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Robust Guaranteed Cost Output-Feedback Gain-Scheduled Controller Design ⋆

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Abstract: In the paper a new robust guaranteed cost output-feedback gain-scheduled PID controller design technique is presented for affine linear parameter-varying systems under polytopic model uncertainty, with the assumption that the scheduled parameters are affected with absolute uncertainty. The proposed centralized or decentralized method is based on the Bellman-Lyapunov equation, guaranteed cost, and parameter-dependent quadratic stability. The robust stability and performance conditions are translated to an optimization problem subject to bilinear matrix inequalities, which can be solved or further linearized. As the main result, the suggested stability and performance conditions without any restrictions on the controller structure are *convex* functions of the scheduling and uncertainty parameters. Hence, there is no need for applying multi-convexity or other relaxation techniques and consequently the proposed solution delivers a less conservative design method. The viability of the novel design technique is demonstrated and evaluated through numerical examples.

Keywords: Robust controller, gain scheduling, linear time-varying system, parameter-dependent Lyapunov function, Bellman-Lyapunov equation.

1. INTRODUCTION

Over the past three decades, gain scheduling and linear parameter-varying (LPV) techniques for nonlinear systems have been actively researched. Lyapunov theory and small-gain theorem are two main (not independent) research directions for testing and synthetizing performance and stability of LPV systems. This paper contributes to controller design techniques using the Lyapunov theory in the affine LPV framework. For a more comprehensive survey of the field, readers are also referred to survey papers (Wei et al., 2014; Leith and Leithead, 2000; Rugh and Shamma, 2000) and references therein.

Within the mentioned framework, convexification in the scheduling and uncertain parameter dependency of the closed-loop conditions ensures obtaining the controller design as an optimization problem subject to some finite number of linear and/or bilinear matrix inequality (LMI/BMI) constraints. The convexity (in the scheduling parameter dependency) can be ensured with some conservativeness by using the multi-convexification technique introduced in (Gahinet et al., 1996) within the affine quadratic stability (AQS) framework. As a result, researchers have started to apply the multi-convexity requirement in affine LPV framework for gain-scheduled controller, filter, and observer design (Veselý and Ilka, 2013; Liu et al., 2014; Sato, 2006; Bara et al., 2001). Furthermore, different relaxation techniques have been deployed to reduce the conservativeness caused by the multi-convexity requirement (Tuan and Apkarian, 1999; Ichihara et al., 2003; Adegas and Stoustrup, 2011; Veselý and Ilka, 2015b). Nevertheless, these relaxations can have significant influence on the performance and can drift the guaranteed cost far away from its optima. Along this line, convexity (in the scheduling parameter dependency) can be ensured also by restricting the closed-loop LPV structure, system or controller to avoid cross term effects of the scheduling parameters. In (Aouani et al., 2013) the convexity under AQS conditions for affine LPV systems is ensured by making the static output-feedback controller parameters independent on the scheduled parameters. Another option is to restrict the system interconnection with the control input $u(t)$, as in (Sato, 2011), to annihilate multiplication of the scheduling parameters. The same idea has been applied in (Sato and Peaucelle, 2013) and (Sato, 2015). Similar idea was presented in papers (Veselý and Ilka, 2015a) and (Emedi and Karimi, 2016) where some of the dynamic output-feedback LPV controller's matrices were restricted to be parameter-independent in order to obtain convex stability conditions regarding the scheduling parameters.

In this paper, we present a new solution to this specific problem, which ensures convex dependency on the scheduling and uncertain parameters without any restrictions in controller matrices. The proposed solution is carried out for fixed-order output-feedback gain-scheduled controllers

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under affine scheduling dependency. The performance metric is defined in a standard LQR fashion. The paper is organized as follows. Section 2 gives preliminaries and problem formulation. Section 3 gives our main results and in Section 4 the proposed method is validated on two examples.

Our notations are standard. $D \in R^{m\times n}$ denotes the set of real $m \times n$ matrices, I_m is an $m \times m$ identity matrix, $P > 0 (P \geq 0)$ is real symmetric, positive definite (semidefinite) matrix. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the continuous-time uncertain linear parameter-varying system in the form

$$\begin{aligned} \dot{x} &= A(\hat{\theta},\xi)x + B(\hat{\theta},\xi)u, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$ denote the state, control input and controlled output, respectively. The linear parameter-varying system matrices are assumed to depend on the scheduled vector parameter $\hat{\theta}$ as follows:

$$\begin{aligned} A(\hat{\theta},\xi) &= A_0(\xi) + \sum_{j=1}^{p} A_j(\xi)\hat{\theta}_j \in \mathbb{R}^{n\times n}, \\ B(\hat{\theta},\xi) &= B_0(\xi) + \sum_{j=1}^{p} B_j(\xi)\hat{\theta}_j \in \mathbb{R}^{n\times m}. \end{aligned} \tag{2}$$

Matrices $A_j(\xi), B_j(\xi), j = 0,1,2,\ldots,p$ belong to the convex set, a polytope with N vertices that can be formally defined as:

$$\begin{aligned} \Psi := \Big\{ &(A_j(\xi) \in \mathbb{R}^{n\times n}, B_j(\xi) \in \mathbb{R}^{n\times m}) = \\ &\sum_{i=1}^{N}(A_{ji}, B_{ji})\xi_i, \sum_{i=1}^{N}\xi_i = 1, \xi_i \geq 0, j = 0,1,2,\ldots,p \Big\}, \end{aligned} \tag{3}$$

where $\xi_i, i = 1,2,\ldots,N$ are constant (or time-varying) but unknown parameters, respective uncertainties in system matrices $A_i(\xi)$, $B_i(\xi)$. Furthermore, A_{ji}, B_{ji}, C are constant matrices of corresponding dimensions. $\hat{\theta} \in \mathbb{R}^p$ is a vector of known constant or time-varying real (measured/estimated) gain-scheduled parameters. In this paper is assumed that the scheduling parameters are affected by uncertainties, i.e. the measured or estimated gain scheduling parameter $\hat{\theta}$ consists of an exact and error part:

$$\hat{\theta}_j = \theta_j + \delta_j,\ j = 1,2,\ldots,p. \tag{4}$$

Furthermore, it is supposed that the uncertain parameters δ_j are independent from each other, as well as from the actual exact values of the gain scheduling parameters θ_j. We assume that both lower and upper bounds are available for these parameters and their variation rates that is:

- each parameter θ_j lies in a known hyperreactangle Ω_θ, which is defined as:

$$\theta_j \in \Omega_\theta := \left\{\theta_j \in \langle \underline{\theta_j}, \overline{\theta_j}\rangle, j = 1,2,\ldots,p\right\}, \tag{5}$$

  furthermore, the rate of variation $\dot{\theta}_j$ is well defined at all times and satisfies:

$$\dot{\theta}_j \in \Omega_t := \left\{\dot{\theta}_j \in \langle \underline{\dot{\theta}_j}, \overline{\dot{\theta}_j}\rangle, j = 1,2,\ldots,p\right\}. \tag{6}$$

- each parameter δ_j lies in a known hyperreactangle Ω_δ, which is defined as:

$$\delta_j \in \Omega_\delta := \left\{\delta_j \in \langle \underline{\delta_j}, \overline{\delta_j}\rangle, j = 1,2,\ldots,p\right\}, \tag{7}$$

  furthermore, the rate of variation $\dot{\delta}_j$ is well defined at all times and satisfies:

$$\dot{\delta}_j \in \Omega_r := \left\{\dot{\delta}_j \in \langle \underline{\dot{\delta}_j}, \overline{\dot{\delta}_j}\rangle, j = 1,2,\ldots,p\right\}. \tag{8}$$

Note 1. The uncertain system (1) consists of three type of vertices. The first one is due to the system uncertainties, N-vertices, the second is due to the gain-scheduled parameter θ with $T_\theta = 2^p$ vertices, and the third set of vertices are due to the scheduled parameter uncertainties δ with $T_\delta = 2^p$.

Triggered by real-time applicability, we hereby focus on the following problem in this paper:
Problem. Design a robust static output-feedback gain-scheduled PID controller with control algorithm:

$$u(t) = K_p(\hat{\theta})y(t) + K_i(\hat{\theta})\int_0^t y(\tau)d\tau + K_d(\hat{\theta})\dot{y}(t), \tag{9}$$

such that the controller (9) ensures the closed-loop parameter-dependent quadratic stability and guaranteed cost with respect to (13).

To achieve the desired goal, a new additional system output variable $z = \int_0^t y(\tau)d\tau \in \mathbb{R}^l$ is introduced. Furthermore, the system (1) can be augmented as:

$$\begin{aligned} \dot{v} &= \begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}A(\hat{\theta},\xi) & 0\\ C & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}B(\hat{\theta},\xi)\\ 0\end{bmatrix}u \\ &= A_v(\hat{\theta},\xi)v + B_v(\hat{\theta},\xi)u, v \in \mathbb{R}^{n+l}, \\ y_a &= \begin{bmatrix}y\\ z\end{bmatrix} = C_v v, \quad C_v = \begin{bmatrix}C & 0\\ 0 & I_m\end{bmatrix}. \end{aligned} \tag{10}$$

The control algorithm (9) then simplifies to:

$$u(t) = K_p(\hat{\theta})C_p v + K_i(\hat{\theta})C_i v + K_d(\hat{\theta})C_d\dot{v}, \tag{11}$$

where $C_p = [C \quad 0] \in \mathbb{R}^{l\times(n+l)}$, $C_i = [0 \quad I_m] \in \mathbb{R}^{m\times(n+m)}$, $C_d = C_p$, furthermore:

$$\begin{aligned} \{(K_p(.), K_i(.), K_d(.)\} &= \{K_{p0}, K_{i0}, K_{d0}\} \\ &+ \sum_{j=1}^{p}\{K_{pj}, K_{ij}, K_{dj}\}\hat{\theta}_j. \end{aligned} \tag{12}$$

To assess the performance quality, a quadratic cost function is defined in a standard LQR fashion:

$$J_c = \int_0^\infty J(v,u,\dot{v},\hat{\theta})dt, \tag{13}$$

where

$$\begin{aligned} J(.) &= v^T Q(\hat{\theta})v + \dot{v}^T S(\hat{\theta})\dot{v} + u^T Ru, \\ \{Q(\hat{\theta}), S(\hat{\theta})\} &= \{Q_0, S_0\} + \sum_{j=1}^{p}\{Q_j, S_j\}\hat{\theta}_j, \end{aligned}$$

and wherein $Q(\hat{\theta})$, $S(\hat{\theta}) \in \mathbb{R}^{(m+l)\times(n+l)}$ are positive definite (semidefinite), and $R \in \mathbb{R}^{m\times m}$ is positive definite matrix, respectively.

3. ROBUST GAIN-SCHEDULED CONTROLLER DESIGN PROCEDURE

In this section, a new robust gain-scheduled PID controller design procedure is developed for the uncertain polytopic system (10) with the assumption that the scheduled parameters are inexact (affected by uncertainty δ_i, $i = 1, 2, ..., p$). The proposed controller design procedure for all $\dot{\theta} \in \Omega_t, \delta \in \Omega_\delta$, and $\dot{\delta} \in \Omega_r$, ensures parameter-dependent quadratic stability and guaranteed cost defined by (13). The main result is built on the following Lemma:

Lemma 1. (Bellman-Lyapunov equation)
Consider the system (10). Control algorithm (11) is the guaranteed cost control law for the closed-loop system if and only if a Lyapunov function $V(v, \hat{\theta}, \xi)$ exists such that the following condition holds:

$$B_e = \frac{dV(.)}{dt} + J(v, u, \dot{v}, \hat{\theta}) = -\varepsilon v^T v, \varepsilon \to 0, \varepsilon \geq 0. \quad (14)$$

Proof 1. Suppose that $P(\hat{\theta}, \xi) > 0$, and the Lyapunov function candidate $V(x, \hat{\theta}, \xi) = x^T P(\hat{\theta}, \xi)x$ is positive definite. From Eq. (14) for $\epsilon \to 0$ we can obtain

$$\dot{V}(.) + J(v, u, \dot{v}, \hat{\theta}) \leq 0, \to \dot{V}(.) \leq -J(v, u, , \dot{v}, \hat{\theta}) \leq 0.$$

Integrating both sides from 0 to ∞ we can obtain:

$$J_c = \int_0^\infty J(v, u, \hat{\theta}, \xi)dt \leq V(0) - V(\infty) \leq V(0) = v_0^T P(\hat{\theta}, \xi)v_0.$$

It follows that if Lemma 1 holds, then the closed-loop system is asymptotically stable with guaranteed cost $J_c \leq v_0^T P(\hat{\theta})v_0$.

Note 2. Equation (14) is known as Bellman-Lyapunov equation and function $V(v, \hat{\theta}, \xi)$ which satisfies (14) is the Lyapunov function. For particular structure of the Lyapunov function the obtained gain-scheduled controller design procedure may reduce from "if and only if" to "if".

The main results of the robust gain-scheduled PID controller design, which ensure robust parameter-dependent quadratic stability and guaranteed cost, are given in the next theorem.

Theorem 1. The uncertain system (10) with gain-scheduled controller (11) is robust parameter-dependent quadratically stable with guaranteed cost for all $\theta \in \Omega_\theta$, $\dot{\theta} \in \Omega_t$, $\delta \in \Omega_\delta$, and $\dot{\delta} \in \Omega_r$, if there exist:

- $2^p + 2^p + N$ symmetric matrices such that $P(\hat{\theta}, \xi) \in \mathbb{R}^{(l+n)\times(l+n)}$ is positive definite,
- 6 auxiliary matrices N_k, $k = 1, 2, \ldots, 6$,

for the given weighting matrices $Q(\hat{\theta})$, $S(\hat{\theta})$, and R, such that the inequalities (15) hold for all $i = 1, 2, \ldots, N$.

$$W_i(\hat{\theta}) = W_{i0} + \sum_{j=1}^{p} W_{ij}(\theta_j + \delta_j) \leq 0, \; i = 1, 2, \ldots, N \quad (15)$$

where $W_{i0} = \{m_{i0kr}\}_{3\times3}$, $W_{ij} = \{m_{ijkr}\}_{3\times3}$, and

$$\begin{aligned}
m_{i011} &= S_0 + N_1^T + N_1 - N_4^T K_{d0} C_d - C_d^T K_{d0}^T N_4,\\
m_{i012} &= P_{0i} - N_1^T A_{v0i} + N_2 - N_4^T (K_{p0} C_p + K_{i0} C_i)\\
&\quad - C_d^T K_{d0}^T N_5,\\
m_{i013} &= -N_1^T B_{v0i} + N_3 + N_4^T - C_d^T K_{d0}^T N_6,\\
m_{i022} &= Q_0 + \textstyle\sum_{j=1}^p P_{ji}(\dot{\theta}_i + \dot{\delta}_i) - N_2^T A_{v0i} - A_{v0i}^T N_2\\
&\quad - N_5^T (K_{p0} C_p + K_{i0} C_i) - (K_{p0} C_p + K_{i0} C_i)^T N_5,\\
m_{i023} &= -N_2^T B_{voi} - A_{v0i}^T N_3 - (K_{p0} C_p + K_{i0} C_i)^T N_6\\
&\quad + N_5^T,\\
m_{i033} &= R - N_3^T B_{v0i} - B_{v0i}^T N_3 + N_6^T + N_6,\\
m_{ij11} &= S_j - N_4^T K_{dj} C_d - C_d^T K_{dj}^T N_4,\\
m_{ij12} &= P_{ji} - N_1^T A_{vji} - N_4^T (K_{pj} C_p + K_{ij} C_i)\\
&\quad - C_d^T K_{dj}^T N_5,\\
m_{ij13} &= -N_1^T B_{vji} - C_d^T K_{dj}^T N_6,\\
m_{ij22} &= Q_j - N_2^T A_{vji} - A_{vji}^T N_2 - N_5^T (K_{pj} C_p + K_{ij} C_i)\\
&\quad - (K_{pj} C_p + K_{ij} C_i)^T N_4,\\
m_{ij23} &= -N_2^T B_{vji} - A_{vji}^T N_3 - (K_{pj} C_p + K_{ij} C_i)^T N_6,\\
m_{ij33} &= -N_3^T B_{vji} - B_{vji}^T N_3.
\end{aligned}$$

Proof 2. Assume that in the Bellman-Lyapunov equation (14) the Lyapunov function $V(v, \hat{\theta}, \xi)$ is given with the following structure:

$$V(x, \hat{\theta}, \xi) = v^T P(\hat{\theta}, \xi)v. \quad (16)$$

where

$$P(\hat{\theta}, \xi) = P_0(\xi) + \sum_{j=1}^{p} P_j(\xi)\hat{\theta}_j,$$

$$P_j(\xi) = \sum_{i=1}^{N} P_{ji}\xi_i, j = 0, 1, 2, \ldots, p.$$

The time derivative of the Lyapunov function (16) is then:

$$\frac{dV(.)}{dt} = [\dot{v}^T \quad v^T \quad u^T] \begin{bmatrix} 0 & P(\hat{\theta}, \xi) & 0 \\ P(\hat{\theta}, \xi) & P_d(.) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ v \\ u \end{bmatrix}. \quad (17)$$

where for time-invariant uncertain parameter ξ the Lyapunov matrix $P_d(.)$ is as follows:

$$P_d(.) = \sum_{j=1}^{p} \sum_{i=1}^{N} P_{ji}\xi_i(\dot{\theta}_j + \dot{\delta}_j). \quad (18)$$

For the case of time-varying uncertain parameter ξ, the Lyapuniv matrix $P_d(.)$ is as follows:

$$P_d(.) = \sum_{j=1}^{p} \sum_{i=1}^{N} \left(\sum_{k=1}^{N} P_{jk}\dot{\xi}_k\hat{\theta}_j + P_{ji}\dot{\hat{\theta}}_j \right) \xi_i. \quad (19)$$

For simplicity, in the rest of the proof we assume that $\dot{\xi}_i = 0$, $i = 1, 2, \ldots, N$. To separate the Lyapunov matrix from the system matrices, new auxiliary matrices $N_i \in \mathbb{R}^{(n+l)\times(n+l)}$, $i = 1, 2$, $N_3 \in \mathbb{R}^{(n+l)\times m}$, $N_i \in \mathbb{R}^{m\times n}$, $i = 4, 5$, and $N_6 \in \mathbb{R}^{m\times m}$ of corresponding dimensions are introduced in the following form:

$$2(N_1\dot{v} + N_2 v + N_3 u)^T M_1(v, \dot{v}, u, \hat{\theta}, \xi) = 0, \quad (20a)$$

$$2(N_4\dot{v} + N_5 v + N_6 u)^T M_2(v, \dot{v}, u, \hat{\theta}) = 0, \quad (20b)$$

where

$$M_1(v, \dot{v}, u, \hat{\theta}, \xi) = \dot{v} - A_v(\hat{\theta}, \xi)v - B_v(\hat{\theta}, \xi)u,$$

$$M_2(v, \dot{v}, u, \hat{\theta}) = u - K(\hat{\theta})C_p v - K_i(\hat{\theta})C_i v - K_d(\hat{\theta})C_d \dot{v}.$$

By summarizing equations (20) with the time derivative of Lyapunov function (17) and with the cost function (13), the following Bellman-Lyapunov function can be obtained:

$$B_e = e^T W(\hat{\theta}, \xi)e \leq 0, \quad e^T = [\dot{v}^T \quad v^T \quad u^T] \quad (21)$$

where $W(\hat{\theta}, \xi) = \{w_{kr}(\hat{\theta}, \xi)\}_{3\times3}$ and

$$\begin{aligned}
w_{11} &= S(\hat{\theta}) + N_1^T + N_1 - N_4^T K_d(\hat{\theta})C_d - C_d^T K_d(\hat{\theta})^T N_4,\\
w_{12} &= P(\hat{\theta}, \xi) - N_1^T A_v(\hat{\theta}, \xi) + N_2 - N_4^T (K_p(\hat{\theta})C_p\\
&\quad + K_i(\hat{\theta})C_i) - C_d^T K_d^T(\hat{\theta})N_5,\\
w_{13} &= -N_1^T B_v(.) + N_3 + N_4^T - C_d^T K_d^T(.)N_6,\\
w_{22} &= P_d(.) - N_2^T A_v(.) - A_v^T(.)N_2 - N_5^T (K_p(.)C_p\\
&\quad + K_i(.)C_i) - (K_p(.)C_p + K_i(.)C_i)^T N_5 + Q(\hat{\theta}),\\
w_{23} &= -N_2^T B_v(.) - A_v^T(.)N_3 - (K_p(.)C_p + K_i(.)C_i)^T N_6\\
&\quad + N_5^T,\\
w_{33} &= R - N_3^T B_v(.) - B_v^T(.)N_3 + N_6^T + N_6.
\end{aligned}$$

From the Bellman-Lyapunov equation (21) the term $W(\hat{\theta}, \xi)$ can be split with respect to uncertain parameter ξ_i as:

$$W(\hat{\theta}, \xi) = \sum_{i=1}^{N} W_i(\hat{\theta})\xi_i, \qquad (22)$$

from which follows that the robust stability condition with guaranteed cost defined in *Theorem 1* holds if and only if

$$W_i(\hat{\theta}) \leq 0, \quad i = 1, 2, \dots, N. \qquad (23)$$

The matrix $W_i(\hat{\theta})$ with respect to $\hat{\theta}$ can be further split as follows:

$$W_i(\hat{\theta}) = W_{i0} + \sum_{j=1}^{p} W_{ij}(\theta_j + \delta_j), \quad i = 1, 2, \dots, N \qquad (24)$$

which proves the *Theorem 1*.

Note 3. Note that inequalities (15) are convex with respect to gain-scheduled parameters θ and gain-scheduled parameter's uncertainties δ. Inequalities (15) hold if and only if they are negative definite for all $i = 1, 2, \dots, N$ $\xi - vertices$, and in the vertices defined by the gain-scheduled uncertainties δ_j and the gain-scheduled parameters θ_j, $j = 1, 2, \dots, p$. To reduce the computation load in equation (15), one could substitute to entries m_{i022} from *Theorem 1* as follows:

$$\sum_{j=1}^{p} P_{ji}(\dot{\theta}_i + \dot{\delta}_i) \leq \sum_{j=1}^{p} P_{ji}\varrho, \quad \max|\dot{\theta}_i + \dot{\delta}_i| \leq \varrho \qquad (25)$$

assuming that $P_{ji} > 0$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, p$.

Note 4. For the case of PI controller design, matrix $C_d = 0$ and controller matrices $K_{di} = 0$, $i = 0, 1, \dots, p$. For P or PD controller design the system is simply not extended with the integral part. Furthermore, if the derivative part of the controller includes some filter, the model of this filter can be included in the system model. In addition, *Theorem 1* can be used also for full/reduced order dynamic output-feedback gain-scheduled controller design, since it can be reformulated to static output-feedback design (Petersson and Löfberg, 2011).

4. EXAMPLES

In order to show the viability of the previous proposed method, the following two examples have been chosen. Numerical solutions have been carried out by PENBMI 2.1 (Henrion et al., 2005) solver under MATLAB 2014b using YALMIP R20150918 (Löfberg, 2004). The simulations were done via MATLAB/SIMULINK.

Example 1. PID like controllers are extensively used in energetics due to their simplicity and performance characteristics. However, with these conventional fixed-gain controllers we could have difficulties to handle nonlinear or time-variant characteristics. To show the viability and applicability of the previous proposed method, a successful implementation in control of a turbo-generator (consisting from a synchronous generator (SG) and a thermal turbine) is presented. Within this example, three controllers are designed, two PI (for excitation and governor control) and one D controller (as power system stabilizer for the excitation controller).

Under the well-known assumptions (Kundur, 1994; Machowski et al., 2008) a model of the synchronous generator can be described as a third order model (E_q, δ, ω) in p.u. as follows:

$$\begin{aligned} U_q &= I_d X_d + E_q - I_q R, \\ U_d &= -I_q X_q - I_d R, \\ U_b k_{G0} &= E_q + T'_{d0}\frac{dE_q}{dt} + T'_{d0}\frac{dI_d}{dt}(X_d - X'_d), \\ T_j\frac{d\omega}{dt} &= P_T - P_e, \\ P_e &= P + P_{as} \quad P_{as} \doteq D\omega, \\ P &= E_q I_q + I_d I_q(X_d - X_q) - R(I_d^2 + I_q^2), \end{aligned} \qquad (26)$$

where I_d, I_q are the currents flowing in the fictitious d and q axis armature coils; E_q is the q-axis component of the internal emf, proportional to the field and excitation current of the SG; U_q, U_d are the voltages across the fictitious d and q axis armature coils; P_e is the total electric power generated by the SG to the system; P_{as} is the damping power; U_b is the input voltage applied to the field winding; T'_{d0} is the open-circuit d-axis transient time constant; T_j is the inertia coefficient of the turbo-generator; X_d, X_q, X'_d are the reactances and the transient reactance of the fictitious d and q axis armature windings; R is the resistance of the armature winding of the SG; $U = \sqrt{(U_d^2 + U_q^2)}$ is the terminal voltage of the generator; $E_Q = E_q + I_d(X_d - X_q)$ is the fictive internal emf. Furthermore, the time derivative of the rotor load angle δ:

$$\frac{d\delta}{dt} = \triangle\omega = \omega - \omega_s, \qquad (27)$$

is the rotor speed deviation in rad/s, wherein ω_s is the power system angular speed. In the rest of the paper the following denotation will be used $\triangle\omega = \omega$.

The thermal turbine model with governor valve in simplified structure (Machowski et al., 2008) is given by the following third order transfer function:

$$G_{T(s)} = \frac{P_{T(s)}}{P_{R(s)}} = \frac{sb_1 + b_0}{s^3 a_3 + s^2 a_2 + sa_1 + 1}, \qquad (28)$$

where $P_{T(s)}$ is the output of the turbine power; $P_{R(s)}$ is the output of the turbine power controller, furthermore:

$$\begin{gathered} b_1 = k_l T_h + k_h T_l, \quad b_0 = k_l + k_h = 1, \quad a_3 = T_s T_l T_h, \\ a_2 = T_l T_h + T_s(T_l + T_h), \quad a_1 = T_l + T_h + T_s, \end{gathered}$$

wherein, $T_s\,[s]$ is the servomotor's time constant, $k_l\,[p.u.]$ is the low pressure gain, $T_l\,[s]$ is the low pressure time constant, $k_h\,[p.u.]$ is the high pressure gain, and $T_h\,[s]$ is the high pressure time constant.

We assume that the machine is connected to a large power system through transmission lines, and that the large power system belongs to the class of infinite bus system with bus {voltage, angular speed}=(U_s, ω_s) (Kundur, 1994; Machowski et al., 2008). The transmission lines can be transformed to the T equivalent circuit with impedances $\bar{z}_1, \bar{z}_2$, and reluctant impedance $\bar{z}_3$. Using Kirchhoff's law for currents on the d and q axis one can obtain:

$$\begin{aligned} I_q = \frac{E_q}{M\tan\varphi_{11}} + \frac{U_s}{z_{12}}\sqrt{\Bigg[\Bigg(1+} \\ + \left(\frac{(X_d - X_q)cos\varphi_{11}}{z_{11} + (X_d - X_q)sin\varphi_{11}}\right)^2\Bigg) sin(\delta + \varphi_{12} - \psi)\Bigg], \end{aligned} \qquad (29)$$

$$I_d = -\frac{E_q}{M} + \frac{U_s}{z_{12}\left(1 + \frac{X_d - X_q}{z_{11}} sin\varphi_{11}\right)} sin(\delta + \varphi_{12}), \quad (30)$$

where

$$M = \frac{z_{11}}{sin\varphi_{11}} + X_d - X_q,$$
$$\psi = arctan\frac{z_{11} + (X_d - X_q)sin\varphi_{11}}{(X_d - X_q)cos\varphi_{11}},$$
$$\bar{z_{11}} = jX_q + \bar{z_1} + \frac{\bar{z_2}\bar{z_3}}{\bar{z_2} + \bar{z_3}} = z_{11}e^{j\varphi_{11}},$$
$$\bar{z_{12}} = jX_q + \bar{z_1} + \frac{\bar{z_2}}{\bar{z_3}}(\bar{z_3} + \bar{z_1} + jX_q) = z_{12}e^{j\varphi_{12}}.$$

A simple LPV model of the previously described turbo-generator model, in the form (1), can be obtained easily using the grid-based LPV modelling technique (Naus, 2009). To obtain low-complexity LPV model, only 3 working points were chosen based on the electrical power $P_z\,[p.u.] = \{0.3,\ 0.7,\ 1\}$ and terminal voltage $U_z = 1\,[p.u.]$. As the uncertain parameters, the $U_s \in \langle U_{smin}, U_{smax}\rangle$ and the reactance $\bar{z}_3 = jx_{v3}, x_{v3} \in \langle x_{v3min}, x_{v3max}\rangle$ were chosen. The nonlinear turbo-generator model then can be linearized at the vertices of these uncertain parameters in all working points:

$$\dot{x} = \bar{A}_{wi}x + \bar{B}_{wi}u,\ w = 1,2,3,\ i = 1,2,3,4,\ y = Cx. \quad (31)$$

The obtained family of linear systems (31) then can be transformed to LPV model with $p = 2$ scheduled parameters $\theta_{1,2} \in \langle -1, 1\rangle$ and $N = 4$ uncertainty vertices.

For the controller design the following plant parameters were identified: $U_s \in \langle 0.9, 1.1\rangle\,[p.u.]$, $x_{v3} \in \langle 0.2, 2\rangle\,[p.u.]$, $\delta_i \in \langle -0.025, 0.025\rangle$, $\dot{\bar{\theta}}_{1,2} = 110\,[p.u.]$, $\dot{\delta}_{1,2,3,4} = 2\,[p.u.]$, $U_z = 1\,[p.u.]$ $T_j = 0.02245\,[s^2]$, $T'_{d0} = 0.4\,[s]$, $X_d = 2\,[p.u.]$, $X'_d = 0.247\,[p.u.]$, $X_q = 1.75\,[p.u.]$, $R = 0$, $X_{v1} = 0.127\,[p.u]$, $X_{v2} = 0.12\,[p.u]$, $T_s = 0.4\,[s]$, $T_l = 5.4\,[s]$, $T_h = 0.25$, $k_l = 0.75$, and $k_n = 0.25$.

Parameters of the designed robust gain-scheduled PI controller for the excitation control using *Theorem 1* with weighting matrices $q_0 = 0.1$, $q_{1,2} = 0$, $Q_i = q_iI$, $s_0 = 0.01$, $s_{1,2} = 0$, $S_i = s_iI$, $r = 1$, $R = rI$, and constraints on the Lyapunov matrix $0 < P(\hat{\theta}) \leq 1000I$, are as follows:

$$\begin{aligned} K_{pEC}(\hat{\theta}) &= 14.184 + 1.7504\,\hat{\theta}_1 + 1.6243\,\hat{\theta}_2, \\ K_{iEC}(\hat{\theta}) &= 2.0447 + 0.2051\,\hat{\theta}_1 + 0.2289\,\hat{\theta}_2. \end{aligned} \quad (32)$$

Parameters of the designed robust gain-scheduled PI controller for the governor control using *Theorem 1*, with the same weighting matrices as for the excitation controller, are as follows:

$$\begin{aligned} K_{pGC}(\hat{\theta}) &= 2.0929 + 0.0576\,\hat{\theta}_1 + 0.0742\,\hat{\theta}_2, \\ K_{iGC}(\hat{\theta}) &= 0.5742 + 0.1542\,\hat{\theta}_1 + 0.1541\,\hat{\theta}_2. \end{aligned} \quad (33)$$

Parameters of the power system stabilizer for the excitation controller as the first derivative of the electrical power using *Theorem 1*, with the same weighting matrices as for the excitation controller, are as follows:

$$K_{dPSS}(\hat{\theta}) = 0.1372 - 0.0799\,\hat{\theta}_1 + 0.0451\,\hat{\theta}_2 \quad (34)$$

Simulation results (Fig. 1 and 2) with the original nonlinear turbo-generators' model with the designed controllers ((32), (33), and (34)) prove that the turbo-generator is stable and that the controllers fulfil the Slovak Transmission System Operator's (STSO – SEPS) requirements. The subsequent two simulation experiments are:

- at time $t = 40\,s$, two phase short circuits were realized in the place of system voltage U_s when within the time=0.25s the voltage U_s shut down from 1 $p.u.$ to 0.2 $p.u.$;
- at time t=60s the terminal voltage setpoint U_z was changed from 1.0 $p.u.$ to 0.95 $p.u.$.

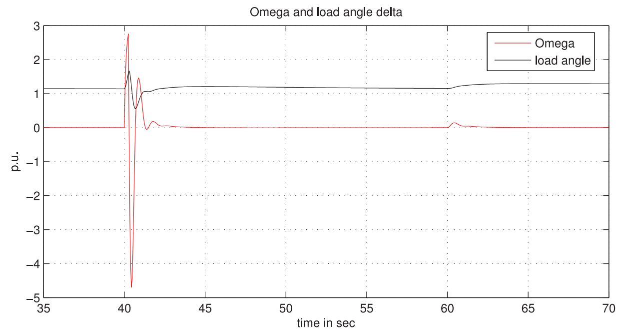

Fig. 1. Simulation results to rotor speed deviation (omega) and SG'load angle

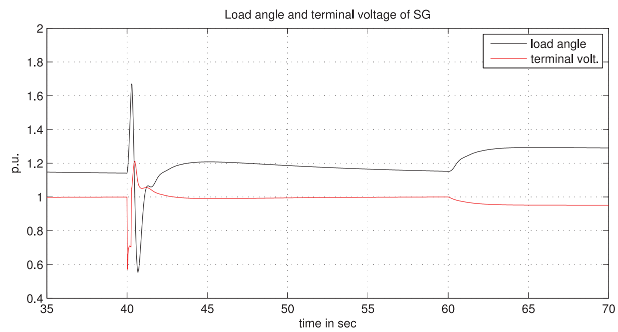

Fig. 2. Simulation results to SG' load angle and terminal voltage

Example 2.
The second example is a simple nonlinear academic model in the form (35). With this example we want to highlight that using the proposed approach it is possible to design a controller even with relatively big maximal rate of change of the scheduled parameters ($\dot{\theta} = 50\,[s^{-1}]$), while other guaranteed cost approaches (with the same weighting matrices) using the multi-convexity requirement (Veselý and Ilka, 2013; Ilka and Veselý, 2014) fail.

$$\dot{x} = -asinx + bu, \quad y = x, \quad (35)$$

where $a \in \langle 0.8, 1\rangle$, when $a = 0.8$ then $b = 1$, and when $a = 1$ then $b = 0.5$. Furthermore, $\dot{\theta} = 50\,[s^{-1}]$, $\delta(j) \in \langle -0.05, 0.05\rangle$, $j = 1, 2$, $\max\dot{\delta}(j) = 0.01[s^{-1}]$.

A simple LPV model can be obtained using the grid-based LPV modelling technique. One can linearize the nonlinear model (35) in three working points $x_0 = \{0, \pi/4, \pi/2\}$. The obtained augmented LPV model for the robust gain-scheduled PID controller design, with $\theta_{1,2} \in \langle -1, 1\rangle$, is as follows:

$$A(\xi,\theta) = \left(\begin{bmatrix} -0.4 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0.117 & 0 \\ 0 & 0 \end{bmatrix}\theta_1 + \begin{bmatrix} -0.28 & 0 \\ 0 & 0 \end{bmatrix}\theta_2 \right)\xi_1$$
$$+ \left(\begin{bmatrix} -0.5 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0.14645 & 0 \\ 0 & 0 \end{bmatrix}\theta_1 + \begin{bmatrix} -0.35355 & 0 \\ 0 & 0 \end{bmatrix}\theta_2 \right)\xi_2,$$
$$B(\xi) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\xi_1 + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}\xi_2, \quad C = [1 \;\; 0].$$

The grid plan for the scheduling parameters is as follows:

$$\begin{array}{lll} x = 0 & \rightarrow & \theta_1 = -1, \ \ \theta_2 = -1, \\ x = pi/4 & \rightarrow & \theta_1 = +1, \ \ \theta_2 = -1, \\ x = pi/2 & \rightarrow & \theta_1 = +1, \ \ \theta_2 = +1, \end{array}$$

The obtained controllers' parameters using *Theorem 1*, with weighting matrices $Q_0 = 0.01I$, $Q_1 = 0.001I$, $Q_3 = 0.005I$, $R = I$, $S_{0,1,2} = 0$, and with the constraint on Lyapunov matrix $0 < P(\hat{\theta}, \xi) \leq 1000I$, are as follows:

$$\begin{aligned} Kp(\hat{\theta}) &= -1.3463 - 0.1573\,\hat{\theta}_1 + 0.3103\,\hat{\theta}_2, \\ Ki(\hat{\theta}) &= -1.3872 - 0.0125\,\hat{\theta}_1 + 0.0195\,\hat{\theta}_2, \\ Kd(\hat{\theta}) &= -0.0450 + 0.0128\,\hat{\theta}_1 - 0.0344\,\hat{\theta}_2. \end{aligned} \tag{36}$$

CONCLUSION

A novel approach for robust guaranteed cost output-feedback gain-scheduled PID controller design is presented in this paper for affine LPV systems with polytopic uncertainty on system matrices and with the assumption that the scheduled parameters are inexact. The obtained design procedure is based on the Bellman-Lyapunov equation, parameter-dependent quadratic stability, and guaranteed cost. The proposed sufficient stability and performance conditions, without any restrictions on the controller's structure, are derived in the form of BMIs which can be efficiently solved or further linearized. The presented solution is directly convex in the scheduled and uncertain parameters, therefore there is no need for applying multi-convexity or other relaxation techniques. The performance metric is defined in a standard LQR fashion, so the proposed approach can be viewed as an extension of the infinite-horizon LQR problem. Future research will focus on the extension of the established results for systems affected by noises and disturbances. In addition, further emphasis will be put on investigation of the possible LMI solution in order to reduce the remained conservativeness caused by limitations of currently available BMI solvers.

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