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Seminormed $*$ -subalgebras of $\ell^\infty(X)$



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ABSTRACT

Arbitrary representations of an involutive commutative unital \mathbb{F} -algebra A as a subalgebra of \mathbb{F}^X are considered, where $\mathbb{F} = \mathbb{C}$ or \mathbb{R} and $X \neq \emptyset$. The Gelfand spectrum of A is explained as a topological extension of X where a seminorm on the image of A in \mathbb{F}^X is present. It is shown that among all seminorms, the sup-norm is of special importance which reduces \mathbb{F}^X to $\ell^\infty(X)$. The Banach subalgebra of $\ell^\infty(X)$ of all Σ -measurable bounded functions on X , $M_b(X, \Sigma)$, is studied for which Σ is a σ -algebra of subsets of X . In particular, we study lifting of positive measures from (X, Σ) to the Gelfand spectrum of $M_b(X, \Sigma)$ and observe an unexpected shift in the support of measures. In the case that Σ is the Borel algebra of a topology, we study the relation of the underlying topology of X and the topology of the Gelfand spectrum of $M_b(X, \Sigma)$.

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1. Introduction

It is common to look at rings and algebras as families of functions over a nonempty set with values in a suitable ring or field. This is especially helpful if one wants to study the ideal structure of a ring or algebra which naturally involves topological notions, mainly compactness.

In this article, we summarize some observations about topological algebras in an abstract manner. One motivation comes from [3] which attempts to represent positive linear functionals on a given commutative unital algebra as an integral with respect to a positive measure on the space of characters of the algebra. This is done by realizing the algebra as a subalgebra of continuous functions over the character space.

During the present article we always assume that A is an involutive commutative algebra over the field $\mathbb{F} = \mathbb{R}$ or \mathbb{C} equipped with a seminorm ρ . In Section 2, first we provide a brief overview of the theory of seminormed algebras and their Gelfand spectrum. Then, we assume that A can be embedded into (\mathbb{F}^X, ρ)

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for a nonempty set X where ρ is a submultiplicative seminorm on a subalgebra of \mathbb{F}^X that contains the image of A . This induces a seminormed structure on A as well. [Theorem 2.5](#) gives a necessary and sufficient condition for X to be dense in the Gelfand spectrum of A , that is, when the topology induced by the seminorm is equivalent to the topology induced by the sup-norm defined in [\(2\)](#).

Motivated by [\[3\]](#), where positive linear functionals on an algebra are presented as integrals with respect to constructibly Radon measures, in [Section 3](#), we consider a measurable structure Σ on X and study the spectrum of the algebra of bounded measurable functions on (X, Σ) , denoted by $M_b(X, \Sigma)$. We prove that positive measures on X lift to positive measures over the spectrum of $M_b(X, \Sigma)$, but this lifting shifts the support of the original measure out of X modulo at most a countable subset of X ([Propositions 3.7 and 3.8](#)). At the end we choose Σ to be the Borel algebra of a topology τ on X and observe some connections between τ and the spectrum of $M_b(X, \Sigma)$ ([Proposition 3.10](#) and [Theorem 3.11](#)).

1.1. Notations

Let X be a non-empty set and \mathfrak{S} be a structure on X which induces a topology on X . We denote this topology by $\tau(X, \mathfrak{S})$. For instance, let \mathfrak{S} be a family of functions, defined on X , with values in a topological space. Then $\tau(X, \mathfrak{S})$ is the coarsest topology on X which makes every function in \mathfrak{S} continuous.

Let (X, τ) be a topological space. We denote the set of all τ -continuous \mathbb{F} -valued functions on X by $C(X, \tau)$ or $C(X)$ if there is no risk of confusion. We use $C_b(X)$ (or $C_b(X, \tau)$) to denote the set of all $f \in C(X)$ which are bounded on X . If (X, τ) is locally compact, $C_0(X)$ denotes the set of all $f \in C_b(X)$ which are vanishing at infinity.

Let $P(X)$ be the power set of X . The σ -algebra of sets induced on X by a set $\Lambda \subseteq P(X)$ is denoted by $\sigma(\Lambda)$. In particular if τ is a topology on X , then $\sigma(\tau)$ is the σ -algebra of all Borel subsets of (X, τ) denoted by \mathcal{B}_τ .

2. Involutive subalgebras of $\ell^\infty(X)$

The set theory which is applied in this paper is ZFC. Throughout this article all algebras are assumed to be involutive (also called $*$ -algebra) and commutative over a field \mathbb{F} (which is either \mathbb{R} or \mathbb{C} as specified). Subsequently, all \mathbb{F} -valued $*$ -algebra homomorphisms are also supposed to be \mathbb{F} -module maps.

Definition 2.1. Let A be a commutative $*$ -algebra. A function $\rho : A \rightarrow [0, \infty]$ is called a *quasi-norm* on A if

- (1) $\forall a \in A \quad \rho(a^*) = \rho(a)$,
- (2) $\forall a, b \in A \quad \rho(a + b) \leq \rho(a) + \rho(b)$ (subadditive),
- (3) $\forall r \in \mathbb{F} \forall a \in A \quad \rho(ra) = |r|\rho(a)$.

ρ is called *submultiplicative* if

- (4) $\forall a, b \in A, \quad \rho(ab) \leq \rho(a)\rho(b)$ where the product of ∞ and 0 is ∞ .

A *quasi-norm* ρ on A is called a *seminorm* if $\rho(a) < \infty$ for every $a \in A$.

Let A be a commutative $*$ -algebra and let ρ be a quasi-norm on A . The set of all elements of A with a finite quasi-norm ρ is denoted by $B_\rho(A)$, i.e.,

$$B_\rho(A) = \{a \in A : \rho(a) < \infty\}.$$

If ρ is a submultiplicative quasi-norm, it is clear that $B_\rho(A)$ is a $*$ -subalgebra of A and the restriction of ρ to $B_\rho(A)$ is a seminorm. A $*$ -algebra A with a seminorm ρ forms a *seminormed algebra* if ρ is submultiplicative. For a seminormed algebra (A, ρ) , the set of all non-zero $*$ -algebra homomorphisms $\alpha : A \rightarrow \mathbb{F}$ is denoted by $\mathcal{X}(A)$. The set $\mathfrak{sp}_\rho(A)$ of all ρ -continuous $*$ -algebra homomorphisms belonging to $\mathcal{X}(A)$ is called the *Gelfand spectrum* of (A, ρ) . Every element $a \in A$ induces a map $\hat{a} : \mathcal{X}(A) \rightarrow \mathbb{F}$ defined by $\hat{a}(\alpha) := \alpha(a)$ for each $\alpha \in \mathcal{X}(A)$. Next, we have a characterization of all ρ -continuous \mathbb{F} -valued $*$ -algebra homomorphisms. The following lemma was proved as Lemma 3.2 in [2].

Lemma 2.2. *Let (A, ρ) be a commutative seminormed $*$ -algebra and $\alpha \in \mathcal{X}(A)$. Then $\alpha \in \mathfrak{sp}_\rho(A)$ if and only if $|\alpha(a)| \leq \rho(a)$, for all $a \in A$.*

The Gelfand spectrum $\mathfrak{sp}_\rho(A)$ (as well as $\mathcal{X}(A)$) naturally carries a Hausdorff topology as a subspace of \mathbb{F}^A with the product topology. For a real number $r > 0$, let $D_r := \{c \in \mathbb{F} : |c| \leq r\}$. According to Lemma 2.2, $\mathfrak{sp}_\rho(A) \subseteq \prod_{a \in A} D_{\rho(a)}$. One simple approximation argument implies that every element in the closure of $\mathfrak{sp}_\rho(A)$ is a $*$ -algebra homomorphism. But it also belongs to $\prod_{a \in A} D_{\rho(a)}$. Therefore, the closure of $\mathfrak{sp}_\rho(A)$ is a subset of $\mathfrak{sp}_\rho(A) \cup \{\mathbf{0}\}$ where $\mathbf{0}$ is the constant linear functional zero on A . From now on, we use $\mathfrak{sp}_\rho(A)$ to denote it as a topological subspace of $\prod_{a \in A} D_{\rho(a)}$. Note that, for each $a \in A$, \hat{a} is an element in $C(\mathcal{X}(A))$ and subsequently, its restriction to $\mathfrak{sp}_\rho(A)$ belongs to $C(\mathfrak{sp}_\rho(A))$.

Note that the difference between the following corollary and [2, Corollary 3.3] is due to the fact that we exclude zero in the definition of $\mathcal{X}(A)$.

Corollary 2.3. *Let (A, ρ) be a commutative seminormed $*$ -algebra. If A is unital then $\mathfrak{sp}_\rho(A)$ is compact. If $\mathfrak{sp}_\rho(A)$ is compact then there exists an element $a_0 \in A$ such that $|\alpha(a_0)| \geq 1$ for every $\alpha \in \mathfrak{sp}_\rho(A)$.*

Proof. If A is unital, one may use the identity element, $\mathbf{1}$, (for which we have $\alpha(\mathbf{1}) = 1$ for every $\alpha \in \mathfrak{sp}_\rho(A)$) to show that $\mathbf{0}$ does not belong to the closure of $\mathfrak{sp}_\rho(A)$. Therefore, $\mathfrak{sp}_\rho(A)$ is indeed a closed set in $\prod_{a \in A} D_{\rho(a)}$, and subsequently, $\mathfrak{sp}_\rho(A)$ is compact.

Now suppose that $\mathfrak{sp}_\rho(A)$ is compact. Therefore, $\mathfrak{sp}_\rho(A)$ is a closed subset of $\prod_{a \in A} D_{\rho(a)}$, not containing $\mathbf{0}$. So, there exist a finite set $\{a_1, \dots, a_m\}$ and $\epsilon > 0$ such that for each α there is an i with $|\alpha(a_i)| \geq \epsilon^{1/2}$. Now set $a := a_1^* a_1 + \dots + a_m^* a_m$. Then, this particular element a , satisfies $|\alpha(a)| \geq \epsilon$ for each $\alpha \in \mathfrak{sp}_\rho(A)$. Let $k = \inf\{|\alpha(a)| : \alpha \in \mathfrak{sp}_\rho(A)\} \geq \epsilon$ and $a_0 := a/k$. The claim follows for a_0 . \square

Remark 2.4. Every non-unital commutative seminormed $*$ -algebra (A, ρ) can be embedded into the unital $*$ -algebra $A_1 := A \oplus \mathbb{F}$ with multiplication $(a, \lambda)(b, \gamma) = (ab + \gamma a + \lambda b, \lambda\gamma)$ and involution $(a, \lambda)^* = (a^*, \bar{\lambda})$. Defining $\rho_1(a + \lambda) = \rho(a) + |\lambda|$ we also obtain a seminorm on A_1 which makes the natural embedding $a \mapsto (a, 0)$ continuous. For each $\alpha \in \mathcal{X}(A)$, define the extension $\alpha'(a, \lambda) = \alpha(a) + \lambda$ which is obviously an element in $\mathcal{X}(A_1)$. So one can regard $\mathcal{X}(A)$ as a subset of $\mathcal{X}(A_1)$. Regarding \mathbb{F} as a commutative algebra, we know that $\mathcal{X}(\mathbb{F})$ has only one element which is the identity map. For an element $\beta \in \mathcal{X}(A_1)$, note that if the restriction $\beta|_A$ is non-zero, it is formed by the element $\beta|_A \in \mathcal{X}(A)$ as described above. Hence, $\mathcal{X}(A_1) \setminus \mathcal{X}(A)$ consists of exactly one element, denoted here by $\hat{\infty}$, which maps (a, λ) to λ for all $a \in A$ and $\lambda \in \mathbb{F}$. Clearly, $\hat{\infty} \in \mathfrak{sp}_{\rho_1}(A_1)$, therefore A is a closed maximal ideal of A_1 . Moreover, if $\mathfrak{sp}_\rho(A)$ is not compact, $\mathfrak{sp}_{\rho_1}(A_1)$ is the one-point compactification of $\mathfrak{sp}_\rho(A)$.

Every $*$ -algebra homomorphism $\phi : A \rightarrow B$ induces a mapping $\phi_* : \mathcal{X}(B) \rightarrow \mathcal{X}(A) \cup \{\mathbf{0}\}$ defined by $\phi_*(\beta) = \beta \circ \phi$ for each $\beta \in \mathcal{X}(B)$. Suppose that B is equipped with a seminorm ρ . The homomorphism ϕ induces a seminorm ρ_ϕ on A defined by $\rho_\phi(a) = \rho(\phi(a))$. If ρ is submultiplicative, then so is ρ_ϕ . The map ϕ as a homomorphism between seminormed $*$ -algebras (A, ρ_ϕ) and (B, ρ) is continuous. Therefore ϕ_* maps $\mathfrak{sp}_\rho(B)$ continuously into $\mathfrak{sp}_{\rho_\phi}(A)$.

Here we are mainly interested in the case where B is a $*$ -subalgebra of \mathbb{F}^X for a non-empty set X where \mathbb{F}^X is the space of all \mathbb{F} -valued functions on X furnished with pointwise multiplication and the canonical \mathbb{F} -conjugate involution. This generally enables us to realize $\mathfrak{sp}(A)$ relative to X as follows.

Let ρ be a submultiplicative quasi-norm on \mathbb{F}^X with $\rho(\mathbf{1}) \geq 1$ where $\mathbf{1}$ denotes the constant function which takes 1 all over the X . There is a natural map $e : X \rightarrow \mathcal{X}(\mathbb{F}^X)$ which, to every $x \in X$, assigns the evaluation map $e_x : \mathbb{F}^X \rightarrow \mathbb{F}$, defined by $e_x(f) := f(x)$. It is clear that $e_x \in \mathcal{X}(\mathbb{F}^X)$. We denote the set of all ρ -continuous evaluations by X_ρ . Note that by Lemma 2.2, for every $x \in X$, $e_x \in X_\rho$ if and only if $e_x \in \mathfrak{sp}_\rho(B_\rho(\mathbb{F}^X))$. In symbols:

$$X_\rho = \{e_x : x \in X, e_x \in \mathfrak{sp}_\rho(B_\rho(\mathbb{F}^X))\}. \tag{1}$$

Let $\iota : A \rightarrow B_\rho(\mathbb{F}^X)$ be a $*$ -algebra homomorphism. By abuse of notation, we use ι_* to denote the induced map $\iota_*|_X : X \rightarrow \mathfrak{sp}_\rho(A)$.

Theorem 2.5. *Let A be a commutative $*$ -algebra and $\iota : A \rightarrow B_\rho(\mathbb{F}^X)$ be a $*$ -algebra homomorphism, where ρ is a submultiplicative quasi-norm on \mathbb{F}^X with $\rho(\mathbf{1}) \geq 1$. Define $\rho_\iota := \rho \circ \iota$ on A . Then $\iota_*(X_\rho)$ is dense in $\mathfrak{sp}_{\rho_\iota}(A)$ if and only if there exists $D > 0$ such that*

$$\rho_\iota(a) \leq D \cdot \sup_{x \in X_\rho} |e_x(\iota a)|,$$

for all $a \in A$.

Proof. Note that by Lemma 2.2, for each $a \in A$,

$$\sup_{\beta \in \mathfrak{sp}_{\rho_\iota}(A)} |\hat{a}(\beta)| \leq \rho_\iota(a).$$

(\Rightarrow) Since $\iota_*(X_\rho)$ is dense in $\mathfrak{sp}_{\rho_\iota}(A)$ we have

$$\sup_{x \in X_\rho} |e_x(\iota a)| = \sup_{\beta \in \mathfrak{sp}_{\rho_\iota}(A)} |\beta(a)|,$$

and it suffices to take $D = 1$.

(\Leftarrow) In contrary, suppose that $\alpha \in \mathfrak{sp}_{\rho_\iota}(A) \setminus \overline{\iota_*(X_\rho)}$. There exists $f \in C(\mathfrak{sp}_{\rho_\iota}(A))$ such that $f(\alpha) = 1$ and $f|_{\iota_*(X_\rho)} = 0$, by Urysohn’s lemma. Since \hat{A} separates points of $\mathfrak{sp}_{\rho_\iota}(A)$, $\hat{a} \in C(\mathfrak{sp}_{\rho_\iota}(A))$, and $\mathfrak{sp}_{\rho_\iota}(A)$ is compact, by Stone–Weierstrass theorem, it is dense in $C(\mathfrak{sp}_{\rho_\iota}(A))$. Therefore, for $\epsilon > 0$, there is $a_\epsilon \in A$ with $\|f - \hat{a}_\epsilon\| < \epsilon$. Take an $\epsilon > 0$ such that $\frac{1-\epsilon}{\epsilon} > D$. Then $|f(\alpha) - \alpha(a_\epsilon)| = |1 - \alpha(a_\epsilon)| < \epsilon$ or $1 - \epsilon < |\alpha(a_\epsilon)| < 1 + \epsilon$. Also $|f(\iota_* e_x) - e_x(\iota a_\epsilon)| = |0 - \iota a_\epsilon(x)| < \epsilon$ for all $x \in X_\rho$. Now

$$\sup_{\beta \in \mathfrak{sp}_{\rho_\iota}(A)} |\beta(a_\epsilon)| \leq \rho_\iota(a_\epsilon) \leq D \sup_{x \in X_\rho} |e_x(\iota a_\epsilon)| \leq D\epsilon < 1 - \epsilon,$$

and hence $|\alpha(a_\epsilon)| < 1 - \epsilon$, a contradiction which completes the proof. \square

The immediate implication of Theorem 2.5 is that if one is to realise a unital commutative algebra as a subalgebra of (\mathbb{F}^X, ρ) the natural choice for ρ is the sup-norm over X which is defined by

$$\|f\|_X = \sup_{x \in X} |f(x)|. \tag{2}$$

We denote $B_{\|\cdot\|_X}(\mathbb{F}^X)$ by $\ell^\infty(X)$. According to Theorem 2.5 the image of X under the map $x \mapsto e_x$ is dense in $\mathfrak{sp}_{\|\cdot\|_X}(\ell^\infty(X))$ and also for $\iota : A \rightarrow \ell^\infty(X)$, we have $\overline{\iota_*(X_{\|\cdot\|_X})}^{\|\cdot\|_{X_\iota}} = \mathfrak{sp}_{\|\cdot\|_{X_\iota}}(A)$.

It is well known that if (X, τ) is a completely regular Hausdorff space, then $\mathfrak{sp}_{\|\cdot\|_{X_t}}(C_b(X))$ is the Stone–Čech compactification of (X, τ) . Moreover, every Hausdorff compactification of (X, τ) is homeomorphic to the spectrum of a unital subalgebra of $C_b(X)$. In the next section we study the algebra of bounded measurable functions for a measurable structure on X .

3. Measurable structures on X

Let Σ be a σ -algebra of subsets of X . Let $M_b(X, \Sigma)$ be the $*$ -algebra of all bounded Σ -measurable \mathbb{F} -valued functions on (X, Σ) . Suppose that $M_b(X, \Sigma)$ separates the points of X . Hence, there is an injection from X onto a dense subset of $\mathfrak{sp}_{\|\cdot\|_X}(M_b(X, \Sigma))$. Although we are assuming that $M_b(X, \Sigma)$ separates points of X , this does not imply that Σ contains singletons as we see in the following example.

Example 3.1. Recall that a topological space (X, τ) is called a T_0 space if for each pair x, y of distinct points of X , either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Then characteristic functions of open sets clearly separate points of X . Let ω_1 be the first uncountable ordinal and $X = \omega_1 + 1$. The family of sets $R_a := \{x \in X : x > a\}$ ($a \in X$) forms a basis for a topology on $Y = X \setminus \{0\}$. This topology is evidently T_0 and satisfies the first axiom of countability at every point except ω_1 . Although $\{\omega_1\} = \bigcap_{\omega_1 > a} R_a$, every countable intersection of sets R_a for $a < \omega_1$ contains ordinals smaller than ω_1 . Thus $\{\omega_1\}$ does not belong to the σ -algebra Σ_r generated by $\{R_a : a \in X\}$, while $M_b(Y, \Sigma_r)$ separates points of Y . Note that the topology of Y in this case is not first countable. Singletons always belong to the σ -algebra of Borel subsets of first countable spaces.

We denote $\mathfrak{sp}_{\|\cdot\|_X}(M_b(X, \Sigma))$ by $\xi_\Sigma X$ which is a compact Hausdorff space. Since $M_b(X, \Sigma)$ separates the points of X , there is an injection $\psi : X \rightarrow \xi_\Sigma X$ such that $\psi(X)$ is a dense subset of $\xi_\Sigma X$. Further, for every bounded Σ -measurable function f on X , the function $f \circ \psi^{-1}$ is continuously extendible over $\xi_\Sigma X$. Also, $\xi_\Sigma X$ is unique (up to a homeomorphism) with this property in the sense that for every other compact Hausdorff space, say γX , with X as a dense subset, so that the elements of $M_b(X, \Sigma)$ are continuously extendible to γX , there is a continuous map $\iota : \gamma X \rightarrow \xi_\Sigma X$ agreeing on the images of X in $\xi_\Sigma X$ and γX .

For $E \in \Sigma$, let χ_E be the characteristic function of E , defined on X . Denoting its continuous extension to $\xi_\Sigma X$ with $\tilde{\chi}_E$ we have:

$$(\tilde{\chi}_E)^2 = (\chi_E^2)^\sim = \tilde{\chi}_E;$$

thus it ranges over $\{0, 1\}$, which implies that $\tilde{\chi}_E$ itself must be the characteristic function of a set, say \tilde{E} in $\xi_\Sigma X$.

Lemma 3.2. *Let $E \in \Sigma$. Then $\overline{E} = \tilde{E}$ where \overline{E} is the closure of E in $\xi_\Sigma X$.*

Proof. It is clear that $\tilde{E} = \tilde{\chi}_E^{-1}(\{1\})$ is closed and $E \subseteq \tilde{E}$. Thus $\overline{E} \subseteq \tilde{E}$. If $z \notin \overline{E}$, then for an open neighbourhood U of z we have $U \cap E = \emptyset$. Therefore there is a function $f \in M_b(X, \Sigma)$ and an open interval I in \mathbb{R} such that $z \in \tilde{f}^{-1}(I) \subseteq U$. Let $F = f^{-1}(I) \in \Sigma$, then $E \cap F = \emptyset$, so $\chi_E \cdot \chi_F = 0$ and $\tilde{\chi}_E \cdot \tilde{\chi}_F = 0$. Since $\tilde{\chi}_F(z) = 1$ the later equation implies $\tilde{\chi}_E(z) = 0$. This contradicts the assumption $z \in \tilde{E}$, therefore $\tilde{E} = \overline{E}$. \square

Using the above lemma, we investigate some properties of X as a subspace of $\xi_\Sigma X$.

Corollary 3.3. *Let Σ be a σ -algebra of subsets of X .*

- (1) \overline{E} is a clopen subset of $\xi_\Sigma X$ for every $E \in \Sigma$;
- (2) $\tilde{\Sigma} := \{\tilde{E} : E \in \Sigma\}$ forms a basis for the topology of $\xi_\Sigma X$;
- (3) $\tilde{\Sigma}$ is the set of all clopen subsets of $\xi_\Sigma X$.

In addition, if Σ contains all singletons, then

- (4) X is an open dense subspace of $\xi_\Sigma X$ whose subspace topology is discrete;
- (5) For a subset $Y \subset X$, $\overline{Y} = Y$ if and only if Y is finite.
- (6) For $x \in X$ and $E \in \Sigma$, $x \in \tilde{E}$ if and only if $x \in E$.

Proof. (1) Since $\overline{E} = \tilde{E} = \tilde{\chi}_E^{-1}(\{1\}) = \tilde{\chi}_E^{-1}(\frac{1}{2}, \infty)$ and $\tilde{\chi}_E$ is continuous, we conclude that \tilde{E} is clopen.

(2) The family $\{\tilde{E} : E = f^{-1}([0, 1]) \text{ for } f \in M_b(X, \Sigma)\}$ forms a basis for the closed subsets of $\xi_\Sigma X$. Note that $E = f^{-1}([0, 1]) \in \Sigma$ and $\tilde{E} = \overline{E} = \tilde{f}^{-1}([0, 1])$ which is clopen by (1) and the conclusion follows.

(3) By (1) and (2), $\xi_\Sigma X$ is totally disconnected. Suppose that $Y \subseteq \xi_\Sigma X$ is clopen. Since $\xi_\Sigma X$ is compact, so is Y . By (2), as an open set, $Y = \bigcup_{i \in I} \tilde{E}_i$ for a family $\{E_i\}_{i \in I} \subset \Sigma$. Therefore, $Y = \tilde{E}_{i_1} \cup \dots \cup \tilde{E}_{i_n}$ for some $i_1, \dots, i_n \in I$, which belongs to $\tilde{\Sigma}$.

(4) By (1), the closure of every element of Σ is open in $\xi_\Sigma X$. Since the topology of $\xi_\Sigma X$ is Hausdorff and Σ contains all singletons, singletons are closed. Therefore $\{x\}$ is clopen for every $x \in X$ and hence X is open in $\xi_\Sigma X$. Moreover, by [Theorem 2.5](#), X is dense in $\xi_\Sigma X$.

(5) If Y is finite, then since the topology of $\xi_\Sigma X$ is Hausdorff, it is also closed. Let Y be an arbitrary subset of X . The set $\overline{Y} \subseteq \xi_\Sigma X$ is compact. If $Y = \overline{Y}$, then $\{\{x\} : x \in Y\}$ is an open cover of Y which will not have a finite subcover, if Y is infinite.

(6) Clearly if $x \in E$ then $x \in \tilde{E}$. Conversely, suppose that $x \in \tilde{E} \setminus E$. Then $E \subseteq \tilde{E} \setminus \{x\}$. The superset is closed since $\{x\}$ and \tilde{E} are both clopen in $\xi_\Sigma X$ by (5) and (1) respectively. Thus $\overline{E} = \tilde{E} \subseteq \tilde{E} \setminus \{x\}$, a contradiction. \square

Remark 3.4. Let Σ be a σ -algebra of subsets of an infinite set X . If there are infinitely many disjoint sets in Σ , then $M_b(X, \Sigma)$ is not separable. The proof is similar to the classical proof of the fact that $\ell^\infty(\mathbb{N})$ is not separable. Hence, in this case $\xi_\Sigma X$ is not metrizable. (It is classically known that for a compact space Y , $C_0(Y)$ is separable if and only if Y is metrizable [[1, Theorem 2.4](#)].)

A topological space is called *extremely disconnected* if the closure of every open set is open. In the following we study this property for $\xi_\Sigma X$. For the relation between Boolean algebras and extremely disconnected spaces see [[6, §3.5](#)] or [[7, 22.4](#)]. Commutative algebras with extremely disconnected Gelfand spectra are forming the commutative class of AW*-algebras where $\mathbb{F} = \mathbb{C}$.

An algebra of sets is said to be *complete* if it is closed under arbitrary union and hence intersection

Proposition 3.5. *Let Σ be a σ -algebra on X including all singletons. Then $\xi_\Sigma X$ is extremely disconnected if and only if Σ is complete.*

Proof. Suppose that $\xi_\Sigma X$ is extremely disconnected and let $\Delta \subseteq \Sigma$. Then $U = \bigcup_{Y \in \Delta} \tilde{Y}$ is open and hence \overline{U} is also open, thus by [Corollary 3.3\(3\)](#), it belongs to $\tilde{\Sigma}$, say $\overline{U} = \tilde{E}$ for some $E \in \Sigma$. We show that $E = \bigcup_{Y \in \Delta} Y$. To do so, first suppose that $\exists x \in (\bigcup_{Y \in \Delta} Y) \setminus E$. Clearly $x \in \overline{U} = \tilde{E}$. This violates [Corollary 3.3\(6\)](#). Conversely, if $\exists x \in E \setminus \bigcup_{Y \in \Delta} Y$, then $\bigcup_{Y \in \Delta} Y \subseteq F$ for $F = E \setminus \{x\} \in \Sigma$. Therefore, $U \subseteq \tilde{F}$. Also, by [Corollary 3.3\(6\)](#), $\tilde{F} \subset \tilde{E} \setminus \{x\}$. On the other hand, since \tilde{F} is clopen,

$$\tilde{E} = \overline{U} \subseteq \tilde{F} \subseteq \tilde{E} \setminus \{x\} \subsetneq \tilde{E},$$

a contradiction and hence the claim is proved.

Now, suppose that Σ is complete and let U be an open set in $\xi_\Sigma X$. Take $\Delta \subset \Sigma$ such that $U = \bigcup_{F \in \Delta} \tilde{F}$. Since Σ is complete, $E = \bigcup_{F \in \Delta} F \in \Sigma$ and $\overline{U} \subseteq \overline{E} = \tilde{E}$. If $\tilde{E} \setminus \overline{U}$, which is open, is not empty, then it contains a nonempty clopen $Y \in \tilde{\Sigma}$. Now $V = \tilde{E} \setminus Y$ is a clopen set such that

$$E \subseteq U \subseteq V = \bar{V} \subsetneq \tilde{E} = \bar{E}$$

which is a contradiction. Thus $\bar{U} = \tilde{E}$ is clopen and hence $\xi_\Sigma X$ is extremely disconnected. \square

Let $\theta : A \rightarrow \ell^\infty(X)$ be an algebra homomorphism and τ be a topology on X . Then one can show that the induced map $\theta_*|_X : (X, \tau) \rightarrow \mathfrak{sp}_{\|\cdot\|_{X^\theta}}(A)$ is continuous if and only if $\theta A \subseteq C_b(X, \tau)$. The following proposition is an analogue of this result for $M_b(X, \Sigma)$ and Σ -measurability.

Proposition 3.6. *Suppose that Σ is a σ -algebra on $X \neq \emptyset$ such that every open subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$. Let $\iota : A \rightarrow \ell^\infty(X)$ be an algebra homomorphism. Then the induced map $\iota_*|_X : (X, \Sigma) \rightarrow \mathfrak{sp}_{\|\cdot\|_{X^\iota}}(A)$ is Σ -measurable if and only if $\iota A \subseteq M_b(X, \Sigma)$.*

Proof. By assumption, every Borel subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$. A basic open set of $\mathfrak{sp}_{\|\cdot\|_{X^\iota}}(A)$ is of the form $\hat{a}^{-1}(O)$ where $O \subseteq \mathbb{F}$ is open and $a \in A$. Looking at the inverse image of $\hat{a}^{-1}(O)$ under ι_* , we have

$$\iota_*|_X^{-1}\hat{a}^{-1}(O) = \hat{\iota a}^{-1}(O) \cap X \tag{3}$$

(\Rightarrow) Suppose that ι_* is Σ -measurable. If in contrary $\iota a \notin M_b(X, \tau)$ for some $a \in A$, then there exists a set $O \subseteq \mathbb{F}$, such that $\hat{\iota a}^{-1}(O) \cap X$ is not Σ -measurable and hence by (3), $\iota_*|_X$ cannot be Σ -measurable which is a contradiction.

(\Leftarrow) If each ιa is Σ -measurable, then $\hat{\iota a}^{-1}(O)$ is Σ -measurable for any open $O \subseteq \mathbb{F}$ and again by (3), $\iota_*|_X$ is Σ -measurable. \square

It is not known to the authors if the assumption “every open subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$ ” in Proposition 3.6 is essential or not. One can show that this assumption rules out some examples including $X = \mathbb{N}$ and $\Sigma = P(\mathbb{N})$, the power set of X .

3.1. Measures on (X, Σ) and $\xi_\Sigma X$

Starting with a measurable structure (X, Σ) such that $M_b(X, \Sigma)$ separates the points of X . We identified X as an open dense subset of a totally disconnected compact space $\xi_\Sigma X$ where every bounded Σ -measurable function on X extends continuously to $\xi_\Sigma X$. This naturally leads one to ask about the relation between measures on (X, Σ) and $\xi_\Sigma X$.

Proposition 3.7. *Let μ be a finite positive measure on (X, Σ) . Then μ extends to a Borel measure ${}^*\mu$ on $\xi_\Sigma X$, satisfying*

$$\forall E \in \Sigma \quad {}^*\mu(\tilde{E}) = \mu(E).$$

Proof. Define a linear functional $L : C(\xi_\Sigma X) \rightarrow \mathbb{F}$ by

$$L(f) = \int_X f|_X \, d\mu, \quad \forall f \in C(\xi_\Sigma X).$$

Clearly L is positive and hence by Riesz representation theorem, there exists a Borel measure ${}^*\mu$ on $\xi_\Sigma X$ such that

$$L(f) = \int_{\xi_\Sigma X} f \, d{}^*\mu, \quad \forall f \in C(\xi_\Sigma X).$$

Note that for every $E \in \Sigma$, ${}^*\mu(\tilde{E}) = \int \tilde{\chi}_E \, d{}^*\mu = L(\tilde{\chi}_E) = \int \chi_E \, d\mu = \mu(E)$. \square

Although the measure ${}^*\mu$ obtained in Proposition 3.7 seems to be mainly supported on X , but in fact, the size of $X \cap \text{supp}({}^*\mu)$ is rather small as it is pointed out in the following proposition.

Proposition 3.8. *Let μ be a finite Borel measure on $\xi_\Sigma X$ and Σ contains all singletons. Then $X \cap \text{supp}(\mu)$ is at most countable.*

Proof. By definition, a point $x \in \xi_\Sigma X$ belongs to $\text{supp}(\mu)$ if and only if for every neighbourhood U of x , $\mu(U) > 0$. Every singleton $\{z\}$ for $z \in X$ is open in $\xi_\Sigma X$, thus for every $z \in X \cap \text{supp}(\mu)$, $\mu(\{z\}) > 0$. Since $\mu(\xi_\Sigma X) < \infty$, $X \cap \text{supp}(\mu)$ cannot be uncountable. \square

Corollary 3.9. *Let μ be a finite positive measure on (X, Σ) where Σ contains all singletons. If $\mu(\{x\}) = 0$, for some $x \in X$, then $x \notin \text{supp}({}^*\mu)$.*

Proof. Since $\{x\} \in \Sigma$ and $\mu(\{x\}) = 0$, $\chi_x \in M_b(X, \Sigma)$ and $\int_X \chi_x d\mu = 0$. Thus ${}^*\mu(\{x\}) = \int_{\xi_\Sigma X} \tilde{\chi}_x d{}^*\mu = 0$. But $\{x\}$ is open and hence $x \notin \text{supp}({}^*\mu)$. \square

3.2. Borel algebra of a topology

Let (X, τ) be a T_1 topological space. Since the topology τ is T_1 , singletons are Borel and hence $M_b(X, \mathcal{B}_\tau)$ separates points of X . Clearly the inclusion $\iota : C_b(X, \tau) \rightarrow M_b(X, \mathcal{B}_\tau)$ is continuous and hence $\iota_* : \xi_{\mathcal{B}_\tau} X \rightarrow \text{sp}_{\|\cdot\|_X}(C_b(X, \tau))$ is onto. Consequently, if τ is completely regular, then βX is a continuous image of $\xi_{\mathcal{B}_\tau} X$ where βX is the Stone–Čech compactification of X (look at [4, 6.5]). If $\mathcal{B}_\tau = \tau$ then $\xi_{\mathcal{B}_\tau}$ and β are identical and ι_* is injective. It is natural to ask if there is any relation between topological structures of (X, τ) and $\xi_{\mathcal{B}_\tau} X$.

Let $x \in X$ and $\mathcal{N}_\tau(x)$ be the family of open neighbourhoods of x in τ and $\tilde{\mathcal{N}}_\tau(x) = \{\tilde{U} : U \in \mathcal{N}_\tau(x)\}$. Define the *halo* of x in $\xi_{\mathcal{B}_\tau} X$ as

$$h(x) := \bigcap \tilde{\mathcal{N}}_\tau(x).$$

The set $h(x)$ is compact and contains all points of $\xi_{\mathcal{B}_\tau} X$ that cannot be distinguished from x via the image of τ . If τ is Hausdorff, then for each $y \in X$ such that $x \neq y$, there are open sets $U_x, U_y \in \tau$ with $U_x \cap U_y = \emptyset$. Thus $\tilde{U}_x \cap \tilde{U}_y = \emptyset$, and therefore $h(x) \cap h(y) = \emptyset$.

Proposition 3.10. *If τ is Hausdorff, then $h(x)$ is open if and only if $\{x\}$ is open in (X, τ) .*

Proof. If $\{x\}$ is open, then $\{x\} \in \mathcal{N}_\tau(x)$. Since $\tilde{\{x\}} = \{x\}$, clearly, $x \in h(x) \subseteq \{x\}$. Conversely, if $h(x)$ is open, then it is clopen and hence, by Corollary 3.3(3), $h(x) = \tilde{E}$ for some $E \in \mathcal{B}_\tau$. If $E \neq \{x\}$, then E contains another point $y \in X$, $y \neq x$. Thus $y \in h(x)$ which implies that $h(x) \cap h(y) \neq \emptyset$, contradicting the above argument before the proposition. \square

Proposition 3.10 can be read as $h(x) = \{x\}$ if and only if $\{x\}$ is open in (X, τ) . The following shows how the compactness of a Borel subset of (X, τ) is reflected in $\xi_{\mathcal{B}_\tau} X$.

Theorem 3.11. *Let $Y \subseteq (X, \tau)$ be a Borel subspace. Then Y is compact if and only if $\tilde{Y} \subseteq \bigcup_{y \in Y} h(y)$.*

Proof. (\Rightarrow) Suppose that Y is compact and let $z \in \xi_{\mathcal{B}_\tau} X \setminus \bigcup_{y \in Y} h(y)$. We show $z \notin \tilde{Y}$. Since $z \notin \bigcup_{y \in Y} h(y)$, for each $y \in Y$, there exists $O_y \in \mathcal{N}_\tau(y)$ such that $z \notin \tilde{O}_y$. Now $\{O_y : y \in Y\}$ is an open cover of the compact set Y . Let $\{O_{y_1}, \dots, O_{y_k}\}$ be such that $Y \subseteq \bigcup_{i=1}^k O_{y_i}$, then $\tilde{Y} \subseteq \bigcup_{i=1}^k \tilde{O}_{y_i}$ which proves $z \notin \tilde{Y}$, and hence $\tilde{Y} \subseteq \bigcup_{y \in Y} h(y)$.

(\Leftarrow) Suppose that $\tilde{Y} \subseteq \bigcup_{y \in Y} h(y)$, but Y is not compact. Then there exists an open cover $\{O_i\}_{i \in I}$ of Y with no finite subcover. So, for every finite subset $\{i_1, \dots, i_n\}$ of I ,

$$Y \cap \left(\bigcap_{k=1}^n O_{i_k}^c \right) \neq \emptyset.$$

Since Y is Borel, \tilde{Y} is compact and hence $\{\tilde{O}_i^c\}_{i \in I}$ forms a basis for an ultrafilter \mathcal{F} in $\xi_{\mathcal{B}_\tau} X$. Clearly $\tilde{Y} \in \mathcal{F}$ and hence $z = \lim \mathcal{F} \in \tilde{Y}$ (for more detail on filters see [8, §12]). For every $y \in Y$, there exists $i \in I$ such that $O_i \in \mathcal{N}_\tau(y)$ and hence $z \notin \tilde{O}_i$. Thus $z \notin h(y) \subseteq \tilde{O}_i$. This proves

$$z \in \tilde{Y} \setminus \bigcup_{y \in Y} h(y),$$

as desired. \square

It is worth mentioning that the results of Subsection 3.2 resemble significant similarities between properties of $\xi_{\mathcal{B}_\tau} X$ and nonstandard extensions of (X, τ) . We can consider $\xi_{\mathcal{B}_\tau} X$ as a nonstandard model of (X, τ) and characterize halos as analogue to monads and so on. In this scope, Theorem 3.11 is the analogue of Robinson's theorem [5, Theorem III.2.2] on nonstandard extensions of compact spaces.

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References

- [1] C-Y. Chou, Notes on the separability of C^* -algebras, *Taiwanese J. Math.* 16 (2) (2012) 555–559.
- [2] M. Ghasemi, S. Kuhlmann, M. Marshall, Application of Jacobi's representation theorem to locally multiplicatively convex topological \mathbb{R} -algebras, *J. Funct. Anal.* 266 (2) (2014) 1041–1049.
- [3] M. Ghasemi, S. Kuhlmann, M. Marshall, Moment problem in infinitely many variables, *Israel J. Math.* 212 (2016) 1012, <http://dx.doi.org/10.1007/s11856-016-1318-5>.
- [4] L. Gillman, M. Jerison, *Rings of Continuous Functions*, Grad. Texts in Math., vol. 43, Springer-Verlag, New York–Heidelberg, 1976, reprint of the 1960 edition.
- [5] A.E. Hurd, P.A. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press Inc., Orlando, Florida, 1985.
- [6] P.T. Johnstone, *Stone Spaces*, Cambridge Stud. Adv. Math., vol. 3, Cambridge University Press, Cambridge, 1986, reprint of the 1982 edition.
- [7] R. Sikorski, *Boolean Algebras*, third edition, *Ergeb. Math. Grenzgeb.*, vol. 25, Springer-Verlag, New York Inc., New York, 1969.
- [8] S. Willard, *General Topology*, Addison–Wesley Publishing Co., Reading, Mass.–London–Don Mills, Ont., 1970.