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Eichler–Shimura isomorphism for complex hyperbolic lattices

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ABSTRACT

We consider the cohomology group $H^1(\Gamma, \rho)$ of a discrete subgroup $\Gamma \subset G = SU(n, 1)$ and the symmetric tensor representation ρ on $S^m(\mathbb{C}^{n+1})$. We give an elementary proof of the Eichler–Shimura isomorphism that harmonic forms $H^1(\Gamma \setminus G/K, \rho)$ are $(0, 1)$-forms for the automorphic holomorphic bundle induced by the representation $S^m(\mathbb{C}^n)$ of K.

1. Introduction

Let B be the unit ball in $\mathbb{C}^n$ considered as the Hermitian symmetric space $B = G/K$ of $G = SU(n, 1)$, $n > 1$. Let Γ be a cocompact torsion free discrete subgroup of G and ρ a finite dimensional representation of G, and $X = \Gamma \setminus B$. The representation ρ of G defines also one for $\Gamma \subset G$. The first cohomology $H^1(\Gamma, \rho)$ is of substantial interests and appears naturally in the study of infinitesimal deformation of Γ in a bigger group $G' \supset G$; see [1–5]. It is a classical result of Raghunathan [6] that the cohomology group $H^1(\Gamma, \rho)$ vanishes except when $\rho = \rho_m$ is the symmetric tensor $S^m(\mathbb{C}^{n+1})$ (or ρ'_m on $S^m(\mathbb{C}^{n+1})'$). In a recent work [1] it is proved that realizing $H^1(X, \rho)$ as harmonic forms, it consists of $(0, 1)$-forms for the symmetric tensor of the holomorphic tangent bundle of $X = \Gamma \setminus B$. The proof in [1] uses a Hodge vanishing theorem and the Koszul complex. In the present paper we shall give a rather elementary proof of the result. We will prove that any harmonic form with values in $S^m(\mathbb{C}^{n+1})$ is $(0, 1)$-form taking values in $S^m(\mathbb{C}^n)$. Let TX and $T'X$ be the holomorphic tangent and cotangent bundles respectively. Let $\mathcal{L}^{-1}$ be the line bundle on X defined so that $\mathcal{L}^{-(n+1)}$ is the canonical line bundle $\mathcal{K} = K_X$. More precisely we shall prove the following, the notations being explained in Section 2,

Theorem 1.1. *Let Γ be a torsion free cocompact lattice of G acting properly discontinuously on B.*

(1) *Let $\alpha \in A^1(\Gamma, B, \rho_m)$ be a harmonic form. Then α is a $(0, 1)$-form on $\Gamma \setminus B$ with values in the holomorphic vector bundle $S^m TX \otimes \mathcal{L}^{-m}$.*
(2) *Let $\alpha \in A^1(\Gamma, B, \rho'_m)$ be a harmonic form. Then α is a $(1, 0)$-form on $\Gamma \setminus B$ with values in the holomorphic vector bundle $S^m T'X \otimes \mathcal{L}^{-m}$ and α is symmetric in all $m + 1$ variables. In particular α is naturally identified with a section of the bundle $S^{m+1} T'X \otimes \mathcal{L}^m$.*

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Corollary 1.2. *Let Γ be as above and assume that $\Gamma \setminus B$ is compact then we have*

$$H^1(\Gamma, \rho_m) = H^1(\Gamma \setminus B, S^m TX \otimes \mathcal{L}^{-m}), \quad H^1(\Gamma, \rho'_m) = H^0(\Gamma \setminus B, S^{m+1} T'X \otimes \mathcal{L}^m),$$

where the cohomology on the right hand side are the Dolbeault cohomology of $\bar\partial$-closed $(0, 1)$-forms of the holomorphic vector bundles.

The case $n = 1$, namely a Riemann surface $\Gamma \setminus B$, is slightly different. In that case the group cohomology $H^1(\Gamma, \rho_{2j})$ of the $2j$th power of the defining representation of $\Gamma \subset SU(1, 1)$ will have both holomorphic and antiholomorphic components, $H^{(1,0)}(\Gamma, \rho_{2j})$, $H^{(0,1)}(\Gamma, \rho_{2j})$, the holomorphic part $H^{(1,0)}(\Gamma, \rho_{2j})$ corresponds to

$$H^{(1,0)}(\Gamma, \rho_{2j}) = H^{(1,0)}(\Gamma \setminus B, \mathcal{K}^{j+1}) = H^0(\Gamma \setminus B, \mathcal{K}^{j+1})$$

of the tensor power of the canonical line bundle. This is known as the Eichler–Shimura correspondence; see [7, THÉORÈME 1] where a concrete construction was given. We can also follow our proof and get an elementary proof of this result.

Our proof is a bit tricky but it is still very akin to the variation of Hodge structures; conceptually we are treating explicitly the filtration of holomorphic bundles defined by the central action of K. It is stated in [1] that the results can be derived from the work of Deligne and Zucker [8,9]. We note here that results of this type that $(0, q)$-forms in the group cohomology $H^q(\Gamma, B, \rho)$ are actually $(0, q)$-forms for a corresponding automorphic bundle have been obtained much earlier by Matsushima and Murakami [10,11]. Such vanishing results are also at our disposal thanks to very general works of Vogan and Zuckerman. And it seems that one can prove the above result by combining the works of [10–13]. This would certainly be of interest as people in rigidity theory are not so familiar with Vogan–Zuckermav's work. But our method is down-to-earth hence we expect that it can be applied to various situations. For example we deal with $n = 1$ case, i.e., surface case in the last section, which is not available in [1]. We will investigate further applications in the near future.

2. Preliminaries

Let $V = \mathbb{C}^{n+1}$ be equipped with the Hermitian inner product $\langle Jv, v\rangle$ of signature $(n, 1)$, where J is the diagonal matrix $J = \mathrm{diag}(1, \ldots, 1, -1)$ and $\langle v, w\rangle = \sum \bar v_i w_i$ the Euclidean form in $\mathbb{C}^{n+1}$. We write $V = V_1 \oplus \mathbb{C}e_{n+1}$ with V_1 being the Euclidean space $\mathbb{C}^n$ with an orthonormal basis $\{e_k, k = 1, \ldots, n\}$. Let $G = SU(n, 1)$ be the group of linear transformations on V preserving the Hermitian form. The maximal compact subgroup of G is

$$K = \{\begin{bmatrix} A & 0 \\ 0 & e^{i\theta}\end{bmatrix}; A \in U(n),\ e^{i\theta} \det A = 1\} = U(n),$$

namely $K = S(U(n) \times U(1)) = U(n)$. The subgroup $SU(n) \subset U(n)$ viewed as a subgroup in K will be denoted by $SU(n) \times e$ to avoid confusion. The Lie algebra $\mathfrak{g} = \mathfrak{su}(n, 1)$ consists of matrices X such that $X^*J + JX = 0$. The symmetric space G/K can be realized as the unit ball B in $V_1 = \mathbb{C}^n$, $B = G/K$ with $x_0 = 0$ being the base point. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ and the subspace $\mathfrak{p} = \{\xi_v; v \in \mathbb{C}^n\}$ with

$$\xi_v = \begin{pmatrix} 0 & v \\ \bar v & 0 \end{pmatrix}.$$

The tangent space $T_{x_0}(B)$ at x_0 will be identified with $\mathfrak{p} = \mathbb{C}^n$ as real spaces.

We fix an element in the center of the maximal compact subalgebra $\mathfrak{k} = \mathfrak{u}(n)$

$$H_0 = (n+1)^{-1}\sqrt{-1}\mathrm{diag}(1, \ldots, 1, -n),$$

which defines the complex structure on B, and we have

$$\mathfrak{sl}(n+1) = \mathfrak{sl}(n) \oplus \mathbb{C}H_0 \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

Then the holomorphic and anti-holomorphic tangent space $\mathfrak{p}^\pm$ consists of upper triangular, respectively lower triangular matrices. We denote

$$\xi_v^+ = \frac{1}{2}(\xi_v - i\xi_{iv}) = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{p}^+,\ \xi_v^- = \frac{1}{2}(\xi_v + i\xi_{iv}) = \begin{pmatrix} 0 & 0 \\ \bar v & 0 \end{pmatrix} \in \mathfrak{p}^-, \tag{2.1}$$

the $\mathbb{C}$- and $\overline{\mathbb{C}}$-linear components of ξ_v.

Let $V_1 = \mathbb{C}^n$ be the defining representation and $\det(A)$ the determinant representation of $U(n)$. We take the diagonal elements as a Cartan algebra of $\mathfrak{gl}(n, \mathbb{C})$ and the upper triangular matrices as positive root vectors. Denote $\omega_1, \ldots, \omega_{n-1}$ the fundamental representations of $U(n)$, so that $\omega_1 = V_1$ is the defining representation above and ω_{n-1} the dual representation. Note that ω_i has the highest weight $L_1 + \cdots + L_i$ where $L_j(\mathrm{diag}(h_1, \ldots, h_n)) = h_j$ is a canonical dual element on the Cartan algebra.

As complex representation of $\mathfrak{u}(n)$ we have

$$\mathfrak{p}^+ = \omega_1 \otimes \det = V_1 \otimes \det,\ \mathfrak{p}^- = \omega_{n-1} \otimes \det^{-1}.$$

This entails that, for $A \in U(n)$,

$$A(\xi_{v_1}^+ \wedge \cdots \wedge \xi_{v_n}^+) = (\det A)^n A\xi_{v_1}^+ \wedge \cdots \wedge A\xi_{v_n}^+ = (\det A)^{n+1}(\xi_{v_1}^+ \wedge \cdots \wedge \xi_{v_n}^+).$$

Hence

$$K_X^{-1} = \wedge^n \mathfrak{p}^+ = (\det)^{n+1} \tag{2.2}$$

and $\mathcal{L} = \det$.

We shall just identify $\mathfrak{p}^+$ with V_1, $\mathfrak{p}^+ = V_1$, when the center action of $U(n)$ is irrelevant.

The defining representation V of G under $\mathfrak{u}(n)$ is

$$V = V_1 \oplus \det{}^{-1}$$

We shall consider its symmetric representation $(S^m(V), \rho_m)$ of G and $\mathfrak{g}$. Note that we have

$$W = S^m(V) = \oplus_{k=0}^m W_k = \oplus_{k=0}^m S^k(V_1) \otimes e_{n+1}^{m-k}, \tag{2.3}$$

and we make the identification of the spaces

$$W_k = S^k(V_1) \otimes e_{n+1}^{m-k} = S^k(V_1)$$

whenever the factor e_{n+1}^{m-k} is irrelevant.

Note that the Euclidean inner product on V induces one on $W = S^m(V)$ and the above decomposition is an orthogonal decomposition. Note also that action of $\rho_m(X)$ is Hermitian for $X \in \mathfrak{p}$ and skew Hermitian for $X \in \mathfrak{k}$.

A representation of G on a finite dimensional real or complex vector space defines also a vector bundle over the quotient space $\Gamma\backslash B$ and we recall briefly its construction following the exposition [10,14] and also some notations there. Let (W, ρ) be a finite dimensional representation of G on a real (or complex) vector space W. Eventually we shall only consider $W = S^m(V)$ as above and its dual $S^m(V')$. We fix on W a K-invariant positive definite inner (respectively Hermitian) product. Let Γ be a torsion free discrete subgroup of G. The restriction of ρ on Γ will also be written as ρ. Suppose Γ acts properly discontinuously on B. Let $\Gamma \times K$ act on $G \times W$ by $(\gamma, \kappa)(g, w) := (\gamma g \kappa^{-1}, \rho(\gamma)w)$. Then $E_\rho = G \times W/\Gamma \times K$ is a vector bundle on $\Gamma \setminus B$. The de Rham operator d is well-defined on E_ρ and we let $\Delta_\rho = dd^* + d^*d$ be the corresponding Hodge Laplacian operator on space of p-forms $\Omega(\Gamma \setminus B, E_\rho)$. We choose its standard realizations as W-valued p-forms on G as follows. Let $A^p(\Gamma, B, \rho)$ be the space of W-valued p-forms α on G satisfying

(a) $\alpha(\gamma g) = \alpha(g), \gamma \in \Gamma$.
(b) $\rho(\kappa)\alpha(g\kappa^{-1}) = \alpha(g), \quad \kappa \in K$.
(c) $\iota(Y)\alpha = 0, Y \in \mathfrak{k}$.

Here $\iota(Y)$ is the pairing of $Y \in \mathfrak{g}$ as left-invariant vector fields on G (by differentiation from right) with a p-form α on G, $\iota(Y)\alpha(Z_1, \ldots, Z_{p-1}) = \alpha(Y, Z_1, \ldots, Z_{p-1})$. Equivalently it can be realized as p-forms on $\Gamma \setminus G$ satisfying $(b) - (c)$ above and $A_0^p(\Gamma, B, \rho)$ denotes the space of W-valued p-forms on $\Gamma \setminus G$. With some abuse of notation we denote Δ_ρ the corresponding Hodge Laplacian on $A_0^p(\Gamma, B, \rho)$.

We shall also need the automorphic bundle defined by representations of K, see [10]. So let (V, τ) be a complex representation of the complexification of $K_{\mathbb{C}}$ and we fix as above a Hermitian inner product on V so that K acts unitarily. The group $\Gamma \times K$ acts on $G \times V$ by $(\gamma, \kappa)(g, w) = (\gamma g \kappa^{-1}, \tau(\kappa)w)$. Then $\mathcal{E}_\rho = \Gamma \times K \setminus G \times V$ defines a holomorphic vector bundle over $\Gamma \setminus B$. The p-forms on the vector bundle can be realized as the space $\mathcal{A}^p(\Gamma, B, \tau)$ (again with some abuse of notation) of p-forms on $\Gamma \setminus G$ satisfying

(b') $\tau(\kappa)\alpha(g\kappa^{-1}) = \alpha(g), \quad \kappa \in K$.
(c') $\iota(Y)\alpha = 0, Y \in \mathfrak{k}$.

When ρ is a complex representation of G and (τ, K) is a sub-representation of ρ restricted to K, then discrete group cohomology $H^{p+q}(\Gamma, B, \rho)$ and automorphic cohomology $H^{(p,q)}(\Gamma, B, \tau)$ are related by the work of [10].

3. The Eichler–Shimura isomorphism

In general, a real linear map B on a complex vector space W decomposes into $\mathbb{C}$-linear part B^+ and $\overline{\mathbb{C}}$-linear part B^- so that $B(w) = B^+(w) + B^-(\bar{w})$ for $w \in W$. For any real linear map $A : \mathfrak{p} \to \mathrm{End}_{\mathbb{R}}(W)$ from $\mathfrak{p}$ to any complex vector space W we let

$$A^+(\xi_v) = \frac{1}{2}(A(\xi_v) - iA(\xi_{iv})), \quad A^-(\xi_v) = \frac{1}{2}(A(\xi_v) + iA(\xi_{iv}))$$

be the $\mathbb{C}$-linear and respectively $\overline{\mathbb{C}}$-linear components. In particular for any complex representation (W, ρ) of G and $\mathfrak{g}$ we have

$$\rho^\pm(\xi_v) = \rho(\xi_v^\pm),$$

where $\xi_v^{\pm}$ are defined in (2.1). Let now $\rho = \rho_m$ be the representation $S^m(V)$ and ρ^m the dual representation $S^m(V')$ of $\mathfrak{g}$. Note that ρ_1 is a defining representation V. We start now with a few simple observations formulated only $\rho = \rho_m$; the corresponding ones hold for ρ^m.

Denote by

$$P_k : W \to W_k = S^k(V_1) \otimes e_{n+1}^{m-k}$$

the orthogonal projection onto the component W_k in (2.3), and write

$$\alpha = \sum_{k=0}^{m} \alpha_k$$

the corresponding decomposition for $\alpha \in W = \sum_{k=0}^{m} W_k$.

Let $\{X_j\}$ be an orthogonal basis of $\mathfrak{p}$ viewed as tangent vectors on $\Gamma \setminus G$ at a fixed point Γg and $\{e_j\}$ be the corresponding orthonormal basis of V_1. Let $T = T_\rho$ and $T^* = T_\rho^*$ be the operator defined on $A^1(\Gamma, B, \rho)$ as follows.

$$T\alpha(X_1, X_2) = \rho(X_1)\alpha(X_2) - \rho(X_2)\alpha(X_1)$$

$$T^*\alpha = \sum_{j=1}^{n} \rho(X_j)\alpha(X_j).$$

We recall the following result [14, Corollary 7.50]

Proposition 3.1. *Suppose $\alpha \in A_0^1(\Gamma, B, \rho)$ is harmonic, $\Delta_\rho \alpha = 0$. Then $T_\rho \alpha = 0$ and $T_\rho^* \alpha = 0$.*

This can be restated as the following (which is also proved in [2] for $S^2(V)$ by using matrix computations).

Corollary 3.2. *Suppose $\alpha \in A_0^1(\Gamma, B, \rho)$ satisfies $T_\rho \alpha = 0$ and $T_\rho^*\alpha = 0$. Then the W-valued $\mathbb{R}$-bilinear form $(X, Y) \mapsto \rho(X)\alpha(Y)$ is symmetric*

$$\rho(\xi_v)\alpha(\xi_u) = \rho(\xi_u)\alpha(\xi_v), \tag{3.1}$$

and trace free

$$\sum_j (\rho(\xi_{e_j})\alpha(\xi_{e_j}) + \rho(\xi_{ie_j})\alpha(\xi_{ie_j})) = 0. \tag{3.2}$$

Our theorem will be an easy consequence of the following proposition, whose proof is based on a few elementary lemmas.

Proposition 3.3.

(1) *Suppose $\alpha \in Hom_{\mathbb{R}}(\mathfrak{p}, W)$ satisfies $T_\rho \alpha = T_\rho^* \alpha = 0$. Then α is $\overline{\mathbb{C}}$-linear and takes value in $W_m = S^m V_1$, that is, $\alpha = \alpha_m = \alpha_m^- \in Hom_{\overline{\mathbb{C}}}(\mathfrak{p}^-, W_m)$.*

(2) *Suppose $\alpha \in Hom_{\mathbb{R}}(\mathfrak{p}, W')$ satisfies $T_{\rho'}\alpha = T_{\rho'}^*\alpha = 0$. Then α is $\mathbb{C}$-linear and takes value in $S^m(V_1')$. Moreover as an element in $(\mathfrak{p}^+)' \otimes S^m(V_1') = (V_1)' \otimes S^m(V_1')$, it is symmetric in all variables, i.e., an element in $S^{m+1}(V_1')$, the leading component in $(V_1)' \otimes S^m(V_1')$.*

Denote $u^i v^{j-i}$ the symmetric tensor power of u and v normalized by

$$(u + v)^j = \otimes^j (u + v) = \sum_{i=0}^{j} \binom{j}{i} u^i v^{j-i}.$$

Note that the representation $\rho = \rho_m$ is the symmetric tensor $S^m(\mathbb{C}^{n+1})$ and $\rho' = \rho_m'$ its dual throughout the paper.

Lemma 3.4.

(1) *Let $1 \le k \le m - 1$. Then for any $0 \ne \xi_v \in \mathfrak{p}$,*

$$\rho(\xi_v) : W_k \to W_{k+1} + W_{k-1}, \ \ \rho(\xi_v^+) : W_k \to W_{k+1}, \ \ \rho(\xi_v^-) : W_k \to W_{k-1},$$

and on each space it is nonzero. Moreover if $w \in W_k$ and $\rho(\xi_v^+)w = 0$ or $\rho(\xi_v^-)w = 0$ for all $\xi_v^{\pm} \in \mathfrak{p}^{\pm}$ then $w = 0$.

(2) *The restriction $\rho(\xi_v)|_{W_m} : W_m \to W_{m-1}$ on the top component W_m of W is $\overline{\mathbb{C}}$-linear in ξ_v, $\rho(\xi_v)|_{W_m} = \rho^-(\xi_v)|_{W_m}$, and $\rho(\xi_v)_{W_0}$ on the bottom component is $\mathbb{C}$-linear in ξ_v, $\rho(\xi_v)_{W_0} = \rho^+(\xi_v)_{W_0}$.*

Proof. The defining representation ρ_1 is just the matrix multiplication and we have

$$\rho_1(\xi_v)u = \langle v, u\rangle e_{n+1}$$

for $u \in V_1$, and

$$\rho_1(\xi_v)e_{n+1} = v.$$

Thus

$$\rho_1(\xi_v^+)u = 0,\ \rho_1(\xi_v^-)u = \langle v, u\rangle e_{n+1},\ \rho_1(\xi_v)e_{n+1} = v,$$

and

$$\rho_1(\xi_v^+)e_{n+1} = v,\ \rho_1(\xi_v^-)e_{n+1} = 0.$$

Taking the tensor power we find

$$\rho(\xi_v^+)e_{n+1}^k = kve_{n+1}^{k-1}, \quad \rho(\xi_v^-)e_j^k = k\overline{v_j}e_{n+1}e_j^{k-1}, \quad 1 \le j \le n,$$

which are non-zero if $v_j \neq 0$. Then

$$\rho(\xi_v^+)(W_k) = \rho(\xi_v^+)(S^k(V_1) \otimes e_{n+1}^{m-k}) = (m-k)ve_{n+1}^{m-(k+1)} \in W_{k+1},$$

$$\rho(\xi_v^-)(W_k) = \rho(\xi_v^-)(S^k(V_1) \otimes e_{n+1}^{m-k}) \in S^{k-1}(V_1) \otimes e_{n+1}^{m-(k-1)} \in W_{k-1}.$$

First note that

$$\rho(\kappa)\rho^{\pm}(\xi_v)\rho(\kappa^{-1}) = \rho^{\pm}(\xi_{\kappa v}), \quad \kappa \in SU(n) \times \{e\}, v \in V_1.$$

If $\rho(\xi_v^{\pm})w = 0$ for all $\xi_v^{\pm} \in \mathfrak{p}^{\pm}$ and for a fixed $w \neq 0$, then

$$\rho(\kappa)\rho(\xi_v^{\pm})\rho(\kappa^{-1})w = \rho(\xi_{\kappa v}^{\pm})w = 0$$

for all $\kappa \in SU(n) \times \{e\}$. Here the action of K on W is via the given representation ρ from G. Hence it is zero for all $\rho(\kappa^{-1})w$, and therefore zero for $w = e_j^k, j = 1, \ldots, n$, contradicting the previous claim.

The second part (2) follows immediately from the above formulas for $\rho(\xi_v^{\pm})$ and fact that $W_{m+1} = 0$ and $W_{-1} = 0$. □

The space $\mathrm{Hom}_{\overline{\mathbb{C}}}(\mathfrak{p}^-, W_j)$ of $\overline{\mathbb{C}}$-linear forms $\beta = \beta^-$ on $\mathfrak{p}^-$ will be identified with the tensor product $(\mathfrak{p}^-)' \otimes W_j$. Using $(\mathfrak{p}^-)' = V_1 \otimes \det$, the tensor product is decomposed under K as [15]

$$\begin{aligned}\mathrm{Hom}_{\overline{\mathbb{C}}}(\mathfrak{p}^-, W_j) &= (\mathfrak{p}^-)' \otimes S^j(V_1) \otimes e_{n+1}^{m-j} \\ &\equiv (S^{j+1}(V_1) \otimes e_{n+1}^{m-j}) \oplus (S^{j-1,1}(V_1) \otimes e_{n+1}^{m-j})\end{aligned} \tag{3.3}$$

with the corresponding highest weights

$$\omega_1 \otimes j\omega_1 = (j+1)\omega_1 + ((j-1)\omega_1 + \omega_2).$$

Lemma 3.5. *If $\rho^-(\xi_u)\beta^-(\xi_v) = \rho^-(\xi_v)\beta^-(\xi_u)$ then β is in the first component $S^{j+1}(V_1)$ in the above decomposition (3.3).*

Proof. Note that the relation $\rho^-(\xi_u)\beta^-(\xi_v) = \rho^-(\xi_v)\beta^-(\xi_u)$ is invariant under the K-action, since

$$\rho(\kappa)\rho^{\pm}(\xi_v)\rho(\kappa^{-1}) = \rho^{\pm}(\xi_{\kappa v}), \quad \kappa \in K, v \in V_1$$

and

$$\rho(\kappa)\beta(g\kappa^{-1}) = \beta(g)$$

for all $\kappa \in K$ (see Section 2), which results in

$$\rho(\kappa)\rho^{\pm}(\xi_v)\beta(g\kappa^{-1}) = \rho^{\pm}(\xi_{\kappa v})\beta(g).$$

Thus if β^- satisfies the relation so is its component in $((j-1)\omega_1 + \omega_2)$. We prove that any element in $((j-1)\omega_1 + \omega_2)$ satisfying the relation must be zero. This space is an irreducible representation of K and we need only to check the relation for its highest weight vector. The highest weight vector of $((j-1)\omega_1 + \omega_2)$ in $V_1 \otimes S^j(V_1)$ is

$$\beta = \epsilon_2 \otimes e_1^j - \epsilon_1 \otimes (e_1^{j-1}e_2)$$

where ϵ_i is a dual vector to $\xi_{e_i}^-$ in $\mathfrak{p}^-$. We check the relation

$$\rho(\xi_{e_2}^-)\beta(\xi_{e_1}^-) = \rho(\xi_{e_1}^-)\beta(\xi_{e_2}^-).$$

The left hand side is $-e_1^{j-1}e_{n+1}$ whereas the right hand side is $je_1^{j-1}e_{n+1}$, and the relation is not satisfied. Hence β should be in the first component $S^{j+1}(V_1)$. □

Note that $\{e_k\}$ is an orthogonal basis of V_1. Observe that for any $\beta \in \mathrm{Hom}_{\overline{\mathbb{C}}}(\mathfrak{p}^-, W_j)$ we have

$$\rho(\xi_v^+)\beta \in \mathrm{Hom}_{\overline{\mathbb{C}}}(\mathfrak{p}^-, W_{j+1}).$$

Lemma 3.6. *Suppose $1 \le j < m$. The map*

$$T : Hom_{\overline{\mathbb{C}}}(\mathfrak{p}^-, W_j) \equiv (j+1)\omega_1 \oplus ((j-1)\omega_1 + \omega_2) \to W_{j+1}, \quad \beta \mapsto \sum_{k=1}^{n} \rho(\xi^+_{e_k})\beta(\xi^-_{e_k})$$

is up to non-zero constant an isometry on the space $(j+1)\omega_1$ where ω_i's are fundamental representations of $U(n)$ introduced in Section 2.

Proof. It is clear that T is a K-intertwining map from $\mathrm{Hom}_{\overline{\mathbb{C}}}(\mathfrak{p}^-, S^j(V_1))$ into W_{j+1}. By Schur's lemma it is either zero or an isometry up to non-zero constant on the irreducible space $(j+1)\omega_1$. To find the constant we take $\beta = \varepsilon_1 \otimes e_1^j e_{n+1}^{m-j}$ where ε_1 is the dual form of $\xi^-_{e_1}$. It is indeed in the first component $(j+1)\omega_1$ and is actually the highest weight vector. Then by direct computation we find

$$T\beta = (m-j)e_1^{j+1}e_{n+1}^{m-j-1},$$

which is nonzero. □

We consider the corresponding symmetry property for the dual representation $\rho' = \rho^m$.

Lemma 3.7. *Suppose $\beta = \beta^+$ is $S^m(V_1')$-valued $\mathbb{C}$-linear form on $\mathfrak{p}^+$. If $\rho'(\xi^+_u)\beta(\xi^+_v) = \rho'(\xi^+_v)\beta(\xi^+_u)$ then β as an element in $(\mathfrak{p}^+)' \otimes S^m(V_1')$ is symmetric in all $m+1$ variables.*

Proof. The statement is equivalent to that $\beta(\xi^+_v)(\xi^+_u, \xi^+_{v_1}, \cdots, \xi^+_{v_{m-1}})$ is symmetric in all $m+1$ variables. However the equality $\rho'(\xi^+_u)\beta(\xi^+_v) = \rho'(\xi^+_v)\beta(\xi^+_u)$ implies that it is symmetric in the first two variables and thus is symmetric in all $m+1$ variables. More precisely, viewing $\rho'(\xi^+_u)\beta(\xi^+_v)$ and $\rho'(\xi^+_v)\beta(\xi^+_u)$ as elements in $S^m(V')$,

$$\rho'(\xi^+_u)\beta(\xi^+_v)(e_{n+1}, \dots, e_{n+1}) = \beta(\xi^+_v)(\rho'(\xi^+_u)e_{n+1}, \dots, \rho'(\xi^+_u)e_{n+1})$$

$$= \rho'(\xi^+_v)\beta(\xi^+_u)(e_{n+1}, \dots, e_{n+1}) = \beta(\xi^+_u)(\rho'(\xi^+_v)e_{n+1}, \dots, \rho'(\xi^+_v)e_{n+1}).$$

Hence from $\rho'(\xi^+_u)e_{n+1} = u$ and $\rho'(\xi^+_v)e_{n+1} = v$ and identifying $\mathfrak{p}^+ = V_1$, we get

$$\beta(\xi^+_v)(\xi^+_u, \dots, \xi^+_u) = \beta(\xi^+_u)(\xi^+_v, \dots, \xi^+_v). \qquad \square$$

We prove now Proposition 3.3.

Proof. We shall prove by induction that all $\alpha_j = 0$ for $k \le m-1$. Let $1 \le k \le m-1$. Taking the kth component of (3.1) we get

$$\rho^+(\xi_v)\alpha^+_{k-1}(\xi_u) = \rho^+(\xi_u)\alpha^+_{k-1}(\xi_v), \tag{3.4}$$

$$\rho^-(\xi_u)\alpha^-_{k+1}(\xi_v) = \rho^-(\xi_v)\alpha^-_{k+1}(\xi_u), \tag{3.5}$$

$$\rho^+(\xi_u)\alpha^-_{k-1}(\xi_v) = \rho^-(\xi_v)\alpha^+_{k+1}(\xi_u). \tag{3.6}$$

We prove first that $\alpha_0 = 0$. Consider the 1-component of the identity

$$T^*_\rho\alpha = \sum_j \left(\rho(\xi_{e_j})\alpha(\xi_{e_j}) + \rho(\xi_{ie_j})\alpha(\xi_{ie_j})\right) = 0 \tag{3.7}$$

and write each term in terms of their $\mathbb{C}$-linear and $\overline{\mathbb{C}}$-linear parts. Note that bilinear $\mathbb{C}$-linear and bilinear $\overline{\mathbb{C}}$-linear terms have their sum zero. Also on the component W_0 the action $\rho(\xi_u) = \rho(\xi^+_u)$ is $\mathbb{C}$-linear, by Lemma 3.4. Thus

$$\sum_j \left(\rho^+(\xi_{e_j})\alpha^-_0(\xi_{e_j}) + \rho^-(\xi_{e_j})\alpha^+_2(\xi_{e_j})\right) = 0.$$

But by the equality of (3.6) for $k=1$ we have $\rho(\xi^-_{e_j})\alpha^+_2(\xi_{e_j}) = \rho(\xi^+_{e_j})\alpha^-_0(\xi_{e_j})$. Namely

$$2\sum_j \rho(\xi^+_{e_j})\alpha^-_0(\xi_{e_j}) = 0. \tag{3.8}$$

Taking inner product with $e_1e_{n+1}^{m-1} \in W_1$, and using the fact that

$$\langle \rho(\xi^+_{e_1})\alpha^-_0(\xi_{e_1}), e_1e_{n+1}^{m-1}\rangle = \langle \alpha^-_0(\xi_{e_1}), \rho(\xi^-_{e_1})(e_1e_{n+1}^{m-1})\rangle = \langle \alpha^-_0(\xi_{e_1}), e_{n+1}^m\rangle$$

and

$$\langle \rho(\xi_{e_j}^+)\alpha_0^-(\xi_{e_j}), e_1 e_{n+1}^{m-1}\rangle = \langle \alpha_0^-(\xi_{e_j}), \rho(\xi_{e_j}^-)(e_1 e_{n+1}^{m-1})\rangle = 0, \quad j \neq 1,$$

we see that $\langle \alpha_0^-(\xi_{e_1}), e_{n+1}^m\rangle = 0$, namely $\alpha_0^-(\xi_{e_1}) = 0$ since it is a scalar multiple of e_{n+1}^m. By the K-invariance of above relation (3.8) we may replace e_1 by any e_j, and get $\alpha_0^-(\xi_{e_j}) = 0$, i.e., $\alpha_0^- = 0$ and α_0 is $\mathbb{C}$-linear, $\alpha_0 = \alpha_0^+$. Now $W_0 = \mathbb{C}e_{n+1}^m$ is one-dimensional and α_0 is thus of the form

$$\alpha_0(\xi_u) = \alpha_0(\xi_u^+) = \langle u_0, u\rangle e_{n+1}^m$$

for some $u_0 \in V_1$. The relation (3.4) implies that

$$\langle u_0, u\rangle v e_{n+1}^{m-1} = \langle u_0, v\rangle u e_{n+1}^{m-1}$$

for all $u, v \in V_1$. This is impossible unless $u_0 = 0$ since $\dim V_1 > 1$, i.e., $\alpha_0 = 0$.

Taking the 0th component of the equality $\rho(\xi_u)\alpha(\xi_v) = \rho(\xi_v)\alpha(\xi_u)$ we get

$$\rho^-(\xi_u)\alpha_1(\xi_v) = \rho^-(\xi_v)\alpha_1(\xi_u).$$

Changing v to iv we find

$$\rho^-(\xi_u)\alpha_1(\xi_{iv}) = -i\rho^-(\xi_v)\alpha_1(\xi_u).$$

Summing the two results we get

$$\rho^-(\xi_u)(\alpha_1(\xi_{iv}) + i\alpha_1(\xi_v)) = 0.$$

Taking further the inner product with $e_{n+1}^m \in W_0$ we have

$$\begin{aligned} 0 &= \langle \rho^-(\xi_u)\alpha_1(\xi_{iv}) + i\alpha_1(\xi_v), e_{n+1}^m\rangle = \langle \alpha_1(\xi_{iv}) + i\alpha_1(\xi_v), \rho^+(\xi_u)e_{n+1}^m\rangle \\ &= \langle \alpha_1(\xi_{iv}) + i\alpha_1(\xi_v), u e_{n+1}^{m-1}\rangle \end{aligned}$$

for all u. Thus $\alpha_1(\xi_{iv}) + i\alpha_1(\xi_v) = 0$, namely α_1 is $\bar{\mathbb{C}}$-linear, $\alpha_1 = \alpha_1^-$. Furthermore it follows from Lemma 3.5 that α_1 is an element in the component $S^2(V_1)$ in $(\mathfrak{p}^-)' \otimes S^1(V_1)$.

We take now the 2-component of the identity (3.7) using again the fact that α_1 is $\overline{\mathbb{C}}$-linear, and find

$$0 = \sum_j \left(\rho^+(\xi_{e_j})\alpha_1(\xi_{e_j}) + \rho^+(\xi_{ie_j})\alpha_1(\xi_{ie_j})\right) = 2\sum_j \left(\rho^+(\xi_{e_j})\alpha_1(\xi_{e_j}^-)\right).$$

But α_1 is in the component $2\omega_1 = S^2(V_1)$ and Lemma 3.6 implies that $\alpha_1 = 0$.

Using the above procedure successively we prove then that $\alpha_j = 0$ for $j \leq m-2$. Consequently we have $\alpha_{m-1}^+ = 0$ and $\alpha_{m-1} = \alpha_{m-1}^-$. Taking the trace of $(m-2)$th component of (3.2) we have again $\sum_j \rho^+(\xi_{e_j})\alpha_{m-1}^-(\xi_{e_j}) = 0$ and $\alpha_{m-1} = 0$ by the same arguments.

Finally we consider the $(m-1)$th component of the equality $\rho(\xi_u)\alpha(\xi_v) = \rho(\xi_v)\alpha(\xi_u)$. We have

$$\rho^-(\xi_u)\alpha_m(\xi_v) = \rho^-(\xi_v)\alpha_m(\xi_u).$$

Replacing u by iu gives

$$-i\rho^-(\xi_u)\alpha_m(\xi_v) = \rho^-(\xi_v)\alpha_m(\xi_{iu}).$$

Thus

$$\rho^-(\xi_v)\alpha_m^+(\xi_u) = \frac{1}{2}\rho^-(\xi_v)\left(\alpha_m(\xi_u) - i\alpha_m(\xi_{iu})\right) = 0.$$

This holds for all $\xi_v \in \mathfrak{p}$. Thus $\alpha_m^+(\xi_u) = 0$ by Lemma 3.4, and α_m is $\overline{\mathbb{C}}$-linear.

The second part (2) of the Proposition on the dual representation can be proved similarly using the similar arguments and Lemma 3.7. □

We prove now Theorem 1.1 and Corollary 1.2.

Proof. The statements in Theorem 1.1 follows from Proposition 3.3. Indeed if $\alpha \in A^1(\Gamma, B, \rho_m)$ is a harmonic form, then it will have values in $S^m(\mathbb{C}^n)$ by Proposition 3.3. By the relation $\mathbb{C}^n = \mathfrak{p}^+ \otimes \det^{-1}$ we have

$$S^m(\mathbb{C}^n) = (\mathfrak{p}^+)^m \otimes (\det)^{-m} = S^m TX \otimes \mathcal{L}^{-m},$$

proving that α is a $(0,1)$-section of $S^m TX \otimes \mathcal{L}^{-m}$. The proof of the second one is similar. The claim that α is $\overline{\mathbb{C}}$-linear is precisely that α is a $(0,1)$-form. This proves the first part, and the second part follows similarly from Proposition 3.3(2).

Let α be a harmonic form representing an element $H^1(\Gamma, \rho)$. Write $\alpha = \sum_{k=0}^m \alpha_k$ according to the decomposition (2.3). It follows then from above that $\alpha_k = 0$ for $k < m$, i.e. $\alpha = \alpha_m$. The isomorphism of the cohomology $H^1(\Gamma, \rho)$ and $H^1(\Gamma \setminus B, S^m TX \otimes \mathcal{L}^{-m})$ is then a consequence of [10, Proposition 4.2 and Theorem 6.1]. The second isomorphism is proved similarly. □

4. The Eichler–Shimura isomorphism for Riemann surfaces and applications

We consider now the case $n = 1$. Keeping the previous notation we consider the group cohomology $H^1(\Gamma, \rho_m)$ of the tensor power $S^m(\mathbb{C}^2)$ of the representation of $\Gamma \subset SU(1,1)$. In this case $H^1(\Gamma, \rho_m)$ has a decomposition as $H^1(\Gamma, \rho_m) = H^{(1,0)}(\Gamma, \rho_m) + H^{(0,1)}(\Gamma, \rho_m)$, and in contrast to the case $n \geq 2$ the component $H^{(1,0)}(\Gamma, \rho_m)$ is not vanishing but it is dual to $H^{(0,1)}(\Gamma, \rho_m)$. This Eichler–Shimura isomorphism further gives a correspondence between $H^{(1,0)}(\Gamma, \rho_m)$ and H^0-cohomology of a line bundle over the Riemann surface $\Sigma := \Gamma\backslash B$. We denote $K_\Sigma = \mathcal{K}$ the holomorphic cotangent bundle, i.e. the canonical line bundle on $\Gamma \setminus B$.

Theorem 4.1. *Realizing $H^1(\Gamma, \rho_m)$ as the space of harmonic forms on $\Gamma \setminus B$ we have $H^1(\Gamma, \rho_m) = H^{(1,0)}(\Gamma, \rho_m) + H^{(0,1)}(\Gamma, \rho_m)$ and furthermore the two space are dual to each other,*

$$H^{(1,0)}(\Gamma, \rho_m) = H^0(\Sigma, \mathcal{K}^{\frac{m}{2}+1}), \quad H^{(0,1)}(\Gamma, \rho_m) = H^0(\Sigma, \mathcal{K}^{\frac{m}{2}+1})^*.$$

Proof. We prove the second isomorphism using the computation in Section 3. Let α be a $(0,1)$ form in $H^{(0,1)}(\Gamma, \rho_m)$. Using $z \in B$ near $z = 0$ as local coordinate as above, let $\alpha = \sum_{j=0}^m \alpha_j$ be the decomposition of α in the decomposition of $S^m(\mathbb{C}^2) = \oplus_{j=0}^m \mathbb{C} e_1^j e_2^{m-j}$. The symmetry condition (3.6) implies that

$$\rho^+(\xi_u)\alpha(\xi_v) = \rho^+(\xi_u)\alpha^-(\xi_v) = \rho^-(\xi_v)\alpha^+(\xi_u) = 0$$

since α is $\overline{\mathbb{C}}$-linear, hence $\alpha = \alpha^-$ and $\alpha^+(\xi_u) = 0$. Thus

$$(\rho^+(\xi_u)\alpha(\xi_v))_j = \rho^+(\xi_u)\alpha(\xi_v)_{j-1} = 0$$

for all $j \geq 1$. But then since $\rho^+(\xi_{e_1})$ maps $e_1^{j-1}e_2^{m-j+1}$ to $e_1^j e_2^{m-j}$ for $1 \leq j \leq m$, the component $\alpha(\xi_v)_{j-1}$ is vanishing for all $1 \leq j \leq m$, and we have $\alpha = \alpha_m$. Now from Eq. (2.2), $K_\Sigma^{-1} = \det^2$. Hence $S^m V_1 = S^m T\Sigma \otimes \det^{-m} = \mathcal{K}^{-m} \otimes \mathcal{K}^{\frac{m}{2}} = \mathcal{K}^{-\frac{m}{2}}$ and we have thus $\alpha \in H^1(\Sigma, \mathcal{K}^{-\frac{m}{2}})$ which is dual to $H^0(\Sigma, \mathcal{K}^{\frac{m}{2}+1})$ by Serre duality. That this map is onto is a consequence of the general results of [10,11], as in the proof of Corollary 1.2. $\square$

We give now an application of the above result computing the tangent space of the Hitchin's Teichmüller component of representations of Γ in a semisimple Lie group G. We shall only treat the case $G = SL(n, \mathbb{R})$ even though much computations can be carried over to other cases. The result might be known to experts but it seems still to provide some novel understanding for the geometry of the component.

We consider two representations of $SL(2, \mathbb{R}) = SU(1,1)$ into the group $SL(n, \mathbb{R})$ and compute the corresponding group cohomologies of $\Gamma \subset SL(2, \mathbb{R})$. We consider first the real representation of $SL(2, \mathbb{R})$ on the symmetric tensor $(S^m(\mathbb{R}^2), \rho_m)$ in the group $SL(m+1, \mathbb{R})$. Let τ_k, $k \leq m$, be the representation ρ_k in $SL(k+1, \mathbb{R})$ considered as a representation in $SL(m+1, \mathbb{R})$. We compute the corresponding cohomologies which can be viewed as the tangent space of the variety at the respective points.

Theorem 4.2. *Realizing the elements in the group cohomologies as harmonic forms we have that $H^1(\Gamma, \rho_m, \mathfrak{sl}(m+1, \mathbb{R}))$ and $H^1(\Gamma, \tau_k, \mathfrak{sl}(m+1, \mathbb{R}))$ are real forms in the space*

$$\sum_{j=1}^m H^0(\Sigma, \mathcal{K}^{j+1}) + H^0(\Sigma, \mathcal{K}^{j+1})^*$$

and

$$\sum_{j=1}^k H^0(\Sigma, \mathcal{K}^{j+1}) + H^0(\Sigma, \mathcal{K}^{j+1})^* + (H^0(\Sigma, \mathcal{K}^{\frac{k}{2}+1}) + H^0(\Sigma, \mathcal{K}^{\frac{k}{2}+1})^*)^{2(m-k)}$$
$$+ (H^0(\Sigma, \mathcal{K}) + H^0(\Sigma, \mathcal{K})^*)^{(m-k)^2}.$$

Proof. We consider the complexification $\mathfrak{sl}(2, \mathbb{C})$ and its representation $S^k\mathbb{C}^2$ in $\mathfrak{sl}(m+1, \mathbb{C})$. The real representation $S^k\mathbb{R}^2$ of $\mathfrak{sl}(2, \mathbb{R})$ in $\mathfrak{sl}(m+1, \mathbb{R})$ is the fixed point of the conjugation $X \to \bar{X}$ of $\mathfrak{sl}(m+1, \mathbb{C})$. Now the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$ under ρ_m in $\mathfrak{sl}(m+1, \mathbb{C})$ is decomposed as [16,17]

$$\mathfrak{sl}(m+1, \mathbb{C}) = \sum_{j=1}^m S^{2j}\mathbb{C}^2$$

with the first component $S^2\mathbb{C}^2$ being $\mathfrak{sl}(2, \mathbb{C})$ itself. Now by Theorem 4.1 we have

$$H^{(1,0)}(\Gamma, S^{2j}\mathbb{C}^2) = H^{(1,0)}(\Sigma, \mathcal{K}^j) = H^0(\Sigma, \mathcal{K}^{j+1}),$$

$$H^{(0,1)}(\Gamma, S^{2j}\mathbb{C}^2) = H^{(0,1)}(\Sigma, \mathcal{K}^{-j}) = H^0(\Sigma, \mathcal{K}^{j+1})^*.$$

The involution $X \mapsto \bar{X}$ on the one-forms is now $f(z)(dz)^j \otimes dz \mapsto \bar{f}(z)\partial_{\bar{z}}^j \otimes d\bar{z}$. Thus the real cohomology $H^1(\Gamma, \mathfrak{sl}(m+1, \mathbb{R}))$ is a real form in the space stated. Now under the action τ_k we have

$$\mathfrak{sl}(m+1, \mathbb{C}) = \sum_{j=1}^{k} S^{2j}\mathbb{C}^2 \oplus (S^k\mathbb{C}^2)^{2(m-k)} \oplus \mathbb{C}^{(m-k)^2}$$

the cohomology of Γ in the $S^{2j}\mathbb{C}^2$ is computed as above. The cohomology in $\mathbb{C}$ is

$$H^1(\Gamma, \mathbb{C}) = H^0(\Sigma, \mathcal{K}) + H^0(\Sigma, \mathcal{K})^*$$

the space of abelian differentials. The rest of the claim follows immediately. $\square$

The set

$$\{\rho_m \circ \phi | \phi : \Gamma \to SL(2, \mathbb{R}) \text{ discrete and faithful}\}/\sim$$

constitutes the Fuchsian locus in the Hitchin component. The above theorem shows that the tangent space of Hitchin component at Fuchsian locus consists of $\sum_{j=2}^{m+1} H^0(\Sigma, K_\Sigma^j)$. When $m = 2$, the Hitchin component is the set of convex real projective structures on a surface. Furthermore it is known that the Hitchin component is a holomorphic vector bundle over Teichmüller space with fibers cubic holomorphic forms [18,19]. In the forthcoming paper [20], we will analyze this case in more details to show that the Hitchin component is a Kähler manifold, and we will elaborate more on local rigidity of complex hyperbolic lattices [21].

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