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Citation for the original published paper (version of record):
Alaghmandan, M., Ghasemi, M. (2017). Seminormed $*$-subalgebras of $\ell^\infty(X)$. Journal of Mathematical Analysis and Applications, 455(1): 212-220.
http://dx.doi.org/10.1016/j.jmaa.2017.05.041

J. Math. Anal. Appl. 455 (2017) 212–220

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Seminormed $*$-subalgebras of $\ell^\infty(X)$

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ARTICLE INFO

Article history:
Received 17 August 2016
Available online 26 May 2017
Submitted by K. Jarosz

Keywords:
Measurable functions
Commutative normed algebras
Function algebras
Gelfand spectrum
Seminormed algebras
Measures on Boolean rings

ABSTRACT

Arbitrary representations of an involutive commutative unital $\mathbb{F}$-algebra A as a subalgebra of $\mathbb{F}^X$ are considered, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ and $X \neq \emptyset$. The Gelfand spectrum of A is explained as a topological extension of X where a seminorm on the image of A in $\mathbb{F}^X$ is present. It is shown that among all seminorms, the sup-norm is of special importance which reduces $\mathbb{F}^X$ to $\ell^\infty(X)$. The Banach subalgebra of $\ell^\infty(X)$ of all Σ-measurable bounded functions on X, $\mathrm{M}_b(X, \Sigma)$, is studied for which Σ is a σ-algebra of subsets of X. In particular, we study lifting of positive measures from (X, Σ) to the Gelfand spectrum of $\mathrm{M}_b(X, \Sigma)$ and observe an unexpected shift in the support of measures. In the case that Σ is the Borel algebra of a topology, we study the relation of the underlying topology of X and the topology of the Gelfand spectrum of $\mathrm{M}_b(X, \Sigma)$.

1. Introduction

It is common to look at rings and algebras as families of functions over a nonempty set with values in a suitable ring or field. This is especially helpful if one wants to study the ideal structure of a ring or algebra which naturally involves topological notions, mainly compactness.

In this article, we summarize some observations about topological algebras in an abstract manner. One motivation comes from [3] which attempts to represent positive linear functionals on a given commutative unital algebra as an integral with respect to a positive measure on the space of characters of the algebra. This is done by realizing the algebra as a subalgebra of continuous functions over the character space.

During the present article we always assume that A is an involutive commutative algebra over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ equipped with a seminorm ρ. In Section 2, first we provide a brief overview of the theory of seminormed algebras and their Gelfand spectrum. Then, we assume that A can be embedded into $(\mathbb{F}^X, \rho)$

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http://dx.doi.org/10.1016/j.jmaa.2017.05.041

for a nonempty set X where ρ is a submultiplicative seminorm on a subalgebra of $\mathbb{F}^X$ that contains the image of A. This induces a seminormed structure on A as well. Theorem 2.5 gives a necessary and sufficient condition for X to be dense in the Gelfand spectrum of A, that is, when the topology induced by the seminorm is equivalent to the topology induced by the sup-norm defined in (2).

Motivated by [3], where positive linear functionals on an algebra are presented as integrals with respect to constructibly Radon measures, in Section 3, we consider a measurable structure Σ on X and study the spectrum of the algebra of bounded measurable functions on (X, Σ), denoted by $\mathrm{M}_b(X, \Sigma)$. We prove that positive measures on X lift to positive measures over the spectrum of $\mathrm{M}_b(X, \Sigma)$, but this lifting shifts the support of the original measure out of X modulo at most a countable subset of X (Propositions 3.7 and 3.8). At the end we choose Σ to be the Borel algebra of a topology τ on X and observe some connections between τ and the spectrum of $\mathrm{M}_b(X, \Sigma)$ (Proposition 3.10 and Theorem 3.11).

1.1. Notations

Let X be a non-empty set and $\mathfrak{S}$ be a structure on X which induces a topology on X. We denote this topology by $\tau(X, \mathfrak{S})$. For instance, let $\mathfrak{S}$ be a family of functions, defined on X, with values in a topological space. Then $\tau(X, \mathfrak{S})$ is the coarsest topology on X which makes every function in $\mathfrak{S}$ continuous.

Let (X, τ) be a topological space. We denote the set of all τ-continuous $\mathbb{F}$-valued functions on X by $\mathrm{C}(X, \tau)$ or $\mathrm{C}(X)$ if there is no risk of confusion. We use $\mathrm{C}_b(X)$ (or $\mathrm{C}_b(X, \tau)$) to denote the set of all $f \in \mathrm{C}(X)$ which are bounded on X. If (X, τ) is locally compact, $\mathrm{C}_0(X)$ denotes the set of all $f \in \mathrm{C}_b X$ which are vanishing at infinity.

Let $P(X)$ be the power set of X. The σ-algebra of sets induced on X by a set $\Lambda \subseteq P(X)$ is denoted by $\sigma(\Lambda)$. In particular if τ is a topology on X, then $\sigma(\tau)$ is the σ-algebra of all Borel subsets of (X, τ) denoted by $\mathcal{B}_\tau$.

2. Involutive subalgebras of $\ell^\infty(X)$

The set theory which is applied in this paper is ZFC. Throughout this article all algebras are assumed to be involutive (also called $*$-algebra) and commutative over a field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$ as specified). Subsequently, all $\mathbb{F}$-valued $*$-algebra homomorphisms are also supposed to be $\mathbb{F}$-module maps.

Definition 2.1. Let A be a commutative $*$-algebra. A function $\rho : A \longrightarrow [0, \infty]$ is called a *quasi-norm* on A if

(1) $\forall a \in A \quad \rho(a^*) = \rho(a)$,
(2) $\forall a, b \in A \quad \rho(a + b) \leq \rho(a) + \rho(b)$ (subadditive),
(3) $\forall r \in \mathbb{F}\ \forall a \in A \quad \rho(ra) = |r|\rho(a)$.

ρ is called *submultiplicative* if

(4) $\forall a, b \in A, \quad \rho(ab) \leq \rho(a)\rho(b)$ where the product of ∞ and 0 is ∞.

A *quasi-norm* ρ on A is called a *seminorm* if $\rho(a) < \infty$ for every $a \in A$.

Let A be a commutative $*$-algebra and let ρ be a quasi-norm on A. The set of all elements of A with a finite quasi-norm ρ is denoted by $\mathrm{B}_\rho(A)$, i.e.,

$$\mathrm{B}_\rho(A) = \{a \in A : \rho(a) < \infty\}.$$

If ρ is a submultiplicative quasi-norm, it is clear that $\mathrm{B}_\rho(A)$ is a $*$-subalgebra of A and the restriction of ρ to $\mathrm{B}_\rho(A)$ is a seminorm. A $*$-algebra A with a seminorm ρ forms a *seminormed algebra* if ρ is submultiplicative. For a seminormed algebra (A, ρ), the set of all non-zero $*$-algebra homomorphisms $\alpha : A \longrightarrow \mathbb{F}$ is denoted by $\mathcal{X}(A)$. The set $\mathfrak{sp}_\rho(A)$ of all ρ-continuous $*$-algebra homomorphisms belonging to $\mathcal{X}(A)$ is called the *Gelfand spectrum* of (A, ρ). Every element $a \in A$ induces a map $\hat{a} : \mathcal{X}(A) \longrightarrow \mathbb{F}$ defined by $\hat{a}(\alpha) := \alpha(a)$ for each $\alpha \in \mathcal{X}(A)$. Next, we have a characterization of all ρ-continuous $\mathbb{F}$-valued $*$-algebra homomorphisms. The following lemma was proved as Lemma 3.2 in [2].

Lemma 2.2. *Let (A, ρ) be a commutative seminormed $*$-algebra and $\alpha \in \mathcal{X}(A)$. Then $\alpha \in \mathfrak{sp}_\rho(A)$ if and only if $|\alpha(a)| \le \rho(a)$, for all $a \in A$.*

The Gelfand spectrum $\mathfrak{sp}_\rho(A)$ (as well as $\mathcal{X}(A)$) naturally carries a Hausdorff topology as a subspace of $\mathbb{F}^A$ with the product topology. For a real number $r > 0$, let $D_r := \{c \in \mathbb{F} : |c| \le r\}$. According to Lemma 2.2, $\mathfrak{sp}_\rho(A) \subseteq \prod_{a\in A} D_{\rho(a)}$. One simple approximation argument implies that every element in the closure of $\mathfrak{sp}_\rho(A)$ is a $*$-algebra homomorphism. But it also belongs to $\prod_{a\in A} D_{\rho(a)}$. Therefore, the closure of $\mathfrak{sp}_\rho(A)$ is a subset of $\mathfrak{sp}_\rho(A) \cup \{\mathbf{0}\}$ where $\mathbf{0}$ is the constant linear functional zero on A. From now on, we use $\mathfrak{sp}_\rho(A)$ to denote it as a topological subspace of $\prod_{a\in A} D_{\rho(a)}$. Note that, for each $a \in A$, $\hat{a}$ is an element in $\mathrm{C}(\mathcal{X}(A))$ and subsequently, its restriction to $\mathfrak{sp}_\rho(A)$ belongs to $\mathrm{C}(\mathfrak{sp}_\rho(A))$.

Note that the difference between the following corollary and [2, Corollary 3.3] is due to the fact that we exclude zero in the definition of $\mathcal{X}(A)$.

Corollary 2.3. *Let (A, ρ) be a commutative seminormed $*$-algebra. If A is unital then $\mathfrak{sp}_\rho(A)$ is compact. If $\mathfrak{sp}_\rho(A)$ is compact then there exists an element $a_0 \in A$ such that $|\alpha(a_0)| \ge 1$ for every $\alpha \in \mathfrak{sp}_\rho(A)$.*

Proof. If A is unital, one may use the identity element, $\mathbf{1}$, (for which we have $\alpha(\mathbf{1}) = 1$ for every $\alpha \in \mathfrak{sp}_\rho(A)$) to show that $\mathbf{0}$ does not belong to the closure of $\mathfrak{sp}_\rho(A)$. Therefore, $\mathfrak{sp}_\rho(A)$ is indeed a closed set in $\prod_{a\in A} D_{\rho(a)}$, and subsequently, $\mathfrak{sp}_\rho(A)$ is compact.

Now suppose that $\mathfrak{sp}_\rho(A)$ is compact. Therefore, $\mathfrak{sp}_\rho(A)$ is a closed subset of $\prod_{a\in A} D_{\rho(a)}$, not containing $\mathbf{0}$. So, there exist a finite set $\{a_1, \dots, a_m\}$ and $\epsilon > 0$ such that for each α there is an i with $|\alpha(a_i)| \ge \epsilon^{1/2}$. Now set $a := a_1^* a_1 + \cdots + a_m^* a_m$. Then, this particular element a, satisfies $|\alpha(a)| \ge \epsilon$ for each $\alpha \in \mathfrak{sp}_\rho(A)$. Let $k = \inf\{|\alpha(a)| : \alpha \in \mathfrak{sp}_\rho(A)\} \ge \epsilon$ and $a_0 := a/k$. The claim follows for a_0. □

Remark 2.4. Every non-unital commutative seminormed $*$-algebra (A, ρ) can be embedded into the unital $*$-algebra $A_1 := A \oplus \mathbb{F}$ with multiplication $(a, \lambda)(b, \gamma) = (ab + \gamma a + \lambda b, \lambda\gamma)$ and involution $(a, \lambda)^* = (a^*, \bar{\lambda})$. Defining $\rho_1(a + \lambda) = \rho(a) + |\lambda|$ we also obtain a seminorm on A_1 which makes the natural embedding $a \mapsto (a, 0)$ continuous. For each $\alpha \in \mathcal{X}(A)$, define the extension $\alpha'(a, \lambda) = \alpha(a) + \lambda$ which is obviously an element in $\mathcal{X}(A_1)$. So one can regard $\mathcal{X}(A)$ as a subset of $\mathcal{X}(A_1)$. Regarding $\mathbb{F}$ as a commutative algebra, we know that $\mathcal{X}(\mathbb{F})$ has only one element which is the identity map. For an element $\beta \in \mathcal{X}(A_1)$, note that if the restriction $\beta|_A$ is non-zero, it is formed by the element $\beta|_A \in \mathcal{X}(A)$ as descried above. Hence, $\mathcal{X}(A_1) \setminus \mathcal{X}(A)$ consists of exactly one element, denoted here by $\hat{\infty}$, which maps (a, λ) to λ for all $a \in A$ and $\lambda \in \mathbb{F}$. Clearly, $\hat{\infty} \in \mathfrak{sp}_{\rho_1}(A_1)$, therefore A is a closed maximal ideal of A_1. Moreover, if $\mathfrak{sp}_\rho(A)$ is not compact, $\mathfrak{sp}_{\rho_1}(A_1)$ is the one-point compactification of $\mathfrak{sp}_\rho(A)$.

Every $*$-algebra homomorphism $\phi : A \longrightarrow B$ induces a mapping $\phi_* : \mathcal{X}(B) \longrightarrow \mathcal{X}(A) \cup \{\mathbf{0}\}$ defined by $\phi_*(\beta) = \beta \circ \phi$ for each $\beta \in \mathcal{X}(B)$. Suppose that B is equipped with a seminorm ρ. The homomorphism ϕ induces a seminorm ρ_ϕ on A defined by $\rho_\phi(a) = \rho(\phi(a))$. If ρ is submultiplicative, then so is ρ_ϕ. The map ϕ as a homomorphism between seminormed $*$-algebras (A, ρ_ϕ) and (B, ρ) is continuous. Therefore ϕ_* maps $\mathfrak{sp}_\rho(B)$ continuously into $\mathfrak{sp}_{\rho_\phi}(A)$.

Here we are mainly interested in the case where B is a $*$-subalgebra of $\mathbb{F}^X$ for a non-empty set X where $\mathbb{F}^X$ is the space of all $\mathbb{F}$-valued functions on X furnished with pointwise multiplication and the canonical $\mathbb{F}$-conjugate involution. This generally enables us to realize $\mathfrak{sp}(A)$ relative to X as follows.

Let ρ be a submultiplicative quasi-norm on $\mathbb{F}^X$ with $\rho(\mathbf{1}) \geq 1$ where $\mathbf{1}$ denotes the constant function which takes 1 all over the X. There is a natural map $e : X \longrightarrow \mathcal{X}(\mathbb{F}^X)$ which, to every $x \in X$, assigns the *evaluation map* $e_x : \mathbb{F}^X \longrightarrow \mathbb{F}$, defined by $e_x(f) := f(x)$. It is clear that $e_x \in \mathcal{X}(\mathbb{F}^X)$. We denote the set of all ρ-continuous evaluations by X_ρ. Note that by Lemma 2.2, for every $x \in X$, $e_x \in X_\rho$ if and only if $e_x \in \mathfrak{sp}_\rho(\mathrm{B}_\rho(\mathbb{F}^X))$. In symbols:

$$X_\rho = \{e_x : \ x \in X, \ e_x \in \mathfrak{sp}_\rho(\mathrm{B}_\rho(\mathbb{F}^X))\}. \tag{1}$$

Let $\iota : A \longrightarrow \mathrm{B}_\rho(\mathbb{F}^X)$ be a $*$-algebra homomorphism. By abuse of notation, we use ι_* to denote the induced map $\iota_*|_X : X \longrightarrow \mathfrak{sp}_\rho(A)$.

Theorem 2.5. *Let A be a commutative $*$-algebra and $\iota : A \longrightarrow \mathrm{B}_\rho(\mathbb{F}^X)$ be a $*$-algebra homomorphism, where ρ is a submultiplicative quasi-norm on $\mathbb{F}^X$ with $\rho(\mathbf{1}) \geq 1$. Define $\rho_\iota := \rho \circ \iota$ on A. Then $\iota_*(X_\rho)$ is dense in $\mathfrak{sp}_{\rho_\iota}(A)$ if and only if there exists $D > 0$ such that*

$$\rho_\iota(a) \leq D \cdot \sup_{x \in X_\rho} |e_x(\iota a)|,$$

for all $a \in A$.

Proof. Note that by Lemma 2.2, for each $a \in A$,

$$\sup_{\beta \in \mathfrak{sp}_{\rho_\iota}(A)} |\hat{a}(\beta)| \leq \rho_\iota(a).$$

($\Rightarrow$) Since $\iota_*(X_{\rho_\iota})$ is dense in $\mathfrak{sp}_{\rho_\iota}(A)$ we have

$$\sup_{x \in X_\rho} |e_x(\iota a)| = \sup_{\beta \in \mathfrak{sp}_{\rho_\iota}(A)} |\beta(a)|,$$

and it suffices to take $D = 1$.

($\Leftarrow$) In contrary, suppose that $\alpha \in \mathfrak{sp}_{\rho_\iota}(A) \setminus \overline{\iota_*(X_\rho)}$. There exists $f \in \mathrm{C}(\mathfrak{sp}_{\rho_\iota}(A))$ such that $f(\alpha) = 1$ and $f|_{\iota_*(X_\rho)} = 0$, by Urysohn's lemma. Since $\hat{A}$ separates points of $\mathfrak{sp}_{\rho_\iota}(A)$, $\hat{a} \in \mathrm{C}(\mathfrak{sp}_{\rho_\iota}(A))$, and $\mathfrak{sp}_{\rho_\iota}(A)$ is compact, by Stone–Weierstrass theorem, it is dense in $\mathrm{C}(\mathfrak{sp}_{\rho_\iota}(A))$. Therefore, for $\epsilon > 0$, there is $a_\epsilon \in A$ with $\|f - \hat{a_\epsilon}\| < \epsilon$. Take an $\epsilon > 0$ such that $\frac{1-\epsilon}{\epsilon} > D$. Then $|f(\alpha) - \alpha(a_\epsilon)| = |1 - \alpha(a_\epsilon)| < \epsilon$ or $1 - \epsilon < |\alpha(a_\epsilon)| < 1 + \epsilon$. Also $|f(\iota_* e_x) - e_x(\iota a_\epsilon)| = |0 - \iota a_\epsilon(x)| < \epsilon$ for all $x \in X_\rho$. Now

$$\sup_{\beta \in \mathfrak{sp}_{\rho_\iota}(A)} |\beta(a_\epsilon)| \leq \rho_\iota(a_\epsilon) \leq D \sup_{x \in X_\rho} |e_x(\iota a)| \leq D\epsilon < 1 - \epsilon,$$

and hence $|\alpha(a_\epsilon)| < 1 - \epsilon$, a contradiction which completes the proof. □

The immediate implication of Theorem 2.5 is that if one is to realise a unital commutative algebra as a subalgebra of $(\mathbb{F}^X, \rho)$ the natural choice for ρ is the sup-*norm* over X which is defined by

$$\|f\|_X = \sup_{x \in X} |f(x)|. \tag{2}$$

We denote $\mathrm{B}_{\|\cdot\|_X}(\mathbb{F}^X)$ by $\ell^\infty(X)$. According to Theorem 2.5 the image of X under the map $x \mapsto e_x$ is dense in $\mathfrak{sp}_{\|\cdot\|_X}(\ell^\infty(X))$ and also for $\iota : A \longrightarrow \ell^\infty(X)$, we have $\overline{\iota_*(X_{\|\cdot\|_X})}^{\|\cdot\|_{X_\iota}} = \mathfrak{sp}_{\|\cdot\|_{X_\iota}}(A)$.

It is well known that if (X,τ) is a completely regular Hausdorff space, then $\mathfrak{sp}_{\|\cdot\|_{X_\iota}}(\mathrm{C}_b(X))$ is the Stone–Čech compactification of (X,τ). Moreover, every Hausdorff compactification of (X,τ) is homeomorphic to the spectrum of a unital subalgebra of $\mathrm{C}_b(X)$. In the next section we study the algebra of bounded measurable functions for a measurable structure on X.

3. Measurable structures on X

Let Σ be a σ-algebra of subsets of X. Let $\mathrm{M}_b(X,\Sigma)$ be the $*$-algebra of all bounded Σ-measurable $\mathbb{F}$-valued functions on (X,Σ). Suppose that $\mathrm{M}_b(X,\Sigma)$ separates the points of X. Hence, there is an injection from X onto a dense subset of $\mathfrak{sp}_{\|\cdot\|_X}(\mathrm{M}_b(X,\Sigma))$. Although we are assuming that $\mathrm{M}_b(X,\Sigma)$ separates points of X, this does not imply that Σ contains singletons as we see in the following example.

Example 3.1. Recall that a topological space (X,τ) is called a T_0 space if for each pair x,y of distinct points of X, either $x\notin\overline{\{y\}}^{\tau}$ or $y\notin\overline{\{x\}}^{\tau}$. Then characteristic functions of open sets clearly separate points of X. Let ω_1 be the first uncountable ordinal and $X=\omega_1+1$. The family of sets $R_a:=\{x\in X: x>a\}$ $(a\in X)$ forms a basis for a topology on $Y=X\setminus\{0\}$. This topology is evidently T_0 and satisfies the first axiom of countability at every point except ω_1. Although $\{\omega_1\}=\bigcap_{\omega_1>a}R_a$, every countable intersection of sets R_a for $a<\omega_1$ contains ordinals smaller than ω_1. Thus $\{\omega_1\}$ does not belong to the σ-algebra Σ_r generated by $\{R_a: a\in X\}$, while $\mathrm{M}_b(Y,\Sigma_r)$ separates points of Y. Note that the topology of Y in this case is not first countable. Singletons always belong to the σ-algebra of Borel subsets of first countable spaces.

We denote $\mathfrak{sp}_{\|\cdot\|_X}(\mathrm{M}_b(X,\Sigma))$ by $\xi_\Sigma X$ which is a compact Hausdorff space. Since $\mathrm{M}_b(X,\Sigma)$ separates the points of X, there is an injection $\psi: X\longrightarrow\xi_\Sigma X$ such that $\psi(X)$ is a dense subset of $\xi_\Sigma X$. Further, for every bounded Σ-measurable function f on X, the function $f\circ\psi^{-1}$ is continuously extendible over $\xi_\Sigma X$. Also, $\xi_\Sigma X$ is unique (up to a homeomorphism) with this property in the sense that for every other compact Hausdorff space, say γX, with X as a dense subset, so that the elements of $\mathrm{M}_b(X,\Sigma)$ are continuously extendible to γX, there is a continuous map $\iota:\gamma X\longrightarrow\xi_\Sigma X$ agreeing on the images of X in $\xi_\Sigma X$ and γX.

For $E\in\Sigma$, let χ_E be the characteristic function of E, defined on X. Denoting its continuous extension to $\xi_\Sigma X$ with $\tilde{\chi}_E$ we have:

$$(\tilde{\chi}_E)^2=\widetilde{(\chi_E^2)}=\tilde{\chi}_E;$$

thus it ranges over $\{0,1\}$, which implies that $\tilde{\chi}_E$ itself must be the characteristic function of a set, say $\tilde{E}$ in $\xi_\Sigma X$.

Lemma 3.2. *Let $E\in\Sigma$. Then $\overline{E}=\tilde{E}$ where $\overline{E}$ is the closure of E in $\xi_\Sigma X$.*

Proof. It is clear that $\tilde{E}=\tilde{\chi}_E^{-1}(\{1\})$ is closed and $E\subseteq\tilde{E}$. Thus $\overline{E}\subseteq\tilde{E}$. If $z\notin\overline{E}$, then for an open neighbourhood U of z we have $U\cap E=\emptyset$. Therefore there is a function $f\in\mathrm{M}_b(X,\Sigma)$ and an open interval I in $\mathbb{R}$ such that $z\in\tilde{f}^{-1}(I)\subseteq U$. Let $F=f^{-1}(I)\in\Sigma$, then $E\cap F=\emptyset$, so $\chi_E\cdot\chi_F=0$ and $\tilde{\chi}_E\cdot\tilde{\chi}_F=0$. Since $\tilde{\chi}_F(z)=1$ the later equation implies $\tilde{\chi}_E(z)=0$. This contradicts the assumption $z\in\tilde{E}$, therefore $\tilde{E}=\overline{E}$. $\square$

Using the above lemma, we investigate some properties of X as a subspace of $\xi_\Sigma X$.

Corollary 3.3. *Let Σ be a σ-algebra of subsets of X.*

(1) *$\overline{E}$ is a clopen subset of $\xi_\Sigma X$ for every $E\in\Sigma$;*
(2) *$\tilde{\Sigma}:=\{\tilde{E}: E\in\Sigma\}$ forms a basis for the topology of $\xi_\Sigma X$;*
(3) *$\tilde{\Sigma}$ is the set of all clopen subsets of $\xi_\Sigma X$.*

In addition, if Σ contains all singletons, then

(4) *X is an open dense subspace of $\xi_\Sigma X$ whose subspace topology is discrete;*
(5) *For a subset $Y \subset X$, $\overline{Y} = Y$ if and only if Y is finite.*
(6) *For $x \in X$ and $E \in \Sigma$, $x \in \tilde{E}$ if and only if $x \in E$.*

Proof. (1) Since $\overline{E} = \tilde{E} = \tilde{\chi}_E^{-1}(\{1\}) = \tilde{\chi}_E^{-1}(\frac{1}{2}, \infty)$ and $\tilde{\chi}_E$ is continuous, we conclude that $\tilde{E}$ is clopen.

(2) The family $\{\tilde{E} : E = f^{-1}([0,1]) \text{ for } f \in \mathrm{M}_b(X,\Sigma)\}$ forms a basis for the closed subsets of $\xi_\Sigma X$. Note that $E = f^{-1}([0,1]) \in \Sigma$ and $\tilde{E} = \overline{E} = \tilde{f}^{-1}([0,1])$ which is clopen by (1) and the conclusion follows.

(3) By (1) and (2), $\xi_\Sigma X$ is totally disconnected. Suppose that $Y \subseteq \xi_\Sigma X$ is clopen. Since $\xi_\Sigma X$ is compact, so is Y. By (2), as an open set, $Y = \bigcup_{i\in I} \tilde{E}_i$ for a family $\{E_i\}_{i\in I} \subset \Sigma$. Therefore, $Y = \tilde{E}_{i_1} \cup \cdots \cup \tilde{E}_{i_n}$ for some $i_1, \ldots, i_n \in I$, which belongs to $\tilde{\Sigma}$.

(4) By (1), the closure of every element of Σ is open in $\xi_\Sigma X$. Since the topology of $\xi_\Sigma X$ is Hausdorff and Σ contains all singletons, singletons are closed. Therefore $\{x\}$ is clopen for every $x \in X$ and hence X is open in $\xi_\Sigma X$. Moreover, by Theorem 2.5, X is dense in $\xi_\Sigma X$.

(5) If Y is finite, then since the topology of $\xi_\Sigma X$ is Hausdorff, it is also closed. Let Y be an arbitrary subset of X. The set $\overline{Y} \subseteq \xi_\Sigma X$ is compact. If $Y = \overline{Y}$, then $\{\{x\} : x \in Y\}$ is an open cover of Y which will not have a finite subcover, if Y is infinite.

(6) Clearly if $x \in E$ then $x \in \tilde{E}$. Conversely, suppose that $x \in \tilde{E} \setminus E$. Then $E \subseteq \tilde{E} \setminus \{x\}$. The superset is closed since $\{x\}$ and $\tilde{E}$ are both clopen in $\xi_\Sigma X$ by (5) and (1) respectively. Thus $\overline{E} = \tilde{E} \subseteq \tilde{E} \setminus \{x\}$, a contradiction. □

Remark 3.4. Let Σ be a σ-algebra of subsets of an infinite set X. If there are infinitely many disjoint sets in Σ, then $\mathrm{M}_b(X,\Sigma)$ is not separable. The proof is similar to the classical proof of the fact that $\ell^\infty(\mathbb{N})$ is not separable. Hence, in this case $\xi_\Sigma X$ is not metrizable. (It is classically known that for a compact space Y, $\mathrm{C}_0(Y)$ is separable if and only if Y is metrizable [1, Theorem 2.4].)

A topological space is called *extremely disconnected* if the closure of every open set is open. In the following we study this property for $\xi_\Sigma X$. For the relation between Boolean algebras and extremely disconnected spaces see [6, §3.5] or [7, 22.4]. Commutative algebras with extremely disconnected Gelfand spectra are forming the commutative class of AW*-algebras where $\mathbb{F} = \mathbb{C}$.

An algebra of sets is said to be *complete* if it is closed under arbitrary union and hence intersection

Proposition 3.5. *Let Σ be a σ-algebra on X including all singletones. Then $\xi_\Sigma X$ is extremely disconnected if and only if Σ is complete.*

Proof. Suppose that $\xi_\Sigma X$ is extremely disconnected and let $\Delta \subseteq \Sigma$. Then $U = \bigcup_{Y\in\Delta} \tilde{Y}$ is open and hence $\overline{U}$ is also open, thus by Corollary 3.3(3), it belongs to $\tilde{\Sigma}$, say $\overline{U} = \tilde{E}$ for some $E \in \Sigma$. We show that $E = \bigcup_{Y\in\Delta} Y$. To do so, first suppose that $\exists x \in \left(\bigcup_{Y\in\Delta} Y\right) \setminus E$. Clearly $x \in \overline{U} = \tilde{E}$. This violates Corollary 3.3(6). Conversely, if $\exists x \in E \setminus \bigcup_{Y\in\Delta} Y$, then $\bigcup_{Y\in\Delta} Y \subseteq F$ for $F = E \setminus \{x\} \in \Sigma$. Therefore, $U \subseteq \tilde{F}$. Also, by Corollary 3.3(6), $\tilde{F} \subset \tilde{E} \setminus \{x\}$. On the other hand, since $\tilde{F}$ is clopen,

$$\tilde{E} = \overline{U} \subseteq \tilde{F} \subseteq \tilde{E} \setminus \{x\} \subsetneq \tilde{E},$$

a contradiction and hence the claim is proved.

Now, suppose that Σ is complete and let U be an open set in $\xi_\Sigma X$. Take $\Delta \subset \Sigma$ such that $U = \bigcup_{F\in\Delta} \tilde{F}$. Since Σ is complete, $E = \bigcup_{F\in\Delta} F \in \Sigma$ and $\overline{U} \subseteq \overline{E} = \tilde{E}$. If $\tilde{E} \setminus \overline{U}$, which is open, is not empty, then it contains a nonempty clopen $Y \in \tilde{\Sigma}$. Now $V = \tilde{E} \setminus Y$ is a clopen set such that

$$E \subseteq U \subseteq V = \bar{V} \subsetneq \tilde{E} = \bar{E}$$

which is a contradiction. Thus $\overline{U} = \tilde{E}$ is clopen and hence $\xi_\Sigma X$ is extremely disconnected. □

Let $\theta : A \longrightarrow \ell^\infty(X)$ be an algebra homomorphism and τ be a topology on X. Then one can show that the induced map $\theta_*|_X : (X, \tau) \longrightarrow \mathfrak{sp}_{\|\cdot\|_{X\theta}}(A)$ is continuous if and only if $\theta A \subseteq \mathrm{C}_b(X, \tau)$. The following proposition is an analogue of this result for $\mathrm{M}_b(X, \Sigma)$ and Σ-measurability.

Proposition 3.6. *Suppose that Σ is a σ-algebra on $X \neq \emptyset$ such that every open subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$. Let $\iota : A \longrightarrow \ell^\infty(X)$ be an algebra homomorphism. Then the induced map $\iota_*|_X : (X, \Sigma) \longrightarrow \mathfrak{sp}_{\|\cdot\|_{X_\iota}}(A)$ is Σ-measurable if and only if $\iota A \subseteq \mathrm{M}_b(X, \Sigma)$.*

Proof. By assumption, every Borel subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$. A basic open set of $\mathfrak{sp}_{\|\cdot\|_{X_\iota}}(A)$ is of the form $\hat{a}^{-1}(O)$ where $O \subseteq \mathbb{F}$ is open and $a \in A$. Looking at the inverse image of $\hat{a}^{-1}(O)$ under ι_*, we have

$$\iota_*|_X^{-1} \hat{a}^{-1}(O) = \widehat{\iota a}^{-1}(O) \cap X \tag{3}$$

($\Rightarrow$) Suppose that ι_* is Σ-measurable. If in contrary $\iota a \notin \mathrm{M}_b(X, \tau)$ for some $a \in A$, then there exists a set $O \subseteq \mathbb{F}$, such that $\widehat{\iota a}^{-1}(O) \cap X$ is not Σ-measurable and hence by (3), $\iota_*|_X$ cannot be Σ-measurable which is a contradiction.

($\Leftarrow$) If each ιa is Σ-measurable, then $\widehat{\iota a}^{-1}(O)$ is Σ-measurable for any open $O \subseteq \mathbb{F}$ and again by (3), $\iota_*|_X$ is Σ-measurable. □

It is not known to the authors if the assumption "every open subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$" in Proposition 3.6 is essential or not. One can show that this assumption rules out some examples including $X = \mathbb{N}$ and $\Sigma = P(\mathbb{N})$, the power set of X.

3.1. Measures on (X, Σ) and $\xi_\Sigma X$

Starting with a measurable structure (X, Σ) such that $M_b(X, \Sigma)$ separates the points of X. We identified X as an open dense subset of a totally disconnected compact space $\xi_\Sigma X$ where every bounded Σ-measurable function on X extends continuously to $\xi_\Sigma X$. This naturally leads one to ask about the relation between measures on (X, Σ) and $\xi_\Sigma X$.

Proposition 3.7. *Let μ be a finite positive measure on (X, Σ). Then μ extends to a Borel measure ${}^*\mu$ on $\xi_\Sigma X$, satisfying*

$$\forall E \in \Sigma \quad {}^*\mu(\tilde{E}) = \mu(E).$$

Proof. Define a linear functional $L : \mathrm{C}(\xi_\Sigma X) \longrightarrow \mathbb{F}$ by

$$L(f) = \int_X f|_X \, d\mu, \quad \forall f \in \mathrm{C}(\xi_\Sigma X).$$

Clearly L is positive and hence by Riesz representation theorem, there exists a Borel measure ${}^*\mu$ on $\xi_\Sigma X$ such that

$$L(f) = \int_{\xi_\Sigma X} f \, d\,{}^*\mu, \quad \forall f \in \mathrm{C}(\xi_\Sigma X).$$

Note that for every $E \in \Sigma$, ${}^*\mu(\tilde{E}) = \int \tilde{\chi}_E \, d\,{}^*\mu = L(\tilde{\chi}_E) = \int \chi_E \, d\mu = \mu(E)$. □

Although the measure ${}^*\mu$ obtained in Proposition 3.7 seems to be mainly supported on X, but in fact, the size of $X \cap \operatorname{supp}({}^*\mu)$ is rather small as it is pointed out in the following proposition.

Proposition 3.8. *Let μ be a finite Borel measure on $\xi_\Sigma X$ and Σ contains all singletons. Then $X \cap supp(\mu)$ is at most countable.*

Proof. By definition, a point $x \in \xi_\Sigma X$ belongs to $\operatorname{supp}(\mu)$ if and only if for every neighbourhood U of x, $\mu(U) > 0$. Every singleton $\{z\}$ for $z \in X$ is open in $\xi_\Sigma X$, thus for every $z \in X \cap \operatorname{supp}(\mu)$, $\mu(\{z\}) > 0$. Since $\mu(\xi_\Sigma X) < \infty$, $X \cap \operatorname{supp}(\mu)$ cannot be uncountable. □

Corollary 3.9. *Let μ be a finite positive measure on (X, Σ) where Σ contains all singletons. If $\mu(\{x\}) = 0$, for some $x \in X$, then $x \notin supp({}^*\mu)$.*

Proof. Since $\{x\} \in \Sigma$ and $\mu(\{x\}) = 0$, $\chi_x \in \mathrm{M}_b(X, \Sigma)$ and $\int_X \chi_x d\mu = 0$. Thus ${}^*\mu(\{x\}) = \int_{\xi_\Sigma X} \tilde{\chi}_x d\, {}^*\mu = 0$. But $\{x\}$ is open and hence $x \notin \operatorname{supp}({}^*\mu)$. □

3.2. Borel algebra of a topology

Let (X, τ) be a T_1 topological space. Since the topology τ is T_1, singletons are Borel and hence $\mathrm{M}_b(X, \mathcal{B}_\tau)$ separates points of X. Clearly the inclusion $\iota : \mathrm{C}_b(X, \tau) \longrightarrow \mathrm{M}_b(X, \mathcal{B}_\tau)$ is continuous and hence $\iota_* : \xi_{\mathcal{B}_\tau} X \longrightarrow \mathfrak{sp}_{\|\cdot\|_X}(\mathrm{C}_b(X, \tau))$ is onto. Consequently, if τ is completely regular, then βX is a continuous image of $\xi_{\mathcal{B}_\tau X}$ where βX is the Stone–Čech compactification of X (look at [4, 6.5]). If $\mathcal{B}_\tau = \tau$ then $\xi_{\mathcal{B}_\tau}$ and β are identical and ι_* is injective. It is natural to ask if there is any relation between topological structures of (X, τ) and $\xi_{\mathcal{B}_\tau} X$.

Let $x \in X$ and $\mathcal{N}_\tau(x)$ be the family of open neighbourhoods of x in τ and $\tilde{\mathcal{N}_\tau}(x) = \{\tilde{U} : U \in \mathcal{N}_\tau(x)\}$. Define the *halo* of x in $\xi_{\mathcal{B}_\tau} X$ as

$$h(x) := \bigcap \tilde{\mathcal{N}_\tau}(x).$$

The set $h(x)$ is compact and contains all points of $\xi_{\mathcal{B}_\tau} X$ that cannot be distinguished from x via the image of τ. If τ is Hausdorff, then for each $y \in X$ such that $x \neq y$, there are open sets $U_x, U_y \in \tau$ with $U_x \cap U_y = \emptyset$. Thus $\tilde{U}_x \cap \tilde{U}_y = \emptyset$, and therefore $h(x) \cap h(y) = \emptyset$.

Proposition 3.10. *If τ is Hausdorff, then $h(x)$ is open if and only if $\{x\}$ is open in (X, τ).*

Proof. If $\{x\}$ is open, then $\{x\} \in \mathcal{N}_\tau(x)$. Since $\widetilde{\{x\}} = \{x\}$, clearly, $x \in h(x) \subseteq \{x\}$. Conversely, if $h(x)$ is open, then it is clopen and hence, by Corollary 3.3(3), $h(x) = \tilde{E}$ for some $E \in \mathcal{B}_\tau$. If $E \neq \{x\}$, then E contains another point $y \in X$, $y \neq x$. Thus $y \in h(x)$ which implies that $h(x) \cap h(y) \neq \emptyset$, contradicting the above argument before the proposition. □

Proposition 3.10 can be read as $h(x) = \{x\}$ if and only if $\{x\}$ is open in (X, τ). The following shows how the compactness of a Borel subset of (X, τ) is reflected in $\xi_{\mathcal{B}_\tau} X$.

Theorem 3.11. *Let $Y \subseteq (X, \tau)$ be a Borel subspace. Then Y is compact if and only if $\tilde{Y} \subseteq \bigcup_{y \in Y} h(y)$.*

Proof. ($\Rightarrow$) Suppose that Y is compact and let $z \in \xi_{\mathcal{B}_\tau} X \setminus \bigcup_{y \in Y} h(y)$. We show $z \notin \tilde{Y}$. Since $z \notin \bigcup_{y \in Y} h(y)$, for each $y \in Y$, there exists $O_y \in \mathcal{N}_\tau(y)$ such that $z \notin \tilde{O}_y$. Now $\{O_y : y \in Y\}$ is an open cover of the compact set Y. Let $\{O_{y_1}, \ldots, O_{y_k}\}$ be such that $Y \subseteq \bigcup_{i=1}^k O_{y_i}$, then $\tilde{Y} \subseteq \bigcup_{i=1}^k \tilde{O}_{y_i}$ which proves $z \notin \tilde{Y}$, and hence $\tilde{Y} \subseteq \bigcup_{y \in Y} h(y)$.

($\Leftarrow$) Suppose that $\tilde{Y} \subseteq \bigcup_{y\in Y} h(y)$, but Y is not compact. Then there exists an open cover $\{O_i\}_{i\in I}$ of Y with no finite subcover. So, for every finite subset $\{i_1, \dots, i_n\}$ of I,

$$Y \cap \left(\bigcap_{k=1}^{n} O_{i_k}^c \right) \neq \emptyset.$$

Since Y is Borel, $\tilde{Y}$ is compact and hence $\{\tilde{O}_i^c\}_{i\in I}$ forms a basis for an ultrafilter $\mathcal{F}$ in $\xi_{\mathcal{B}_\tau} X$. Clearly $\tilde{Y} \in \mathcal{F}$ and hence $z = \lim \mathcal{F} \in \tilde{Y}$ (for more detail on filters see [8, §12]). For every $y \in Y$, there exists $i \in I$ such that $O_i \in \mathcal{N}_\tau(y)$ and hence $z \notin \tilde{O}_i$. Thus $z \notin h(y) \subseteq \tilde{O}_i$. This proves

$$z \in \tilde{Y} \setminus \bigcup_{y\in Y} h(y),$$

as desired. □

It is worth mentioning that the results of Subsection 3.2 resemble significant similarities between properties of $\xi_{\mathcal{B}_\tau} X$ and nonstandard extensions of (X, τ). We can consider $\xi_{\mathcal{B}_\tau} X$ as a nonstandard model of (X, τ) and characterize halos as analogue to monads and so on. In this scope, Theorem 3.11 is the analogue of Robinson's theorem [5, Theorem III.2.2] on nonstandard extensions of compact spaces.

Acknowledgments

The authors thank Reza Koushesh and Nico Spronk for their productive comments on the first draft of this manuscript. Parts of this research were carried out while the first author was visiting University of Saskatchewan and when the second author was visiting University of Waterloo. We are grateful for these kind hospitalities. The authors would like to express their gratitudes to the referee for valuable comments.

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