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Temporal vs. spatial formulation of autonomous overtaking algorithms

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Abstract— **This paper concerns optimally controlling an autonomous vehicle to perform safe and comfortable overtaking of a slower moving leading vehicle using model predictive control. The contribution of this paper is to further analyze the convex relaxation that was introduced in [1] in order to see how it compares with the standard formulation. The main difference between the formulations is that the sampling is done in the temporal domain in the standard formulation, but in the spatial domain for the other. It is shown that it is easy to convert one formulation to the other. Further, it is shown that the formulations are identical under the assumption of constant longitudinal velocity of the controlled vehicle. However, when the longitudinal velocity is not fixed they are not identical, in fact, the temporal formulation becomes a mixed integer problem which is hard to convexify. On the other hand, the spatial formulation can be easily convexified using standard methods. Therefore, in the case of changing velocity, we have to rely on the spatial formulation to achieve good solutions in short computation time.**

I. INTRODUCTION

Aims such as decreasing the number of traffic accidents motivate the introduction of partially or fully automated systems [2]. Many partially or fully automated systems, such as adaptive cruise control [3], are already standard in production vehicles today. Recently attention has been drawn towards fully automated overtaking maneuvers where the vehicle needs to decide how to perform overtaking of a slow moving leading vehicle in an optimal way. The goal is to obtain position and velocity trajectories which the vehicle can follow in a way that is *i)* safe, *ii)* comfortable and *iii)* does not deviate from the prefered speed more than necessary. This should be done using an algorithm that *a)* is executable in real time, *b)* does not rely heavily on heuristics and *c)* should be able to adapt to unforseen events (e.g., a vehicle showing up in the adjacent lane).

There have been several suggestions on how to solve the problem stated above in the literature. In many cases the task of overtaking has been divided into three steps: moving from the current lane to the adjacent lane, move straight in the adjacent lane and finally go back to the original lane once the leading vehicle is passed (e.g., [4] and [5]). In [4] the overtaking is made using a so called shadow vehicle and the RG (Rendezvous -guidance) technique while in [5] by approximating the lane change manoeuvre using a fifth order minimal jerk trajectory [6]. Other approaches include grid/graph based search [7], [8] but these methods rely on the availability of good heuristics.

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Considering the requirements *i-iii* another natural approach is that of model predictive control (MPC) [9]. This is because of its ability to handle constraints and nonlinearities. MPC relies on iterative solutions of optimal control problems. These optimal control problems are in general non-convex which make them computationally heavy to solve. In our specific application, as can be seen in [9], we have the general case of a non-convex optimal control problem. In order to decrease the computation time, and thus fulfill the requirement *a*, attempts have been made to convexify the optimal control problem [9], [10]. This is advantageous since there are several efficient algorithms developed for solving convex optimization problems [11]. However, most attempts of introducing a convex formulation of the optimal control problem has reduced the flexibility of the algorithm. For example in the method presented in [10] a full overtaking trajectory cannot be calculated in each MPC stage. Not having a full overtaking trajectory could be a serious problem if we for some reason fail to reoptimize while it is impossible to terminate the overtaking due to surrounding vehicles. This issue was brought up in [1] where it was shown that one possible way of turning the model into a convex one is to transform the problem into the spatial domain. The advantages of formulating the problem in the spatial domain, as we shall see, is that it is easy to find a good convex relaxation that plans the full trajectory. In this way, the requirement *i-iii* and *a-c* can all be met at the same time. The safety requirements *i* are implemented as hard constraints while the comfort requirements *ii* are implemented in a softer way by punishing sudden change in velocity and acceleration in the objective function. These choices seem natural because it prioritize safety over comfort.

The focus of this paper is to investigate the approximations made in order to obtain the convex control problem introduced in [1] and investigate whether the solutions are comparable to the original time formulation. Further, we investigate the advantages and disadvantages of the space formulation compared to the time formulation. We will see that when the longitudinal velocity is constant the formulations result in identical convex problems. However, if the longitudinal velocity is not constant the temporal formulation will be a mixed integer optimization problem, which means its hard to convexify, while the space formulation can be convexified using simple linearization techniques.

The paper is organized as follows. In Section II both the temporal and spatial formulation of the optimal control problem are introduced. Then, in Section III the models are analyzed and compared, first under the assumption of

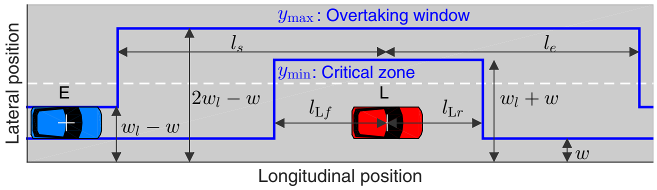

Fig. 1. Scenario where the ego vehicle (E) is overtaking a leading vehicle (L) on a road with two lanes. The center of gravity of the ego vehicle is allowed to reside between the limits $y_{\min}$ and $y_{\max}$, depicted by the thick solid lines.

constant velocity and then without this assumption. To support the discussion a case study consisting of two different overtaking scenarios is made. Finally, conclusions are drawn in Section IV.

II. Optimal control problem

A situation is studied where an automated vehicle, refered to as the ego vehicle (E) should overtake a slower moving leading vehicle (L). The ego vehicle is controlled by an MPC in which the optimal control problem is reoptimized in consecutive stages. Here, however, we consider only one stage of the optimization. Inside the prediction horizon of this optimization stage the road is assumed to be straight and have two lanes. Further, it is assumed that the leading vehicle is travelling at a constant velocity. In order to ensure that the overtaking can be performed safely a critical zone is defined around the leading vehicle into which the ego vehicle is not allowed to enter. See Fig. 1 for an illustration.

A. Problem formulation in time domain

Let

$$\mathbf{x}_{\mathrm{E}}(t) = [x_{\mathrm{E}}(t),\ \dot{x}_{\mathrm{E}}(t),\ \ddot{x}_{\mathrm{E}}(t)\ y_{\mathrm{E}}(t),\ \dot{y}_{\mathrm{E}}(t),\ \ddot{y}_{\mathrm{E}}(t)]^T,$$
$$\mathbf{u}_{\mathrm{E}}(t) = [\dddot{x}_{\mathrm{E}}(t),\ \dddot{y}_{\mathrm{E}}(t)]^T$$

be the state and control vector, respectively. Here, $x_{\mathrm{E}}(t)$ denote the longitudinal position of the ego vehicle and $y_{\mathrm{E}}(t)$ the lateral position. Since the vehicles are modeled as point mass systems the ego vehicle can be described by the state space model

$$\dot{\mathbf{x}}_{\mathrm{E}}(t) = A\mathbf{x}_{\mathrm{E}}(t) + B\mathbf{u}_{\mathrm{E}}(t) \tag{1}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2}$$

The following bounds are introduced on the state and control vectors

$$\mathbf{x}_{\mathrm{E}}(t) \in [\mathbf{x}_{\min}, \mathbf{x}_{\max}], \quad \mathbf{u}_{\mathrm{E}}(t) \in [\mathbf{u}_{\min}, \mathbf{u}_{\max}]$$

with the limits defined as

$$\mathbf{x}_{\min}(\cdot) = [0,\ v_{\mathrm{L}} + \epsilon,\ a_{x\min}(t),\ y_{\min}(\cdot),\ v_{y\min}(t),\ a_{y\min}(t)]^T$$
$$\mathbf{x}_{\max}(\cdot) = [\mathrm{f}, v_{x\max}(t), a_{x\max}(t), y_{\max}(\cdot), v_{y\max}(t), a_{y\max}(t)]^T$$
$$\mathbf{u}_{\min}(t) = [j_{x\min}(t),\ j_{y\min}(t)]^T$$
$$\mathbf{u}_{\max}(t) = [j_{x\max}(t),\ j_{y\max}(t)]^T$$

where f stands for free and the small positive number ϵ makes sure that the ego vehicle always has a higher speed than the leading vehicle. Further, the limits satisfy $v_{y\min}(t), a_{y\min}(t), a_{x\min}(t), j_{x\min}(t) \leq 0$ and $v_{y\max}(t), a_{y\min}(t), a_{x\max}(t), j_{x\max}(t) \geq 0$. The state bounds $y_{\min}(\cdot)$ and $y_{\max}(\cdot)$ keep the ego vehicle on the road and see to that overtaking is made at a safe distance. This is done by modeling the lateral limits as a rectangular critical zone around the leading vehicle, see Fig. 1. This gives the lateral constraints

$$\begin{aligned} y_{\min}(\cdot) &= \begin{cases} w_l + w, & x_{\mathrm{E}}(t) \in x_{\mathrm{L}}(t) + [-l_{\mathrm{L}f}, l_{\mathrm{L}r}] \\ w, & \text{otherwise} \end{cases} \\ y_{\max}(\cdot) &= \begin{cases} 2w_l - w, & x_{\mathrm{E}}(t) \in x_{\mathrm{L}}(t) + [-l_s, l_e] \\ w_l - w, & \text{otherwise} \end{cases} \end{aligned} \tag{3}$$

where x_L is the longitudinal position of the leading vehicle, w_l is the lane width and w is the lateral safety limit we wish to keep from the edge of the road. The constants $l_{\mathrm{L}f}$, $l_{\mathrm{L}r}$ define the size of the critical zone while l_s, l_e define an overtaking window, see again Fig. 1. In addition to these constraints a slip constraint is introduced to make sure that the optimal trajectory is actually possible to follow for a real vehicle. This constraint takes the form of a restriction on the lateral velocity in the following way

$$\dot{y}_{\mathrm{E}}(t) \in [s_{\min}, s_{\max}]\dot{x}_{\mathrm{E}}(t),$$

where $s_{\min} = -\tan(\beta)$, $s_{\max} = \tan(\beta)$ and β denotes the maximum slip angle. In the general case the angle β should be the summation of the heading angle and the slip angle but we make the assumption that the heading angle is zero at all times. This implies that the vehicle is moving "sideways" when switching lanes. With the introduction of the objective function $J(\mathbf{x}_{\mathrm{E}}(t), \mathbf{u}_{\mathrm{E}}(t))$, which will be detailed in Section (II-C), the time formulation of the optimal control problem is

$$\underset{\mathbf{x}_{\mathrm{E}}(t),\mathbf{u}_{\mathrm{E}}(t)}{\text{minimize}}\ J(\mathbf{x}_{\mathrm{E}}(t), \mathbf{u}_{\mathrm{E}}(t)) \tag{4a}$$
subject to
$$\dot{\mathbf{x}}_{\mathrm{E}}(t) = A\mathbf{x}_{\mathrm{E}}(t) + B\mathbf{u}_{\mathrm{E}}(t) \tag{4b}$$
$$\mathbf{x}_{\mathrm{E}}(\cdot) \in [\mathbf{x}_{\min}(\cdot), \mathbf{x}_{\max}(\cdot)] \tag{4c}$$
$$\mathbf{u}_{\mathrm{E}}(t) \in [\mathbf{u}_{\min}(t), \mathbf{u}_{\max}(t)] \tag{4d}$$
$$\dot{y}_{\mathrm{E}}(t) \in [s_{\min}, s_{\max}]\dot{x}_{\mathrm{E}}(t) \tag{4e}$$
$$\mathbf{x}_{\mathrm{E}}(0) = \mathbf{x}_{\mathrm{E}0} \tag{4f}$$

where the constraints (4b)-(4e) are implemented for all $t = 1, \ldots, t_f$ and t_f denotes the length of the prediction horizon. The initial state values are denoted by $\mathbf{x}_{\mathrm{E}0} = [x_{\mathrm{E}0}, v_{\mathrm{E}x0}, a_{\mathrm{E}x0}, y_{\mathrm{E}0}, v_{\mathrm{E}y0}, a_{\mathrm{E}y0}]$ and satisfy $\mathbf{x}_{\mathrm{E}0} \in$

$[\mathbf{x}_{\min}(0), \mathbf{x}_{\max}(0)]$. Note that there are no constraints on the final time t_f or the final position, but these are possible to introduce, if needed.

B. Problem formulation in relative space domain

Problem (4) is reformulated here into a model where the longitudinal velocity of the ego vehicle is measured relative to the leading vehicle and sampling is made into the relative longitudinal distance instead of time. For a detailed derivation of the model presented below, see [1]. If the velocity of the leading vehicle is denoted by v_L and $\tilde{x} = x_E - v_L t$ is the longitudinal sample variable, the state and control vectors can be written as

$$\hat{\mathbf{x}}_{\mathrm{E}}(\tilde{x}) = [\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x}),\ \dot{\tilde{x}}'_{\mathrm{E}}(\tilde{x}),\ y_{\mathrm{E}}(\tilde{x}),\ y'_{\mathrm{E}}(\tilde{x}),\ y''_{\mathrm{E}}(\tilde{x})]^T$$
$$\hat{\mathbf{u}}_{\mathrm{E}}(\tilde{x}) = [\dot{\tilde{x}}''_{\mathrm{E}}(\tilde{x}),\ y'''_{\mathrm{E}}(\tilde{x})]^T$$

where $\dot{\tilde{x}}(\tilde{x}) = \dot{x}(\tilde{x}) - v_L$ and $(\cdot)'$ denotes the derivative with respect to $\tilde{x}$, e.g., $y' = dy/d\tilde{x}$. The "hat" is added to distinguish the spatial state and control vectors from the temporal ones. The longitudinal distance, x_E, is not present in this new state vector since the relative distance is the new sampling variable. Time could be included as a state but we have chosen not to do so for simplicity. This leads to that the state space model is now given by the expression

$$\hat{\mathbf{x}}'_{\mathrm{E}}(\tilde{x}) = A\hat{\mathbf{x}}_{\mathrm{E}}(\tilde{x}) + B\hat{\mathbf{u}}_{\mathrm{E}}(\tilde{x})$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5}$$

and the modified limits

$$\begin{aligned} \hat{\mathbf{x}}_{\min}(\tilde{x}) &= [\epsilon,\ \frac{a_{x\min}(\tilde{x})}{\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x})},\ y_{\min}(\tilde{x}),\ \frac{v_{y\min}(\tilde{x})}{\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x})},\ \frac{a_{y\min}(\tilde{x})}{\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x})}]^T, \\ \hat{\mathbf{x}}_{\max}(\tilde{x}) &= [\tilde{v}_{x\max}(\tilde{x}),\ \frac{a_{x\max}(\tilde{x})}{\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x})},\ y_{\max}(\tilde{x}),\ \frac{v_{y\max}(\tilde{x})}{\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x})},\ \frac{a_{y\max}(\tilde{x})}{\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x})}]^T, \\ \hat{\mathbf{u}}_{\min}(\tilde{x}) &= \mathbf{u}_{\min}(\tilde{x})/\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x}), \\ \hat{\mathbf{u}}_{\max}(\tilde{x}) &= \mathbf{u}_{\max}(\tilde{x})/\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x}). \end{aligned} \tag{6}$$

where $\tilde{v}_{x\max}(\tilde{x}) = v_{x\max}(\tilde{x}) - v_{\mathrm{L}}$, $\mathbf{u}_{\min}(\tilde{x}) = [j_{x\min}(\tilde{x}), j_{y\min}(\tilde{x})]$ and $\mathbf{u}_{\max}(\tilde{x}) = [j_{x\max}(\tilde{x}), j_{y\max}(\tilde{x})]$. The lateral limits are in the spatial domain translated to

$$\begin{aligned} y_{\min}(\tilde{x}) &= \begin{cases} w_l + w, & \tilde{x} \in x_{\mathrm{L0}} + [-l_{\mathrm{L}f}, l_{\mathrm{L}r}] \\ w, & \text{otherwise} \end{cases} \\ y_{\max}(\tilde{x}) &= \begin{cases} 2w_l - w, & \tilde{x} \in x_{\mathrm{L0}} + [-l_s, l_e] \\ w_l - w, & \text{otherwise.} \end{cases} \end{aligned} \tag{7}$$

while the slip constraints become

$$y'_{\mathrm{E}}(\tilde{x}) \in [s_{\min}, s_{\max}](1 + v_{\mathrm{L}}/\dot{\tilde{x}}_{\mathrm{E}}(\tilde{x})). \tag{8}$$

This gives the following complete model for the space formulation

$$\min_{\hat{\mathbf{u}}_{\mathrm{E}}(\tilde{x})} \tilde{J}(\hat{\mathbf{x}}_{\mathrm{E}}(\tilde{x}), \hat{\mathbf{u}}_{\mathrm{E}}(\tilde{x})) \tag{9a}$$

subject to

$$\dot{\hat{\mathbf{x}}}_{\mathrm{E}}(\tilde{x}) = A\hat{\mathbf{x}}_{\mathrm{E}}(\tilde{x}) + B\hat{\mathbf{u}}_{\mathrm{E}}(\tilde{x}) \tag{9b}$$
$$\hat{\mathbf{x}}_{\mathrm{E}}(\tilde{x}) \in [\hat{\mathbf{x}}_{\min}(\tilde{x}), \hat{\mathbf{x}}_{\max}(\tilde{x})] \tag{9c}$$
$$\hat{\mathbf{u}}_{\mathrm{E}}(\tilde{x}) \in [\mathbf{u}_{\min}(\tilde{x}), \mathbf{u}_{\max}(\tilde{x})]\frac{1}{\dot{\tilde{x}}(\tilde{x})} \tag{9d}$$
$$y'_{\mathrm{E}}(\tilde{x}) \in [s_{\min}, s_{\max}]\left(1 + \frac{v_L}{\dot{\tilde{x}}(\tilde{x})}\right) \tag{9e}$$
$$\hat{\mathbf{x}}_{\mathrm{E}}(0) = \hat{\mathbf{x}}_{\mathrm{E0}} \tag{9f}$$

where (9b)-(9e) are imposed for all $\tilde{x} \in [0, \tilde{x}_f]$ and $\tilde{x}_f$ denote the final longitudinal position of the ego vehicle.

Before moving on to discuss the objective functions of the two formulations (4) and (9) let us remark on the choice of states in the space formulation (9). The direct transformation of the longitudinal acceleration from time to relative space domain is

$$\ddot{x}_E = \dot{\tilde{x}}'_E \dot{x}_E. \tag{10}$$

Notice that the control signals chosen here is not the vehicle acceleration but instead the space derivative of the longitudinal speed. Similarly for the lateral speed the direct transformation is

$$\dot{y}_E = y'_E x_E \tag{11}$$

which motivates the choice of space derivative of position in the lateral direction.

C. Quadratic objective function

Several different objective functions can be under consideration for (4a) and (9a). In this paper, as in [1], a quadratic objective function has been chosen that penalizes deviations from a longitudinal reference velocity v_r, deviations from a reference lateral position y_r and control actions. With $\|x\|_A := x^T A x$ the objective functions are formulated as

$$J(\cdot) = \int_0^{t_f} \|\mathbf{x}_{\mathrm{E}}(t) - \mathbf{x}_{\mathrm{r}}(t)\|_Q^2 dt + \int_0^{t_f} \|\mathbf{u}_{\mathrm{E}}(t)\|_R^2 dt \tag{12}$$

and

$$\tilde{J}(\cdot) = \int_0^{\tilde{x}_f} \|\hat{\mathbf{x}}_{\mathrm{E}}(\tilde{x}) - \hat{\mathbf{x}}_{\mathrm{r}}(\tilde{x})\|_{\tilde{Q}}^2 d\tilde{x} + \int_0^{\tilde{x}_f} \|\hat{\mathbf{u}}_{\mathrm{E}}(\tilde{x})\|_{\tilde{R}}^2 d\tilde{x} \tag{13}$$

where $\mathbf{x}_{\mathrm{r}}(t) = [0, v_r(t), 0, y_r(t), 0, 0]^T$ and $\hat{\mathbf{x}}_{\mathrm{r}}(\tilde{x}) = [\tilde{v}_r(\tilde{x}), 0, y_r(\tilde{x}), 0, 0]^T$ are the vectors of reference trajectories and Q, $\tilde{Q}$, R and $\tilde{R}$ are semidefinite block diagonal matrices containing the penalty weights.

III. Comparison of the two methods

The case of constant and non-constant longitudinal velocity of the ego vehicle will be studied in order to compare and analyze the two optimization problems introduced in Section II. In Scenario 1 the ego vehicle has a constant longitudinal velocity while overtaking a slower moving leading vehicle, i.e., $\dot{\tilde{x}} = v_{\mathrm{E}x0} = v_r$, and it is shown that the two formulations are equivalent and convex under this assumption. To illustrate this a simulation is included which show that the space formulation (9) gives the same solution as the time formulation (4). This scenario is studied in Section III-A. In scenario 2, discussed in section III-B, the ego vehicle starts with a velocity lower than its prefered velocity, i.e., $v_{\mathrm{E}x0} < v_r$, which makes it accelerate during overtaking. It is shown that the space formulation (with some approximations) is still a convex optimization problem under varying speed and that it yields safe and comfortable solutions also in this case. In practice, a change in ego vehicle speed may also be required for safety reasons, e.g., when the overtaking is performed in the neighborhood of other surrounding vehicles [1].

In the implementations the optimal control problems are first discretized with step sizes of $0.15\,\mathrm{s}$ and $5/6\,\mathrm{m}$ and then solved using a customized solver generated by the Forces Pro [12]. Forces pro is a software for Predictive Control on embedded systems. In contrast to, for example, the CVX modeling language [13], [14] Forces Pro do not prefer to have the problem given as a standard QP, but rather as a multi-stage formulation, where one stage refers to one step in the prediction horizon.

A. Scenario 1: $v_{Ex0} = v_r$

Since the longitudinal velocity of the ego vehicle is constant in this scenario no variables are needed to describe the longitudinal state of the leading vehicle. This will lead to simplification for both of the models discussed above. For the time formulation (4) introduced in Section II-A this implies that the lateral limits are simplified since the position of the ego vehicle is known at each instant. All constraints of the time formulation (4) are linear under this assumption, so together with the quadratic objective function (12) the problem is a convex Quadratic program (QP). Likewise, the space formulation (9) is a convex QP, using the objective function (13). So, in this case the problems are identical convex QPs.

1) Translation of objective function: For simplicity of notation let $\dot{y} = v_{\mathrm{E}y}$ and $\dot{x} = v_{\mathrm{E}x}$ and assume that the objective function to the time formulation (4) is given by

$$\int_0^{t_f} \Big(w_y (y_{\mathrm{E}}(t) - y_r(t))^2 + w_v v_{\mathrm{E}y}(t)^2 + w_a \dot{v}_{\mathrm{E}y}(t)^2 + w_j \ddot{v}_{\mathrm{E}y}(t)^2 \Big) dt. \tag{14}$$

Further, let the objective function to the space formulation (9) be given by

$$\int_0^{\tilde{x}_f} \Big(\tilde{w}_y (y_{\mathrm{E}}(\tilde{x}) - y_r(\tilde{x}))^2 + \tilde{w}_v v_{\mathrm{E}y}(\tilde{x})^2 + \tilde{w}_a v'_{\mathrm{E}y}(\tilde{x})^2 + \tilde{w}_j v''_{\mathrm{E}y}(\tilde{x})^2 \Big) d\tilde{x}. \tag{15}$$

Let us now determine the weights of (15) in terms of the weights of (14). To do so the following manipulations of the derivatives are made

$$\begin{aligned} v_{\mathrm{E}y} &= \frac{dy_{\mathrm{E}}}{dt} = \frac{dy_{\mathrm{E}}}{d\tilde{x}}\frac{d\tilde{x}}{dt} = y'_{\mathrm{E}} v_{\mathrm{E}x} \\ \dot{v}_{\mathrm{E}y} &= v'_{\mathrm{E}y}\frac{d\tilde{x}}{dt} = (y'_{\mathrm{E}} v_{\mathrm{E}x})' v_{\mathrm{E}x} = y''_{\mathrm{E}} v_{\mathrm{E}x}^2 + y'_{\mathrm{E}} v'_{\mathrm{E}x} v_{\mathrm{E}x} \\ \ddot{v}_{\mathrm{E}y} &= y'''_{\mathrm{E}} v_{\mathrm{E}x}^3 + 3 y''_{\mathrm{E}} v_{\mathrm{E}x}^2 v'_{\mathrm{E}x} + y'_{\mathrm{E}} (v''_{\mathrm{E}x} v_{\mathrm{E}x} + (v'_{\mathrm{E}x})^2) \end{aligned} \tag{16}$$

where the expression for $\ddot{v}_{\mathrm{E}y}$ is derived in the same fashion as the expressions for $v_{\mathrm{E}y}$ and $\dot{v}_{\mathrm{E}y}$. Since $v_{\mathrm{E}x}$ is constant $v'_{\mathrm{Ex}} = 0$, which gives $v_{\mathrm{E}y} = y'_{\mathrm{E}} v_{\mathrm{E}x}$, $\dot{v}_{\mathrm{E}y} = y''_{\mathrm{E}} v_{\mathrm{E}x}^2$ and $\ddot{v}_{\mathrm{E}y} = y''' v_{\mathrm{E}x}^3$. Setting this into (14) and changing the integration variable from t to $\tilde{x}$ yields

$$\int_0^{\tilde{x}_f} \Big(w_y (y_{\mathrm{E}}(\tilde{x}) - y_r(\tilde{x}))^2 + w_v (y'_{\mathrm{E}}(\tilde{x}))^2 v_{\mathrm{E}x}^2 + w_a v_{\mathrm{E}x}^4 y''_{\mathrm{E}}(\tilde{x})^2 + w_j v_{\mathrm{E}x}^6 (y'''(\tilde{x}))^2 \Big) d\tilde{x} \frac{1}{v_{\mathrm{E}x}}$$

which by comparison with (15) gives us the weights

$$\tilde{w}_y = \frac{w_y}{v_{\mathrm{E}x}},\ \tilde{w}_v = w_v v_{\mathrm{E}x},\ \tilde{w}_a = w_a v_{\mathrm{E}x}^3,\ \tilde{w}_j = w_j v_{\mathrm{E}x}^5 \tag{17}$$

So, (15) tells us how the weights should be chosen to make the optimization problem equivalent in the temporal and spatial domain. Note that it is of course possible to generalise this to even more derivatives on y if necessary.

2) Simulation: The ego vehicles initial and reference velocity are both chosen to be $70\,\mathrm{km/h}$, i.e., $v_{\mathrm{E}x0} = v_r = 70\,\mathrm{km/h}$, while the leading vehicle is traveling at a speed of $v_{\mathrm{L}} = 50\,\mathrm{km/h}$. The initial distance between the ego vehicle and leading vehicle is $75\,\mathrm{m}$. The reference position of the ego vehicle is chosen to be in the middle of its own lane except above the critical zone of the leading vehicle where it is instead chosen to be in the middle of the adjacent lane as depicted in Fig. 2. All the problem data are summarized in Table I.

Note that the weight matrices presented in Table I are for the time formulation (4) only. The corresponding weights for the space formulation (9) are chosen according to the weight transformation (17).

The result of the simulation is shown in Fig. 2. It is seen that the two models yield more or less the same solutions. The small differences in lateral velocity and lateral acceleration that are present are due to numerical errors. It may also be worth noting that all of the plots, the lateral position, velocity and acceleration are smooth. Thus, it can be concluded that the two methods yield the same safe and comfortable solution in scenario 1.

B. Scenario 2: $v_{Ex0} < v_r$

In this scenario the ego vehicle has an initial velocity $v_{\mathrm{E}x0}$ than its prefered velocity v_r which will make the ego vehicle accelerate during overtaking. Such a situation might appear in real applications if there, for instance, is an oncoming vehicle in the overtaking lane. If this oncoming vehicle is at an adequate distance the ego vehicle could decide to speed up and overtake the leading vehicle before meeting the oncoming one. Adding an oncoming vehicle, however, requires the addition of extra constraints [1] which

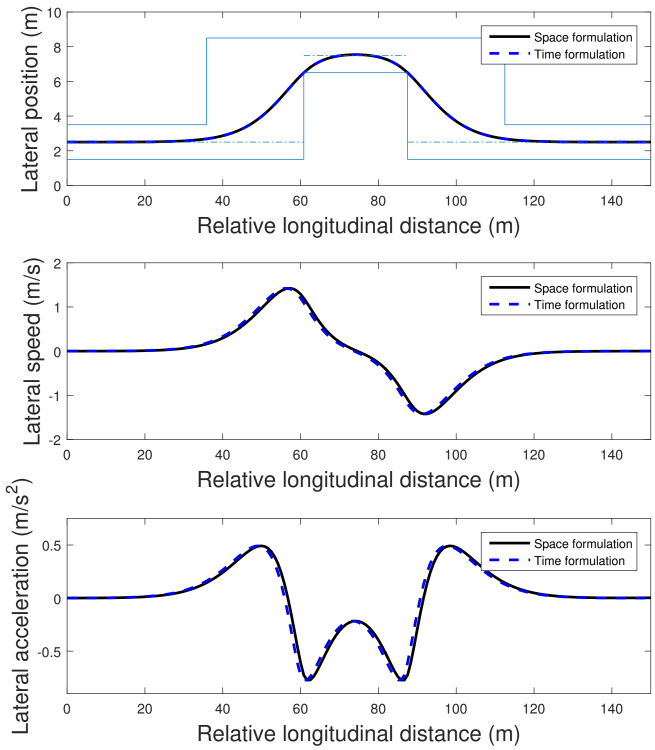

Fig. 2. Scenario where the ego vehicle with a fixed velocity is overtaking a leading vehicle on a road with two lanes. The center of gravity of the ego vehicle is allowed to reside between the limits y_{min} and y_{max}, depicted by the solid lines. The dash-dotted line is the reference position of the ego vehicle.

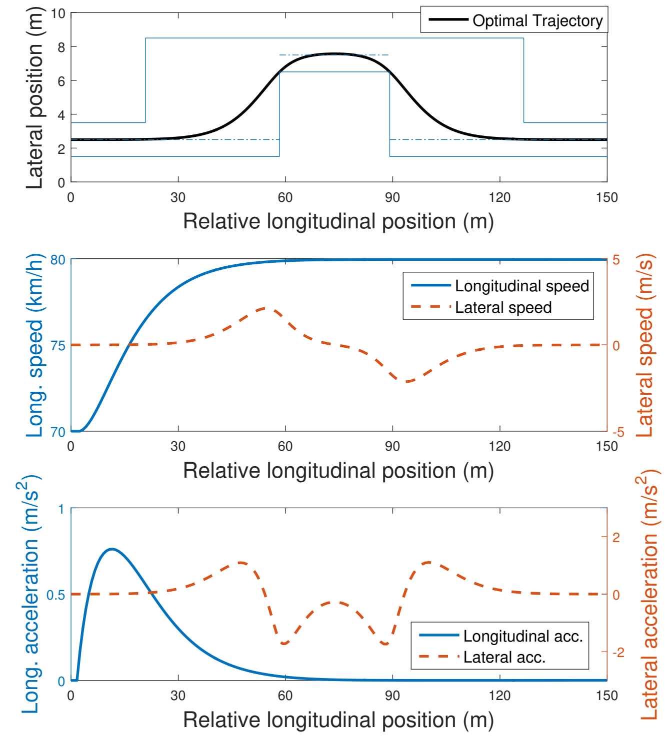

Fig. 3. Scenario where the ego vehicle is overtaking a leading vehicle on a road with two lanes while increasing its velocity from 70 km/h to the prefered 80 km/h. The center of gravity of the ego vehicle is allowed to reside between the limits y_{min} and y_{max}, depicted by the solid lines. The dash-dotted line is the reference position of the ego vehicle.

TABLE I
PROBLEM DATA.

Ego vehicle	Leading vehicle	Problem specific parameters
$x_{\text{E}0} = 0\,\text{m}$	$x_{L0} = 75\,\text{m}$	$\tilde{x}_f = 150\,\text{m}$
$y_{\text{E}0} = 2.5\,\text{m}$	$y_{L0} = 2.5\,\text{m}$	$t_f = 27\,\text{s}$
$v_r = 70\,\text{km/h}$	$l_{Lf} = 15\,\text{m}$	$Q = \text{diag}(0, 0.1, 0.5, 1, 2, 1.5)$
$v_{x0} = 70\,\text{km/h}$	$l_{Lr} = 12.3\,\text{m}$	$R = \text{diag}(0.5, 0.3)$
$v_{x\text{min}} = 0\,\text{km/h}$	$l_s = 40\,\text{m}$	$w = 1.5\,\text{m}$
$v_{x\text{max}} = 80\,\text{km/h}$	$l_e = 37.3\,\text{m}$	$w_l = 5\,\text{m}$
$v_{y\text{min}} = -4\,\text{m/s}$		$\epsilon = 0.01$
$v_{y\text{max}} = 4\,\text{m/s}$		
$a_{x\text{min}} = -4\,\text{m/s}^2$		
$a_{x\text{max}} = 1\,\text{m/s}^2$		
$\beta = 10^\circ$		
$v_L = 50\,\text{km/h}$		

complicate the analysis and therefore we choose to force the ego vehicle to speed up by using the reference velocity instead.

The layout for the analysis of this scenario follows the layout of scenario 1. So, in the following subsections convexity of the optimization problems in the time formulation (4) and space formulation (9), translation of the objective function and simulation results are discussed.

1) Convexity discussion: The time sampling optimization problem (4) will no longer be convex. The non-convexity arises in the state limits (4c) or, more precisely, in the lateral limits (3). Since these limits are mixed integer there is no straightforward way of transforming the time formulation (4) into a convex problem.

Neither the space sampling optimization problem (9) is a convex problem for this scenario. This is not because of to the lateral limits (7) but instead due to the fact that part of the constraints in (9c) and (9d)-(9e) are non-convex (see (6)). Notice that whether these constraints are convex or not depend on the sign of the limits, which were defined in Section II-A. Further, as we shall see bellow, non-constant velocity also introduces non-linearities in the objective function for the space formulation (9). However, in contrast to the time formulation, the space formulation contains no integer variables so it can be turned into a convex problem by applying standard linearization techniques on the constraints and straightforward approximations on the objective function, as will be shown in the next two sections.

2) Linearization of constraints: As proposed in [1] the constraints can be modeled as convex just by linearizing them around a reference velocity $v_r(\tilde{x})$, i.e,

$$\frac{1}{\dot{\tilde{x}}(\tilde{x})} \approx \frac{1}{\tilde{v}_r(\tilde{x})}\left(2 - \frac{\dot{\tilde{x}}(\tilde{x})}{\tilde{v}_r(\tilde{x})}\right). \tag{18}$$

The important thing to note is that these linearizations are inner approximations of the feasible set since $\frac{1}{x}$ is a convex function for $x > 0$. This quarantees that the solutions obtained from this new problem is feasible in the original problem, i.e., in the time formulation (4). Of course, an optimal solution to the convexified version of the space formulation does not guarantee an optimal solution in the original problem since the optimal solution may lie in the part of the feasible set which is cut away by the inner approximations.

3) Approximation of objective function: The objective function of the space formulation (9) is non-convex since the longitudinal velocity $v_{\mathrm{E}x}$ in transformations of the derivatives (16) is no longer constant and should be replaced by the state $\dot{\tilde{x}}$, which when substituted into (15) will result in a non-convex objective function. In order to convexify the objective function it is argued that the variable $\dot{\tilde{x}}(\tilde{x})$ can be substituted for the mean velocity of the ego vehicle, v_m. This approximation leads to a simplification of (16) to

$$v_{\mathrm{E}y} = y'_{\mathrm{E}} v_m, \quad \dot{v}_{\mathrm{E}y} = y''_{\mathrm{E}y} v_m^2, \quad \ddot{v}_{\mathrm{E}y} = y''' v_m^3.$$

In turn this implies that the weight transformation formula for the case of non-constant longitudinal velocity becomes almost identical to , i.e.,

$$\tilde{w}_y = \frac{w_y}{v_m},\ \tilde{w}_v = w_v v_m,\ \tilde{w}_a = w_a v_m^3,\ \tilde{w}_j = w_j v_m^5. \quad (19)$$

If we compare this to the weight transformation for scenario 1, (17), the only difference is that the constant velocity $v_{\mathrm{E}x}$ has been exchanged for the mean velocity v_m. This means that the size of change is dependent only on the change of the relative velocity.

4) Simulation: The problem data is the same as presented in Table I except that $v_r = 80\,\mathrm{km/h}$. The result for the space formulation (9) is shown in Fig. 3 (note that no simulation is presented for the time formulation (4)). As can be seen the resulting trajectory and the lateral velocity and acceleration are smooth curves also in this scenario. Further, the longitudinal velocity is increasing smoothly from its initial value to its reference velocity. Comparing with the results for the first scenario one can note that the bell shape of the lateral trajectory is narrower for the second scenario. This is expected since the ego vehicle should spend a shorter time overtaking the leading vehicle. Even though this may not be the optimal solution to the original time sampled non-convex problem (4) it is a reasonable feasible solution for a low computation cost.

IV. Conclusions

In this paper the non-convex optimal control problem that arise when taking an MPC approach to the overtaking problem has been discussed. A convexification of the problem, introduced in [1], has been analyzed and compared with the non-convex standard formulation. It was first shown that, via translation of the objective function weights, the two optimization problems are identical convex QPs under the assumption of constant longitudinal velocity. This was illustrated by the simple scenario of overtaking a slower moving leading vehicle where both vehicles had fixed, but different, longitudinal velocities. Further, it was shown that the space sampling formulation can easily be convexified in the case when the longitudinal velocity of the ego vehicle is not fixed. This is done by standard linearizations of some constraints and simple approximations in the translation of the weights for the objective function. A simulation of a scenario where the ego vehicle wants to accelerate while overtaking a slower moving leading vehicle was made for the convexified problem to show that smooth speed and acceleration trajectories can be obtained even when penalizing space derivatives of position and velocity.

Future research can include investigation on other convex formulations of the problem by, for instance, considering certain variable changes. More practical issues would be to test the model in an actual MPC approach in the presence of noise and disturbance.

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