Thesis for the Degree of Licentiate of Philosophy

Symplectic methods for Hamiltonian isospectral flows and 2D incompressible Euler equations on a sphere

Milo Viviani

CHALMERS | GÖTEBORGS UNIVERSITET

Division of Applied Mathematics and Statistics
Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg
Göteborg, Sweden 2018

Symplectic methods for Hamiltonian isospectral flows and 2D incompressible Euler equations on a sphere
Milo Viviani

© Milo Viviani, 2018.

Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg
SE-412 96 GÖTEBORG, Sweden
Phone: +46 (0)31 772 1000

Author e-mail: viviani@chalmers.se

Typeset with LaTeX.
Department of Mathematical Sciences
Printed in Göteborg, Sweden 2018

Symplectic methods for Hamiltonian isospectral flows and 2D incompressible Euler equations on a sphere

Milo Viviani

Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg

Abstract

The numerical solution of non-canonical Hamiltonian systems is an active and still growing field of research. At the present time, the biggest challenges concern the realization of structure preserving algorithms for differential equations on infinite dimensional manifolds. Several classical PDEs can indeed be set in this framework. In this thesis, I develop a new class of numerical schemes for Hamiltonian isospectral flows, in order to solve the hydrodynamical Euler equations on a sphere. The results are presented in two papers.

In the first one, we derive a general framework for the isospectral flows, providing then a class of numerical methods of arbitrary order, based on the Lie–Poisson reduction of Hamiltonian systems. Avoiding the use of any constraint, we obtain a large class of numerical schemes for Hamiltonian and non-Hamiltonian isospectral flows. One of the advantages of these methods is that, together with the isospectrality, they have near conservation of the Hamiltonian and, indeed, they are Lie–Poisson integrators.

In the second paper, using the results of the first one, we present a numerical method based on the geometric quantization of the Poisson algebra of the smooth functions on a sphere, which gives an approximate solution of the Euler equations with a number of discrete first integrals which is consistent with the level of discretization.

Keywords: Geometric integration, Symplectic methods, Structure preserving algorithms, Lie–Possion systems, Hamiltonian systems, Isospectral flows, Euler equations, Fluid dynamics.

List of appended papers

Paper I K. Modin, **M. Viviani**.
Lie–Poisson methods for isospectral flows.
Preprint

Paper II **M. Viviani**.
A structure preserving scheme for the Euler equations on a (rotating) sphere.
Preprint

My contribution to the appended papers:

Paper I: I have developed the numerical methods and most of the theoretical framework, drafted the manuscript and, after consultation, produced the final manuscript, except for the first two introductory sections. I have implemented the code, under the supervision of professor K. Modin.

Paper II: Independently developed and written.

Acknowledgements

I would like to thank my supervisor, professor Klas Modin, for the interesting and enlightening discussions and suggestions during the last two and half years, for the good moments around the world.

Heartfelt thanks to professor Stig Larsson, for having always been a reference point in these years.

Thank you Anders, a great scientist and a better friend. Thank you Efthymios for your $\varphi\iota\lambda\iota\alpha$ and your precious teachings. A great thanks to the "lunch group" and colleagues: Maximilian, Olof, Simone, Carl, Valentina, Medet, Helga, Christian, Jimmy, Adam, Mårten, Johannes, Hanna, Marco, Hung, Geir, Dawan, Matteo, Peter. A special mention, for all the professors, the students and the staff of Chalmers, with whom I have shared the intellectual hill of Johanneberg.

Thank you also to all the friends in Göteborg: Carlo III, Marco, Antonio, Layla, and all the others that I have met in these years. I remember all of you.

To my family and friends still close, no matter how far, thank you, I simply owe it all to you.

Milo Viviani
Gothenburg, May 2018

Contents

Introduction **1**
History . . . 1
Motivation . . . 2

1 Lie–Poisson systems **3**
1.1 Poisson structures and Hamiltonian systems . . . 3
1.2 Lie–Poisson systems . . . 5
1.3 Co-adjoint representation . . . 6
1.4 Momentum maps and Lie–Poisson reduction . . . 8
1.4.1 Lie–Poisson reduction . . . 9
1.5 Lie–Poisson systems on $\mathfrak{gl}(n, \mathbb{C})^*$ . . . 10
1.5.1 ad vs ad^* . . . 10
1.5.2 Euler–Poincaré equations and their representations . . . 10
1.5.3 Lie–Poisson maps on $\mathfrak{gl}(n, \mathbb{C})^*$ . . . 12

2 Isospectral flows and their numerical solution **14**
2.1 Isospectral flows and their properties . . . 14
2.1.1 Restriction to a subspace of $\mathfrak{gl}(n, \mathbb{C})$ . . . 15
2.2 Numerical approximation of the isospectral flows . . . 15

3 2D Euler equations on the sphere and their numerical solution **17**
3.1 Hydrodynamical Euler equations . . . 17
3.2 Geometric structure of the Euler equations . . . 18
3.3 L_α-convergence . . . 20
3.4 The reduced system . . . 22
3.4.1 With the Coriolis force . . . 22

4 Summary of the results in the papers **22**
4.1 Paper I: Lie–Poisson methods for isospectral flows . . . 22
4.2 Paper II: A structure preserving scheme for the Euler equations on a (rotating) sphere . . . 24

5 Proposals for future work **27**
5.1 Paper I: Lie–Poisson methods for isospectral flows . . . 27
5.2 Paper II: A structure preserving scheme for the Euler equations on a (rotating) sphere . . . 27

Bibliography **28**

Introduction

History

The following thesis would like to summarize the last two and half years spent on the study of the numerical solution of Lie–Poisson Hamiltonian systems and their connection with other fields of mathematics and applications.

The starting point of the present research dates back to my master thesis, defended in September 2015. In that work I was interested in the numerical solution of the hydrodynamical Euler equations on a rotating sphere with continuous and singular (point vortices) vorticity fields. The aim of that thesis was to get a numerical method which retained the main geometric properties of the continuous equations in the discrete case. The hydrodynamical Euler equations are indeed a classical example of Lie–Poisson Hamiltonian system. This means that the equations encode a lot of symmetries and therefore conservation laws. What had motivated my research was that there was not yet an established and efficient way to integrate the Euler equations respecting those symmetries.

Eventually that thesis did not give a satisfactory result and the research had to be continued during my PhD studies, started in October 2015 under the supervision of prof. Klas Modin. During the first one and half years the results obtained were quite satisfactory but still not really innovative. The main reason was that our simulations of the Euler equations required very large matrices and the algorithm developed still had too many implicit equations to be solved in order to be really applicable.

Finally, we came to a turning point. In our approach, it was clear that, to retain the first integrals, we needed to put some constraints on the equations. However, what if the constraints could be instead intrinsically encoded into the numerical method? This was not in general a feasible approach but surprisingly it turned out that, in this case, aiming for simplicity was rewarding. Working with this idea in mind it was possible to generate several numerical methods much simpler and efficient than the previous ones. Moreover a lot of Lie–Poisson systems could then be solved with the same approach and in fact, for any quadratic Lie algebra, it was easy to derive a Lie–Poisson integrator of any order. An encouraging fact was also that the methods developed looked like to be the natural ones, requiring only the information coming from the Lie algebra.

Here I present these results in the following way. In the first section I will introduce the general framework of Lie–Poisson Hamiltonian systems and some remarks on the Poisson reduction. In section two, the theory and the numerics of isospectral flows will be presented and discussed. In section three I will focus on the numerical solution of the Euler equations on a sphere, which had been the main source and aim of the whole research. Finally, I will conclude with a summary of the papers and the aim of future research. In appendix, paper I-II

are attached.

Motivation

The problems here presented are a classical and widely studied topic among the geometric integration community. However, it may (or may not) be surprising that several questions are still unsolved. It will be clear while reading the thesis that the work here presented aims to connect different threads, either to conclude or complete several papers found in literature. In particular, the two main branches of the thesis, i.e., the Hamiltonian isospectral flows and the incompressible Euler equations, will be connected. The first one will be focused on the possibility of having intrinsic arbitrarily high order methods for Hamiltonian isospectral flows. The positive answer obtained will lead to a direct application to the second one and will provide a better understanding of the possible advantages of having a Casimir functions-preservation discretization scheme.

1 Lie–Poisson systems

Since its foundation, mathematical physics has been built up from the language and the concepts coming from geometry. The mechanics of Giuseppe Lodovico Lagrangia, Leonard Euler and William Hamilton tried instead to develop an analytical formulation of the fundamental laws of the Universe. Therefore it was not expected that the same equations were hiding even more geometry than before. Sophus Lie, Emmy Nöether and lately Vladimir Arnold showed that the natural language of physics was indeed the differential geometry.

In this section I want to introduce and discuss one of the most intriguing and ubiquitous structure arising in differential geometry and mathematical physics, which is named after the French mathematician Siméon Denis Poisson.

1.1 Poisson structures and Hamiltonian systems

Definition 1 (Poisson bracket)**.** *Let M be a smooth manifold and $C^\infty(M)$ the real vector space of smooth real valued functions defined on M. The Poisson bracket is a bilinear operation $\{\cdot,\cdot\}$ on $C^\infty(M)$, satisfying the following conditions:*

- $\{F,G\} = -\{G,F\}$ *skew symmetry;*
- $\{F,G\cdot H\} = \{F,G\}\cdot H + \{F,H\}\cdot G$ *Leibniz rule;*
- $\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0$ *Jacobi identity.*

A mainfold M equipped with a Poisson bracket is said to be a *Poisson manifold*. The Poisson bracket can be represented by a form $\mathrm{P}\in \bigwedge^2 TM$ by[1]:

$$\{F,G\}(x) = \mathrm{P}_x(dF(x), dG(x)) \tag{1}$$

for any $x \in M$.

Definition 2 (Symplectic form)**.** *Let M be a smooth manifold. $\omega \in \bigwedge^2 M$ is said to be a symplectic form if it is closed and non degenerate, i.e., for any $p \in M$, $v \in T_pM$, if for all $w \in T_pM$ $\omega_p(v,w) = 0$, then $v = 0$.*

A manifold M equipped with a symplectic form ω is said to be a *symplectic manifold*, and it is denoted as (M,ω).

[1]Note that we will denote by $\bigwedge^2 TM$ the space of the sections from M to the alternating 2-tensor on the tangent bundle of M while by $\bigwedge^2 T^*M =: \bigwedge^2 M$ the space of the sections from M to the the alternating 2-tensor on the cotangent bundle of M, which are the usual 2-forms.

Remark 1. *We observe that M always admits a trivial Poisson bracket, i.e., the zero one, but not always a symplectic form. In fact M has to be of even dimension and orientable (e.g., $\mathbb{R}^{2n}$, $n > 0$). Moreover, if M is compact, then the second group of De Rahm cohomology of M must be non zero (e.g., $\mathbb{S}^2$ and $\mathbb{T}^n$ are symplectic but neither $\mathbb{RP}^2$ nor $\mathbb{S}^{2n}$ for $n > 1$ are). Furthermore, a symplectic form induces a canonical Poisson bracket as we will see below.*

Definition 3 (Hamiltonian vector field)**.** *Let (M, ω) be a symplectic manifold. A vector field $X \in TM$ is said to be Hamiltonian if there exists a function $H \in C^\infty(M)$ such*

$$\iota_X \omega = dH \tag{2}$$

where $\iota_X \omega$ is the contraction of ω by X, i.e., $\iota_X : \bigwedge^2 M \to \bigwedge^1 M$ such that, for all $p \in M$ and $v \in T_pM$, $\iota_X \omega_p(v) = \omega_p(X_p, v)$.

Remark 2. *We observe that (2) are nothing else than the Hamilton equations. In fact, for the sake of simplicity, assume $M = \mathbb{R}^{2n}$. Then a symplectic form can be represented in the canonical coordinates $q_1, ..., q_n, p_1, ..., p_n$ by the constant skew matrix $J \in \mathbb{M}(2n, \mathbb{R})$ with coefficients as follows: $J_{ij} = 0$ if $i, j \leq n$ or $i, j > n$, $J_{ij} = \delta_{ij}$ if $i > n, j \leq n$ and $J_{ij} = -\delta_{ij}$ if $i \leq n, j > n$. The $dH_{q,p}$ can be written as $\nabla H_{q,p}$ and $X(q, p) = (\dot{q}, \dot{p})$, where (q, p) are the flow line of X from some initial values. So (2) becomes*

$$J \cdot (\dot{q}, \dot{p}) = \nabla H_{q,p}, \tag{3}$$

which are the Hamilton equations after inversion of J.

Furthermore, we observe that ω induces a diffeomorphism $\hat{\omega} : TM \to T^*M$ defined as $\hat{\omega}(v) = \omega_p(v, \cdot)$ for every $v \in T_pM$. So, given F, we define the *Hamiltonian vector field associate to F* as $X_F = \hat{\omega}^{-1}(dF)$. Finally, given a symplectic manifold (M, ω), we define, for every $F, G \in C^\infty(M)$ the following Poisson bracket:

$$\{F, G\} = \omega(X_F, X_G), \tag{4}$$

where everything is defined pointwise. To compute (4) in local coordinates we need the following basic theorem.

Theorem 1 (Darboux)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then for every $p \in M$, there exists a local chart $(V, \varphi = (q_1, ..., q_n, p_1, ..., p_n))$ centred in p, such that:*

$$\omega_{|V} = \textstyle\sum_{i=1}^n dq_i \wedge dp_i,$$

i.e., ω is represented by the matrix J defined above.

Such coordinates are called *canonical* or *Darboux coordinates*. Now we can write (4) in coordinates. Let (V, φ) be a chart given by the Darboux theorem, then, in this chart, $X_F = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}$. A similar expression holds for X_G. Then, a straightforward computation leads to write (4) as:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}, \tag{5}$$

where the relations $dq_i(\partial_{q_i}) = 1, dq_i(\partial_{p_i}) = 0, dp_i(\partial_{q_i}) = 0, dp_i(\partial_{p_i}) = 1$, for $i = 1, ...n$, have been used. The canonical coordinates satisfy:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \text{ and } \{q_i, p_j\} = -\{p_j, q_i\} = \delta_{ij} \text{ for } i, j = 1, ..., n.$$

An obvious consequence is that, for every $F \in C^\infty(M)$, and any Hamiltonian vector field X_H we have:

$$X_H(F) = \{F, H\}.$$

So for p_i, q_i, $i = 1, ..., n$ integral curves of X_H we have that:

$$\dot{q_i} = X_H(q_i) = \{q_i, H\} \text{ and } \dot{p_i} = X_H(p_i) = \{p_i, H\} \text{ for } i = 1, ..., n,$$

which is an other formulation of the Hamilton equations (3).

Definition 4. *Let (M, ω, H) be a Hamiltonian system, i.e., a symplectic manifold with a Hamiltonian function H. A function $f \in C^\infty(M)$ constant on any integral curve of X_H is said to be a first integral of the system. A vector field $X \in TM$ is said to be an infinitesimal symmetry if both ω and H are invariant under the flow of X.*

Theorem 2 (Noether theorem)**.** *Let (M, ω, H) be a Hamiltonian system.*

- *if f is a first integral, then X_f is an infinitesimal symmetry;*
- *on the other hand, if $H^1(M) = 0$ (where $H^1(M)$ is the first group of De Rham cohomology of M), then any infinitesimal symmetry is a Hamiltonian vector field of a first integral, uniquely defined, except for an additive constant for any connected component of M.*

1.2 Lie–Poisson systems

A remarkable Poisson structure can be naturally given to the vector spaces that are also the dual of a Lie algebra. Consider a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, not

necessarily of finite dimension, and let $\mathfrak{g}^*$ be its dual. Then on $C^\infty(\mathfrak{g}^*)$ we have the following (canonical) Poisson bracket:[2]

$$\{F, G\}_\pm(v) = \pm\langle v, [dF(v), dG(v)]\rangle \tag{6}$$

where $v \in \mathfrak{g}^*$ and we have identified $\mathfrak{g} \cong \mathfrak{g}^{**}$.

The Lie–Poisson bracket is a very important example of a generally neither trivial nor symplectic Poisson bracket. In this case, the Poisson form $\mathrm{P} \in \bigwedge^2 T\mathfrak{g}^*$ is linear and can be expressed by:

$$\mathrm{P}_{ij}(v) = \pm C_{ij}^k v_k \tag{7}$$

where C_{ij}^k are the structure constants of $\mathfrak{g}$. Let H be a smooth function on $\mathfrak{g}^*$. Then, the system:

$$\begin{aligned} \dot{F}(v(t)) &= \{F, H\}_\pm(v(t)) \\ F(v(0)) &= F(v_0) \end{aligned} \tag{8}$$

which has to be satisfied for any $F \in C^\infty(\mathfrak{g}^*)$, it is said to be *Lie–Poisson system* with Hamiltonian function H (the bracket is the one defined above and $v(t) \in \mathfrak{g}^*$, for any $t \in \mathbb{R}$). Because of the anti-commutativity of the Poisson bracket, it is clear that in a Lie–Poisson system the Hamiltonian is a conserved quantity in time. Moreover, depending on the rank of the form defining the bracket, we have a certain number of first integrals of the motion, that are the same for any Hamiltonian. These functions that commute with any other one, i.e., $\{C, \cdot\} = 0$ are called *Casimir functions*. As we will discuss in the next sections, the preservation of the Casimir functions and the Hamiltonian by a numerical method is crucial in the applications in order to guarantee good predictions for long times.

1.3 Co-adjoint representation

We want now express (8) in terms of the co-adjoint representation of a Lie algebra. We have first to recall some definitions.

Let G be a Lie group, and consider the map

$$\begin{aligned} C : G \times G &\longrightarrow G \\ (g, h) &\mapsto C_g(h) := ghg^{-1}. \end{aligned}$$

Then, for each $g \in G$, we have the internal automorphism C_g. If we take the differential of this map in the identity e we get the *adjoint representation*, *Ad* of G in $\mathrm{End}(\mathfrak{g})$, that is defined by:

[2]The $\pm$ sign depends on the fact that the Poisson bracket here defined can also be obtained via the reduction of the canonical ones on the left (-) or right (+) invariant functions on T^*G (see section 1.4.1 below).

$$Ad_g(X) = \frac{d}{dt}_{|t=0}(g \exp^{tX} g^{-1}), \quad \forall g \in G, X \in \mathfrak{g}.$$

Finally, differentiating $Ad : G \to \mathrm{End}(\mathfrak{g})$ and identifying $\mathrm{End}(\mathfrak{g})$ with its tangent, we obtain the map:

$$\begin{aligned} ad : &\mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}) \\ &X \mapsto ad_X = [X, \cdot]. \end{aligned}$$

Let us now define the dual of the adjoint representation, i.e., the representation of the group G and the Lie algebra $\mathfrak{g}$ on the endomorphism of the dual of the Lie algebra $\mathfrak{g}^*$. We define the *co-adjoint representation* $Ad^* : G \to \mathrm{End}(\mathfrak{g}^*)$ by:

$$\langle Ad^*(g)(\phi), X\rangle = \langle \phi, Ad(g^{-1})X\rangle \quad \forall g \in G, X \in \mathfrak{g}, \phi \in \mathfrak{g}^*.$$

Proceeding as before, one can find the infinitesimal version $ad^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, given by $ad^*_X = -(ad_X)^*$, i.e.,

$$\langle ad^*_X(\phi), Y\rangle = \langle \phi, -ad_X(Y)\rangle \;\; \forall X, Y \in \mathfrak{g}, \phi \in \mathfrak{g}^*.$$

Let $\mathcal{O}$ be an orbit of the co-adjoint action $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$. It holds the remarkable fact that the co-adjoint orbits have a canonical symplectic structure, called *Kirillov-Kostant-Souriau form.* Let $p \in \mathcal{O}$ and $X, Y \in \mathfrak{g}$, then the two form:

$$\omega_p(\mathrm{ad}^*_X(p), \mathrm{ad}^*_Y(p)) = \langle p, [X, Y]\rangle,$$

is a symplectic form on $\mathcal{O}$, where we have used the canonical identification of $\mathfrak{g}^{**}$ with $\mathfrak{g}$, from which we have obtained that $\mathrm{T}^*\mathfrak{g}^* \simeq \mathfrak{g}^* \times \mathfrak{g}$. We conclude noticing that the co-adjoint orbits are immersed submanifold[3] where the Casimir functions are constant. However, in general, the Casimir functions don't characterize the co-adjoint orbits[4].

Let us go back to the Lie–Poisson system (8). We notice that the bracket can be expressed in terms of the co-adjoint representation of $\mathfrak{g}$:

$$\pm \langle v, [dF(v), dH(v)]\rangle = \mp\langle v, \mathrm{ad}_{dH(v)}(dF(v))\rangle = \pm\langle \mathrm{ad}^*_{dH(v)}(v), dF(v)\rangle. \quad (9)$$

We want to remark that a Lie–Poisson system evolves precisely on the co-adjoint orbits given by the Ad^* action. In fact let consider $F = F(v(t))$ where $x(t)$ is a curve in $\mathfrak{g}^*$, and $v(0) = v_0$. Applying the chain rule we get:

$$dF(\dot{v}) = \pm\langle \mathrm{ad}^*_{dH(v)}(v), dF(v)\rangle$$

[3]If the action of the group G is also proper, e.g., G compact, then the co-adjoint orbits are embedded submanifold.

[4][13], pag. 479.

for any $F \in C^\infty(\mathfrak{g}^*)$. Hence it is true that:

$$\dot{v} = \pm \mathrm{ad}^*_{dH(v)}(v). \tag{10}$$

Integrating this system we get:

$$v(t) = \mathrm{Ad}^*_{\exp(\pm \int_0^t dH(v(s))ds)}(v_0).$$

1.4 Momentum maps and Lie–Poisson reduction

In this section we will briefly recall the concept and the main properties of the *momentum map* of a group action on a Poisson manifold. For further details we refer to [12] and [13].

Let G be a Lie group acting to the left on a Poisson manifold P, such that for any $g \in G$ the action Φ_g is a Poisson map, i.e., $\{\cdot,\cdot\} \circ \Phi_g = \{\cdot \circ \Phi_g, \cdot \circ \Phi_g\}$. Let the *infinitesimal action* of G be the map $\rho : \mathfrak{g} \times \mathrm{P} \to \mathrm{TP}$ defined by:

$$\rho_\xi(p) = \frac{d}{dt}_{|t=0} \exp(t\xi)p, \tag{11}$$

for any $\xi \in \mathfrak{g}, p \in P$. Hence, ρ_ξ is a vector field on P. Furthermore, we assume the G-action to be Hamiltonian, i.e., there exists a function $J_\xi \in C^\infty(P)$ such that $\rho_\xi = \{\cdot, J_\xi\}$. Then we define the momentum map $\mu : \mathrm{P} \to \mathfrak{g}^*$ by:

$$\langle \mu(p), \xi \rangle = J_\xi(p). \tag{12}$$

We remark that, if the Poisson bracket is induced by a symplectic form ω, then the momentum map can be defined by the formula:

$$d\langle \mu(p), \xi \rangle = \iota_{\rho_\xi(p)} \omega_p. \tag{13}$$

For a right action one can repeat exactly the same calculations. The main difference, as one can easily check, is that the map $J : \mathfrak{g} \to C^\infty(\mathrm{P})$ is a Lie algebra homomorphism for the left action and a Lie algebra anti-morphism for the right action (cfr. [12]).

Let us now denote μ_L (respectively μ_R) the momentum map coming from the left (respectively right) G-action on P. Let also $\mathfrak{g}^*_-$ (respectively $\mathfrak{g}^*_+$) be the dual of the Lie algebra $\mathfrak{g}$ endowed with the $-$ (respectively $+$) Lie–Poisson bracket.

The main property of the momentum maps is stated in the following proposition:

Proposition 1 (Prop 2.1, [12])**.** *Let $\mu_L : P \to \mathfrak{g}^*_+$ (respectively $\mu_R : P \to \mathfrak{g}^*_-$) be the momentum map defined above. Then μ_L (respectively μ_R) is a Poisson map.*

Proof. Let consider the left case. By definition of the Lie–Poisson bracket:

$$\begin{aligned}\{F,G\}_+(\mu(p)) &= \langle \mu(p), [dF(\mu(p)), dG(\mu(p))]\rangle \\ &= J_{[dF(\mu(p)),dG(\mu(p))]}(p).\end{aligned}$$

Now, since $J : \mathfrak{g} \to C^\infty(\mathrm{P})$ is a Lie algebra homomorphism, we have that:

$$J_{[dF(\mu(p)),dG(\mu(p))]}(p) = \{J_{dF(\mu(p))}, J_{dG(\mu(p))}\}_+(p).$$

Finally, by the definition of the Poisson bracket, it is enough to prove that:

$$d(J_{dF(\mu(p))}) = d(F \circ \mu)(p)-$$

Indeed, we have:

$$\begin{aligned}\langle d(F\circ\mu)(p), v_p\rangle &= \langle dF(\mu(p)) \circ d\mu(x), v_p\rangle \\ &= \langle d\langle \mu(p), dF(\mu(p))\rangle, v_p\rangle \\ &= \langle d(J_{dF(\mu(p))}), v_p\rangle.\end{aligned}$$

for any $v_p \in T_pP$. □

1.4.1 Lie–Poisson reduction

Now that we have introduced the momentum maps, we can show how Lie–Poisson systems are related to the canonical Hamilton equations.

Let $(P, \{\cdot,\cdot\}, H)$ be a Poisson Hamiltonian system and let (M, ω, H_ψ) be a Hamiltonian system, where $H_\psi = H \circ \psi$ and $\psi : M \to P$ is a Poisson map.

Consider G a Lie group with a Hamiltonian left (resp. right) action[5] on M and such that H_ψ is left (respectively right) G-invariant and G is transitive on the fibres of ψ. Suppose that there exists a left momentum map $\mu : M \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the associated Lie algebra of G. Then, by Proposition 1, we know that μ is a Poisson map between the canonical Poisson bracket on M and the Lie–Poisson bracket on $\mathfrak{g}^*_+$.

Since H_ψ is $G-$invariant, the momentum map μ is a conserved quantity of the dynamical system. It is shown in [12] that, assuming there are no singularities in the quotient with respect to the group action, given a co-adjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$, the map $\Psi_\mathcal{O} = \psi_{|\mu^{-1}\mathcal{O}}$ induces an embedding $\widetilde{\Psi}_\mathcal{O} : \mu^{-1}\mathcal{O}/G \to P$ to a symplectic leaf of P[6].

[5] With Hamiltonian action, we understand an action such that its infinitesimal action is an Hamiltonian vector field.

[6] It is a general fact that any Poisson manifold is a union of symplectic submanifolds, called "symplectic leaves". The trajectory of X_H starting in a particular leaf necessarily stays there. For $\mathfrak{g}^*_\pm$ the symplectic leaves coincide with the respective co-adjoint orbits.

In particular, when $M = T^*G$ and $P = \mathfrak{g}^*_-$ (resp. $P = \mathfrak{g}^*_+$), we can take $\Psi = \mu_R$ (resp. μ_L) and $\mu = \mu_L$ (resp. μ_R). Then, the canonical Hamilton equations in T^*G w.r.t to the Hamiltonian $\widetilde{H}$ become the equations (8) on $\mathfrak{g}^*_-$ (resp. $\mathfrak{g}^*_+$) with respect to to the Hamiltonian H on $\mathfrak{g}^*$, defined by $H \circ \mu_L = \widetilde{H}$ (resp. $H \circ \mu_R = \widetilde{H}$).

1.5 Lie–Poisson systems on $\mathfrak{gl}(n, \mathbb{C})^*$

In this section, in view of the applications, we want to remark some facts about Lie–Poisson systems on the dual of the general matrix Lie algebra $\mathfrak{gl}(n, \mathbb{C})^*$. In particular, we want to clarify in detail the meaning of the identification between $\mathfrak{gl}(n, \mathbb{C})^*$ and $\mathfrak{gl}(n, \mathbb{C})$ and how this affects the representation of the equations of a Lie–Poisson system.

1.5.1 ad vs ad*

Considering the adjoint representation of $\mathfrak{gl}(n, \mathbb{C})$ on itself:

$$\mathrm{ad}_A(B) = [A, B] = AB - BA,$$

for any $A, B \in \mathfrak{gl}(n, \mathbb{C})$.

Let us now look at the co-adjoint representation of $\mathfrak{gl}(n, \mathbb{C})$ on $\mathfrak{gl}(n, \mathbb{C})^*$. Consider the two different identifications of $\mathfrak{gl}(n, \mathbb{C})^*$ with $\mathfrak{gl}(n, \mathbb{C})$:

$$\begin{aligned} \langle A, B \rangle_1 &= \mathrm{Tr}(AB) \\ \langle A, B \rangle_2 &= \mathrm{Tr}(A^\dagger B), \end{aligned}$$

for $A, B \in \mathfrak{gl}(n, \mathbb{C})$. The second one comes from the Frobenius inner product on $\mathfrak{gl}(n, \mathbb{C})$ (in terms of basis, the first one says that the dual element of a given one is its complex adjoint whereas the second one says that it is itself)[7].

Recalling that $\mathrm{ad}^*_A = -(\mathrm{ad}_A)^*$, the respective co-adjoint representations are:

$$\begin{aligned} \mathrm{ad}^{*_1}_A B &= -[B, A] = \mathrm{ad}_A B \\ \mathrm{ad}^{*_2}_A B &= [B, A^\dagger] = -\,\mathrm{ad}_{A^\dagger} B. \end{aligned}$$

1.5.2 Euler–Poincaré equations and their representations

In literature, equations (10), for quadratic Hamiltonian functions, are often called Euler–Poincaré equations. In this paragraph we want to show that the

[7] Here with † we understand the complex adjoint.

dynamics generated is independent from the identification of $\mathfrak{gl}(n,\mathbb{C})^*$ with $\mathfrak{gl}(n,\mathbb{C})$.

To define the Euler–Poincaré equations, we need a symmetric positive-definite linear map $\mathcal{A} : \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})^*$. An explicit form of this map depends on the way we identify the algebra with its dual. Let us denote $\mathcal{A}$ with $\mathcal{A}$ and $\widetilde{\mathcal{A}}$, the respective form, with respect to ad^{*_2} and respectively, ad^{*_1}. We then have that the following inner products are identically defined:

$$\langle A, B\rangle_{\mathcal{A}} := \langle \mathcal{A}A, B\rangle_2 = \mathrm{Tr}(((\mathcal{A}A)^\dagger B)$$

$$\|$$

$$\langle A, B\rangle_{\widetilde{\mathcal{A}}} := \langle \widetilde{\mathcal{A}}A, B\rangle_1 = \mathrm{Tr}(\widetilde{\mathcal{A}}AB).$$

for $A, B \in \mathfrak{gl}(n,\mathbb{C})$. Therefore, we have to have that $\widetilde{\mathcal{A}} = {}^\dagger \circ \mathcal{A}$. Then, for $\Psi \in \mathfrak{gl}(n,\mathbb{C})$, the Lagrangian function can be defined as:

$$\mathcal{L}(\Psi) = \frac{1}{2}\langle \Psi, \Psi\rangle_{\mathcal{A}} = \frac{1}{2}\langle \Psi, \Psi\rangle_{\widetilde{\mathcal{A}}}.$$

The respective momentum variables in $\mathfrak{gl}(n,\mathbb{C})^*$ are:

$$\Omega_{\mathcal{A}} = \frac{\partial \mathcal{L}(\Psi)}{\partial \Psi} = \mathcal{A}\Psi$$
$$\Omega_{\widetilde{\mathcal{A}}} = \left(\frac{\partial \mathcal{L}(\Psi)}{\partial \Psi}\right)^\dagger = \widetilde{\mathcal{A}}\Psi$$

and we observe that $(\Omega_{\mathcal{A}})^\dagger = \Omega_{\widetilde{\mathcal{A}}}$. From these calculations, we get the Hamiltonian functions:

$$H_{\mathcal{A}}(\Omega_{\mathcal{A}}) = \frac{1}{2}\langle \Omega_{\mathcal{A}}, \mathcal{A}^{-1}\Omega_{\mathcal{A}}\rangle_2$$
$$H_{\widetilde{\mathcal{A}}}(\Omega_{\widetilde{\mathcal{A}}}) = \frac{1}{2}\langle \Omega_{\widetilde{\mathcal{A}}}, \widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}}\rangle_1.$$

So we have the identities:

$$\frac{\partial H_{\mathcal{A}}(\Omega_{\mathcal{A}})}{\partial \Omega_{\mathcal{A}}} = \mathcal{A}^{-1}\Omega_{\mathcal{A}} = \Psi$$
$$\frac{\partial H_{\widetilde{\mathcal{A}}}(\Omega_{\widetilde{\mathcal{A}}})}{\partial \Omega_{\widetilde{\mathcal{A}}}} = \widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}} = \Psi.$$

Finally, we get the equation of motion ([13], Chapt. 13):

$$\langle \dot{\Psi}, Y\rangle_{\mathcal{A}} = -\langle \Psi, \mathrm{ad}_\Psi Y\rangle_{\mathcal{A}} = \langle \mathcal{A}^{-1}\,\mathrm{ad}^{*_2}_\Psi\, \mathcal{A}\Psi, Y\rangle_{\mathcal{A}},$$
$$\langle \dot{\Psi}, Y\rangle_{\widetilde{\mathcal{A}}} = \langle \Psi, \mathrm{ad}_\Psi Y\rangle_{\widetilde{\mathcal{A}}} = -\langle \widetilde{\mathcal{A}}^{-1}\,\mathrm{ad}^{*_1}_\Psi\, \widetilde{\mathcal{A}}\Psi, Y\rangle_{\widetilde{\mathcal{A}}},$$

for any $Y \in \mathfrak{gl}(n, \mathbb{C})$. These can also be written in the strong form as:

$$\dot{\Psi} = \mathcal{A}^{-1} \operatorname{ad}^{*_2}_{\Psi} \mathcal{A}\Psi = \mathcal{A}^{-1}[\mathcal{A}\Psi, \Psi^\dagger],$$
$$\dot{\Psi} = \widetilde{\mathcal{A}}^{-1} \operatorname{ad}^{*_1}_{\Psi} \widetilde{\mathcal{A}}\Psi = -\widetilde{\mathcal{A}}^{-1}[\widetilde{\mathcal{A}}\Psi, \Psi],$$

or, considering the dual version for $\Omega_{\mathcal{A}}, \Omega_{\widetilde{\mathcal{A}}}$:

$$\dot{\Omega}_{\mathcal{A}} = \operatorname{ad}^{*_2}_{\mathcal{A}^{-1}\Omega_{\mathcal{A}}} \Omega_{\mathcal{A}} = [\Omega_{\mathcal{A}}, (\mathcal{A}^{-1}\Omega_{\mathcal{A}})^\dagger],$$
$$\dot{\Omega}_{\widetilde{\mathcal{A}}} = \operatorname{ad}^{*_1}_{\widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}}} \Omega_{\widetilde{\mathcal{A}}} = -[\Omega_{\widetilde{\mathcal{A}}}, \widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}}].$$

Remark If we transpose the second equation, we get:

$$\dot{\Omega}^\dagger_{\widetilde{\mathcal{A}}} = [\Omega^\dagger_{\widetilde{\mathcal{A}}}, (\widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}})^\dagger],$$

and, using the fact that $(\Omega_{\mathcal{A}})^\dagger = \Omega_{\widetilde{\mathcal{A}}}$, and $\widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}} = \mathcal{A}^{-1}\Omega_{\mathcal{A}}$, we see that the Euler–Poincaré equations are independent from the choice of the pairing.

1.5.3 Lie–Poisson maps on $\mathfrak{gl}(n, \mathbb{C})^*$

Consider the identification of $\mathfrak{gl}(n, \mathbb{C})$ with its dual, via the Frobenius pairing. After this identification, the Lie–Poisson structure on $\mathfrak{gl}(n, \mathbb{C})^*$ is completely determined by the structure constants of $\mathfrak{gl}(n, \mathbb{C})$. Therefore any Lie algebra morphism of $\mathfrak{gl}(n, \mathbb{C})$ will be a *Lie Poisson map* on $\mathfrak{gl}(n, \mathbb{C})^*$ and viceversa.

We now want to check how it looks with respect to ad^*. Let consider $a : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ invertible Lie algebra morphism, $A, B \in \mathfrak{gl}(n, \mathbb{C})$ and $\phi \in \mathfrak{gl}(n, \mathbb{C})^* \equiv \mathfrak{gl}(n, \mathbb{C})$ (via the Frobenius identification). Then we get:

$$\begin{aligned}
\operatorname{Tr}((a \operatorname{ad}^*_A(\phi))^\dagger B) &= -\operatorname{Tr}((a[A^\dagger, \phi])^\dagger B) \\
&= -\operatorname{Tr}(\phi^\dagger [A, a^\dagger B]) \\
&= -\operatorname{Tr}((a\phi)^\dagger [a^{-T} A, B]) \\
&= -\operatorname{Tr}(([A^\dagger a^{-1}, a\phi])^\dagger B) \\
&= \operatorname{Tr}((\operatorname{ad}^*_{a^{-T}A}(a\phi))^\dagger B).
\end{aligned}$$

So we have the formula:

$$a \operatorname{ad}^*_A(\phi) = \operatorname{ad}^*_{a^{-T}A}(a\phi). \tag{14}$$

Consider A to be equal to $\nabla H(\phi)$, for a smooth function H defined on $\mathfrak{gl}(n, \mathbb{C})^*$, i.e., we have a Lie–Poisson Hamiltonian system. Then the action on an invertible linear map on the (Lie–Poisson) Hamiltonian vector field is:

$$a \cdot X_H := Da \circ X_H \circ a^{-1},$$

where $X_H(\phi) = \mathrm{ad}^*_{\nabla H(\phi)}(\phi)$. Then, using the formula (14), we get:

$$\begin{aligned} a \cdot X_H(\phi) &= a\,\mathrm{ad}^*_{\nabla H(a^{-1}\phi)}(a^{-1}\phi) \\ &= \mathrm{ad}^*_{a^{-T}\nabla H(a^{-1}\phi)}(\phi) \\ &= \mathrm{ad}^*_{\nabla (H\circ a^{-1})(\phi)}(\phi) \\ &= X_{H\circ a^{-1}}(\phi), \end{aligned}$$

which is again a Lie–Poisson Hamiltonian system.

2 Isospectral flows and their numerical solution

2.1 Isospectral flows and their properties

The isospectral flows are a central class of dynamical systems with symmetries. They arise in fact in different contexts: Lie–Poisson reduction, matrix factorization, Lax pairs of integrable systems, et cetera [8],[10],[16]. As the name suggests, isospectral flows represent a dynamical system on linear operators such that the spectrum of operator is fixed during the whole evolution. If the operators are diagonalizable, then, at each time, the operator is similar to the initial one.

Let the flow

$$\begin{aligned}\Phi : [0,\infty) \times \mathcal{L}(V) &\to \mathcal{L}(V)\\ (t, W) &\mapsto \Phi_t(W)\end{aligned}$$

be an isospectral flow on $\mathcal{L}(V)$, where V is a finite dimensional vector space of dimension n. Let W_0 be the initial value. Then, for any $t \geq 0$, there exists $U(t)$ such that:

$$W(t) = \Phi_t(W_0) = U(t)^{-1} W_0 U(t). \tag{15}$$

By differentiation of (15), one find that W is the solution of:

$$\begin{aligned}\dot{W} &= [B(W), W]\\ W(0) &= W_0,\end{aligned} \tag{16}$$

where $B(W) = U^{-1}\dot{U}$ and the bracket is the usual matrix commutator $[A, B] = AB - BA$.

Other than the eigenvalues of the operator, one can choose a different set of generators for the first integrals of (16). This is provided by the momentum of W. In fact:

$$\frac{d}{dt}\operatorname{Tr}(W^k) = \operatorname{Tr}(W^{k-1}[B(W), W]) = \operatorname{Tr}(B(W)[W^{k-1}, W]) = 0,$$

for $k = 1, 2, \ldots$. Since W is represented by a $n \times n$ matrix, its first n momenta are independent, then they are related by the Cayley–Hamilton theorem (in fact $\operatorname{Tr}(W^k) = \sum_{i=1}^{n} \lambda_i^k$, for λ_i the eigenvalues of W).

When $B(W)$ takes the form of (the transpose of) a gradient of a function, the equation (16) will be said *Hamiltonian-Isospectral flow.* The word Hamiltonian is because the function H such that $B = -\nabla H^\dagger$ is a conserved quantity of (16). In fact:

$$\frac{d}{dt}H(W) = -\operatorname{Tr}(\nabla H(W)^\dagger[\nabla H(W)^\dagger, W]) = -\operatorname{Tr}(W[\nabla H(W)^\dagger, \nabla H(W)^\dagger]) = 0.$$

A further reason to use the word Hamiltonian is that $\mathcal{L}(V)$, endowed with the bracket $[\cdot,\cdot]$, can be seen as the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ and the equations (16) as the reduced form of a canonical Hamiltonian system, as shown in section 1.4.1.

Indeed, if we identify the dual of $\mathfrak{gl}(n,\mathbb{C})$ with itself, via the Frobenius inner product $\langle A,B\rangle = \mathrm{Tr}(A^\dagger B)$, the equations (16) above form a Lie–Poisson Hamiltonian system with respect to the co-adjoint representation of $\mathfrak{gl}(n,\mathbb{C})$ on $\mathfrak{gl}(n,\mathbb{C})^*$ given by:

$$\mathrm{ad}^*_A B = [B, A^\dagger] = -\,\mathrm{ad}_{A^\dagger} B$$

for $A \in \mathfrak{gl}(n,\mathbb{C}), B \in \mathfrak{gl}(n,\mathbb{C})^* \cong \mathfrak{gl}(n,\mathbb{C})$.

2.1.1 Restriction to a subspace of $\mathfrak{gl}(n,\mathbb{C})$

It is interesting, both for theoretical and practical purposes, to analyse the case when W evolves on a subspace S of $\mathfrak{gl}(n,\mathbb{C})$. It is clear that if $W \in S$ then $B(W)$ has to be in $\mathfrak{n}(S)$, i.e., the normalizer of S in $\mathfrak{gl}(n,\mathbb{C})$, which is the largest subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ such that $[\mathfrak{n}(S), S] \subseteq S$. This framework is used in Paper I to encompass at the same time the "classical" isospectral flows, e.g., $W \in Sym(n), B(W) \in \mathfrak{o}(n)$, and the Lie–Poisson systems on reductive Lie-algebras.

2.2 Numerical approximation of the isospectral flows

As we have shown above, the main feature of the isospectral flows is to have a set of first integrals that can be expressed as polynomials of a certain order. A direct application of a Runge–Kutta method to (16) would not preserve these invariants. It has actually been proved that in general none will work for this purpose [8].

A popular method to overcome this issue is the so called Runge-Kutta-Munthe-Kaas scheme [10],[16]. The idea is to solve

$$\dot{U} = UB(U^{-1}W_0U) \tag{17}$$

for U and then find W using (15). Since U is in a Lie group G, the Munthe-Kaas method consists in lifting (17) to its Lie algebra $\mathfrak{g}$ via some map from $\mathfrak{g} \to G$ (e.g., $\exp, \mathrm{Cay}$). Then, on $\mathfrak{g}$, any classical Runge-Kutta method can be applied. This method allows to preserve the isospectrality of the flow but in general not its Lie–Poisson structure and therefore, for example, we cannot expect (near) conservation of the Hamiltonian H. An other disadvantage is that the lifting can be expensive to compute. However, a huge advantage is that it provides explicit isospectral methods.

A related technique is given by the symplectic Lie group methods on T^*G as developed by Bogfjellmo and Martinsen [4]. These methods rely on an invertible mapping between the Lie algebra and (an identity neighbourhood of) the Lie group, such as the exponential map (works in general) or the Cayley map (works for quadratic Lie groups).

An other approach for solving (16) is given by the so called RATTLE method [10]. RATTLE is a general method for Hamiltonian systems with constraints. To use RATTLE for (15), one has to pull-back the equations from $\mathfrak{g}^*$ to T^*G and then solve the constrained Hamiloninan system. It indeed provides a Lie–Poisson integrator for (18) but with the burden of solving implicit equations to constrain the system on the right manifold. Some attempts of removing the constraints can be found for example in [15].

Our approach, presented in Paper I, has (independently) followed exactly that thread. Indeed, starting from some simple cases, it was not hard to realize that, with some manipulations of the canonical symplectic Runge-Kutta methods, in many cases the removal of the constraints was possible. This has led to a large class of isospectral methods directly defined on the Lie algebra.

3 2D Euler equations on the sphere and their numerical solution

3.1 Hydrodynamical Euler equations

Consider a homogeneous, incompressible, inviscid, two-dimensional fluid which is constrained to move on a spherical surface, embedded in the standard Euclidean $\mathbb{R}^3$, which is rotating with constant angular speed, with respect to a fixed normal axis. The equations of motion of such a fluid are given by the well known Euler equations of hydrodynamics:

$$\begin{aligned} &\dot{v} + v \cdot \nabla v = -\nabla p - 2\widetilde{\Omega} \times v \\ &\nabla \cdot v = 0 \end{aligned} \tag{18}$$

where v is the velocity vector field of the fluid, p is its internal pressure and $\widetilde{\Omega} = (\Omega \cdot n)n$ is the projection of the angular rotation of the sphere Ω to the normal n at a point of the sphere. The last term in the first equation of (18), $F_c = -2\widetilde{\Omega} \times v$ is called *Coriolis force*.

The geometry behind this system turns out to play a central role in understanding the behaviour of the fluid [2], [3], [12] and in the investigation of numerical methods to solve it [1], [14], [17], [18]. In particular the Euler equations (18) can be equivalently expressed in terms of the one form $v^\flat$ as a Lie–Poisson system on the dual of the infinite-dimensional Lie-algebra of divergence-free vector fields. The respective Poisson tensor is degenerate so that there is an infinite number of independent first integrals (Casimir functions) [3].

On the other hand, an equivalent formulation of (18) is given in terms of the vorticity $\omega = (\nabla \times v) \cdot n$. We notice that by the Stokes' theorem it must be that $\int \omega = 0$. Then the Euler equations (18) can be written as:

$$\begin{aligned} &\dot{\omega} = \{\psi, \omega\} \\ &\Delta\psi = \omega - f, \end{aligned} \tag{19}$$

where $f = 2\Omega \cdot n$ and ψ is the unique solution to the Poisson equation in $C^\infty(\mathbb{S}^2)$, such that $\int \psi = 0$.

In this form the Euler equations are a Lie–Poisson system on the smooth functions on the sphere which integrate to 0. The Hamiltonian is given by

$$H(\omega) = \frac{1}{2} \int (\omega - f)\psi.$$

The (infinitely many) Casimir functions are given, for any smooth f, by $F(\omega) = \int f(\omega)$. In fact, it is easy to check:

$$\frac{d}{dt} \int f(\omega) = -\int f'(\omega) v \cdot \nabla\omega = -\int v \cdot \nabla f(\omega) = \int (\nabla \cdot v) f(\omega) = 0,$$

where we have used the following identity:

$$\{\psi, \cdot\}_p = (X_\psi)_p(\cdot) = p \cdot (\nabla\psi \times \nabla\cdot) = (p \times \nabla\psi) \cdot \nabla\cdot = -v_p \cdot \nabla \cdot .$$

The presence of all these first integrals turns out to be the leading point in giving a suitable discretization of (19).

3.2 Geometric structure of the Euler equations

The geometric picture of fluid dynamics dates back to Arnold [2]. The velocity vector field of a 2D incompressible fluid moving on a symplectic surface (S, α), embedded in the Euclidean $\mathbb{R}^3$, may indeed be seen as a trajectory in the Lie algebra of divergence free vector fields, denoted by $\mathrm{sdiff}(S)$. The Euler equations (18) can be seen in this picture as a Lie–Poisson system on the dual space of $\mathrm{sdiff}(S)$. Consider the standard pairing of 1-forms and vector fields, i.e.,

$$\langle \beta, X \rangle = \int_S \beta(X)\alpha,$$

where $\beta \in \bigwedge^1 S$ and X is a vector field on S. Then one gets that, for $X \in \mathrm{sdiff}(S)$, the pairing is invariant with respect to any exact translation of β.[8] Therefore we have that $\mathrm{sdiff}^*(S) = \bigwedge^1 S / d \bigwedge^0 S$.

Let us continue to work on $S = \mathbb{S}^2$. In [3] it's shown that the Euler equations (18) are equivalent to a Lie–Poisson system on $\bigwedge^1 \mathbb{S}^2 / d \bigwedge^0 \mathbb{S}^2 = \mathrm{sdiff}^*(\mathbb{S}^2)$ (which is isomorphic to the kernel of the 1-form divergence operator δ), with respect to the Hamiltonian function:

$$H([\eta]) = \frac{1}{2}\langle \eta - c^\flat, \eta^\sharp - c \rangle,$$

which represents the kinetic energy in the non inertial frame. Here $\eta = (v + c)^\flat$, $[\eta]$ is its respective class in $\bigwedge^1 \mathbb{S}^2 / d \bigwedge^0 \mathbb{S}^2$ and c is the velocity due to the rotation of the sphere. The Lie–Poisson system can be written as:

$$\dot{F}([\eta]) = \langle \mathrm{ad}^*_{dH}([\eta]), dF \rangle,$$

for any $F : \mathrm{sdiff}^*(\mathbb{S}^2) \to \mathbb{R}$, where $\mathrm{ad}^* : \mathrm{sdiff}(\mathbb{S}^2) \to \mathrm{End}(\mathrm{sdiff}^*(\mathbb{S}^2))$ is the co-adjoint representation of $\mathrm{sdiff}(\mathbb{S}^2)$. Equivalently (see I.6-7, [3]):

$$\dot{[\eta]} = -L_{dH}([\eta]).$$

[8] This is easily checked as $\int df(X)\alpha = \int (\iota_X) df \alpha = \int (L_X f)\alpha = \int f(L_X \alpha) = 0$, where $f \in C^\infty(S)$ and we have used the fact that X is volume preserving.

where L is the Lie derivative. In our case, we have $dH = \eta^\sharp - c = v$. Hence:

$$\dot{[\eta]} = -L_v([\eta]). \tag{20}$$

Note that Lie–Poisson system above defined is respect to the dual pairing in $\mathrm{sdiff}(\mathbb{S}^2)$, being $dF \in (\mathrm{sdiff}^*(\mathbb{S}^2))^* \cong \mathrm{sdiff}(\mathbb{S}^2)$.

At this point, to get rid off the equivalence class, one just needs to take the exterior derivative of (20) and, using the fact that $[L_v, d] = 0$, get the Euler equations in the vorticity form:

$$\dot{\beta} = -L_v\beta, \tag{21}$$

where $\beta = d[\eta] \in \bigwedge^2 \mathbb{S}^2$ represents the vorticity of v.

We write the vorticity in terms of the volume form α such that $\beta = q\alpha$, where the $q \in C^\infty(\mathbb{S}^2)$ and has zero mean. Then we get

$$L_v\beta = L_v(q\alpha) = (L_v q)\alpha + qL_v\alpha = (L_v q)\alpha,$$

being v volume preserving.

By taking the Hodge star of (21), via the identification $\bigwedge^2 \mathbb{S}^2 \cong \bigwedge^0 \mathbb{S}^2 = C^\infty(\mathbb{S}^2)$, we can understand (21) in $C_0^\infty(\mathbb{S}^2)$, i.e., the space of smooth functions which integrate to 0. Hence, we get a map $*d : \mathrm{sdiff}^*(\mathbb{S}^2) \to C_0^\infty(\mathbb{S}^2)$ between a Lie–Poisson algebra and a Poisson algebra. If we call ad^* the Lie–Poisson structure in $\mathrm{sdiff}^*(\mathbb{S}^2)$ and ad the Poisson structure in $C_0^\infty(\mathbb{S}^2)$, then we have:

Lemma 1. *The map $\pi \equiv *d : \mathrm{sdiff}^*(\mathbb{S}^2) \to C_0^\infty(\mathbb{S}^2)$ is such that*

$$\pi_* \,\mathrm{ad}^* = \mathrm{ad}\,.$$

Proof. Let $v \in \mathrm{sdiff}(\mathbb{S}^2)$ and $[\eta] \in \mathrm{sdiff}^*(\mathbb{S}^2)$. Let call $q = d[\eta]$ and, as above, again $*q = q$.

$$\pi \circ \mathrm{ad}_v^*[\eta] = - * dL_v[\eta] = - * L_v d[\eta] = -L_v q = L_{X_\psi} q = \{\psi, q\} = \mathrm{ad}_\psi\, q,$$

where ψ is the only function in $C_0^\infty(\mathbb{S}^2)$ such that $X_\psi = -v$ (it exists being v divergence free and being $\mathbb{S}^2$ simply connected.). □

It is now important to notice that, via the L^2 pairing, we can identify the dual of $C_0^\infty(\mathbb{S}^2)$ with itself. We can also endow it with a Lie–Poisson structure which coincides with ad in $C_0^\infty(\mathbb{S}^2)$. Let call $p : C_0^\infty(\mathbb{S}^2) \to C_0^\infty(\mathbb{S}^2)^*$ the identification. Then we have:

Theorem 3. *The map $p \circ \pi : \mathrm{sdiff}^*(\mathbb{S}^2) \to C_0^\infty(\mathbb{S}^2)^*$ is a Lie–Poisson isomorphism.*

Proof. This follows immediately from the Lemma above and the fact that $\mathrm{ad}^*_\psi\, q = -(\mathrm{ad}_\psi\, q)^* = \mathrm{ad}_\psi\, q$, which is due to the fact:

$$\int (v \cdot \nabla f) g + \int f (v \cdot \nabla g) = \int v \cdot \nabla (fg) = -\int \nabla \cdot v (fg) = 0$$

and the equivalence:

$$\langle (\mathrm{ad}_\psi\, q)^*, g \rangle = -\int (v \cdot \nabla q) g = \int q (v \cdot \nabla g) = \langle q, -\mathrm{ad}_\psi\, g \rangle.$$

□

As we have seen in the previous paragraph, a consequence of the Euler equations of being a Lie–Poisson system is that there exists an infinite number of independent first integrals or Casimir functions. This fact turns out to be the leading point in giving a suitable spatial discretization of the system. In fact, while solving the equations with a numerical scheme, we cannot expect to preserve all the infinite first integrals but what we do want to get is having an increasing number of first integrals with respect to the size of the discrete problem. This cannot be achieved by simply considering a truncated spectral decomposition of the vorticity [17], [18].

Instead we used the approach proposed by Zeitlin in [18], based on the theory of geometric quantization of compact Kähler manifolds [6], [5], [11]. It provides a sequence of finite-dimensional twisted-representations of the infinite-dimensional Lie algebra of divergence-free vector fields, $\mathrm{sdiff}(\mathbb{S}^2)$. This sequence can also be seen as a finite dimensional approximation of $\mathrm{sdiff}(\mathbb{S}^2)$, in the sense of the L_α-convergence, which will be explained below. Then, for any of those quasi-representations, we get a finite dimensional analogue of (19), i.e., a Lie–Poisson Hamiltonian system on $\mathfrak{su}(n)$ (or $\mathfrak{sl}(n, \mathbb{C})$), for any $n \geq 1$, with $n-1$ independent Casimir functions.

3.3 L_α-convergence

Let us consider a Lie-algebra $(\mathfrak{g}, [\cdot,\cdot])$ and a family of labelled Lie algebras $(\mathfrak{g}_\alpha, [\cdot,\cdot]_\alpha)_{\alpha \in I}$, where $\alpha \in I = \mathbb{N}$ or $\mathbb{R}$. Furthermore, assume then that to any element of this family it is associated a distance d_α and a surjective projection map $p_\alpha : \mathfrak{g} \to \mathfrak{g}_\alpha$. Then we will say that $(\mathfrak{g}, [\cdot,\cdot])$ is an L_α-approximation of $(\mathfrak{g}_\alpha, [\cdot,\cdot]_\alpha)_{\alpha \in I}$ if:

- if $x, y \in \mathfrak{g}$ and $d_\alpha(p_\alpha(x), p_\alpha(y)) \to 0$, for $\alpha \to \infty$, then $x = y$,
- for all $x, y \in \mathfrak{g}$ we have $d_\alpha(p_\alpha([x,y]), [p_\alpha(x), p_\alpha(y)]_\alpha) \to 0$, for $\alpha \to \infty$,
- all p_α, for $\alpha \gg 0$, are surjective.

The above definition is given in [5] and it is a quite weak requirement to get a limit for a sequence of Lie algebras. Indeed the same sequence may converge in the L_α sense to different Lie algebras [6]. Much depends on the choice of the projections that are not canonical. However, for our purposes, since we have already a target and we need a suitable sequence to approximate it, we won't need more than that.

Let us now consider the smooth complex functions with 0 mean on the sphere, and denote them with $C_0^\infty(\mathbb{S}^2, \mathbb{C})$. This vector space can be canonically endowed by a Poisson structure given by the respective Hamiltonian vector fields of two functions and a symplectic form α on $\mathbb{S}^2$. We have, for any $f, g \in C_0^\infty(\mathbb{S}^2, \mathbb{C})$:

$$\{f, g\} = \alpha(X_f, X_g). \tag{22}$$

With this bracket, $C_0^\infty(\mathbb{S}^2, \mathbb{C})$ becomes an infinite dimensional Poisson algebra. A basis is given by the complex spherical harmonics, which will be denoted in the standard notation and azimuthal-inclination coordinates (ϕ, θ) as:

$$Y_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi},$$

for $l \geq 1$ and $m = -l, \ldots, l$. In this basis it has been built up by J. Hoppe [11] and fully proved (even in a more general contest) by M. Bordemann, E. Meinrenken and M. Schlichenmaier [5] an explicit L_α-approximating sequence, given by the matrix Lie algebra $(\mathfrak{sl}(n, \mathbb{C}), [\cdot, \cdot]_n)_{n \in \mathbb{N}}$, where $[\cdot, \cdot]_n = n^{3/2}[\cdot, \cdot]$, the rescaled usual commutator of matrices.

The distances are given by a suitable matrix norm and the projections are defined by associating to any spherical harmonic a respective matrix, for any $n \in \mathbb{N}$, i.e., $p_n : Y_{lm} \mapsto T_{lm}^n$, where

$$(T_{lm}^n)_{m_1 m_2} = (-1)^{n/2 - m_1} \sqrt{2l+1} \begin{pmatrix} n/2 & l & n/2 \\ -m_1 & m & m_2 \end{pmatrix},$$

where the round bracket is the Wigner 3j-symbols. The result can be summarized as:

Theorem 4 (Bordemann, Hoppe, Meinrenken, Schlichenmaier [6],[5])**.** *Let us consider the Poisson algebra $(C_0^\infty(\mathbb{S}^2, \mathbb{C}), \{\cdot, \cdot\})$ whose pairing is defined in (22). Then with respect to p_n defined above and d_n any matrix norm, we have that $(C_0^\infty(\mathbb{S}^2, \mathbb{C}), \{\cdot, \cdot\})$ is an L_α-approximation of $(\mathfrak{sl}(n, \mathbb{C}), [\cdot, \cdot]_n)_{n \in \mathbb{N}}$.*

3.4 The reduced system

We can now derive the spatial discretization of the Euler equations via the L_α-approximation. We first present the system without the Coriolis force.

For any $n \in \mathbb{N}$, we get an analogous of the Euler equations (19):

$$\dot{W} = [\Delta_n^{-1} W, W]_n, \tag{23}$$

where $W \in \mathfrak{sl}(n, \mathbb{C})$ and Δ_n^{-1} is the inverse of the discrete Laplacian as defined in [18]. The crucial property of Δ_n^{-1} is that $\Delta_n^{-1} T_{lm}^n = (-l(l+1))^{-1} T_{lm}^n$, for any $l = 1, ..., n$, $m = -l, ..., l$.

We remark that, for a real valued vorticity, W is actually in $\mathfrak{su}(n)$, which means that $\overline{W}_{lm} = (-1)^m W_{l-m}$.

The discrete Hamiltonian takes the following form:

$$H(W) = \frac{1}{2} \operatorname{Tr}(\Delta_n^{-1} W W^\dagger).$$

The discrete system has the following independent $n-1$ of first integrals[9]

$$F_n(W) = \operatorname{Tr}(W^k) \text{ for k=2,..,n}$$

which, up to a normalization constant dependent on n, converge to the powers of the continuous vorticity.

3.4.1 With the Coriolis force

In the case with the Coriolis force, the discrete system is:

$$\dot{W} = [\Delta_n^{-1}(W - F), W]_n,$$

where $F = 2\Omega T_{10}^n$ represents the discrete Coriolis force.

The discrete Hamiltonian in this case takes the following form:

$$H(W) = \frac{1}{2} \operatorname{Tr}(\Delta_n^{-1}(W - F)(W - F)^\dagger).$$

4 Summary of the results in the papers

4.1 Paper I: Lie–Poisson methods for isospectral flows

In paper I, we treat isospectral flows and Lie–Poisson systems together, which lead to a general recipe for solving both numerically, capturing their main geometrical features.

[9] One should notice that by definition $\operatorname{Tr}(W) = 0$ for all $W \in \mathfrak{sl}(n, \mathbb{C})$ and $\operatorname{Tr}(W^n)$ can be replaced by $\det(W)$, by the Cayley–Hamilton theorem.

Consider the following Hamiltonian isospectral flow for $W \in \mathfrak{g}$, Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ and H smooth function on $\mathfrak{g}$:

$$\begin{aligned} \dot{W} &= [\nabla H(W)^\dagger, W] \\ W(0) &= W_0. \end{aligned} \tag{24}$$

Then we have the following fact:

Proposition 2. *If $\mathfrak{g}$ is a semisimple Lie algebra then $\nabla H(W)^\dagger \in \mathfrak{g}$ and, via the Frobenius norm identification, (24) is a Lie–Poisson system on $\mathfrak{g}^*$.*

If we just have a general isospectral flow

$$\begin{aligned} \dot{W} &= [B(W), W] \\ W(0) &= W_0, \end{aligned} \tag{25}$$

for $W \in S$, linear subspace of $\mathfrak{gl}(n, \mathbb{C})$, then, for being (25) well defined, $B(W)$ has to belong to $\mathfrak{n}(S)$, i.e., the $\mathfrak{gl}(n, \mathbb{C})$-normalizer of S.

In both cases, we require the following assumption to hold (the Hamiltonian case is for $B = \nabla H^\dagger$):

Assumption 1. *Given $\varepsilon > 0$, let S_ε be a ε-neighbourhood of S in $\mathfrak{gl}(n, \mathbb{C})$. Then we assume that $B = B(Z)$ can be extended on S_ε, such that $B(Z) \in \mathfrak{n}(S)$ for all $Z \in S_\varepsilon$, where $\mathfrak{n}(S)$ is the $\mathfrak{gl}(n, \mathbb{C})$-normalizer of S.*

This assumption is not restrictive. For example one can extend $B = B(Z)$ invariantly with respect to the S-orthogonal directions. Notice that S is a linear space, therefore this extension of the gradient of the Hamiltoninan requires only an orthogonal projection of Z to S.

Finally consider the lifted equations on $T^*GL(n, \mathbb{C})$, for $(Q, P) \in T^*GL(n, \mathbb{C})$ such that $W = Q^\dagger P$ satisfies (25):

$$\begin{aligned} \dot{Q} &= QB(Q^\dagger P) \\ \dot{P} &= -PB(Q^\dagger P)^\dagger. \end{aligned} \tag{26}$$

Then Q has to belong to $N(S)$, the $GL(n, \mathbb{C})$-normalizer of S. If this is preserved by a numerical method, we have obtained an isospectral integrator:

Theorem 5. *Let $W = W(t)$ be the solution of (25) in some linear subspace S of $\mathfrak{gl}(n, \mathbb{C})$ and let Assumption 1 hold. Then a symplectic numerical method applied to (26) descends to an isospectral integrator on S for (25) if:*

"there exists a fixed $G \in GL(n, \mathbb{C})$ such that $GQ^\dagger \in N(S)$"

is a first integral of the discrete flow.

Moreover, if $B = \nabla H^\dagger$ and $S = \mathfrak{g}$, semisimple (or reductive) Lie algebra, the method is a Lie–Poisson integrator for (24).

The second constructive result is that any symplectic Runge-Kutta method gives a Lie–Poisson integrator for $\mathfrak{gl}(n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C})$ and any of their quadratic reductive subalgebras. The general s-stage methods is given by the following scheme.

Given a Butcher tableau:

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

of a s-stage symplectic Runge-Kutta method with time step h, we get the following Lie–Poisson integrator:

$$X_i = -h(W_n + \textstyle\sum_{j=1}^s a_{ij} X_j)\nabla H(\widetilde{W}_i)^\dagger, \qquad \text{for } i = 1, ..., s.$$

$$Y_i = h\nabla H(\widetilde{W}_i)^\dagger (W_n + \textstyle\sum_{j=1}^s a_{ij} Y_j), \qquad \text{for } i = 1, ..., s.$$

$$K_{ij} = h\nabla H(\widetilde{W}_i)^\dagger (\textstyle\sum_{j'=1}^s (a_{ij'} X_{j'} + a_{jj'} K_{ij'})), \qquad \text{for } i, j = 1, ..., s.$$

$$\widetilde{W}_i = W_n + \textstyle\sum_{j=1}^s a_{ij}(X_j + Y_j + K_{ij}), \qquad \text{for } i = 1, ..., s.$$

$$W_{n+1} = W_n + h\textstyle\sum_{i=1}^s b_i [\nabla H(\widetilde{W}_i)^\dagger, \widetilde{W}_i],$$

where the unknowns are X_i, Y_i, K_{ij} for $i, j = 1, ..., s$ and the last two lines are explicit.

In the paper, it is shown how it can be simplified in several cases. We conclude the article by presenting several applications of the method to the rigid body equations, the point vortex equations, the Heisenberg spin chain equations, the Euler equations, the Toda lattice and the Toeplitz inverse problem.

In the figure below we show the results for one of our methods applied to the generalized rigid body equations.

4.2 Paper II: A structure preserving scheme for the Euler equations on a (rotating) sphere

In this paper we present a new class of numerical schemes to discretize the Euler equations, both in time and space . These methods are obtained by combining the results of Paper I and the geometric quantization reduction proposed by Zeitlin (cf. Section 3), which leads to an ODE in $\mathfrak{su}(n)$:

$$\dot{W} = [\Delta_n^{-1} W, W]_n. \tag{27}$$

where Δ_n^{-1} is the discrete Laplacian operator.

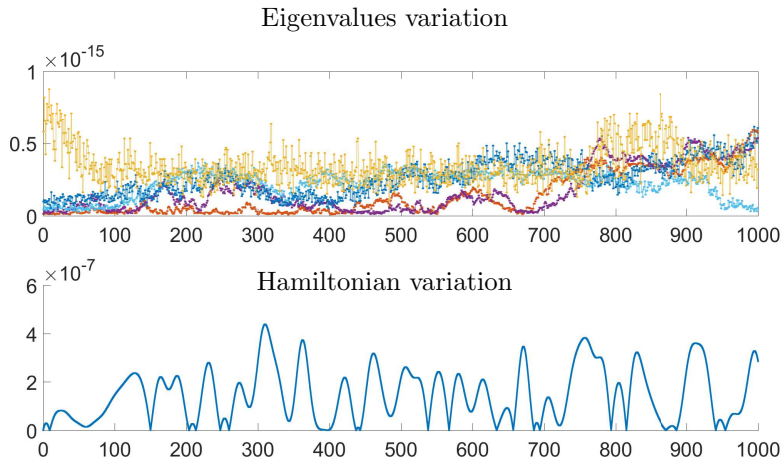

Figure 1: Generalized 45-dimensional rigid body in $\mathfrak{so}(10)$. Eigenvalues (which occur in pair) and Hamiltonian variation; $h = 10^{-1}$; inertia tensor $\mathbb{I} = diag(1:10)$; initial value $(W_0)_{ij} = 1/10$ if $i < j$, $(W_0)_{ij} = -1/10$ if $i > j$, $(W_0)_{ij} = 0$ if $i = j$.

The numerical methods developed have the advantage to preserve, up to machine precision, the discrete Casimir functions, nearly conserving the Hamiltonian and obtaining a discrete flow qualitatively similar to the original Euler equations.

The simplest scheme for the quantized Euler equations (27) that we propose is the 2nd order isospectral midpoint rule. With time step h, it is:

$$\begin{aligned} X &= -h(W_n + \tfrac{1}{2}X)\Delta_n^{-1}\widetilde{W} \\ K &= \tfrac{h}{2}\Delta_n^{-1}\widetilde{W}(X+K) \\ \widetilde{W} &= W_n + \tfrac{1}{2}(X - X^\dagger + K) \\ W_{n+1} &= W_n + X - X^\dagger + K - K^\dagger. \end{aligned} \tag{28}$$

In Figures 2-3, we show the results of two simulations, where we have applied (28) to two examples studied in [9] and [7].

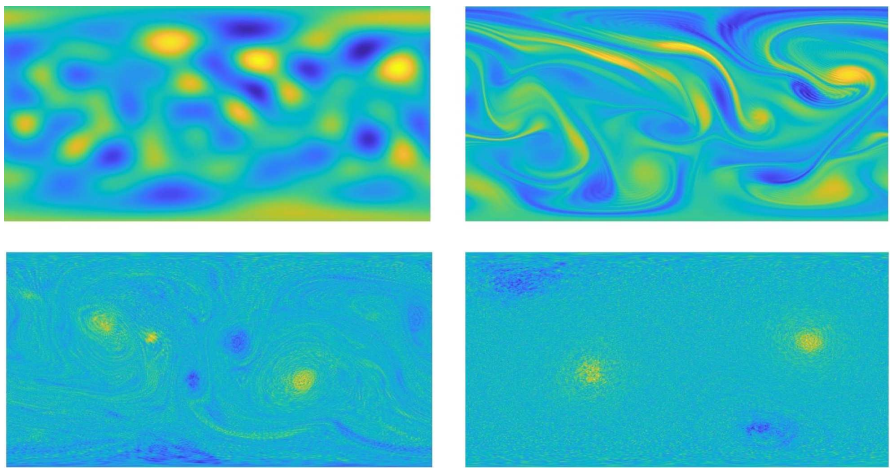

Figure 2: Vorticity $\omega(x,t)$ from the top-left at $t = 0s, 4s, 40s, 140s$, for the intial data in [9]. The horizontal axis is the azimuth $\varphi \in [0, 2\pi]$ and the vertical axis is minus the inclination $\theta \in [0, \pi]$. Spatial discretization in $\mathfrak{su}(501)$.

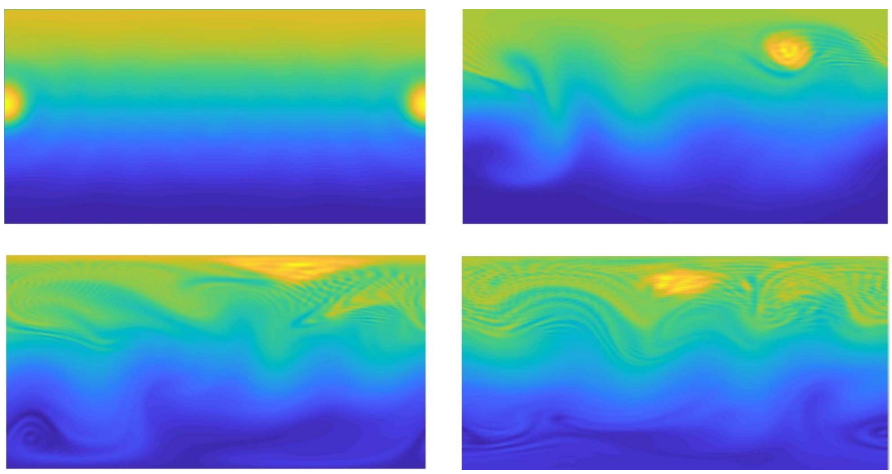

Figure 3: Vorticity $\omega(x,t)$ from the top-left at iteration$= 1, 150, 300, 450$, for the intial data in [7]. The horizontal axis is the azimuth $\varphi \in [0, 2\pi]$ and the vertical axis is minus the inclination $\theta \in [0, \pi]$. . Spatial discretization in $\mathfrak{su}(201)$.

5 Proposals for future work

5.1 Paper I: Lie–Poisson methods for isospectral flows

In Paper I, we presented a general approach for solving numerically Lie–Poisson systems on reductive Lie algebras. In view of the Levi decomposition of finite dimensional Lie algebras, i.e., that any of them can be decomposed into a semi-direct product of a semisimple and a solvable Lie subalgebra, it would be interesting to develop analogous results for Lie–Poisson systems on solvable Lie algebras.

5.2 Paper II: A structure preserving scheme for the Euler equations on a (rotating) sphere

In paper II, encouraging results in the study of the Euler equations have been shown. However, our simulations, despite good, should be implemented with higher resolution, in order to give more reliable predictions.

Analogously, a full analysis of the convergence of the quantized equations to the original one is still missing and it should be accomplished to understand the quality of such an approximation.

Last but not least, simulations of coupled continuous and singular vorticity (point vortices) have yet to be done.

Bibliography

[1] A. Arakawa. "Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I". *Journal of Computational Physics* 1.1 (1966), pp. 119–143.

[2] V.I. Arnold. "Sur la géometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique del fluids parfaits". *Annales de l'institut Fourier* 16.1 (1966), pp. 319–361.

[3] V.I. Arnold and B.A. Khesin. *Topological Methods in Hydrodynamics.* Vol. 125. Springer-Verlag New York, 1998, pp. XV, 376. ISBN: 978-0-387-94947-5.

[4] G. Bogfjellmo and H. Marthinsen. "High-Order Symplectic Partitioned Lie Group Methods". *Foundations of Computational Mathematics* 16.2 (2016), pp. 493–530.

[5] M. Bordemann, E. Meinrenken, and M. Schlichenmaier. "Toeplitz Quantization of Kähler Manifolds and $\mathfrak{gl}(N), N \to \infty$ limits". *Communications in Mathematical Physics* 165.2 (1994), pp. 281–296.

[6] M. Bordemann et al. "$\mathfrak{gl}(\infty)$ and geometric quantization". *Communications in Mathematical Physics* 138.2 (1991), pp. 209–244.

[7] P. Bosler et al. "A Lagrangian particle/panel method for the barotropic vorticity equations on a rotating sphere". *The Japan Society of Fluid Mechanics and IOP Publishing Ltd* 46.3 (2014).

[8] M.P. Calvo, A. Iserles, and A. Zanna. "Numerical Solution of Isospectral Flows". *Mathematics of Computation Volume* 66.220 (1997), pp. 1461–1486.

[9] D.G. Dritschel, W. Qi, and J.B. Marston. "On the late-time behaviour of a bounded, inviscid two-dimensional flow". *Journal of Fluid Mechanics* 783 (2015), pp. 1–22.

[10] E. Hairer, C. Lubich, and G. Wanner. *Geometric Numerical Integration.* Vol. 31. Springer, Berlin, Heidelberg, 2006, pp. XVII, 644. ISBN: 978-3-540-30663-4.

[11] J. Hoppe. "Ph.D. thesis MIT Cambridge" (1982).

[12] J. Marsden and A. Weinstein. "Co-adjoint orbits, vortices, and Clebsch variables for incompressible fluids". *Physica D* 7.1–3 (1983), pp. 305–323.

[13] J.E. Marsden and T.S. Ratiu. *Introduction to Mechanics and Symmetry.* Vol. vol. 17 of *Texts in Applied Mathematics.* Springer-Verlag, New York, 1999. ISBN: 978-0-387-21792-5.

[14] R. McLachlan. "Explicit Lie–Poisson Integration and the Euler Equations". *Physical review letters* 71.19 (1993), pp. 3043–3046.

[15] R. McLachlan and C. Scovel. "Equivariant constrained symplectic integration". *Journal of Nonlinear Science* 5.3 (1995), pp. 233–256.

[16] M. Webb. "Isospectral algorithms, Toeplitz matrices and orthogonal polynomials, Ph.D. thesis" (2017).

[17] V. Zeitlin. "Finite-mode analogues of 2D ideal hydrodynamics: Coadjoint orbits and local canonical structure". *Physica D* 49.3 (1991), pp. 353–362.

[18] V. Zeitlin. "Self-Consistent-Mode Approximation for the Hydrodynamics of an Incompressible Fluid on Non rotating and Rotating Spheres". *Physical review letters* 93.26 (2004), pp. 353–362.

LIE–POISSON METHODS FOR ISOSPECTRAL FLOWS

KLAS MODIN AND MILO VIVIANI

Abstract. The theory of isospectral flows encompasses a large class of dynamical systems, e.g., gradient flows, Lie–Poisson Hamiltonian systems, etc.. Their numerical resolution is a classical problem in geometric integration. The main issue is due to the fact that the preservation of the spectra in the discrete flow requires the conservation of high order polynomials and, for example, no Runge-Kutta method can preserve arbitrary quadratic and cubic invariants at the same time. However, there exist high order numerical methods (e.g., RK-MK) which can do this, but in general they are not Lie–Poisson integrators. In this paper we present a general framework for isospectral flows and we provide a class of numerical methods of arbitrary order, based on the Lie–Poisson reduction of Hamiltonian systems. Avoiding the use of any constraint, we provide a large class of numerical schemes for Hamiltonian and non-Hamiltonian isospectral flows. One of the advantages of these methods is that, together with the isospectrality, they have near conservation of the Hamiltonian and indeed they are Lie–Poisson integrators.

Contents

1. Introduction 2
2. Main results 4
3. Reduction theory for isospectral Lie–Poisson integrators 7
4. Isospectral symplectic Runge–Kutta methods 9
4.1. Symplectic Runge-Kutta methods 10
4.2. Partitioned symplectic Runge-Kutta methods 13
4.3. Linear equivariance of the schemes 15
5. Examples 17
5.1. A Hamiltonian integrable system in the momentum coordinates: the free rigid body in n dimensions 17
5.2. A Hamiltonian integrable system in the Lax pair coordinates: the Toda lattice 18
5.3. A Hamiltonian non-integrable system: vorticity Euler equations on a sphere 18
5.4. Hamiltonian systems on a product of Lie algebras: Point vortices on a sphere and Heisenberg spin chain 20
5.5. A non-Hamiltonian vector field: Toeplitz inverse eigenvalue problem 21
Bibliography 23

1. Introduction

Lie–Poisson systems and isospectral flows are two well-studied classes of dynamical systems. The former appear as Poisson reductions of Hamiltonian systems for which the configuration and symmetry space is a Lie group (see the monograph [11] and references therein). The classical example is the free rigid body as viewed by Poincaré [18]. The latter, isospectral flows, appear as Lax formulations of integrable systems (see the survey papers [22, 2, 21] and references therein). The classical example is the Toda lattice as viewed by Flaschka [20, 5].

The study of numerical methods for the two classes are by now classical subjects in numerical analysis. The motivation for such schemes came through the strong connection between matrix factorizations in numerical linear algebra and isospectral flows (see the survey papers [3, 17]). This was initiated by the remarkable discovery that the iterative QR-algorithm for computing eigenvalues is a discretization of the (non-periodic) Toda flow [19, 4].

The general form of an isospectral flow is

$$\dot{W} = [B(W), W], \quad W \in S \subset \mathfrak{gl}(n, \mathbb{C}). \tag{1}$$

Here, $[\cdot, \cdot]$ denotes the matrix commutator, S is a linear subspace of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$, and the function $B\colon S \to \mathfrak{n}(S)$ maps into the normalizer algebra $\mathfrak{n}(S)$ (see Section 3 below for details). The most studied setting is when $S = \mathrm{Sym}(n, \mathbb{R})$ is the space of symmetric real matrices, for which the normalizer is the Lie algebra of skew-symmetric real matrices $\mathfrak{n}(S) = \mathfrak{so}(n)$. Another setting is when $S = \mathfrak{g}$ is a Lie subgroup of $\mathfrak{gl}(n, \mathbb{C})$, for which the normalizer is the subalgebra itself $\mathfrak{n}(S) = \mathfrak{g}$.

The predominant example connecting isospectral flows and Lie–Poisson systems is Manakov's n-dimensional rigid body [10]. Let us now discuss this connection.

Recall that a Lie–Poisson system evolves on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. Given a Hamiltonian function H on $\mathfrak{g}^*$, the flow $W(t) \in \mathfrak{g}^*$ is given by

$$\dot{W} = \mathrm{ad}^*_{dH(W)}(W), \tag{2}$$

where the operator ad^* is defined by

$$\langle \mathrm{ad}^*_U(W), V \rangle = \langle W, [U, V] \rangle \qquad \forall\, U, V \in \mathfrak{g}. \tag{3}$$

Without loss of generality we may assume that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. To identify $\mathfrak{gl}(n, \mathbb{C})^*$ with $\mathfrak{gl}(n, \mathbb{C})$ we use the Frobenius inner product

$$\langle W, V \rangle = \mathrm{Tr}(W^\dagger V),$$

where $W^\dagger$ denotes the conjugate transpose. In this way we also identify $\mathfrak{g}^*$ with the subspace $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$. Next we extend the Hamiltonian to all of $\mathfrak{gl}(n, \mathbb{C})$ by taking it to be constant on the affine spaces given by translations of the orthogonal complement of $\mathfrak{g}$. Then dH corresponds to ∇H. From the definition (3) and the identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ we get

$$\mathrm{ad}^*_W(M) = \Pi\, [W^\dagger, M],$$

where Π is the orthogonal projection $\mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{g}$. We thus arrive at an explicit formulation of the Lie–Poisson system (2), namely

$$\dot{W} = \Pi\, [\nabla H(W)^\dagger, W]. \tag{4}$$

Now, the key observation is that if the representation of $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ is closed under conjugate transpose, then equation (4) becomes the isospectral flow

$$\dot{W} = [\nabla H(W)^{\dagger}, W]. \tag{5}$$

Such a representation is possible if and only if $\mathfrak{g}$ is a *semisimple Lie algebra* (see Section 2–3 below for details). Thus we arrive at the statement that Lie–Poisson systems for any reductive[1] Lie algebra can be viewed as isospectral flows. Recall that most classical Lie algebras are reductive, for example $\mathfrak{gl}(n,\mathbb{C}), \mathfrak{gl}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{u}(n)$, $\mathfrak{su}(n)$, $\mathfrak{so}(n)$, and $\mathfrak{sp}(n)$.

An interesting consequence of equation (5) is that whenever the function $B(W)$ in the isospectral flow (1) can be written as $B(W) = \nabla H(W)^{\dagger}$, then it can be extended to a Lie–Poisson system on $\mathfrak{gl}(n,\mathbb{C})$ (or possibly a smaller reductive Lie algebra containing S). Indeed, just extend the Hamiltonian function H to be constant of the affine fibres orthogonal to S. In this way we obtain an extended system foliated into invariant affine subspaces generated by S. The Toda-flow is an example where this construction is possible (see Section 5.2 below).

The key feature of isospectral flows is, of course, that the eigenvalues of W are preserved. Equivalently, given any analytic function f extended to matrices, the function

$$F(W) = \mathrm{Tr}(f(W))$$

is a first integral regardless of the choice of $B(W)$ in (1). From the perspective of Lie–Poisson systems (5), this means that $F(W)$ is a *Casimir function* associated with the Lie–Poisson structure (3). Although there are infinitely many Casimir functions, only a finite number of them can be functionally independent.

In this paper we develop spectral preserving numerical methods for flows of the form (1) which, in the case of Hamiltonian isospectral flows (5), also preserves the Lie–Poisson structure. There already exist at least four ways to achieve this:

- If the Hamiltonian can be written as a sum of explicitly integrable Hamiltonians one can use *splitting method* (see [15] and references therein).
- The Lie–Poisson system on $\mathfrak{g}^* \simeq \mathfrak{g}$ can be extended to a constrained canonical Hamiltonian system on $T^*G \simeq TG \subset T\mathrm{GL}(n,\mathbb{C})$. One can then use the symplectic RATTLE method (or higher order versions of it) for the constrained system (see [8, 16]).
- One can use symplectic Lie group methods on T^*G as developed in [1]. These methods rely on an invertible mapping between the Lie algebra and (an identity neighbourhood of) the Lie group, such as the exponential map (works in general) or the Cayley map (works for quadratic Lie groups).
- One can, in some cases, use *collective symplectic integrators*, which rely on Clebsch variables originating from a Hamiltonian action of G on a symplectic vector space (see [13, 14] for details).

Compared to these methods our approach is: (i) simpler since the algorithms are formulated directly on the algebra $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$; (ii) free of constraints; (iii) free of algebra-to-group maps, such as the exponential or Cayley map; (iv) generic as they apply to any isospectral Hamiltonian flow. Furthermore, through the framework

[1] A Lie algebra is said to be reductive if it is isomorphic to the direct sum of a semisimple Lie algebra and an abelian Lie algebra. Since the abelian part does not affect the bracket, and the transpose of a reductive Lie algebra is still reductive, the results we have obtained are still valid for reductive Lie algebras.

of Poisson reduction (cf. [11]) our methods are directly related to classical symplectic Runge–Kutta methods (or partitioned symplectic Runge–Kutta methods). We therefore believe they merit the designation *Isospectral Symplectic Runge–Kutta* (IsoSyRK) methods.

2. Main results

Recall that a Runge–Kutta method is defined by its *Butcher tableau* (cf. [7])

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} \tag{6}$$

Also recall that if

$$b_i a_{ij} + b_j a_{ji} = b_i b_j, \tag{7}$$

for $i, j = 1, ..., s$, then the corresponding Runge–Kutta method is symplectic when applied to canonical Hamiltonian systems on $\mathbb{R}^{2n}$. However, directly applying a symplectic Runge–Kutta method to the Hamiltonian isospectral flow (5) does *not* yield a Poisson integrator nor does it, in general, preserve the isospectral property, as it is well known.

Definition 1 (IsoSyRK)**.** *Given a Butcher tableau* (6) *fulfilling the symplectic condition* (7)*, the corresponding* Isospectral Symplectic Runge–Kutta method *for the flow* (1) *is the map*

$$\Phi_{hB} \colon \mathfrak{gl}(n, \mathbb{C}) \ni W_k \longmapsto W_{k+1} \in \mathfrak{gl}(n, \mathbb{C})$$

defined by

$$\begin{aligned} X_i &= -\Big(W_n + \sum_{j=1}^{s} a_{ij} X_j\Big) hB(\widetilde{W}_i) \\ Y_i &= hB(\widetilde{W}_i)\Big(W_k + \sum_{j=1}^{s} a_{ij} Y_j\Big) \\ K_{ij} &= hB(\widetilde{W}_i)\Big(\sum_{j'=1}^{s} (a_{ij'} X_{j'} + a_{jj'} K_{ij'})\Big) \\ \widetilde{W}_i &= W_k + \sum_{j=1}^{s} a_{ij}(X_j + Y_j + K_{ij}) \\ W_{k+1} &= W_k + \sum_{i=1}^{s} b_i [hB(\widetilde{W}_i), \widetilde{W}_i], \end{aligned}$$

for $i, j = 1, ..., s$, where $h > 0$ denotes the step size.

Theorem 1. *The method in Definition 1 fulfills the following properties:*

(1) *It has the same order as the underlying Runge–Kutta method.*
(2) *It is isospectral; for any analytic function f extended to matrices*

$$\mathrm{Tr}(f(W_{k+1})) = \mathrm{Tr}(f(W_k)).$$

(3) *It is equivariant with respect to Lie algebra morphisms; if $\mathcal{A} \colon \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ is a linear invertible mapping fulfilling for all $X, Y \in \mathfrak{gl}(n, \mathbb{C})$*

$$\mathcal{A}[X, Y] = [\mathcal{A}X, \mathcal{A}Y], \tag{8}$$

then the following diagram commutes

$$\begin{array}{ccc} W_k & \xrightarrow{\mathcal{A}} & W'_k \\ \downarrow{\scriptstyle \Phi_{hB}} & & \downarrow{\scriptstyle \Phi_{h\mathcal{A}\circ B\circ\mathcal{A}^{-1}}} \\ W_{k+1} & \xrightarrow{\mathcal{A}} & W'_{k+1} \end{array}$$

(4) *It is a Lie–Poisson integrator if the isospectral flow is Hamiltonian, i.e., of the form* (5).
(5) *It restricts to a Lie–Poisson integrator for any Lie subalgebra* $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ *defined by*

$$W \in \mathfrak{g} \iff W^{\dagger} J + JW = 0, \tag{9}$$

where $J^2 = cI$ *for some* $c \in \mathbb{R}\backslash\{0\}$.
(6) *It restricts to a Lie–Poisson integrator for any Lie subalgebra given by arbitrary intersections of* $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{C})$, *and Lie algebras of the form* (9).
(7) *It extends to a Lie–Poisson integrator for direct products of Lie algebras of the form in item* (6).
(8) *It restricts to an isospectral integrator on the orthogonal complement* $\mathfrak{g}^{\perp} \subset \mathfrak{gl}(n, \mathbb{C})$ *of any Lie algebra* $\mathfrak{g}$ *of the form in item* (6).

Remark 1. *Items (5)–(7) of Theorem 1 implies that the IsoSyRK methods constitute Lie–Poisson integrators for the classical Lie algebras* $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{so}(n)$, $\mathfrak{u}(n)$, $\mathfrak{su}(n)$, $\mathfrak{sp}(n, \mathbb{C})$, *and* $\mathfrak{sp}(n, \mathbb{R})$. *Item (8) implies that they also preserve the classical isospectral setting as flows on symmetric or Hermitian matrices, since, for example,* $\mathfrak{so}(n)^{\perp} = \mathrm{Sym}(n, \mathbb{R})$.

We also have a weaker result for the partitioned symplectic Runge–Kutta mehtods. A partitioned Runge–Kutta method is defined by two *Butcher tableau* (cf. [7])

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} \qquad\qquad \begin{array}{c|c} \widehat{\mathbf{c}} & \widehat{\mathbf{A}} \\ \hline & \widehat{\mathbf{b}}^T \end{array} \tag{10}$$

Also recall that if

$$\begin{aligned} &b_i \widehat{a}_{ij} + \widehat{b}_j a_{ji} = b_i \widehat{b}_j, \\ &\widehat{b}_i = b_i, \end{aligned} \tag{11}$$

for $i, j = 1, ..., s$, then the corresponding partitioned Runge–Kutta method is symplectic when applied to canonical Hamiltonian systems on $\mathbb{R}^{2n}$.

Definition 2 (IsoSyPartRK)**.** *Given a Butcher tableau* (10) *fulfilling the symplectic condition* (11)*, the corresponding* Isospectral Symplectic Runge–Kutta method *for the flow* (1) *is the map*

$$\Phi_{hB} \colon \mathfrak{gl}(n, \mathbb{C}) \ni W_k \longmapsto W_{k+1} \in \mathfrak{gl}(n, \mathbb{C})$$

defined by

$$X_i = -h\Big(W_n + \sum_{j=1}^{s} a_{ij} X_j\Big)\nabla H(\widetilde{W}_i)^\dagger$$

$$Y_i = h\nabla H(\widetilde{W}_i)^\dagger\Big(W_n + \sum_{j=1}^{s} \widehat{a}_{ij} Y_j\Big)$$

$$K_{ij} = h\nabla H(\widetilde{W}_i)^\dagger\Big(\sum_{j'=1}^{s}(a_{ij'}X_{j'} + \widehat{a}_{jj'}K_{ij'})\Big)$$

$$\widetilde{W}_i = W_n + \sum_{j=1}^{s} a_{ij}X_j + \widehat{a}_{ij}(Y_j + K_{ij})$$

$$W_{n+1} = W_n + h\sum_{i=1}^{s} b_i[\nabla H(\widetilde{W}_i)^\dagger, \widetilde{W}_i].$$

for $i, j = 1, ..., s$, where $h > 0$ denotes the step size.

Theorem 2. *The method in Definition 2 fulfills the following properties:*

(1) *It has the same order as the underlying Runge–Kutta method.*
(2) *It is isospectral; for any analytic function f extended to matrices*

$$\mathrm{Tr}(f(W_{k+1})) = \mathrm{Tr}(f(W_k)).$$

(3) *It is equivariant with respect to Lie algebra morphisms; if $\mathcal{A}\colon \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$ is a linear invertible mapping fulfilling for all $X, Y \in \mathfrak{gl}(n,\mathbb{C})$*

$$\mathcal{A}[X,Y] = [\mathcal{A}X, \mathcal{A}Y], \tag{12}$$

then the following diagram commutes

$$\begin{array}{ccc} W_k & \xrightarrow{\mathcal{A}} & W'_k \\ \downarrow{\scriptstyle \Phi_{hB}} & & \downarrow{\scriptstyle \Phi_{h\mathcal{A}\circ B\circ\mathcal{A}^{-1}}} \\ W_{k+1} & \xrightarrow{\mathcal{A}} & W'_{k+1} \end{array}$$

(4) *It is a Lie–Poisson integrator if the isospectral flow is Hamiltonian, i.e., of the form* (5).
(5) *If it restricts to a Lie–Poisson integrator for a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$ defined by*

$$W \in \mathfrak{g} \iff W^\dagger J + JW = 0, \tag{13}$$

where $J^2 = cI$ for some $c \in \mathbb{R}\backslash\{0\}$ and $b_i \neq 0$ for $i = 1, ..., s$, then the two Butcher tableau coincide and the partitioned Runge–Kutta method is already a Runge–Kutta method.
(6) *If it restricts to a Lie–Poisson integrator for a Lie subalgebra given by arbitrary intersections of $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{C})$, and Lie algebras of the form* (13) *and $b_i \neq 0$ for $i = 1, ..., s$ then the two Butcher tableau coincide and the partitioned Runge–Kutta method is already a Runge–Kutta method.*

3. Reduction theory for isospectral Lie–Poisson integrators

Let us consider the Lie–Poisson Hamitonian systems of the form (5) and let $W = W(t)$ be its solution in some linear subspace S of $\mathfrak{gl}(n, \mathbb{C})$. Then the condition for the equations (5) to be well defined is that $\nabla H(W)^\dagger$ is in the $\mathfrak{gl}(n, \mathbb{C})$-normalizer of S. We recall here its definition:

Definition 3. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let then $S \subseteq \mathfrak{g}$ be a linear subspace. Then the two sets:*

$$N(S) = \{g \in G | g^{-1} S g \subseteq S\}$$
$$\mathfrak{n}(S) = \{n \in \mathfrak{g} | [n, S] \subseteq S\}$$

are respectively called the $G-$normalizer and the $\mathfrak{g}-$normalizer of S. Notice that $N(S)$ is a subgroup of G and $\mathfrak{n}(S)$ is a Lie subalgebra of $\mathfrak{g}$.

We now want to give some examples of Definition 3. We first give the following definition.

Definition 4. *Let $\mathfrak{g} \subseteq \mathfrak{sl}(n, \mathbb{C})$ be a Lie algebra and $J \in GL(n, \mathbb{C})$. Then $\mathfrak{g}$ is said to be a $J-$quadratic Lie algebra with respect to J if $A^\dagger J + JA = 0$, for any $A \in \mathfrak{g}$.*

Examples:

(1) $S = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{n}(S) = \mathfrak{gl}(n, \mathbb{C})$

(2) $S = \mathfrak{g} \subset \mathfrak{sl}(n, \mathbb{C})$ J-quadratic Lie subalgebra such that $\mathfrak{n}(S) = \mathfrak{g} \oplus \mathbb{C}Id$, e.g., $S = \mathfrak{su}(n)$ and $\mathfrak{n}(S) = \mathfrak{u}(n)$, with $J = Id$.

(3) $S = \mathfrak{g}^\perp$, where $\mathfrak{g} \subset \mathfrak{sl}(n, \mathbb{C})$ $J-$quadratic Lie subalgebra such that $\mathfrak{n}(S) = \mathfrak{g} \oplus \mathbb{C}Id$.[2], e.g., $S = Sym(n, \mathbb{R})$ and $\mathfrak{n}(S) = \mathfrak{o}(n)$, with $J = Id$.

Remark 2. *A crucial fact to consider is that the semisimple Lie algebras are, up to representation, those that are closed under transposition ([9], Prop. 6.28). This means that, after the identification with the dual of the Lie algebra $\mathfrak{g}$ with itself, a Lie–Poisson system on a semisimple Lie algebra $\mathfrak{g}$ coincides with a Lie–Poisson system on $\mathfrak{gl}(n, \mathbb{C})$ restricted to $\mathfrak{g}$. Therefore equations (5) on $\mathfrak{g}$ form a Hamiltonian Lie–Poisson system on $\mathfrak{g}$, for $\mathfrak{g}$ semisimple (or reductive) Lie algebra and H smooth function on $\mathfrak{g}$.*

We proceed taking the extending the equations (5) to $T^*GL(n, \mathbb{C})$, in such a way that they transform into the canonical Hamiltonian equations on $T^*GL(n, \mathbb{C})$. To do so, we will use the inverse process of the Poisson reduction, via the associated momentum map with respect to the left cotangent lifted action of $GL(n, \mathbb{C})$ on $T^*GL(n, \mathbb{C})$:

$$G.(Q, P) = (G^{-1}Q, G^T P), \tag{14}$$

for $G, Q \in GL(n, \mathbb{C})$ and $P \in T^*_Q GL(n, \mathbb{C})$, [7].

The fact that the momentum map is a Poisson map between $T^*GL(n, \mathbb{C})$ and $\mathfrak{gl}(n, \mathbb{C})^*$ means that a symplectic map in $T^*GL(n, \mathbb{C})$ can be mapped to a Poisson map in $\mathfrak{gl}(n, \mathbb{C})^*$. Therefore, in terms of numerical analysis, this means that a Poisson integrator on $\mathfrak{gl}(n, \mathbb{C})^*$ can be obtained from a symplectic integrator in $T^*GL(n, \mathbb{C})$.

[2]In general, when on $\mathfrak{gl}(n, \mathbb{C})$ is given the Frobenius inner product, we have that if $\mathfrak{n}(S^\perp) = \mathfrak{n}(S)^\dagger$.

Let $\mu : T^*GL(n,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$ be the momentum map with respect to the cotangent lifted action of $GL(n,\mathbb{C})$ on $T^*GL(n,\mathbb{C})$. It is given by $\mu(Q,P) = Q^\dagger P$, for $Q \in GL(n,\mathbb{C})$ and $P \in T^*_Q GL(n,\mathbb{C})$, [7]. Using this momentum map, we lift the equation (5) to the cotangent bundle $T^*GL(n,\mathbb{C})$, obtaining the Hamiltonian system:

$$\begin{aligned} \dot{Q} &= Q\nabla H(Q^\dagger P) \\ \dot{P} &= -P\nabla H(Q^\dagger P)^\dagger. \end{aligned} \tag{15}$$

We need to seek some conditions which provide the equivalence of (15) and (5). We stress again the fact that we are identifying $\mathfrak{g}^*$ with $\mathfrak{g}$, via the Frobenius pairing.

Proposition 1. *Consider the Hamilton equations (15), with $(Q,P) \in T^*GL(n,\mathbb{C})$, and $S \subseteq \mathfrak{gl}^*(n,\mathbb{C})$ a linear subspace. Then the following three statements are equivalent:*

1. *$(Q^\dagger P)(t) \in S$ for any $t \geq 0$;*
2. *$Q_0^\dagger P_0 \in S$ and $\nabla H(Z)^\dagger(t) \in \mathfrak{n}(S)$ for any $Z \in S$;*
3. *$Q_0^\dagger P_0 \in S$ and there exists a fixed $G \in GL(n,\mathbb{C})$ such that $GQ^\dagger(t) \in N(S)$ for any $t \geq 0$.*[3]

Proof. 1) $\Rightarrow$ 2) Clearly $Q_0^\dagger P_0 \in S$, moreover $S \ni \frac{dQ^\dagger P}{dt} = [\nabla H(Q^\dagger P)^\dagger, Q^\dagger P]$, since S is a linear space and $(Q^\dagger P)(t)$ is in it for any $t \geq 0$. But this exactly means that $\nabla H(Q^\dagger P)^\dagger$ has to be in $\mathfrak{n}(S)$.

2) $\Rightarrow$ 1) We have that:

$$(Q^\dagger P)(t) = \exp(\textstyle\int_0^t \nabla H(Q^\dagger P)^\dagger(s)ds) Q_0^\dagger P_0 \exp(-\int_0^\dagger \nabla H(Q^\dagger P)^\dagger(s)ds),$$

which proves the statement, being $N(S) \supseteq \exp(\mathfrak{n}(S))$.

2) $\Rightarrow$ 3) WLOG we can assume Q_0 already in $N(S)$. Then we have that:

$$Q^\dagger(t) = \exp(\textstyle\int_0^t \nabla H(Q^\dagger P)^\dagger(s)ds) Q_0^\dagger,$$

which proves the statement.

3) $\Rightarrow$ 2) By the formula above we have that:

$$Q^\dagger(t) Q_0^{-T} = \exp(\textstyle\int_0^t \nabla H(Q^\dagger P)^\dagger(s)ds).$$

Since the LHS is in $N(S)$ we must have, by the arbitrarily of t, that $\nabla H(Q^\dagger P)^\dagger(t) \in \mathfrak{n}(S)$ for any $t \geq 0$. □

Before proceeding with the theory, we recall the definition of weak and strong first integral.

Definition 5. *Let M be a smooth manifold and $N \subset M$ a smooth submanifold. Consider the following dynamical system on N:*

$$\begin{aligned} \dot{y} &= f(y) \\ y(0) &= y_0, \end{aligned} \tag{16}$$

with f smooth vector field on N and $y_0 \in N$. Then assume that f can be extended on a $\varepsilon-$neighbourhood N_ε of N in M.

[3] This statement could be replaced by assuming $Q^\dagger(t) \in N(S)$ for any $t \geq 0$, provided that $Q_0^\dagger$ is already in $N(S)$.

Then a differentiable function $I : N_\varepsilon \to \mathbb{C}$ is said to be a weak, respectively, strong first integral of (16) if:

$$\begin{aligned} &\nabla I(y) f(y) = 0 \text{ for all } y \in N \\ &\nabla I(y) f(y) = 0 \text{ for all } y \in N_\varepsilon. \end{aligned} \tag{17}$$

Remark 3. *In numerical analysis, the difference between strong and weak first integrals is crucial. In general, numerical methods cannot be fully constrained to take values at any stage only on a prescribed manifold. Therefore they are expected to conserve at most the strong first integrals.*

Motivated by this remark we need to make the following:

Assumption 1. *Given $\varepsilon > 0$, let S_ε be a $\varepsilon-$neighbourhood of S in $\mathfrak{gl}(n, \mathbb{C})$. Then we assume that $\nabla H^\dagger = \nabla H(Z)^\dagger$ can be extended on S_ε, such that $\nabla H(Z)^\dagger \in \mathfrak{n}(S)$ for all $Z \in S_\varepsilon$, where $\mathfrak{n}(S)$ is the $\mathfrak{gl}(n, \mathbb{C})$-normalizer of S.*

For example, one can extend $\nabla H^\dagger = \nabla H(Z)^\dagger$ invariantly with respect to the S-orthogonal directions. Notice that S is a linear space, therefore this extension of the gradient of the Hamiltoninan requires only an orthogonal projection of Z to S. In many cases we will see that this is not even necessary, provided that the algorithm is well implemented.

Under the assumption 1, the proposition 1 says that $(Q^\dagger P) \in S$ is just a *weak first integral* of the Hamiltonian system (15), provided that the gradient of the Hamiltonian is in $\mathfrak{n}(S)$. In fact having $(Q^\dagger P) \in S$ is equivalent to have $[\nabla H(Q^\dagger P)^\dagger, Q^\dagger P] \in S$ which is not in general true for $Q^\dagger P$ in an $\varepsilon-$neighbourhood of S. Instead an equivalent *strong first integral* is given by the third statement, which says that there exists a fixed matrix G such that $GQ^\dagger \in N(S)$ at any time. Therefore only the numerical methods that have $GQ^\dagger \in N(S)$ as a discrete invariant can descend to an integrator on S.

For example, since symplectic Runge-Kutta methods preserve general quadratic first integrals, in the case that $N(S)$ is a quadratic Lie group and S its Lie algebra, they will satisfy $(Q^\dagger P)(t) \in S$, for any $t \geq 0$. On the other hand, only the symplectic partitioned Runge-Kutta methods that in the $Q-$component are already quadratic-first integral preserving can be expected to fulfil $(Q^\dagger P)(t) \in S$, for any $t \geq 0$.

We summarize these results in the following theorem.

Theorem 3. *Let $W = W(t)$ be the solution of (5) in some linear subspace S of $\mathfrak{gl}(n, \mathbb{C})$ and let assumption 1 hold. Then a symplectic numerical method applied to (15) descends to a Lie–Poisson integrator on S for (5) if:*

"there exists a fixed $G \in GL(n, \mathbb{C})$ such that $GQ^\dagger \in N(S)$"

is a first integral of the discrete flow.

4. Isospectral symplectic Runge–Kutta methods

In this section, we will specify theorem (3) for the symplectic Runge-Kutta and partitioned Runge-Kutta methods. We will derive some practical numerical schemes for solving the equation (5), for the examples stated in the previous section.

4.1. **Symplectic Runge-Kutta methods.** Given a Butcher tableau:

$$\begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array}$$

the associated Runge-Kutta method for (15) is given by:

$$K_i^Q = (Q_n + h\sum_{j=1}^s a_{ij}K_j^Q)\nabla H((Q_n + h\sum_{j=1}^s a_{ij}K_j^Q)^\dagger(P_n + h\sum_{j=1}^s a_{ij}K_j^P))$$

$$K_i^P = -(P_n + h\sum_{j=1}^s a_{ij}K_j^P)\nabla H((Q_n + h\sum_{j=1}^s a_{ij}K_j^P)^\dagger(P_n + h\sum_{j=1}^s a_{ij}K_j^P))^\dagger$$

$$Q_{n+1} = Q_n + h\sum_{i=1}^s b_iK_i^Q$$

$$P_{n+1} = P_n + h\sum_{i=1}^s b_iK_i^P,$$

for $i, j = 1, ..., s$. The Runge-Kutta method is *symplectic*, i.e., the discrete flow is a symplectic map, if it holds $b_ia_{ij} + b_ja_{ji} = b_ib_j$ for any $i, j = 1...s$.

Theorem 4. *Given a Butcher tableau:*

$$\begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{b}^T \end{array}$$

of a symplectic s-stages Runge-Kutta method we have that:

(1) *the method descends to an implicit Lie–Poisson integrator for (5) in* $S = \mathfrak{sl}(N, \mathbb{C})$ *in* $2s + s^2$ *unknowns,*
(2) *it also descends to an implicit Lie–Poisson integrator for (5) in* S *equals to any* $J-$*quadratic Lie subalgebra or the orthogonal space to a* $J-$*quadratic Lie subalgebra of* $\mathfrak{sl}(N, \mathbb{C})$ *in* $s + s^2$ *unknowns.*

Moreover, the definition of the descended maps is completely constructive.

Remark 4. *No modification in the definition of gradient of the Hamiltonian is needed to fulfil assumption 1, as shown in the proof of Theorem 4.*

We get in particular the following schemes:

1. $S = \mathfrak{sl}(n, \mathbb{C})$.

$$X_i = -h(W_n + \sum_{j=1}^s a_{ij}X_j)\nabla H(\widetilde{W}_i)^\dagger$$

$$Y_i = h\nabla H(\widetilde{W}_i)^\dagger(W_n + \sum_{j=1}^s a_{ij}Y_j)$$

$$K_{ij} = h\nabla H(\widetilde{W}_i)^\dagger(\sum_{j'=1}^s(a_{ij'}X_{j'} + a_{jj'}K_{ij'}))$$

$$\widetilde{W}_i = W_n + \sum_{j=1}^s a_{ij}(X_j + Y_j + K_{ij})$$

$$W_{n+1} = W_n + h\sum_{i=1}^s b_i[\nabla H(\widetilde{W}_i)^\dagger, \widetilde{W}_i],$$

for $i, j = 1, ..., s$, where the unknowns are X_i, Y_i, K_{ij} for $i, j = 1, ..., s$ and the last two lines are explicit.

2. $S = \mathfrak{g} \subseteq \mathfrak{sl}(n, \mathbb{C})$ *$J-$quadratic.*

$$X_i = -h(W_n + \textstyle\sum_{j=1}^s a_{ij} X_j)\nabla H(\widetilde{W}_i)^\dagger$$

$$K_{ij} = h\nabla H(\widetilde{W}_i)^\dagger(\textstyle\sum_{j'=1}^s (a_{ij'} X_{j'} + a_{jj'} K_{ij'}))$$

$$\widetilde{W}_i = W_n + \textstyle\sum_{j=1}^s a_{ij}(X_j - J^{-1}X_j^\dagger J + K_{ij})$$

$$W_{n+1} = W_n + h\textstyle\sum_{i=1}^s b_i[\nabla H(\widetilde{W}_i)^\dagger, \widetilde{W}_i],$$

for $i, j = 1, ..., s$, where the unknowns are X_i, K_{ij} for $i, j = 1, ..., s$ and the last two lines are explicit. The last line is also equivalent to:

$$W_{n+1} = W_n + \sum_{i=1}^s b_i(X_i - J^{-1}X_j^\dagger J + K_{ii} - J^{-1}K_{ii}^\dagger J).$$

3. $S = \mathfrak{g}^\perp$, $\mathfrak{g} \subseteq \mathfrak{sl}(n, \mathbb{C})$ *$J-$quadratic.*

$$X_i = -h(W_n + \textstyle\sum_{j=1}^s a_{ij} X_j)\nabla H(\widetilde{W}_i)^\dagger$$

$$K_{ij} = h\nabla H(\widetilde{W}_i)^\dagger(\textstyle\sum_{j'=1}^s (a_{ij'} X_{j'} + a_{jj'} K_{ij'}))$$

$$\widetilde{W}_i = W_n + \textstyle\sum_{j=1}^s a_{ij}(X_j + J^{-1}X_j^\dagger J + K_{ij})$$

$$W_{n+1} = W_n + h\textstyle\sum_{i=1}^s b_i[\nabla H(\widetilde{W}_i)^\dagger, \widetilde{W}_i],$$

for $i, j = 1, ..., s$, where the unknowns are X_i, K_{ij} for $i, j = 1, ..., s$ and the last two lines are explicit. The last line is also equivalent to:

$$W_{n+1} = W_n + \sum_{i=1}^s b_i(X_i + J^{-1}X_j^\dagger J + K_{ii} + J^{-1}K_{ii}^\dagger J).$$

Proof. [thm. 4]

(1) For $S := \mathfrak{sl}(n, \mathbb{C})$ we have that $\mathfrak{n}(S) = \mathfrak{gl}(n, \mathbb{C})$ and $N(S) = GL(n, \mathbb{C})$. Therefore the hypothesis of theorem 3 are trivially satisfied.

To get the explicit construction, we look at the argument of the gradient of the Hamiltonian which suggests to define:

$$\begin{aligned}
W_{n+1} &:= Q_{n+1}^\dagger P_{n+1} \\
W_n &:= Q_n^\dagger P_n \\
X_i &:= hQ_n^\dagger K_i^P \\
Y_i &:= h(K_i^Q)^\dagger P_n \\
K_{ij} &:= h^2 \sum_{j'=1}^s a_{ij'}(K_j^Q)^\dagger K_{j'}^P \\
\widetilde{W}_i &:= W_n + \sum_{j=1}^s a_{ij}(X_j + Y_j + K_{ij}),
\end{aligned}$$

for $i, j = 1, ..., s$. The equations for X_i, Y_i are quite straightforward (consider the equations of the Runge-Kutta method defined at the beginning

of the paragraph and just transpose the first one and multiply by P_n and multiply the second one by $Q_n^\dagger$, respectively).

To get those of K_{ij}, we first transpose the first equation, then we multiply it (indexed now by j) by $h^2 a_{ij'} K^P_{j'}$ and then we sum over j'. Therefore we get:

$$K_{ij} = h\nabla H(\widetilde{W}_i)^\dagger(\sum_{j'=1}^{s}(a_{ij'}X_{j'} + a_{ij'}\widetilde{K}_{jj'})) \qquad \text{for } i,j = 1,...,s,$$

where

$$\widetilde{K}_{ij} := h^2 \sum_{j'=1}^{s} a_{ij'}(K^Q_{j'})^\dagger K^P_j.$$

Multiplying the second equation (indexed now by j') by $h^2 a_{ij'}(K^Q_{j'})^\dagger$ and then summing over j', obtaining:

$$\widetilde{K}_{ij} = -h(\sum_{j'=1}^{s}(a_{ij'}Y_{j'} + a_{ij'}K_{jj'}))\nabla H(\widetilde{W}_i)^\dagger \qquad \text{for } i,j = 1,...,s.$$

Using then the (obvious) fact that:

$$\sum_{j'=1}^{s}\sum_{j''=1}^{s} a_{ij'}a_{jj''}(K^Q_{j''})^\dagger K^P_{j'} = \sum_{j'=1}^{s}\sum_{j''=1}^{s} a_{jj'}a_{ij''}(K^Q_{j'})^\dagger K^P_{j''},$$

for $i = 1,...,s$, we get that:

$$\sum_{j'=1}^{s} a_{ij'}\widetilde{K}_{jj'} = \sum_{j'=1}^{s} a_{jj'}K_{ij'} \qquad \text{for } i,j = 1,...,s.$$

Therefore the $\widetilde{K}_{ij}$ depend completely on the K_{ij} and so we can neglect them, obtaining the desired equations for the K_{ij}.

Finally, to get the equation for W_{n+1}, we multiply the third transposed with the fourth and we get:

$$W_{n+1} = W_n + \sum_{i=1}^{s} b_i(X_i + Y_i) + h^2 \sum_{i,j=1}^{s} b_i b_j (K^Q_i)^\dagger K^P_j.$$

Using the symplecticity of the method, the last term becomes:

$$h^2 \sum_{i,j=1}^{s}(b_i a_{ij} + b_j a_{ji})(K^Q_i)^\dagger K^P_j = \sum_{i=1}^{s} b_i(K_{ii} + \widetilde{K}_{ii}).$$

Therefore, we have:

$$W_{n+1} = W_n + \sum_{i=1}^{s} b_i(X_i + Y_i + K_{ii} + \widetilde{K}_{ii}).$$

Now substituting the equations found for $X_i, Y_i, K_{ii}, \widetilde{K}_{ii}$ we get the desired equation for W_{n+1}.

(2) Symplectic Runge-Kutta methods preserve exactly the strong quadratic first integrals of a dynamical system. In particular, when S is one of the spaces in the examples 2-3, they preserve $N(S) = \{Q \in GL(n,\mathbb{C}) | Q^\dagger J Q = J\}$. Therefore, by theorem 3, they descend to a Lie–Poisson integrator on S.

It is also easy to check that, if we assume $\nabla H^\dagger$ to be in $\mathfrak{n}(S)$, we get $Y_i = -J^{-1}X_i^\dagger J$. Moreover, from the definition of K_{ij} and $\widetilde{K}_{ij}$ and the equations:

$$K_{ij} = h\nabla H(\widetilde{W}_i)^\dagger(\sum_{j'=1}^{s}(a_{ij'}X_{j'} + a_{jj'}K_{ij'})) \qquad \text{for } i,j = 1,...,s,$$

$$\widetilde{K}_{ij} = -h(\sum_{j'=1}^{s}(a_{ij'}Y_{j'} + a_{ij'}K_{jj'}))\nabla H(\widetilde{W}_i)^\dagger \qquad \text{for } i,j = 1,...,s,$$

we get also that

$$\sum_{j=1}^{s} a_{ij}(-J^{-1}K_{ij}^\dagger J) = \sum_{j=1}^{s} a_{ij}\widetilde{K}_{ij} = \sum_{j=1}^{s} a_{ij}K_{ij}.$$

This implies that $\widetilde{W} \in S$ and therefore we don't need to do any extension to the gradient of Hamiltonian. □

Remark 5. *The following observation is valid for the examples 2) and 3) but for simplicity we consider $S = \mathfrak{su}(n)$. If one looks at the methods above may notice that it might be possible to only calculate the skew part of the X_j. In fact, it is quite easy to check (by some identities shown in the proof) that $\sum_{j=1}^{s} a_{ij}K_{ij}$ is always skew symmetric. However, if now we assume the X_j to be skew, the method becomes exactly the underling Runge-Kutta scheme. This is impossible since that is not isospectral. In fact one can find that the symmetric part of the X_j paired with $\nabla H(\widetilde{W}_i)^\dagger$, in the form $X\nabla H(\widetilde{W}_i)^\dagger + X\nabla H(\widetilde{W}_i)^\dagger$, is determinant in defining the skew symmetric part of the X_j. Therefore, there is no hope to present the schemes above only in terms of the matrix bracket.*

4.2. **Partitioned symplectic Runge-Kutta methods.** Given two Butcher tableau:

$$\begin{array}{c|ccc} \widehat{c}_1 & \widehat{a}_{11} & \dots & \widehat{a}_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{c}_s & \widehat{a}_{s1} & \dots & \widehat{a}_{ss} \\ \hline & \widehat{b}_1 & \dots & \widehat{b}_s \end{array} \qquad \begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array}$$

the associated partitioned Runge-Kutta method for (15) is given by:

$$K_i^Q = (Q_n + h\textstyle\sum_{j=1}^{s}\widehat{a}_{ij}K_j^Q)\nabla H((Q_n + h\sum_{j=1}^{s}\widehat{a}_{ij}K_j^Q)^\dagger(P_n + h\sum_{j=1}^{s}a_{ij}K_j^P))$$

$$K_i^P = -(P_n + h\textstyle\sum_{j=1}^{s}a_{ij}K_j^P)\nabla H((Q_n + h\sum_{j=1}^{s}\widehat{a}_{ij}K_j^Q)^\dagger(P_n + h\sum_{j=1}^{s}a_{ij}K_j^P))^\dagger$$

$$Q_{n+1} = Q_n + h\textstyle\sum_{i=1}^{s}\widehat{b}_iK_i^Q$$

$$P_{n+1} = P_n + h\textstyle\sum_{i=1}^{s}b_iK_i^P,$$

for $i,j = 1,...,s$. The partitioned Runge-Kutta method is *symplectic*, i.e., the discrete flow is a symplectic map, if it holds $b_i\widehat{a}_{ij} + \widehat{b}_ja_{ji} = b_i\widehat{b}_j$ and $\widehat{b}_i = b_i$, for any $i,j = 1...s$.

Theorem 5. *Given two Butcher tableau:*

$$\begin{array}{c|c} \widehat{\boldsymbol{c}} & \widehat{\boldsymbol{A}} \\ \hline & \widehat{\boldsymbol{b}}^T \end{array} \qquad\qquad \begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{b}^T \end{array}$$

of a symplectic partitioned s-stages Runge-Kutta method we have that:

(1) *the method descends to an implicit Lie–Poisson integrator for (5) in* $S = \mathfrak{sl}^*(N, \mathbb{C})$ *in* $2s + s^2$ *unknowns,*
(2) *if it also descends to an implicit Lie–Poisson integrator for (5) in* S *equals to any* $J-$*quadratic Lie subalgebra or the orthogonal space to a* $J-$*quadratic Lie subalgebra of* $\mathfrak{sl}(N, \mathbb{C})$ *(in* $s + s^2$ *unknowns), then* $b_i \neq 0$ *implies* $a_{ij} = \widehat{a}_{ij}$*, for* $i, j = 1, ..., s$*, i.e., the partitioned symplectic Runge-Kutta method is already a symplectic Runge-Kutta method.*

Moreover, the definition of the descended map is completely constructive.

We get in particular the following scheme:

$S = \mathfrak{sl}(n, \mathbb{C})$.

$$X_i = -h(W_n + \textstyle\sum_{j=1}^s a_{ij} X_j)\nabla H(\widetilde{W}_i)^\dagger$$

$$Y_i = h\nabla H(\widetilde{W}_i)^\dagger(W_n + \textstyle\sum_{j=1}^s \widehat{a}_{ij} Y_j)$$

$$K_{ij} = h\nabla H(\widetilde{W}_i)^\dagger(\textstyle\sum_{j'=1}^s (a_{ij'} X_{j'} + \widehat{a}_{jj'} K_{ij'}))$$

$$\widetilde{W}_i = W_n + \textstyle\sum_{j=1}^s a_{ij} X_j + \widehat{a}_{ij}(Y_j + K_{ij})$$

$$W_{n+1} = W_n + h\textstyle\sum_{i=1}^s b_i[\nabla H(\widetilde{W}_i)^\dagger, \widetilde{W}_i],$$

for $i, j = 1, ..., s$, where the unknowns are X_i, Y_i, K_{ij} for $i, j = 1, ..., s$ and the last two lines are explicit.

Proof. [thm. 5]

(1) The proof is, mutatis mutandis, identical to the one of the previous theorem. We have just to change accordingly the following definitions:

$$\widetilde{K}_{ij} := h^2 \sum_{j'=1}^s \hat{a}_{ij'} (K_j^Q)^\dagger K_{j'}^P$$

$$K_{ij} := h^2 \sum_{j'=1}^s a_{ij'} (K_j^Q)^\dagger K_{j'}^P$$

$$\widetilde{W}_i := W_n + \sum_{j=1}^s a_{ij} X_j + \widehat{a}_{ij}(Y_j + K_{ij}),$$

and pointing out the following identity:

$$\sum_{j'=1}^s a_{ij'} \widetilde{K}_{jj'} = \sum_{j'=1}^s \widehat{a}_{jj'} K_{ij'} \qquad \text{for } i, j = 1, ..., s.$$

Finally we just use the condition of symplecticity for partitioned Runge-Kutta methods.

(2) Partitioned symplectic Runge-Kutta methods preserve exactly the strong quadratic first integrals of a dynamical system if they are on the form $Q^\dagger AP$, for a fixed matrix A. In particular, when S is one of the spaces in the examples 2-3, to preserve $N(S) = \{Q \in GL(n, \mathbb{C}) | Q^\dagger JQ = J\}$ the method associated to Q-part has to preserve already the quadratic first integrals. This fact, together the condition of symplecticity of the partitioned Runge-Kutta methods, implies that $a_{ij} = \widehat{a}_{ij}$, for $i, j = 1, ..., s$ whenever $b_i \neq 0$. □

Remark 6. *The order of convergence of the descended methods is the same as the underlying Runge-Kutta ones (see Figure 1), since if* $Q_n = Q(nh) + \mathcal{O}(h^p)$ *and* $P_n = P(nh) + \mathcal{O}(h^p)$*, then* $W_n = Q_n^\dagger P_n = W(nh) + \mathcal{O}(h^p) = Q(nh)^\dagger P(nh) + \mathcal{O}(h^p)$.

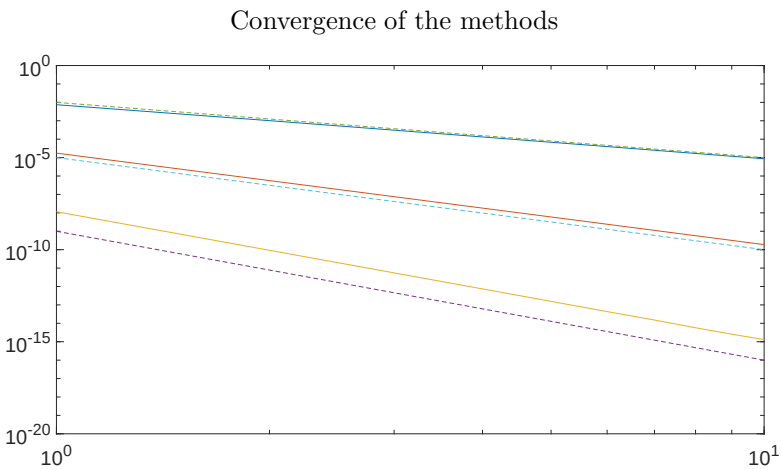

FIGURE 1. Convergence estimate for the methods, in loglog scale. The x axis is $i = 1, ..., 10$, such that the time step is $h = 0.1/i$, the y axis is the error of the numerical methods. The continuous lines represent the error respectively for a 2nd, 4th, 6th order symplectic Runge-Kutta method. The dashed lines are the functions $f(i) = i^3, f(i) = i^5, f(i) = i^7$.

4.3. **Linear equivariance of the schemes.** Here we show that the numerical methods above are equivariant with respect to any invertible linear Lie–Poisson map, where here it is understood as a linear Lie–Poisson map such that the form of the equations remains the same after the transformation, but possibly with a different Hamiltonian (see Theorem 11.5.1., [11]). Consider the numerical scheme above defined as a map F between the space of Hamiltonian Lie–Poisson vector fields on $\mathfrak{g}^*$ and the Lie–Poisson maps of $\mathfrak{g}^*$ into itself.

Recall that we have the following actions of an affine map a:

$$a \cdot X := Da \circ X \circ a^{-1}$$
$$a \cdot \Phi := a \circ \Phi \circ a^{-1},$$

for any vector field X and diffeomorphism Φ.

Under our usual identification of $\mathfrak{g}$ with its dual an invertible Lie–Poisson map a on $\mathfrak{g}^*$ has to satisfy:

$$a \cdot [\nabla H(W)^\dagger, W] = A[\nabla H(A^{-1}W)^\dagger, A^{-1}W] = [\nabla(H \circ a^{-1})(W)^\dagger, W].$$

The numerical scheme F is then said *Lie–Poisson linear equivariant* if:

$$a \cdot F(f) = F(a \cdot f), \tag{18}$$

for any Hamiltonian Lie–Poisson field f on $\mathfrak{g}^*$ and any linear Lie–Poisson map a.

Let us now consider equation (18) for the partitioned symplectic Runge-Kutta schemes. The same conclusion for the symplectic Runge-Kutta method will follow straightforwardly from this. We want to check if for any $W_n \in \mathfrak{sl}(n, \mathbb{C})$ and affine map a on $\mathfrak{sl}(n, \mathbb{C})$ it holds:

$$a(F(f)(W_n)) = F(Da \circ f \circ a^{-1})(a(W_n)). \tag{19}$$

The RHS looks like:

$$X_i = -hA(A^{-1}(AW_n + \textstyle\sum_{j=1}^s a_{ij}X_j))\nabla H(A^{-1}\widetilde{W}_i)^\dagger$$

$$Y_i = hA\nabla H(A^{-1}\widetilde{W}_i)^\dagger(A^{-1}(AW_n + \textstyle\sum_{j=1}^s \widehat{a}_{ij}Y_j))$$

$$K_{ij} = hA\nabla H(A^{-1}\widetilde{W}_i)^\dagger(\textstyle\sum_{j'=1}^s(A^{-1}(a_{ij'}X_{j'} + \widehat{a}_{jj'}K_{ij'})))$$

$$\widetilde{W}_i = AW_n + \textstyle\sum_{j=1}^s a_{ij}X_j + \widehat{a}_{ij}(Y_j + K_{ij})$$

$$W_{n+1} = AW_n + h\textstyle\sum_{i=1}^s b_i A[\nabla H(A^{-1}\widetilde{W}_i)^\dagger, A^{-1}\widetilde{W}_i],$$

for $i, j = 1, ..., s$, which is equivalent to

$$A^{-1}X_i = -h(W_n + \textstyle\sum_{j=1}^s a_{ij}A^{-1}X_j)\nabla H(A^{-1}\widetilde{W}_i)^\dagger$$

$$A^{-1}Y_i = h\nabla H(A^{-1}\widetilde{W}_i)^\dagger(W_n + \textstyle\sum_{j=1}^s \widehat{a}_{ij}A^{-1}Y_j)$$

$$A^{-1}K_{ij} = h\nabla H(A^{-1}\widetilde{W}_i)^\dagger(\textstyle\sum_{j'=1}^s(a_{ij'}A^{-1}X_{j'} + \widehat{a}_{jj'}A^{-1}K_{ij'}))$$

$$A^{-1}\widetilde{W}_i = W_n + \textstyle\sum_{j=1}^s a_{ij}A^{-1}X_j + \widehat{a}_{ij}(A^{-1}Y_j + A^{-1}K_{ij})$$

$$W_{n+1} = A(W_n + h\textstyle\sum_{i=1}^s b_i[\nabla H(A^{-1}\widetilde{W}_i)^\dagger, A^{-1}\widetilde{W}_i]),$$

for $i, j = 1, ..., s$. Relabelling $X_i := A^{-1}X_i, Y_i = A^{-1}Y_i, K_{ij} := A^{-1}K_{ij}, \widetilde{W}_i := A^{-1}\widetilde{W}_i$ we get:

$$X_i = -h(W_n + \textstyle\sum_{j=1}^s a_{ij}X_j)\nabla H(\widetilde{W}_i)^\dagger$$

$$Y_i = h\nabla H(\widetilde{W}_i)^\dagger(W_n + \textstyle\sum_{j=1}^s \widehat{a}_{ij}Y_j)$$

$$K_{ij} = h\nabla H(\widetilde{W}_i)^\dagger(\textstyle\sum_{j'=1}^s(a_{ij'}X_{j'} + \widehat{a}_{jj'}K_{ij'}))$$

$$\widetilde{W}_i = W_n + \textstyle\sum_{j=1}^s a_{ij}X_j + \widehat{a}_{ij}(Y_j + K_{ij})$$

$$W_{n+1} = A(W_n + h\textstyle\sum_{i=1}^s b_i[\nabla H(\widetilde{W}_i)^\dagger, \widetilde{W}_i]),$$

for $i, j = 1, ..., s$, which is exactly the LHS of (18), which proves the linear equivariance.

5. Examples

In this section, we mention some important examples of Lie–Poisson dynamical systems and we show that applying the numerical methods above presented we get the expected results of near conservation of the Hamiltonian and exact conservation (up to round off) of the Casimir functions.

5.1. **A Hamiltonian integrable system in the momentum coordinates: the free rigid body in n dimensions.** The core example among Hamiltonian systems is the free rigid body dynamics. It is known that in any dimension the equations of motions form a complete integrable system in $\mathfrak{so}(n)$, [10]. The Hamiltonian is given by:

$$H(W) = \frac{1}{2}\operatorname{Tr}((\mathbb{I}^{-1}W)^{\dagger}W),$$

where $W \in \mathfrak{so}(n)$ and $\mathbb{I} : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is a symmetric positive definite inertia tensor. The equations of motion are:

$$\begin{aligned} \dot{W} &= -[\mathbb{I}^{-1}W, W] \\ W(0) &= W_0. \end{aligned} \tag{20}$$

We have solved (20), with the method 2 of theorem 4, descending from the midpoint rule. As shown in figure 2, we have that the Hamiltonian is nearly conserved and the Casimir functions are conserved up to machine precision.

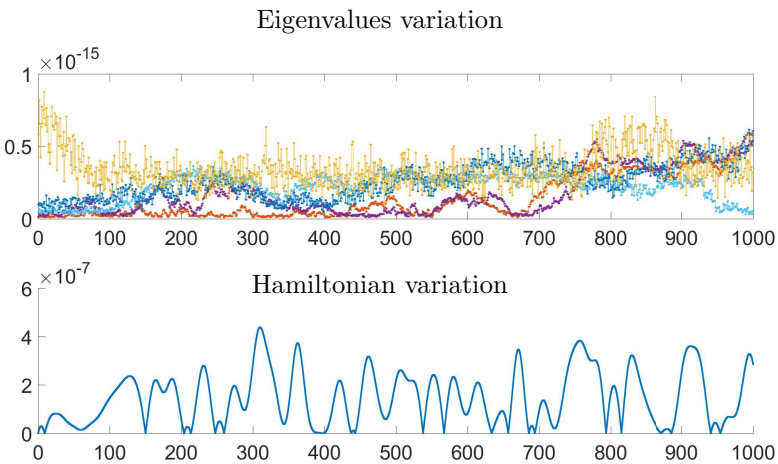

Figure 2. Generalized 45-dimensional rigid body in $\mathfrak{so}(10)$. Eigenvalues (which occur in pair) and Hamiltonian variation; h=10^{-1}; inertia tensor $\mathbb{I} = diag(1:10)$; initial value $(W_0)_{ij} = 1/10$ if $i < j$, $(W_0)_{ij} = -1/10$ if $i > j$, $(W_0)_{ij} = 0$ if $i = j$.

5.2. **A Hamiltonian integrable system in the Lax pair coordinates: the Toda lattice.** Among the Hamiltonian integrable systems the Toda lattice is a well known and studied example. It represents a system of particles interacting pairwise with exponential forces. The equations of motion are determined by the Hamiltonian:

$$H(p,q) = \sum_{k=1}^{n} \left(\frac{1}{2} p_k^2 + \exp(q_k - q_{k+1}) \right),$$

where (q_i, p_i) are the canonical coordinates of the n particles. Independently, Hénon, Flaschka and Manakov proved that the Toda system is integrable when $q_n = q_{n+1}$ (periodic boundary conditions), [7]. To do that, they found a Lax pair formulation of the Toda flow. They applied the following change of variables:

$$a_k = -\frac{1}{2} p_k, \qquad b_k = \frac{1}{2} \exp\left(\frac{1}{2}(q_k - q_{k+1}) \right),$$

to get an equivalent isospectral flow:

$$\dot{L} = [B(L), L], \tag{21}$$

where

$$L = \begin{bmatrix} a_1 & b_1 & 0 & ... & b_n \\ b_1 & a_2 & b_2 & ... & 0 \\ 0 & b_2 & a_3 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & ... & a_n \end{bmatrix}, \quad B(L) = \begin{bmatrix} 0 & b_1 & 0 & ... & -b_n \\ -b_1 & 0 & b_2 & ... & 0 \\ 0 & -b_2 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & ... & 0 \end{bmatrix}.$$

In these coordinates the Hamiltonian is simply $H(L) = 2\,\mathrm{Tr}(L^2)$. Let us now extend $B(L)$ to any matrix L, such that $B(L) = diag(L,1) - diag(L,-1) + diag(L,-n) - diag(L,n)$. We notice that we can take as a new Hamiltonian the function $\widetilde{H}(W) = -1/2\,\mathrm{Tr}(L^\dagger B(L)) + H(L)$, which generates the isospectral flow (21) and has the same level sets of $H(L)$. In fact, since $B(L) \in \mathfrak{so}(n)$ when $L \in Sym(n,\mathbb{R})$, $\mathrm{Tr}(L^\dagger B(L)) = 0$ for $L \in Sym(n,\mathbb{R})$ and $\nabla\,\mathrm{Tr}(L^\dagger B(L)) = 2B(L)$, when extended to any matrix L, being $B(\cdot)$ symmetric with respect to the Frobenius inner product. Moreover, since $H(L)$ is a Casimir function for (21) its gradient doesn't affect the dynamics.

We can apply the methods 3 of theorem 4, descending from the midpoint rule. Since the $H(L)$ is one of the Casimir functions of the flow, we have that it is preserved up to machine precision (see figure 3).

5.3. **A Hamiltonian non-integrable system: vorticity Euler equations on a sphere.** On the 2-sphere the hydrodynamical Euler equations have an equivalent formulation in terms of vorticity of the velocity vector field of a 2-D incompressible, inviscid, homogeneous fluid. These look like:

$$\begin{aligned} \dot{\omega} &= \{\Delta^{-1}\omega, \omega\} \\ \omega(0) &= \omega_0, \end{aligned} \tag{22}$$

where ω is a smooth function on $\mathbb{S}^2$ with zero mean and, Δ^{-1} is the inverse of the Laplace-Beltrami operator on the sphere (it is invertible because the kernel are just the constant functions) and the bracket is the one defined by the volume form on the sphere. This system is an infinite dimensional Lie–Poisson system on $sdiff^*(\mathbb{S}^2)$, the dual of the algebra of the volume preserving diffeomorphisms

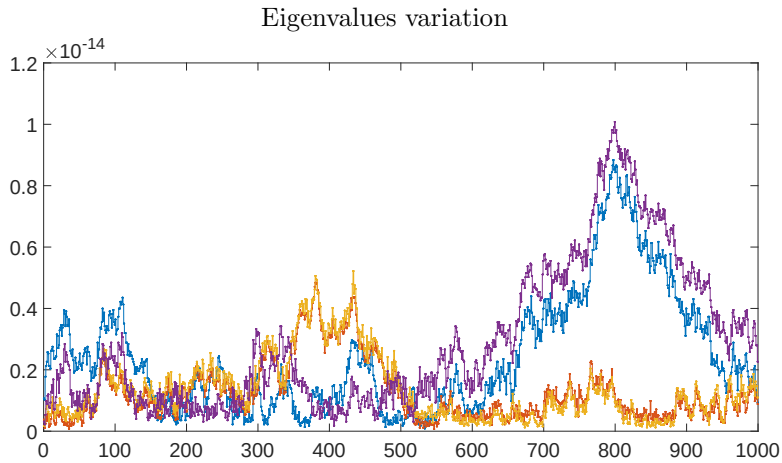

FIGURE 3. Periodic 8-dimensional Toda Lattice. Eigenvalues variation; h=10^{-1}; initial values $a_i = b_i = (-1)^i$, for $i = 1, .., 4$.

on the sphere. Moreover, the integral of any smooth function f on $sdiff^*(\mathbb{S}^2)$ is a conserved quantity of (22), which means that there exist an infinite number of independent first integrals.

It is well known that a quantization of (22) gives the so called *sine-bracket* Euler equations on $\mathfrak{su}(n)$ (or $\mathfrak{sl}(n, \mathbb{C})$), for any $n \geq 1$, whose solutions converge pointwise to those of (22) for $n \to \infty$ (see [6], for a global convergence result for the Euler equations on a 2-dimensional torus). The nice property of the *sine-bracket* Euler equations is that they better resemble (22) compared to a general spectral approximation, having a increasing number of first integrals converging to those of (22) when $n \to \infty$. Specifically we have that the *sine-bracket* Euler equations look like:

$$\begin{aligned} \dot{W} &= [\Delta_n^{-1} W, W] \\ W(0) &= W_0, \end{aligned} \tag{23}$$

where $W \in \mathfrak{su}(n)$ and Δ_n^{-1} is the inverse of the discrete Laplace-Beltrami operator on the sphere (it is invertible because the kernel are just the multiple of the identity matrix).

The equations (23) are a Lie–Poisson Hamiltonian system with respect to the Hamiltonian $H(W) = \frac{1}{2}\operatorname{Tr}((\Delta_n^{-1} W)^\dagger W)$. The Casimir functions of (23) are the momenta of W or equivalently, the eigenvalues of W.

We present the numerical solution of (23) using the numerical scheme (2) of theorem 4, descending from the midpoint rule. As seen in figure 4, we have machine precision conservation of the Casimir functions and near conservation of the Hamiltonian. Moreover the definition of Δ_n allows to have a complexity of $\mathcal{O}(n^3)$. The RHS approximation of the Euler equations has an accuracy of $\mathcal{O}(n^2)$ which is much less than the spectral accuracy of the classical spectral methods which however don't preserve the Casimir functions. We will discuss this topic in details in a forthcoming paper.

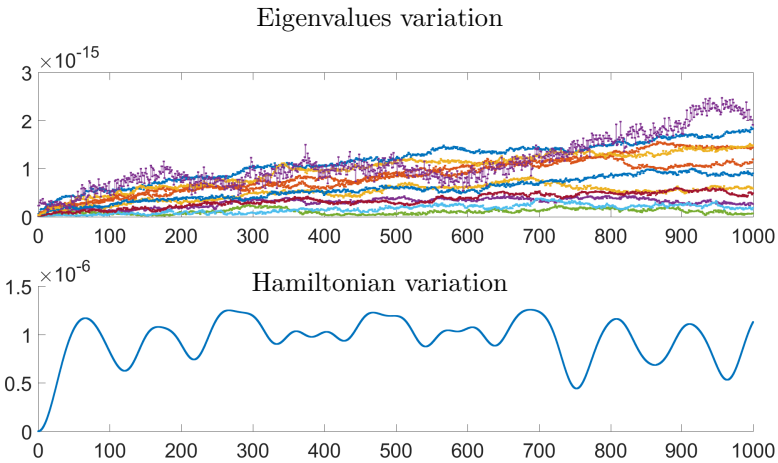

FIGURE 4. Euler equations on $\mathfrak{su}(n)$, for $n = 11$. Eigenvalues and Hamiltonian variation; h=10^{-1}; initial value W_0 the normalized and trace free matrix from $A_{lm} = i$ if $l \neq m$, $A_{lm} = 2i$ if $l = m$.

5.4. **Hamiltonian systems on a product of Lie algebras: Point vortices on a sphere and Heisenberg spin chain.** The methods above presented can be extended to a product space. For example we can deal with $(\mathfrak{su}(2)^*)^n$, where n is the number of vortices or spin particles in the point-vortices equation and, respectively, in the Heisenberg spin chain.

The Hamiltonian for the point vortices is:

$$H(W_1, W_2, ..., W_n) = -\frac{1}{4\pi} \sum_{\substack{i,j=1 \\ i<j}}^{n} \Gamma_i \Gamma_j \log\left(1 - \frac{\mathrm{Tr}(W_i^\dagger W_j)}{\|W_i\|^2 \|W_j\|^2}\right),$$

where $W_1, ..., W_n$ are the positions of the point vortices and $\Gamma_1, ..., \Gamma_n$ the respective magnitude, and for the Heisenberg spin chain:

$$H(W_1, W_2, ..., W_n) = \sum_{i=1}^{n} \mathrm{Tr}(W_i^\dagger W_{i+1}),$$

where $W_1, ..., W_n$ are the spins of the particles and $W_1 = W_{n+1}$ [12].

For this systems a new first integral arises, due to the $SU(2)$ symmetry of the Hamiltonians:

$$G.H(W_1, W_2, ..., W_n) = H(GW_1G^{-1}, GW_2G^{-1}, ..., GW_nG^{-1}) = H(W_1, W_2, ..., W_n),$$

for any $G \in SU(2)$. This is given by the (weighted) sum of the vortices/spin particle:

$$M(W_1, W_2, ..., W_n) = \sum_{i=1}^{n} \Gamma_i W_i.$$

Here we show in figure 5 that together with the norm of each particle, this first integral is also preserved up to machine precision.

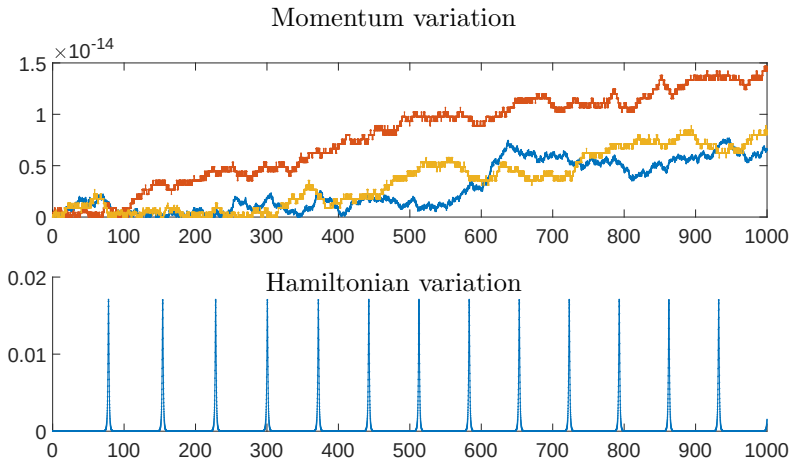

FIGURE 5. 4 point vortex on a sphere. Three compontents of the momentum and Hamiltonian variation; h=10^{-1}; initial values: $x_1 = [1\ 0\ 0], x_2 = [-1\ 0\ 0], x_3 = [0\ 1\ 0], x_4 = [0\ \ -1\ 0]$.

5.5. **A non-Hamiltonian vector field: Toeplitz inverse eigenvalue problem.** The methods introduced can be applied also to non-Hamiltonian systems. As an example, we consider the Chu's flow on symmetric real matrices. Here instead of $\nabla H(W)^\dagger$ we take $B(W)$, where:

$$B(W) = \begin{bmatrix} 0 & W_{1,1} - W_{2,2} & W_{1,2} - W_{2,3} & \dots & W_{1,n-1} - W_{2,n} \\ W_{2,2} - W_{1,1} & 0 & W_{2,2} - W_{3,3} & \dots & W_{2,n-1} - W_{3,n} \\ W_{3,2} - W_{2,1} & W_{3,3} - W_{2,2} & 0 & \dots & W_{3,n-1} - W_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_{n,2} - W_{n-1,1} & W_{n,3} - W_{n-1,2} & W_{n,4} - W_{n-1,3} & \dots & 0 \end{bmatrix}$$

Notice that when $W \in Sym(n, \mathbb{R})$, $B(W) \in \mathfrak{so}(n)$. The Chu's flow was introduced to solve in practice the Toeplitz inverse eigenvalue problem, i.e., given a certain set of eigenvalues, find a Toeplitz matrix with that prescribed spectra (we recall that a Toeplitz matrix is a symmetric matrix whit constant elements on the diagonals). Toeplitz had in fact established with a non constructive proof that, for any given spectra, there exists a Toeplitz matrix with those eigenvalues. Chu instead proved that the fixed point of its flow are Toeplitz matrices, provided that the eigenvalues are distinct [23].

The Chu's flow is very interesting even from a numerical point of view because there exist periodic orbits in it. This means that the convergence is not automatically ensured. However these orbits are unstable and because of the drift in the numerical methods is true that the Chu's flow in practice always converge to a Toeplitz matrix, when the starting point has distinct eigenvalues [23].

However, a qualitatively better simulation of the Chu's flow can be obtain if it is restricted to the centrosymmetric matrices. A matrix is said to be centrosymmetric if it is invariant with respect to a rotation of the components of π grade. In other

words a matrix A is centrosymmetric if:

$$A^{\dagger}E - EA = 0,$$

where

$$E = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Let us denote with Centro(n) the Lie algebra of centrosymmetric matrices of dimension n. In particular we have that the Toeplitz matrices are centrosymmetric and the tensor $B(W)$ is centrosymmetric when W is symmetric and centrosymmetric [23]. Therefore, for the Toeplitz problem, the Chu's flow can be restricted to the symmetric-centrosymmetric matrices. With this restriction the periodic orbits are numerically preserved and therefore the simulation of the flow is more realistic.

We show in figure (6)-(7) the difference of the behaviour of the flow with and without the restriction to the centrosymmetric matrices.

Notice that by theorem 3, since the flow is in S determined by quadratic constraints, we have that the methods (5) descend to an isospectral integrator on the symmetric-centrosymmetric matrices, provided that $B(W)$ is in its normalizer (which is $\mathfrak{so}(n) \cap \text{Centro}(n)$).

We have taken the following initial value proposed in [23], confirming the results there presented.

$$W_0 = \begin{bmatrix} 0.1336 & 0 & 0 & 0.5669 \\ 0 & -0.1336 & 0.378 & 0 \\ 0 & 0.378 & -0.1336 & 0 \\ 0.5669 & 0 & 0 & 0.1336 \end{bmatrix}$$

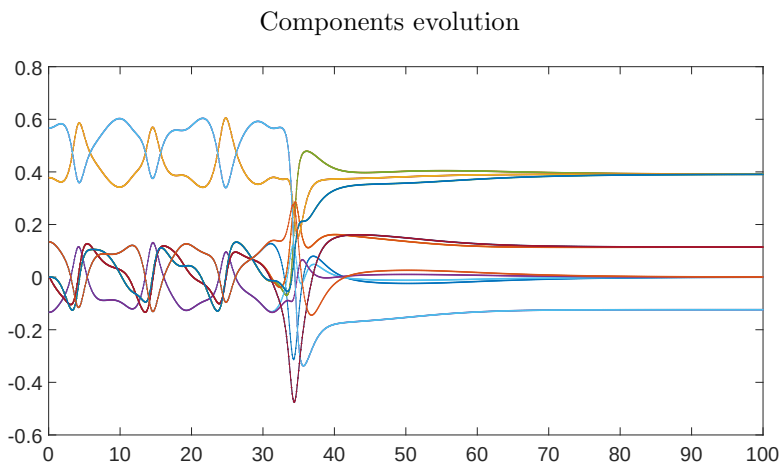

FIGURE 6. Chu's flow without forcing centrosymmetry of $B(W)$. Components evolution; h=10^{-1}; initial value W_0.

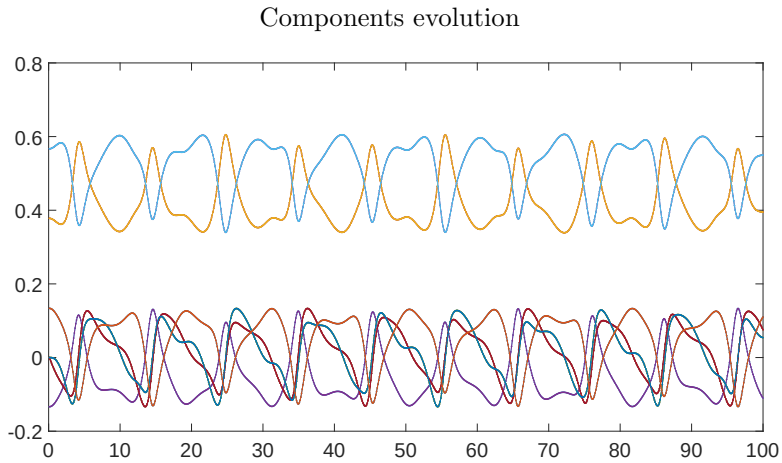

FIGURE 7. Chu's flow forcing centrosymmetry of $B(W)$. Components evolution; h=10^{-1}; initial value W_0.

BIBLIOGRAPHY

[1] G. Bogfjellmo and H. Marthinsen. "High-Order Symplectic Partitioned Lie Group Methods". *Foundations of Computational Mathematics* 16.2 (2016), pp. 493–530.

[2] M. T. Chu. "A list of matrix flows with applications". *Fields Institute Communications* 3 (1994), pp. 87–97.

[3] M. T. Chu. "Linear algebra algorithms as dynamical systems". *Acta Numer.* 17 (2008), pp. 1–86.

[4] P. Deift, T. Nanda, and C. Tomei. "Ordinary differential equations and the symmetric eigenvalue problem". *SIAM J. Numer. Anal.* 20 (1983), pp. 1–22.

[5] H. Flaschka. "The Toda lattice. II. Existence of integrals". *Physical Review B* 9 (1974), p. 1924.

[6] I. Gallagher. "Mathematical analysis of a structure-preserving approximation of the bidimensional vorticity equation". *Numerische Mathematik* 91 (2002), pp. 223–236.

[7] E. Hairer, C. Lubich, and G. Wanner. *Geometric Numerical Integration.* Vol. 31. Springer, Berlin, Heidelberg, 2006, pp. XVII, 644. ISBN: 978-3-540-30663-4.

[8] L. Jay. "Symplectic partitioned Runge-Kutta methods for constrained Hamiltonian systems". *SIAM J. Numer. Anal.* 33(1) (1996), pp. 368–387.

[9] A.W. Knapp. *Lie groups beyond an introduction.* Birkhäuser, 1996. ISBN: 978-1-4757-2453-0.

[10] S.V. Manakov. "Note on the integration of the Euler's equations of the dynamics of a n-dimensional rigid body". *Functional Anal. Appl.* 10 (1976), pp. 328–329.

[11] J.E. Marsden and T.S. Ratiu. *Introduction to Mechanics and Symmetry.* Vol. vol. 17 of *Texts in Applied Mathematics.* Springer-Verlag, New York, 1999. ISBN: 978-0-387-21792-5.

[12] R. McLachlan, K. Modin, and O. Verdier. "A minimal-variable symplectic integrator on spheres". *Math. Comp.* 86(307) (2017), pp. 2325–2344.

[13] R. McLachlan, K. Modin, and O. Verdier. "Collective symplectic integrators". *Nonlinearity* 27(6) (2014), pp. 1525–1542.

[14] R. McLachlan, K. Modin, and O. Verdier. "Symplectic integrators for spin systems". *Phys. Rev. E* 89:061301 (2014).

[15] R. McLachlan and G.R.W. Quispel. "Splitting methods". *Acta Numerica* 11 (2002), pp. 341–434.

[16] R. McLachlan et al. "Geometric Generalisations of SHAKE and RATTLE". *Found. Comput. Math. (FoCM)* 14(2) (2014), pp. 339–370.

[17] K. Modin. "Geometry of Matrix Decompositions Seen Through Optimal Transport and Information Geometry". *J. Geom. Mech.* 9(3) (2017), pp. 335–390.

[18] H. Poincaré. "Sur une forme nouvelle des équations de la mécanique". *C.R. Acad. Sci.* 132 (1901), pp. 369–371.

[19] W. Symes. "The QR algorithm and scattering for the finite nonperiodic Toda lattice". *Phys. D* 4 (1982), pp. 275–280.

[20] M. Toda. "Waves in nonlinear lattice". *Progress of Theoretical Physics Supplement* 45 (1970), pp. 174–200.

[21] C. Tomei. "The Toda lattice, old and new". *J. Geom. Mech.* 5 (2013), pp. 511–530.

[22] D.S. Watkins. "Isospectral flows". *SIAM Rev.* 26 (1984), pp. 379–391.

[23] M. Webb. "Isospectral algorithms, Toeplitz matrices and orthogonal polynomials, Ph.D. thesis" (2017).

A STRUCTURE-PRESERVING SCHEME FOR THE EULER EQUATIONS ON A ROTATING SPHERE

MILO VIVIANI

Abstract. The Euler equations for an incompressible fluid on a sphere are a fundamental model of oceanic and atmospheric dynamics on Earth. Their solution has been a long standing challenge which concerns several different approaches, both from an analytical and numerical point of view. From a numerical point of view, the main issue is to take as much as possible of the geometry of the problem into account when discretizing the original system. Currently, there is not yet an a fully satisfactory way to do it. However, a possible approach is provided by the quantization of the Poisson algebra of the smooth functions on a sphere. Combining this this technique with the symplectic integration of Lie–Poisson systems, we present a new numerical method which gives an approximate solution of the Euler equations with a number of discrete first integrals consistent with the level of discretization. The method is then tested on two examples found in literature, showing good agreement with the results there found.

Contents

1. Introduction 1
2. Numerical integration of the Euler equations 3
2.1. Spatial discretisation via $\mathfrak{sl}(N,\mathbb{C})$ 3
2.2. With the Coriolis force 5
2.3. Time discretisation by Isospectral-RK methods 5
3. Simulations 7
3.1. Random initial vorticity given in [10] 7
3.2. Gaussian vortex on a rotating sphere given in [8] 9
4. Complexity and time scaling 11
4.1. Complexity 11
4.2. Time scaling 11
Bibliography 13

1. Introduction

The motion of a fluid on a surface is a wide and well established field of study [2], [3], [4], [5], [14], whose applications in oceanography, weather forecasting and astrophysics have motivated extensive research from a numerical point of view [1], [8], [9], [10], [15], [17], [18]. The geometry encoded in the equations describing this phenomenon plays a central role in understanding the behaviour of the fluid [3], [4], [14], and in the investigation of numerical methods to simulate it [1], [15], [17], [18].

Following the references above, in this paper we consider a homogeneous, incompressible, inviscid, two-dimensional fluid which is constrained to move on a spherical surface, embedded in the standard Euclidean $\mathbb{R}^3$, and possibly rotating with constant angular speed Ω, with respect to a fixed normal axis. The equations of motion of such a fluid are given by the well-known Euler equations of hydrodynamics:

$$\begin{aligned} &\dot{v} + v \cdot \nabla v = -\nabla p - 2\widetilde{\Omega} \times v \\ &\nabla \cdot v = 0, \end{aligned} \tag{1}$$

where v is the velocity vector field of the fluid, p is its internal pressure, and $\widetilde{\Omega} = (\Omega \cdot n)n$ is the projection of the angular rotation of the sphere Ω to the normal n at a point of the sphere. The last term in the first equation of (1), $F_{\mathrm{c}} = -2\widetilde{\Omega} \times v$, is called *Coriolis force*. Equations (1) can be equivalently expressed in terms of the one form $v^\flat$ as a Lie–Poisson system on the dual of the infinite-dimensional Lie-algebra of divergence-free vector fields. The respective Poisson tensor is degenerate so that there is an infinite number of independent first integrals (Casimir functions) [4].

An equivalent formulation of (1) is given in terms of the vorticity $\omega = (\nabla \times v) \cdot n$. We notice that by the Stokes' theorem it must be that $\int \omega = 0$. Then the Euler equations (1) can be written as:

$$\begin{aligned} &\dot{\omega} = \{\psi, \omega\} \\ &\Delta\psi = \omega - f, \end{aligned} \tag{2}$$

where $f = 2\Omega \cdot n$, Δ is the Laplace-Beltrami operator on the sphere (which is minus the Hodge Laplacian and hence negative semidefinite), and ψ is the unique solution to the Poisson equation in $C^\infty(\mathbb{S}^2)$, such that $\int \psi = 0$.

In this form the Euler equations are a Lie–Poisson system on the smooth functions on the sphere which integrate to 0. The Hamiltonian is given by

$$H(\omega) = \frac{1}{2} \int (\omega - f)\psi.$$

The (infinitely many) Casimir functions are given, for any smooth real function g, by $F(\omega) = \int g(\omega)$. This is easy to check:

$$\frac{d}{dt} \int g(\omega) = -\int g'(\omega) v \cdot \nabla\omega = -\int v \cdot \nabla g(\omega) = \int (\nabla \cdot v) g(\omega) = 0,$$

where we have used the fact that

$$\{\psi, \cdot\}_p = (X_\psi)_p(\cdot) = p \cdot (\nabla\psi \times \nabla\cdot) = (p \times \nabla\psi) \cdot \nabla\cdot = -v_p \cdot \nabla \cdot .$$

The presence of all these first integrals turns out to be the leading point in giving a suitable discretization of (2). In fact, while solving it with a numerical scheme, we cannot expect to preserve all the infinitely many first integrals, but what we do want is that the number of conserved quantities increases with the size of the discrete problem. As shown in [17], [18], this cannot be achieved by simply considering a truncated spectral decomposition of the vorticity.

We consider instead the approach proposed by Zeitlin [18] based on the theory of geometric quantization of compact Kähler manifolds [7],[6],[12]. It provides a sequence of finite-dimensional twisted representations of the infinite-dimensional Lie algebra of divergence free vector fields, which converges[1] to the latter Lie algebra.

[1]Here the term converge is referred to L_α-convergence, which is explained later.

Then, for any of these twisted-representations, we get a finite dimensional analogue to (2), i.e., a Hamiltonian Lie–Poisson system on $\mathfrak{su}(n)$ (or $\mathfrak{sl}(n,\mathbb{C})$), for any $n \geq 1$, with $n-1$ Casimir functions. To solve the obtained equations, i.e., to discretize the equations in time, we apply a Lie–Poisson integrator presented in [16], such that there is conservation of the discrete first integrals and near conservation of the Hamiltonian.

This method is then applied to two examples given in literature. The first one is taken from [10]. In this example the sphere is non rotating and the main interest is in the long time behaviour of the fluid. In [10], the author claims that eventually a non steady state of four vortices is reached. The second one is given in [8]. In this case, the sphere is rotating and the main qualitative behaviour is the north-west movement of a Gaussian vortex, given as initial value for the vorticity. The results obtained with our method have a good qualitative agreement with those in [10] and [8], but respect to the methods used there, ours also exhibit conservation of the Enstrophy (together with the other discrete first integrals) and near conservation of the Hamiltonian.

2. Numerical integration of the Euler equations

In this section we propose a numerical scheme to solve the equations (2). For the discretization in space, we consider the system of differential equations developed by Zeitlin [18], based on the work of Hoppe et al. on the approximation of infinite dimensional Lie algebras [6],[7]. The main idea is that in order to respect the infinite dimensional features of the Euler equations, we need to have a consistent discretization of the system, such that we get a qualitatively coherent behaviour of the system. In our case this means that we want a number of discrete Casimir functions increasing with respect to the level of discretization. It has been found that there exists a sequence of finite dimensional matrix algebras whose respective Lie–Poisson systems have the desired properties. The sequence is given by $\{\mathfrak{sl}(N,\mathbb{C})\}_{N\in\mathbb{N}}$ and $\{\mathfrak{su}(N,\mathbb{C})\}_{N\in\mathbb{N}}$, which are an approximation (L_α-approximation) of $C^\infty(\mathbb{S}^2,\mathbb{C})$ and $C^\infty(\mathbb{S}^2,\mathbb{R})$, respectively. To integrate the equations obtained after the discretization in space, we consider a numerical scheme presented in [16]. Finally, the space-time numerical method derived has the desired properties of nearly conserving the Hamiltonian and exactly (up to machine precision) the discrete Casimir functions.

2.1. Spatial discretisation via $\mathfrak{sl}(N,\mathbb{C})$.

This section is devoted to the technique used to get a finite dimension analogous of the Euler equations on a sphere. The main theoretical concept behind this approach is the so called L_α-approximation.

2.1.1. *L_α-approximation.* Let us consider a Lie-algebra $(\mathfrak{g},[\cdot,\cdot])$ and a family of labelled Lie algebras $(\mathfrak{g}_\alpha,[\cdot,\cdot]_\alpha)_{\alpha\in I}$, where $\alpha \in I = \mathbb{N}$ or $\mathbb{R}$. Furthermore, assume then that to any element of this family, a distance d_α and a surjective projection map $p_\alpha : \mathfrak{g} \to \mathfrak{g}_\alpha$ are associated. Then we say that $(\mathfrak{g},[\cdot,\cdot])$ is an L_α-approximation of $(\mathfrak{g}_\alpha,[\cdot,\cdot]_\alpha)_{\alpha\in I}$ if:

- if $x,y \in \mathfrak{g}$ and $d_\alpha(p_\alpha(x),p_\alpha(y)) \to 0$, for $\alpha \to \infty$, then $x = y$,
- for all $x,y \in \mathfrak{g}$ we have $d_\alpha(p_\alpha([x,y]),[p_\alpha(x),p_\alpha(y)]_\alpha) \to 0$, for $\alpha \to \infty$,
- all p_α, for $\alpha \gg 0$, are surjective.

The above definition is given in [6], and it is a quite weak requirement to obtain a limit for a sequence of Lie algebras. Indeed the same sequence may converge

in the L_α sense to different Lie algebras [7]. Much depends on the choice of the projections, that are not canonical.
However, for our purpose, since we have already a target and we need a suitable sequence to approximate it, we do not need more than that.

Let us now consider the smooth complex functions with 0 mean on the sphere, and denote them by $C_0^\infty(\mathbb{S}^2,\mathbb{C})$. This vector space can be canonically endowed by a Poisson structure given by the respective Hamiltonian vector fields of two functions and a symplectic form α on $\mathbb{S}^2$. We have for any $f,g\in C_0^\infty(\mathbb{S}^2,\mathbb{C})$:

$$\{f,g\}=\alpha(X_f,X_g). \tag{3}$$

With this bracket, $C_0^\infty(\mathbb{S}^2,\mathbb{C})$ becomes an infinite dimensional Poisson algebra. A basis is given by the complex spherical harmonics, which is denoted in the standard notation and azimuthal-inclination coordinates (ϕ,θ) as:

$$Y_{lm}=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi},$$

for $l\geq 1$ and $m=-l,\dots,l$. In the spherical harmonics basis, an explicit L_α-approximating sequence for $C_0^\infty(\mathbb{S}^2,\mathbb{C})$ has been built up by J. Hoppe [12], and fully proved (in a more general contest) by M. Bordemann, E. Meinrenken and M. Schlichenmaier [6]. It is given by the matrix Lie algebras $(\mathfrak{sl}(N,\mathbb{C}),[\cdot,\cdot]_N)_{N\in\mathbb{N}}$, where $[\cdot,\cdot]_N=N^{3/2}[\cdot,\cdot]$, the rescaled usual commutator of matrices.

The distances are given by a suitable matrix norm, and the projections are defined associating to any spherical harmonic a respective matrix in $\mathfrak{sl}(N,\mathbb{C})$, for any $N\in\mathbb{N}$, i.e., $p_N:Y_{lm}\mapsto T^N_{lm}$, where

$$(T^N_{lm})_{m_1m_2}=(-1)^{n/2-m_1}\sqrt{2l+1}\begin{pmatrix} N/2 & l & N/2\\ -m_1 & m & m_2\end{pmatrix},$$

where the round bracket is the Wigner 3j-symbols. The result can be summarized as:

Theorem 1 (Bordemann, Hoppe, Meinrenken, Schlichenmaier [7],[6])**.** *Let us consider the Poisson algebra $(C_0^\infty(\mathbb{S}^2,\mathbb{C}),\{\cdot,\cdot\})$, whose pairing is defined in (3). Then with respect to p_N above defined, and d_N any matrix norm, $(C_0^\infty(\mathbb{S}^2,\mathbb{C}),\{\cdot,\cdot\})$ is an L_α-approximation of $(\mathfrak{sl}(N,\mathbb{C}),[\cdot,\cdot]_N)_{N\in\mathbb{N}}$.*

2.1.2. *The reduced system.* We can now derive the spatial discretization of the Euler equations via the L_α-approximation. We first present the system without the Coriolis force.

For any $N\in\mathbb{N}$, we get an analogue of the Euler equations (2):

$$\dot W=[\Delta_N^{-1}W,W]_N, \tag{4}$$

where $W\in\mathfrak{sl}(N,\mathbb{C})$ and Δ_N^{-1} is the inverse of the discrete Laplacian as defined in [18]. The crucial property of Δ_N^{-1} is that $\Delta_N^{-1}T^N_{lm}=(-l(l+1))^{-1}T^N_{lm}$, for any $l=1,...,N$, $m=-l,...,l$.

We remark that, for a real valued vorticity, W is actually in $\mathfrak{su}(N,\mathbb{C})$, which means that $\overline{W}_{lm}=(-1)^mW_{l-m}$.

The discrete Hamiltonian takes the following form:

$$H(W)=\frac{1}{2}\operatorname{Tr}(\Delta_N^{-1}WW^\dagger).$$

The discrete system has the following independent $N-1$ of first integrals[2]

$$F_n(W) = \mathrm{Tr}(W^n) \text{ for n=2,..,N}$$

which, up to a normalization constant dependent on N, converge to the powers of the continuous vorticity.

2.2. **With the Coriolis force.** In the case with the Coriolis force, the discrete system looks like:

$$\dot{W} = [\Delta_N^{-1}(W-F), W]_N,$$

where $F = 2\Omega T_{10}^N$ represents the discrete Coriolis force. The discrete Hamiltonian in this case takes the following form:

$$H(W) = \frac{1}{2}\,\mathrm{Tr}(\Delta_N^{-1}(W-F)(W-F)^\dagger).$$

2.3. **Time discretisation by Isospectral-RK methods.** The final step in the discretization consists in solving (4) with a suitable numerical scheme. To take advantage of the quantization of the original equations, we have to solve (4) with a Lie–Poisson integrator. In this way we would get an exact conservation of the Casimir functions and a near conservation of the Hamiltonian. Since (4) is a Hamiltonian isospectral flow, we can apply one of the methods presented in [16]. The simplest method is the 2nd order isospectral midpoint rule. Given a time step disctretization h, it looks like:

$$\begin{aligned} X &= -h(W_n + \tfrac{1}{2}X)\Delta_N^{-1}\widetilde{W} \\ K &= \tfrac{h}{2}\Delta_N^{-1}\widetilde{W}(X+K) \\ \widetilde{W} &= W_n + \tfrac{1}{2}(X - X^T + K) \\ W_{n+1} &= W_n + X - X^T + K - K^T. \end{aligned} \tag{5}$$

In the numerical scheme X, K are $N \times N$ auxiliary matrix variables implicitly defined in the first two lines of (5). For the derivation and the theory behind (5), we refer to [16]. We recall that (5) preserves the Casimir functions of (4) up to machine precision and nearly conserve the Hamiltonian H.

It is now important to remark that since the matrices T_{lm}^N, with the Frobenius inner product, share the same orthogonality properties of Y_{lm}, with the $L^2(\mathbb{S}^2)$ inner product. Therefore, if the initial vorticity ω_0 is represented by a finite linear combination of spherical harmonics, then choosing N sufficiently large, the Hamitlonian and the Enstrophy, i.e., the integral of the square of the vorticity, coincide with the quantized ones.

[2]One should notice that by definition $\mathrm{Tr}(W) = 0$ for all $W \in \mathfrak{sl}(N, \mathbb{C})$ and $\mathrm{Tr}(W^N)$ can be replaced by $\det(W)$, by the Cayley-Hamilton theorem.

2.3.1. *With the Coriolis force.* In presence of the Coriolis force term F the scheme becomes:

$$
\begin{aligned}
X &= -h(W_n + \tfrac{1}{2}X)\Delta_N^{-1}(\widetilde{W} - F) \\
K &= \tfrac{h}{2}\Delta_N^{-1}(\widetilde{W} - F)(X + K) \\
\widetilde{W} &= W_n + \tfrac{1}{2}(X - X^T + K) \\
W_{n+1} &= W_n + X - X^T + K - K^T.
\end{aligned}
\tag{6}
$$

3. Simulations

3.1. **Random initial vorticity given in [10].** We have tested our method on the initial data proposed in [10]. One main claim in [10] is that no steady state is reached after long time simulations but rather four main vortices are created and move around the sphere. In this subsection we show and compare the results obtained with our method with those in [10].

We have run our simulation in MATLAB, taking $W_0 \in \mathfrak{su}(N)$, for $N = 501$ and normalized, time-step $h = 0.1$, and solving (5) with a Newton type iteration with tolerance of 10^{-13}.

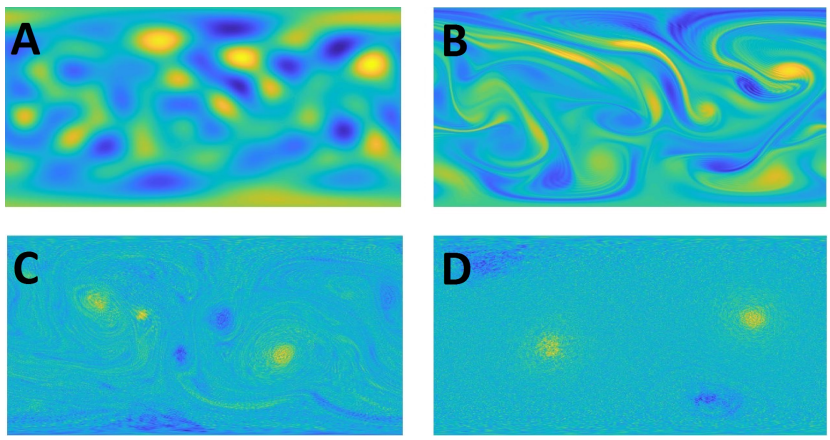

Figure 1. Simulation with method (5). Vorticity $\omega(x,t)$ from the top-left at $t = 0s, 4s, 40s, 140s$, for the intial data in [10]. The horizontal axis is the azimuth $\varphi \in [0, 2\pi]$ and the vertical axis is minus the inclination $\theta \in [0, \pi]$.

We can see that at time $t = 4s$, both our simulation and those in [10] give very similar results, with the beginning of some "artificial" ripples in ours due to a less detail in the solution. In the simulation in [10], the level of truncation is in fact at $N = 2800$. However, as shown in [10], the early-intermediate vorticity, i.e., at time $t = 40s$, is already different according to different simulations. In fact ours is very divergent to any of the ones in [10], as shown in figure 1-2.

For longer times, at the end of our simulation, i.e., at $t = 140s$, we have the creation of four large vortices. We have also found that the two vortices with positive vorticity, i.e., the bright ones, are quite stable in their position, whereas the two negatives are drifting away.

Here we want to summarize the main conclusions obtained from our simulation.

- A direct comparison of Figure 1B and Figure 2B shows that they are almost indistinguishable. Figure 1C and Figure 2C exhibit instead a significant divergence, also due to the different resolution in which the simulations have been carried. However, in Figure 1D, we also get four main persistent non steady large vortices, as shown in Figure 2D [10].

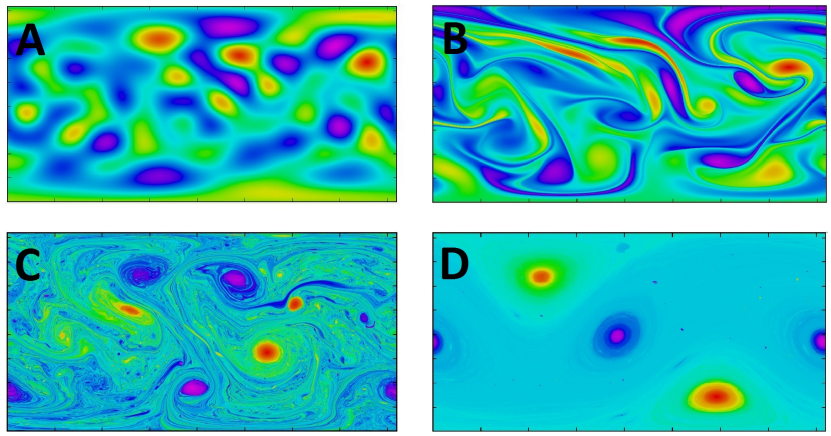

FIGURE 2. Simulation in [10]. Vorticity $\omega(x,t)$ from the top-left at $t = 0s, 4s, 40s, 400s$, for the intial data in [10]. The horizontal axis is the azimuth $\varphi \in [0, 2\pi]$ and the vertical axis is minus the inclination $\theta \in [0, \pi]$.

- From the theory, we know that our method has a number of discrete conservation laws proportional with the level of truncation. In the implementation, we have solved the method (5) with a Newton iteration, with tolerance parameter equals to 10^{-13}. Hence, we have that the Enstrophy and the other Casimir functions (i.e., the eigenvalues of W) are conserved up to a fluctuation of the order of 10^{-13}.
- In figure 3, we can also see the near conservation of the Hamiltonian that is conserved up to order 10^{-6}.
- However, we cannot claim that our simulation is a proof for any realistic prediction for long time evolution of the vorticity for the following reasons: 1. it is still not proved a complete analytical result about the convergence in time of the L_α-approximation of the Euler equations on the sphere (for the torus we have [11]); 2. our simulation was carried with good detail, but a higher detailed one would be necessary for direct comparison with the results in [10].
- One difficulty in extracting information from our simulations depends on the fact that we have to express W in the T^N_{lm} basis, which for being calculated requires the evaluation of the 3j-Wigner symbols for very large numbers. A fast computation of those or a full library is not known by the author.

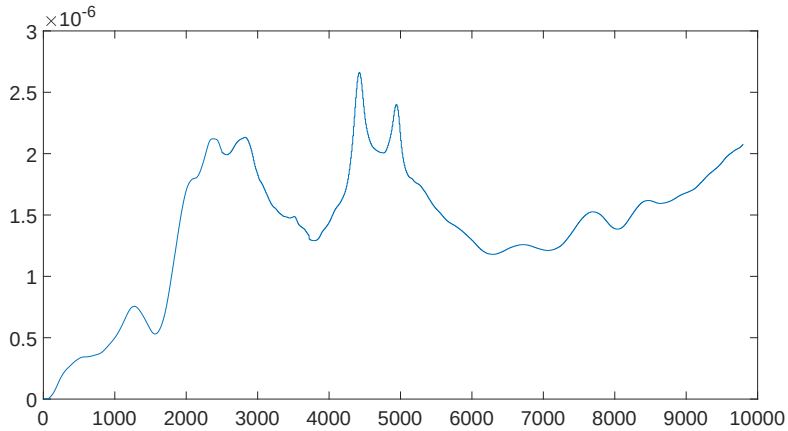

FIGURE 3. Hamiltonian variation $|H - H_0|$ in simulation with method (5). The horizontal axis is the number of records (each record is 200 iterations). Each 70 records corresponds to a real time second.

3.2. **Gaussian vortex on a rotating sphere given in [8].** The second example we present is done taking the initial condition defined in [8], which is a Gaussian vortex on a rotating sphere. A Gaussian vortex centred in $x_c = [\sin(\pi/20)\ 0\ \cos(\pi/20)]$ is defined in [8] as:

$$\omega_0(x) = 4\pi \exp(-16|x - x_c|^2) + C, \tag{7}$$

for $x \in \mathbb{S}^2$ and C such that the integral of ω_0 is 0. Here we show the behaviour obtained with $N = 201$, $h = 0.1$ and angular speed of the sphere $\Omega = 2\sqrt{3\pi}$. To derive a formula for W_0, we have developed (7) in the complex spherical harmonics Y_{lm}, using a least square approximation, and then we have summed up the components found in the T^N_{lm} basis. In figure 4, we show the output of the simulation. We can see the vortex moving north west (i.e., top-left) and stretching during its migration, as observed in [8].

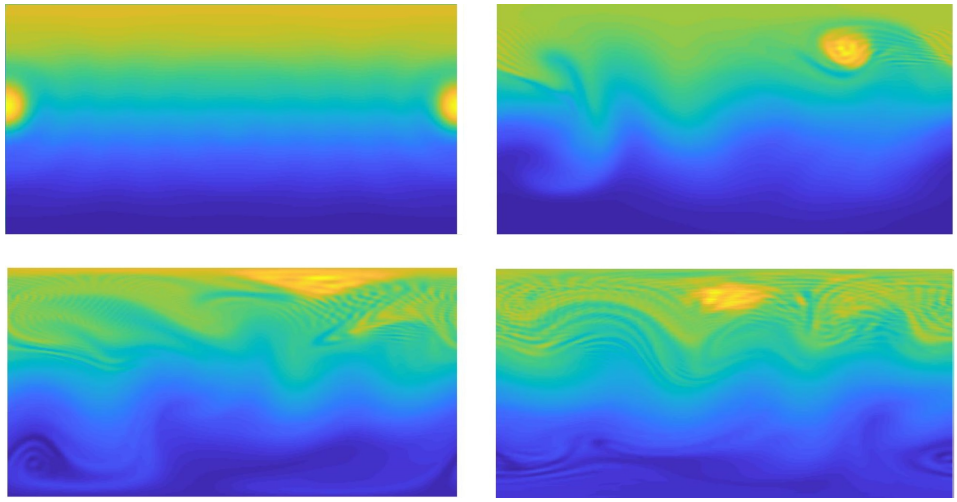

FIGURE 4. Simulation with method (6). Vorticity $\omega(x,t)$ from the top-left at record= $1, 150, 300, 450$ (each record is 200 iterations), for the intial data in [8]. The horizontal axis is the azimuth $\varphi \in [0, 2\pi]$ and the vertical axis is minus the inclination $\theta \in [0, \pi]$.

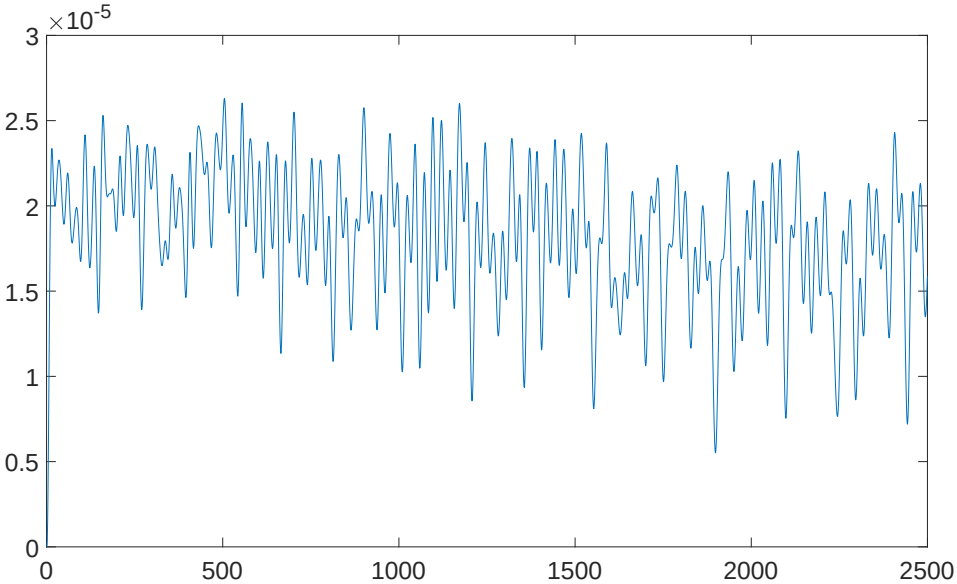

FIGURE 5. Hamiltonian variation $|H - H_0|$ in simulation with method (6). The horizontal axis is the number of records (each record is 200 iterations).

4. Complexity and time scaling

4.1. **Complexity.** The most demanding computational operation is, a priori, the inversion of the discrete Laplacian Δ_N. It is indeed a 4th order tensor defined on a particular basis of $\mathfrak{sl}(n,\mathbb{C})$. Fortunately a deep study has been done in [13], where and simpler formulas are provided. In particular, we use the following expression in components:

$$\begin{aligned}(\Delta_N)^{M_1' M_2'}_{M_1 M_2} =& 2\delta^{M_1'}_{M_1}\delta^{M_2'}_{M_2}(s(s+1)M_1M_2)\\ &- \delta^{M_1'}_{M_1+1}\delta^{M_2'}_{M_2+1}\sqrt{s(s+1)M_1(M_1+1)}\sqrt{s(s+1)M_2(M_2+1)}\\ &- \delta^{M_1'}_{M_1-1}\delta^{M_2'}_{M_2-1}\sqrt{s(s+1)M_1(M_1-1)}\sqrt{s(s+1)M_2(M_2-1)},\end{aligned}$$

for $M_1, M_1', M_2, M_2' = 1, ..., N$ and $s = (N-1)/2$. Moreover, to make Δ_N invertible we require that $\Delta_N Id = Id$. This of course does not affect the dynamics being Id in the centralizer of $\mathfrak{sl}(N,\mathbb{C})$. We store Δ_N in a sparse matrix of full size $N^2 \times N^2$, and then to have Δ_N^{-1}, we apply the sparse LU-factorization, which gives two sparse L, U matrices to be inverted to calculate Δ_N^{-1}. We remark that the definition of Δ_N and its LU-factorization can be done once and for all. In conclusion, the operation of multiplication by Δ_N^{-1} has a complexity of $O(N^2)$, as shown in figure 6.

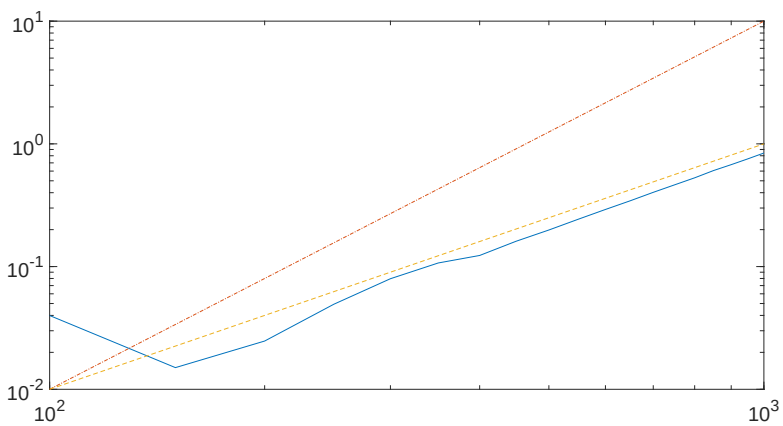

Figure 6. Complextiy of Δ_N^{-1} multiplication for $N = 101, ..., 1001$ in loglog scale. $f(N) = N^3$ in red dashed-dot, $f(N) = N^2$ in yellow dashed, Δ_N^{-1} multiplication in continuous blue line.

We have solved (5) with a Newton iteration type. Since the number of iterations is independent of N, the global complexity of the algorithm is as much as two full matrix multiplication, which is $O(N^3)$, as shown in figure 7.

4.2. **Time scaling.** The equations (4) are quadratic, and therefore are not invariant under rescaling of W. The rescaling $W \mapsto cW$ is equivalent to multiplying the bracket by the same constant c. In particular, we notice that for all $N \geq 1$ the matrices T^N_{lm} are orthonormal with respect to the Frobenius norm. Therefore, the constant $N^{3/2}$ in front of the matrix commutator implies that the step-size h of

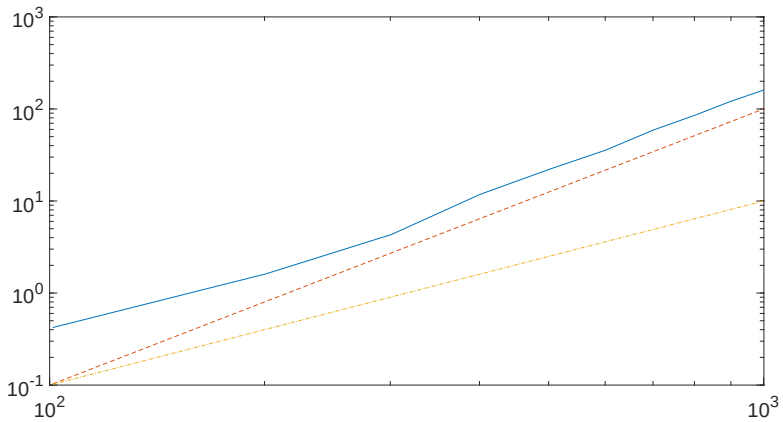

FIGURE 7. Complextiy of the whole algorithm for $N = 101, ..., 1001$ in loglog scale. $f(N) = N^3$ in red dashed-dot, $f(N) = N^2$ in yellow dashed, Δ_N^{-1} multiplication in continuous blue line.

discretization (or equivalently the length of the simulations) is scaled of a factor $N^{3/2}$. As local accuracy, i.e., the dimension of the discrete vorticity, goes as N^2, an improvement of a factor c in the spatial accuracy, at a certain time, requires $\sqrt{c}^3 \sqrt{c}^{3/2} = c^2 \sqrt[4]{c}$ more operations.

Bibliography

[1] A. Arakawa. "Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I". *Journal of Computational Physics* 1.1 (1966), pp. 119–143.

[2] H. Aref. "Point vortex dynamics: A classical mathematics playground". *Journal of Mathematical Physics* 48, 065401 (2007).

[3] V.I. Arnold. "Sur la géometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique del fluids parfaits". *Annales de l'institut Fourier* 16.1 (1966), pp. 319–361.

[4] V.I. Arnold and B.A. Khesin. *Topological Methods in Hydrodynamics*. Vol. 125. Springer-Verlag New York, 1998, pp. XV, 376. ISBN: 978-0-387-94947-5.

[5] V.A. Bogomolov. "Dynamics of vorticity at sphere". *Fluid Dyn.* 16 ().

[6] M. Bordemann, E. Meinrenken, and M. Schlichenmaier. "Toeplitz Quantization of Kähler Manifolds and $\mathfrak{gl}(N), N \to \infty$ limits". *Communications in Mathematical Physics* 165.2 (1994), pp. 281–296.

[7] M. Bordemann et al. "$\mathfrak{gl}(\infty)$ and geometric quantization". *Communications in Mathematical Physics* 138.2 (1991), pp. 209–244.

[8] P. Bosler et al. "A Lagrangian particle/panel method for the barotropic vorticity equations on a rotating sphere". *The Japan Society of Fluid Mechanics and IOP Publishing Ltd* 46.3 (2014).

[9] H.-B. Cheong. "Double Fourier Series on a Sphere: Applications to Elliptic and Voricity Equations". *Journal of Computational Physics* 157 (2000), pp. 327–349.

[10] D.G. Dritschel, W. Qi, and J.B. Marston. "On the late-time behaviour of a bounded, inviscid two-dimensional flow". *Journal of Fluid Mechanics* 783 (2015), pp. 1–22.

[11] I. Gallagher. "Mathematical analysis of a structure-preserving approximation of the bidimensional vorticity equation". *Numerische Mathematik* 91 (2002), pp. 223–236.

[12] J. Hoppe. "Ph.D. thesis MIT Cambridge" (1982).

[13] J. Hoppe and S.-T. Yau. "Some properties of matrix harmonics on S2". *Commun. Math. Phys.* 195 (1998), pp. 66–77.

[14] J. Marsden and A. Weinstein. "Co-adjoint orbits, vortices, and Clebsch variables for incompressible fluids". *Physica D* 7.1–3 (1983), pp. 305–323.

[15] R. McLachlan. "Explicit Lie–Poisson Integration and the Euler Equations". *Physical review letters* 71.19 (1993), pp. 3043–3046.

[16] K. Modin and M. Viviani. "Lie–Poisson numerical schemes for isospectral flows". *Arxiv e-prints* (June 2018).

[17] V. Zeitlin. "Finite-mode analogues of 2D ideal hydrodynamics: Coadjoint orbits and local canonical structure". *Physica D* 49.3 (1991), pp. 353–362.

[18] V. Zeitlin. "Self-Consistent-Mode Approximation for the Hydrodynamics of an Incompressible Fluid on Non rotating and Rotating Spheres". *Physical review letters* 93.26 (2004), pp. 353–362.