

THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

**Symplectic methods for Hamiltonian isospectral
flows and 2D incompressible Euler equations on
a sphere**

MILO VIVIANI



CHALMERS



GÖTEBORGS UNIVERSITET

*Division of Applied Mathematics and Statistics
Department of Mathematical Sciences*

CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG
Göteborg, Sweden 2018

Symplectic methods for Hamiltonian isospectral flows and 2D incompressible Euler equations on a sphere

Milo Viviani

© Milo Viviani, 2018.

Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg
SE-412 96 GÖTEBORG, Sweden
Phone: +46 (0)31 772 1000

Author e-mail: viviani@chalmers.se

Typeset with L^AT_EX.
Department of Mathematical Sciences
Printed in Göteborg, Sweden 2018

Symplectic methods for Hamiltonian isospectral flows and 2D incompressible Euler equations on a sphere

Milo Viviani

Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg

Abstract

The numerical solution of non-canonical Hamiltonian systems is an active and still growing field of research. At the present time, the biggest challenges concern the realization of structure preserving algorithms for differential equations on infinite dimensional manifolds. Several classical PDEs can indeed be set in this framework. In this thesis, I develop a new class of numerical schemes for Hamiltonian isospectral flows, in order to solve the hydrodynamical Euler equations on a sphere. The results are presented in two papers.

In the first one, we derive a general framework for the isospectral flows, providing then a class of numerical methods of arbitrary order, based on the Lie–Poisson reduction of Hamiltonian systems. Avoiding the use of any constraint, we obtain a large class of numerical schemes for Hamiltonian and non-Hamiltonian isospectral flows. One of the advantages of these methods is that, together with the isospectrality, they have near conservation of the Hamiltonian and, indeed, they are Lie–Poisson integrators.

In the second paper, using the results of the first one, we present a numerical method based on the geometric quantization of the Poisson algebra of the smooth functions on a sphere, which gives an approximate solution of the Euler equations with a number of discrete first integrals which is consistent with the level of discretization.

Keywords: Geometric integration, Symplectic methods, Structure preserving algorithms, Lie–Poisson systems, Hamiltonian systems, Isospectral flows, Euler equations, Fluid dynamics.

List of appended papers

Paper I K. Modin, **M. Viviani**.
Lie–Poisson methods for isospectral flows.
Preprint

Paper II **M. Viviani**.
A structure preserving scheme for the Euler equations on a (rotating) sphere.
Preprint

My contribution to the appended papers:

Paper I: I have developed the numerical methods and most of the theoretical framework, drafted the manuscript and, after consultation, produced the final manuscript, except for the first two introductory sections. I have implemented the code, under the supervision of professor K. Modin.

Paper II: Independently developed and written.

Acknowledgements

I would like to thank my supervisor, professor Klas Modin, for the interesting and enlightening discussions and suggestions during the last two and half years, for the good moments around the world.

Heartfelt thanks to professor Stig Larsson, for having always been a reference point in these years.

Thank you Anders, a great scientist and a better friend. Thank you Efthymios for your *φιλια* and your precious teachings. A great thanks to the "lunch group" and colleagues: Maximilian, Olof, Simone, Carl, Valentina, Medet, Helga, Christian, Jimmy, Adam, Mårten, Johannes, Hanna, Marco, Hung, Geir, Dawan, Matteo, Peter. A special mention, for all the professors, the students and the staff of Chalmers, with whom I have shared the intellectual hill of Johanneberg.

Thank you also to all the friends in Göteborg: Carlo III, Marco, Antonio, Layla, and all the others that I have met in these years. I remember all of you.

To my family and friends still close, no matter how far, thank you, I simply owe it all to you.

Milo Viviani
Gothenburg, May 2018

Contents

Introduction	1
History	1
Motivation	2
1 Lie–Poisson systems	3
1.1 Poisson structures and Hamiltonian systems	3
1.2 Lie–Poisson systems	5
1.3 Co-adjoint representation	6
1.4 Momentum maps and Lie–Poisson reduction	8
1.4.1 Lie–Poisson reduction	9
1.5 Lie–Poisson systems on $\mathfrak{gl}(n, \mathbb{C})^*$	10
1.5.1 ad vs ad^*	10
1.5.2 Euler–Poincaré equations and their representations	10
1.5.3 Lie–Poisson maps on $\mathfrak{gl}(n, \mathbb{C})^*$	12
2 Isospectral flows and their numerical solution	14
2.1 Isospectral flows and their properties	14
2.1.1 Restriction to a subspace of $\mathfrak{gl}(n, \mathbb{C})$	15
2.2 Numerical approximation of the isospectral flows	15
3 2D Euler equations on the sphere and their numerical solution	17
3.1 Hydrodynamical Euler equations	17
3.2 Geometric structure of the Euler equations	18
3.3 L_α -convergence	20
3.4 The reduced system	22
3.4.1 With the Coriolis force	22
4 Summary of the results in the papers	22
4.1 Paper I: Lie–Poisson methods for isospectral flows	22
4.2 Paper II: A structure preserving scheme for the Euler equations on a (rotating) sphere	24
5 Proposals for future work	27
5.1 Paper I: Lie–Poisson methods for isospectral flows	27
5.2 Paper II: A structure preserving scheme for the Euler equations on a (rotating) sphere	27
Bibliography	28

Introduction

History

The following thesis would like to summarize the last two and half years spent on the study of the numerical solution of Lie–Poisson Hamiltonian systems and their connection with other fields of mathematics and applications.

The starting point of the present research dates back to my master thesis, defended in September 2015. In that work I was interested in the numerical solution of the hydrodynamical Euler equations on a rotating sphere with continuous and singular (point vortices) vorticity fields. The aim of that thesis was to get a numerical method which retained the main geometric properties of the continuous equations in the discrete case. The hydrodynamical Euler equations are indeed a classical example of Lie–Poisson Hamiltonian system. This means that the equations encode a lot of symmetries and therefore conservation laws. What had motivated my research was that there was not yet an established and efficient way to integrate the Euler equations respecting those symmetries.

Eventually that thesis did not give a satisfactory result and the research had to be continued during my PhD studies, started in October 2015 under the supervision of prof. Klas Modin. During the first one and half years the results obtained were quite satisfactory but still not really innovative. The main reason was that our simulations of the Euler equations required very large matrices and the algorithm developed still had too many implicit equations to be solved in order to be really applicable.

Finally, we came to a turning point. In our approach, it was clear that, to retain the first integrals, we needed to put some constraints on the equations. However, what if the constraints could be instead intrinsically encoded into the numerical method? This was not in general a feasible approach but surprisingly it turned out that, in this case, aiming for simplicity was rewarding. Working with this idea in mind it was possible to generate several numerical methods much simpler and efficient than the previous ones. Moreover a lot of Lie–Poisson systems could then be solved with the same approach and in fact, for any quadratic Lie algebra, it was easy to derive a Lie–Poisson integrator of any order. An encouraging fact was also that the methods developed looked like to be the natural ones, requiring only the information coming from the Lie algebra.

Here I present these results in the following way. In the first section I will introduce the general framework of Lie–Poisson Hamiltonian systems and some remarks on the Poisson reduction. In section two, the theory and the numerics of isospectral flows will be presented and discussed. In section three I will focus on the numerical solution of the Euler equations on a sphere, which had been the main source and aim of the whole research. Finally, I will conclude with a summary of the papers and the aim of future research. In appendix, paper I-II

are attached.

Motivation

The problems here presented are a classical and widely studied topic among the geometric integration community. However, it may (or may not) be surprising that several questions are still unsolved. It will be clear while reading the thesis that the work here presented aims to connect different threads, either to conclude or complete several papers found in literature. In particular, the two main branches of the thesis, i.e., the Hamiltonian isospectral flows and the incompressible Euler equations, will be connected. The first one will be focused on the possibility of having intrinsic arbitrarily high order methods for Hamiltonian isospectral flows. The positive answer obtained will lead to a direct application to the second one and will provide a better understanding of the possible advantages of having a Casimir functions-preservation discretization scheme.

1 Lie–Poisson systems

Since its foundation, mathematical physics has been built up from the language and the concepts coming from geometry. The mechanics of Giuseppe Lodovico Lagrangia, Leonard Euler and William Hamilton tried instead to develop an analytical formulation of the fundamental laws of the Universe. Therefore it was not expected that the same equations were hiding even more geometry than before. Sophus Lie, Emmy Nöether and lately Vladimir Arnold showed that the natural language of physics was indeed the differential geometry.

In this section I want to introduce and discuss one of the most intriguing and ubiquitous structure arising in differential geometry and mathematical physics, which is named after the French mathematician Siméon Denis Poisson.

1.1 Poisson structures and Hamiltonian systems

Definition 1 (Poisson bracket). *Let M be a smooth manifold and $C^\infty(M)$ the real vector space of smooth real valued functions defined on M . The Poisson bracket is a bilinear operation $\{\cdot, \cdot\}$ on $C^\infty(M)$, satisfying the following conditions:*

- $\{F, G\} = -\{G, F\}$ *skew symmetry;*
- $\{F, G \cdot H\} = \{F, G\} \cdot H + \{F, H\} \cdot G$ *Leibniz rule;*
- $\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0$ *Jacobi identity.*

A manifold M equipped with a Poisson bracket is said to be a *Poisson manifold*. The Poisson bracket can be represented by a form $P \in \bigwedge^2 TM$ by¹:

$$\{F, G\}(x) = P_x(dF(x), dG(x))$$

for any $x \in M$.

Definition 2 (Symplectic form). *Let M be a smooth manifold. $\omega \in \bigwedge^2 M$ is said to be a symplectic form if it is closed and non degenerate, i.e., for any $p \in M$, $v \in T_p M$, if for all $w \in T_p M$ $\omega_p(v, w) = 0$, then $v = 0$.*

A manifold M equipped with a symplectic form ω is said to be a *symplectic manifold*, and it is denoted as (M, ω) .

¹Note that we will denote by $\bigwedge^2 TM$ the space of the sections from M to the alternating 2-tensor on the tangent bundle of M while by $\bigwedge^2 T^*M =: \bigwedge^2 M$ the space of the sections from M to the the alternating 2-tensor on the cotangent bundle of M , which are the usual 2-forms.

Remark 1. We observe that M always admits a trivial Poisson bracket, i.e., the zero one, but not always a symplectic form. In fact M has to be of even dimension and orientable (e.g., \mathbb{R}^{2n} , $n > 0$). Moreover, if M is compact, then the second group of De Rahm cohomology of M must be non zero (e.g., \mathbb{S}^2 and \mathbb{T}^n are symplectic but neither $\mathbb{R}\mathbb{P}^2$ nor \mathbb{S}^{2n} for $n > 1$ are). Furthermore, a symplectic form induces a canonical Poisson bracket as we will see below.

Definition 3 (Hamiltonian vector field). Let (M, ω) be a symplectic manifold. A vector field $X \in TM$ is said to be Hamiltonian if there exists a function $H \in C^\infty(M)$ such

$$\iota_X \omega = dH$$

where $\iota_X \omega$ is the contraction of ω by X , i.e., $\iota_X : \bigwedge^2 M \rightarrow \bigwedge^1 M$ such that, for all $p \in M$ and $v \in T_p M$, $\iota_X \omega_p(v) = \omega_p(X_p, v)$.

Remark 2. We observe that (3) are nothing else than the Hamilton equations. In fact, for the sake of simplicity, assume $M = \mathbb{R}^{2n}$. Then a symplectic form can be represented in the canonical coordinates $q_1, \dots, q_n, p_1, \dots, p_n$ by the constant skew matrix $J \in \mathbb{M}(2n, \mathbb{R})$ with coefficients as follows: $J_{ij} = 0$ if $i, j \leq n$ or $i, j > n$, $J_{ij} = \delta_{ij}$ if $i > n, j \leq n$ and $J_{ij} = -\delta_{ij}$ if $i \leq n, j > n$. The $dH_{q,p}$ can be written as $\nabla H_{q,p}$ and $X(q, p) = (\dot{q}, \dot{p})$, where (q, p) are the flow line of X from some initial values. So (3) becomes

$$J \cdot (\dot{q}, \dot{p}) = \nabla H_{q,p},$$

which are the Hamilton equations after inversion of J .

Furthermore, we observe that ω induces a diffeomorphism $\hat{\omega} : TM \rightarrow T^*M$ defined as $\hat{\omega}(v) = \omega_p(v, \cdot)$ for every $v \in T_p M$. So, given F , we define the Hamiltonian vector field associate to F as $X_F = \hat{\omega}^{-1}(dF)$. Finally, given a symplectic manifold (M, ω) , we define, for every $F, G \in C^\infty(M)$ the following Poisson bracket:

$$\{F, G\} = \omega(X_F, X_G),$$

where everything is defined pointwise. To compute (1.1) in local coordinates we need the following basic theorem.

Theorem 1 (Darboux). Let (M, ω) be a symplectic manifold of dimension $2n$. Then for every $p \in M$, there exists a local chart $(V, \varphi = (q_1, \dots, q_n, p_1, \dots, p_n))$ centred in p , such that:

$$\omega|_V = \sum_{i=1}^n dq_i \wedge dp_i,$$

i.e., ω is represented by the matrix J defined above.

Such coordinates are called *canonical* or *Darboux coordinates*. Now we can write (1.1) in coordinates. Let (V, φ) be a chart given by the Darboux theorem, then, in this chart, $X_F = \sum_{i=1}^n \frac{\partial F}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial}{\partial p_i}$. A similar expression holds for X_G . Then, a straightforward computation leads to write (1.1) as:

$$\{F, G\} = \sum_{i=1}^n \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i},$$

where the relations $dq_i(\partial_{q_i}) = 1, dq_i(\partial_{p_i}) = 0, dp_i(\partial_{q_i}) = 0, dp_i(\partial_{p_i}) = 1$, for $i = 1, \dots, n$, have been used. The canonical coordinates satisfy:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \text{ and } \{q_i, p_j\} = -\{p_j, q_i\} = \delta_{ij} \text{ for } i, j = 1, \dots, n.$$

An obvious consequence is that, for every $F \in C^\infty(M)$, and any Hamiltonian vector field X_H we have:

$$X_H(F) = \{F, H\}.$$

So for $p_i, q_i, i = 1, \dots, n$ integral curves of X_H we have that:

$$\dot{q}_i = X_H(q_i) = \{q_i, H\} \text{ and } \dot{p}_i = X_H(p_i) = \{p_i, H\} \text{ for } i = 1, \dots, n,$$

which is an other formulation of the Hamilton equations (2).

Definition 4. Let (M, ω, H) be a Hamiltonian system, i.e., a symplectic manifold with a Hamiltonian function H . A function $f \in C^\infty(M)$ constant on any integral curve of X_H is said to be a first integral of the system. A vector field $X \in TM$ is said to be an infinitesimal symmetry if both ω and H are invariant under the flow of X .

Theorem 2 (Noether theorem). Let (M, ω, H) be a Hamiltonian system.

- if f is a first integral, then X_f is an infinitesimal symmetry;
- on the other hand, if $H^1(M) = 0$ (where $H^1(M)$ is the first group of De Rham cohomology of M), then any infinitesimal symmetry is a Hamiltonian vector field of a first integral, uniquely defined, except for an additive constant for any connected component of M .

1.2 Lie–Poisson systems

A remarkable Poisson structure can be naturally given to the vector spaces that are also the dual of a Lie algebra. Consider a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, not

necessarily of finite dimension, and let \mathfrak{g}^* be its dual. Then on $C^\infty(\mathfrak{g}^*)$ we have the following (canonical) Poisson bracket:²

$$\{F, G\}_\pm(v) = \pm \langle v, [dF(v), dG(v)] \rangle$$

where $v \in \mathfrak{g}^*$ and we have identified $\mathfrak{g} \cong \mathfrak{g}^{**}$.

The Lie–Poisson bracket is a very important example of a generally neither trivial nor symplectic Poisson bracket. In this case, the Poisson form $P \in \bigwedge^2 T\mathfrak{g}^*$ is linear and can be expressed by:

$$P_{ij}(v) = \pm C_{ij}^k v_k$$

where C_{ij}^k are the structure constants of \mathfrak{g} . Let H be a smooth function on \mathfrak{g}^* . Then, the system:

$$\begin{aligned} \dot{F}(v(t)) &= \{F, H\}_\pm(v(t)) \\ F(v(0)) &= F(v_0) \end{aligned}$$

which has to be satisfied for any $F \in C^\infty(\mathfrak{g}^*)$, it is said to be *Lie–Poisson system* with Hamiltonian function H (the bracket is the one defined above and $v(t) \in \mathfrak{g}^*$, for any $t \in \mathbb{R}$). Because of the anti-commutativity of the Poisson bracket, it is clear that in a Lie–Poisson system the Hamiltonian is a conserved quantity in time. Moreover, depending on the rank of the form defining the bracket, we have a certain number of first integrals of the motion, that are the same for any Hamiltonian. These functions that commute with any other one, i.e., $\{C, \cdot\} = 0$ are called *Casimir functions*. As we will discuss in the next sections, the preservation of the Casimir functions and the Hamiltonian by a numerical method is crucial in the applications in order to guarantee good predictions for long times.

1.3 Co-adjoint representation

We want now express (1.2) in terms of the co-adjoint representation of a Lie algebra. We have first to recall some definitions.

Let G be a Lie group, and consider the map

$$\begin{aligned} C : G \times G &\longrightarrow G \\ (g, h) &\mapsto C_g(h) := ghg^{-1}. \end{aligned}$$

Then, for each $g \in G$, we have the internal automorphism C_g . If we take the differential of this map in the identity e we get the *adjoint representation*, Ad of G in $\text{End}(\mathfrak{g})$, that is defined by:

²The \pm sign depends on the fact that the Poisson bracket here defined can also be obtained via the reduction of the canonical ones on the left (-) or right (+) invariant functions on T^*G (see section 1.4.1 below).

$$Ad_g(X) = \frac{d}{dt}\Big|_{t=0} (g \exp^{tX} g^{-1}), \quad \forall g \in G, X \in \mathfrak{g}.$$

Finally, differentiating $Ad : G \rightarrow \text{End}(\mathfrak{g})$ and identifying $\text{End}(\mathfrak{g})$ with its tangent, we obtain the map:

$$\begin{aligned} ad : \mathfrak{g} &\longrightarrow \text{End}(\mathfrak{g}) \\ X &\mapsto ad_X = [X, \cdot]. \end{aligned}$$

Let us now define the dual of the adjoint representation, i.e., the representation of the group G and the Lie algebra \mathfrak{g} on the endomorphism of the dual of the Lie algebra \mathfrak{g}^* . We define the *co-adjoint representation* $Ad^* : G \rightarrow \text{End}(\mathfrak{g}^*)$ by:

$$\langle Ad^*(g)(\phi), X \rangle = \langle \phi, Ad(g^{-1})X \rangle \quad \forall g \in G, X \in \mathfrak{g}, \phi \in \mathfrak{g}^*.$$

Proceeding as before, one can find the infinitesimal version $ad^* : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*)$, given by $ad_X^* = -(ad_X)^*$, i.e.,

$$\langle ad_X^*(\phi), Y \rangle = \langle \phi, -ad_X(Y) \rangle \quad \forall X, Y \in \mathfrak{g}, \phi \in \mathfrak{g}^*.$$

Let \mathcal{O} be an orbit of the co-adjoint action $Ad^* : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$. It holds the remarkable fact that the co-adjoint orbits have a canonical symplectic structure, called *Kirillov-Kostant-Souriau form*. Let $p \in \mathcal{O}$ and $X, Y \in \mathfrak{g}$, then the two form:

$$\omega_p(\text{ad}_X^*(p), \text{ad}_Y^*(p)) = \langle p, [X, Y] \rangle,$$

is a symplectic form on \mathcal{O} , where we have used the canonical identification of \mathfrak{g}^{**} with \mathfrak{g} , from which we have obtained that $T^*\mathfrak{g}^* \simeq \mathfrak{g}^* \times \mathfrak{g}$. We conclude noticing that the co-adjoint orbits are immersed submanifold³ where the Casimir functions are constant. However, in general, the Casimir functions don't characterize the co-adjoint orbits⁴.

Let us go back to the Lie-Poisson system (1.2). We notice that the bracket can be expressed in terms of the co-adjoint representation of \mathfrak{g} :

$$\pm \langle v, [dF(v), dH(v)] \rangle = \mp \langle v, \text{ad}_{dH(v)}(dF(v)) \rangle = \pm \langle \text{ad}_{dH(v)}^*(v), dF(v) \rangle.$$

We want to remark that a Lie-Poisson system evolves precisely on the co-adjoint orbits given by the Ad^* action. In fact let consider $F = F(v(t))$ where $x(t)$ is a curve in \mathfrak{g}^* , and $v(0) = v_0$. Applying the chain rule we get:

$$dF(\dot{v}) = \pm \langle \text{ad}_{dH(v)}^*(v), dF(v) \rangle$$

³If the action of the group G is also proper, e.g., G compact, then the co-adjoint orbits are embedded submanifold.

⁴[13], pag. 479.

for any $F \in C^\infty(\mathfrak{g}^*)$. Hence it is true that:

$$\dot{v} = \pm \text{ad}_{dH(v)}^*(v).$$

Integrating this system we get:

$$v(t) = \text{Ad}_{\exp(\pm \int_0^t dH(v(s)) ds)}^*(v_0).$$

1.4 Momentum maps and Lie–Poisson reduction

In this section we will briefly recall the concept and the main properties of the *momentum map* of a group action on a Poisson manifold. For further details we refer to [12] and [13].

Let G be a Lie group acting to the left on a Poisson manifold P , such that for any $g \in G$ the action Φ_g is a Poisson map, i.e., $\{\cdot, \cdot\} \circ \Phi_g = \{\cdot \circ \Phi_g, \cdot \circ \Phi_g\}$. Let the *infinitesimal action* of G be the map $\rho : \mathfrak{g} \times P \rightarrow TP$ defined by:

$$\rho_\xi(p) = \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi)p,$$

for any $\xi \in \mathfrak{g}, p \in P$. Hence, ρ_ξ is a vector field on P . Furthermore, we assume the G -action to be Hamiltonian, i.e., there exists a function $J_\xi \in C^\infty(P)$ such that $\rho_\xi = \{\cdot, J_\xi\}$. Then we define the momentum map $\mu : P \rightarrow \mathfrak{g}^*$ by:

$$\langle \mu(p), \xi \rangle = J_\xi(p).$$

We remark that, if the Poisson bracket is induced by a symplectic form ω , then the momentum map can be defined by the formula:

$$d\langle \mu(p), \xi \rangle = \iota_{\rho_\xi(p)} \omega_p.$$

For a right action one can repeat exactly the same calculations. The main difference, as one can easily check, is that the map $J : \mathfrak{g} \rightarrow C^\infty(P)$ is a Lie algebra homomorphism for the left action and a Lie algebra anti-morphism for the right action (cfr. [12]).

Let us now denote μ_L (respectively μ_R) the momentum map coming from the left (respectively right) G -action on P . Let also \mathfrak{g}_- (respectively \mathfrak{g}_+) be the dual of the Lie algebra \mathfrak{g} endowed with the $-$ (respectively $+$) Lie–Poisson bracket.

The main property of the momentum maps is stated in the following proposition:

Proposition 1 (Prop 2.1, [12]). *Let $\mu_L : P \rightarrow \mathfrak{g}_+^*$ (respectively $\mu_R : P \rightarrow \mathfrak{g}_-^*$) be the momentum map defined above. Then μ_L (respectively μ_R) is a Poisson map.*

Proof. Let consider the left case. By definition of the Lie–Poisson bracket:

$$\begin{aligned}\{F, G\}_+(\mu(p)) &= \langle \mu(p), [dF(\mu(p)), dG(\mu(p))] \rangle \\ &= J_{[dF(\mu(p)), dG(\mu(p))]}(p).\end{aligned}$$

Now, since $J : \mathfrak{g} \rightarrow C^\infty(P)$ is a Lie algebra homomorphism, we have that:

$$J_{[dF(\mu(p)), dG(\mu(p))]}(p) = \{J_{dF(\mu(p))}, J_{dG(\mu(p))}\}_+(p).$$

Finally, by the definition of the Poisson bracket, it is enough to prove that:

$$d(J_{dF(\mu(p))}) = d(F \circ \mu)(p) -$$

Indeed, we have:

$$\begin{aligned}\langle d(F \circ \mu)(p), v_p \rangle &= \langle dF(\mu(p)) \circ d\mu(x), v_p \rangle \\ &= \langle d\langle \mu(p), dF(\mu(p)) \rangle, v_p \rangle \\ &= \langle d(J_{dF(\mu(p))}), v_p \rangle.\end{aligned}$$

for any $v_p \in T_pP$. □

1.4.1 Lie–Poisson reduction

Now that we have introduced the momentum maps, we can show how Lie–Poisson systems are related to the canonical Hamilton equations.

Let $(P, \{\cdot, \cdot\}, H)$ be a Poisson Hamiltonian system and let (M, ω, H_ψ) be a Hamiltonian system, where $H_\psi = H \circ \psi$ and $\psi : M \rightarrow P$ is a Poisson map.

Consider G a Lie group with a Hamiltonian left (resp. right) action⁵ on M and such that H_ψ is left (respectively right) G -invariant and G is transitive on the fibres of ψ . Suppose that there exists a left momentum map $\mu : M \rightarrow \mathfrak{g}^*$, where \mathfrak{g} is the associated Lie algebra of G . Then, by Proposition 1, we know that μ is a Poisson map between the canonical Poisson bracket on M and the Lie–Poisson bracket on \mathfrak{g}_+^* .

Since H_ψ is G -invariant, the momentum map μ is a conserved quantity of the dynamical system. It is shown in [12] that, assuming there are no singularities in the quotient with respect to the group action, given a co-adjoint orbit \mathcal{O} in \mathfrak{g}^* , the map $\Psi_{\mathcal{O}} = \psi|_{\mu^{-1}\mathcal{O}}$ induces an embedding $\Psi_{\mathcal{O}} : \mu^{-1}\mathcal{O}/G \rightarrow P$ to a symplectic leaf of P ⁶.

⁵With Hamiltonian action, we understand an action such that its infinitesimal action is an Hamiltonian vector field.

⁶It is a general fact that any Poisson manifold is a union of symplectic submanifolds, called "symplectic leaves". The trajectory of X_H starting in a particular leaf necessarily stays there. For \mathfrak{g}_\pm^* the symplectic leaves coincide with the respective co-adjoint orbits.

In particular, when $M = T^*G$ and $P = \mathfrak{g}_-^*$ (resp. $P = \mathfrak{g}_+^*$), we can take $\Psi = \mu_R$ (resp. μ_L) and $\mu = \mu_L$ (resp. μ_R). Then, the canonical Hamilton equations in T^*G w.r.t to the Hamiltonian \tilde{H} become the equations (1.2) on \mathfrak{g}_-^* (resp. \mathfrak{g}_+^*) with respect to to the Hamiltonian H on \mathfrak{g}^* , defined by $H \circ \mu_L = \tilde{H}$ (resp. $H \circ \mu_R = \tilde{H}$).

1.5 Lie–Poisson systems on $\mathfrak{gl}(n, \mathbb{C})^*$

In this section, in view of the applications, we want to remark some facts about Lie–Poisson systems on the dual of the general matrix Lie algebra $\mathfrak{gl}(n, \mathbb{C})^*$. In particular, we want to clarify in detail the meaning of the identification between $\mathfrak{gl}(n, \mathbb{C})^*$ and $\mathfrak{gl}(n, \mathbb{C})$ and how this affects the representation of the equations of a Lie–Poisson system.

1.5.1 ad vs ad^*

Considering the adjoint representation of $\mathfrak{gl}(n, \mathbb{C})$ on itself:

$$\text{ad}_A(B) = [A, B] = AB - BA,$$

for any $A, B \in \mathfrak{gl}(n, \mathbb{C})$.

Let us now look at the co-adjoint representation of $\mathfrak{gl}(n, \mathbb{C})$ on $\mathfrak{gl}(n, \mathbb{C})^*$. Consider the two different identifications of $\mathfrak{gl}(n, \mathbb{C})^*$ with $\mathfrak{gl}(n, \mathbb{C})$:

$$\langle A, B \rangle_1 = \text{Tr}(AB)$$

$$\langle A, B \rangle_2 = \text{Tr}(A^\dagger B),$$

for $A, B \in \mathfrak{gl}(n, \mathbb{C})$. The second one comes from the Frobenius inner product on $\mathfrak{gl}(n, \mathbb{C})$ (in terms of basis, the first one says that the dual element of a given one is its complex adjoint whereas the second one says that it is itself)⁷.

Recalling that $\text{ad}_A^* = -(\text{ad}_A)^*$, the respective co-adjoint representations are:

$$\text{ad}_A^{*1} B = -[B, A] = \text{ad}_A B$$

$$\text{ad}_A^{*2} B = [B, A^\dagger] = -\text{ad}_{A^\dagger} B.$$

1.5.2 Euler–Poincaré equations and their representations

In literature, equations (1.3), for quadratic Hamiltonian functions, are often called Euler–Poincaré equations. In this paragraph we want to show that the

⁷Here with \dagger we understand the complex adjoint.

dynamics generated is independent from the identification of $\mathfrak{gl}(n, \mathbb{C})^*$ with $\mathfrak{gl}(n, \mathbb{C})$.

To define the Euler–Poincaré equations, we need a symmetric positive-definite linear map $\mathcal{A} : \mathfrak{gl}(n, \mathbb{C}) \rightarrow \mathfrak{gl}(n, \mathbb{C})^*$. An explicit form of this map depends on the way we identify the algebra with its dual. Let us denote \mathcal{A} with \mathcal{A} and $\tilde{\mathcal{A}}$, the respective form, with respect to ad^{*2} and respectively, ad^{*1} . We then have that the following inner products are identically defined:

$$\langle A, B \rangle_{\mathcal{A}} := \langle \mathcal{A}A, B \rangle_2 = \text{Tr}((\mathcal{A}A)^\dagger B)$$

||

$$\langle A, B \rangle_{\tilde{\mathcal{A}}} := \langle \tilde{\mathcal{A}}A, B \rangle_1 = \text{Tr}(\tilde{\mathcal{A}}AB).$$

for $A, B \in \mathfrak{gl}(n, \mathbb{C})$. Therefore, we have to have that $\tilde{\mathcal{A}} = \cdot^\dagger \circ \mathcal{A}$. Then, for $\Psi \in \mathfrak{gl}(n, \mathbb{C})$, the Lagrangian function can be defined as:

$$\mathcal{L}(\Psi) = \frac{1}{2} \langle \Psi, \Psi \rangle_{\mathcal{A}} = \frac{1}{2} \langle \Psi, \Psi \rangle_{\tilde{\mathcal{A}}}.$$

The respective momentum variables in $\mathfrak{gl}(n, \mathbb{C})^*$ are:

$$\begin{aligned} \Omega_{\mathcal{A}} &= \frac{\partial \mathcal{L}(\Psi)}{\partial \Psi} = \mathcal{A}\Psi \\ \Omega_{\tilde{\mathcal{A}}} &= \left(\frac{\partial \mathcal{L}(\Psi)}{\partial \Psi} \right)^\dagger = \tilde{\mathcal{A}}\Psi \end{aligned}$$

and we observe that $(\Omega_{\mathcal{A}})^\dagger = \Omega_{\tilde{\mathcal{A}}}$. From these calculations, we get the Hamiltonian functions:

$$\begin{aligned} H_{\mathcal{A}}(\Omega_{\mathcal{A}}) &= \frac{1}{2} \langle \Omega_{\mathcal{A}}, \mathcal{A}^{-1} \Omega_{\mathcal{A}} \rangle_2 \\ H_{\tilde{\mathcal{A}}}(\Omega_{\tilde{\mathcal{A}}}) &= \frac{1}{2} \langle \Omega_{\tilde{\mathcal{A}}}, \tilde{\mathcal{A}}^{-1} \Omega_{\tilde{\mathcal{A}}} \rangle_1. \end{aligned}$$

So we have the identities:

$$\begin{aligned} \frac{\partial H_{\mathcal{A}}(\Omega_{\mathcal{A}})}{\partial \Omega_{\mathcal{A}}} &= \mathcal{A}^{-1} \Omega_{\mathcal{A}} = \Psi \\ \frac{\partial H_{\tilde{\mathcal{A}}}(\Omega_{\tilde{\mathcal{A}}})}{\partial \Omega_{\tilde{\mathcal{A}}}} &= \tilde{\mathcal{A}}^{-1} \Omega_{\tilde{\mathcal{A}}} = \Psi. \end{aligned}$$

Finally, we get the equation of motion ([13], Chapt. 13):

$$\begin{aligned} \langle \dot{\Psi}, Y \rangle_{\mathcal{A}} &= -\langle \Psi, \text{ad}_{\Psi} Y \rangle_{\mathcal{A}} = \langle \mathcal{A}^{-1} \text{ad}_{\Psi}^{*2} \mathcal{A}\Psi, Y \rangle_{\mathcal{A}}, \\ \langle \dot{\Psi}, Y \rangle_{\tilde{\mathcal{A}}} &= \langle \Psi, \text{ad}_{\Psi} Y \rangle_{\tilde{\mathcal{A}}} = -\langle \tilde{\mathcal{A}}^{-1} \text{ad}_{\Psi}^{*1} \tilde{\mathcal{A}}\Psi, Y \rangle_{\tilde{\mathcal{A}}}, \end{aligned}$$

for any $Y \in \mathfrak{gl}(n, \mathbb{C})$. These can also be written in the strong form as:

$$\begin{aligned}\dot{\Psi} &= \mathcal{A}^{-1} \operatorname{ad}_{\Psi}^* \mathcal{A} \Psi = \mathcal{A}^{-1} [\mathcal{A} \Psi, \Psi^\dagger], \\ \dot{\Psi} &= \tilde{\mathcal{A}}^{-1} \operatorname{ad}_{\Psi}^* \tilde{\mathcal{A}} \Psi = -\tilde{\mathcal{A}}^{-1} [\tilde{\mathcal{A}} \Psi, \Psi],\end{aligned}$$

or, considering the dual version for $\Omega_{\mathcal{A}}, \Omega_{\tilde{\mathcal{A}}}$:

$$\begin{aligned}\dot{\Omega}_{\mathcal{A}} &= \operatorname{ad}_{\mathcal{A}^{-1} \Omega_{\mathcal{A}}}^* \Omega_{\mathcal{A}} = [\Omega_{\mathcal{A}}, (\mathcal{A}^{-1} \Omega_{\mathcal{A}})^\dagger], \\ \dot{\Omega}_{\tilde{\mathcal{A}}} &= \operatorname{ad}_{\tilde{\mathcal{A}}^{-1} \Omega_{\tilde{\mathcal{A}}}}^* \Omega_{\tilde{\mathcal{A}}} = -[\Omega_{\tilde{\mathcal{A}}}, \tilde{\mathcal{A}}^{-1} \Omega_{\tilde{\mathcal{A}}}]^\dagger.\end{aligned}$$

Remark If we transpose the second equation, we get:

$$\dot{\Omega}_{\tilde{\mathcal{A}}}^\dagger = [\Omega_{\tilde{\mathcal{A}}}^\dagger, (\tilde{\mathcal{A}}^{-1} \Omega_{\tilde{\mathcal{A}}})^\dagger],$$

and, using the fact that $(\Omega_{\mathcal{A}})^\dagger = \Omega_{\tilde{\mathcal{A}}}$, and $\tilde{\mathcal{A}}^{-1} \Omega_{\tilde{\mathcal{A}}} = \mathcal{A}^{-1} \Omega_{\mathcal{A}}$, we see that the Euler–Poincaré equations are independent from the choice of the pairing.

1.5.3 Lie–Poisson maps on $\mathfrak{gl}(n, \mathbb{C})^*$

Consider the identification of $\mathfrak{gl}(n, \mathbb{C})$ with its dual, via the Frobenius pairing. After this identification, the Lie–Poisson structure on $\mathfrak{gl}(n, \mathbb{C})^*$ is completely determined by the structure constants of $\mathfrak{gl}(n, \mathbb{C})$. Therefore any Lie algebra morphism of $\mathfrak{gl}(n, \mathbb{C})$ will be a *Lie–Poisson map* on $\mathfrak{gl}(n, \mathbb{C})^*$ and viceversa.

We now want to check how it looks with respect to ad^* . Let consider $a : \mathfrak{gl}(n, \mathbb{C}) \rightarrow \mathfrak{gl}(n, \mathbb{C})$ invertible Lie algebra morphism, $A, B \in \mathfrak{gl}(n, \mathbb{C})$ and $\phi \in \mathfrak{gl}(n, \mathbb{C})^* \equiv \mathfrak{gl}(n, \mathbb{C})$ (via the Frobenius identification). Then we get:

$$\begin{aligned}\operatorname{Tr}((a \operatorname{ad}_A^*(\phi))^\dagger B) &= -\operatorname{Tr}((a[A^\dagger, \phi])^\dagger B) \\ &= -\operatorname{Tr}(\phi^\dagger [A, a^\dagger B]) \\ &= -\operatorname{Tr}((a\phi)^\dagger [a^{-T} A, B]) \\ &= -\operatorname{Tr}([A^\dagger a^{-1}, a\phi]^\dagger B) \\ &= \operatorname{Tr}((\operatorname{ad}_{a^{-T} A}^*(a\phi))^\dagger B).\end{aligned}$$

So we have the formula:

$$a \operatorname{ad}_A^*(\phi) = \operatorname{ad}_{a^{-T} A}^*(a\phi).$$

Consider A to be equal to $\nabla H(\phi)$, for a smooth function H defined on $\mathfrak{gl}(n, \mathbb{C})^*$, i.e., we have a Lie–Poisson Hamiltonian system. Then the action on an invertible linear map on the (Lie–Poisson) Hamiltonian vector field is:

$$a \cdot X_H := Da \circ X_H \circ a^{-1},$$

where $X_H(\phi) = \text{ad}_{\nabla H(\phi)}^*(\phi)$. Then, using the formula (1.5.3), we get:

$$\begin{aligned} a \cdot X_H(\phi) &= a \text{ad}_{\nabla H(a^{-1}\phi)}^*(a^{-1}\phi) \\ &= \text{ad}_{a^{-T}\nabla H(a^{-1}\phi)}^*(\phi) \\ &= \text{ad}_{\nabla(H \circ a^{-1})(\phi)}^*(\phi) \\ &= X_{H \circ a^{-1}}(\phi), \end{aligned}$$

which is again a Lie–Poisson Hamiltonian system.

2 Isospectral flows and their numerical solution

2.1 Isospectral flows and their properties

The isospectral flows are a central class of dynamical systems with symmetries. They arise in fact in different contexts: Lie–Poisson reduction, matrix factorization, Lax pairs of integrable systems, et cetera [8],[10],[16]. As the name suggests, isospectral flows represent a dynamical system on linear operators such that the spectrum of operator is fixed during the whole evolution. If the operators are diagonalizable, then, at each time, the operator is similar to the initial one.

Let the flow

$$\begin{aligned} \Phi : [0, \infty) \times \mathcal{L}(V) &\rightarrow \mathcal{L}(V) \\ (t, W) &\mapsto \Phi_t(W) \end{aligned}$$

be an isospectral flow on $\mathcal{L}(V)$, where V is a finite dimensional vector space of dimension n . Let W_0 be the initial value. Then, for any $t \geq 0$, there exists $U(t)$ such that:

$$W(t) = \Phi_t(W_0) = U(t)^{-1}W_0U(t).$$

By differentiation of (2.1), one find that W is the solution of:

$$\begin{aligned} \dot{W} &= [B(W), W] \\ W(0) &= W_0, \end{aligned}$$

where $B(W) = U^{-1}\dot{U}$ and the bracket is the usual matrix commutator $[A, B] = AB - BA$.

Other than the eigenvalues of the operator, one can choose a different set of generators for the first integrals of (2.1). This is provided by the momentum of W . In fact:

$$\frac{d}{dt} \text{Tr}(W^k) = \text{Tr}(W^{k-1}[B(W), W]) = \text{Tr}(B(W)[W^{k-1}, W]) = 0,$$

for $k = 1, 2, \dots$. Since W is represented by a $n \times n$ matrix, its first n momenta are independent, then they are related by the Cayley–Hamilton theorem (in fact $\text{Tr}(W^k) = \sum_{i=1}^n \lambda_i^k$, for λ_i the eigenvalues of W).

When $B(W)$ takes the form of (the transpose of) a gradient of a function, the equation (2.1) will be said *Hamiltonian-Isospectral flow*. The word Hamiltonian is because the function H such that $B = -\nabla H^\dagger$ is a conserved quantity of (2.1). In fact:

$$\frac{d}{dt} H(W) = -\text{Tr}(\nabla H(W)^\dagger [\nabla H(W)^\dagger, W]) = -\text{Tr}(W[\nabla H(W)^\dagger, \nabla H(W)^\dagger]) = 0.$$

A further reason to use the word Hamiltonian is that $\mathcal{L}(V)$, endowed with the bracket $[\cdot, \cdot]$, can be seen as the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ and the equations (2.1) as the reduced form of a canonical Hamiltonian system, as shown in section 1.4.1.

Indeed, if we identify the dual of $\mathfrak{gl}(n, \mathbb{C})$ with itself, via the Frobenius inner product $\langle A, B \rangle = \text{Tr}(A^\dagger B)$, the equations (2.1) above form a Lie–Poisson Hamiltonian system with respect to the co-adjoint representation of $\mathfrak{gl}(n, \mathbb{C})$ on $\mathfrak{gl}(n, \mathbb{C})^*$ given by:

$$\text{ad}_A^* B = [B, A^\dagger] = -\text{ad}_{A^\dagger} B$$

for $A \in \mathfrak{gl}(n, \mathbb{C}), B \in \mathfrak{gl}(n, \mathbb{C})^* \cong \mathfrak{gl}(n, \mathbb{C})$.

2.1.1 Restriction to a subspace of $\mathfrak{gl}(n, \mathbb{C})$

It is interesting, both for theoretical and practical purposes, to analyse the case when W evolves on a subspace S of $\mathfrak{gl}(n, \mathbb{C})$. It is clear that if $W \in S$ then $B(W)$ has to be in $\mathfrak{n}(S)$, i.e., the normalizer of S in $\mathfrak{gl}(n, \mathbb{C})$, which is the largest subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ such that $[\mathfrak{n}(S), S] \subseteq S$. This framework is used in Paper I to encompass at the same time the "classical" isospectral flows, e.g., $W \in \text{Sym}(n), B(W) \in \mathfrak{o}(n)$, and the Lie–Poisson systems on reductive Lie-algebras.

2.2 Numerical approximation of the isospectral flows

As we have shown above, the main feature of the isospectral flows is to have a set of first integrals that can be expressed as polynomials of a certain order. A direct application of a Runge–Kutta method to (2.1) would not preserve these invariants. It has actually been proved that in general none will work for this purpose [8].

A popular method to overcome this issue is the so called Runge-Kutta-Munthe-Kaas scheme [10],[16]. The idea is to solve

$$\dot{U} = UB(U^{-1}W_0U)$$

for U and then find W using (2.1). Since U is in a Lie group G , the Munthe-Kaas method consists in lifting (2.2) to its Lie algebra \mathfrak{g} via some map from $\mathfrak{g} \rightarrow G$ (e.g., \exp, Cay). Then, on \mathfrak{g} , any classical Runge-Kutta method can be applied. This method allows to preserve the isospectrality of the flow but in general not its Lie–Poisson structure and therefore, for example, we cannot expect (near) conservation of the Hamiltonian H . An other disadvantage is that the lifting can be expensive to compute. However, a huge advantage is that it provides explicit isospectral methods.

A related technique is given by the symplectic Lie group methods on T^*G as developed by Bogfjellmo and Martinsen [4]. These methods rely on an invertible mapping between the Lie algebra and (an identity neighbourhood of) the Lie group, such as the exponential map (works in general) or the Cayley map (works for quadratic Lie groups).

An other approach for solving (2.1) is given by the so called RATTLE method [10]. RATTLE is a general method for Hamiltonian systems with constraints. To use RATTLE for (2.1), one has to pull-back the equations from \mathfrak{g}^* to T^*G and then solve the constrained Hamiltonian system. It indeed provides a Lie–Poisson integrator for (3.1) but with the burden of solving implicit equations to constrain the system on the right manifold. Some attempts of removing the constraints can be found for example in [15].

Our approach, presented in Paper I, has (independently) followed exactly that thread. Indeed, starting from some simple cases, it was not hard to realize that, with some manipulations of the canonical symplectic Runge-Kutta methods, in many cases the removal of the constraints was possible. This has led to a large class of isospectral methods directly defined on the Lie algebra.

3 2D Euler equations on the sphere and their numerical solution

3.1 Hydrodynamical Euler equations

Consider a homogeneous, incompressible, inviscid, two-dimensional fluid which is constrained to move on a spherical surface, embedded in the standard Euclidean \mathbb{R}^3 , which is rotating with constant angular speed, with respect to a fixed normal axis. The equations of motion of such a fluid are given by the well known Euler equations of hydrodynamics:

$$\begin{aligned}\dot{v} + v \cdot \nabla v &= -\nabla p - 2\tilde{\Omega} \times v \\ \nabla \cdot v &= 0\end{aligned}$$

where v is the velocity vector field of the fluid, p is its internal pressure and $\tilde{\Omega} = (\Omega \cdot n)n$ is the projection of the angular rotation of the sphere Ω to the normal n at a point of the sphere. The last term in the first equation of (3.1), $F_c = -2\tilde{\Omega} \times v$ is called *Coriolis force*.

The geometry behind this system turns out to play a central role in understanding the behaviour of the fluid [2], [3], [12] and in the investigation of numerical methods to solve it [1], [14], [17], [18]. In particular the Euler equations (3.1) can be equivalently expressed in terms of the one form v^\flat as a Lie–Poisson system on the dual of the infinite-dimensional Lie-algebra of divergence-free vector fields. The respective Poisson tensor is degenerate so that there is an infinite number of independent first integrals (Casimir functions) [3].

On the other hand, an equivalent formulation of (3.1) is given in terms of the vorticity $\omega = (\nabla \times v) \cdot n$. We notice that by the Stokes' theorem it must be that $\int \omega = 0$. Then the Euler equations (3.1) can be written as:

$$\begin{aligned}\dot{\omega} &= \{\psi, \omega\} \\ \Delta \psi &= \omega - f,\end{aligned}$$

where $f = 2\Omega \cdot n$ and ψ is the unique solution to the Poisson equation in $C^\infty(\mathbb{S}^2)$, such that $\int \psi = 0$.

In this form the Euler equations are a Lie–Poisson system on the smooth functions on the sphere which integrate to 0. The Hamiltonian is given by

$$H(\omega) = \frac{1}{2} \int (\omega - f)\psi.$$

The (infinitely many) Casimir functions are given, for any smooth f , by $F(\omega) = \int f(\omega)$. In fact, it is easy to check:

$$\frac{d}{dt} \int f(\omega) = - \int f'(\omega)v \cdot \nabla \omega = - \int v \cdot \nabla f(\omega) = \int (\nabla \cdot v)f(\omega) = 0,$$

where we have used the following identity:

$$\{\psi, \cdot\}_p = (X_\psi)_p(\cdot) = p \cdot (\nabla\psi \times \nabla\cdot) = (p \times \nabla\psi) \cdot \nabla\cdot = -v_p \cdot \nabla\cdot.$$

The presence of all these first integrals turns out to be the leading point in giving a suitable discretization of (3.1).

3.2 Geometric structure of the Euler equations

The geometric picture of fluid dynamics dates back to Arnold [2]. The velocity vector field of a 2D incompressible fluid moving on a symplectic surface (S, α) , embedded in the Euclidean \mathbb{R}^3 , may indeed be seen as a trajectory in the Lie algebra of divergence free vector fields, denoted by $\text{sdiff}(S)$. The Euler equations (3.1) can be seen in this picture as a Lie–Poisson system on the dual space of $\text{sdiff}(S)$. Consider the standard pairing of 1-forms and vector fields, i.e.,

$$\langle \beta, X \rangle = \int_S \beta(X)\alpha,$$

where $\beta \in \bigwedge^1 S$ and X is a vector field on S . Then one gets that, for $X \in \text{sdiff}(S)$, the pairing is invariant with respect to any exact translation of β .⁸ Therefore we have that $\text{sdiff}^*(S) = \bigwedge^1 S/d \bigwedge^0 S$.

Let us continue to work on $S = \mathbb{S}^2$. In [3] it's shown that the Euler equations (3.1) are equivalent to a Lie–Poisson system on $\bigwedge^1 \mathbb{S}^2/d \bigwedge^0 \mathbb{S}^2 = \text{sdiff}^*(\mathbb{S}^2)$ (which is isomorphic to the kernel of the 1-form divergence operator δ), with respect to the Hamiltonian function:

$$H([\eta]) = \frac{1}{2} \langle \eta - c^\flat, \eta^\sharp - c \rangle,$$

which represents the kinetic energy in the non inertial frame. Here $\eta = (v + c)^\flat$, $[\eta]$ is its respective class in $\bigwedge^1 \mathbb{S}^2/d \bigwedge^0 \mathbb{S}^2$ and c is the velocity due to the rotation of the sphere. The Lie–Poisson system can be written as:

$$\dot{F}([\eta]) = \langle \text{ad}_{dH}^*([\eta]), dF \rangle,$$

for any $F : \text{sdiff}^*(\mathbb{S}^2) \rightarrow \mathbb{R}$, where $\text{ad}^* : \text{sdiff}(\mathbb{S}^2) \rightarrow \text{End}(\text{sdiff}^*(\mathbb{S}^2))$ is the co-adjoint representation of $\text{sdiff}(\mathbb{S}^2)$. Equivalently (see I.6-7, [3]):

$$\dot{[\eta]} = -L_{dH}([\eta]).$$

⁸This is easily checked as $\int df(X)\alpha = \int (\iota_X)df\alpha = \int (L_X f)\alpha = \int f(L_X \alpha) = 0$, where $f \in C^\infty(S)$ and we have used the fact that X is volume preserving.

where L is the Lie derivative. In our case, we have $dH = \eta^\# - c = v$. Hence:

$$[\dot{\eta}] = -L_v([\eta]).$$

Note that Lie–Poisson system above defined is respect to the dual pairing in $\text{sdiff}(\mathbb{S}^2)$, being $dF \in (\text{sdiff}^*(\mathbb{S}^2))^* \cong \text{sdiff}(\mathbb{S}^2)$.

At this point, to get rid off the equivalence class, one just needs to take the exterior derivative of (3.2) and, using the fact that $[L_v, d] = 0$, get the Euler equations in the vorticity form:

$$\dot{\beta} = -L_v\beta,$$

where $\beta = d[\eta] \in \bigwedge^2 \mathbb{S}^2$ represents the vorticity of v .

We write the vorticity in terms of the volume form α such that $\beta = q\alpha$, where the $q \in C^\infty(\mathbb{S}^2)$ and has zero mean. Then we get

$$L_v\beta = L_v(q\alpha) = (L_vq)\alpha + qL_v\alpha = (L_vq)\alpha,$$

being v volume preserving.

By taking the Hodge star of (3.2), via the identification $\bigwedge^2 \mathbb{S}^2 \cong \bigwedge^0 \mathbb{S}^2 = C^\infty(\mathbb{S}^2)$, we can understand (3.2) in $C_0^\infty(\mathbb{S}^2)$, i.e., the space of smooth functions which integrate to 0. Hence, we get a map $*d : \text{sdiff}^*(\mathbb{S}^2) \rightarrow C_0^\infty(\mathbb{S}^2)$ between a Lie–Poisson algebra and a Poisson algebra. If we call ad^* the Lie–Poisson structure in $\text{sdiff}^*(\mathbb{S}^2)$ and ad the Poisson structure in $C_0^\infty(\mathbb{S}^2)$, then we have:

Lemma 1. *The map $\pi \equiv *d : \text{sdiff}^*(\mathbb{S}^2) \rightarrow C_0^\infty(\mathbb{S}^2)$ is such that*

$$\pi_* \text{ad}^* = \text{ad}.$$

Proof. Let $v \in \text{sdiff}(\mathbb{S}^2)$ and $[\eta] \in \text{sdiff}^*(\mathbb{S}^2)$. Let call $q = d[\eta]$ and, as above, again $*q = q$.

$$\pi \circ \text{ad}_v^*[\eta] = - * dL_v[\eta] = - * L_v d[\eta] = -L_v q = L_{X_\psi} q = \{\psi, q\} = \text{ad}_\psi q,$$

where ψ is the only function in $C_0^\infty(\mathbb{S}^2)$ such that $X_\psi = -v$ (it exists being v divergence free and being \mathbb{S}^2 simply connected). \square

It is now important to notice that, via the L^2 pairing, we can identify the dual of $C_0^\infty(\mathbb{S}^2)$ with itself. We can also endow it with a Lie–Poisson structure which coincides with ad in $C_0^\infty(\mathbb{S}^2)$. Let call $p : C_0^\infty(\mathbb{S}^2) \rightarrow C_0^\infty(\mathbb{S}^2)^*$ the identification. Then we have:

Theorem 3. *The map $p \circ \pi : \text{sdiff}^*(\mathbb{S}^2) \rightarrow C_0^\infty(\mathbb{S}^2)^*$ is a Lie–Poisson isomorphism.*

Proof. This follows immediately from the Lemma above and the fact that $\text{ad}_\psi^* q = -(\text{ad}_\psi q)^* = \text{ad}_\psi q$, which is due to the fact:

$$\int (v \cdot \nabla f)g + \int f(v \cdot \nabla g) = \int v \cdot \nabla(fg) = - \int \nabla \cdot v(fg) = 0$$

and the equivalence:

$$\langle (\text{ad}_\psi q)^*, g \rangle = - \int (v \cdot \nabla q)g = \int q(v \cdot \nabla g) = \langle q, -\text{ad}_\psi g \rangle.$$

□

As we have seen in the previous paragraph, a consequence of the Euler equations of being a Lie–Poisson system is that there exists an infinite number of independent first integrals or Casimir functions. This fact turns out to be the leading point in giving a suitable spatial discretization of the system. In fact, while solving the equations with a numerical scheme, we cannot expect to preserve all the infinite first integrals but what we do want to get is having an increasing number of first integrals with respect to the size of the discrete problem. This cannot be achieved by simply considering a truncated spectral decomposition of the vorticity [17], [18].

Instead we used the approach proposed by Zeitlin in [18], based on the theory of geometric quantization of compact Kähler manifolds [6], [5], [11]. It provides a sequence of finite-dimensional twisted-representations of the infinite-dimensional Lie algebra of divergence-free vector fields, $\text{sdiff}(\mathbb{S}^2)$. This sequence can also be seen as a finite dimensional approximation of $\text{sdiff}(\mathbb{S}^2)$, in the sense of the L_α -convergence, which will be explained below. Then, for any of those quasi-representations, we get a finite dimensional analogue of (3.1), i.e., a Lie–Poisson Hamiltonian system on $\mathfrak{su}(n)$ (or $\mathfrak{sl}(n, \mathbb{C})$), for any $n \geq 1$, with $n - 1$ independent Casimir functions.

3.3 L_α -convergence

Let us consider a Lie-algebra $(\mathfrak{g}, [\cdot, \cdot])$ and a family of labelled Lie algebras $(\mathfrak{g}_\alpha, [\cdot, \cdot]_\alpha)_{\alpha \in I}$, where $\alpha \in I = \mathbb{N}$ or \mathbb{R} . Furthermore, assume then that to any element of this family it is associated a distance d_α and a surjective projection map $p_\alpha : \mathfrak{g} \rightarrow \mathfrak{g}_\alpha$. Then we will say that $(\mathfrak{g}, [\cdot, \cdot])$ is an L_α -approximation of $(\mathfrak{g}_\alpha, [\cdot, \cdot]_\alpha)_{\alpha \in I}$ if:

- if $x, y \in \mathfrak{g}$ and $d_\alpha(p_\alpha(x), p_\alpha(y)) \rightarrow 0$, for $\alpha \rightarrow \infty$, then $x = y$,
- for all $x, y \in \mathfrak{g}$ we have $d_\alpha(p_\alpha([x, y]), [p_\alpha(x), p_\alpha(y)]_\alpha) \rightarrow 0$, for $\alpha \rightarrow \infty$,
- all p_α , for $\alpha \gg 0$, are surjective.

The above definition is given in [5] and it is a quite weak requirement to get a limit for a sequence of Lie algebras. Indeed the same sequence may converge in the L_α sense to different Lie algebras [6]. Much depends on the choice of the projections that are not canonical. However, for our purposes, since we have already a target and we need a suitable sequence to approximate it, we won't need more than that.

Let us now consider the smooth complex functions with 0 mean on the sphere, and denote them with $C_0^\infty(\mathbb{S}^2, \mathbb{C})$. This vector space can be canonically endowed by a Poisson structure given by the respective Hamiltonian vector fields of two functions and a symplectic form α on \mathbb{S}^2 . We have, for any $f, g \in C_0^\infty(\mathbb{S}^2, \mathbb{C})$:

$$\{f, g\} = \alpha(X_f, X_g).$$

With this bracket, $C_0^\infty(\mathbb{S}^2, \mathbb{C})$ becomes an infinite dimensional Poisson algebra. A basis is given by the complex spherical harmonics, which will be denoted in the standard notation and azimuthal-inclination coordinates (ϕ, θ) as:

$$Y_{lm} = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi},$$

for $l \geq 1$ and $m = -l, \dots, l$. In this basis it has been built up by J. Hoppe [11] and fully proved (even in a more general contest) by M. Bordemann, E. Meinrenken and M. Schlichenmaier [5] an explicit L_α -approximating sequence, given by the matrix Lie algebra $(\mathfrak{sl}(n, \mathbb{C}), [\cdot, \cdot]_n)_{n \in \mathbb{N}}$, where $[\cdot, \cdot]_n = n^{3/2}[\cdot, \cdot]$, the rescaled usual commutator of matrices.

The distances are given by a suitable matrix norm and the projections are defined by associating to any spherical harmonic a respective matrix, for any $n \in \mathbb{N}$, i.e., $p_n : Y_{lm} \mapsto T_{lm}^n$, where

$$(T_{lm}^n)_{m_1 m_2} = (-1)^{n/2-m_1} \sqrt{2l+1} \begin{pmatrix} n/2 & l & n/2 \\ -m_1 & m & m_2 \end{pmatrix},$$

where the round bracket is the Wigner 3j-symbols. The result can be summarized as:

Theorem 4 (Bordemann, Hoppe, Meinrenken, Schlichenmaier [6],[5]). *Let us consider the Poisson algebra $(C_0^\infty(\mathbb{S}^2, \mathbb{C}), \{\cdot, \cdot\})$ whose pairing is defined in (3.3). Then with respect to p_n defined above and d_n any matrix norm, we have that $(C_0^\infty(\mathbb{S}^2, \mathbb{C}), \{\cdot, \cdot\})$ is an L_α -approximation of $(\mathfrak{sl}(n, \mathbb{C}), [\cdot, \cdot]_n)_{n \in \mathbb{N}}$.*

3.4 The reduced system

We can now derive the spatial discretization of the Euler equations via the L_α -approximation. We first present the system without the Coriolis force.

For any $n \in \mathbb{N}$, we get an analogous of the Euler equations (3.1):

$$\dot{W} = [\Delta_n^{-1}W, W]_n,$$

where $W \in \mathfrak{sl}(n, \mathbb{C})$ and Δ_n^{-1} is the inverse of the discrete Laplacian as defined in [18]. The crucial property of Δ_n^{-1} is that $\Delta_n^{-1}T_{lm}^n = (-l(l+1))^{-1}T_{lm}^n$, for any $l = 1, \dots, n$, $m = -l, \dots, l$.

We remark that, for a real valued vorticity, W is actually in $\mathfrak{su}(n)$, which means that $\overline{W}_{lm} = (-1)^m W_{l-m}$.

The discrete Hamiltonian takes the following form:

$$H(W) = \frac{1}{2} \text{Tr}(\Delta_n^{-1}WW^\dagger).$$

The discrete system has the following independent $n - 1$ of first integrals⁹

$$F_n(W) = \text{Tr}(W^k) \text{ for } k=2, \dots, n$$

which, up to a normalization constant dependent on n , converge to the powers of the continuous vorticity.

3.4.1 With the Coriolis force

In the case with the Coriolis force, the discrete system is:

$$\dot{W} = [\Delta_n^{-1}(W - F), W]_n,$$

where $F = 2\Omega T_{10}^n$ represents the discrete Coriolis force.

The discrete Hamiltonian in this case takes the following form:

$$H(W) = \frac{1}{2} \text{Tr}(\Delta_n^{-1}(W - F)(W - F)^\dagger).$$

4 Summary of the results in the papers

4.1 Paper I: Lie–Poisson methods for isospectral flows

In paper I, we treat isospectral flows and Lie–Poisson systems together, which lead to a general recipe for solving both numerically, capturing their main geometrical features.

⁹One should notice that by definition $\text{Tr}(W) = 0$ for all $W \in \mathfrak{sl}(n, \mathbb{C})$ and $\text{Tr}(W^n)$ can be replaced by $\det(W)$, by the Cayley–Hamilton theorem.

Consider the following Hamiltonian isospectral flow for $W \in \mathfrak{g}$, Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ and H smooth function on \mathfrak{g} :

$$\begin{aligned}\dot{W} &= [\nabla H(W)^\dagger, W] \\ W(0) &= W_0.\end{aligned}$$

Then we have the following fact:

Proposition 2. *If \mathfrak{g} is a semisimple Lie algebra then $\nabla H(W)^\dagger \in \mathfrak{g}$ and, via the Frobenius norm identification, (4.1) is a Lie–Poisson system on \mathfrak{g}^* .*

If we just have a general isospectral flow

$$\begin{aligned}\dot{W} &= [B(W), W] \\ W(0) &= W_0,\end{aligned}$$

for $W \in S$, linear subspace of $\mathfrak{gl}(n, \mathbb{C})$, then, for being (4.1) well defined, $B(W)$ has to belong to $\mathfrak{n}(S)$, i.e., the $\mathfrak{gl}(n, \mathbb{C})$ -normalizer of S .

In both cases, we require the following assumption to hold (the Hamiltonian case is for $B = \nabla H^\dagger$):

Assumption 1. *Given $\varepsilon > 0$, let S_ε be a ε -neighbourhood of S in $\mathfrak{gl}(n, \mathbb{C})$. Then we assume that $B = B(Z)$ can be extended on S_ε , such that $B(Z) \in \mathfrak{n}(S)$ for all $Z \in S_\varepsilon$, where $\mathfrak{n}(S)$ is the $\mathfrak{gl}(n, \mathbb{C})$ -normalizer of S .*

This assumption is not restrictive. For example one can extend $B = B(Z)$ invariantly with respect to the S -orthogonal directions. Notice that S is a linear space, therefore this extension of the gradient of the Hamiltonian requires only an orthogonal projection of Z to S .

Finally consider the lifted equations on $T^*GL(n, \mathbb{C})$, for $(Q, P) \in T^*GL(n, \mathbb{C})$ such that $W = Q^\dagger P$ satisfies (4.1):

$$\begin{aligned}\dot{Q} &= QB(Q^\dagger P) \\ \dot{P} &= -PB(Q^\dagger P)^\dagger.\end{aligned}$$

Then Q has to belong to $N(S)$, the $GL(n, \mathbb{C})$ -normalizer of S . If this is preserved by a numerical method, we have obtained an isospectral integrator:

Theorem 5. *Let $W = W(t)$ be the solution of (4.1) in some linear subspace S of $\mathfrak{gl}(n, \mathbb{C})$ and let Assumption 1 hold. Then a symplectic numerical method applied to (4.1) descends to an isospectral integrator on S for (4.1) if:*

”there exists a fixed $G \in GL(n, \mathbb{C})$ such that $GQ^\dagger \in N(S)$ ”

is a first integral of the discrete flow.

Moreover, if $B = \nabla H^\dagger$ and $S = \mathfrak{g}$, semisimple (or reductive) Lie algebra, the method is a Lie–Poisson integrator for (4.1).

The second constructive result is that any symplectic Runge-Kutta method gives a Lie–Poisson integrator for $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$ and any of their quadratic reductive subalgebras. The general s -stage method is given by the following scheme.

Given a Butcher tableau:

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

of a s -stage symplectic Runge-Kutta method with time step h , we get the following Lie–Poisson integrator:

$$X_i = -h(W_n + \sum_{j=1}^s a_{ij} X_j) \nabla H(\widetilde{W}_i)^\dagger, \quad \text{for } i = 1, \dots, s.$$

$$Y_i = h \nabla H(\widetilde{W}_i)^\dagger (W_n + \sum_{j=1}^s a_{ij} Y_j), \quad \text{for } i = 1, \dots, s.$$

$$K_{ij} = h \nabla H(\widetilde{W}_i)^\dagger (\sum_{j'=1}^s (a_{ij'} X_{j'} + a_{jj'} K_{ij'})), \quad \text{for } i, j = 1, \dots, s.$$

$$\widetilde{W}_i = W_n + \sum_{j=1}^s a_{ij} (X_j + Y_j + K_{ij}), \quad \text{for } i = 1, \dots, s.$$

$$W_{n+1} = W_n + h \sum_{i=1}^s b_i [\nabla H(\widetilde{W}_i)^\dagger, \widetilde{W}_i],$$

where the unknowns are X_i, Y_i, K_{ij} for $i, j = 1, \dots, s$ and the last two lines are explicit.

In the paper, it is shown how it can be simplified in several cases. We conclude the article by presenting several applications of the method to the rigid body equations, the point vortex equations, the Heisenberg spin chain equations, the Euler equations, the Toda lattice and the Toeplitz inverse problem.

In the figure below we show the results for one of our methods applied to the generalized rigid body equations.

4.2 Paper II: A structure preserving scheme for the Euler equations on a (rotating) sphere

In this paper we present a new class of numerical schemes to discretize the Euler equations, both in time and space. These methods are obtained by combining the results of Paper I and the geometric quantization reduction proposed by Zeitlin (cf. Section 3), which leads to an ODE in $\mathfrak{su}(n)$:

$$\dot{W} = [\Delta_n^{-1} W, W]_n.$$

where Δ_n^{-1} is the discrete Laplacian operator.

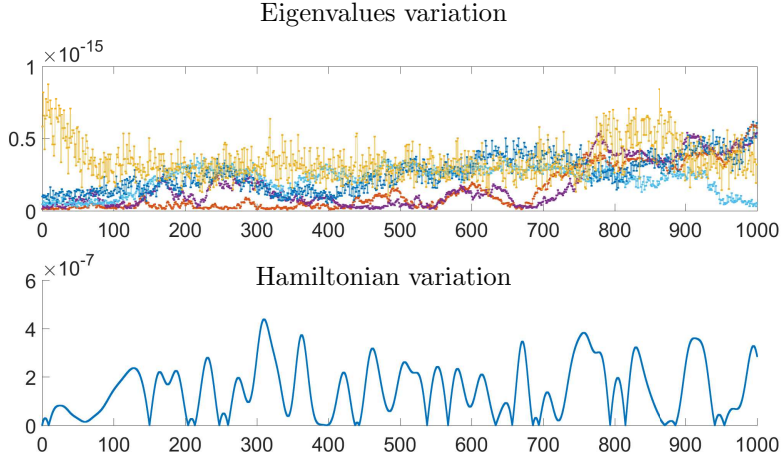


Figure 1: Generalized 45-dimensional rigid body in $\mathfrak{so}(10)$. Eigenvalues (which occur in pair) and Hamiltonian variation; $h = 10^{-1}$; inertia tensor $\mathbb{I} = \text{diag}(1 : 10)$; initial value $(W_0)_{ij} = 1/10$ if $i < j$, $(W_0)_{ij} = -1/10$ if $i > j$, $(W_0)_{ij} = 0$ if $i = j$.

The numerical methods developed have the advantage to preserve, up to machine precision, the discrete Casimir functions, nearly conserving the Hamiltonian and obtaining a discrete flow qualitatively similar to the original Euler equations.

The simplest scheme for the quantized Euler equations (4.2) that we propose is the 2nd order isospectral midpoint rule. With time step h , it is:

$$\begin{aligned}
 X &= -h(W_n + \frac{1}{2}X)\Delta_n^{-1}\widetilde{W} \\
 K &= \frac{h}{2}\Delta_n^{-1}\widetilde{W}(X + K) \\
 \widetilde{W} &= W_n + \frac{1}{2}(X - X^\dagger + K) \\
 W_{n+1} &= W_n + X - X^\dagger + K - K^\dagger.
 \end{aligned}$$

In Figures 2-3, we show the results of two simulations, where we have applied (4.2) to two examples studied in [9] and [7].

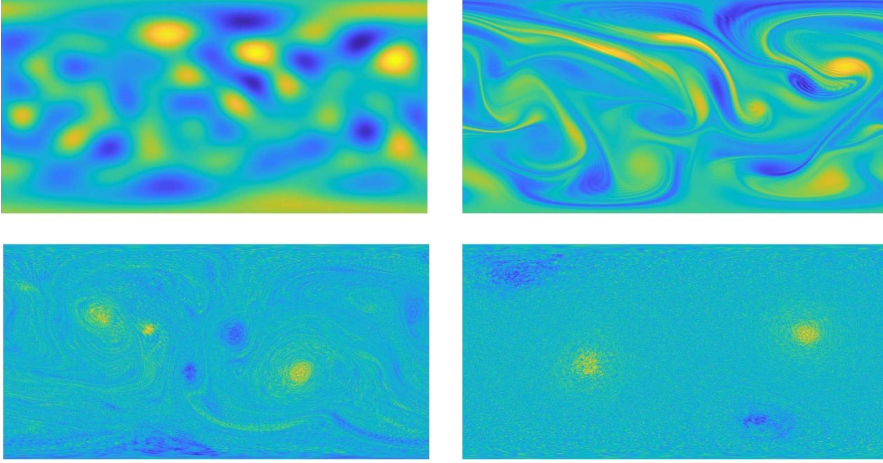


Figure 2: Vorticity $\omega(x, t)$ from the top-left at $t = 0s, 4s, 40s, 140s$, for the initial data in [9]. The horizontal axis is the azimuth $\varphi \in [0, 2\pi]$ and the vertical axis is minus the inclination $\theta \in [0, \pi]$. Spatial discretization in $\mathfrak{su}(501)$.

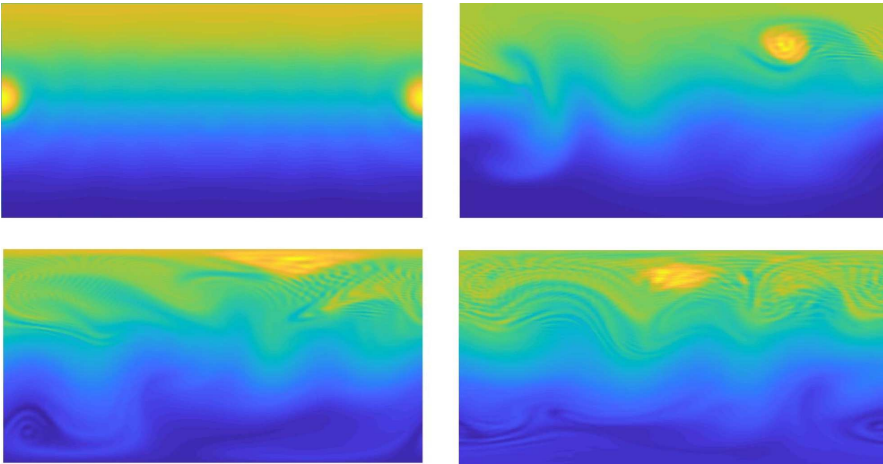


Figure 3: Vorticity $\omega(x, t)$ from the top-left at iteration= 1, 150, 300, 450, for the initial data in [7]. The horizontal axis is the azimuth $\varphi \in [0, 2\pi]$ and the vertical axis is minus the inclination $\theta \in [0, \pi]$. . Spatial discretization in $\mathfrak{su}(201)$.

5 Proposals for future work

5.1 Paper I: Lie–Poisson methods for isospectral flows

In Paper I, we presented a general approach for solving numerically Lie–Poisson systems on reductive Lie algebras. In view of the Levi decomposition of finite dimensional Lie algebras, i.e., that any of them can be decomposed into a semi-direct product of a semisimple and a solvable Lie subalgebra, it would be interesting to develop analogous results for Lie–Poisson systems on solvable Lie algebras.

5.2 Paper II: A structure preserving scheme for the Euler equations on a (rotating) sphere

In paper II, encouraging results in the study of the Euler equations have been shown. However, our simulations, despite good, should be implemented with higher resolution, in order to give more reliable predictions.

Analogously, a full analysis of the convergence of the quantized equations to the original one is still missing and it should be accomplished to understand the quality of such an approximation.

Last but not least, simulations of coupled continuous and singular vorticity (point vortices) have yet to be done.

Bibliography

- [1] A. Arakawa. “Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I”. *Journal of Computational Physics* 1.1 (1966), pp. 119–143.
- [2] V.I. Arnold. “Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses applications a l’hydrodynamique del fluids parfaits”. *Annales de l’institut Fourier* 16.1 (1966), pp. 319–361.
- [3] V.I. Arnold and B.A. Khesin. *Topological Methods in Hydrodynamics*. Vol. 125. Springer-Verlag New York, 1998, pp. XV, 376. ISBN: 978-0-387-94947-5.
- [4] G. Bogfjellmo and H. Marthinsen. “High-Order Symplectic Partitioned Lie Group Methods”. *Foundations of Computational Mathematics* 16.2 (2016), pp. 493–530.
- [5] M. Bordemann, E. Meinrenken, and M. Schlichenmaier. “Toeplitz Quantization of Kähler Manifolds and $\mathfrak{gl}(N)$, $N \rightarrow \infty$ limits”. *Communications in Mathematical Physics* 165.2 (1994), pp. 281–296.
- [6] M. Bordemann et al. “ $\mathfrak{gl}(\infty)$ and geometric quantization”. *Communications in Mathematical Physics* 138.2 (1991), pp. 209–244.
- [7] P. Bosler et al. “A Lagrangian particle/panel method for the barotropic vorticity equations on a rotating sphere”. *The Japan Society of Fluid Mechanics and IOP Publishing Ltd* 46.3 (2014).
- [8] M.P. Calvo, A. Iserles, and A. Zanna. “Numerical Solution of Isospectral Flows”. *Mathematics of Computation Volume* 66.220 (1997), pp. 1461–1486.
- [9] D.G. Dritschel, W. Qi, and J.B. Marston. “On the late-time behaviour of a bounded, inviscid two-dimensional flow”. *Journal of Fluid Mechanics* 783 (2015), pp. 1–22.
- [10] E. Hairer, C. Lubich, and G. Wanner. *Geometric Numerical Integration*. Vol. 31. Springer, Berlin, Heidelberg, 2006, pp. XVII, 644. ISBN: 978-3-540-30663-4.
- [11] J. Hoppe. “Ph.D. thesis MIT Cambridge” (1982).
- [12] J. Marsden and A. Weinstein. “Co-adjoint orbits, vortices, and Clebsch variables for incompressible fluids”. *Physica D* 7.1–3 (1983), pp. 305–323.
- [13] J.E. Marsden and T.S. Ratiu. *Introduction to Mechanics and Symmetry*. Vol. vol. 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1999. ISBN: 978-0-387-21792-5.

- [14] R. McLachlan. “Explicit Lie–Poisson Integration and the Euler Equations”. *Physical review letters* 71.19 (1993), pp. 3043–3046.
- [15] R. McLachlan and C. Scovel. “Equivariant constrained symplectic integration”. *Journal of Nonlinear Science* 5.3 (1995), pp. 233–256.
- [16] M. Webb. “Isospectral algorithms, Toeplitz matrices and orthogonal polynomials, Ph.D. thesis” (2017).
- [17] V. Zeitlin. “Finite-mode analogues of 2D ideal hydrodynamics: Coadjoint orbits and local canonical structure”. *Physica D* 49.3 (1991), pp. 353–362.
- [18] V. Zeitlin. “Self-Consistent-Mode Approximation for the Hydrodynamics of an Incompressible Fluid on Non rotating and Rotating Spheres”. *Physical review letters* 93.26 (2004), pp. 353–362.