepsilon)}$

$$|T_\delta(\varphi)v| \leq T\, \|v\|_{\mathcal{Y}^{(\varepsilon)}} \|\varphi\|_{H^{(\pi)}}.$$

The last three estimates show that there exist unique extensions $T_\delta \in \mathcal{L}(\mathcal{W} \hat{\otimes}_\pi H; \mathcal{Y}^{(\varepsilon)\prime})$, $T_\delta \in \mathcal{L}(H \hat{\otimes}_\pi \mathcal{W}; \mathcal{Y}^{(\varepsilon)\prime})$, and $T_\delta \in \mathcal{L}(H^{(\pi)}; \mathcal{Y}^{(\varepsilon)\prime})$ and complete the proof. □

In addition to T_δ we define the operator $R_t\colon H \to \mathcal{Y}'$ for $t \in \mathbb{T}$ by

$$R_t(\varphi)v := \langle \varphi, v(t)\rangle_H, \quad v \in \mathcal{Y}. \tag{3.6}$$

The next lemma shows that we obtain a well-defined operator $R_{s,t} \in \mathcal{L}(H^{(\pi)}; \mathcal{Y}^{(\varepsilon)\prime})$ by setting $R_{s,t} := R_s \hat{\otimes}_\pi R_t$ for $s, t \in \mathbb{T}$.

Lemma 3.9. *The operator R_t defined for $t \in \mathbb{T}$ in (3.6) is bounded and satisfies $\|R_t\|_{\mathcal{L}(H;\mathcal{Y}')} \leq 1$. Furthermore, for $s, t \in \mathbb{T}$ the operator $R_{s,t} : H \otimes H \to \mathcal{Y}' \otimes \mathcal{Y}'$ defined for $\varphi \in H \otimes H$ by*

$$R_{s,t}(\varphi) := (R_s \otimes R_t)(\varphi) = \sum_{k=1}^{N} R_s(\varphi_k^1) \otimes R_t(\varphi_k^2), \tag{3.7}$$

where $\sum_{k=1}^N \varphi_k^1 \otimes \varphi_k^2$ *is a representation of* $\varphi \in H \otimes H$, *admits a unique extension to a bounded linear operator* $R_{s,t} \in \mathcal{L}(H^{(\pi)}; \mathcal{Y}^{(\varepsilon)\prime})$.

Proof. For $t \in \mathbb{T}$ and $\varphi \in H$ we calculate by using the Cauchy–Schwarz inequality and Lemma 3.6,

$$|R_t(\varphi)v| = |\langle \varphi, v(t)\rangle_H| \le \|\varphi\|_H \|v(t)\|_H \le \|\varphi\|_H \|v\|_{C^0(\mathbb{T};H)} \le \|\varphi\|_H \|v\|_{\mathcal{Y}}$$

for all $v \in \mathcal{Y}$. This proves that $R_t(\varphi) \in \mathcal{Y}'$ for all $\varphi \in H$ with $\|R_t(\varphi)\|_{\mathcal{Y}'} \le \|\varphi\|_H$, which implies the assertion $R_t \in \mathcal{L}(H; \mathcal{Y}')$ with $\|R_t\|_{\mathcal{L}(H;\mathcal{Y}')} \le 1$ for all $t \in \mathbb{T}$.

By Lemma 3.1 (i) there exists a unique continuous extension $R_{s,t} \in \mathcal{L}(H \hat{\otimes}_\pi H; \mathcal{Y}' \hat{\otimes}_\pi \mathcal{Y}')$ of $R_{s,t} \colon H \otimes H \to \mathcal{Y}' \otimes \mathcal{Y}'$ defined in (3.7) for $s, t \in \mathbb{T}$. The fact that $\mathcal{Y}^{(\varepsilon)\prime}$ is isometrically isomorphic to $\mathcal{Y}' \hat{\otimes}_\pi \mathcal{Y}'$, cf. Lemma 3.2, completes the proof. □

3.4. A weak Itô isometry

In this subsection the diagonal trace operator is used to formulate an isometry for the expectation of the product of two weak stochastic integrals driven by the same Lévy process. This isometry is an essential component in the derivation of the deterministic variational problems for the second moment and the covariance in Sections 4 and 6.

Lemma 3.10. *For a predictable process* $\Phi \in L^2(\Omega \times \mathbb{T}; \mathcal{L}(U; H))$ *and the covariance kernel* $q \in U^{(2)}$ *in (3.1) the function* $\mathbb{E}[\Phi(\cdot) \otimes \Phi(\cdot)]q$ *is a well-defined element in the space* $\mathcal{W}^{(\pi)}$. *The weak stochastic integral, cf. (2.4), satisfies for* $v_1, v_2 \in \mathcal{Y}$

$$\mathbb{E}\left[\int_0^T \langle v_1(s), \Phi(s)\,\mathrm{d}L(s)\rangle_H \int_0^T \langle v_2(t), \Phi(t)\,\mathrm{d}L(t)\rangle_H\right] = T_\delta(\mathbb{E}[\Phi(\cdot) \otimes \Phi(\cdot)]q)(v_1 \otimes v_2).$$

Proof. In order to prove that $\mathbb{E}[\Phi(\cdot) \otimes \Phi(\cdot)]q$ is a well-defined element in the space $\mathcal{W}^{(\pi)}$, it suffices to show that $\Phi(\cdot) \otimes \Phi(\cdot) \in L^1(\Omega; \mathcal{L}(U^{(\pi)}; \mathcal{W}^{(\pi)}))$, and, hence, $\mathbb{E}[\Phi(\cdot) \otimes \Phi(\cdot)] \in \mathcal{L}(U^{(\pi)}; \mathcal{W}^{(\pi)})$, since $q \in U^{(\pi)}$ by Lemma 3.3. To this end, we estimate

$$\begin{aligned}
\|\Phi(\cdot) \otimes \Phi(\cdot)\|_{L^1(\Omega;\mathcal{L}(U^{(\pi)};\mathcal{W}^{(\pi)}))} &= \mathbb{E}\left[\|\Phi(\cdot) \otimes \Phi(\cdot)\|_{\mathcal{L}(U^{(\pi)};\mathcal{W}^{(\pi)})}\right] = \mathbb{E}\left[\|\Phi(\cdot)\|^2_{\mathcal{L}(U;\mathcal{W})}\right] \\
&= \mathbb{E}\left[\sup_{\substack{\psi \in U \\ \|\psi\|_U = 1}} \int_0^T \|\Phi(t)\psi\|_H^2 \,\mathrm{d}t\right] \le \mathbb{E}\left[\int_0^T \sup_{\substack{\psi \in U \\ \|\psi\|_U = 1}} \|\Phi(t)\psi\|_H^2 \,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T \|\Phi(t)\|^2_{\mathcal{L}(U;H)} \,\mathrm{d}t\right] = \|\Phi\|^2_{L^2(\Omega\times\mathbb{T};\mathcal{L}(U;H))} < \infty.
\end{aligned}$$

In order to justify that the weak stochastic integrals are well-defined, we note that the following embedding holds,

$$L^2(\Omega \times \mathbb{T}; \mathcal{L}(U; H)) \hookrightarrow L^2(\Omega \times \mathbb{T}; \mathcal{L}_2(\mathcal{H}; H))$$

with embedding constant $\sqrt{\operatorname{tr}(Q)} < \infty$, since

$$\begin{aligned}\|\Phi\|^2_{L^2(\Omega\times\mathbb{T};\mathcal{L}_2(\mathcal{H};H))} &= \mathbb{E}\int_0^T \|\Phi(t)\|^2_{\mathcal{L}_2(\mathcal{H};H)}\,\mathrm{d}t = \mathbb{E}\int_0^T \sum_{j\in\mathcal{I}} \|\Phi(t) f_j\|^2_H\,\mathrm{d}t \\ &\le \mathbb{E}\int_0^T \sum_{j\in\mathcal{I}} \|\Phi(t)\|^2_{\mathcal{L}(U;H)}\|f_j\|^2_U\,\mathrm{d}t \\ &= \operatorname{tr}(Q)\,\mathbb{E}\int_0^T \|\Phi(t)\|^2_{\mathcal{L}(U;H)}\,\mathrm{d}t = \operatorname{tr}(Q)\,\|\Phi\|^2_{L^2(\Omega\times\mathbb{T};\mathcal{L}(U;H))},\end{aligned}$$

where $f_n := \sqrt{\gamma_n}\, e_n$ and $\mathcal{I} := \{j \in \mathbb{N} : \gamma_j \neq 0\}$ for an orthonormal eigenbasis $(e_n)_{n\in\mathbb{N}} \subset U$ of Q with corresponding eigenvalues $(\gamma_n)_{n\in\mathbb{N}}$. For this reason, the weak stochastic integrals $\int_0^T \langle v_\ell(t), \Phi(t)\,\mathrm{d}L(t)\rangle_H$ are well-defined for $v_\ell \in \mathcal{Y} \subset C^0(\mathbb{T}; H)$, $\ell \in \{1, 2\}$. Recalling the definition of the weak stochastic integral in (2.4) yields the equality

$$\int_0^T \langle v_\ell(t), \Phi(t)\,\mathrm{d}L(t)\rangle_H = \int_0^T \Psi_\ell(t)\,\mathrm{d}L(t), \quad \ell = 1, 2,$$

where for $\ell \in \{1, 2\}$ the stochastic process $\Psi_\ell \in L^2(\Omega \times \mathbb{T}; \mathcal{L}(U; \mathbb{R}))$ is defined by

$$\Psi_\ell(t)\colon z \mapsto \langle v_\ell(t), \Phi(t)z\rangle_H \quad \forall z \in \mathcal{H}$$

for all $t \in \mathbb{T}$. Applying Itô's isometry, see [8, Corollary 8.17], along with the polarization identity, yields

$$\mathbb{E}\Big[\int_0^T \Psi_1(t)\,\mathrm{d}L(t) \int_0^T \Psi_2(t)\,\mathrm{d}L(t)\Big] = \int_0^T \mathbb{E}\big[\langle \Psi_1(t), \Psi_2(t)\rangle_{\mathcal{L}_2(\mathcal{H};\mathbb{R})}\big]\,\mathrm{d}t,$$

where $\langle\cdot,\cdot\rangle_{\mathcal{L}_2(\mathcal{H};\mathbb{R})}$ denotes the Hilbert–Schmidt inner product, i.e.,

$$\langle \tilde{\Phi}, \tilde{\Psi}\rangle_{\mathcal{L}_2(\mathcal{H};\mathbb{R})} = \sum_{n\in\mathbb{N}} \tilde{\Phi}(\tilde{f}_n)\,\tilde{\Psi}(\tilde{f}_n)$$

for $\tilde{\Phi}, \tilde{\Psi} \in \mathcal{L}_2(\mathcal{H}; \mathbb{R})$, where $(\tilde{f}_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. By choosing the orthonormal basis $(f_j)_{j\in\mathcal{I}}$ from above we obtain

$$\mathbb{E}\Big[\int_0^T \langle v_1(s), \Phi(s)\,\mathrm{d}L(s)\rangle_H \int_0^T \langle v_2(t), \Phi(t)\,\mathrm{d}L(t)\rangle_H\Big] = \int_0^T \mathbb{E}\big[\langle \Psi_1(t), \Psi_2(t)\rangle_{\mathcal{L}_2(\mathcal{H};\mathbb{R})}\big]\,\mathrm{d}t$$

$$= \int_0^T \mathbb{E}\Big[\sum_{j\in\mathcal{I}} \langle v_1(t), \Phi(t) f_j\rangle_H \langle v_2(t), \Phi(t) f_j\rangle_H\Big] \mathrm{d}t$$

$$= \int_0^T \mathbb{E}\Big[\sum_{n\in\mathbb{N}} \gamma_n \langle v_1(t), \Phi(t) e_n\rangle_H \langle v_2(t), \Phi(t) e_n\rangle_H\Big] \mathrm{d}t$$

$$= \int_0^T \mathbb{E}\Big[\sum_{n\in\mathbb{N}} \langle v_1(t)\otimes v_2(t), [\Phi(t)\otimes\Phi(t)]\gamma_n (e_n\otimes e_n)\rangle_{H^{(2)}}\Big] \mathrm{d}t$$

$$= \int_0^T \mathbb{E}\Big[\langle v_1(t)\otimes v_2(t), [\Phi(t)\otimes\Phi(t)]\sum_{n\in\mathbb{N}} \gamma_n (e_n\otimes e_n)\rangle_{H^{(2)}}\Big] \mathrm{d}t$$

$$= \int_0^T \mathbb{E}\big[\langle v_1(t)\otimes v_2(t), [\Phi(t)\otimes\Phi(t)] q\rangle_{H^{(2)}}\big] \mathrm{d}t.$$

By Proposition 3.8 the diagonal trace $T_\delta(\mathbb{E}[\Phi(\cdot)\otimes\Phi(\cdot)]q)(v_1\otimes v_2)$ is well-defined, since $\mathbb{E}[\Phi(\cdot)\otimes\Phi(\cdot)]q \in \mathcal{W}^{(\pi)}$. With the introduced notion of the operator T_δ we can rewrite the above expression as

$$\mathbb{E}\Big[\int_0^T \langle v_1(s), \Phi(s)\,\mathrm{d}L(s)\rangle_H \int_0^T \langle v_2(t), \Phi(t)\,\mathrm{d}L(t)\rangle_H\Big]$$

$$= \int_0^T \langle v_1(t)\otimes v_2(t), \mathbb{E}[\Phi(t)\otimes\Phi(t)]q\rangle_{H^{(2)}}\,\mathrm{d}t = T_\delta(\mathbb{E}[\Phi(\cdot)\otimes\Phi(\cdot)]q)(v_1\otimes v_2),$$

which completes the proof. □

4. The second moment

After having introduced the stochastic partial differential equation of interest and its mild solution in Section 2, the aim of this section is to derive a well-posed deterministic variational problem, which is satisfied by the second moment of the mild solution.

The second moment of a random variable $Y \in L^2(\Omega; H_1)$ taking values in a Hilbert space H_1 is denoted by $\mathbb{M}^{(2)}Y := \mathbb{E}[Y\otimes Y]$. We recall the Bochner spaces $\mathcal{W} = L^2(\mathbb{T}; H)$, $\mathcal{X} = L^2(\mathbb{T}; V)$ and $\mathcal{Y} = L^2(\mathbb{T}; V)\cap H^1_{0,\{T\}}(\mathbb{T}; V^*)$. It follows immediately from the definition of the mild solution that its second moment is an element of the tensor space $\mathcal{W}^{(2)}$. Under the assumptions made above we can prove even more regularity.

Theorem 4.1. *Let Assumption 2.1 (i)–(iv) be satisfied. Then the second moment of the mild solution X defined in (2.3) satisfies $\mathbb{M}^{(2)}X \in \mathcal{X}^{(\pi)} = \mathcal{X}\,\hat{\otimes}_\pi\,\mathcal{X}$.*

Proof. First, we remark that

$$\|\mathbb{M}^{(2)}X\|_{\mathcal{X}^{(\pi)}} = \|\mathbb{E}[X\otimes X]\|_{\mathcal{X}^{(\pi)}} \le \mathbb{E}\|X\otimes X\|_{\mathcal{X}^{(\pi)}} = \mathbb{E}\Big[\|X\|^2_{\mathcal{X}}\Big].$$

Hence, we may estimate as follows:

$$\begin{aligned}
\|\mathbb{M}^{(2)}X\|_{\mathcal{X}^{(\pi)}} &\le \mathbb{E}\int_0^T \Big\| S(t)X_0 + \int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s)\Big\|_V^2 \,\mathrm{d}t \\
&\le 2\,\mathbb{E}\int_0^T \Big[\|S(t)X_0\|_V^2 + \Big\|\int_0^t S(t-s)G(X(s))\,\mathrm{d}L(s)\Big\|_V^2\Big]\mathrm{d}t \\
&= 2\,\mathbb{E}\Big[\int_0^T \|A^{\frac12}S(t)X_0\|_H^2\,\mathrm{d}t\Big] + 2\int_0^T \mathbb{E}\Big\|\int_0^t A^{\frac12}S(t-s)G(X(s))\,\mathrm{d}L(s)\Big\|_H^2\,\mathrm{d}t.
\end{aligned}$$

Since the generator $-A$ of the semigroup $(S(t), t\ge 0)$ is self-adjoint and negative definite, we can bound the first integral from above by using the inequality

$$\int_0^T \|A^{\frac12}S(t)\varphi\|_H^2\,\mathrm{d}t \le \frac12\|\varphi\|_H^2, \qquad \varphi\in H, \tag{4.1}$$

and for the second term we use Itô's isometry, cf. [8, Corollary 8.17], as well as the affine structure of the operator G to obtain

$$\begin{aligned}
\|\mathbb{M}^{(2)}X\|_{\mathcal{X}^{(\pi)}} &\le \mathbb{E}\|X_0\|_H^2 + 2\int_0^T \mathbb{E}\int_0^t \|A^{\frac12}S(t-s)G(X(s))\|^2_{\mathcal{L}_2(\mathcal{H};H)}\,\mathrm{d}s\,\mathrm{d}t \\
&\le \mathbb{E}\|X_0\|_H^2 + 4\int_0^T\int_0^t \|A^{\frac12}S(t-s)G_2\|^2_{\mathcal{L}_2(\mathcal{H};H)}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + 4\int_0^T \mathbb{E}\int_0^t \|A^{\frac12}S(t-s)G_1(X(s))\|^2_{\mathcal{L}_2(\mathcal{H};H)}\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

By Assumption 2.1 (i)–(iii) as well as Theorem 2.3 there exists a regularity exponent $r\in[0,1]$ such that the mild solution satisfies $X\in L^\infty(\mathbb{T}; L^2(\Omega;\dot{H}^r))$. In addition, by Assumption 2.1 (iv) it holds that $A^{1/2}S(\cdot)G_1 \in L^2(\mathbb{T};\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H)))$. Then we estimate as follows,

$$\|\mathbb{M}^{(2)}X\|_{\mathcal{X}^{(\pi)}} \le \mathbb{E}\|X_0\|_H^2 + 4\sum_{n\in\mathbb{N}}\int_0^T\int_0^t \|A^{\frac12}S(t-s)G_2 f_n\|_H^2\,\mathrm{d}s\,\mathrm{d}t$$

$$+ 4 \int_0^T \int_0^t \|A^{\frac{1}{2}} S(t-s) G_1\|^2_{\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H))} \mathbb{E}\|X(s)\|^2_{\dot{H}^r} \, \mathrm{d}s \, \mathrm{d}t$$

for an orthonormal basis $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$. Applying (4.1) again with upper integral bound t instead of T yields

$$\|\mathbb{M}^{(2)} X\|_{\mathcal{X}^{(\pi)}} \le \|X_0\|^2_{L^2(\Omega;H)} + 2T\|G_2\|^2_{\mathcal{L}_2(\mathcal{H};H)} + 4T\|X\|^2_{L^\infty(\mathbb{T};L^2(\Omega;\dot{H}^r))} \|A^{\frac{1}{2}} S(\cdot) G_1\|^2_{L^2(\mathbb{T};\mathcal{L}(\dot{H}^r;\mathcal{L}_2(\mathcal{H};H)))},$$

which is finite under our assumptions and completes the proof. □

We define the bilinear form $\mathcal{B}\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ by

$$\mathcal{B}(u, v) := \int_0^T {}_V\langle u(t), (-\partial_t + A^*) v(t)\rangle_{V^*} \, \mathrm{d}t, \qquad u \in \mathcal{X}, v \in \mathcal{Y}, \tag{4.2}$$

and the mean function m of the mild solution X in (2.3) by

$$m(t) := \mathbb{E}X(t) = S(t)\mathbb{E}X_0, \quad t \in \mathbb{T}. \tag{4.3}$$

Note that due to the mean zero property of the stochastic integral the mean function depends only on the initial value X_0 and not on the operator G. Furthermore, applying inequality (4.1) shows the regularity $m \in \mathcal{X}$, and m can be interpreted as the unique function satisfying

$$m \in \mathcal{X}: \quad \mathcal{B}(m, v) = \langle \mathbb{E}X_0, v(0)\rangle_H \quad \forall v \in \mathcal{Y}. \tag{4.4}$$

Well-posedness of this problem follows from [11, Theorem 2.3].

In addition, we introduce the operator $\mathbb{B}\colon \mathcal{X} \to \mathcal{Y}'$ associated with the bilinear form $\mathcal{B}$, i.e., $\mathbb{B}u := \mathcal{B}(u, \cdot) \in \mathcal{Y}'$ for $u \in \mathcal{X}$. Then this linear operator is bounded, $\mathbb{B} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ and $\mathbb{B} \otimes \mathbb{B}\colon \mathcal{X} \otimes \mathcal{X} \to \mathcal{Y}' \otimes \mathcal{Y}'$ defined by

$$(\mathbb{B} \otimes \mathbb{B})\Big(\sum_{k=1}^N u_k^1 \otimes u_k^2\Big) := \sum_{k=1}^N \mathbb{B}u_k^1 \otimes \mathbb{B}u_k^2 = \sum_{k=1}^N \mathcal{B}(u_k^1, \cdot) \otimes \mathcal{B}(u_k^2, \cdot)$$

admits a unique extension to a bounded linear operator $\mathbb{B}^{(\pi)} \in \mathcal{L}(\mathcal{X}^{(\pi)}; (\mathcal{Y}')^{(\pi)})$ satisfying $\mathbb{B}^{(\pi)} = \mathbb{B} \otimes \mathbb{B}$ on $\mathcal{X} \otimes \mathcal{X}$ and $\|\mathbb{B}^{(\pi)}\|_{\mathcal{L}(\mathcal{X}^{(\pi)};(\mathcal{Y}')^{(\pi)})} = \|\mathbb{B}\|^2_{\mathcal{L}(\mathcal{X};\mathcal{Y}')}$ by Lemma 3.1 (i). With these definitions and preliminaries we are now able to show that the second moment of the mild solution solves a deterministic variational problem.

Theorem 4.2. *Let all conditions of Assumption 2.1 be satisfied and let X be the mild solution to (2.2). Then the second moment $\mathbb{M}^{(2)} X \in \mathcal{X}^{(\pi)}$ solves the following variational problem*

$$u \in \mathcal{X}^{(\pi)}: \quad \widetilde{\mathcal{B}}^{(\pi)}(u, v) = f(v) \quad \forall v \in \mathcal{Y}^{(\varepsilon)}, \tag{4.5}$$

where for $u \in \mathcal{X}^{(\pi)}$ and $v \in \mathcal{Y}^{(\varepsilon)}$

$$\widetilde{\mathcal{B}}^{(\pi)}(u,v) := \mathbb{B}^{(\pi)}(u)v - T_\delta((G_1 \otimes G_1)(u)q)v, \tag{4.6}$$
$$f(v) := R_{0,0}\big(\mathbb{M}^{(2)}X_0\big)v + T_\delta((G_1(m)\otimes G_2)q)v + T_\delta((G_2\otimes G_1(m))q)v + T_\delta((G_2\otimes G_2)q)v$$

with the operators T_δ and $R_{0,0}$ defined in (3.5) and (3.7) and the mean function $m \in \mathcal{X}$ in (4.3).

Proof. First, we remark that $\widetilde{\mathcal{B}}^{(\pi)}(u,v)$ is well-defined for $u \in \mathcal{X}^{(\pi)}$ and $v \in \mathcal{Y}^{(\varepsilon)}$, since the tensor spaces $\mathcal{Y}' \hat{\otimes}_\pi \mathcal{Y}'$ and $(\mathcal{Y} \hat{\otimes}_\varepsilon \mathcal{Y})'$ are isometrically isomorphic by Lemma 3.2 and, hence, $\mathbb{B}^{(\pi)}u - T_\delta((G_1\otimes G_1)(u)q) \in \mathcal{Y}^{(\varepsilon)\prime}$ for all $u \in \mathcal{X}^{(\pi)}$ by the definition of $\mathbb{B}^{(\pi)}$ and Proposition 3.8.

Let $v_1, v_2 \in C^1_{0,\{T\}}(\mathbb{T}; \mathcal{D}(A^*)) = \{\phi \in C^1(\mathbb{T}; \mathcal{D}(A^*)) : \phi(T) = 0\}$. Then we obtain

$$\begin{aligned}
\mathbb{B}^{(\pi)}\big(\mathbb{M}^{(2)}X\big)(v_1\otimes v_2) &= \mathbb{B}^{(\pi)}(\mathbb{E}[X\otimes X])(v_1\otimes v_2) = \mathbb{E}\big[\mathbb{B}^{(\pi)}(X\otimes X)(v_1\otimes v_2)\big] \\
&= \mathbb{E}\big[(\mathbb{B}(X)\otimes\mathbb{B}(X))(v_1\otimes v_2)\big] = \mathbb{E}[\mathcal{B}(X,v_1)\,\mathcal{B}(X,v_2)] \\
&= \mathbb{E}\big[\langle X, (-\partial_t + A^*)v_1\rangle_{L^2(\mathbb{T};H)}\langle X, (-\partial_t + A^*)v_2\rangle_{L^2(\mathbb{T};H)}\big].
\end{aligned}$$

Due to the regularity of v_1 and v_2 we may take the inner product on $L^2(\mathbb{T}; H)$ in this calculation. Now, since X is the mild solution of (2.2), Lemma 2.4 yields

$$\begin{aligned}
\mathbb{B}^{(\pi)}\big(\mathbb{M}^{(2)}X\big)(v_1\otimes v_2) &= \mathbb{E}\Big[\Big(\langle X_0, v_1(0)\rangle_H + \int_0^T \langle v_1(s), G(X(s))\,\mathrm{d}L(s)\rangle_H\Big) \\
&\qquad\cdot\Big(\langle X_0, v_2(0)\rangle_H + \int_0^T \langle v_2(t), G(X(t))\,\mathrm{d}L(t)\rangle_H\Big)\Big] \\
&= \mathbb{E}\left[\langle X_0, v_1(0)\rangle_H \langle X_0, v_2(0)\rangle_H\right] \\
&\quad + \mathbb{E}\Big[\langle X_0, v_1(0)\rangle_H \int_0^T \langle v_2(t), G(X(t))\,\mathrm{d}L(t)\rangle_H\Big] \\
&\quad + \mathbb{E}\Big[\langle X_0, v_2(0)\rangle_H \int_0^T \langle v_1(s), G(X(s))\,\mathrm{d}L(s)\rangle_H\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T \langle v_1(s), G(X(s))\,\mathrm{d}L(s)\rangle_H \int_0^T \langle v_2(t), G(X(t))\,\mathrm{d}L(t)\rangle_H\Big].
\end{aligned}$$

The $\mathcal{F}_0$-measurability of $X_0 \in L^2(\Omega; H)$ along with the independence of the stochastic integral with respect to $\mathcal{F}_0$ and its mean zero property imply that the second and the third term vanish: For $\ell \in \{1,2\}$ we define the $\mathcal{L}_2(\mathcal{H};\mathbb{R})$-valued stochastic process Ψ_ℓ $\mathbb{P}$-almost surely by

$$\Psi_\ell(t)\colon w \mapsto \langle v_\ell(t), G(X(t))w\rangle_H \quad \forall w \in \mathcal{H}$$

for $t \in \mathbb{T}$. Then we obtain $\|\Psi_\ell(t)\|^2_{\mathcal{L}_2(\mathcal{H};\mathbb{R})} = \|G(X(t))^* v_\ell(t)\|^2_{\mathcal{H}}$ $\mathbb{P}$-almost surely with the adjoint $G(X(t))^* \in \mathcal{L}(H;\mathcal{H})$ of $G(X(t))$ and

$$\mathbb{E}\Big[\langle X_0, v_\ell(0)\rangle_H \int_0^T \langle v_\ell(t), G(X(t))\,\mathrm{d}L(t)\rangle_H\Big] = \mathbb{E}\Big[\langle X_0, v_\ell(0)\rangle_H \int_0^T \Psi_\ell(t)\,\mathrm{d}L(t)\Big]$$

$$= \mathbb{E}\Big[\langle X_0, v_\ell(0)\rangle_H \mathbb{E}\Big[\int_0^T \Psi_\ell(t)\,\mathrm{d}L(t)\,\Big|\,\mathcal{F}_0\Big]\Big] = 0$$

by the definition of the weak stochastic integral, cf. [8, p. 151], the independence of the stochastic integral with respect to $\mathcal{F}_0$, and the fact that the stochastic integral has mean zero. For the first term we calculate by using the operator $R_{0,0}$ defined in (3.7) and its continuity $R_{0,0} \in \mathcal{L}(H^{(\pi)}; \mathcal{Y}^{(\varepsilon)\prime})$, cf. Lemma 3.9,

$$\mathbb{E}[\langle X_0, v_1(0)\rangle_H \langle X_0, v_2(0)\rangle_H] = \mathbb{E}\left[R_{0,0}(X_0 \otimes X_0)(v_1 \otimes v_2)\right] = R_{0,0}(\mathbb{E}[X_0 \otimes X_0])(v_1 \otimes v_2)$$

$$= R_{0,0}\big(\mathbb{M}^{(2)} X_0\big)(v_1 \otimes v_2).$$

Finally, the predictability of X together with the continuity assumptions on G imply the predictability of $G(X)$ and we may use Lemma 3.10 for the last term yielding

$$\mathbb{E}\Big[\int_0^T \langle v_1(s), G(X(s))\,\mathrm{d}L(s)\rangle_H \int_0^T \langle v_2(t), G(X(t))\,\mathrm{d}L(t)\rangle_H\Big]$$

$$= T_\delta(\mathbb{E}[G(X) \otimes G(X)]q)(v_1 \otimes v_2)$$

$$= T_\delta(\mathbb{E}[G_1(X) \otimes G_1(X)]q)(v_1 \otimes v_2) + T_\delta((\mathbb{E}[G_1(X)] \otimes G_2)q)(v_1 \otimes v_2)$$
$$+ T_\delta((G_2 \otimes \mathbb{E}[G_1(X)])q)(v_1 \otimes v_2) + T_\delta((G_2 \otimes G_2)q)(v_1 \otimes v_2)$$

$$= T_\delta((G_1 \otimes G_1)(\mathbb{M}^{(2)} X)q)(v_1 \otimes v_2) + T_\delta((G_1(m) \otimes G_2)q)(v_1 \otimes v_2)$$
$$+ T_\delta((G_2 \otimes G_1(m))q)(v_1 \otimes v_2) + T_\delta((G_2 \otimes G_2)q)(v_1 \otimes v_2).$$

Since $C^1_{0,\{T\}}(\mathbb{T}; \mathcal{D}(A^*)) \subset \mathcal{Y}$ is a dense subset, the claim follows. □

5. Existence and uniqueness

Before we extend the results of Section 4 for the second moment to the covariance of the mild solution in Section 6, we investigate in this section well-posedness of the variational problem (4.5) satisfied by the second moment.

To this end, we first take a closer look at the variational problem (4.4) satisfied by the mean function $m = \mathbb{E}X$ of the solution process X. The bilinear form $\mathcal{B}$ arising in this problem is known to satisfy an inf-sup and a surjectivity condition on $\mathcal{X} \times \mathcal{Y}$, cf. the second part of [11, Theorem 2.2].

Theorem 5.1. *For the bilinear form $\mathcal{B}$ in (4.2) the following hold:*

$$\beta := \inf_{u\in\mathcal{X}\setminus\{0\}} \sup_{v\in\mathcal{Y}\setminus\{0\}} \frac{\mathcal{B}(u,v)}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} > 0, \tag{5.1}$$

$$\forall v \in \mathcal{Y}\setminus\{0\}: \quad \sup_{u\in\mathcal{X}} \mathcal{B}(u,v) > 0.$$

For proving well-posedness of the variational problem (4.5) satisfied by the second moment of the mild solution, we need a lower bound on the inf-sup constant β in (5.1). In order to derive this bound, we first recall the Nečas theorem, cf. [4, Theorem 2.2, p. 422].

Theorem 5.2 *(Nečas theorem). Let H_1 and H_2 be two separable Hilbert spaces and $\mathfrak{B}\colon H_1 \times H_2 \to \mathbb{R}$ a continuous bilinear form. Then the variational problem*

$$u \in H_1: \quad \mathfrak{B}(u,v) = f(v) \quad \forall v \in H_2 \tag{5.2}$$

admits a unique solution $u \in H_1$ for all $f \in H_2'$, which depends continuously on f, if and only if the bilinear form $\mathfrak{B}$ satisfies one of the following equivalent inf-sup conditions:

(i) *It holds*

$$\inf_{v_1\in H_1\setminus\{0\}} \sup_{v_2\in H_2\setminus\{0\}} \frac{\mathfrak{B}(v_1,v_2)}{\|v_1\|_{H_1}\|v_2\|_{H_2}} > 0, \qquad \inf_{v_2\in H_2\setminus\{0\}} \sup_{v_1\in H_1\setminus\{0\}} \frac{\mathfrak{B}(v_1,v_2)}{\|v_1\|_{H_1}\|v_2\|_{H_2}} > 0.$$

(ii) *There exists $\gamma > 0$ such that*

$$\inf_{v_1\in H_1\setminus\{0\}} \sup_{v_2\in H_2\setminus\{0\}} \frac{\mathfrak{B}(v_1,v_2)}{\|v_1\|_{H_1}\|v_2\|_{H_2}} = \inf_{v_2\in H_2\setminus\{0\}} \sup_{v_1\in H_1\setminus\{0\}} \frac{\mathfrak{B}(v_1,v_2)}{\|v_1\|_{H_1}\|v_2\|_{H_2}} = \gamma.$$

In addition, the solution u of (5.2) satisfies the stability estimate

$$\|u\|_{H_1} \le \gamma^{-1}\|f\|_{H_2'}.$$

By using the equivalence of the Conditions (i) and (ii) in the Nečas theorem we are able to calculate a lower bound on β in the following lemma.

Lemma 5.3. *The inf-sup constant β in (5.1) satisfies $\beta \ge 1$.*

Proof. Combining the results of Theorem 5.1 with the equivalence of (i) and (ii) in Theorem 5.2 yields the equality

$$\beta = \inf_{u\in\mathcal{X}\setminus\{0\}} \sup_{v\in\mathcal{Y}\setminus\{0\}} \frac{\mathcal{B}(u,v)}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} = \inf_{v\in\mathcal{Y}\setminus\{0\}} \sup_{u\in\mathcal{X}\setminus\{0\}} \frac{\mathcal{B}(u,v)}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}}.$$

To derive a lower bound for β, we proceed as in [10,12]. Fix $v \in \mathcal{Y}\setminus\{0\}$, and define $u := v - (A^*)^{-1}\partial_t v$, where $(A^*)^{-1}$ is the right-inverse of the surjection $A^* \in \mathcal{L}(V;V^*)$. Then $u \in \mathcal{X} = L^2(\mathbb{T};V)$, since $(A^*)^{-1} \in \mathcal{L}(V^*;V)$, and we calculate as follows:

$$\begin{aligned}\|u\|_{\mathcal{X}}^2 &= \int_0^T \|u(t)\|_V^2 \,\mathrm{d}t = \int_0^T {}_V\langle u(t), A^* u(t)\rangle_{V^*} \,\mathrm{d}t \\ &= \int_0^T {}_V\langle v(t) - (A^*)^{-1}\partial_t v(t), A^* v(t) - \partial_t v(t)\rangle_{V^*} \,\mathrm{d}t \\ &= \int_0^T {}_V\langle v(t), A^* v(t)\rangle_{V^*} \,\mathrm{d}t + \int_0^T {}_V\langle (A^*)^{-1}\partial_t v(t), \partial_t v(t)\rangle_{V^*} \,\mathrm{d}t \\ &\quad - \int_0^T {}_V\langle v(t), \partial_t v(t)\rangle_{V^*} \,\mathrm{d}t - \int_0^T {}_V\langle (A^*)^{-1}\partial_t v(t), A^* v(t)\rangle_{V^*} \,\mathrm{d}t.\end{aligned}$$

Now the symmetry of the inner product $\langle\cdot,\cdot\rangle_V$ on V yields

$$\begin{aligned}{}_V\langle (A^*)^{-1}\partial_t v(t), A^* v(t)\rangle_{V^*} &= \langle (A^*)^{-1}\partial_t v(t), v(t)\rangle_V = \langle v(t), (A^*)^{-1}\partial_t v(t)\rangle_V \\ &= {}_V\langle v(t), \partial_t v(t)\rangle_{V^*},\end{aligned}$$

and by inserting the identity $A^*(A^*)^{-1}$, using $\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_H^2 = 2\,{}_V\langle v(t), \partial_t v(t)\rangle_{V^*}$ and $v(T) = 0$ we obtain

$$\begin{aligned}\|u\|_{\mathcal{X}}^2 &= \|v\|_{\mathcal{X}}^2 + \|(A^*)^{-1}\partial_t v\|_{\mathcal{X}}^2 - \int_0^T 2\,{}_V\langle v(t), \partial_t v(t)\rangle_{V^*} \,\mathrm{d}t \\ &= \|v\|_{\mathcal{X}}^2 + \|(A^*)^{-1}\partial_t v\|_{\mathcal{X}}^2 + \|v(0)\|_H^2 \\ &\geq \|v\|_{\mathcal{X}}^2 + \|(A^*)^{-1}\partial_t v\|_{\mathcal{X}}^2 = \|v\|_{\mathcal{X}}^2 + \|\partial_t v\|_{L^2(\mathbb{T};V^*)}^2 = \|v\|_{\mathcal{Y}}^2.\end{aligned}$$

In the last line, we used that $\|w\|_{V^*} = \|(A^*)^{-1} w\|_V$ for every $w \in V^*$, since

$$\begin{aligned}\|w\|_{V^*} &= \sup_{v\in V\setminus\{0\}} \frac{{}_V\langle v, w\rangle_{V^*}}{\|v\|_V} \\ &= \sup_{v\in V\setminus\{0\}} \frac{{}_V\langle v, A^*((A^*)^{-1}w)\rangle_{V^*}}{\|v\|_V} = \sup_{v\in V\setminus\{0\}} \frac{\langle v, (A^*)^{-1}w\rangle_V}{\|v\|_V} = \|(A^*)^{-1}w\|_V.\end{aligned}$$

Hence, we obtain for any fixed $v \in \mathcal{Y}$ and $u = v - (A^*)^{-1}\partial_t v$ that $\|u\|_{\mathcal{X}} \geq \|v\|_{\mathcal{Y}}$. In addition, we estimate

$$\mathcal{B}(u, v) = \int_0^T {}_V\langle u(t), (-\partial_t + A^*)v(t)\rangle_{V^*} \,\mathrm{d}t$$

$$= \int_0^T {}_V\langle v(t) - (A^*)^{-1}\partial_t v(t), A^*(v(t) - (A^*)^{-1}\partial_t v(t))\rangle_{V^*}\, \mathrm{d}t$$

$$= \int_0^T \|v(t) - (A^*)^{-1}\partial_t v(t)\|_V^2\, \mathrm{d}t = \|v - (A^*)^{-1}\partial_t v\|_{\mathcal{X}}^2 = \|u\|_{\mathcal{X}}^2 \geq \|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}$$

and, therefore,

$$\sup_{w\in\mathcal{X}\setminus\{0\}} \frac{\mathcal{B}(w,v)}{\|w\|_{\mathcal{X}}} \geq \|v\|_{\mathcal{Y}} \quad \forall v \in \mathcal{Y}.$$

This shows the assertion

$$\beta = \inf_{w\in\mathcal{X}\setminus\{0\}} \sup_{v\in\mathcal{Y}\setminus\{0\}} \frac{\mathcal{B}(w,v)}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} = \inf_{v\in\mathcal{Y}\setminus\{0\}} \sup_{w\in\mathcal{X}\setminus\{0\}} \frac{\mathcal{B}(w,v)}{\|w\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} \geq 1. \qquad \square$$

The result on the inf-sup constant β in Lemma 5.3 above can be formulated in terms of the operator $\mathbb{B} \in \mathcal{L}(\mathcal{X}; \mathcal{Y}')$ associated with the bilinear form $\mathcal{B}$ as follows: For every $u \in \mathcal{X}$ it holds

$$\|\mathbb{B}u\|_{\mathcal{Y}'} = \sup_{v\in\mathcal{Y}\setminus\{0\}} \frac{\mathcal{B}(u,v)}{\|v\|_{\mathcal{Y}}} \geq \|u\|_{\mathcal{X}}, \tag{5.3}$$

i.e., $\mathbb{B}$ is injective and by Theorem 5.1 also surjective and, hence, boundedly invertible with $\|\mathbb{B}^{-1}\|_{\mathcal{L}(\mathcal{Y}';\mathcal{X})} \leq 1$.

These preliminary observations on the operator $\mathbb{B}$ associated with the bilinear form $\mathcal{B}$ yield the following result on the operator $\mathbb{B}^{(\pi)} = \mathbb{B} \hat{\otimes}_\pi \mathbb{B}$ mapping from the tensor product space $\mathcal{X}^{(\pi)}$ to the tensor product space $(\mathcal{Y}')^{(\pi)}$.

Lemma 5.4. *The unique operator $\mathbb{B}^{(\pi)} \in \mathcal{L}(\mathcal{X}^{(\pi)}; (\mathcal{Y}')^{(\pi)})$ satisfying $\mathbb{B}^{(\pi)}(u^1 \otimes u^2) = \mathbb{B}u^1 \otimes \mathbb{B}u^2$ for all $u^1, u^2 \in \mathcal{X}$ is injective and, moreover, it holds*

$$\|\mathbb{B}^{(\pi)}(u)\|_{(\mathcal{Y}')^{(\pi)}} \geq \|u\|_{\mathcal{X}^{(\pi)}} \tag{5.4}$$

for all $u \in \mathcal{X}^{(\pi)}$.

Proof. Let $u \in \mathcal{X} \otimes \mathcal{X}$ and $\sum_{k=1}^N u_k^1 \otimes u_k^2$ be a representation of u and $\sum_{\ell=1}^M f_\ell^1 \otimes f_\ell^2$ be a representation of $\mathbb{B}^{(\pi)}u$.

Since $\mathbb{B}$ is boundedly invertible, $\sum_{\ell=1}^M \mathbb{B}^{-1} f_\ell^1 \otimes \mathbb{B}^{-1} f_\ell^2$ is a well-defined element in $\mathcal{X} \otimes \mathcal{X}$ and, furthermore, it is a representation of u, since

$$u = \sum_{k=1}^N u_k^1 \otimes u_k^2 = \sum_{k=1}^N \left(\mathbb{B}^{-1}\mathbb{B}u_k^1\right) \otimes \left(\mathbb{B}^{-1}\mathbb{B}u_k^2\right) = \left(\mathbb{B}^{-1} \otimes \mathbb{B}^{-1}\right)\left(\sum_{k=1}^N \mathbb{B}u_k^1 \otimes \mathbb{B}u_k^2\right)$$

$$= \left(\mathbb{B}^{-1} \otimes \mathbb{B}^{-1}\right)\left(\mathbb{B}^{(\pi)}u\right) = \left(\mathbb{B}^{-1} \otimes \mathbb{B}^{-1}\right)\left(\sum_{\ell=1}^M f_\ell^1 \otimes f_\ell^2\right) = \sum_{\ell=1}^M \mathbb{B}^{-1} f_\ell^1 \otimes \mathbb{B}^{-1} f_\ell^2.$$

With this observation we can estimate

$$\|u\|_{\mathcal{X}^{(\pi)}} \leq \sum_{\ell=1}^{M} \|\mathbb{B}^{-1} f_\ell^1\|_{\mathcal{X}} \|\mathbb{B}^{-1} f_\ell^2\|_{\mathcal{X}} \leq \sum_{\ell=1}^{M} \|f_\ell^1\|_{\mathcal{Y}'} \|f_\ell^2\|_{\mathcal{Y}'},$$

since $\|\mathbb{B}^{-1}\|_{\mathcal{L}(\mathcal{Y}';\mathcal{X})} \leq 1$. This calculation shows $\|u\|_{\mathcal{X}^{(\pi)}} \leq \|\mathbb{B}^{(\pi)} u\|_{(\mathcal{Y}')^{(\pi)}}$ for all $u \in \mathcal{X}^{(\pi)}$ and the assertion is proven. □

By using this lemma together with the properties of the operator T_δ, which we have derived in Section 3.3, we now prove well-posedness of the variational problem satisfied by the second moment of the mild solution.

Theorem 5.5. *Suppose that*

$$\|G_1\|_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))} < 1. \tag{5.5}$$

Then the variational problem

$$w \in \mathcal{X}^{(\pi)}: \quad \widetilde{\mathcal{B}}^{(\pi)}(w, v) = f(v) \quad \forall v \in \mathcal{Y}^{(\varepsilon)} \tag{5.6}$$

admits at most one solution $w \in \mathcal{X}^{(\pi)}$ for every $f \in \mathcal{Y}^{(\varepsilon)\prime}$. In particular, there exists a unique solution $u \in \mathcal{X}^{(\pi)}$ satisfying (4.5).

Proof. It suffices to show that only $u = 0$ solves the homogeneous problem

$$u \in \mathcal{X}^{(\pi)}: \quad \widetilde{\mathcal{B}}^{(\pi)}(u, v) = 0 \quad \forall v \in \mathcal{Y}^{(\varepsilon)}.$$

For this purpose, let $u \in \mathcal{X}^{(\pi)}$ be a solution to the homogeneous problem. Then it holds

$$0 = \widetilde{\mathcal{B}}^{(\pi)}(u, v) = \mathbb{B}^{(\pi)}(u)v - T_\delta((G_1 \otimes G_1)(u)q)v$$

for all $v \in \mathcal{Y}^{(\varepsilon)}$ and, hence,

$$\|\mathbb{B}^{(\pi)} u - T_\delta((G_1 \otimes G_1)(u)q)\|_{\mathcal{Y}^{(\varepsilon)\prime}} = 0.$$

We calculate by using the estimate (5.4) of Lemma 5.4 as well as Lemma 3.2 as follows,

$$\begin{aligned}
\|u\|_{\mathcal{X}^{(\pi)}} &\leq \|\mathbb{B}^{(\pi)} u\|_{(\mathcal{Y}')^{(\pi)}} = \|\mathbb{B}^{(\pi)} u\|_{\mathcal{Y}^{(\varepsilon)\prime}} \\
&\leq \|\mathbb{B}^{(\pi)} u - T_\delta((G_1 \otimes G_1)(u)q)\|_{\mathcal{Y}^{(\varepsilon)\prime}} + \|T_\delta((G_1 \otimes G_1)(u)q)\|_{\mathcal{Y}^{(\varepsilon)\prime}} \\
&= \|T_\delta((G_1 \otimes G_1)(u)q)\|_{\mathcal{Y}^{(\varepsilon)\prime}}
\end{aligned}$$

In addition, Proposition 3.8 and Estimate (3.4) in Lemma 3.5 (i) yield

$$\begin{aligned}
\|u\|_{\mathcal{X}^{(\pi)}} &\leq \|T_\delta\|_{\mathcal{L}(\mathcal{W}^{(\pi)};\mathcal{Y}^{(\varepsilon)\prime})} \|(G_1 \otimes G_1)(u)q\|_{\mathcal{W}^{(\pi)}} \\
&\leq \|(G_1 \otimes G_1)(\cdot)q\|_{\mathcal{L}(\mathcal{X}^{(\pi)};\mathcal{W}^{(\pi)})} \|u\|_{\mathcal{X}^{(\pi)}} \leq \|G_1\|^2_{\mathcal{L}(V;\mathcal{L}_2(\mathcal{H};H))} \|u\|_{\mathcal{X}^{(\pi)}}.
\end{aligned}$$

Therefore, $u = 0$, if G_1 satisfies Condition (5.5), and the variational problem (5.6) has at most one solution. Under Assumption 2.1 on X_0 and the affine operator $G(\cdot) = G_1(\cdot) + G_2$ there exists a unique (up to modification) mild solution X to the stochastic partial differential equation (2.2) with second moment $\mathbb{M}^{(2)}X \in \mathcal{X}^{(\pi)}$ satisfying the variational problem (4.5), cf. Theorems 2.3, 4.1, and 4.2. Therefore, we obtain existence and uniqueness of a solution to (5.6) for the right-hand side

$$\begin{aligned} f(v) = R_{0,0}\big(\mathbb{M}^{(2)}X_0\big)v + T_\delta((G_1(m) \otimes G_2)q)v \\ + T_\delta((G_2 \otimes G_1(m))q)v + T_\delta((G_2 \otimes G_2)q)v, \end{aligned}$$

where $m = \mathbb{E}X$ and the variational problem (4.5) is well-posed. □

To conclude, we have shown in this section that there exists a variational problem that has the second moment of the mild solution (2.3) as its unique solution.

6. From the second moment to the covariance

In the previous sections, we have seen that the second moment $\mathbb{M}^{(2)}X$ of the mild solution X to the stochastic partial differential equation (2.2) satisfies a well-posed deterministic variational problem. As a consequence of this result we derive another deterministic problem in this section, which is satisfied by the covariance $\operatorname{Cov}(X)$ of the solution process. For this purpose, we remark first that

$$\begin{aligned} \operatorname{Cov}(X) &= \mathbb{E}[(X - \mathbb{E}X) \otimes (X - \mathbb{E}X)] \\ &= \mathbb{E}[(X \otimes X) - (\mathbb{E}X \otimes X) - (X \otimes \mathbb{E}X) + (\mathbb{E}X \otimes \mathbb{E}X)] \\ &= \mathbb{M}^{(2)}X - \mathbb{E}X \otimes \mathbb{E}X \end{aligned}$$

and $\operatorname{Cov}(X) \in \mathcal{X}^{(\pi)}$, since $\mathbb{M}^{(2)}X \in \mathcal{X}^{(\pi)}$ by Theorem 4.1 and $m = \mathbb{E}X \in \mathcal{X}$. By using this relation we can immediately deduce the following result for the covariance $\operatorname{Cov}(X)$ of the mild solution.

Theorem 6.1. *Let all conditions of Assumption 2.1 be satisfied and let X be the mild solution to (2.2). Then the covariance $\operatorname{Cov}(X) \in \mathcal{X}^{(\pi)}$ solves the well-posed problem*

$$u \in \mathcal{X}^{(\pi)}: \quad \widetilde{\mathcal{B}}^{(\pi)}(u, v) = g(v) \quad \forall v \in \mathcal{Y}^{(\varepsilon)} \tag{6.1}$$

with $\widetilde{\mathcal{B}}^{(\pi)}$ as in (4.6) and for $v \in \mathcal{Y}^{(\varepsilon)}$

$$g(v) := R_{0,0}(\operatorname{Cov}(X_0))v + T_\delta((G(m) \otimes G(m))q)v,$$

where T_δ and $R_{0,0}$ are the operators defined in (3.5) and (3.7) and $m \in \mathcal{X}$ denotes the mean function introduced in (4.3).

Proof. The covariance of the mild solution satisfies that $\operatorname{Cov}(X) = \mathbb{M}^{(2)}X - \mathbb{E}X \otimes \mathbb{E}X$ by the remark above. By using the result of Theorem 4.2 for the second moment $\mathbb{M}^{(2)}X$ as well as (4.4) for the mean function $m = \mathbb{E}X$ we calculate for $v_1, v_2 \in \mathcal{Y}$:

$$
\begin{aligned}
&\widetilde{\mathcal{B}}^{(\pi)}(\mathrm{Cov}(X), v_1 \otimes v_2) = \widetilde{\mathcal{B}}^{(\pi)}(\mathbb{M}^{(2)} X, v_1 \otimes v_2) - \widetilde{\mathcal{B}}^{(\pi)}(\mathbb{E}X \otimes \mathbb{E}X, v_1 \otimes v_2) \\
&\quad = R_{0,0}\big(\mathbb{M}^{(2)} X_0\big)(v_1 \otimes v_2) + T_\delta((G_2 \otimes G_2)q)(v_1 \otimes v_2) \\
&\qquad + T_\delta((G_1(m) \otimes G_2)q)(v_1 \otimes v_2)as + T_\delta((G_2 \otimes G_1(m))q)(v_1 \otimes v_2) \\
&\qquad - \langle \mathbb{E}X_0, v_1(0)\rangle_H \langle \mathbb{E}X_0, v_2(0)\rangle_H + T_\delta((G_1(m) \otimes G_1(m))q)(v_1 \otimes v_2) \\
&\quad = R_{0,0}\big(\mathbb{M}^{(2)} X_0\big)(v_1 \otimes v_2) - R_{0,0}(\mathbb{E}X_0 \otimes \mathbb{E}X_0)(v_1 \otimes v_2) + T_\delta((G(m) \otimes G(m))q)(v_1 \otimes v_2).
\end{aligned}
$$

Hence,

$$
\widetilde{\mathcal{B}}^{(\pi)}(\mathrm{Cov}(X), v_1 \otimes v_2) = g(v_1 \otimes v_2) \quad \forall v_1, v_2 \in \mathcal{Y}
$$

and this observation completes the proof, since the subset $\mathrm{span}\{v_1 \otimes v_2 : v_1, v_2 \in \mathcal{Y}\} \subset \mathcal{Y}^{(\varepsilon)}$ is dense and well-posedness of (6.1) follows from the existence of the mild solution X to (2.2) as well as its covariance $\mathrm{Cov}(X) \in \mathcal{X}^{(\pi)}$ and Theorem 5.5. □

Remark 6.2. Theorem 6.1 shows that, if only the covariance of the mild solution to the stochastic partial differential equation (2.2) needs to be computed, then one can do this by solving sequentially two deterministic variational problems: first, the more or less standard parabolic problem (4.4) for the mean function and afterwards problem (6.1) for the covariance, which is posed on non-reflexive tensor product spaces.

Acknowledgments

This work was supported in part by the Knut and Alice Wallenberg Foundation as well as the Swedish Research Council under Reg. No. 621-2014-3995. The authors thank Roman Andreev, Arnulf Jentzen, and Christoph Schwab for fruitful discussions and helpful comments.

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