Computation of Low-Complexity Control-Invariant Sets for Systems with Uncertain Parameter Dependence

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Abstract

This paper proposes two new algorithms to compute low-complexity robust control-invariant (LC-RCI) sets along with associated static linear state-feedback laws. The RCI set is assumed to be symmetric around the origin and described by the same number of affine inequalities as twice the dimension of the state vector. The proposed algorithms are applicable to systems with rational parameter dependence, which cannot be handled by the existing algorithms in the literature without introducing additional conservatism. The state and control input constraints are reformulated as simple scalar inequalities, while the invariance condition is relaxed into two alternative sets of (standard and dilated) linear matrix inequality (LMI) conditions. Based on the tractable formulations of the system constraints and invariance condition, both one-step and iterative algorithms are developed for the computation of LC-RCI sets of desirably large/small volumes. The iterative algorithms are constructed in a way to ensure recursive feasibility and convergence to a stationary point. The potential benefits of the proposed algorithms are demonstrated with reference to the existing literature via an illustrative example.

Key words: Linear matrix inequalities (LMI); Invariant set; Semi-definite program, Linear fractional transformation (LFT).

1 Introduction

Robust control-invariant (RCI) sets are generally used for closed-loop stability analysis and constrained control of dynamical systems [5, 6]. They are also employed for the prediction of constraint violation in reference governor systems for constrained tracking [9]. In schemes like model predictive control (MPC), RCI sets are used to define the terminal constraint in a way to guarantee the persistent feasibility of the underlying optimization problem [14]. The size of the feasibility set and the complexity of the resulting MPC controller are directly related to the volume and the number of inequalities defining the RCI set [15]. Thus, for such applications it is desirable to have RCI sets with the least possible level of complexity and with a sufficiently large volume. The RCI set with the minimum volume also has an important role in robust control. For example in [4, 12], it is used to generate an invariant tube around a nominal trajectory. In such an application, the goal would be too keep the actual trajectory of the system very close to the nominal trajectory. Calculating the minimum volume RCI set has always been a challenge, since this set is, in general, not finitely determined. Computation is possible in finite number of steps only if the system is nilpotent. Hence, for systems which are not nilpotent, an approximate minimum volume set can be calculated; see [19] and the references therein.

Most of the initial methods to find an RCI set for a system were based on the geometric approach, a detailed survey of which can be found in [5, 6]. A major drawback of this class of algorithms is that they do not guarantee termination in a finite number of steps. Moreover, there is no control on the representational complexity of the set resulting from the algorithm. Thus, many approaches proposed later were directed towards reducing the computational complexity of the algorithms that are used to calculate the maximal/minimal RCI set or finding an approximation of the set with a relatively less complex representation; see [16, 18, 19] and the references therein.

There are two common RCI set descriptions that would facilitate computation in a finite number of steps: ellipsoidal and low-complexity polytopic sets with symmetry around the origin. The main advantage of using el-
lipsoidal sets is that only one quadratic inequality is required for the set representation and that the set can be computed by using a semi-definite program (SDP), i.e. optimization under linear matrix inequality (LMI) conditions [1, 26]. But due to the quadratic constraints, an ellipsoidal RCI set would not be suitable for use in online control strategies like MPC [7, 15]. Indeed, the overall optimization problem would then become an SDP, which is computationally more demanding (for consideration in most online applications) if compared to a linear program (LP) or quadratic program (QP). To overcome this drawback, many recent works shifted the focus to the direction of low-complexity robust control-invariant (LC-RCI) set computation. In [3], an approach is proposed to calculate a control-invariant set of desired complexity in terms of the number of vertices. The approach is developed in this work solely for systems without any parametric uncertainty. Moreover, this is a somewhat indirect approach to reduce complexity in MPC schemes, which is influenced directly by the number of inequalities describing the RCI set.

A recent body of works deal with polytopic LC-RCI sets identified by their edges [7, 8, 23]. In all of these works, the considered LC-RCI set is symmetric around the origin and the number of the associated affine inequalities is equal to twice the state dimension. In [7, 8], an iterative approach is proposed to find an LC-RCI set for an uncertain parameter-dependent system, with the parameter vector assumed to lie in a known polytopic region. An SDP is solved to find an RCI set with a desirably large volume together with an associated state-feedback gain. Due to the polytopic uncertainty description, the number of LMIs for the invariance condition grows exponentially. In [23], a system with norm-bounded uncertainty is considered and the parameter-dependent conditions resulting from the state, control input constraints and invariance condition were relaxed by using the D-G scales in the way commonly employed within the robust control literature. The block-diagonal, norm-bounded uncertainty characterization is somewhat restrictive as it disregards any possible correlation between different parameters. Moreover, relaxation with block-diagonal D-G scales is potentially conservative for multiple blocks if compared to full-block scaling matrices. Another limitation of [23] is that the derivations are restricted to systems with affine parameter dependence. The recent work [13] facilitates the adjustment of the level of complexity of the computed RCI set, but is not applicable to uncertain systems. In [24], a one step linear program (LP) is solved to find a minimal RCI set of desired complexity for a closed-loop system.

In this paper, the considered LC-RCI set computation problem for a broader class of uncertain systems if compared to [23] and employ more advanced relaxation schemes to arrive at tractable problem formulations. The considered systems are assumed to depend rationally on an uncertain parameter vector and this dependence is represented in the form of a specific linear fractional transformation (LFT). The uncertain parameter vector is allowed to vary arbitrarily within a known polytopic region. Our derivations are based on a state transformation that maps the invariant set into a hyper-rectangular region. This facilitates an equivalent expression of state and control input constraints as simple scalar inequalities. On the other hand, the invariance condition is relaxed into two alternative sets of (standard and dilated) matrix inequality conditions with rational parameter dependence. Sufficient conditions are then derived in the form of finitely many LMIs by an application of the full-block S-procedure [20] followed by a Pólya-type relaxation inherited from [11]. Tractable formulations of the system constraints and invariance condition are then used to develop one-step as well as iterative algorithms for the computation of LC-RCI sets of desirably large/small volumes. The proposed iterative algorithms rely on novel methods for iterative maximization/minimization of the determinant of a non-symmetric matrix under LMI conditions. Volume increase is achieved by determinant maximization problems referred to as generalized SPDs in [25], while volume decrease is obtained by standard SDPs, i.e. linear objective minimization under LMI conditions. It is demonstrated by an illustrative example that the proposed algorithms deliver improved results if compared to [23], especially thanks to the use of full-block multipliers.

The paper is organized as follows. In Section 2, we first reformulate the full-block S-procedure and present the Pólya-type relaxation that will be used to express the main results. A precise statement of the LC-RCI set computation problem is given in Section 3. Section 4 is focused towards representing the system constraints and invariance condition in the form of tractable conditions. In Section 5, we present the one-step and the iterative algorithms that can be used to calculate LC-RCI sets with desirably large/small volumes. The effectiveness of the proposed algorithms is demonstrated via an illustrative example in Section 6. The paper is then concluded with a summary of our contributions and a brief discussion about future research directions.

## 2 Preliminaries

In this section, we recall some existing results about parameter-dependent matrix inequalities and ways to obtain tractable sufficient conditions thereof in the form of LMIs.

### 2.1 Full-Block S-Procedure

In order to derive the key results of our paper, we formulate a lemma which is in fact a simple corollary of the following explicit version of the full-block S-procedure:
Lemma 1. (Full Block S-Procedure [20])
Consider the following statements for real-valued matrices $M$ and $\Psi$ with $\Psi_{22} \leq 0$:

i. 
$$
\Psi \star \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} = M_{22} + M_{21} \Psi (I - M_{11} \Psi)^{-1} M_{12}
$$

is well-posed, i.e. $I - M_{11} \Psi$ is non-singular, $\forall \Psi \in \Psi$.

ii. 
$$
\begin{bmatrix}
I \\
\Psi \star M
\end{bmatrix}
\begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{T12} & Y_{22}
\end{bmatrix}
\begin{bmatrix}
I \\
\Psi \star M
\end{bmatrix} = 0,
\forall \Psi \in \Psi.
$$

These two statements are true $\forall \Psi \in \Psi$, if there exists a scaling matrix $P$ satisfying the following conditions:

$$
\begin{bmatrix}
\Psi^T & Q & S \\
I & \Psi \star M & P
\end{bmatrix}
\begin{bmatrix}
I \\
0
\end{bmatrix}
\begin{bmatrix}
\Psi \\
0
\end{bmatrix} \leq 0,
\forall \Psi \in \Psi,
$$

(3)

$$
\begin{bmatrix}
I & 0 \\
M_{11} & M_{12} \\
0 & I \\
M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
Q & S \\
S^T & R \\
\Psi \star M
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
M_{11} & M_{12} \\
0 & I \\
M_{21} & M_{22}
\end{bmatrix} \geq 0.
$$

(4)

The converse statement also holds true if $\Psi$ is compact (i.e. closed and bounded).

In the usually intended applications of this lemma, $\Psi$ is a matrix of uncertain parameters and $\Psi$ represents a known set that describes the uncertainty. It is common to consider LFTs as in (2) with block-diagonal $\Psi$, although the lemma is valid without any structural restriction on $\Psi$. We now formulate a dual version of the full block S-procedure expressed for a particular $\Psi$. This will serve as the key to our major derivations in this paper.

Lemma 2. The LFT $\Delta \star \Psi$ is well-posed $\forall \Delta \in \Delta$ and

$$
\text{He}[\Delta \star \Psi] \triangleq \Delta \star \Psi + (\Delta \star \Psi)^T \geq 0,
\forall \Delta \in \Delta
$$

holds if there exists a scaling matrix $P$ satisfying the following conditions:

$$
\begin{bmatrix}
\Delta^T \\
I
\end{bmatrix}
\begin{bmatrix}
Q & S \\
S^T & R \\
\Psi \star M & P
\end{bmatrix}
\begin{bmatrix}
\Delta^T \\
I
\end{bmatrix} \leq 0,
\forall \Delta \in \Delta,
$$

(6)

Throughout the paper, $*$ will be used to represent matrix entries that are uniquely identifiable from symmetry.

Proof. This result is a direct corollary of Lemma 1 with $\Psi = \Delta^T$ and $M = \Psi^T$. In order to see this, we first employ the push-through rule to recall an alternative expression of the LFT:

$$
\Delta \star \Psi = \Psi_{22} + \Psi_{21} (I - \Delta \Psi_{11})^{-1} \Delta \Psi_{12}.
$$

It then follows with $\Psi = \Delta^T$ and $M_{11} = \Psi_{11}^T$, $i, j = 1, 2$ that

$$
(\Delta \star \Psi)^T = \Psi_{22}^T + \Psi_{12}^T \Delta^T (I - \Psi_{11}^T \Delta^{-1}) \Psi_{11}^{-1} = \Psi \star M.
$$

This allows to express condition (5) equivalently as

$$
\text{He}[\Delta \star \Psi] = \begin{bmatrix}
I \\
\Psi \star M
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
I \\
\Psi \star M
\end{bmatrix} \geq 0,
\forall \Delta \in \Delta.
$$

(7)

This condition is amenable to an application of Lemma 1 with $\Psi = \Delta^T$, $M = \Psi^T$ and the indicated $\Psi$. The statement of Lemma 2 is thus established after some standard matrix manipulations to express (4) equivalently as in (7).

Lemma 2 allows us to replace the matrix inequality (5) that has rational parameter dependence with conditions consisting of two LMI's which are easier to deal with. Indeed (7) is a single (i.e. parameter-independent) matrix inequality, while (6) has only quadratic parameter dependence. Nevertheless, (6) has to be satisfied for all possible values of $\Delta \in \Delta$, which results in infinitely many LMI conditions. Hence, in order to obtain tractable sufficient conditions for (5), we need to employ some relaxation methods to derive finitely many LMI conditions that ensure (6). There are various relaxation methods in the literature serving this purpose depending upon the type of the uncertainty region. For polytopic regions, one can consider the so-called convex-hull relaxation or Pólya’s method, while for regions described by polynomial inequalities one might use the sum-of-squares (SOS) approach (see e.g. [11,21,22] and the references therein). The recent work [23] relies on the so-called D-G scales for norm-bounded parametric uncertainties, which correspond to a very specific choice of $P$ in our setting. In this paper, we opt for an approach based on Pólya’s method, which offers significant reduction in conservatism together with the use of a full-block $P$ matrix.
2.2 A Relaxation based on Pólya’s Method

In this subsection, we provide a set of sufficient LMI conditions for (6) assuming that \( \Delta \) is a polytopic region. These conditions are adapted from [11], where the underlying method is referred to as the zeroth-order Pólya relaxation. Let us first introduce the set of all multipliers that satisfy (6) as

\[
P \triangleq \left\{ P : (6) \right\}. \tag{8}
\]

Our goal is to characterize an inner approximation of this set in terms of finitely many LMI conditions. We consider a polytopic uncertainty set that is expressed as the convex-hull of a known set of finitely many given matrices \( \Delta_c = \{ \Delta_1, \ldots, \Delta_n \} \):

\[
\Delta = \text{conv}(\Delta_c) \triangleq \left\{ \sum_{j=1}^{n} \alpha_j \Delta_j : \sum_{j=1}^{n} \alpha_j = 1, \alpha_j \in [0, 1] \right\}. \tag{9}
\]

For the sake of concise expressions in the sequel, we now introduce

\[
\Omega_{ij}(P) \triangleq \left( \Delta^T \right)^T P \left( \Delta^T \right), i, j = 1, \ldots, \eta. \tag{10}
\]

The idea behind the Pólya approach is to express the left hand side of (6) as a polynomial matrix that is homogeneous in \( \alpha \). In the simplest case, we can achieve this with a polynomial order of two:

\[
\sum_{j=1}^{n} \alpha_j^2 \Omega_{jj} + \sum_{j=1}^{n} \sum_{i=j+1}^{n} \alpha_j \alpha_i \left( \Omega_{ji} + \Omega_{ij}^T \right) \succ 0. \tag{11}
\]

With \( \alpha_j \in [0, 1] \), we can now state a set of sufficient LMI conditions for (6) as follows:

\[
\Omega_{jj}(P) \preceq 0, j = 1, \ldots, \eta \tag{12}
\]

\[
\text{He} \{ \Omega_{ij}(P) \} \preceq 0, j = 1, \ldots, \eta; i = j + 1, \ldots, \eta \tag{13}
\]

As justified in [11], these conditions are potentially less conservative than the convex-hull approach in which the upper LMI in (12) is accompanied by \( Q \succ 0 \).

We have thus obtained an inner approximation to the original set of multipliers \( P \) as follows:

\[
P_{\text{pol}} \triangleq \left\{ P : (12) \right\} \subseteq P. \tag{13}
\]

Pólya relaxation can be realized by first multiplying (11) with \( \left( \sum_{j=1}^{n} \alpha_j \right)^{n_{\text{pol}}} = 1 \). With the same reasoning as before, sufficient LMI conditions would then be obtained by enforcing that all coefficient matrices are non-negative definite. With increased \( n_{\text{pol}} \) choices, we could potentially reduce conservatism at the cost of an exponential increase in the number of LMI conditions.

3 Problem Statement

Consider a discrete-time linear parameter-varying (LPV) system whose dynamics are described by

\[
x(k+1) = A(\Delta(k))x(k) + B(\Delta(k))u(k) + E(\Delta(k))w(k), \tag{14}
\]

where \( x, u \) and \( w \) are the state, control input and the (additive) disturbance vectors respectively. The system matrices depend rationally on an uncertain (possibly) time-varying parameter matrix \( \Delta \) that satisfies

\[
\Delta(k) \in \Delta, \forall k \geq 0, \tag{15}
\]

where \( \Delta \) is a known compact uncertainty set. In this paper, we restrict our interest to polytopic regions described as in (9). The dependence of the system matrices on the parameter matrix is expressed compactly with an LFT of the form

\[
\begin{bmatrix}
A(\Delta) & B(\Delta) & E(\Delta)
\end{bmatrix}
= \begin{bmatrix}
A & B & E
\end{bmatrix}
+ B_p \Delta(I - D_p \Delta)^{-1} \begin{bmatrix}
A_d & B_d & E_d
\end{bmatrix}, \tag{16}
\]

where \( A, B, E, A_d, B_d, E_d, B_p \) and \( D_p \) are known system matrices of appropriate dimensions. Note that affine parameter dependence, as considered in e.g. [23], is just a special case that corresponds to \( D_p = 0 \) in our system model. There are systematic methods for obtaining an LFT representation as in (16) (see e.g. [22]; Chapter 6). These methods would usually lead to structured \( \Delta \) matrices (e.g. block-diagonal). Nevertheless, the main results of our paper do not require a particular structure in \( \Delta \) as they are derived from Lemma 2. For a meaningful problem formulation, \( (I - D_p \Delta)^{-1} \) needs to be well-defined for all \( \Delta \in \Delta \), which would also be guaranteed thanks to Lemma 2.

The system (16) is subject to a number of state and control input constraints. Moreover, the additive disturbances are assumed to be bounded. All these constraints are expressed concisely by introducing the sets

\[
\mathcal{X} = \left\{ x \in \mathbb{R}^m : Hx \leq 1 \right\}, \tag{16}
\]

\[
\mathcal{U} = \left\{ u \in \mathbb{R}^m : Gu \leq 1 \right\}, \tag{17}
\]

\[
\mathcal{W} = \left\{ w \in \mathbb{R}^d : |Dw| \leq 1 \right\},
\]
where \( H \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times m}, D \in \mathbb{R}^{n \times d} \) are given matrices and \( 1 \) represents the vector of ones of compatible dimension. We emphasize that the inequalities in (17) are to be interpreted element-wise. It is important to observe that \( X \) and \( U \) are allowed to be non-symmetric, whereas \( \mathcal{W} \) is assumed to be symmetric around origin.

The synthesis goal in this paper is to ensure invariance in an LC-RCI set via a static state feedback controller as
\[
    u(k) = K x(k),
\]
where \( K \in \mathbb{R}^{m \times n} \) is the feedback gain matrix to be found. The resulting closed-loop dynamics can be expressed as
\[
    x^+ = (A(\Delta) + B(\Delta)K)x + E(\Delta)w,
\]
where the \( k \) dependence is dropped and \( x(k+1) \) is represented by \( x^+ \) for simplicity. Along the lines of the recent works \([7, 8, 23]\), we assume the LC-RCI set to be symmetric with respect to origin and to be described as
\[
    \mathcal{C} = \{ x \in \mathbb{R}^n : -1 \leq Mx \leq 1 \},
\]
where \( M \in \mathbb{R}^{n \times n} \) is a square matrix to be found. Note that, once the set is constrained to be symmetric with respect to origin, the normalization of lower/upper bounds to \( \pm 1 \) is no loss of generality.

Now we are ready to formulate the problem for finding the LC-RCI set. Since the state and control input constraints are to be satisfied by the elements of \( \mathcal{C} \), this implies \( \mathcal{C} \subseteq \mathcal{X} \) and \( KC \subseteq \mathcal{U} \), which can be expressed as
\[
    x \in \mathcal{C} \Rightarrow x \in \mathcal{X}, \quad x \in \mathcal{C} \Rightarrow u = Kx \in \mathcal{U}.
\]

Let us now recall the definition of robust invariance and express the corresponding condition in a similar way. A subset \( \mathcal{C} \) of the state-space is referred to as robustly invariant for the system (19), if the following condition is satisfied:
\[
    x \in \mathcal{C} \Rightarrow x^+ \in \mathcal{C}, \quad \forall w \in \mathcal{W}, \quad \forall \Delta \in \Delta. \tag{23}
\]

The mathematical formulation of the problem for finding an RCI set together with a static state-feedback gain can now be stated as follows:

**Problem 1.** Given the discrete-time system in (14) subject to the constraints (17) for the polytopic uncertainty set described by (9), find \((M, K)\) such that:

1. All elements of the set \( \mathcal{C} \) in (20) satisfy the state and control input constraints (21) and (22);
2. The controlled system in (19) satisfies the robust invariance condition (23).

We are typically interested in an invariant set whose volume is as large/small as possible. It is well known that the volume of a set of the form (20) is proportional to \(|\det(M^{-1})|\) \([7]\). Although this motivates us to frame Problem 1 as a determinant maximization/minimization, we have restricted the interest to feasibility for the following reasons. In \([25]\) it has been shown that determinant maximization problem is concave only for a symmetric matrix. Enforcing symmetry on \( M \) would possibly lead to a conservative result, since the resulting set might not be the one with the largest volume achievable. On the other hand, determinant minimization is not a convex problem in general. Therefore iterative approaches will be proposed in the sequel to obtain invariant sets with desirably large/small volumes. In order to conveniently express the basic results used to build the iterative schemes, we opted for a formulation in the form of a feasibility problem.

## 4 Feasibility Conditions for Invariant Set

In this section, we derive tractable feasibility conditions for the solvability of Problem 1. The system constraints (for the state and the control input) discussed in the first part are scalar inequality conditions and are exact. On the other hand, the invariance conditions discussed afterwards are obtained in the form of matrix inequalities by employing S-procedure arguments \([17]\) and are hence potentially conservative.

### 4.1 System Constraints

Let us first recall that the system constraints are formulated as in (21) and (22). To express them in the form of inequalities, we first introduce a state transformation as
\[
    \theta = Mx \Leftrightarrow x = W\theta, \quad \text{where } W \triangleq M^{-1}. \tag{24}
\]

The invertibility of \( M \) is assumed, which would in fact be guaranteed by the LMI conditions for invariance. This allows us to express the LC-RCI set of (20) as
\[
    \mathcal{C} = \{ W\theta \in \mathbb{R}^n : \theta \in \Theta \}, \tag{25}
\]
where \( \Theta \) is a hyper-rectangular region defined as follows:
\[
    \Theta \triangleq \{ \theta \in \mathbb{R}^n : -1 \leq \theta \leq 1 \}. \tag{26}
\]

This set can also be expressed as the convex hull of the vertices of the hyper-rectangular region, which we represent as \( \theta^i \):
\[
    \Theta = \text{conv} \left( \left\{ \theta^1, \ldots, \theta^m \right\} \right). \tag{27}
\]

We have thus introduced the \( \theta \)-state-space in which the candidate invariant region is a hyper-rectangle. The corresponding polytopic set in the \( x \)-state-space will clearly
be determined by the choice of \( W \). Since the state and control input constraints identified from (17) are both affine in \( x = W \theta \), it will be enough to satisfy them at the extreme points \( \theta^j \) in order to ensure that they are satisfied over the whole set \( \Theta \). Due to symmetry, we can always order \( \theta^j \) to have

\[
\theta^{j+2n-1} = -\theta^j, \quad j = 1, \ldots, 2n-1. \tag{28}
\]

In view of this, we can express (21) in terms of \( W \) as follows:

\[
HW \theta \leq 1, \forall \theta \in \Theta \iff -1 \leq HW \theta^j \leq 1, \quad j = 1, \ldots, 2n-1. \tag{29}
\]

In order to express the control input constraints in a similar way, we first introduce a new matrix variable as

\[
N \triangleq KW \iff K = NW^{-1}. \tag{30}
\]

The control input constraints in (22) then read as

\[
GN \theta \leq 1, \forall \theta \in \Theta \iff -1 \leq GN \theta^j \leq 1, \quad j = 1, \ldots, 2n-1. \tag{31}
\]

We stress that both (29) and (31) read as (respectively \( n_x \times 2^n \) and \( n_u \times 2^n \)) scalar constraints.

### 4.2 Invariance Conditions

In this subsection, we first derive for invariance simple LMI conditions that facilitate a one-step solution to Problem 3. We then obtain dilated LMI conditions for invariance which will be used in the next section to develop iterative algorithms for potential reduction of conservatism.

#### 4.2.1 Simple LMI Conditions for Invariance

In reference to the transformations (24) and (30) introduced above, we first express the controlled system dynamics of (19) as follows:

\[
x^+ = \underbrace{(A(\Delta)W + B(\Delta)N)}_{\mathcal{F}(\Delta,W,N)} \theta + \mathcal{E}(\Delta)w. \tag{32}
\]

Note that a new term is introduced in this expression as \( \mathcal{F}(\Delta, W, N) \). In the sequel, we will suppress the \((W, N)\) dependence in \( \mathcal{F} \) and the \( \Delta \) dependence in both \( \mathcal{F} \) and \( \mathcal{E} \) for notational simplicity. Now, let us express the system dynamics in the \( \theta \)-state-space as follows:

\[
W \theta^+ = \mathcal{F} \theta + \mathcal{E}w. \tag{33}
\]

From (23) and (25), each element \( \theta^+_i \) of the vector \( \theta^+ \) should satisfy the following condition to ensure invariance:

\[
1 - (\theta^+_i)^2 = 1 - (\epsilon_i^T \theta^+)^2 \geq 0, \quad i = 1, \ldots, n. \tag{34}
\]

\( \forall \theta \in \Theta, \forall w \in \mathcal{W}, \forall \Delta \in \Delta \). In this expression, \( \epsilon_i \) represents the \( i \)’th column of the identity matrix of size \( n \times n \). A sufficient condition for (34) can be obtained by first multiplying this inequality with a positive scalar variable \( \phi_i \) and then imposing this product to be greater than or equal to an expression that is known to be non-negative within the region \( \Theta \) and for a disturbance input that satisfies \( w \in \mathcal{W} \). In this fashion, we obtain a sufficient condition that is expressed as follows:

\[
\phi_i (1 - (\epsilon_i^T \theta^+)^2) \geq 2(\theta^+)^T (\mathcal{F} \theta + \mathcal{E}w - W \theta^+) \geq 0 \tag{35}
\]

where \( \mathcal{F} = \begin{bmatrix} 0 & \mathcal{E} & \mathcal{W} \end{bmatrix} \). In order to ensure invariance, we need to have (35) satisfied for any \( \mathcal{F} \). With \( \mathcal{F} = 0 \), we must then have

\[
\phi_i \geq 1^T \Lambda_i \mathbf{1} + 1^T \Gamma_i \mathbf{1}, \quad i = 1, \ldots, n. \tag{38}
\]
In order to ensure (37) for any $x \neq 0$, it would then be enough to impose
\[
\begin{bmatrix}
    \Lambda_i \\
    0 & * \\
    0 & D^T \Gamma_i D & * \\
    F & \mathcal{E} & W + W^T - \phi_i e_i e_i^T
\end{bmatrix} \succeq 0,
\]
which reads as a parameter-dependent LMI. By finally employing Lemma 2, we obtain a solution to Problem 1 as follows:

**Theorem 3.** Problem 1 is feasible, if there exist $W \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{m \times n}$ and $\phi_i \in \mathbb{R}_+$, $\Lambda_i \in \mathbb{R}^{n \times n}$, $\Gamma_i \in \mathbb{R}^{n_w \times n_w}$ as in (36), $P_i \in \mathcal{P}_{\text{pol}}$, for $i = 1, \ldots, n$ such that (29), (31), (38) and (40) are satisfied. A state-feedback gain that solves the problem can be constructed as in (30).

**Proof.** The $i$’th invariance condition is expressed as the scalar inequality of (38) and the LMI in (39). We note that (39) depends on the parameter matrix $\Delta$ as can be identified from (1, 3), (2, 3), (3, 1) and (3, 2) blocks. We now make use of Lemma 2 by expressing (39) as in (5) with $Y$ and its sub-blocks chosen as
\[
Y = \begin{bmatrix}
    D_p & A_d W + B_d N & E_d \\
    0 & \frac{1}{2} A_i & 0 & 0 \\
    0 & 0 & \frac{1}{2} D^T \Gamma_i D & 0 \\
    B_p & A W + B N & E & W - \frac{1}{2} \phi_i e_i e_i^T
\end{bmatrix}.
\]

By now applying the full-block S-procedure as in Lemma 2, we get the single LMI condition as in (40). To ensure invariance, this single LMI is to be accompanied by (6), which can be relaxed into the conditions presented in (12). Thus a $(W, N)$ couple satisfying (29), (31), (38) and (40) together with the scaling matrices $P_i \in \mathcal{P}_{\text{pol}}$ serves as a feasible solution to Problem 1. \qed

Having thus obtained tractable formulations of both system constraints and invariance condition, let us now evaluate these in reference to the recent works [13, 23]. The conditions derived in the previous subsection for the system constraints are exact in terms of $W$ and $N$, while the conditions in [13, 23] are expressed in terms of additional variables introduced via an application of the S-procedure. In our LMI conditions for invariance, parameter-dependent LMIs are relaxed by using full-block scaling matrices (by favor of Lemma 2 and Pólya’s method) as opposed to D-G scales used in [23]. The D-G scales are constructed as block-diagonal multipliers of commuting structure for block-diagonal $\Delta$’s with norm-bounded sub-blocks. When we limit the discussion to $\Delta$’s with scalar-repeated sub-blocks, we can note two drawbacks in connection with the use of D-G scales. The first one is that this necessarily restricts the interest to hyper-rectangular uncertainty regions, with which possible correlations among the parameters are disregarded. Second, D-G scales are obtained by restricting the choices of $P_i$’s in a way to satisfy $R_i = -Q_i \succeq 0$ and $S_i = -S_i^T$ in addition to the block-diagonal structure. In conclusion our result not only offers potential reduction in conservatism but also facilitates the consideration of possible correlation among the uncertain parameters. It is straightforward to extend this result in a way to handle additional norm-bounded full-blocks via simple block-diagonal concatenation of multipliers.

### 4.2.2 Dilated LMI Conditions for Invariance

We start our derivations by introducing new matrix variables $V_i \in \mathbb{R}^{n \times n}$ and new signals as $x_i \triangleq V_i^{-1} x^+$. With $x^+ = V_i x_i$, it then follows from the system dynamics (32) that
\[
\mathcal{F} \theta + \mathcal{E} w - V_i x_i = 0.
\]
In this case, we can express the sufficient condition of (35) as
\[
\phi_i (1 - (e_i^T W^{-1} V_i e_i)^2) \geq 2 \xi_i^T (\mathcal{F} \theta + \mathcal{E} w - V_i x_i) + (1 + \theta)^T \Lambda_i (1 - \theta) + (1 + D w)^T \Gamma_i (1 - D w) \geq 0.
\]

Along similar lines to the previous subsection, we can obtain the $i$’th matrix inequality condition for invariance that accompanies (38) as follows:
\[
\begin{bmatrix}
    \Lambda_i & 0 & * \\
    0 & D^T \Gamma_i D & * \\
    F & \mathcal{E} & V_i + V_i^T - \phi_i V_i^T W - T e_i e_i^T W^{-1} V_i
\end{bmatrix} \succeq 0.
\]

Note that, if we select $V_i = W$, $\forall i = 1, \ldots, n$ in (44), we get (39). This shows that (44) is potentially less conservative than (39). Nevertheless, the (3, 3) block clearly has nonlinear dependence on the new matrix variables, which need to be handled in some way to arrive at an LMI optimization problem. Instead of such an LMI formulation, we present the second main result of our paper in a way to serve as the basis of an iterative algorithm in the next section:

**Theorem 4.** Problem 1 is feasible, if there exist $W \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{m \times n}$ and $\phi_i \in \mathbb{R}_+$, $\Lambda_i \in \mathbb{R}^{n \times n}$, $\Gamma_i \in \mathbb{R}^{n_w \times n_w}$ as in (36), $X_i = X_i^T \in \mathbb{R}^{n \times n}$, $V_i \in \mathbb{R}^{n \times n}$, $P_i \in \mathcal{P}_{\text{pol}}$, for $i = 1, \ldots, n$ with which (29), (31), (38), (45) and (46) are satisfied. A state-feedback gain that solves the problem can be constructed as in (30).

**Proof.** With the intention to resolve the nonlinear variable dependence in the (3, 3) block of (44), we first introduce a matrix variable $X_i$ that satisfies
\[
X_i^{-1} - \phi_i W - T e_i e_i^T W^{-1} \succeq 0.
\]
Then by applying the Schur complement to this inequality followed by a congruence transformation, we obtain the equivalent condition (45). With $X_i$, satisfying (47), a condition that implies (44) can be formulated as

$$
\begin{bmatrix}
\Lambda_i & 0 & * & * \\
0 & D^T \Gamma_i D & * & * \\
AW+BN & E & V_i+V_i^T+B_i R_i B_i^T & * \\
A_d W + B_d N & E_d & S_i B_i^T + D_p R_i B_i^T & Q_i + D_p S_i^T + S_i D_p^T + D_p R_i D_p^T \\
\end{bmatrix} \succeq 0, \quad i = 1, \ldots, n.
$$

(48)

Again, by a standard application of the Schur complement, (48) can be expressed equivalently as

$$
\begin{bmatrix}
\Lambda_i & 0 & * & * \\
0 & D^T \Gamma_i D & * & * \\
F & E & V_i+V_i^T & * \\
0 & 0 & V_i & X_i \\
\end{bmatrix} \succeq 0.
$$

(49)

This inequality is amenable to an application of Lemma 2 with the choice of $Y$ and its sub-blocks as follows:

$$
Y = \begin{bmatrix}
D_p & A_d W + B_d N & E_d & 0 & 0 \\
0 & \frac{1}{2} \Lambda_i & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} D^T \Gamma_i D & 0 & 0 \\
B_p & AW + BN & E & V_i & 0 \\
0 & 0 & 0 & V_i & \frac{1}{2} X_i \\
\end{bmatrix}.
$$

(50)

This application leads to the LMI condition in (46). In conclusion (45) and (46) imply (44) together with $P_i \in \mathcal{P}_{\text{pot}}$.

Note that Theorem 4 does not facilitate a single-step LMI optimization since the (1, 1) block of (45) has nonlinear dependence on some matrix variables. For the convenience of our presentation, the resolution of this problem is deferred to Section 5.2.1, where iterative algorithms will be developed.

5 Algorithms for the Computation of Low-Complexity Robust Control-Invariant Sets

In this section, we develop algorithms to obtain control invariant sets of desirably large/small volumes. We first present one-step algorithms based on the standard LMI conditions for invariance (38) and (40). This is followed by iterative schemes which are developed primarily based on the dilated LMI conditions for invariance (38), (45) and (46).

5.1 One-Step Algorithms

In this section we present one-step approaches to volume maximization and minimization based on the LMI invariance conditions derived in Section 4.2.1. As noted in [7], the volume of $C$ is proportional to $|\det(W)|$. We hence need to deal with determinant maximization and minimization problems.

5.1.1 Volume Maximization

In this case, we are interested in finding the largest set $C$ described as in (20) satisfying the state constraints (29), the input constraints (31) and the sufficient conditions for invariance (38) and (40). As a result, we can easily formulate a determinant maximization problem under LMI conditions. Nevertheless, efficient numerical solution of such a problem is possible only when $W$ is required to be symmetric [25]. By enforcing this extra condition and exploiting the positive-definiteness of $W \succ 0$...
that would then be ensured by (40), we propose a one-step algorithm for obtaining an invariant set with a desirably large volume as follows:

Algorithm 1A: Computation of $\mathcal{C}$ with a Desirably Large Volume

$$\begin{align*}
\max & \quad \log \det(W) \\
n & \text{subject to: } (12), (29), (31), (38), (40) \text{ and } W = W^T
\end{align*}$$

5.1.2 Volume Minimization

Determinant minimization is also known to be a non-convex problem. In this case, we adapt the approach proposed by [23]. Where, first, a new matrix variable $W > 0$ introduced and is required to satisfy

$$W^T W \preceq W \iff \begin{bmatrix} W^T & W \end{bmatrix} \succeq 0.$$  \hspace{0.5cm} (51)

As shown in [23], $\log \det(W) \leq \text{trace}(W)$. Hence, in the next step, the cost to be minimized is changed to $\text{trace}(W)$. In conclusion, we propose the following minimization to obtain an LC-RCI set with a desirably small volume.

Algorithm 1B: Computation of $\mathcal{C}$ with a Desirably Small Volume

$$\begin{align*}
\min & \quad \text{trace}(W) \\
n & \text{subject to: } (12), (29), (31), (38), (40) \text{ and } (51)
\end{align*}$$

5.2 Iterative Algorithms

In this section, we present iterative algorithms for computing LC-RCI sets of desirably large and small volumes. Though the algorithms are formulated based on the dilated LMI conditions for invariance (38), (45) and (46) presented in Section 4.2.2, brief remarks are also made to show how similar formulations can be obtained by using the standard LMI conditions of Section 4.2.1. Before proposing these algorithms, we need to recall that (45) still has nonlinearity in the (1, 1) block. An approach for linearization is presented first. We then propose a modification in conditions for iterative reduction of conservatism introduced due to linearization. For a clear presentation of the algorithms, we add a superscript $k$ to the solutions obtained from the $k$'th step, e.g. $W$ is the generic matrix variable, $W^k$ is its value obtained from the $k$’th optimization.

5.2.1 Linearization via an Updated Slack Variable

To linearize the (1,1) block of (45), we make use of a slack variable identity [2]. This approach is based on the implication

$$W^T X^{-1} W = (W - X_i Y_i)^T X_i^{-1} (W - X_i Y_i)$$

$$\leq Y_i^T W + W^T Y_i - Y_i^T X_i Y_i$$

$$\geq Y_i^T W + W^T Y_i - Y_i^T X_i Y_i, \quad (52)$$

where $Y_i$ is an arbitrary matrix of compatible dimension. This inequality allows us to obtain a sufficient condition for (45) by replacing its (1, 1) block with $X_i$. The sufficient condition that we obtain in this fashion will be an LMI only if $Y_i$ is a fixed matrix. For this linearization approach to be useful, we hence need to specify a suitable choice for $Y_i$. A straightforward choice is $Y_i = \psi I$ with a positive scalar $\psi \in \mathbb{R}_+$. With different choices selected over a certain grid, we would obtain different sufficient conditions for (45), which can be used in volume maximization or minimization problems. Although we would thus be able to search for regions with larger/smaller volumes, we would not have control over the reduction of conservatism introduced by relaxing (45).

In order to reduce the conservatism introduced by (52), we propose to update $Y_i$ within an iterative scheme. An appealing choice is $Y_i = X_i^{-1} W$ since this leads to $Z_i = 0$. Nevertheless, the (1, 1) block of (45) would be unchanged. In order to obtain a sufficient LMI condition that can be updated within an iterative scheme, we propose choosing $Y_i$ at the $k+1$st step based on the solutions at the $k$'th step as follows:

$$Y_i = Y_i^k = (X_i^k)^{-1} W^k. \quad (53)$$

Hence, a sufficient LMI condition for (45) can be used at the $k+1$st step of an iterative scheme as follows:

$$\begin{bmatrix} (Y_i^k)^T W + W^T Y_i - (Y_i^k)^T X_i Y_i e_i \phi_i \\ \phi_i e_i^T \phi_i \end{bmatrix} \succeq 0. \quad (54)$$

Such an iterative scheme helps to reduce the conservatism introduced by (52) because $Z_i \rightarrow 0$ as $X_i^k \rightarrow X_i$ and $W^k \rightarrow W$ and the term $Z_i^T X_i^{-1} Z_i$ in (52) becomes negligible. This is a result of the fact that (54) is feasible with $(W, X_i) = (W^k, X_i^k)$, which follows from

$$W^T Y_i + (Y_i^k)^T W - (Y_i^k)^T X_i Y_i ig|_{(W, X_i) = (W^k, X_i^k)} = (W^k)^T (X_i^k)^{-1} W^k, \quad (55)$$

and the fact that (45) is necessarily feasible with
\((W^k, X^k)\). Lastly, we also note that (52) and (54) together imply that \(W\) is invertible.

### 5.2.2 Iterative Volume Maximization

In the sequel, we develop an algorithm that will guarantee iterative volume increase under the system constraints and invariance conditions. For the simplicity of our presentation, we express the LMI constraints in (12), (29), (31), (38), (46) and (54) concisely (via block-diagonal concatenation) as

\[
\mathcal{L}(W, L, Y^k) \succeq 0, \quad (56)
\]

where \(L\) represents all the variables other than \(W\) and \(Y^k\) denotes the block-diagonal concatenation of \(Y^k\) introduced in (53). As will be crucial for the applicability of our iterative algorithm, we emphasize at this point that

\[
\mathcal{L}(W^k, L^k, Y^{k-1}) \succeq 0 \Rightarrow \mathcal{L}(W^k, L^k, Y^k) \succeq 0. \quad (57)
\]

This simply follows from (54) and the ensuing statement in the previous subsection.

The main motivation behind an iterative scheme is to avoid the requirement that \(W\) is symmetric. We hence introduce a new symmetric matrix variable \(T\) whose determinant is to be maximized. This maximization is to be formulated in such a way that the solutions obtained at consecutive steps fulfill the condition

\[
|\det(W^{k+1})| \geq |\det(W^k)|. \quad (58)
\]

To this end, the new matrix variable is required to satisfy

\[
W^T W \succ T \succ 0. \quad (59)
\]

It then follows from Minkowski determinant inequality that

\[
\det(W^T W) = |\det(W)|^2 \geq \det(T). \quad (60)
\]

Since (59) is not an LMI, we consider replacing it with

\[
W^T W^k + (W^k)^T W - (W^k)^T W \succ T. \quad (61)
\]

By showing that this is a sufficient condition for (59), we can establish the following lemma, which will be the basis for our iterative algorithm:

**Lemma 5.** Consider an optimization problem formulated at the \(k + 1\)st step of an iterative scheme as follows:

\[
\begin{align*}
\max_{W, L, T} & \quad \log \det(T) \\
\text{subject to:} & \quad (56) \text{ and } (61)
\end{align*}
\]

The solutions of this optimization problem satisfy the determinant condition in (58) provided that (57) is true.

**Proof.** It follows directly from

\[
\begin{align*}
W^T W &= \left(\begin{array}{ccc}
(W - W^k) & (W - W^k) & (W - W^k) \\
(W - W^k) & (W - W^k) & (W - W^k) \\
(W - W^k) & (W - W^k) & (W - W^k)
\end{array}\right) \\
&= W^T W^k + (W^k)^T W - (W^k)^T W^k \succeq T
\end{align*}
\]

that (61) implies (59). As follows from (60), we will then have for the solution of the \(k + 1\)st optimization that

\[
|\det(W^{k+1})|^2 \geq \det(T^{k+1}). \quad (63)
\]

On the other hand, when (57) is true, \((W, L, T) = (W^k, L^k, (W^k)^T W^k)\) will be a feasible solution of the \(k + 1\)st optimization problem since (61) would then also hold true with equality. With the problem formulated as the maximization of \(|\det(T)|\), this implies that

\[
\det(T^{k+1}) \geq \det((W^k)^T W^k) = |\det(W^k)|^2. \quad (64)
\]

In conclusion, the determinant increase condition of (58) is necessarily satisfied.

Lemma 5 is quite general in nature and can in fact be used for the maximization of the determinant of a matrix that is subject to a set of LMI conditions. For our specific problem of LC-RCI set computation, an iterative algorithm can be developed based on this lemma as follows:

**Algorithm 2A: Iterative Computation of \(C\) with a Desirably Large Volume Optimization at \(k + 1\)st step:**

\[
\begin{align*}
\max & \quad \log \det(T) \\
\text{subject to:} & \quad (12), (29), (31), (38), (46), (54), (61)
\end{align*}
\]

**Initial Optimization to Compute \(W^0\):** Condition (61) is removed; (54) is imposed with \(Y^k_i = \psi I\) and \(\log \det(T)\) is changed to \(\log \det(W)\).

**Termination:** \(|\det(W^{k+1})|/|\det(W^k)| \leq 1 + \epsilon \) with a desirable small \(\epsilon > 0\).

**Remark 1.** An iterative algorithm can be developed based on the standard LMI conditions for invariance simply by removing the variables \(X_i, V_i\) as well as the condition (54) and then replacing condition (46) by (40).

### 5.2.3 Iterative Volume Minimization

Along similar lines to the previous subsection, we first use Minkowski inequality to write down a sufficient con-
Optimization at (66) is replaced by (51); (54) is imposed with $Y$, satisfying (54) and (51) simultaneously.

**Initial Optimization to Compute $W$ and $\rho$**

**Algorithm 2B: Iterative Computation of $C$ with a Desirably Small Volume**

Optimization at $k + 1$st step:

$$
\min \rho
\text{subject to: } (12), (29), (31), (38), (46), (54), (66)
$$

Initial Optimization to Compute $W^0$: Condition (66) is replaced by (51); (54) is imposed with $Y^k \rightarrow \psi^k$ and $\rho$ is changed to $\text{trace}(W)$.

Termination: $|\det(W^{k+1})|/|\det(W^k)| \geq 1 - \epsilon$ with a desirable small $\epsilon \in (0, 1)$.

**Remark 2.** Ideally we will have $\rho^k \rightarrow 1$ as $k \rightarrow \infty$ since the $k$'th solution is feasible in the $k + 1$st optimization with $\rho = 1$. An alternative criterion for termination in practice would hence be $\rho^k \geq 1 - \epsilon$ with a desirable small $\epsilon \in (0, 1)$.

**Remark 3.** An iterative algorithm can be developed based on the standard LMI conditions in the same way as in Remark 1.

### 6 Simulations

In this section, the proposed algorithms are demonstrated with a discrete-time double integrator that has rational parameter dependence. The system dynamics are described as in (14) with

$$
A = \begin{bmatrix} 1 + \delta_1 & 1 + \delta_1 \\ 0 & 1 + \delta_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (67)
$$

where $\delta_1$ and $\delta_2$ represent the uncertain parameters. The state and control input constraints are expressed as

$$
\begin{bmatrix}
0.10 & -0.10 \\
0.15 & -0.15
\end{bmatrix}^T x \leq 1, \quad |u| \leq 3. \quad (68)
$$

In the sequel, we first illustrate the proposed algorithm and compare it with an existing one [23] by using specific bounds on the uncertain parameters and the disturbance in (67). We restrict our comparison of the algorithms to max-volume LC-RCI set computations only. We then provide a more comprehensive comparison by using the same example and varying the bounds on the uncertain parameters and disturbance.

All algorithms are implemented with CVX [10] by using the solver SeDuMi. In all the iterative algorithms, the initial optimization was solved using $Y^0 = I$ in (54) and the termination criteria were applied with $\epsilon = 10^{-3}$.

#### 6.1 RCI Set Computation with Different Uncertainty Descriptions

In this subsection, we provide two alternative representations of the uncertain system (67) by assuming fixed bounds on the uncertain parameters and the disturbance as $|\delta_i| \leq 0.2$, $i = 1, 2$ and $|u| \leq 0.1$. The first is an LFT representation that captures exactly the original system as well as the uncertainty description. The second representation is obtained (solely for purposes of comparison) with a more conservative uncertainty description by introducing a new parameter on which the dependence is affine.

An LFT representation of system (67) is obtained as in (16) with the following matrices and uncertainty description:

$$
\begin{bmatrix}
A & B & E & \bar{B}_p \\
A_d & B_d & E_d & \bar{D}_p
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & 0 & 1 & 1 \\
0 & 1 & 0 & -1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\Delta = \{\Delta = \text{diag}(\delta_1, \delta_1, \delta_1, \delta_2) : |\delta_i| \leq 0.2, i = 1, 2\}. \quad (69)
$$

By using Algorithm 2 for this system, the maximum and minimum volume LC-RCI sets and the associated state...
feedback gain matrices are computed as

\[
W_{\text{max}} = \begin{bmatrix} 5.2049 & -1.7930 \\ 0.6013 & 2.6026 \end{bmatrix}, \quad K_{\text{max}} = -\begin{bmatrix} 0.1673 \\ 0.7582 \end{bmatrix},
\]

\[
W_{\text{min}} = \begin{bmatrix} 0.2541 & -0.0260 \\ -0.0042 & 0.4119 \end{bmatrix}, \quad K_{\text{min}} = -\begin{bmatrix} 0.8147 \\ 1.0498 \end{bmatrix},
\]

respectively. We emphasize that existing algorithms in the literature cannot be applied directly to systems that have rational parameter dependence.

In order to be able make comparisons with the existing algorithm [23], we first introduce a new parameter as \( \delta_3 = \delta_2/(1 + \delta_1) \) and thus view (67) as a system that has affine dependence on \((\delta_1, \delta_3)\). In the new uncertainty description, one typically needs to use a region that covers the original uncertainty set, which is shaded with green color (dashed line) in Figure 1. Hence, an uncertain LFT representation can be obtained in this case as

\[
\begin{bmatrix} A & B & E \\ A_d & B_d & E_d \end{bmatrix} \Delta = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta = \text{diag}(\delta_1, \delta_3) : |\delta_1| \leq 0.2, |\delta_3| \leq 0.25.
\]

We like to clarify that the uncertainty region is formed in this representation as rectangular in order to be able to use the algorithm from [23]. It is visible that this approach will be quite conservative, since the red region in Figure 1 is added to the uncertainty set unnecessarily. The maximum and minimum volume LC-RCI sets computed with the algorithm of [23] and Algorithm 2 for the new uncertainty description are identified as follows:

Existing Algorithm of [23]:

\[
W_{\text{max}} = \begin{bmatrix} 5.9549 & -0.8738 \\ 0.0970 & 2.2112 \end{bmatrix}, \quad K_{\text{max}} = -\begin{bmatrix} 0.1853 \\ 0.8917 \end{bmatrix},
\]

\[
W_{\text{min}} = \begin{bmatrix} 0.2524 & -0.0209 \\ 0.0044 & 0.4210 \end{bmatrix}, \quad K_{\text{min}} = -\begin{bmatrix} 0.8533 \\ 1.0381 \end{bmatrix},
\]

Algorithm 2:

\[
W_{\text{max}} = \begin{bmatrix} 5.6165 & -1.0681 \\ 0.1520 & 2.3622 \end{bmatrix}, \quad K_{\text{max}} = -\begin{bmatrix} 0.1878 \\ 0.8535 \end{bmatrix},
\]

\[
W_{\text{min}} = \begin{bmatrix} 0.2550 & -0.0259 \\ 0.0003 & 0.4116 \end{bmatrix}, \quad K_{\text{min}} = -\begin{bmatrix} 0.8183 \\ 1.0512 \end{bmatrix},
\]

It is observed from (70), (73) and (75) that the maximum-volume LC-RCI set is obtained by Algorithm 2A (\( \det(W) = 14.6244 \)) applied with the system’s original uncertainty description in (69). It is also important to note that Algorithm 2A delivers a larger set (\( \det(W) = 13.4296 \)) if compared to [23] (\( \det(W) = 13.2522 \)) when both are applied with the modified uncertain system representation in (72). Plotted in Figure 2 in red (dotted line), green (solid line), yellow (dot-dashed line) and blue (dashed line) colors are the set of admissible states \( \mathcal{X} \) in (71), maximum volume LC-RCI set from Algorithm 2A with (69), Algorithm 2A with (72) and [23] with (72), respectively. To demonstrate the invariance of the set obtained from Algorithm 2A, the state trajectories are plotted in Figure 2 in black by randomly varying the input disturbance and the uncertain parameters within the chosen bounds. In Figure 3, the convergence of the three algorithms are shown. Though Algorithm 2A requires more iterations to converge, it already reaches larger volumes in fewer iterations if compared to the existing algorithm from [23].

6.2 Comparison of Different Algorithms for Varying Uncertainty and Disturbance Bounds

To make a comprehensive comparison between various algorithms, maximum volume LC-RCI sets were calcu-
Fig. 2. Admissible states \( X \) (red) and maximum volume LC-RCI set from Algorithm 2A with (69) (green), Algorithm 2A with (72) (yellow) and [23] with (72) (blue)

11.5
12
12.5
13
13.5
14
14.5
15

Algorithm 2A with (66)
Algorithm 2A with (69)
Existing Algorithm [23] with (69)
Algorithm 1A (Iterative) with (66)
Algorithm 1A (One-Step) with (66)

Fig. 3. Convergence plot for the shown algorithms

It can be seen from Figure 4-5 that, in the presence of parametric uncertainties, Algorithm 2A with (69) gives significantly larger RCI sets if compared to the other algorithms. This demonstrates the benefit of the proposed direct way of handling the uncertainty instead of using modified form. It is also visible from the same figures that Algorithm 2A gives slightly better results if compared to [23] when both algorithms are applied to the same system description (72). Furthermore, from Figure 4 and Figure 6, it can be observed that Algorithm 2A gives feasible solutions for larger parameter or disturbance bounds. Note that that in Figure 6 only one case of Algorithm 2A is considered, since parameters are assumed to be zero. We also observe from this figure that Algorithm 2A is slightly better than [23], especially for disturbance bounds around \( a = 0.6 \). Though seemingly more conservative in this example as revealed from all three figures, Algorithm 1A is observed to be computationally more efficient in terms of the number of iterations and computation time thanks to the reduced number of variables in the related SDP programs. Lastly, recall that in Algorithms 1 and 2, zeroth-order Pólya relaxation was used to render (6) tractable. By using higher order Pólya or various other relaxations as indicated earlier in Section 2, we may compute even larger LC-RCI sets.

7 Conclusion

We have developed in this paper algorithms to compute LC-RCI sets together with associated state-feedback gains for systems that have rational dependence on uncertain parameters. By using the full-block S-procedure and a state transformation, novel conditions have been derived for invariance both as standard and dilated LMIs. These conditions are used to develop one-step as well as iterative algorithms for the computation of LC-RCI sets of desirably large/small volumes. The iterative algorithms are built on novel recursive approaches...
to the maximization/minimization of the determinant of a non-symmetric matrix that is subject to a set of LMI conditions. The potential benefits of the iterative algorithms in reducing the conservatism are shown by an illustrative example. An immediate next step would be to allow for an adjustment of the level of complexity, i.e. the number inequalities with which the RCI set is described, in a way similar to [13]. It is possible to develop similar algorithms with parameter-dependent feedback gains. It would be more challenging to develop synthesis methods based on nonlinear state feedback.

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References


