Thesis for the degree of Doctor of Technology

Real and complex Monge-Ampère equations, statistical mechanics and canonical metrics

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Abstract

Recent decades has seen a strong trend in complex geometry to study canonical metrics and the way they relate to geometric analysis, algebraic geometry and probability theory. This thesis consists of four papers each contributing to this field. The first paper sets up a probabilistic framework for real Monge-Ampère equations on tori. We show that solutions to a large class of real Monge-Ampère equations arise as the many particle limit of certain permanental point processes. The framework can be seen as a real, compact analog of the probabilistic framework for Kähler-Einstein metrics on Kähler manifolds. The second paper introduces a variational approach in terms of optimal transport to real Monge-Ampère equations on compact Hessian manifolds. This is applied to prove existence and uniqueness results for various types of canonical Hessian metrics. The results can, on one hand, be seen as a first step towards a probabilistic approach to canonical metrics on Hessian manifolds and, on the other hand, as a remark on the Gross-Wilson and Kontsevich-Soibelmann conjectures in Mirror symmetry. The third paper introduces a new type of canonical metrics on Kähler manifolds, called coupled Kähler-Einstein metrics, that generalises Kähler-Einstein metrics. Existence and uniqueness theorems are given as well as a proof of one direction of a generalised Yau-Tian-Donaldson conjecture, establishing a connection between this new notion of canonical metrics and stability in algebraic geometry. The fourth paper gives a necessary and sufficient condition for existence of coupled Kähler-Einstein metrics on toric manifolds in terms of a collection of associated polytopes, proving this generalised Yau-Tian-Donaldson conjecture in the toric setting.

Keywords: Real Monge-Ampère equations, Complex Monge-Ampère equations, Kähler-Einstein metrics, Kähler Geometry, Canonical metrics, Hessian manifolds, Optimal Transport, Point Processes, Statistical Mechanics

Preface

This thesis consists of the following papers:

- Jakob Hultgren
  "Permanental Point Processes on Real Tori, Theta Functions and Monge–Ampère Equations"
  To appear in Annales de la faculté des sciences de Toulouse

- Jakob Hultgren and Magnus Önnheim
  "An optimal transport approach to Monge-Ampère equations on compact Hessian manifolds"
  Preprint

- Jakob Hultgren and David Witt Nyström
  "Coupled Kähler-Einstein metrics"
  International Mathematics Research Notices
  rnx298
  https://doi.org/10.1093/imrn/rnx298

- Jakob Hultgren
  "Coupled Kähler-Ricci solitons on toric Fano manifolds"
  Preprint

The second and third paper are based on collaborations where both authors partook on equal terms. The original idea for the third paper is due to David Witt Nyström.

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I am very grateful for my time as a PhD student in the complex analysis group at Chalmers. My first supervisor, Robert Berman, is a constant source of knowledge and inspiration. He supervised me during the first years of my PhD, exposing me to as diverse fields as probability theory, algebraic geometry, pluripotential theory and optimal transport. Thank you, Robert. I might not have grasped all you tried to teach me (or even a small part of it) but your open attitude and willingness to experiment and learn really changed the way I think about mathematics. Over time our relationship has developed into a relationship of friends and on a personal level I find you, to mention one thing, a very inspiring fellow parent who illustrates that it is more than possible to have a successful mathematical career and still be 100% present in your family life.

Funnily, the same thing goes for my second supervisor, David Witt Nyström. We started out as friends. David defended his PhD thesis at roughly the same time as I started studying for mine. When he returned to Gothenburg after a few years abroad I asked if he would supervise me during my final two years and this is something I never regretted. Thank you, David. Both working with you on our joint paper, and your more general advice on which directions to pursue and what problems to work on has been extremely rewarding. Your constant drive and your many ideas are truly inspiring and being your student I have always felt that there is a pair of watchful eyes looking out for me.

Another person I would like to thank, and who played a big role in my choice to study mathematics in the first place, is my father. When my siblings and I grew up, he would expose us to math and engineering problems on a regular basis. He had (and I presume still has) a conviction that under the right circumstances anything could be understood in 15 seconds. Guided by this he would sometimes stop us in whatever we were doing and show us something. Perhaps he would draw a curve and ask us what the slope of the curve was at its minimal point and we would stop for a moment (15 seconds to be precise) and think about it. I still believe this 15 seconds principle is true but even after these years as a PhD student I haven't been able to pin down exactly what the "right circumstances" are.

In the complex analysis group I want to thank Bo who supervised my masters thesis and has remained a great support during my years as a PhD student and Elizabeth for being a great friend and role model in academia. I want to thank my fellow PhD students and the postdocs in the group for company and support: Hossein, Richard, Mattias, Antonio, Martin, Lucas, Chinh and Xu. Thank you, Mats and Håkan, for being good bosses. There are also many teachers at the department whose courses I gained a lot from, for example Ulf Persson, Per Salberger, Jan Stevens, Jan-Alve Svensson, Ulla Dinger and Maria Roginskaya.

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I also want to mention (and there is some overlap here) Neil, Simon and Hannes. I feel very privileged to work with you on our projects outside academia and I am confident the enthusiasm I have for this has spilled over onto my work with this thesis.

Finally, I want to thank the two people who are closest to me in the world: Elina and Noa.

Jakob Hultgren

Oslo, March, 2018

Contents

Part I

INTRODUCTION

Introduction

1. The Einstein Field Equations

Although this is not a thesis in physics, a good starting point for it is the Einstein Field Equations. Probably one of the most famous systems of equations in history, it appeared in Einstein's publication about general relativity in 1915. The purpose of these equations is to describe the curvature of space time. Solutions constitute components of a metric tensor. As such, they allow us to measure the length of tangent vectors. Together with the geodesic equation they define paths of shortest distance and thus the trajectories of freely moving (or falling) particles.

Simply put, the Einstein field equations uses geometry to describe gravity and they have given rise to a very active research area in mathematical physics. What is perhaps more surprising, though, is the theories they have inspired in mathematics. Taking them out of the four dimensional context of space-time they have turned out to be a very useful tool in geometry. Here, a manifold endowed with a metric satisfying the Einstein Field Equations in vacuum is called an Einstein manifold. The resulting equations, and the less restrictive equations achieved by taking trace of both sides (defining metrics of constant scalar curvature), has during the last decades been studied extensively in complex geometry. Here, they define canonical choices of metrics: Kähler-Einstein metrics and constant scalar curvature Kähler (cscK) metrics. Such canonical choices of metrics have turned out to provide an intersection point between several very different point of views in geometry. Perhaps most famous is the connection between Geometric Invariant Theory in algebraic geometry and differential geometry that is given by the Yau-Tian-Donaldson conjecture. Another one is the connection to probability theory given by probabilistic constructions of Kähler-Einstein metrics using determinantal point processes.

This thesis consists of four contributions to this field. While most of the tools that are used come from geometric analysis (Paper II-IV) and probability theory (Paper I), the main motivation is given by the connections between the different points of view in geometry mentioned above.

From the perspective of geometric analysis, Kähler-Einstein metrics are studied in terms of certain nonlinear elliptic scalar equations called complex Monge-Ampère equations. On n-dimensional Kähler manifolds with an $(S^1)^n$-symmetry, these equations reduce to so called real Monge-Ampère equations. All the papers in the thesis are concerned with either complex or real Monge-Ampère equations.

2. The Uniformization Theorem

The Uniformization Theorem might be the first historical instance of canonical metrics. The theorem was proved by Poincaré and Koebe in 1917 and says that any complete, simply connected Riemannian surface is conformally equivalent to one of the following three: The Riemann sphere with its standard metric, the complex plane with the Euclidean metric or the unit disc with the Poincaré metric. Two Riemannian surfaces are comformally equivalent if there is an angle-preserving diffeomorphism from one to the other and the significance of the three Riemannian surfaces above is that they are the unique simply connected Riemannian surfaces of constant curvature 1, 0 and -1 respectively.

If we instead look at the more general class of complete Riemannian surfaces, X, then this theorem has an interesting consequence. The universal covering space of X satisfies the assumptions of the Uniformization Theorem and is thus conformally equivalent to one of the three options above. Moreover, the deck transformations of this covering will form a discrete group of conformal (angle-preserving) maps on the covering space. The amount of conformal maps on these three spaces is rather restricted and by doing a case by case examination you can show that:

- Any discrete group of conformal maps on one of the three Riemannian surfaces above consists of isometries.
- On the Riemann sphere any discrete group of conformal maps is trivial, on the complex plane any discrete group of conformal maps is abelian and on the unit disc any discrete group of conformal maps is non-abelian.

The first point means that the metric on the covering spaces defines a metric on X, thus any complete Riemannian surface is conformally equivalent to a Riemannian surface of constant curvature. The second point means that the fundamental group of X determines whether this metric is of curvature 1, 0 or -1. Put differently: Existence of metrics with certain properties (or canonical metrics) hold topological information about the surface. Knowing this it should not be surprising that existence of canonical metrics has consequences in other areas of geometry as well.

3. Complex projective manifolds

The original setting for the Kähler-Einstein problem and the cscK problem is given by a complex projective manifold. A complex projective manifold is a submanifold of complex projective space $\mathbb{P}^n$ cut out by homogenous polynomials. A good example is given by

$$X = \left\{(x : y : z) \in \mathbb{P}^2, x^3 y + y^3 z + z^3 x = 0\right\}. \tag{1}$$

There are many different approaches to study complex projective manifolds. In the complex geometric approach one would focus on the fact that X is a complex manifold, in other words it is possible cover X with complex valued coordinate charts such that the transition functions are holomorphic. Moreover, the standard (Fubini-Study) metric on $\mathbb{P}^n$ restricts to a metric on X. This means X is a so called Kähler manifold (see Section 5). However, which metric you get this way depends on the embedding of X in $\mathbb{P}^n$ and there is no reason to believe that this metric has good properties. In a nutshell, the Kähler-Einstein problem and the cscK problem consists in deforming this metric (in a suitable sense) into a Kähler-Einstein metric or a cscK metric.

When studying X from the point of view of algebraic geometry, one would focus on the spaces of regular functions on X. For example, restricting to the set $\{z \neq 0\} \subset \mathbb{P}^2$, we get a ring of polynomials

$$\mathbb{C}[x, y]/(x^3 y + y^3 + x)$$

defined on $X \cap \{z \neq 0\}$. Similarly, you get a coordinate ring for any Zariski open subset of X. From the algebraic point of view, there is a stability condition in terms of Geometric Invariant Theory that is closely related to the Kähler-Einstein problem and the cscK problem (see Section 8).

From a third and more recent perspective, the metric acquired by restricting the Fubini-Study metric to X determines a volume form on X. This defines a way of randomly sampling points on X. Similarly as the metric, this volume form depends on the embedding. However, in the case when the embedding of X is canonical (this is true for the example above), the homogenous functions on $\mathbb{P}^n$ defines n-forms on X and these can be used to define canonical point processes on X. In other words, there is a canonical way of randomly sampling *sets of points* on X. As will be explained in Section 15, the many particle limit of these point processes gives a direct connection to Kähler-Einstein metrics.

Often you want a more intrinsic way than (1) to describe a projective manifold. This can be attained by considering the pair (X, L) where X is the abstract complex manifold (or algebraic variety) defined by (1) and L is the line bundle over X given by the restriction of the the hyper plane bundle over $\mathbb{P}^N$. By construction, the space of holomorphic sections of this bundle will be spanned by the pullback of the coordinate functions in $\mathbb{P}^N$. This means we can reconstruct the embedding (up to linear transformations on $\mathbb{C}^{N+1}$) using only the space of holomorphic sections of L.

Finally, from an analytical point of view, the pair (X, L) is often replaced by (X, α) where α is the $(1,1)$-Dolbeault cohomology class given by the first Chern class of L.

4. Complex structures and one of their features

A complex n-dimensional manifold is a smooth manifold of real dimension $2n$ with a holomorphic coordinate system, i.e. a coordinate system with coordinate charts in $\mathbb{C}^n$ and holomorphic transition functions. Each choice of coordinates induce an isomorphism between the tangent space of the manifold and the tangent space of $\mathbb{C}^n$. This gives an operation on the tangent space of the manifold that is analogous to multiplying by $\sqrt{-1}$. This operation, which is called the *complex structure*, is usually denoted by J. While different choices of coordinates gives different isomorphisms between the tangent space of the manifold and the tangent space of $\mathbb{C}^n$, the fact that the transition functions are holomorphic ensures J is well-defined.

Using J lets us express many concepts on a complex manifold in a convenient way. First of all, the derivative df of a smooth complex valued function f at a point p on a complex manifold is by definition a real linear map from the tangent space at p to the tangent space of $\mathbb{C}$. The function is holomorphic if this map is also complex linear. Moreover, by elementary linear algebra, any real linear map between complex vector spaces can be written as a sum of one complex linear map and one complex anti-linear map (meaning $L(iv) = -iL(v)$). This means df, can be decomposed into one part that captures its complex linear part, ∂f, and one part that captures its complex anti-linear part, $\bar{\partial} f$.

Incidentally, this gives a compact way of writing that a function is holomorphic, namely that the complex anti-linear part of its derivative vanishes:

$$\bar{\partial} f = 0.$$

What is perhaps more surprising, though, is what happens if we compose these two operators. It turns out that $\sqrt{-1}\partial\bar{\partial}$ is a non-trivial second order differential operator that is invariant under holomorphic changes of coordinates. Usually, one considers $\sqrt{-1}\partial\bar{\partial}$ for real valued functions f. When $n = 1$ it takes the following form in complex coordinates $(z) = (x + \sqrt{-1}y)$:

$$\sqrt{-1}\partial\bar{\partial} f = \left(\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}\right) dx \wedge dy. \tag{2}$$

In the general case it is a 2-form that encodes the Laplacian type expression (2) of $f|_Y$ for any smooth complex curve Y in X. This means we have an operator which (in some sense) captures the second derivative of a function. It is interesting to compare this to the Laplace-Beltrami operator which is only defined after fixing a Riemannian metric.

In the next section we will show how this operator is used when studying Riemannian geometry on X.

5. Kähler geometry

In this section f will always be a smooth real valued function. The 2-form $\omega_f = \sqrt{-1}\partial\bar{\partial} f$ can (by construction) be written as a sum of tensor products of a complex linear form and a complex anti-linear form (this is usually referred to as having bi-degree (1,1)). Using this one can show that the bilinear form on the tangent space

$$g_f = \omega_f(\cdot, J\cdot)$$

is symmetric. For some f, g_f is also positive definite and thus defines a Riemannian metric. Those f that satisfies this are called plurisubharmonic functions, the 2-forms ω_f that (locally) arise in this manner are called Kähler forms and the associated Riemannian metrics g_f are called Kähler metrics.

In fact, the Kähler metrics are precisely the Riemannian metrics that "respect the complex structure" in the sense that $g(J\cdot, J\cdot) = g$ and J is parallell with respect to the connection determined by g.

When studying Kähler metrics you often work with the Kähler form rather than the Kähler metric itself. The Kähler form is necessarily a closed form. As such it defines a de Rham cohomology class on X. This class is often referred to as the cohomology class of the Kähler metric. Conversely, any de Rham class which contains at least one Kähler form is called a Kähler class.

Not all complex manifolds admit a Kähler metric. While you can always construct a Kähler metric locally, patching together the local pieces is a non-trivial problem. Manifolds that admit Kähler metrics are called Kähler manifolds and existence of Kähler metrics have some striking implications. One is that the cohomology of the $\bar{\partial}$-operator (the Dolbeault cohomology) on a Kähler manifold is a refinement of the deRahm cohomology. One implication of this is the following crucial fact in Kähler geometry: two Kähler forms ω_1 and ω_2 represent the same de Rham class if and only if $\omega_1 = \omega_2 + \sqrt{-1}\partial\bar{\partial}f$ for some smooth function f. This means that, fixing a Kähler form ω, the Kähler forms that define the same de Rham class as ω can be studied in terms of the space of Kähler potentials

$$\{f : \omega + \sqrt{-1}\partial\bar{\partial}f \text{ is a Kähler form}\}.$$

6. Canonical Kähler metrics

The two types of canonical metrics most studied in Kähler geometry are Kähler-Einstein metrics and cscK metrics. As mentioned above, the first of these comes from considering the Einstein Field Equations in vacuum on a Kähler manifold. In this case the equations reduce to the following

$$\operatorname{Ric}\omega = \lambda\omega \tag{3}$$

where ω is the Kähler form of the metric, $\operatorname{Ric}\omega$ is the Ricci curvature form of the metric and $\lambda \in \mathbb{R}$. By homogenity of $\operatorname{Ric}\omega$ with respect to ω, it suffices to consider the cases $\lambda \in \{-1, 0, 1\}$. The Ricci curvature form of a Kähler metric always represents the first Chern class $c_1(X)$ of the manifold. This provides a necessary topological condition on X for existence of Kähler-Einstein metrics, namely that (corresponding to the cases $\lambda = 1$, $\lambda = -1$ and $\lambda = 0$) either $c_1(X)$ or $-c_1(X)$ contains a Kähler form or $c_1(X)$ vanishes. In the first two cases any solution of (3) represents $\lambda^{-1}c_1(X)$. Fixing a Kähler form ω in the appropriate class, we may represent any Kähler form in this class by an ω-plurisubharmonic function f. With respect to this, (3) reduces to the following differential equation in f:

$$\omega_f^n = e^{-\lambda f + h}\omega^n \tag{4}$$

where $\omega_f^n = \omega + i\partial\bar{\partial}f$ and h is a smooth function determined by ω. This is a complex Monge-Ampère equation. The left hand side of this is the complex Monge-Ampère operator. It is fully non-linear and of second order in f.

The solvability of (4) for $\lambda = 0$ was proved by Yau and solvability in the case of $\lambda = -1$ was proved independently by Aubin and Yau (see [**2**], [**15**], [**16**]). The case $\lambda = 1$ turned out to be more difficult. It was quickly understood that there are obstructions to this problem. For example, $\mathbb{P}^2$ blown up in one point has positive first Chern class (thus corresponds to the case $\lambda = 1$) but does not admit a Kähler-Einstein metric. Over time it was realised that the obstructions to Kähler-Einstein metrics on manifolds with positive first Chern class are subtle and deep. It turned out that one had to look towards Geometric Invariant Theory in algebraic geometry to find the right conditions. This will be explained in Section 8.

The other type of canonical metrics widely studied in Kähler geometry, cscK metrics, are defined by the equation

$$s(\omega) = c \tag{5}$$

where s is the scalar curvature of ω and c is a constant (the average scalar curvature of the class of ω). This is the equation you get if you apply the trace operator to both sides of (3), hence any Kähler-Einstein metric is a cscK metric. Moreover, Equation (5) does not imply a topological condition like the one above, in other words it makes sense to look for cscK metrics in any Kähler class. However, it can be shown that any cscK metric in a class proportional to $c_1(X)$ is also a Kähler-Einstein metric. This means (5) has two uses: It gives an alternative way of characterising Kähler-Einstein metrics when the class of ω is proportional to $c_1(X)$ and it provides a good generalisation of Kähler-Einstein metrics when the topological condition above does not hold.

In terms of a complex projective manifold (X, L) (see Section 3), the associated cscK problem is to determine whether the Kähler class $c_1(L)$ on X contains a cscK metric.

The main purpose of Paper III is to introduce a new type of canonical metrics: coupled Kähler-Einstein metrics. These provide a different generalisation to Kähler-Einstein metrics than cscK metrics and has much in common with Kähler-Einstein metrics in terms of the tools used to study them.

7. Quotients in algebraic geometry

Consider the action by $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $\lambda.(z_1, z_2) = (\lambda z_1, \lambda z_2)$. Now, a naive way to construct a coordinate ring for the quotient of $\mathbb{C}^2$ by this action would be to consider the invariant elements of the coordinate ring of $\mathbb{C}^2$. However, the only elements in $\mathbb{C}[x, y]$ that are invariant under this action are the constant polynomials, which suggests an orbit space consisting of just one point. However, the true orbit space is much bigger than that. In particular, each punctured line through the origin in $\mathbb{C}^2$ is a distinct orbit.

The example above is naive, but it illustrates that taking quotients in algebraic geometry is a subtle problem. One way to approach this problem is through Geometric Invariant Theory, a set of ideas developed by Mumford and published in his 1965 book with the same name. His motivation was to construct moduli spaces in algebraic geometry as quotients of algebraic varieties. Any details about this is beyond the scope of this thesis but the main idea is to differentiate between points depending on their properties with respect to the action. Building on works of Hilbert, Mumford defines three loci of points, the stable points the semi-stable points and the unstable points (where the first is a subset of the second) and proposes forming moduli spaces of the semi-stable points. Ideally, the quotient will have good properties on the stable locus and the rest of the semi-stable locus will serve to compactify the quotient space.

8. The Yau-Tian-Donaldson conjecture

The definitions of Mumford inspired Tian [**13**] and Donaldson [**10**] to develop a stability notion in Kähler geometry: K-stability. As explained in Section 3, the basic object here is a pair (X, L). The definition of K-semistability and K-stability involves so called test configurations. Loosely speaking, a test configuration for (X, L) is a family $\pi : \mathcal{X} \to \mathbb{C}$ such that the generic fiber $\pi^{-1}(\tau)$ for $\tau \neq 0$ is isomorphic to X, together with a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that its restriction to a generic fiber is L and a $\mathbb{C}^*$-action on $\mathcal{L}$ that respects the standard action on $\mathbb{C}$. Associated to each test configuration is an invariant called the Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X},\mathcal{L})$. The pair (X,L) is defined to be K-semistable if $\mathrm{DF}(\mathcal{X},\mathcal{L}) \geq 0$ for all test configurations and K-stable if it is K-semistable and $\mathrm{DF}(\mathcal{X},\mathcal{L}) = 0$ implies a certain triviality condition on $(\mathcal{X},\mathcal{L})$.

The Yau-Tian-Donaldson conjecture states that (X,L) admits a canonical metric (a Kähler-Einstein metric if L is the canonical or anti-canonical line bundle and a general cscK metric otherwise) if and only if (X,L) is K-stable.

One important motivation for this conjecture is the Kobayashi-Hitchin correspondence (also called the Donaldson-Uhlenbeck-Yau Theorem), stating that a hermitian vector bundle with trivial determinant bundle over a compact Kähler manifold admits a Einstein-Hermitian metric if and only if it is K-semistable in the sense of Geometric Invariant Theory. The Yau-Tian-Donaldson conjecture is sometimes described as a non-linear version of this.

The Yau-Tian-Donaldson conjecture was proven in the case when L is the anti-canonical line bundle by Chen, Donaldson and Sun in 2013 [7]. This constituted a major breakthrough in the field. However, the general case remains open.

In Paper III, we adapt the the definition of K-stability to the setting of coupled Kähler-Einstein metrics and formulate a generalised Yau-Tian-Donaldson conjecture. We also prove one direction of this conjecture, namely that existence of coupled Kähler-Einstein metrics imply K-stability.

9. Variational approaches

Both Equation (3) and Equation (5) have variational interpretations. These are in terms of two functionals: the Ding functional and the Mabuchi K-energy. Their significance is that (3) and (5) arise as the Euler-Lagrange equations of these functionals, i.e. Kähler-Einstein metrics are stationary points of the Ding functional and constant scalar curvature Kähler metrics are stationary points of the Mabuchi K-energy.

The important terms in these functionals are essentially given by fixing a reference ω_0 and integrating the relevant differential operator along a curve in the space of ω-plurisubharmonic functions. We get

$$E(f,\omega_0) = \int_{t=0}^{1} \int_X ((1-t)\omega_0 + t\omega_f)^n f \, dt \tag{6}$$

and

$$\mathcal{K}(f,\omega_0)=\int_{t=0}^{1}\int_X\Big(c-s\big((1-t)\omega_0+t\omega_f\big)\Big)\big((1-t)\omega_0+t\omega_f\big)^n f\,dt \tag{7}$$

Here we are integrating along straight lines from ω_0 to ω_f. However, it can be shown that the integrals are independent of the choices of curves. It follows that if f_t is a curve in the space of Kähler potentials defined for t near 1, then

$$\left.\frac{dE(f_t,\omega_0)}{dt}\right|_{t=1}=\int_X \dot{f}_1\omega_{f_1}^n$$

and

$$\left.\frac{d\mathcal{K}(f_t,\omega_0)}{dt}\right|_{t=1}=\int_X \dot{f}_1\left(c-s\left(\omega_{f_1}\right)\right)\omega_{f_1}^n.$$

The functional given by (7) is the Mabuchi K-energy and its stationary points are cscK metrics. The Ding functional is

$$D(f)=E(f,\omega_0)+\frac{1}{\lambda}\log\int_X e^{-\lambda f+h}\omega_0^n \tag{8}$$

and its stationary points are Kähler-Einstein metrics.

Both E and $\mathcal{K}$ can be rewritten into expressions that are easier to work with (see for example Equation (2.2) in Paper III and Equation 4.41 in [**9**]) and they play a major role in the study of Kähler-Einstein metrics and cscK metrics.

One way to understand the new type of metrics introduced in Paper III is to compare its generalised Ding functional (see Definition 113 in Paper III) to (8).

10. Toric geometry and beyond

An n-dimensional toric manifold is an n-dimensional complex manifold with a $(\mathbb{C}^*)^n$-action that admits an open, dense and free orbit. When studying these from the point of view of Kähler geometry you often restrict attention to those Kähler metrics that are invariant under the action of the subgroup $(S^1)^n \subset (\mathbb{C}^*)^n$. When doing this, much of the complex analytic framework described above reduces to convex analysis on $\mathbb{R}^n$. For example, an $(S^1)^n$-invariant plurisubharmonic function f on $(\mathbb{C}^*)^n$ can be represented by a convex function ϕ on $\mathbb{R}^n$. Moreover, in this case the complex Monge-Ampère operator can be expressed by the

Hessian determinant of ϕ:

$$(\sqrt{-1}\partial\bar{\partial}f)^n = \det\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right) dVol, \tag{9}$$

where $(x_1,\ldots,x_n)$ are the standard coordinates on $\mathbb{R}^n$ and $dVol$ is the Euclidean volume form on $\mathbb{C}^n$. The right hand side of this is the real Monge-Ampère operator. Using this, equation (4) can, on a toric manifold, be reduced to the real Monge-Ampère equation on $\mathbb{R}^n$

$$\det\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right) = e^{-\lambda\phi}, \tag{10}$$

subject to a certain boundary condition determined by the toric manifold. This convex analytic framework has been successfully used to study both Kähler-Einstein metrics and cscK metrics on toric manifolds. In Paper IV we use it to study the new type of canonical metrics introduced in Paper III, coupled Kähler-Einstein metrics, on toric manifolds.

A real torus is a quotient of $\mathbb{R}^n$ by a rank n lattice. There is a similar procedure as the one for toric manifolds to reduce complex Monge-Ampère equations on Abelian manifolds to real Monge-Ampère equations on real tori. In Paper I we study real Monge-Ampère equations on real tori and one motivation for this is their relationship to complex Monge-Ampère equations on Abelian manifolds.

11. Optimal transport

The real Monge-Ampère operator is part of a historical context that precedes the complex Monge-Ampère operator and Kähler geometry. It is related to a certain optimisation problem figuring in the 1781 memoirs of Gaspard Monge, namely optimal transport. This problem is defined by two probability measures on $\mathbb{R}^n$, one *source* measure μ and one *target* measure ν, together with a cost function $c(\cdot,\cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$. The problem is then to minimise the cost

$$C(T) = \int_{\mathbb{R}^n} c(x, T(x))d\mu \tag{11}$$

over all maps $T : \mathbb{R}^n \to \mathbb{R}^n$ that "transports" μ to ν in the sense that the push forward of μ under T is ν.

This problem has been revisited several times in history. Important contributors are Kantorovich, who in the 1940's reformulated it into a linear problem with respect to certain probability measures on $\mathbb{R}^n \times \mathbb{R}^n$

and determined its dual in terms of linear programming, Wasserstein who in 1979 used the optimal values of (11) to define a metric $W(\cdot,\cdot)^2$ on the space of probability measures on $\mathbb{R}^n$ and Knott and Smith in 1984 and Brenier in 1987 who showed that, in the case when the cost function is the squared distance function on $\mathbb{R}^n$, the optimal map T arise as the gradient map of a solution to a certain real Monge-Ampère equation. Given this, and the connection between Kähler-Einstein metrics and real Monge-Ampère equations explained in the previous section, it is not surprising that there are links between Kähler-Einstein metrics and optimal transport. One of these links is that if ϕ is a solution to (10), then the real Monge-Ampère measure of ϕ minimises the following functional defined on the space of probability measures on $\mathbb{R}^n$:

$$G(\mu) = -\lambda C(\mu) + Ent(\mu). \tag{12}$$

where

$$C(\mu) = W(dp,\mu)^2 - \int_{\mathbb{R}^n} |x|^2 \mu - c.$$

Here, dp is a certain measure on $\mathbb{R}^n$ determined by the toric manifold, c is a constant and $Ent(\mu)$ is the entropy of μ. A connection to the variational approaches in Section 9 is given by the following: If μ is the volume form of a Kähler metric in the first Chern class of a toric manifold, then G is the Mabuchi K-energi of this Kähler metric.

Variants of G play a crucial role in both Paper I and Paper II.

12. Hessian manifolds

A Kähler manifold is a complex manifold with a Riemannian metric that, loosely speaking, can be expressed locally as the second derivative of a function. On a toric manifold with a $(S^1)^n$-invariant plurisubharmonic function f, and the associated convex function ϕ on $\mathbb{R}^n$ we have that the Kähler metric g_f is essentially given by extending the metric

$$\sum_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i \otimes dx_j. \tag{13}$$

on $\mathbb{R}^n$. In other words, on toric manifolds many Kähler metrics can be expressed using the (real) Hessian of a convex function on $\mathbb{R}^n$. Turning this around, a natural question to ask is what extra structure on a smooth manifold is needed for the tensor (13) to be well-defined. Doing this we arrive at so called affine manifolds. An affine manifold is a smooth manifold with a distinguished atlas such that the transition functions are affine maps on $\mathbb{R}^n$. Indeed, it is easy to verify that (13) is invariant under affine transformations. A Riemannian metric on an affine manifold that locally can be expressed as (13) is called a Hessian metric and an affine manifold that admits a Hessian metric is called a Hessian manifold. Hessian manifolds provide a real analog of Kähler manifolds. Interest in these comes from questions in Mirror Symmetry and tropical geometry, in particular in the framework of the Strominger-Yau-Zaslow, Gross-Wilson and Kontsevich-Soibelman conjecures (see [**1**]). Here they appear as "large complex limits" of complex manifolds and in this context, solutions to real Monge-Ampère equations on Hessian manifolds are expected to appear as limits of Kähler-Einstein metrics on complex manifolds.

13. Hessian metrics and Monge-Ampère equations (Paper II)

A Hessian manifold is called special if the transition functions in the distinguished atlas are volume preserving. It is easy to verify that in this case the corresponding real Monge-Ampère measure

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) dx_1 \wedge \ldots \wedge dx_n \tag{14}$$

is well defined.

Motivated by the conjectures in Mirror Symmetry mentioned in the previous section, Paper II, which is written together with Magnus Önnheim, sets up a variational framework for real Monge-Ampère equations on compact special Hessian manifolds and use it to prove various existence and uniqueness results. This generalises results by Cheng and Yau [**8**] to a very singular setting (in fact, the results of Paper II can be applied to homogenous real Monge-Ampère equations where the right hand side is given by a sum of Dirac masses). We also show that there is a natural extension of the definition of the Monge-Ampère measure (14) to general (not necessarily special) Hessian manifolds and provide existence and uniqueness results for the resulting equation. This extension can be interpreted in terms of optimal transport and the main technical tool we develop for this is a type of Legendre transform that is invariant under affine coordinate transformation. The definition of this generalised Legendre transform is inspired by constructions in [**1**] and [**12**].

The approach provide a conceptually interesting picture in the same spirit as many of the results about canonical metrics in Kähler geometry since, starting with an object analogous to a polarisation of a projective variety in algebraic geometry, we use this data to define a corresponding ”cost function” in terms of optimal transport (an object which is normally associated with analysis and probability).

A natural but challenging continuation of this work would be to extend it to the singular and possibly non-compact settings of the conjectures in Mirror Symmetry above.

14. N-particle point processes

The main background for the probabilistic part of this thesis, i.e. Paper I, is Berman's work on Kähler-Einstein metrics and determinantal point processes. Berman shows how a projective manifold (X, L) where L is either K_X or $-K_X$ defines a sequence of point processes on X. Point processes on X are ways of randomly sampling points on X. Formally, the point processes will be defined by symmetric probability measures μ_N on the configuration space X^N where N is the number of points. The word symmetric means that the probability measures are invariant under permutations of the coordinates in X^N. If the points are though of as the positions of particles on X, then the symmetry property reflects the fact that the particles considered have identical properties. Moreover, the fact that the point processes can be defined in terms of probability measures on the configuration space X^N means we are considering point processes where the number of points in each outcome is fixed (as N). This is what we mean by N-particle point processes.

Before we outline Berman's work we will explain some basic concepts related to convergence of point processes. We will be interested in what happens when the number of points N tend to infinity. Since each probability measure, μ_N, is defined on a separate space, it is not immediately clear how to speak about the limit of μ_N as $N \to \infty$. One way to deal with this is to consider the *marginals* of μ_N. Let d be a positive integer. The d'th marginal of μ_N is a probability measure on X^d. It is denoted $(\mu_N)_d$ and it is defined by

$$(\mu_N)_d(E) = \mu_N(E \times X^{N-d})$$

for any measurable $E \subset X^d$. Loosely speaking, the first marginal describes what happens if we sample points according to μ_N and restrict attention to the first particle. Similarly, the second marginal describes what happens if we restrict attention to the first two particles. The point is that $(\mu_N)_d \in \mathcal{M}_1(X^d)$ for every N. Hence, given a probability measure

μ_* on X which we believe in some sense describes the "limit" of μ_N, there are two natural questions to ask:

- Does $(\mu_N)_1 \to \mu_*$?

and more generally, given any integer $d \geq 1$,

- Does $(\mu_N)_d \to \mu_*^{\otimes d}$?

The convergence here is in terms of the weak* topology on the space of probability measures on X and X^N respectively. If the first point holds, then sampling a large number of points and restricting our attention to the first point is close to sampling a point according to μ_*. If the second point holds, then sampling a large number of points and restricting our attention to the first d points is close to sampling d points independently and according to μ_*. If the second point holds for all positive d, then $\{\mu_N\}$ is said to be μ_*-chaotic.

One way of thinking about convergence of the first marginal is that it encodes the limiting behaviour of the points as a "cloud" but ignores any interaction between the particles. A good example to consider is the sequence of probability measures on $\{-1,1\}^N$ defined by

$$\mu_N = \delta_{(-1,\ldots,-1)}/2 + \delta_{(1,\ldots,1)}/2.$$

In words, this can be described as an experiment where, at each level N, the outcome is either N particles at -1 or (with the same probability) N particles at 1. These point processes exhibits high dependence between the variables. There is a strong attracting interaction between the particles. Now, μ_N satisfies

$$(\mu_N)_1 = \delta_{-1}/2 + \delta_1/2$$

for each N, thus we have convergence of the first marginal to $\delta_{-1}/2+\delta_1/2$. However, the second marginal takes the form

$$(\mu_N)_2 = \delta_{(-1,-1)}/2 + \delta_{(1,1)}/2$$

for each N, which is different from

$$(\delta_{-1}/2 + \delta_1/2)^{\otimes 2} = \delta_{(-1,-1)}/4 + \delta_{(-1,1)}/4 + \delta_{(1,-1)}/4 + \delta_{(1,1)}/4,$$

in other words the second marginals of μ_N don't converge to $(\delta_{-1}/2 + \delta_1/2)^{\otimes 2}$. The example above shows that convergence of first and second marginals are not equivalent.

An alternative way to deal with the problem that μ_N are defined on different spaces is to map the spaces X^N into the space of probability measures on X, $\mathcal{M}_1(X)$. As mentioned above, we may think of an element $x = (x_1, \ldots, x_N)$ in X^N as representing the position of N particles. Since we don't care about the order of the particles we might just as well represent them by the measure

$$\delta^{(N)}(x) = \frac{1}{N}\sum_{i}^{N} \delta_{x_i}$$

on X. We get a map $\delta^{(N)} : X^N \to \mathcal{M}_1(X)$. Each probability measure μ_N defines a random variable $x^{(N)}$ on X^N and we can think of $\delta^{(N)}(x^{(N)})$ as a random measure on X. It is called the *empirical measure*. The law of $\delta^{(N)}(x^{(N)})$ is given by the push-forward measure

$$\Gamma_{(N)} = \left(\delta^{(N)}\right)_* \mu_N \in \mathcal{M}_1(\mathcal{M}_1(X)).$$

Since $\Gamma_{(N)} \in \mathcal{M}_1(\mathcal{M}(X))$ for all N, we can ask the following question with respect to a possible "limit" μ_*:

- Does $\Gamma_{(N)} \to \delta_{\mu_*}$.

Here δ_{μ_*} denotes the Dirac measure at μ_* and the convergence is in terms of the weak* topology on $\mathcal{M}_1(\mathcal{M}_1(X))$), i.e. the space of probability measures on the space of probability measures on X. In terms of the random measure $\frac{1}{N}\sum \delta_{x_i}$, this convergence means $\frac{1}{N}\sum \delta_{x_i}$ converges *in law* towards the deterministic measure μ_*.

It is easy to show that convergence of the d'th marginal implies convergence of the j'th marginal for any $j < d$. On the other hand, by the example above, convergence of the first marginal is strictly weaker than convergence of the second marginal. In addition to this, the relationship between the different types of convergence discussed here is summarised in the following proposition:

Proposition 1 (See Proposition 2.2 in [**11**])**.** *In the notation above, the following are equivalent:*

- *The random measure $\frac{1}{N}\sum \delta_{x_i}$ converges in law to the deterministic measure μ_*, i.e. $\Gamma_N \to \delta_{\mu_*}$*
- *$\{\mu_N\}$ is μ_*-chaotic, i.e. $(\mu_N)_d \to \mu_*^{\otimes d}$ for all integers $d \geq 1$*
- *$(\mu_N)_2 \to \mu_*^{\otimes 2}$*

15. Canonical determinantal point processes

In a series of papers (see in particular [**3**] and [**4**]) Berman presents a method to construct canonical point processes in various settings of

Kähler geometry (for example Fano, anti-Fano, log-Fano etc). We will outline this in the (anti-Fano) case of (X, L) where L is the canonical line bundle K_X. Here it follows from the Aubin-Yau theorem that X admits a unique Kähler-Einstein metric.

A crucial observation in Berman's framework is the following: If we let N_k, for $k \in \mathbb{N}$, be the dimension of the space $H^0(X, kK_X)$ of holomorphic sections of kK_X, then a generator $\det(s_j)$ of the top exterior product $\wedge^{N_k} H^0(X, kK_X)$ defines (up to a multiplicative constant) a symmetric measure on X^{N_k} by

$$\mu^{(N_k)} = |\det(s_j)|^{2/k}. \tag{15}$$

One concrete way of seeing that this defines a measure is that, using local coordinates $(z_1, \ldots, z_{N_k})$ and local holomorphic representations of a basis $s_1, \ldots, s_{N_k}$ of $H^0(X, kK_X)$, the right hand side of (15) can be expressed as

$$|\det(s_i(z_j))|^{2/k}$$

where each s_i transforms as the k'th power of an $(n, 0)$-form. It follows that (15) transforms as a volume form on x^{N_k}. Since it is symmetric it defines (after normalisation) a symmetric probability measure on X^{N_k}. In other words, for each $k \in \mathbb{N}$ we get a canonical N_k-particle point process on X. As explained in the previous section, by identifying a point $(x_1, \ldots, x_{N_k})$ in X^{N_k} with the corresponding normalised sum of point masses we get a random measure $\frac{1}{N_k} \sum \delta_{x_j}$.

An illuminating example which is not from this setting but the Fano setting is $X = \mathbb{P}^1$. Choosing the standard coordinates excluding the point at infinity, $H^0(X, kK_X)$ can be represented by the space of polynomials on $\mathbb{C}$ of degree at most $N_k - 1 = 2k$. The density of (15) with respect to the Euclidean volume form on $\mathbb{C}^{N_k}$ is then given by the $-2/k'th$ power of the Vandermonde determinant

$$\det \begin{pmatrix} 1 & z_0 & \ldots & z_0^{N_k-1} \\ 1 & z_1 & \ldots & z_1^{N_k-1} \\ \vdots & & \ddots & \vdots \\ 1 & z_{N_k-1} & \ldots & z_{N_k-1}^{N_k-1} \end{pmatrix}.$$

One of the main results of Berman is:

Theorem 2 ([4]). *Assume X is a compact Kähler manifold and K_X is ample. Then the random measure defined by* (15) *converges in law towards $(\omega_{KE})^n$ where ω_{KE} is the unique Kähler-Einstein metric on X.*

In other words, for large k, these point processes approximate the volume form of the unique Kähler-Einstein metric on X.

16. Point processes on real tori (Paper I)

One development of the work in the previous section is a paper by Berman where he, by looking at the toric case, develops a corresponding probabilistic framework for real Monge-Ampr̀e equations on $\mathbb{R}^n$. As explained in Section 10, in the toric setting Kähler-Einstein metrics are given by solutions to certain real Monge-Ampère equations on $\mathbb{R}^n$. Moreover, the point processes considered by Berman in this case are not determinantal but given by so called permanents.

The purpose of Paper I is to set up a probabilistic framework for real Monge-Ampère equations on compact tori $\mathbb{R}^n/\Lambda$, where Λ is a rank n lattice in $\mathbb{R}^n$. The main result of the paper says that if the Monge-Ampère equations admits a unique solution, then the associated point processes converge to (the Monge-Ampère measures of) this solution. The paper elaborates further on a special case analogous to a certain complex Monge-Ampr̀e equation defining twisted Kähler-Einstein metrics on Abelian varieties with positive curvature everywhere except along a divisor. This is because one advantage with this setting compared to that of toric geometry is that the case of positive curvature (which in the probabilistic setting corresponds to *negative* temperature) can be approached in a more straight forward manner.

17. Large deviation principles

Both Paper I and Bermans original papers on the subject use a tool called Large Deviation Principles. We will explain it here since it ties in well with the tools from geometric analysis used in the thesis. In fact, large deviation principles can be thought of as a probabilistic analog of the variational approach to partial differential equations.

Essentially, a large deviation principle for a sequence of probability measures consists of bounds on how fast the probability of ”unlikely events”, or events that deviates a lot from what is expected, tends to zero. In our setting an example of an ”unlikely event” is when the random measure $\frac{1}{N}\sum \delta_{x_i}$ deviates a lot from its expected limit μ_*. These bounds are encoded in a *rate* and a *rate function*. The rate is a sequence of real numbers that tend to ∞ and the rate function is a non-negative real valued function on the sample space. A good but somewhat imprecise way of thinking of a large deviation principle for a sequence of measures $\{\Gamma_N\}$ is that, for large N, the probability measures Γ_N behave roughly as the densities

$$\Gamma_N \sim e^{-r_N G}$$

where r_N is the rate and G is the rate function. This captures the important fact that Γ_N is, for large N, concentrated around the minimisers of G.

The precise definition is as follows:

Definition 3. Let χ be a topological space, $\{\Gamma_N\}$ a sequence of probability measures on χ, G a lower semi continuous function on χ and r_N a sequence of numbers such that $r_N \to \infty$. Then $\{\Gamma_N\}$ satisfies a *large deviation principle* with rate function G and rate r_N if, for all measurable $E \subset \chi$,

$$(16) \qquad -\inf_{E^\circ} G \leq \liminf_{N\to\infty} \frac{1}{r_N} \log \Gamma_N(E) \leq \limsup_{N\to\infty} \frac{1}{r_N} \log \Gamma_N(E) \leq -\inf_{\bar{E}} G$$

where E° and $\bar{E}$ are the interior and the closure of E.

Note that putting $E = \chi$ in (16) gives

$$-\inf_{\chi} G \leq 0 \leq -\inf_{\chi} G$$

hence $\inf_\chi G = 0$. Assume G admits a unique minimiser, μ_*. In other words, μ_* is the unique point where $G = 0$. Then an "unlikely event" is a subset E of χ such that $\mu_* \notin \overline{E}$. By the lower semi-continuity of G we get $\inf_{\overline{E}} G > 0$. A large deviation principle then provides a bound on the probability of the event E in the sense that for any $\delta < \inf_{\overline{E}} G$ we get

$$\frac{1}{r_N} \log \Gamma_N(E) \leq -\delta,$$

or equivalently

$$\Gamma_N(E) \leq e^{-r_N \delta},$$

for large enough N.

As explained above, a large deviation principle with rate function G guarantees that for large N the probability measures are concentrated around the minimisers of G. In fact, if G admits a unique minimiser μ_* and $\{\Gamma_N\}$ satisfies a large deviation principle with G as rate function then the convergence $\Gamma_N \to \delta_{\mu_*}$ follows, which in our setting means the random measure $\frac{1}{N}\sum \delta_{x_i}$ converges in law to μ_*.

In Equation (12) we defined a functional closely related to the Mabuchi K-energy whose minimisers are the Monge-Ampère measures of solutions to real Monge-Ampère equations. The main result of Paper I is derived from a large deviation principle where the rate function is given by this functional.

18. A new type of canonical metrics (Paper III and Paper IV)

The problem of cscK metrics in complex geometry was originally proposed by Calabi in 1982 [**6**]. Given a compact Kähler manifold X with a Kähler class α Calabi suggested looking for a Kähler metric of constant scalar curvature ω_{cscK} (i.e. a cscK metric) in α. As explained above, in 2013 it was shown, as a major breakthrough, that a compact complex manifold with an anti-canonical polarisation $(X, -K_X)$ admits a Kähler-Einstein metric if and only if $(X, -K_X)$ is stable in the appropriate sense [**7**].

Paper III, which is written together with David Witt Nyström, introduces a new type of canonical metrics, providing a "hybrid" between general cscK metrics and the more understood Kähler-Einstein metrics, namely coupled Kähler-Einstein metrics. While the data defining Calabi's problem of constant scalar curvature is given by a pair (X, α) where α is a Kähler class, the data defining the problem of coupled Kähler-Einstein metrics is a compact Kähler manifold X together with a k-tuple of Kähler classes $(\alpha_i) = (\alpha_1, \ldots, \alpha_k)$ such that $\sum \alpha_i$ equals plus or minus the first Chern class of X. We look for k-tuples of Kähler metrics $\omega_1, \ldots, \omega_k$ representing these classes and satisfying

$$\operatorname{Ric} \omega_1 = \ldots = \operatorname{Ric} \omega_k = \pm \sum_i \omega_i. \tag{17}$$

Loosely speaking, the motivation for this is to bring some of the tools that is available for the Kähler-Einstein problem, and which played a major role in [**7**], to a more general setting. Without being too technical, it is worth mentioning that the regularity theory developed for the Kähler-Einstein problem can be applied to coupled Kähler-Einstein metrics as well. Moreover, Cheeger-Colding theory and its implications for Kähler-Einstein metrics, resting on positive lower bounds on the Ricci curvature, has a good chance of extending to coupled Kähler-Einstein metrics and the *Ding functional*, providing a useful alternative in the Kähler-Einstein case to the more general *Mabuchi functional* used for general cscK metrics, extends in a straight forward manner to coupled Kähler-Einstein metrics.

In Paper III we also extend a number of uniqueness and existence results for classical Kähler-Einstein metrics to the coupled setting and, perhaps most importantly, we prove that if a k-tuple of Kähler classes (α_i) as above admits a coupled Kähler-Einstein metric, then it satisfies a stability condition similar to the one in the Yau-Tian-Donaldson conjecture. This shows that the deep connection between canonical metrics and algebraic geometry is manifested through coupled Kähler-Einstein metrics and motivated by this, we made a similar conjecture as the one for cscK metrics, namely that a k-tuple of Kähler classes (α_i) as above admits a coupled Kähler-Einstein metric if and only if it is stable in the appropriate sense.

Paper IV, confirms this generalised Yau-Tian-Donaldson conjecture in the toric case. Here, as is the case for both Kähler-Einstein metrics (see [**14**]) and cscK metrics (see [**10**]), the algebraic stability condition in question simplifies to a concrete condition in terms of the associated polytopes in $\mathbb{R}^n$. More precisely, a k-tuple of Kähler classes (α_i) on a toric manifold X such that $\sum_i \alpha_i = c_1(X)$ defines a set of polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^n$ such that their (Minkowski) sum $\sum P_i$ equals the polytope P_{-K_X} associated to $-K_X$. Given a polytope P in $\mathbb{R}^n$, let $b(P)$ be its normalized barycenter

$$b(P) = \frac{1}{\mathrm{Vol}(P)} \int_P p\, dp,$$

where dp is the standard volume form in $\mathbb{R}^n$ and $\mathrm{Vol}(P)$ is the volume of P with respect to dp. The main result in Paper IV is that on a toric Fano manifold, existence of coupled Kähler-Einstein metrics with respect to a given k-tuple of Kähler classes (α_i) is equivalent to the following condition on the associated polytopes:

$$\sum_i b(P_i) = 0.$$

Moreover, both of these are equivalent to the appropriate notions of stability of (α_i). Aside from strengthening the plausibility of the general conjecture, this provides a good source of example and opens up for comparisons with the stability conditions for Kähler-Einstein metrics and general cscK metrics. The first of these is given by

$$b(P_{-K_x}) = 0,$$

and the second is (at least in the surface case) also given by a barycenter condition, but this time involving a certain measure on the boundary of the polytope (see [**10**] for details).

19. A coupled probabilistic framework (future project)

As explained in Section 15, Berman showed that Kähler-Einstein metrics can be given a probabilistic interpretation. It turns out that from this perspective, coupled Kähler-Einstein arise in a very natural way. This is one the projects I have in mind for the nearest future and to tie up this introduction we will outline it in the case when K_X is ample. It is proven in Paper III, as a generalisation of the Aubin-Yau theorem, that any k-tuple of Kähler classes such that $\sum_i \alpha_i = -c_1(X)$ admits a unique coupled Kähler-Einstein metric. In other words, (17) has a unique solution (ω_i) such that $[\omega_i] = \alpha_i$ for all i. Moreover, an important observation is that if (ω_i) is a coupled Kähler-Einstein metric, then

$$(\omega_1)^n = \ldots = (\omega_k)^n.$$

Fixing a k-tuple of line bundles (or more generally, $\mathbb{R}$-line bundles) $L_1, \ldots, L_k$ over X such that $\sum L_i = K_X$, we let N_m^i for each $i \in \{1, \ldots, k\}$ and $m \in \mathbb{N}$, be the dimension of the space $H^0\left(X, \left\lfloor \frac{m}{c_i} \right\rfloor L_i\right)$ where $c_i = \mathrm{Vol}(L_i)^{1/n}$ and $\lfloor \cdot \rfloor$ is the standard floor function. It follows that the quantities $N_m^1, \ldots, N_m^k$ grows with the same speed when $m \to \infty$. Moreover, we let $\det(s_j)_i$ be a generator of the top exterior product

$$\bigwedge^{N_m^i} H^0\left(X, \left\lfloor \frac{m}{c_i} \right\rfloor L_i\right)$$

over $X^{N_m^i}$. Assuming for simplicity that $N_m^1 = \ldots = N_m^k =: N_m$ for infinitely many m (I am confident that the general case can be handled by scaling and approximation arguments) one may verify that for these m

$$\mu^{(N_m)} = \prod_{i=1}^{k} |\det(s_j)_i|^{2/\left\lfloor \frac{m}{c_i} \right\rfloor} \tag{18}$$

defines (up to a multiplicative constant) a symmetric measure on X^{N_m}. Normalising, we get a canonical symmetric probability measure on X^{N_m}. This defines a canonical point process on X with N_m points. Equivalently, by identifying a point $(x_1, \ldots, x_{N_m})$ in X^{N_m} with the corresponding normalised sum of point masses gives a random measure $\frac{1}{N_m}\sum_j \delta_{x_j}$. I believe the following is true:

Conjecture 4. *Assume X is a compact Kähler manifold such that K_X is ample and $L_1, \ldots, L_k$ are $\mathbb{R}$-line bundles over X such that $\sum L_i = K_X$, then the random measure defined by* (18) *converge in law towards the volume form*

$$\mu^{cKE} = (\omega_1)^n = \ldots = (\omega_k)^n$$

where (ω_i) is the unique coupled Kähler-Einstein metric on X such that $[\omega_i] = c_1(L_i)$ for all i.

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Part II

PAPERS