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(article starts on next page)

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ABSTRACT: We present a unified and completely general formulation of extended geometry, characterised by a Kac-Moody algebra and a highest weight coordinate module. Generalised diffeomorphisms are constructed, as well as solutions to the section constraint. Generically, additional ("ancillary") gauge transformations are present, and we give a concrete criterion determining when they appear. A universal form of the (pseudo-)action determines the dynamics in all cases without ancillary transformations, and also for a restricted set of cases based on the adjoint representation of a finite-dimensional simple Lie group. Our construction reproduces (the internal sector of) all previously considered cases of double and exceptional field theories.

Contents

1 Introduction

During the last years, much research has been devoted to models of gravity together with other fields, obtained by extending d-dimensional space to some module of a structure group G. The structure group plays an important rôle for the gauge symmetries of these models, the generalised diffeomorphisms. Locally, d-dimensional space and the group $\mathrm{GL}(d) \subset G \times \mathbb{R}^+$ are recovered through a section constraint, an algebraic condition on allowed momenta. In this paper, we let the term *extended geometry* denote any such model. The most important classes of extended geometry are double field theory and exceptional field theory, motivated by dualities in string theory.

Recently, universal expressions for the invariant tensors used in the construction of extended geometries have been uncovered [1, 2], pointing towards a unified treatment. These expressions constitute a generalisation of a generic identity fulfilled for elements in a minimal orbit, that has appeared in the mathematics literature [3]. Although double field theory [4–28] has a generic form, thanks to the tensor formalism of $O(d,d)$, exceptional field theory [29–49] has typically relied on a case by case treatment.

It has not been clear what the most general situation is. Some important examples of exceptional field theory for high rank groups (E_8, E_9) exhibit the presence of additional constrained local transformations [2, 42, 45, 46, 49], for which we use the term *ancillary transformations*. However, the classifications in refs. [50, 51] rely on the absence of such transformations, so it is relevant to reconsider the general setting.

In the present paper, we develop a general formalism for extended geometry based on a choice of a Kac-Moody algebra $\mathfrak{g}$ and an irreducible highest weight module $R(\lambda)$. As our only restrictions on the choice, we require the Cartan matrix of $\mathfrak{g}$ to be indecomposable and symmetrisable, and the module $R(\lambda)$ to be integrable (so that the Dynkin labels of

the highest weight λ are non-negative integers) [52]. All expressions for generalised diffeomorphisms, section constraints etc., are universal, and the formalism allows for a unified treatment of all extended geometries. We limit ourselves to local properties; it is not clear if the treatment of global issues may be facilitated by our methods. Our investigation is also limited to what in a compactification would be the "internal" space, and the fields for which we formulate the dynamics are those that from an "external" point of view are scalars.

In section 2, the basic concepts of extended coordinates, section constraints and generalised diffeomorphisms are introduced. We use the quadratic Casimir to obtain the general form of the section constraint, and apply it in the construction of generalised diffeomorphisms. The result shows that ancillary transformations occur in many cases. In section 3 we show how the general form of the section constraint may be derived from bosonic and fermionic extensions of the structure algebra. This alternative approach is then used in section 4, where we give a concrete criterion in terms of the structure algebra and coordinate module for whether ancillary transformations appear or not. A pseudo-action, encoding the full dynamics for a limited set of cases, is given in section 5. In section 6, examples are examined, some of which are well known, and some new. We conclude with a summary and outlook in section 7.

2 Extended space, sections, and generalised diffeomorphisms

We consider models based on a Kac-Moody algebra $\mathfrak{g}$, exponentiated to a group G (extending the structure group $\mathrm{GL}(d)$ in ordinary geometry), and an irreducible and integrable highest weight module $R(\lambda)$ (extending the ordinary coordinate module). Derivatives will thus transform in the dual module $\overline{R(\lambda)}$.

We denote the Cartan matrix of $\mathfrak{g}$ by a_{ij} ($i,j = 1, \ldots, r$) if it is invertible. Otherwise, if the corank of the Cartan matrix is $m \geq 1$ (in particular if $\mathfrak{g}$ is affine, $m = 1$), then we extend the range of indices to $i,j = 1, \ldots, n$, where $n = r + m$, and let a_{ij} be an invertible $n \times n$ matrix obtained by adding m rows and columns to the Cartan matrix (corresponding to a "realisation" of it [52]). We will always assume that a is symmetrisable, which means that there is a diagonal matrix d with nonzero entries d_j such that $(da)_{ij}$ is a symmetric matrix. This gives rise to a non-degenerate symmetric bilinear form on the Cartan subalgebra, and a corresponding metric on the weight space.[1] We use the conventions that d expresses half the lengths squared of the simple roots, $d_i = \frac{(\alpha_i, \alpha_i)}{2}$, and da their mutual inner products, $(da)_{ij} = (\alpha_i, \alpha_j)$. Thus $a_{ij} = (\alpha_i^\vee, \alpha_j)$, where $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$ is the coroot of α_i. It is often convenient to symmetrise a from the right with the inverse of d, so that the symmetrised matrix is $\hat{a}_{ij} = (ad^{-1})_{ij} = (\alpha_i^\vee, \alpha_j^\vee)$. With weights expressed as $\lambda = \sum_i \lambda_i \Lambda_i$, where the fundamental weights Λ_i are given by $(\Lambda_i, \alpha_j^\vee) = \delta_{ij}$, the weight space metric is then given by $\hat{a}^{-1}$, so that $(\lambda, \lambda) = \sum_{ij} (\hat{a}^{-1})^{ij} \lambda_i \lambda_j$. The Dynkin labels λ_i of the (highest) weights λ that we consider are (non-negative) integers. We note that d is not uniquely given by a, but only up to an overall factor, and that all the entries have the same sign. The (standard)

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[1]If the Cartan matrix is not invertible, $m \geq 1$, then the set of simple roots $\alpha_1, \ldots, \alpha_r$ in the weight space is supplemented by m additional basis elements $\alpha_{r+1}, \ldots, \alpha_n$, which strictly speaking are no roots. However, for simplicity we will refer to all basis elements $\alpha_1, \ldots, \alpha_n$ as simple roots.

convention is that the longest simple roots have $(\alpha_i, \alpha_i) = 2$. Throughout the paper, we will take the real form of $\mathfrak{g}$ to be the split one, i.e., the one associated with the weight space decomposition.

Following ref. [38], we consider generalised diffeomorphisms of the form

$$\mathscr{L}_\xi V^M = \xi^N \partial_N V^M - \partial_N \xi^M V^N + Y^{MN}{}_{PQ} \partial_N \xi^P V^Q \,, \tag{2.1}$$

where Y is a $\mathfrak{g}$-invariant tensor that needs to satisfy certain identities in order for the generalised diffeomorphisms to close into an algebra according to

$$[\mathscr{L}_\xi, \mathscr{L}_\eta] = \mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi \eta - \mathscr{L}_\eta \xi)} \,, \tag{2.2}$$

and to be covariant with respect to themselves. The first of these identities is simply the section constraint

$$Y^{MN}{}_{PQ}(\partial_M \otimes \partial_N) = 0 \tag{2.3}$$

and, as shown in ref. [1], all but one of the other identities can be conveniently combined into the single "fundamental" identity

$$\begin{aligned}(Z^{NT}{}_{SM} Z^{QS}{}_{RP} - Z^{QT}{}_{SP} Z^{NS}{}_{RM} \\ - Z^{NS}{}_{PM} Z^{QT}{}_{RS} + Z^{ST}{}_{RP} Z^{NQ}{}_{SM})(\partial_N \otimes \partial_Q) = 0 \,, \end{aligned} \tag{2.4}$$

where $Z^{MN}{}_{PQ} = Y^{MN}{}_{PQ} - \delta^M_P \delta^N_Q$.

Already demanding that the last two terms in the transformation (2.1) represent a local transformation on V in the structure algebra $\mathfrak{g} \oplus \mathbb{R}$ constrains Y to the form

$$Y^{MN}{}_{PQ} = -k\eta_{\alpha\beta} T^{\alpha N}{}_P T^{\beta M}{}_Q + \beta \delta^N_P \delta^M_Q + \delta^M_P \delta^N_Q \,, \tag{2.5}$$

where $T^{\alpha N}{}_P$ are the representation matrices for $R(\lambda)$ in a basis T^α of $\mathfrak{g}$ and $\eta_{\alpha\beta}$ is the inverse of the invariant bilinear form. Then it is easily checked that, for any values of the constants k and β in (2.5), the condition (2.4) is automatically satisfied if (2.3) is (with both conditions depending on k and β). This condition is however not sufficient for closure of the generalised diffeomorphisms, but leads to a remaining term

$$\left([\mathscr{L}_\xi, \mathscr{L}_\eta] - \mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi \eta - \mathscr{L}_\eta \xi)}\right) V^M = \frac{1}{2} Z^{MP}{}_{QN} Y^{QR}{}_{ST} \xi^T \partial_P \partial_R \eta^S V^N - (\xi \leftrightarrow \eta) \,, \tag{2.6}$$

that vanishes in some cases (e.g., $\mathfrak{g} = E_r$, $r \leq 7$) but not in others (e.g., $\mathfrak{g} = E_8$). As we will see later, whether it vanishes or not does not depend on the constants k and β, but is entirely a property of the Lie algebra $\mathfrak{g}$ and the module $R(\lambda)$. What does determine the constants k and β (although not always fully) is the requirement that the "extended geometry" indeed is an extension of ordinary geometry, in the sense that we recover ordinary geometry when we solve the section constraint (2.3). This means that the solutions ("sections") must be d-dimensional linear subspaces of $R(\lambda)$ related to each other by rotations in G, and with a stability group containing $\mathrm{GL}(d)$, such that the generalised diffeomorphisms in a section reduce to ordinary diffeomorphisms together with transformations of other gauge fields. We would like to stress that the closure, or "almost closure" according to eq. (2.6), for a

bad choice of the constant k, is quite uninteresting, as it relies on a section constraint whose solutions are 1-dimensional sections. This follows from the analysis below. As regards the constant β, it can be seen as a weight. A field can transform with any weight, but β is the canonical weight associated with the parameters themselves. We will turn to a closer inspection of the section constraint, after a brief interlude on notation.

The relations above are written in tensor notation. However, we will often use an index-free notation, where the derivatives and vector fields are represented by bra and ket states, respectively. The advantage of using the bra-ket notation is, besides making many equations less cluttered, that it goes well together with the standard treatment of highest/lowest weight modules of Lie algebras. Thus, $\xi^M \leftrightarrow |\xi\rangle$ and $\partial_M \leftrightarrow \langle\partial|$. The section constraint (2.3) takes the index-free form

$$\langle\partial| \otimes \langle\partial|Y = 0\,, \tag{2.7}$$

while the expression (2.5) for the Y tensor reads

$$\sigma Y = -k\eta_{\alpha\beta}T^\alpha \otimes T^\beta + \beta + \sigma\,, \tag{2.8}$$

where σ is the permutation operator, $\sigma|a\rangle \otimes |b\rangle = |b\rangle \otimes |a\rangle$, or, on operators, $\sigma A \otimes B = B \otimes A\,\sigma$.

The section constraint can be introduced in two steps. The first one (which sometimes goes under the name "weak section constraint") demands that momenta (derivatives) lie in a minimal orbit under the action of G. This is equivalent to the statement that the symmetrised product ∂^2 only contains the module $\overline{R(2\lambda)}$, dual to the highest module in the tensor product $\otimes^2 R(\lambda)$. The term "highest" here refers to the partial ordering of (highest) weights. The solution of the weak constraint is not a linear space, but a cône.

In the second step, we demand that the product of any two momenta only contains the lowest symmetric and antisymmetric modules in the tensor product. While obviously the highest module in the symmetrised product $\vee^2 R(\lambda)$ is $R(2\lambda)$, the highest module in the antisymmetrised product $\wedge^2 R(\lambda)$ is in general reducible, and consists of the sum of all irreducible highest weight modules $R(2\lambda - \alpha_i)$, where the simple roots α_i are the ones with $\lambda_i = (\lambda, \alpha_i^\vee) \neq 0$. This is easily seen from an expansion of $R(\lambda)$, starting from the highest weight state $|\lambda\rangle$, followed by the states $f_i|\lambda\rangle = |\lambda - \alpha_i\rangle$ for $\lambda_i \neq 0$. The highest antisymmetric highest weight states then are

$$||2\lambda - \alpha_i\rangle\rangle = |\lambda\rangle \otimes |\lambda - \alpha_i\rangle - |\lambda - \alpha_i\rangle \otimes |\lambda\rangle\,. \tag{2.9}$$

As will soon be clear, not all of these should survive for a solution to the section constraint, but only those with $\lambda_i = 1$ and $\frac{2}{(\alpha_i,\alpha_i)} = k$ for some given k. We denote the corresponding index set by

$$\mathcal{I}_k \subset \left\{ i \,|\, \lambda_i = 1, \, \frac{2}{(\alpha_i, \alpha_i)} = k \right\}. \tag{2.10}$$

The section constraint will thus be equivalent to

$$(\partial \otimes \partial)\Big|_{\overline{R_2 \oplus \widetilde{R}_2}} = 0\,, \tag{2.11}$$

where

$$R_2 = \vee^2 R(\lambda) \ominus R(2\lambda)\,,$$
$$\widetilde{R}_2 = \wedge^2 R(\lambda) \ominus \bigoplus_{i \in \mathcal{I}_k} R(2\lambda - \alpha_i)\,. \tag{2.12}$$

The modules R_2 and $\widetilde{R}_2$, which do not need to be irreducible, have a natural interpretation in terms of extended algebras, as explained in section 3.

As we will see, the constant k in $\mathcal{I}_k$ is the same as the one in eq. (2.5). To get an as sensible and less restrictive model as possible, we set $k = \frac{2}{(\alpha_i, \alpha_i)}$ if all simple roots α_i have the same length squared (α_i, α_i), or if there is only one simple root α_i such that $\lambda_i = 1$. If there is more than one simple root with $\lambda_i = 1$, and they have different lengths, then we set $k = \frac{2}{(\alpha_i, \alpha_i)}$ for one of them (thus k is not fully determined by $\mathfrak{g}$ and λ in this case). If $\lambda_i \neq 1$ for all simple roots α_i, then we let k be such that $k \geq \frac{2}{(\alpha_i, \alpha_i)}$ for all simple roots α_i (i.e., $k \geq 1$ with the convention that the longest simple roots α_i have $(\alpha_i, \alpha_i) = 2$) and otherwise undetermined.

Let us investigate what the section constraint (2.11) implies in terms of algebraic conditions. We will use the quadratic Casimir operator, which is well defined on highest weight modules of any Kac-Moody algebra with a symmetrisable Cartan matrix. It is

$$C_2 = \frac{1}{2}\eta_{\alpha\beta} : T^\alpha T^\beta : +(h, \varrho) = \sum_{\alpha \in \Delta_+} e_{-\alpha} e_\alpha + \frac{1}{2}(h, h) + (h, \varrho)\,. \tag{2.13}$$

Here, ϱ is the Weyl vector, defined so that $(\varrho, \alpha_i^\vee) = 1$. The term (h, ϱ) is a normal ordering term, which for finite-dimensional algebras can be absorbed into a symmetric ordering, since then $\varrho = \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha$. Since C_2 takes the same value on all elements in an irreducible module $R(\lambda)$, it is enough to evaluate it on the highest weight state, where one gets

$$C_2(R(\lambda)) = \frac{1}{2}(\lambda, \lambda + 2\varrho)\,. \tag{2.14}$$

From this expression we also immediately obtain

$$C_2(R(2\lambda)) = 2C_2(R(\lambda)) + (\lambda, \lambda)\,,$$
$$C_2(R(2\lambda - \alpha_i)) = 2C_2(R(\lambda)) + (\lambda, \lambda) - \lambda_i(\alpha_i, \alpha_i)\,, \tag{2.15}$$

where α_i is any simple root such that $\lambda_i = (\lambda, \alpha_i^\vee) > 0$.

Now, consider a representative in the minimal orbit of $R(\lambda)$, which can be chosen proportional to the highest weight state $|\lambda\rangle$ itself. Following section 6.2 of ref. [2], we then have

$$
\begin{aligned}
0 &= [C_2(R(2\lambda)) - 2C_2(R(\lambda)) - (\lambda, \lambda)] \, |\lambda\rangle \otimes |\lambda\rangle \\
&= \left[\frac{1}{2}\eta_{\alpha\beta} : T^\alpha T^\beta : +(h, \varrho)\right] |\lambda\rangle \otimes |\lambda\rangle - \left(\left[\frac{1}{2}\eta_{\alpha\beta} : T^\alpha T^\beta : +(h, \varrho)\right] |\lambda\rangle\right) \otimes |\lambda\rangle \\
&\quad - |\lambda\rangle \otimes \left[\frac{1}{2}\eta_{\alpha\beta} : T^\alpha T^\beta : +(h, \varrho)\right] |\lambda\rangle - (\lambda, \lambda)|\lambda\rangle \otimes |\lambda\rangle \\
&= \left[\eta_{\alpha\beta} T^\alpha \otimes T^\beta - (\lambda, \lambda)\right] |\lambda\rangle \otimes |\lambda\rangle\,.
\end{aligned}
\tag{2.16}
$$

Similarly, for i such that $\lambda_i = 1$, we obtain

$$
\begin{aligned}
0 &= \left[C_2(R(2\lambda - \alpha_i)) - 2C_2(R(\lambda)) - (\lambda, \lambda) + (\alpha_i, \alpha_i) \right] ||2\lambda - \alpha_i\rangle\rangle \\
&= \left[\eta_{\alpha\beta} T^\alpha \otimes T^\beta - (\lambda, \lambda) + (\alpha_i, \alpha_i) \right] ||2\lambda - \alpha_i\rangle\rangle .
\end{aligned}
\tag{2.17}
$$

Next, consider two vectors in a section. Without loss of generality, take one of them to be $|\lambda\rangle$ and the other one $|q\rangle = \sum_{\ell \geq 0} |q\rangle_\ell$, with components given by $(h, \lambda)|q\rangle_\ell = ((\lambda, \lambda) - \ell)|q\rangle_\ell$ in the grading induced by λ. If the right hand side of eq. (2.16) annihilates symmetric products of vectors in a section, we must have

$$
\left[\eta_{\alpha\beta} T^\alpha \otimes T^\beta - (\lambda, \lambda) \right] (|\lambda\rangle \otimes |q\rangle + |q\rangle \otimes |\lambda\rangle) = 0 .
\tag{2.18}
$$

Since the equation is linear in $|q\rangle$, we can treat the terms in the grading decomposition separately. We get

$$
\begin{aligned}
& \left[\eta_{\alpha\beta} T^\alpha \otimes T^\beta - (\lambda, \lambda) \right] (|\lambda\rangle \otimes |q\rangle_\ell + |q\rangle_\ell \otimes |\lambda\rangle) \\
& = (1 + \sigma)\left(\sum_{\alpha \in \Delta_+} e_\alpha |q\rangle_\ell \otimes e_{-\alpha}|\lambda\rangle - \ell|\lambda\rangle \otimes |q\rangle_\ell \right) .
\end{aligned}
\tag{2.19}
$$

This shows that, for $\ell \geq 1$, we must have $|q\rangle_\ell \propto e_{-\alpha}|\lambda\rangle$, for some positive root α. In order for the sum to give a single term, we must have $e_{\alpha'}|q\rangle_\ell = 0$ or $e_{-\alpha'}|\lambda\rangle = 0$ for all positive roots $\alpha' \neq \alpha$. The root α must thus be $\alpha_i + \beta$, where α_i is a simple root with $\lambda_i \neq 0$ and $(\lambda, \beta) = 0$. Eq. (2.18) is then satisfied if $\ell = \lambda_i$.

We also need to verify that $|q\rangle$ itself satisfies the symmetric constraint, i.e., that it lies in the minimal orbit. This condition is not linear, and terms with different degree may mix. Therefore, we consider the maximal value of ℓ. For simplicity, we only display the calculation for $\alpha = \alpha_i$. Then

$$
\begin{aligned}
& \left[\eta_{\alpha\beta} T^\alpha \otimes T^\beta - (\lambda, \lambda) \right] |q\rangle_\ell \otimes |q\rangle_\ell \\
& = \lambda_i \left[|\lambda\rangle \otimes f_i^2|\lambda\rangle + f_i^2|\lambda\rangle \otimes |\lambda\rangle \right] + \left[(\lambda - \alpha_i, \lambda - \alpha_i) - (\lambda, \lambda) \right] f_i|\lambda\rangle \otimes f_i|\lambda\rangle ,
\end{aligned}
\tag{2.20}
$$

and thus $f_i|\lambda\rangle$ is in the minimal orbit only if $f_i^2|\lambda\rangle = 0$ and $(\lambda - \alpha_i, \lambda - \alpha_i) = (\lambda, \lambda)$. Both conditions give $\lambda_i = 1$. The same holds for $|q\rangle_1 = e_{-\alpha_i - \beta}|\lambda\rangle$ as above. This consideration shows why the antisymmetric highest weight modules $R(2\lambda - \alpha_i)$ are allowed to be non-vanishing if $\lambda_i = 1$. Note that this condition is obtained by demanding that the symmetric constraint is satisfied by vectors in a linear subspace.

Continuation of this construction yields a linear subspace, a section, of $R(\lambda)$ that behaves as a $\mathrm{GL}(d)$ module. Both the symmetric and antisymmetric products of vectors in the section contain a single irreducible module, $R(2\lambda)$ and $R(2\lambda - \alpha_i)$, respectively, depending on the choice of simple root with $\lambda_i = 1$. The possible (representatives of) sections are immediately read off from the Dynkin diagram: in addition to $|\lambda\rangle$, one chooses a $f_i|\lambda\rangle$ with $\lambda_i = 1$. Then the section is sequentially enlarged with a neighbouring node (here labelled $i + 1$) as $f_{i+1} f_i|\lambda\rangle$ as long as $\lambda_{i+1} = 0$, and so on. If a branching in the Dynkin diagram is encountered, one chooses one branch to proceed along. The process

stops at an end of the diagram, or before one encounters another node j with $\lambda_j \neq 0$, or before one encounters a node with a multiple connection. This corresponds to choosing a maximal (i.e., non-extendable) line of d simply connected nodes of equal length in the extended Dynkin diagram (see section 3), an "extended gravity line".

From this, it follows that a representative of the section is identified as the states in $R(\lambda)$ at degree 0 in the grading with respect to the node (or one of the nodes) lying next to the gravity line in the Dynkin diagram. Call this node number j. The degree of a weight μ in this grading is given by $\frac{2}{(\alpha_j, \alpha_j)}(\Lambda_j, \mu)$. The representative section is specified by Λ_j, and other sections by elements in the orbit of Λ_j. It follows that a section is specified by an element in the minimal orbit of Λ_j [38], i.e., a $\phi \in R(\Lambda_j)$ such that $\phi^2 \in R(2\Lambda_j)$. A familiar example is the parametrisation of isotropic subspaces in the fundamental representation of D_r (sections in double field theory) in terms of pure spinors.

All vectors $|p\rangle$, $|q\rangle$ in a section thus satisfy the universal constraint

$$Y|p\rangle \otimes |q\rangle = 0 \,, \tag{2.21}$$

where

$$\sigma Y = k \left[-\eta_{\alpha\beta} T^\alpha \otimes T^\beta + (\lambda, \lambda) \right] + \sigma - 1 \,, \tag{2.22}$$

which is (the index-free version of) the expression (2.5) with $\beta = k(\lambda, \lambda) - 1$. In all situations where there is a section of dimension larger than 1, the constant k has the fixed value $k = \frac{2}{(\alpha_i, \alpha_i)}$, where $\lambda_i = 1$.

Note that it is enough to consider the symmetric part of the condition, the antisymmetric is then automatically satisfied (for a given choice of section). The Y tensor commutes with σ, which means that it can be decomposed as $Y^{MN}{}_{PQ} = Y^{(MN)}{}_{(PQ)} + Y^{[MN]}{}_{[PQ]}$. The section constraint is imposed on derivatives as in eq. (2.7).

In ref. [2] this necessary form of the Y tensor was shown for the cases where λ is a fundamental weight dual to a coroot to a long simple root (i.e., $(\alpha_i, \alpha_i) = 2$); the present treatment holds for arbitrary Kac-Moody groups with a symmetrisable Cartan matrix and arbitrary coordinate modules $R(\lambda)$. Many cases seem quite uninteresting. Any highest weight with all $\lambda_i \neq 1$ leads to a 1-dimensional section, spanned by $|\lambda\rangle$. Most of these should be unphysical or uninformative. An exception is SL(2) with the adjoint as coordinate module. This corresponds to the Ehlers symmetry arising on reduction of gravity from 4 to 3 dimensions, and the section should indeed be 1-dimensional. It also seems like "gravity lines" containing short roots ($(\alpha_i, \alpha_i) < 2$) typically do not correspond to interesting situations. The typical example would be when the coordinate module is the fundamental of an algebra C_r. This example is mentioned in section 6. In section 4, we will further investigate the consequences of different possible choices.

Let us now come back to the generalised diffeomorphisms (2.1), which in index-free notation take the form

$$\mathscr{L}_\xi |V\rangle = \langle \partial_V |\xi\rangle \otimes |V\rangle + \langle \partial_\xi |(-k\eta_{\alpha\beta} T^\alpha \otimes T^\beta + k(\lambda, \lambda) - 1)|\xi\rangle \otimes |V\rangle \,. \tag{2.23}$$

– 7 –

Commuting two such transformations leads to the remainder term (2.6) (antisymmetrised in ξ and η), which after some standard calculation becomes

$$\left([\mathscr{L}_\xi, \mathscr{L}_\eta] - \mathscr{L}_{\frac{1}{2}(\mathscr{L}_\xi \eta - \mathscr{L}_\eta \xi)} \right) |V\rangle = \Sigma_\alpha T^\alpha |V\rangle \,, \tag{2.24}$$

where the element $\Sigma = \Sigma_\alpha T^\alpha \in \mathfrak{g}$ is given as

$$\Sigma^\alpha = \frac{k}{2} \langle \partial_\eta | \otimes \langle \partial_\eta | S^\alpha |\xi\rangle \otimes |\eta\rangle - (\xi \leftrightarrow \eta) \,,$$
$$\text{with } \ S^\alpha = -k f^\alpha{}_{\beta\gamma} T^\beta \otimes T^\gamma + T^\alpha \otimes 1 - 1 \otimes T^\alpha \tag{2.25}$$

(adjoint indices raised and lowered with η). The operator S^α has the symmetry $\sigma S^\alpha = -S^\alpha \sigma$, meaning that it decomposes as $S^{\alpha MN}{}_{PQ} = S^{\alpha(MN)}{}_{[PQ]} + S^{\alpha[MN]}{}_{(PQ)}$. Only the first term contributes in Σ^α. Without indices it can be written $\frac{1+\sigma}{2} S^\alpha = S^\alpha \frac{1-\sigma}{2}$, and obtained as

$$\frac{1+\sigma}{2} S^\alpha = \frac{1+\sigma}{2} [1 \otimes T^\alpha, Y] \,. \tag{2.26}$$

Therefore, it follows that (in Σ^α) the operator S^α may be replaced by

$$S^\alpha \to (1 \otimes T^\alpha) Y_- \,, \tag{2.27}$$

where $Y_- = Y \frac{1-\sigma}{2}$.

We see that, in the generic situation, the generalised diffeomorphisms will not close among themselves when acting on a vector. There will be additional transformations present, which are local transformations in $\mathfrak{g}$ of a restricted type. We use the term "ancillary transformations" for these extra gauge symmetries. Such transformations have already been shown to be important in a number of situations [2, 42, 45, 46, 49]. Eq. (2.25) provides a very simple expression for the generated ancillary transformation. The usual (previous) form of the extra remaining term on the right hand side of eq. (2.24) contains the Y tensor quadratically as YZ (see eq. (2.6)). The new form (2.25) is only quadratic in generators, and should be much easier to deal with. It will appear naturally in the variations of the action of section 5.

We will examine the remainder term closer in section 4, where a simple criterion for its presence or absence will be given. The derivation relies in part on the concept of extended algebras, which are introduced in the following section.

3 Extended algebras

From the coordinate module $R(\lambda)$ as R_1 and $\widetilde{R}_1$, and from R_2 and $\widetilde{R}_2$ defined in (2.12), representations R_p and $\widetilde{R}_p$ can be defined for all positive integers p (possibly trivial for all but finitely many p) by extending the Lie algebra $\mathfrak{g}$ in a certain way, and then decomposing the adjoint representation of the extended algebra under $\mathfrak{g}$. In the case of exceptional geometry, the sequence R_p was in ref. [38] shown to encode the infinite reducibility of the generalised diffeomorphisms, and it agrees with tensor hierarchies in gauged supergravity and exceptional field theory [43–45, 53].

The form of the Y tensor in exceptional geometry was constructed from the extended algebras in ref. [1]. In this section we review the construction, but for the general Kac-Moody algebra $\mathfrak{g}$ rather than explicitly for E_r. The generalisation from E_r to $\mathfrak{g}$ is straightforward as long as the Cartan matrices of $\mathfrak{g}$ and of the fermionic and bosonic extensions $\mathscr{A}$ and $\mathscr{B}$ below are invertible. We will assume this until the end of the section, where we briefly describe the general case.

First we extend the Cartan matrix a_{ij} $(i, j = 1, \ldots, r)$ of $\mathfrak{g}$ to the Cartan matrix B_{IJ} $(I, J = 0, 1, \ldots, r)$ of a contragredient Lie superalgebra [54] $\mathscr{B}$ by another row and column such that

$$B_{00} = 0\,, \qquad B_{i0} = -\lambda_i\,, \qquad B_{ij} = a_{ij}\,, \tag{3.1}$$

and such that $(DB)_{IJ}$ is a symmetric matrix, where D is a diagonal matrix with entries $D_0 = 1/k$ and $D_i = d_i$. Thus $B_{0i} = kd_i B_{i0} = -kd_i\lambda_i$.

In the construction of $\mathscr{B}$ from the Cartan matrix B one starts with the Lie superalgebra generated by two odd elements e_0, f_0 and $3r+1$ even elements e_i, f_i, h_I modulo the relations

$$[h_I, e_J] = B_{IJ}e_J\,, \qquad [h_I, f_J] = -B_{IJ}f_J\,, \qquad [e_I, f_J] = \delta_{IJ}h_J\,, \tag{3.2}$$

and factors out the maximal ideal that intersects the Cartan subalgebra (spanned by the h_I) trivially. In this case the resulting contragredient Lie superalgebra $\mathscr{B}$ is a Borcherds superalgebra, and the ideal is generated by the additional (Serre) relations [55]

$$[e_0, e_0] = [f_0, f_0] = 0, \qquad i \neq J \;\Rightarrow\; (\mathrm{ad}\, e_i)^{1-B_{iJ}}(e_J) = (\mathrm{ad}\, f_i)^{1-B_{iJ}}(f_J) = 0\,. \tag{3.3}$$

The Lie superalgebra $\mathscr{B}$ can be decomposed as $\mathscr{B} = \bigoplus_{p\in\mathbb{Z}} \mathscr{B}_p$, where $e_0 \in \mathscr{B}_1$ and $f_0 \in \mathscr{B}_{-1}$, and all other generators belong to $\mathscr{B}_0$. This is a (consistent) $\mathbb{Z}$-grading, which means $[\mathscr{B}_p, \mathscr{B}_q] \subseteq \mathscr{B}_{p+q}$. The even subalgebra $\mathscr{B}_0$ is (as a Lie algebra) the direct sum of $\mathfrak{g}$ and a one-dimensional center spanned by an element c. With a normalisation such that $[c, e_0] = e_0$, the components of c in the basis h_I are given by $c = \sum_I (B^{-1})^{0I} h_I$.

The subspaces $\mathscr{B}_{\pm 1}$ are irreducible $\mathfrak{g}$-modules under the adjoint action of $\mathscr{B}_0$, and f_0 is a highest weight vector of $\mathscr{B}_{-1}$ since $[e_i, f_0] = 0$. The Dynkin labels are given by

$$[h_i, f_0] = -B_{i0}f_0 = \lambda_i f_0\,. \tag{3.4}$$

Thus $\mathfrak{g}$ acts on $\mathscr{B}_{-1}$ and $\mathscr{B}_1$ in the representations $R(\lambda)$ and $\overline{R(\lambda)}$, respectively. With bases E^M and F_M of $\mathscr{B}_1$ and $\mathscr{B}_{-1}$, respectively, this means

$$[T^\alpha, E^M] = -T^{\alpha M}{}_N E^N\,, \qquad [T^\alpha, F_M] = T^{\alpha N}{}_M F_N\,. \tag{3.5}$$

In general, we denote the representation of $\mathfrak{g}$ corresponding to $\mathscr{B}_{-p}$ by R_p. Thus $R_1 = R(\lambda)$, and it follows from the Serre relations (3.3), which set the highest weight vector $[f_0, f_0]$ of $R(2\lambda)$ to zero, that $R_2 = \vee^2 R(\lambda) \ominus R(2\lambda)$.

The additional row and column in the Cartan matrix correspond to an additional (odd) simple root β_0 in an extended weight space with metric given by $(DB)_{IJ} = (\beta_I, \beta_J)$, where $\beta_i = \alpha_i$. In particular β_0 is a null rot, $(\beta_0, \beta_0) = 0$. The corresponding invariant bilinear form on $\mathscr{B}$ is given by $(h_I, h_J) = (\beta_I{}^\vee, \beta_J{}^\vee)$ on the Cartan subalgebra, where $\beta_0{}^\vee = k\beta_0$. It

is not symmetric on the whole of $\mathscr{B}$, but has a $\mathbb{Z}_2$-graded symmetry, consistent with the $\mathbb{Z}_2$-grading of $\mathscr{B}$. In particular, $(e_0, f_0) = -(f_0, e_0) = k$. We choose a relative normalisation of the bases of $\mathscr{B}_1$ and $\mathscr{B}_{-1}$ such that $(E^M, F_N) = -(F_N, E^M) = \delta_N^M$. On $\mathfrak{g}$ we have $(T^\alpha, T^\beta) = \eta^{\alpha\beta}$ as before. The length squared of the element c is $(c, c) = k(B^{-1})^{00}$. Now we have

$$[E^M, F_N] = -\eta_{\alpha\beta} T^{\alpha M}{}_N T^\beta + \frac{1}{k(B^{-1})^{00}} \delta_N^M c \tag{3.6}$$

and

$$f^M{}_N{}^P{}_Q \equiv ([[E^M, F_N], E^P], F_Q) = \eta_{\alpha\beta} T^{\alpha M}{}_N T^{\beta P}{}_Q + \frac{1}{k(B^{-1})^{00}} \delta_N^M \delta_Q^P. \tag{3.7}$$

Consider now the matrix A_{IJ} given by $A_{00} = 2$, and $A_{IJ} = B_{IJ}$ otherwise, i.e., if not $I = J = 0$. In the same way as the contragredient Lie superalgebra $\mathscr{B}$ is constructed from B, we can construct a contragredient Lie algebra [56] $\mathscr{A}$ from A. We thus replace B_{IJ} by A_{IJ} in the relations (3.2), and let all generators be even. For example, if $\mathfrak{g} = E_r$, and λ is the highest weight of the coordinate module in exceptional geometry, then $\mathscr{A}$ is the Kac-Moody algebra E_{r+1}.

Similarly to $\mathscr{B}$, the Lie algebra $\mathscr{A}$ can be decomposed as $\mathscr{A} = \bigoplus_{p \in \mathbb{Z}} \mathscr{A}_p$, where, as $\mathfrak{g}$-modules, $\mathscr{A}_p$ is isomorphic to $\mathscr{B}_p$ for $p = 0, \pm 1$. However, since we need to distinguish them from each other we denote the basis elements of $\mathscr{A}_1$ and $\mathscr{A}_{-1}$ by $\widetilde{E}^M$ and $\widetilde{F}_M$, respectively. We also denote the generators of $\mathscr{A}$ corresponding to e_0 and f_0 in $\mathscr{B}$ by $\widetilde{e}_0$ and $\widetilde{f}_0$. For the other generators e_i, f_i, h_I there is no need to make this distinction. We thus have an isomorphism $\mathscr{B}_{\pm 1} \to \mathscr{A}_{\pm 1}$ mapping a general element $U = U_M E^M$ to $\widetilde{U} = U_M \widetilde{E}^M$, and $V = V^M F_M$ to $\widetilde{V} = V^M \widetilde{F}_M$. Note that whereas the elements U, V are odd (fermionic) in the Lie superalgebra $\mathscr{B}$, the corresponding elements $\widetilde{U}, \widetilde{V}$ are ordinary even (bosonic) elements in the Lie algebra $\mathscr{A}$. Accordingly, the invariant bilinear form is now symmetric, like in $\mathfrak{g}$, so that $(\widetilde{E}^M, \widetilde{F}_N) = (\widetilde{F}_N, \widetilde{E}^M) = \delta_N^M$.

In the same way as for $\mathscr{B}$ we can now compute

$$[\widetilde{E}^M, \widetilde{F}_N] = -\eta_{\alpha\beta} T^{\alpha M}{}_N T^\beta + \frac{1}{k(A^{-1})^{00}} \delta_N^M c \tag{3.8}$$

and

$$\widetilde{f}^M{}_N{}^P{}_Q \equiv ([[\widetilde{E}^M, \widetilde{F}_N], \widetilde{E}^P], \widetilde{F}_Q) = \eta_{\alpha\beta} T^{\alpha M}{}_N T^{\beta P}{}_Q + \frac{1}{k(A^{-1})^{00}} \delta_N^M \delta_Q^P. \tag{3.9}$$

We also define tensors

$$g^{MN}{}_{PQ} = ([E^M, E^N], [F_P, F_Q]), \qquad \widetilde{g}^{MN}{}_{PQ} = ([\widetilde{E}^M, \widetilde{E}^N], [\widetilde{F}_P, \widetilde{F}_Q]), \tag{3.10}$$

which, by invariance of the bilinear form and the Jacobi identity, can be related to (anti-)symmetrisations of f and $\widetilde{f}$ as

$$g^{MN}{}_{PQ} = -2f^{(M}{}_P{}^{N)}{}_Q, \qquad \widetilde{g}^{MN}{}_{PQ} = 2\widetilde{f}^{[M}{}_P{}^{N]}{}_Q. \tag{3.11}$$

The weight λ is an element in the weight space of the original Kac-Moody algebra $\mathfrak{g}$, and can be written in the basis of simple roots as

$$\lambda = \sum_j \frac{(B^{-1})^{j0}}{(B^{-1})^{00}} \alpha_j \quad \Rightarrow \quad (\alpha_i{}^\vee, \lambda) = \sum_j B_{ij} \frac{(B^{-1})^{j0}}{(B^{-1})^{00}} = -B_{i0} = \lambda_i. \tag{3.12}$$

For its length squared we then get

$$
\begin{aligned}
(\lambda, \lambda) &= \sum_{ij} d_i B_{ij} \frac{(B^{-1})^{i0}}{(B^{-1})^{00}} \frac{(B^{-1})^{j0}}{(B^{-1})^{00}} = -\sum_i d_i B_{i0} \frac{(B^{-1})^{i0}}{(B^{-1})^{00}} \\
&= -\sum_i \frac{B_{0i}(B^{-1})^{i0}}{k(B^{-1})^{00}} = -\frac{1}{k(B^{-1})^{00}} \, .
\end{aligned}
\tag{3.13}
$$

Also $(A^{-1})^{00}$ can be related to $(B^{-1})^{00}$, by

$$
\frac{1}{(A^{-1})^{00}} = \frac{\det A}{\det a} = \frac{2\det a + \det B}{\det a} = 2 + \frac{\det B}{\det a} = 2 + \frac{1}{(B^{-1})^{00}} \, .
\tag{3.14}
$$

Inserting (3.14) into (3.9), and using (3.13) and (3.11) now gives

$$
\frac{k}{2}(g^{MN}{}_{PQ} - \widetilde{g}^{MN}{}_{PQ}) = -k\eta_{\alpha\beta}T^{\alpha M}{}_P T^{\beta N}{}_Q + k(\lambda,\lambda)\delta^M_P \delta^N_Q - \delta^M_P \delta^N_Q + \delta^N_P \delta^M_Q \, ,
\tag{3.15}
$$

or, in index-free notation,

$$
\frac{k}{2}(g - \widetilde{g}) = -k\eta_{\alpha\beta}T^{\alpha} \otimes T^{\beta} + k(\lambda,\lambda) - 1 + \sigma \, .
\tag{3.16}
$$

We see that this expression has the form (2.5), and agrees with eq. (2.22). Thus the Y tensor can be derived as

$$
\sigma Y = \frac{k}{2}(g - \widetilde{g}) \, .
\tag{3.17}
$$

In the general case, when possibly any of the involved Cartan matrices is not invertible, it is convenient to go one step further and extend the (invertible) matrix a_{ij} $(i,j = 1, \ldots, n,$ where $n \geq r$, see the beginning of section 2) by *two* rows and columns to a symmetrisable matrix C_{IJ} $(I, J = -1, 0, 1, \ldots, n)$ such that

$$
C_{(-1)0} = C_{0(-1)} = 1 \, , \qquad C_{i0} = -\lambda_i \, , \qquad C_{0i} = -kd_i\lambda_i \, , \qquad C_{ij} = a_{ij} \, ,
\tag{3.18}
$$

and all other entries are zero. Note that $\det C = -\det a$. Let $\mathscr{C}$ be the Lie superalgebra constructed from C in the same way as $\mathscr{B}$ is constructed from B, with odd generators e_{-1}, e_0, f_{-1}, f_0 and even generators e_i, f_i, h_j, where $i = 1, \ldots, r$ and $j = 1, \ldots, n$. Like $\mathscr{B}$, it can be decomposed as $\mathscr{C} = \bigoplus_{p \in \mathbb{Z}} \mathscr{C}_p$, where $e_0 \in \mathscr{C}_1$ and $f_0 \in \mathscr{C}_{-1}$ and all other generators belong to $\mathscr{C}_0$. However, this is not a consistent $\mathbb{Z}$-grading; $\mathscr{C}_{\pm 1}$ is the direct sum of an even and an odd subspace, which can be identified with $\mathscr{A}_{\pm 1}$ and $\mathscr{B}_{\pm 1}$, respectively, by $\widetilde{E}^M = -[e_{-1}, E^M]$ and $\widetilde{F}_M = [f_{-1}, F_M]$. We then get

$$
\begin{aligned}
([[E^M, F_N], E^P], F_Q) &= \eta_{\alpha\beta}T^{\alpha M}{}_N T^{\beta P}{}_Q - \frac{(C^{-1})_{(-1)(-1)}}{k}\delta^M_N \delta^P_Q \, , \\
([[\widetilde{E}^M, \widetilde{F}_N], \widetilde{E}^P], \widetilde{F}_Q) &= \eta_{\alpha\beta}T^{\alpha M}{}_N T^{\beta P}{}_Q + \frac{2 - (C^{-1})_{(-1)(-1)}}{k}\delta^M_N \delta^P_Q \, ,
\end{aligned}
\tag{3.19}
$$

and similarly to eq. (3.13) we have

$$
\begin{aligned}
\frac{(C^{-1})^{(-1)(-1)}}{k} &= -\sum_i \frac{C_{0i}(C^{-1})^{i(-1)}}{k(C^{-1})^{0(-1)}} = -\sum_i d_i C_{i0} \frac{(C^{-1})^{i(-1)}}{(C^{-1})^{0(-1)}} \\
&= \sum_{ij} d_i C_{ij} \frac{(C^{-1})^{i(-1)}}{(C^{-1})^{0(-1)}} \frac{(C^{-1})^{j(-1)}}{(C^{-1})^{0(-1)}} = (\lambda, \lambda) \, .
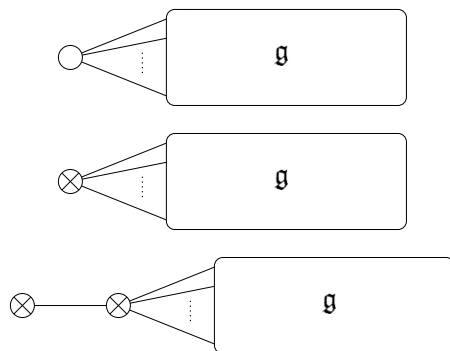\end{aligned}
\tag{3.20}
$$

Figure 1. Dynkin diagrams for the extended (super)algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. The box represents the Dynkin diagram of $\mathfrak{g}$.

Thus (3.17) holds also in the general case.

In figure 1 the Dynkin diagrams of the extended algebras $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are displayed, where the odd simple roots of zero length squared are represented by "gray" nodes. It is obvious from the figure that $\mathscr{B}$ is a subalgebra of $\mathscr{C}$. By performing an "odd reflection" with respect to the outermost gray node in the Dynkin diagram of $\mathscr{C}$ one can obtain an equivalent Dynkin diagram, where instead the embedding of $\mathscr{A}$ into $\mathscr{C}$ is manifest [1].

4 Ancillary transformations

We will now show how the formalism of ref. [1], reviewed and generalised in section 3, can be used to determine in which cases the ancillary transformations appear or not. As explained in the end of section 2, the ancillary transformations vanish if and only if the operator $\frac{1+\sigma}{2}(1 \otimes T^\alpha)Y_-$ does when it acts from the right on derivatives. With indices, using the results in the preceding section, it can be written

$$\left(\frac{1+\sigma}{2}(1 \otimes T^\alpha)Y_-\right)^{MN}{}_{PQ} = T^{\alpha(M}{}_R\delta^{N)}{}_S Y_-^{RS}{}_{PQ} = \frac{k}{2}T^{\alpha(M}{}_R\delta^{N)}{}_S\widetilde{g}^{RS}{}_{PQ}$$
$$= \frac{k}{2}([[T^\alpha, \widetilde{E}^{(M}], \widetilde{E}^{N)}], [\widetilde{F}_P, \widetilde{F}_Q]).$$
(4.1)

If this expression is nonzero, and the indices M and N have any part in $R(2\lambda)$ for any T^α, then it will not vanish when contracted with two derivatives, since the section constraint only removes the part in $R_2 = \vee^2 R(\lambda) \ominus R(2\lambda)$. Since indices M and N of the anti-commutator $[E^M, E^N]$ in $\mathscr{B}$ are automatically projected on R_2, a necessary and sufficient condition for the presence of ancillary transformations is the existence of an element $x \in \mathfrak{g}$ and a symmetric tensor X_{MN} such that $X_{MN}[E^M, E^N] = 0$ but $X_{MN}[\widetilde{E}^M, [\widetilde{E}^N, x]] \neq 0$, or equivalently, a set $\mathscr{S}$ of pairs (U, V) of elements $U, V \in \mathscr{B}_1$ such that

$$\sum_{(U,V)\in\mathscr{S}} [U, V] = 0 \qquad \text{and} \qquad \sum_{(U,V)\in\mathscr{S}} \left([\widetilde{U}, [\widetilde{V}, x]] + [\widetilde{V}, [\widetilde{U}, x]]\right) \neq 0 \qquad (4.2)$$

in $\mathscr{B}_2$ and $\mathscr{A}_2$, respectively.

Since $[e_0, e_0]$ is always zero in $\mathscr{B}$, it is sufficient to find an x such that $[\widetilde{e}_0, [\widetilde{e}_0, x]] \neq 0$. If there is a positive root α of $\mathfrak{g}$ such that $(\alpha_0{}^\vee, \alpha) < -1$, then this condition is satisfied by a corresponding root vector, $x = e_\alpha$, since

$$[\widetilde{f}_0, [\widetilde{e}_0, e_\alpha]] = -[h_0, e_\alpha] = -(\alpha_0{}^\vee, \alpha)e_\alpha \neq 0 \,, \tag{4.3}$$

implying that $[\widetilde{e}_0, e_\alpha] \neq 0$, and then

$$[\widetilde{f}_0, [\widetilde{e}_0, [\widetilde{e}_0, e_\alpha]]] = -[\widetilde{h}_0, [\widetilde{e}_0, e_\alpha]] - [\widetilde{e}_0, [\widetilde{h}_0, e_\alpha]] = (-2 - 2(\alpha_0{}^\vee, \alpha))[\widetilde{e}_0, e_\alpha] \neq 0 \,, \tag{4.4}$$

implying that $[\widetilde{e}_0, [\widetilde{e}_0, e_\alpha]] \neq 0$. To find a root α such that $(\alpha_0{}^\vee, \alpha) < -1$, we set $\alpha = \sum_i a_i \alpha_i$ so that

$$(\alpha_0{}^\vee, \alpha) = \sum_i (\alpha_0{}^\vee, \alpha_i)a_i = \sum_i k \frac{(\alpha_i, \alpha_i)}{2}(\alpha_i{}^\vee, \alpha_0)a_i = -\sum_i k \frac{(\alpha_i, \alpha_i)}{2}\lambda_i a_i \,. \tag{4.5}$$

If λ is not a fundamental weight (that is, if $\sum_i \lambda_i \geq 2$), then $-\sum_i k\frac{(\alpha_i,\alpha_i)}{2}\lambda_i a_i < -a_j$ for some j such that $\lambda_j \geq 1$, and we can choose $\alpha = \alpha_j$. Thus ancillary transformations appear in this case.

Let us now assume that λ is a fundamental weight, $\lambda_i = \delta_{ij}$ for some j. Then we have $-\sum_i k\frac{(\alpha_i,\alpha_i)}{2}\lambda_i a_i = -a_j$, and ancillary transformations will again appear if there is a root $\alpha = \sum_i a_i \alpha_i$ of $\mathfrak{g}$ with $a_j \geq 2$. This includes all infinite-dimensional cases. For finite-dimensional $\mathfrak{g}$, we can consider the highest root $\theta = \sum_i c_i \alpha_i$ where c_i are the Coxeter labels, and it follows that if the ancillary transformations are absent, then we must have $c_j = 1$. Conversely, if $c_j = 1$, then $[\widetilde{e}_0, [\widetilde{e}_0, e_\theta]] = 0$, and by the adjoint action of the raising operators e_i (which commute with e_θ) we get $X_{MN}[\widetilde{E}^M, [\widetilde{E}^N, e_\theta]] = 0$ for all symmetric tensors X_{MN} such that $X_{MN}[E^M, E^N] = 0$, since these correspond to an irreducible representation with lowest weight -2λ. Acting with the lowering operators f_i we can then step down again from e_θ to any $x \in \mathfrak{g}$ and show that $X_{MN}[\widetilde{E}^M, [\widetilde{E}^N, x]] = 0$. Thus the condition $c_j = 1$ is not only necessary for the absence of ancillary transformations, but also sufficient. (This can also be shown by studying the involved tensor product decompositions in all cases with $\lambda = \Lambda_j$ and $c_j = 1$. In many cases, $\mathscr{A}_2$ is zero- or one-dimensional, and it follows immediately that there is no room for ancillary transformations.)

We conclude that *the ancillary transformations vanish if and only if $\mathfrak{g}$ is finite-dimensional, λ is a fundamental weight Λ_j, and the corresponding Coxeter label c_j is equal to 1.*

Note that this can never happen at a node corresponding to a short root, since the Coxeter label of a short root α_i always is larger than 1. The complete list of situations where ancillary transformations are absent is thus:

- $\mathfrak{g} = A_r$, $\lambda = \Lambda_p$, $p = 1, \ldots, r$ (p-form representations);

- $\mathfrak{g} = B_r$, $\lambda = \Lambda_1$ (the vector representation);

- $\mathfrak{g} = C_r$, $\lambda = \Lambda_r$ (the symplectic-traceless r-form representation);

- $\mathfrak{g} = D_r$, $\lambda = \Lambda_1, \Lambda_{r-1}, \Lambda_r$ (the vector and spinor representations);

- $\mathfrak{g} = E_6$, $\lambda = \Lambda_1, \Lambda_5$ (the fundamental representations);

- $\mathfrak{g} = E_7$, $\lambda = \Lambda_1$ (the fundamental representation).

The commutator between a generalised diffeomorphism with parameter ξ and an ancillary transformation with parameter $\Sigma = \Sigma_\alpha T^\alpha$ is trivially given by an ancillary transformation with parameter $\mathcal{L}_\xi \Sigma$. Commuting two ancillary transformations does not manifestly give a new transformation of the form (2.25). It is however easy to give another argument for their closure. In at least one of the possible section-adapted gradings (see section 2), the highest root vectors in $\mathscr{A}_{-2}$ appear at degree $-p$, where $p > 0$. Thus the lower indices in $Y^{RS}_{-PQ} = \frac{k}{2} \widetilde{g}^{RS}_{PQ}$ correspond to degree $-p$ or lower. Since at the same time, by the section constraint, derivatives are nonzero only at degree 0, the degree of the parameter Σ is also $-p$ or lower, and the commutator will be of degree $-2p$ or lower. We will see examples of this in section 6.

5 Dynamics

We now want to investigate if it is possible to write a pseudo-action for fields in $G/K \times \mathbb{R}^+$. Due to the section constraint, we do not yet consider the "actions" obtained here and earlier as proper actions, unless integration is performed over some specified section. They provide, however, an efficient means of encoding the classical dynamics. In this section, we limit ourselves to transformations obtained by normalisation of the Y tensor corresponding to a long root, since all interesting cases are obtained this way (see section 6). We thus use

$$\sigma Y = -\eta_{\alpha\beta} T^\alpha \otimes T^\beta + (\lambda, \lambda) - 1 + \sigma. \tag{5.1}$$

The fields in the coset $G/K \times \mathbb{R}^+$ are parametrised by a generalised metric G_{MN}, which is at the same time a group element in $G \times \mathbb{R}^+$ and a symmetric matrix. The inverse metric will be denoted G^{MN}. Let G_{MN} transform as a tensor density with a weight $-2w$, which does not necessarily equal $-2((\lambda, \lambda) - 1)$, the canonical weight of a tensor with two lower indices. We are looking for a density $\mathcal{L}$, containing two derivatives, that is invariant under generalised diffeomorphisms, up to total derivatives. The weight w will be determined. The result should be compared to known cases.

The introduction of a metric implies a preferred involution on $\mathfrak{g}$, i.e., a local choice of embedding of the maximal compact subalgebra $\mathfrak{k}$. Acting on generators in $R(\lambda)$, it is

$$T^\alpha \mapsto -T^{\star\alpha} = -(GT^\alpha G^{-1})^t. \tag{5.2}$$

The local generators in $\mathfrak{k}$ and $\mathfrak{g} \ominus \mathfrak{k}$ are $T - T^\star$ and $T + T^\star$, respectively. Let

$$(G^{-1}\partial_M G)^N{}_P = \Pi_{M\alpha} T^{\alpha N}{}_P + \Pi_M \delta_P^N. \tag{5.3}$$

When checking the transformations under generalised diffeomorphisms, it is enough to check the inhomogeneous part, $\Delta_\xi X \equiv \delta_\xi X - \mathcal{L}_\xi X$. Using the form of the generalised diffeomorphism above (with the appropriate weight), we obtain

$$\begin{aligned}
\Delta_\xi \Pi_M &= -2w \partial_M \partial_N \xi^N, \\
\Delta_\xi \Pi_{\alpha M} &= \eta_{\alpha\beta} (T^\beta + T^{\star\beta})^N{}_P \partial_M \partial_N \xi^P.
\end{aligned} \tag{5.4}$$

It is often convenient to use the fact that the Killing metric is invariant under the involution, so that $\eta_{\alpha\beta}T^{\alpha} \otimes T^{\beta} = \eta_{\alpha\beta}T^{\star\alpha} \otimes T^{\star\beta}$, and that the adjoint index in $\Pi_{M\alpha}$ takes values in $\mathfrak{g} \ominus \mathfrak{k}$, i.e.,

$$\Pi_{M\alpha}T^{\alpha} = \Pi_{M\alpha}T^{\star\alpha} \,. \tag{5.5}$$

This immediately implies that the invariant tensors Y and Z fulfil

$$G_{MM'}G_{NN'}Z^{M'N'}{}_{P'Q'}G^{P'P}G^{Q'Q} = Z^{PQ}{}_{MN} \,, \tag{5.6}$$

and the corresponding identity for Y. A calculation then shows that the combination

$$\mathcal{L}_0 = \frac{1}{2}A - B - 2C - \frac{(\lambda,\lambda)}{(\lambda,\lambda) - \frac{1}{2}}D \,, \tag{5.7}$$

is necessary for the cancellation of various terms, where

$$\begin{aligned}
A &= G^{MN}\eta^{\alpha\beta}\Pi_{\alpha M}\Pi_{\beta N} \,, \\
B &= G^{PQ}T^{\alpha M}{}_P T^{\beta N}{}_Q \Pi_{\alpha N}\Pi_{\beta M} \,, \\
C &= (T^{\alpha}G^{-1})^{MN}\Pi_{\alpha M}\Pi_N \,, \\
D &= G^{MN}\Pi_M\Pi_N \,.
\end{aligned} \tag{5.8}$$

Note that the terms A and B, which are the ones not containing the scale variation Π_M, have a universal relative coefficient. In previous formulations, based on traces in the fundamental rather than using the Killing metric, the relative coefficient has been determined on a case by case basis.

The remaining inhomogeneous transformation, modulo total derivatives, is

$$\Delta_\xi \mathcal{L}_0 = -2S^{\alpha MN}{}_{PQ}G^{PS}\Pi_{S\alpha}\partial_M\partial_N\xi^Q \,, \tag{5.9}$$

where S is the tensor of eq. (2.25). This shows that $\mathcal{L}_0$ gives the complete dynamics in all cases where ancillary transformations are absent.

By cancellations of inhomogeneous transformations, the weight w is also determined to be $w = (\lambda,\lambda) - \frac{1}{2}$. Note that this implies that the total weight of $\mathcal{L}$ is

$$-2((\lambda,\lambda) - 1) + 2\left((\lambda,\lambda) - \frac{1}{2}\right) = 1 \,, \tag{5.10}$$

where the first term comes from the two derivatives and the second one from an inverse metric. This is the correct weight for partial integration, in the sense that a divergence formed with a naked derivative is covariant. Namely, consider a vector V^M with weight w^+, Using the section constraint, it is straightforward to show that

$$\begin{aligned}
\partial_M \mathscr{L}_\xi V^M &= \xi^N\partial_N\partial_M V^M + (w^+ - (\lambda,\lambda) + 1)\partial_N\xi^N\partial_M V^M \\
&\quad + (w^+ - (\lambda,\lambda))\partial_M\partial_N\xi^N V^M \,.
\end{aligned} \tag{5.11}$$

If $w^+ = (\lambda,\lambda)$, the $\partial^2\xi$ term vanishes, and this equals $\mathscr{L}_\xi\partial_M V^M$. The divergence $\partial_M V^M$ then is a scalar density of weight $w^+ - (\lambda,\lambda) + 1 = 1$.

We do not know if it is possible in general to add terms to the Lagrangian in order to cancel the remainder in eq. (5.9). It was done for E_8 in ref. [45]. That construction can be extended to the class of models where $R(\lambda)$ is the adjoint of a finite-dimensional Lie algebra (finite dimension is needed for the adjoint to be a highest weight representation). Such extended geometries are relevant in connection with compactification to 3 dimensions, and provide geometrisations of extensions of Ehlers symmetry.

In the adjoint cases, using the generic form of the Y tensor with $T^{\alpha\beta}{}_\gamma = -f^{\alpha\beta}{}_\gamma$, the symmetry (5.5) of Π can be expressed as follows. Let $G_{\alpha\beta} = \phi \widetilde{G}_{\alpha\beta}$, where $\widetilde{G}^{-1} d\widetilde{G}$ takes values in $\mathfrak{g}$ (i.e., $\det \widetilde{G} = 1$), and ϕ is a scalar density of the same weight as G (i.e., $\phi = (\det G)^{1/\dim \mathfrak{g}}$). Then,

$$\widetilde{G}^{\alpha\gamma} \Pi_{\gamma\beta} = \eta^{\alpha\gamma} \Pi_{\gamma\beta} \,. \tag{5.12}$$

The remainder term of eq. (5.9) can in these cases be simplified using the section constraint of the form

$$\eta_{\kappa\lambda} f^{\kappa\gamma}{}_\alpha f^{\lambda\delta}{}_\beta \partial_{(\gamma} \partial_{\delta)} = 2\partial_{(\alpha} \partial_{\beta)} \,. \tag{5.13}$$

Using the Jacobi identity in the operator S^α then gives

$$S_\alpha{}^{\delta\epsilon}{}_{\beta\gamma} \partial_{(\delta} \partial_{\epsilon)} = f^\delta{}_{\beta\gamma} \partial_{(\alpha} \partial_{\delta)} \,, \tag{5.14}$$

where the two derivatives act on the same (suppressed) parameter. The remainder term is

$$\Delta_\xi \mathcal{L}_0 = -4 f^\beta{}_{\gamma\delta} G^{\gamma\epsilon} \Pi_\epsilon{}^\alpha \partial_\alpha \partial_\beta \xi^\delta \,, \tag{5.15}$$

where, as earlier, indices (except on G) are raised and lowered with η, and only the presence of G or its inverse are indicated explicitly. It is cancelled by the variation of

$$\mathcal{L}_1 = G^{\alpha\beta} \eta^{\gamma\delta} \Pi_{\alpha\gamma} \Pi_{\delta\beta} \,. \tag{5.16}$$

This gives a complete (local) description of the dynamics in these cases.

The characterisation of the ancillary transformations in cases where $\mathfrak{g}$ is an affine algebra [2] also relied on a specific rewriting of them, in those cases using the coset Virasoro generator L_{-1}^{coset}. If some $\mathcal{L}_1$ is to be formed, it seems likely that it will rely on that rewriting. We have no further insight in how to obtain an invariant Lagrangian in other infinite-dimensional, e.g. hyperbolic, algebras.

A comment on unimodular versus non-unimodular generalised metrics: in e.g. refs [43–45], unimodular generalised metrics are used. The density is provided by the "external" metric, and any invariant expression will contain derivatives of (at least the determinant of) the external metric. This scale can, if one wants, be absorbed in the definition of a non-unimodular metric as above. In the present context, we prefer to include the scale in the generalised metric, since we are in a general situation where we do not want to commit to a specific number of "external" coordinates. This applies as long as the generalised metric, as defined here, carries a non-trivial weight, i.e., as long as $(\lambda, \lambda) \neq \frac{1}{2}$.
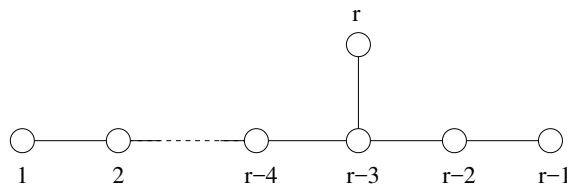
Figure 2. The Dynkin diagram for E_r.

6 Examples

We will take the opportunity to give some examples, some connecting to known models, and some illustrating how our formalism goes beyond already investigated cases.

The first example is the E series, with $\lambda = \Lambda_1$. The series contains the well-known exceptional geometries up to E_8, and continues with E_9, where generalised diffeomorphisms have been constructed [2] but the dynamics remains to be given. The cases of E_{10} and E_{11} are of special interest.

The highest irreducible module in $\wedge^2 R(\Lambda_1)$ is $R(2\Lambda_1 - \alpha_1) = R(\Lambda_2)$. It is obvious that no other highest weight module occurs at the same height. The next to highest one appears (for $r \geq 7$) at highest weight $2\Lambda_1 - \beta$, where $\beta = \sum_{i=1}^{n} b_i \alpha_i$, $b_i = (2, \ldots, 2, 3, 4, 5, 6, 4, 2, 3)$ is the lowest root at level 2 with respect to node 1 (for $r \geq 8$). The state

$$||2\Lambda_1 - \beta\rangle\rangle = |\Lambda_1\rangle \wedge e_{-\alpha_1} e_{-\beta+\alpha_1} |\Lambda_1\rangle - e_{-\alpha_1} |\Lambda_1\rangle \wedge e_{-\beta+\alpha_1} |\Lambda_1\rangle \tag{6.1}$$

in $\wedge^2 R(\Lambda_1)$ consists of two terms which are both annihilated by all e_i, $i = 2, \ldots, r$. The relative coefficient assures that also $e_1 ||2\Lambda_1 - \beta\rangle\rangle = 0$. This is the highest weight state in the next-to-highest module $R(2\Lambda_1 - \beta) \in \wedge^2 R(\Lambda_1)$. (This may be refined by stating the actual multiplicities of $2\Lambda_1 - \beta$ in $\wedge^2 R(\Lambda_1)$ and in $R(\Lambda_2)$, and show that they differ by 1, but it is not necessary for demonstrating that $R(2\Lambda_1 - \beta) \subset \wedge^2 R(\Lambda_1)$.)

This means that ancillary transformations begin at degree -3 in the M-theory grading, and at degree -4 in the type IIB grading, since $(\beta, \Lambda_r) = 3$ and $(\beta, \Lambda_{r-2}) = 4$. This illustrates how the local transformations close for e.g. E_{10}, although we lack an action.

As a second example, take $\mathfrak{g} = A_r$, $\lambda = \Lambda_p$. The case $\lambda = \Lambda_2$ was described in refs. [50, 57]. We only need to consider $p \leq [\frac{r+1}{2}]$, higher p are related to lower by an outer automorphism. There are (generically) two possible sections, one $(p+1)$-dimensional ($p \geq 2$) and one $(r - p + 2)$-dimensional, arising from following a gravity line to the left and to the right. If we decide going to the right (the $(r - p + 2)$-dimensional section), all cases are covered by also including higher p. There is an R-symmetry A_{p-2} when $p \geq 3$.

The field content is obtained by inspecting the grading of the adjoint with respect to node $p - 1$, i.e., its branching under $A_{p-2} \oplus A_{r-p+1} \subset A_r$. In addition to gravity, the fields are scalars in the R-symmetry coset $\mathrm{SL}(p - 1)/\mathrm{SO}(p - 1)$ and $(p - 1)$ $(r + 1 - p)$-form potentials. The Lagrangians[2] are given by eq. (5.7).

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[2]We note that the case $r = 3$, $p = 2$ was excluded in the analysis of ref. [57]. For these values, one has $(\lambda, \lambda) = 1$, implying that the canonical weight of a vector is 0. Therefore, if one starts from a generalised metric transforming as a tensor, it is not possible to form a density with non-zero weight for integration. We do not experience this problem, since we start from a generalised metric which carries a non-tensorial weight.
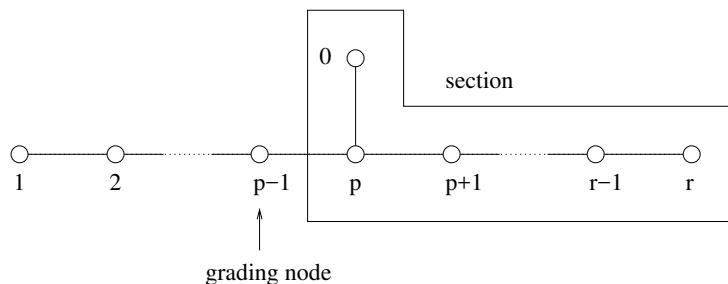
Figure 3. The Dynkin diagram of $\mathscr{A}$ for $\mathfrak{g} = A_r$, $\lambda = \Lambda_p$, with the subdiagram corresponding to one of the two sections.

Another example where ancillary transformations are absent is $\mathfrak{g} = C_r$, $\lambda = \Lambda_r$. The section is 2-dimensional. The R-symmetry, obtained by deleting node $r - 1$ and including the affine node, is C_{r-1}. The field content in the coset is, apart from 2-dimensional gravity, scalars in the coset $\mathrm{Sp}(2(r-1))/\mathrm{SU}(r-1)$ and $2(r-1)$ 1-form potentials in the fundamental of $\mathrm{Sp}(2(r-1))$.

A word on solutions to the section constraint obtained as a gravity line of short roots. Among such cases, there is no situation where ancillary transformations are absent. The simplest example is when $\mathfrak{g} = C_r$ and $\lambda = \Lambda_1$, so that $R(\lambda)$ is the fundamental representation. There is an r-dimensional section. This section is at degree 0 in the grading with respect to node r, which is the decomposition into $\mathrm{GL}(r)$ modules. Since $c_r = 1$, this is a 3-grading. Ancillary transformations will appear already at degree -1, and remove everything except gravity from the coset $\mathrm{Sp}(2r)/\mathrm{SU}(r)$. This seems uninteresting.

The cases where the coordinate module is the adjoint of a finite-dimensional Lie algebra are the only ones where ancillary transformations are present and a Lagrangian is known. Given a choice of section corresponding to a gravity line, both the gauge parameters and the fields are deduced from the corresponding grading. This grading is defined by the node or nodes outside of, but connected to, the gravity line. As in section 2, call this node (or one of them) number j. The relevant next-to-highest module in $\wedge^2 R(\lambda)$, with $\lambda = \theta$, is $R(\theta)$ itself. Ancillary transformations appear at degree $-c_j$, where c_j is the degree of θ in the grading, i.e., the Coxeter label of the grading node j. This is also the lowest degree appearing in the adjoint representation, which implies that ancillary transformations commute. The ancillary transformations are then always a 1-form (with respect to the section), which is equivalent to a section-constrained adjoint element. The corresponding element in a (linearised) coset can always be shifted away by an ancillary transformation. The parameters of generalised diffeomorphisms at degree 0 and lower become irrelevant; such transformations can always be absorbed in an ancillary transformation. The pattern from E_8 exceptional geometry [45, 46] is repeated. The precise content of "matter" fields depends on details.

For example, the E_7 theory with the 7-dimensional section (the same gravity line as ordinary exceptional E_7 geometry, but with the coordinate module in the opposite end) contains a 4-form gauge potential in addition to gravity. Since the grading of the adjoint with respect to the exceptional node is a 5-grading, there are no other physical fields.

7 Conclusions

We have presented a unified formalism for dealing with extended geometry, only relying on the choice of structure group and coordinate module. The treatment is so far local.

Inclusion of an "external" space with ordinary diffeomorphisms has intentionally been left out, since the actual field content depends on other input, e.g. supersymmetry or a knowledge of the duality group present in a certain compactification. Other fields than the ones described here (for dimension of the external space $D \geq 4$) would then be introduced through a tensor hierarchy in cases where ancillary transformations are absent. The techniques are straightforward.

The generalisation of the formalism to include supergeometries [58] should be straightforward. The Dynkin diagram is replaced by the Dynkin diagram of a superalgebra, which plays the rôle of structure algebra for the generalised superdiffeomorphisms. A super-section constraint restrict the super-section to a "supergravity line" corresponding to the structure superalgebra $\mathfrak{g}l(d|s)$ of ordinary supergeometry. However, to be concrete about field content and symmetries, information from the representation theory of Kac-Moody superalgebras will be needed, and not much seems to be known even concerning the (infinite-dimensional) super-extensions of E_r. Neither is it clear how dynamics should be formulated.

Many questions remain to be investigated. Among the most pressing ones is to determine the dynamics, if not in all cases, at least in some important ones, such as affine and hyperbolic cases. It is also desirable to obtain a better understanding of finite transformations, which in double field theory are reasonably well understood [22, 24–26], but which have been elusive in the exceptional cases, for fundamental or technical reasons. A generic treatment is desirable.

In section 5, we noted that there seems to be a singularity for $(\lambda, \lambda) = \frac{1}{2}$. In general, this is not serious, but only a sign that the generalised metric (defined as a density, as in section 5) carries no weight. If some external space is present, a density may be introduced that compensates, and (part of) an action may be written. A peculiar observation is that this happens precisely for the case E_{11}, where no external space should be present. The interpretation is not clear to us, but may point towards a difficulty with formulating an action in that case.

The treatment in the present paper has its focus on transformations and their closure, together with invariant dynamics in some cases. Closure of the algebra of generalised diffeomorphisms (i.e., absence of ancillary transformations) ensures covariance [46], but when ancillary transformations are present tensors cannot be defined unless the transformations are made field-dependent. This was done for E_8 in ref. [46], and can easily be extended to all the finite-dimensional adjoint cases with the methods of the present paper. We do not know if this is possible in general. Without a concept of covariance, it is not possible to define truly geometric entities like torsion, curvature etc., which would be desirable. Again, the most urgent cases are E_r, $r \geq 9$.

Our formalism is based on the weight space decomposition of modules of the Lie algebra $\mathfrak{g}$, which provides a real basis when $\mathfrak{g}$ is of split real form. This is the real form typically used in double and exceptional field theory. We have not tried to analyse in detail what happens

for other choices of real form, like the orthogonal groups in ref. [49]. For a real representation there should be no problems with the definition of the generalised diffeomorphisms and the section constraint, which are built using real invariant tensors. The solution of the section constraint in terms of a real gravity line seems to demand that the real form is obtained from the split one through redefinitions of generators not affecting the $\mathfrak{g}l(d)$ subalgebra. The dynamics must be completely reconsidered when the compact subgroup is changed.

The fermionic extensions of the Kac-Moody algebra $\mathfrak{g}$ used in the present work contain $\mathfrak{g}$ itself at level 0, the coordinate module $R(\lambda) = R_1$ at level -1, and the section constraint module R_2 at level -2. They are contragredient (in particular, Borcherds superalgebras), which means that the modules at the positive levels are dual to those at the corresponding negative levels. Alternative extensions, essentially agreeing at positive levels, but with different (larger) modules at negative levels, are the tensor hierarchy algebras [59, 60]. These algebras seem to have a deeper connection to extended geometry. A peculiar aspect of the tensor hierarchy algebras corresponding to infinite-dimensional extended geometries is the appearance of certain additional elements at level 0, i.e., together with the algebra $\mathfrak{g}$. They have already been demonstrated to be important [2, 61]. It is quite possible that such transformations should be considered part of the structure algebra, when generalised diffeomorphisms are constructed, and there might be a subtle connection between these elements and the ancillary transformations. Issues like these have been ignored in the present work, but seem to point towards interesting potential development.

Acknowledgments

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