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Dedicated to the memory of two great mathematicians,
M.S. Agranovich and M.Z. Solomyak

EIGENVALUE ASYMPTOTICS FOR POTENTIAL TYPE OPERATORS ON LIPSCHITZ SURFACES OF CODIMENSION GREATER THAN 1

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Abstract. For potential type integral operators on a Lipschitz submanifold the asymptotic formula for eigenvalues is proved. The reasoning is based upon the study of the rate of operator convergence as smooth surfaces approximate the Lipschitz one.

Keywords: integral operators, potential theory, eigenvalue asymptotics.

Mathematics Subject Classification: 47G40, 35P20.

1. INTRODUCTION

The present article adjoins the paper [18] by the authors. In the latter paper the asymptotics of the eigenvalues of weakly polar integral operators on Lipschitz surfaces in the Euclidean space was studied. Now we consider integral operators on Lipschitz submanifolds of an arbitrary co-dimension.

Potential type operators play an important part in the theory of partial differential equations and mathematical physics from rather early stages. They can be used to reduce boundary value problems to integral equations on the boundary, thus decreasing the dimension of the problem. The spectral theory of such operators is also important, both in theoretical analysis and in the calculations in electrodynamics, diffraction and quantum physics, see, e.g., [1, 2, 4, 19], and references therein. For the case of a smooth surface, the potential type operators are operators with kernels, smooth everywhere on the surface outside the diagonal and having a weak homogeneous singularity at the diagonal. Therefore they can be represented as pseudodifferential operators (not necessarily elliptic) on the surface, and this reduction takes care of the main spectral properties, such like estimates and asymptotics of eigenvalues, convergence and summability of spectral expansions etc. Such reduction encounters considerable obstacles when the surface is not sufficiently smooth. The kernels, expressed in local co-ordinates, are not smooth any more outside the diagonal, and no theory of pseudodifferential operators can take care of them. The critical case here are Lipschitz surfaces, where lots of facts of general theory, valid for more smooth cases, break down, while many interesting cases involve Lipschitz surfaces, which cannot be reduced to more smooth ones. This is one of the reasons why the theory of boundary problems in Lipschitz domains required a considerable effort in its development.

In the paper [18] we addressed the question on the *eigenvalue asymptotics* for potential type operators on Lipschitz surfaces. For general integral operators with weakly polar kernels, the fundamental results were obtained in the paper [5]; they have been further generalized in [16, 17]. These results, being applied to potential type operators, cease to be valid for Lipschitz surfaces. A crucial development in this problem was made in the paper [3] where an upper estimate for eigenvalues, having the same order as in the smooth case, was proved. This result was further used in the study of a number of spectral problems for elliptic operators and systems in [1]. It was noticed in [3] that although the kernel of the operator, restricted to the surface, is not smooth in the local coordinates on this surface, one still can use the fact that it is a restriction of a smooth function. However, this study reached a little bit short of the eigenvalue asymptotics. The asymptotic formula for eigenvalues was justified in [3] only for a special subclass of Lipschitz surfaces, the *almost smooth* ones, i.e. those that are smooth outside a closed set of measure zero. The question on the eigenvalue asymptotics for Lipschitz surfaces without additional conditions was formulated in [2]. In [18] we answered this question and found the asymptotics of eigenvalues of potential type operators on arbitrary compact Lipschitz surfaces. To do this we implemented the approach, mentioned as possible but not used in [3], of approximating the Lipschitz surface by smooth ones and studying the convergence of the corresponding operators. For the latter task we needed a certain improvement of the existing methods for obtaining eigenvalue estimates.

In the present paper we extend the results of [18] to the case of potential type integral operators acting on Lipschitz surfaces of dimension d in $\mathbb{R}^{d+\mathfrak{d}}$ (or, equivalently, of co-dimension $\mathfrak{d} > 1$; the case $\mathfrak{d} = 1$ corresponds to the usual surfaces.) In certain aspects, the reasoning follows the pattern of [3, 18], and in these places we mostly point out which changes one needs to make (the authors strongly recommend the readers who are interested in details, to study [3], [18] before this paper.) In some steps of the construction, certain additional considerations are required, and they are explained in more detail.

In Section 2 we describe the setting of our problem and introduce important quantities entering in the main formulas in the paper. We also describe the crucial procedure of approximation of a Lipschitz surface of codimension $\mathfrak{d} > 1$ by smooth ones and study the rate of convergence of measure densities under this approximation. Next, in Section 3, we start studying the spectral distribution for integral operators on Lipschitz surfaces. Here we demonstrate the piecewise polynomial approximation procedure, extending the one used in the codimension 1 case, leading to order sharp spectral estimates. The next two sections deal with the study how the approximation of surfaces leads to the closeness of the corresponding operators in local patches. First, we consider the contribution of the strip near the diagonal, where the kernel has a singularity, and then the contribution of the region away from the diagonal, where the kernel is nice. Finally, in Section 6, we collect the spectral estimates and prove the eigenvalue asymptotics formula.

2. SETTING OF THE PROBLEM. APPROXIMATION

Our initial setting follows the pattern of [3]. We, however, need some formulas, not of a common usage. Let A be a classical Hermitian pseudodifferential operator in $\mathbb{R}^{d+\mathfrak{d}}$, having the negative order $l-\mathfrak{d}$, $l<0$ (not necessarily integer). The symbol of A is a function

$$a(X,\Xi)\sim\sum_{\nu=0}^{\infty}a_\nu(X,\Xi), \tag{2.1}$$

where $a_\nu(X,\Xi)$ is a function, smooth in $(X,\Xi)\in\mathbb{R}^{d+\mathfrak{d}}\times(\mathbb{R}^{d+\mathfrak{d}}\setminus\{0\})$ and positive-homogeneous of order $l-\mathfrak{d}-\nu<0$ in Ξ. We will also suppose that the symbol (i.e., all terms in (2.1)) have support in some compact set $\mathbb{K}\subset\mathbb{R}^{d+\mathfrak{d}}$. The sum is understood, as usual, in the sense of asymptotic expansions as $\Xi\to\infty$. We will study the restriction of this operator to a compact Lipschitz submanifold $\mathcal{S}\subset\mathbb{R}^{d+\mathfrak{d}}$ of codimension $\mathfrak{d}$, i.e., of dimension d.

It is well known that such operator A (up to an infinitely smoothing term) acts on functions with compact support as an integral operator with kernel $\mathcal{H}(X,Y)=\mathcal{K}(X,Y;X-Y)$, $X,Y\in\mathbb{R}^{d+\mathfrak{d}}$. It is convenient to cite [14], Section 7, especially, Section 7.1.4, where the relations between the symbol and the the kernel are presented in a comprehensive way. In particular,

$$\mathcal{H}(X,Y)=(2\pi)^{-d-\mathfrak{d}}\int\limits_{\mathbb{R}^{d+\mathfrak{d}}}a(X,\Xi)e^{i(X-Y)\Xi}d\Xi. \tag{2.2}$$

The kernel $\mathcal{H}(X,Y)$ is smooth for $X\neq Y$ but has a singularity at the diagonal $X=Y$: written as $\mathcal{K}(X,Y;X-Y)$, it is polyhomogeneous in the last variable. This means that $\mathcal{K}(X,Y;X-Y)$ can be represented as an asymptotic sum

$$\mathcal{H}(X,Y)=\mathcal{K}(X,Y;X-Y)\sim\sum_{\nu=0}^{\infty}(\mathcal{K}_\nu(X,Y;X-Y)+'\mathcal{K}_{\nu,\log}(X,Y;X-Y)). \tag{2.3}$$

Functions $\mathcal{K}_\nu(X,Y;Z)$ in (2.3) are positively homogeneous of degree $-l-d+\nu=m+\nu$ in Z variable. So, in the leading term with $\nu=0$ the homogeneity order is $m=-d-l>-d$. If, for some ν, the expression $\mathcal{K}_\nu(X,Y;Z)$ is a polynomial in Z (this may happen only if l is a negative integer), the extra term $\mathcal{K}_{\nu,\log}(X,Y;X-Y)$ may be present, having the form $\mathcal{K}_{\nu,\log}(X,Y;Z)=\mathcal{Q}_\nu(X,Y;Z)\log|Z|$ with $\mathcal{Q}_\nu$ being a homogeneous polynomial of order $l+\nu$ in Z. This optional presence of logarithmic terms is expressed by writing $+'$ in (2.3). Note that in this case the first term, $\mathcal{K}_{\nu,\log}(X,Y;X-Y)$, produces an operator with smooth kernel, and therefore does not influence the leading term in eigenvalue asymptotics, at least in the power scale.

The expansion (2.3) is understood in the sense of asymptotics in smoothness: the difference between the kernel $\mathcal{K}$ and the partial sum of the series in (2.3) becomes arbitrarily smooth in all variables, as the number of terms in the partial sum grows:

$$\begin{aligned}\mathcal{K}(X,Y;X-Y) - \sum_{\nu=0}^{M}(\mathcal{K}_\nu(X,Y;X-Y)\\ +'\,\mathcal{K}_{\nu,\log}(X,Y;X-Y)) \in C^{m+M/2}(\mathbb{R}^{d+\delta}\times\mathbb{R}^{d+\delta}\times(\mathbb{R}^{d+\delta}\setminus\{0\}).\end{aligned}\tag{2.4}$$

Having this in mind, we can re-arrange the terms in (2.3). For each term there, we can write its Taylor expansion in the second variable at $Y=X$, so such term expands into an asymptotic series. We collect the terms of the same homogeneity degree (and log-homogeneity, if these are present) in this sum, arriving at

$$\mathcal{H}(X,Y) = \mathcal{K}(X,Y;X-Y) \sim \sum_{\nu=0}^{\infty}(\tilde{\mathcal{K}}_\nu(X;X-Y) +'\,\tilde{\mathcal{K}}_{\nu,\log}(X;X-Y)), \tag{2.5}$$

where all the dependence on Y variable is absorbed by the dependence on $X-Y$. This, equivalent, expansion of the kernel is somewhat more convenient, and we will use it further on. For the sake of the simplicity of notation, we will dispose of the tilde sign.

In our conditions, it is only the behavior of the kernels when approaching the diagonal that determine the asymptotic character of eigenvalues. Moreover, the terms with $\nu>0$ have a weaker singularity than the leading term, the one with $\nu=0$, with homogeneity order $m>-d$, or, for m being a nonnegative integer, the term $\mathcal{K}_{0,\log}$, having the order $\mathcal{Q}(X;Z)\log|Z|$, with $\mathcal{Q}$ being a polynomial of degree m. And, therefore, it is only the leading term that determines the eigenvalue asymptotics. This fact, well known for the smooth submanifolds, was established for Lipschitz ones in [3] for the case $\mathfrak{d}=1$, see Proposition 2.7 there, and it is extended easily to the general case (see Section 3).

We suppose that the initial symbol $a(X,\Xi)$ in (2.1) is infinitely smooth in X,Ξ, which leads to the smoothness of the kernel $\mathcal{H}(X;Z)$ away from the diagonal. Our results can be carried over to the case of considerably weaker regularity conditions imposed on the kernel concerning its smoothness in X,Z, and this may be important for applications. However, some more advanced machinery is needed here, and we are going to deal with this topic on some other occasion.

In the opposite direction, any integral operator with smooth weakly polar kernel having the form (2.3) or (2.5), can be represented as a pseudodifferential operator with certain symbol (2.1). There exist formulas enabling one to re-calculate the components of the symbol from the homogeneous (or log-homogeneous) components of the kernel. One cannot derive these formulas directly by inverting the Fourier transform in (2.2), because, for a positive homogeneity order, these formulas would involve integrals of functions growing at infinity. Applying the Fourier transform in the sense of distributions would lead to a pseudodifferential operator with distributional symbol, which is highly inconvenient. The "correct" transformation formulas can be found, for example, in [14, Section 7.1.4], where the "finite part of the divergent integral" in the Cauchy sense is used. An alternative approach was applied by M. Birman and M. Solomyak in [5], by means of the Riesz summation method, which disposes of singularities in divergent integrals of this kind. In order to avoid these technical complications, we follow the approach of [3], where the pseudodifferential operator A is the primary object and the recalculation from the kernel to the symbol is not needed. This is also convenient because the characteristics of eigenvalue asymptotics are much more comfortably expressed in the term of the symbol, and not in the terms of the kernel, compare with [5].

In $\mathbb{R}^{d+\mathfrak{d}}$ we consider a compact Lipschitz submanifold $\mathcal{S}$ of dimension d, i.e., of codimension $\mathfrak{d}$ (later on we will (mis)use the word "surface" for $\mathcal{S}$). This means that there exists a finite collection of local co-ordinate patches Ω_j in $\mathbb{R}^{d+\mathfrak{d}}$, covering $\mathcal{S}$, such that in each of them, with a proper choice of local co-ordinates $X=(x,z)\in\Omega_j\subset\mathbb{R}^{d+\mathfrak{d}}$, $x\in\mathcal{D}_j\subset\mathbb{R}^d, z\in\mathbb{R}^{\mathfrak{d}}$, the submanifold is described in this patch by the equation $z=\phi_j(x)$ with a Lipschitz vector-function $\phi_j:\mathcal{D}_j\to\mathbb{R}^{\mathfrak{d}}$; to be more precise, the surface is locally the graph in $\mathbb{R}^{d+\mathfrak{d}}$ of the mapping ϕ_j, so we can define the continuous mapping $\boldsymbol{\phi}_j:\mathcal{D}_j\to\mathcal{S}$, $\boldsymbol{\phi}_j(x)=(x,\phi_j(x))$. We recall now that due to the Rademacher theorem, see, e.g., [10, Section 3.1.2], each of the mappings ϕ_j is differentiable almost everywhere with respect to the Lebesgue measure on $\mathbb{R}^d$; the points in $\mathbb{R}^d$, where all components of ϕ_j are differentiable, will be called regular. So, at any regular point $x\in\mathcal{D}_j$, the mapping $\boldsymbol{\phi}_j$ is differentiable, with differential, a linear mapping $D(x)\equiv D\boldsymbol{\phi}_j(x):\mathbb{R}^d\to\mathbb{R}^{d+\mathfrak{d}}$. This linear mapping, a matrix of the size $d\times(d+\mathfrak{d})$ has the unit matrix in its first d rows and $\nabla\phi_j(x)$ in the remaining $\mathfrak{d}$ rows.

The surface $\mathcal{S}$ is equipped with an intrinsic d-dimensional Hausdorff measure $d\mu$, induced by the Lebesgue measure in $\mathbb{R}^{d+\mathfrak{d}}$. This measure is absolutely continuous with respect to the push-forward $\boldsymbol{\phi}_j^*(dx)$ of the Lebesgue measure dx on $\mathbb{R}^d$ under the mapping $\boldsymbol{\phi}_j$. The density of $d\mu$ with respect to dx equals $\sigma_j(x)=\sqrt{\det(D(x)^*D(x))}$ (see, e.g., Theorem 3, Section 3.3.2 in [10]). We recall the structure of the differential $D(x)$, to obtain

Proposition 2.1. *If $\mathcal{S}\subset\mathbb{R}^{d+\mathfrak{d}}$ is a d-dimensional Lipschitz surface then*

$$d\mu=\sigma_j(x)dx;\sigma_j(x)=[\det(1+(\nabla\phi_j(x))^*(\nabla\phi_j(x))]^{\frac{1}{2}}. \tag{2.6}$$

in the piece of the submanifold $\mathcal{S}$ parametrized by the mapping ϕ_j on $\mathcal{D}_j$.

Further on, as long as our considerations are local, we will usually omit the subscript j in the above and similar notations, provided this does not create confusion.

We consider the restriction of the operator A to the surface $\mathcal{S}$. This restriction is the operator $\mathfrak{K}:L_2(\mathcal{S})\to L_2(\mathcal{S})$ acting, locally, on the functions $U\in L_2(\mathcal{S})$ as

$$(\mathfrak{K}U)(X)=\int\limits_{\mathcal{S}}\mathcal{K}(X;X-Y)U(Y)d\mu(Y),\quad X\in\mathcal{S}. \tag{2.7}$$

If $\mathfrak{d}=1$, such operators arise, for example, as single, double, and multiple layer potentials corresponding to elliptic differential operators. In particular, if A is the inverse for the polyharmonic operator $(-\Delta)^N$ in $\mathbb{R}^{d+1}$, $\mathcal{K}(X,;X-Y)$ is the Green kernel for $(-\Delta)^N$ and then it equals $C_{d,l}|X-Y|^{-d-1+2N}\log|X-Y|$ for a non-negative even integer $-d-1+2N$ and equals $C_{d,l}|X-Y|^{-d-1+2N}$ otherwise. A potential theory for a fractional Laplacian was investigated, for example, in [7], where potential type integral operators with fractional singularity order arise. For an arbitrary $\mathfrak{d}$, such operators appear in the study of potentials corresponding to electric or magnetic densities located on submanifolds, such as in the Aharonov-Bohm type magnetic field.

For a smooth surface $\mathcal{S}$, the standard considerations in local co-ordinates, see, e.g., [1], show that $\mathfrak{K}$ is a classical pseudodifferential operator on $\mathcal{S}$ of order $l=-d-m<0$. One can find an explicit expression of the principal symbol $\mathfrak{a}_0$ of this operator in the local coordinates at a point $X=(x,z)\in\mathcal{S}$. This expression, for $\mathfrak{d}=1$, is derived, e.g., in [3, Proposition 3.5]:

$$\mathfrak{a}_0(X,\xi)=(2\pi)^{-1}\int_{-\infty}^{\infty}a_0(X,\xi,\zeta)d\zeta, \tag{2.8}$$

where $(X,\xi)\in T^*(\mathcal{S})$ and a_0 is the principal symbol of the operator A, expressed in the co-ordinates $(X;\xi,\zeta)$ in $T^*(\mathbb{R}^{d+1})$, such that the co-vector $(0,\zeta)$ is orthogonal to the tangent space to $\mathcal{S}$ at X.

For an arbitrary codimension $\mathfrak{d}>1$, the formula looks similarly.

Proposition 2.2. *Let $\mathcal{S}$ be a smooth surface of codimension $\mathfrak{d}$, (x,z) be the local orthogonal co-ordinate system near the point $X_0\in\mathcal{S}$, such that x lies in the tangent plane to $\mathcal{S}$ at X_0 and the $z-$co-ordinate is orthogonal to this plane, so that $X_0=(0,0)$ in these co-ordinates; (ξ,ζ) are the dual co-ordinates to (x,z). Then for the operator $\mathfrak{K}$ the principal symbol at this point equals*

$$\mathfrak{a}_0(X_0,\xi)=(2\pi)^{-\mathfrak{d}}\int_{\mathbb{R}^{\mathfrak{d}}}a_0(X_0;\xi,\zeta)d\zeta. \tag{2.9}$$

where a_0 is the principal symbol of the operator A in (2.1) *expressed in the co-ordinates $(x,z;\xi,\zeta)$.*

Proof. The proof follows mainly the reasoning in Proposition 3.5 in [3], with natural modifications caused by the change in the codimension. First, note that the integral in (2.9) converges for $\xi\neq 0$, due to the limitation on the order of A – it is $l-\mathfrak{d}$, $l<0$. Since all considerations are local, we can, without losing in generality, suppose that the symbol a_0 vanishes outside some small neighborhood of the point $X\in\mathcal{S}$. The operator $\mathfrak{K}$ can be written as $\mathfrak{K}=\Gamma A\Gamma^*$, where Γ is the operator of restriction to $\mathcal{S}$ of a function in $\mathbb{R}^{d+\mathfrak{d}}$, i.e., $\Gamma: u(x,z)\mapsto u(x,\phi(x))$, Γ^* is its adjoint operator, assigning to a function on $\mathcal{S}$ its extension by zero. Since Γ in the Fourier representation acts as $\Gamma^{\mathcal{F}}:\hat{u}(\xi,\zeta)\mapsto(2\pi)^{-\mathfrak{d}}\int_{\mathbb{R}^{\mathfrak{d}}}\hat{u}d\zeta$, we arrive at (2.9). ☐

The above local co-ordinate system rotates in $\mathbb{R}^{d+\mathfrak{d}}$ when the point X_0 is moving along $\mathcal{S}$, smoothly, provided the surface is smooth. However, if the surface is only Lipschitz, the expression in (2.9) is, generally, no longer even continuous in $X_0\in\mathcal{S}$, and even if it happens to be regular, it is not a "symbol" for the operator $\mathfrak{K}$, because the latter fails to be a pseudodifferential operator on $\mathcal{S}$ in any reasonable sense. Nevertheless, the expression (2.8) makes sense almost everywhere on $\mathcal{S}$ and, as we prove it later, it determines the eigenvalue asymptotics.

In the course of the proof of the main asymptotic formula, we will only work with local pieces of the surface and the corresponding pieces of operators in co-ordinate patches; only on the last stage, in Section 6, we will glue these pieces together. So, since we are going to approximate the Lipschitz surface by smooth ones, in order to be able to compare operators acting on different surfaces, it is convenient to pass locally to operators acting on functions defined in $\mathcal{D} \subset \mathbb{R}^d$ (where $\mathcal{D}$ is one of the domains $\mathcal{D}_j$ in the local representations of the surface $\mathcal{S}$). Let $\mathcal{S}_\phi$ be the piece of the surface $\mathcal{S}$, the graph of the function ϕ over $\mathcal{D}$. We include two bounded measurable weight functions φ, ψ defined in $\mathcal{D}$ and consider the operator in $L_2(\mathcal{S})$,

$$\mathfrak{K}_{\varphi,\psi} : U(x,\phi(x)) \mapsto \varphi(x)(\mathfrak{K}[\psi(y)U(y,\phi(y))])(x), \quad x, y \in \mathcal{D},$$

which is unitarily equivalent to the operator

$$\varphi\mathbf{K}\psi : u(x) \mapsto \int_{\mathcal{D}} \sigma(x)\varphi(x)\mathcal{K}(x,\phi(x);x-y,\phi(x)-\phi(y))\psi(y)u(y)\sigma(y)dy \tag{2.10}$$

in $L_2(\mathcal{D})$ with Lebesgue measure (here $\sigma(x)$ is the density in Proposition 2.1, $u(x) = U(x,\phi(x))$, with the subscript j dropped).

We will observe the following font convention henceforth (in fact, we have already started to do this): operators and symbols on surfaces in $\mathbb{R}^{d+\mathfrak{d}}$ will be denoted by the Fraktur font, the corresponding unitarily equivalent operators in a domain in $\mathbb{R}^d$ are denoted by the same symbol, but in boldface, and the regular font will be used to denote operators and symbols in the whole $\mathbb{R}^{d+\mathfrak{d}}$.

For a *smooth* surface $\mathcal{S}$ defined by the equation $z = \phi(x)$, $x \in \mathcal{D}$, and for functions $\varphi, \psi \in C_0^\infty(\mathcal{D})$ the operator $\varphi\mathbf{K}\psi$ is a pseudodifferential operator in $\mathcal{D}$ and the expression for its principal symbol $\mathbf{a}_0(x,\xi)$ observes the general transformation rules for principal symbols of pseudodifferential operators,

$$\mathbf{a}_0(x,\xi) = \varphi(x)\psi(x)\mathfrak{a}_0(x, J(x,\phi)^{-1}\xi), \ (x,\xi) \in \mathcal{D} \times \mathbb{R}^d, \tag{2.11}$$

where $\mathfrak{a}_0$ is the symbol defined in (2.8) and $J(x,\phi)$ is the differential of the projection $(x,\phi(x)) \mapsto (x,0)$ of the surface $\mathcal{S}$ to $\mathbb{R}^d$. For a smooth surface $\mathcal{S}$, the expression (2.8), (2.11) depends smoothly on x and the general theory of pseudodifferential operators applies. If the surface is only Lipschitz, the tangent space to $\mathcal{S}$ still exists almost everywhere in $\mathcal{S}$, so one can stil write the expression (2.8), (2.11) however, again, it will generally be discontinuous in x and the operator $\varphi\mathbf{K}\psi$ will not fit into any reasonable pseudodifferential theory.

We conclude this preparatory section by discussing the approximation topic. It is well known (see, e.g., [21]) that a Lipschitz submanifold $\mathcal{S}$ of co-dimension 1 (i.e., the one which is traditionally called "surface") can be rather well approximated by smooth surfaces. In the language of functions, if ϕ is a Lipschitz function (a scalar one, here) on an arbitrarily domain $\mathcal{D} \subseteq \mathbb{R}^d$, then for any $\epsilon > 0$ it can be approximated by a smooth (say, C^∞) function ϕ_ϵ, so that $\|\phi - \phi_\epsilon\|_{L_\infty(\mathcal{D})} < \epsilon$, and, for a fixed $q < \infty$, $\|\nabla\phi - \nabla\phi_\epsilon\|_{L_q(\mathcal{M})} < \epsilon$ on any bounded measurable set $\mathcal{M} \subset \mathbb{R}^d$. This function ϕ_ϵ can be constructed in the following way. First, it is possible to extend the function ϕ from $\mathcal{D}$ to the whole of $\mathbb{R}^d$, as, again, a Lipschitz function $\tilde{\phi}$, with Lipschitz constant not greater than $\sqrt{d}$ times the one for ϕ; this construction is presented, for example, in [10], Theorem 1 in Section 3.1. Thus, in particular, the function $\tilde{\phi}$ is bounded on any bounded set in $\mathbb{R}^d$, and its gradient $\nabla\tilde{\phi}$, existing by the Rademacher theorem almost everywhere, is bounded in $\mathbb{R}^d$, on the set where it exists. Now, we fix a smooth function $\omega(x)$ with compact support, satisfying $\int \omega(x)dx = 1$, and set $\omega_\epsilon(x) = \epsilon^{-d}\omega(x/\epsilon)$. Then the convolution $\phi_\epsilon = \tilde{\phi} * \omega_\epsilon$ is smooth and tends to $\tilde{\phi}$ as $\epsilon \to 0$, uniformly on $\mathbb{R}^d$, because $\tilde{\phi}$ is uniformly continuous. Since $\nabla\tilde{\phi} \in L_\infty(\mathbb{R}^d)$, we have $\nabla\tilde{\phi} \in L_q$ on any bounded set, for any $q < \infty$. Therefore, this (vector-)function is uniformly continuous in the L_q sense on any bounded set. Since the gradient $\nabla\phi_\epsilon$ equals $(\nabla\tilde{\phi}) * \omega_\epsilon$, $\nabla\phi_\epsilon$ converges to $\nabla\tilde{\phi}$ in L_q in any bounded set, as $\epsilon \to 0$. (Note that we needed to extend our function ϕ to the whole of $\mathbb{R}^d$ in order to be able to use the relation between the convolution and the derivative.)

Now we return to our surface of codimension $\mathfrak{d}$, defined locally by a Lipschitz vector-function $\phi(x)$, $x \in \mathcal{D} \subset \mathbb{R}^d$ with values in $\mathbb{R}^{\mathfrak{d}}$. We can apply the procedure described above to each component of the vector-function $\phi(x)$. Each such component can be extended to a Lipschitz function on $\mathbb{R}^d$, with a controllable Lipschitz constant. Then, by the Rademacher theorem, this extended component is differentiable on some subset of full measure in $\mathbb{R}^d$, so, on the intersection of these subsets, which again is of full measure, all components of $\tilde{\phi}(x)$ are differentiable. We now apply the convolution as above, to obtain a family of vector-functions ϕ_ϵ converging in $C(\mathbb{R}^d)$ to $\tilde{\phi}$, with gradients converging to $\operatorname{grad}\tilde{\phi}$ in L_q on any bounded set. This produces the required local approximation of our Lipschitz surface $\mathcal{S}$ by smooth ones.

Remark 2.3. Note that, geometrically, the approximation of a Lipschitz surface by smooth ones, described above, is only local. By some additional tedious geometrical work, it is possible to arrange gluing together these local approximations, thus obtaining a globally approximating smooth surface. We do not need such construction since local considerations are sufficient for our needs.

We note finally here that the possibility of choosing the exponent q above to be arbitrarily large plays a crucial role in our considerations. Here we note just one important fact. For a family of approximating surfaces $\mathcal{S}_\epsilon$ over $\mathcal{D}$, parametrized by the functions ϕ_ϵ, we consider the measure densities σ_ϵ, defined similar to $\sigma(x) = \sigma_j(x)$ in (2.6), i.e.,

$$\sigma_\epsilon(x) = [\det(1 + (\nabla\phi_\epsilon)^*(\nabla\phi_\epsilon))]^{\frac{1}{2}} . \tag{2.12}$$

Proposition 2.4. *In the above approximation, the densities $\sigma_\epsilon(x)$ converge to $\sigma(x)$ in $L_q(\mathcal{D})$ for any $q < \infty$.*

Proof. The determinant in (2.12), fully expanded, is the sum of finitely many terms, each being a product of no more than $2\mathfrak{d}$ first order partial derivatives of components of the mapping ϕ_ϵ. For a given $q < \infty$, we can take $\hat{q} = 2\mathfrak{d}q < \infty$, and the convergence of $\operatorname{grad}\phi_\epsilon$ to $\operatorname{grad}\phi$ in $L_{\hat{q}}$ implies the convergence of $\det(1 + (\nabla\phi_\epsilon)^*(\nabla\phi_\epsilon))$ to $\det(1 + (\nabla\phi)^*(\nabla\phi))$ in L_q. Finally, since $\sigma_\epsilon(x) > 1$, this convergence is preserved under taking the square root. Quantitatively, this convergence can be expressed as the inequality

$$\|\sigma_\epsilon - \sigma\|_{L_q} \leq C \min(1, \|\nabla\phi_\epsilon - \nabla\phi\|_{L_{\hat{q}}}). \tag{2.13}$$

□

3. SPECTRAL ESTIMATES

In establishing eigenvalue asymptotics, it is highly important to prove first the eigenvalues (or s-numbers) estimates of correct order. The perturbation approach, developed by M. Birman and M. Solomyak, consists in finding the spectral asymptotics first for a certain class of nice operators, where this task may be rather simple, and then extending the formulas to a more general class performing a kind of "closure". This latter step requires having spectral estimates of correct order, under rather weak regularity conditions. This approach has been used for various problems, including, of course, weakly polar integral operators (see, especially, [3, 5, 6, 16, 17], and later [9, 13]). In this section we discuss eigenvalue estimates in our setting.

Having a surface of dimension d, the expected asymptotics of decay of s-numbers is $s_n \sim Cn^{-\gamma}$, $\gamma = 1 + m/d > 0$, where $m > -d$ is the homogeneity order of the kernel of the operator. In the smooth case this asymptotics was established, e.g., in [5, 6]. Note that the order of asymptotics does not involve the codimension of the surface. For 'almost smooth' Lipschitz, and then for general Lipschits surfaces of codimension $\mathfrak{d} = 1$, such asymptotic formulas with the same order were proved in [3] and [18]. We expect that the eigenvalue asymptotics is of the same order $n^{-\gamma}$, $\gamma = 1 + m/d$ for general Lipschitz surfaces with $\mathfrak{d} > 1$; to justify these expectation, we need order sharp spectral estimates.

In the task of finding spectral estimates for integral operators, the following two cases are naturally distinguished. For operators in the Hilbert-Schmidt class $\mathfrak{S}_2$ and worse, no smoothness is required for the kernel, and order sharp eigenvalue estimates are obtained usually from pointwise estimates for the kernel. This case, the one with rather strong singularity, corresponds to $\gamma < \frac{1}{2}$, or, in other terms, $-d < m < -d/2$. Eigenvalue estimates are proved here in a rather simple way, since the pointwise estimates survive under the restriction to Lipschitz surfaces. The opposite case deals with a weaker singularity of the kernel and, correspondingly, with operators with a faster decay of s-numbers, $\gamma \geq \frac{1}{2}$. Here the spectral estimates require a more advanced machinery, because the smoothness of the kernel is not preserved under passing to the local representation.

In application to our setting, the first case is resolved by the direct reference to the theorem that already exists.

Theorem 3.1. *Let the kernel $\mathcal{K}(X;Z)$ (defined for X in a neighborhood of $\mathcal{S}$) of the operator $\mathfrak{K}$ on a compact Lipschitz surface $\mathcal{S}$ is smooth in X and in $Z \in \mathbb{R}^{d+\mathfrak{d}} \setminus \{0\}$ and is positive-homogeneous in Z of order $m \in (-d, -d/2)$. Let φ, ψ be L_2 functions on $\mathcal{S}$ with compact support, with L_∞ norm not exceeding 1. Then the singular numbers $s_n(\varphi\mathfrak{K}\psi)$ of the operator $\varphi\mathfrak{K}\psi$ satisfy*

$$s_n(\varphi\mathfrak{K}\psi) \le Cn^{-\gamma} \max(\|\varphi\|, \|\varphi\|^\gamma) \max(\|\psi\|, \|\psi\|^\gamma), \quad \gamma = 1 + \frac{m}{d},$$

with constant C not depending on the functions φ, ψ. ($\|\cdot\|$ denotes the L_2 norm.)

Proof. Theorem 4.1 in [3] establishes this estimate for operators, acting on a surface $\mathcal{S}^1$ of codimension 1, having (in our notations) the form

$$\mathfrak{K}^1 u(X) = \int\limits_{\mathcal{S}^1} \varphi(X)\chi(X,Y)|X-Y|^m \psi(Y)U(Y)d\mu^1(Y), \tag{3.1}$$

with bounded function $\chi(X,Y)$. (Here μ^1 is the surface measure on $\mathcal{S}^1$.)

We will show that for the case of a surface $\mathcal{S}$ of codimension $\mathfrak{d}$, the operator $\varphi\mathfrak{K}\psi$ can be represented in the form (3.1). We choose a local co-ordinate system where a piece of the surface $\mathcal{S}$ is represented as $z = \phi(x)$, $x \in \mathcal{D}$. We separate one of z-coordinates, $z = (z_1, z')$, $\phi(x) = (\phi_1(x), \phi'(x))$. Now consider the projection of $\mathbb{R}^{d+\mathfrak{d}}$ to the subspace $\mathbb{R}^{d+1}$, $(x, z_1, z') \mapsto (x, z_1)$. Under this projection, the surface $\mathcal{S}$ projects, in bi-Lipschitzian way, onto the surface $\mathcal{S}_1 : z_1 = \phi_1(x)$; the surface $\mathcal{S}_1$ is, again, Lipschitz. Note also that the distance between points cannot increase under the projection, therefore $|(x,\phi(x)) - (y,\phi(y))|^m \le |(x,\phi_1(x)) - (y,\phi_1(y))|^m$ for close points x, y, and therefore, the kernel of the projected operator can be written in the form (3.1) with bounded $\chi(X,Y)$. It remains to apply Theorem 4.1 in [3]. □

It is important to note that in this, easy, case, we, in fact, used only the properties of the restriction of the kernel to the surface $\mathcal{S}$. In this way, we sacrificed the assumed smoothness of the initial kernel $\mathcal{K}$ in $\mathbb{R}^{d+\mathfrak{d}}$, since the restriction of a smooth function to a nonsmooth surface is not smooth in local co-ordinates.

The opposite case, the one of a weak singularity, $m \ge -\frac{d}{2}$ leads to operators with a faster rate of decay of s-numbers, as $n^{-\gamma}$, $\gamma \ge \frac{1}{2}$, i.e., to operators in some narrower Schatten classes than $\mathfrak{S}_2$. It is known that the membership to such classes requires, generally, certain smoothness of the kernel; the higher smoothness corresponding to a narrower class of operators, with a faster eigenvalue decay. Here, it is insufficient to consider the restriction of the kernel to the surface due to the loss of smoothness under such restriction. The approach elaborated in [3] and developed further in [18] enables us to exploit the intrinsic smoothness of the kernel in the whole of $\mathbb{R}^{d+\mathfrak{d}}$, or, more exactly, in the neighborhood of $\mathcal{S}$.

In [3] the following result on the singular numbers estimate for potential type operators on surfaces ($\mathfrak{d} = 1$) was proved (see Proposition 2.3 in [3]). We describe it in our present notations.

Theorem 3.2. *Let $\mathfrak{d} = 1$ and suppose that the kernel $\mathcal{K}(X;Z)$ of the operator $\mathfrak{K}$ on a compact Lipschitz surface $\mathcal{S}$ is smooth in X and in Z for $Z \neq 0$ and is positive-homogeneous in Z of order $m \geq -d/2$. Let φ, ψ be bounded functions on $\mathcal{D}$ with compact support. Then the singular numbers $s_n(\varphi\mathfrak{K}\psi)$ of the operator $\varphi\mathfrak{K}\psi$ satisfy*

$$\limsup_{n\to\infty} n^\gamma s_n(\varphi\mathfrak{K}\psi) \leq C \min(\mu_\varphi^\gamma, \mu_\psi^\gamma), \tag{3.2}$$

where $\gamma = 1 + \frac{m}{d}$ and μ_φ, μ_ψ are measures of the supports of the functions φ, ψ. The constant C is determined by the surface $\mathcal{S}$ and the homogeneity order of the kernel $\mathcal{K}$ but is independent of the cut-off functions φ and ψ.

Our generalization of this statement to operators on surfaces of co-dimension $\mathfrak{d}$ is the following.

Theorem 3.3. *Let the kernel $\mathcal{K}(X;Z)$ of the operator $\mathfrak{K}$ on a compact Lipschitz surface $\mathcal{S}$ of codimension $\mathfrak{d}$ in $\mathbb{R}^{d+\mathfrak{d}}$ is smooth in a neighborhood of $\mathcal{S}$ in X variable, smooth in Z for $Z \neq 0$ and positive-homogeneous in Z order $m \geq -d/2$. Suppose that φ, ψ are bounded functions on $\mathcal{S}$ with compact support. Then the singular numbers $s_n(\varphi\mathfrak{K}\psi)$ of the operator $\varphi\mathfrak{K}\psi$ satisfy* (3.2)*, again with $\gamma = 1 + \frac{m}{d}$.*

So, the order of decay in the singular numbers estimate is determined by the dimension of the surface and the homogeneity order, but not by the codimension $\mathfrak{d}$.

Proof. The reasoning goes similar to the one in [3]. The spectral estimates follow for the operator $H = \varphi\mathfrak{K}\psi$ from the variational principle via Lemma 2.1 in [3], which, being a version of the variational description of the singular numbers, states the following.

If H_N is a family of finite-rank operators such that $\operatorname{rank} H_N \leq c_1 N^\theta$ and $\|H - H_N\| \leq c_2 N^{-\alpha}$ for sufficiently large N, then

$$\limsup s_n(H) n^{\alpha/\theta} \leq c_2 c_1^{\alpha/\theta}.$$

We apply this Lemma, with H being the piece of our operator $\varphi\mathfrak{K}\psi$ in the local chart of $\mathcal{S}$ over $\mathcal{D}$, and the operator H_N taken as the integral operator with degenerate kernel $\varphi(X)\mathcal{K}_N(X,Y)\psi(Y)$ constructed in the following way. Let us write the kernel $\mathcal{K}(X; X-Y)$ as $\boldsymbol{\mathcal{K}}(X,Y)$. Suppose, for simplicity of notation, that the piece of surface (we still call it $\mathcal{S}$) is contained in a unit cube $\mathbf{Q}_0 \subset \mathbb{R}^{d+\mathfrak{d}}$. For a fixed N, set $h = N^{-1}$ and cut the cube $\mathbf{Q}_0$ into $N^{d+\mathfrak{d}}$ equal small cubes V^α with edge h, with multiindex $\alpha = (\alpha_1, \dots, \alpha_{d+\mathfrak{d}})$, $\alpha \in (1, \dots, N)^{d+\mathfrak{d}}$, numerating these small cubes. Some of these cubes intersect $\mathcal{S}$, others do not. Denote by Υ_N the set of cubes V^α that have a nontrivial intersection with $\mathcal{S}$. The Lipschitz condition implies that, for a certain constant c_0, at least one point of the cube $V^\alpha \in \Upsilon_N$ lies further than $c_0 h$ from $\mathcal{S}$. We denote this point X_α. We fix some nonnegative integer r, and for any cube $V^\alpha \in \Upsilon_N$, construct for $X \in V^\alpha$ the kernel

$$\boldsymbol{\mathcal{K}}_N(X,Y;V^\alpha) = \sum_{|\beta|\leq r} \boldsymbol{\kappa}_{\alpha,\beta}(Y)(X - X_\alpha)^\beta,\ X \in V^\alpha,\ Y \in \mathcal{S}, \tag{3.3}$$

where $\boldsymbol{\kappa}_{\alpha,\beta}$ are the Taylor coefficients of the function $\boldsymbol{\mathcal{K}}(X,Y)$ with respect to X variable, calculated at the point X_α. These coefficients depend on Y and may grow as Y approaches X^α, but in a controlled way:

$$|\boldsymbol{\kappa}_{\alpha,\beta}(Y)| \le C\mathrm{dist}\,(X_\alpha,\mathcal{S})^{m-|\beta|} \le C(c_0h)^{m-|\beta|}.$$

Now we compose the kernel $\boldsymbol{\mathcal{K}}_N(X,Y)$ using the kernels $\boldsymbol{\mathcal{K}}_N(X,Y;V^\alpha)$, by setting

$$\boldsymbol{\mathcal{K}}_N(X,Y) = \boldsymbol{\mathcal{K}}_N(X,Y;V^\alpha), X\in V^\alpha;\ \boldsymbol{\mathcal{K}}_N(X,Y) = 0, X \notin \bigcup_{V^\alpha\in\Upsilon_N} V^\alpha.$$

Actually, we do not need to define the kernel $\boldsymbol{\mathcal{K}}_N(X,Y)$ for X outside $\bigcup_{V^\alpha\in\Upsilon_N} V^\alpha$, since such X are not present in the action of our integral operators on the surface $\mathcal{S}$.

We take as the approximation H_N the integral operator with kernel $\varphi(X)\boldsymbol{\mathcal{K}}_N(X,Y)\psi(Y)$. This kernel is the sum of kernels of the form (3.3), supported on the sets $V^\alpha\times Q$, the sum being spread over the cubes $V^\alpha\in\Upsilon_N$. Each of these operators has rank not greater than the dimension of the space of polynomials in $d+\mathfrak{d}$ variables with degree $\le r$. We denote this rank by $\mathbf{r}=\mathbf{r}(d+\mathfrak{d},r)$. Therefore, the rank of the whole approximating operator H_N is not greater than $\mathbf{r}|\Upsilon_N|$. Based upon this formula and the rate of approximation of a function by the starting fragment of its Taylor expansion, the estimate for the norm of the $H-H_N$ was evaluated in [3], in the proof of Lemma 2.1, with the reasoning in the key Lemma 2.3 not depending on the codimension of $\mathcal{S}$. What, actually, might depend on the codimension $\mathfrak{d}$, is the quantity $|\Upsilon_N|$, entering into the expression for the rank of the approximating operator. Recall that $|\Upsilon_N|$ is defined above as the number of cubes V^α with edge $h=N^{-1}$, which have nontrivial intersection with $\mathcal{S}$. We will show now that, actually, the order of growth of $|\Upsilon_N|$ as $N\to\infty$ is the same for any codimension $\mathfrak{d}>0$. Having established this property, we can repeat all other reasoning in [3], arriving at the same result.

So, suppose that a cube V^α of the size N^{-1} intersects $\mathcal{S}$. This means that the whole cube lies inside the $\delta_N = N^{-1}\sqrt{d+\mathfrak{d}}$-neighborhood of some point in $\mathcal{S}$. Therefore, all the cubes in Υ_N lie inside the δ_N - neighborhood of $\mathcal{S}$ in $\mathbb{R}^{d+\mathfrak{d}}$. We denote the volume of this neighborhood by $\mathbf{v}(\delta_N)$ and thus $|\Upsilon_N|\le\delta_N^{-(d+\mathfrak{d})}\mathbf{v}(\delta_N)$. For a Lipschitz surface, this volume satisfies

$$\limsup_{\delta_N\to0}(\delta_N^{-\mathfrak{d}}\mathbf{v}(\delta_N)) \le C\mu_d(\mathcal{S}),$$

where $\mu_d(\mathcal{S})$ is the d-dimensional Hausdorff measure of $\mathcal{S}$ and the constant C depends on the Lipschitz constant of the surface $\mathcal{S}$. Therefore, the number of cubes in Υ_N satisfies

$$\begin{aligned}\limsup_{N\to\infty}(N^{-d}|\Upsilon_N|) &\le \limsup_{N\to\infty}((N^{-d}\delta_N^{-(d+\mathfrak{d})}\mathbf{v}(\delta_N))\\ &\le C\limsup_{N\to\infty}(\delta_N^{-\mathfrak{d}}\mathbf{v}(\delta_N)) \le C\mu_d(\mathcal{S}).\end{aligned} \tag{3.4}$$

And this is the estimate we need. □

The same reasoning establishes singular numbers estimates for integral operators with smooth kernel. These estimates were, in fact, already used in Section 2, when we declared that the smooth remainder term in the expansion (2.4) of the kernel has s-numbers, decaying faster than the ones for the leading term. Such statement, for operators acting on Lipschitz surfaces of codimension 1, was established, by means of the above piecewise-polynomial approximation procedure, in [3, Proposition 2.7]. In our case, the same reasoning applies, with the above crucial remark on the growth order of $|\Upsilon_N|$ in (3.4), showing that the rank of the approximating operator asymptotically, as $N \to \infty$, does not depend on the codimension of the surface.

4. ESTIMATES AT THE DIAGONAL

The spectral properties of the operator $\mathfrak{K}$ are determined by the properties of its localizations to smaller parts of the surface. The main contribution to the spectral behavior for the latter operators is made by the neighborhood of the diagonal $X = Y$, where the singularities lie. In the present section we study the rate of operator approximation near the diagonal when the Lipschitz surface is approximated by smooth ones. While following mostly the reasoning in [18], we indicate the places where this reasoning should be modified due to the change in codimension.

We consider a part $\mathcal{S} = \mathcal{S}_\phi \subset \mathbb{R}^{d+\mathfrak{d}}$ of the Lipschitz surface, the graph of a Lipschitz (vector-)function ϕ on a bounded domain $\mathcal{D} \subset \mathbb{R}^d$ and its smooth approximation $\mathcal{S}_\epsilon$ parametrized by the smooth function ϕ_ϵ.

Let $\mathcal{K}(X; Z)$, $(X; Z) \in \mathbb{R}^{d+\mathfrak{d}} \times (\mathbb{R}^{d+\mathfrak{d}} \setminus \{0\})$ be, as before, a kernel with homogeneity m, $m > -d$ in Z variable, smooth in all variables. Thus, the differentiation in X leads to a function with the same homogeneity order, and each differentiation in Z lowers the homogeneity order by 1, i.e., increases the singularity at the diagonal. We fix bounded measurable functions φ, ψ on $\mathcal{D}$ with compact support in $\mathcal{D}$, $\|\varphi\|_{L_\infty}, \|\psi\|_{L_\infty} \le 1$, and consider the operators $\varphi\mathfrak{K}\psi$ and $\varphi\mathfrak{K}_\epsilon\psi$ defined on the surfaces $\mathcal{S}$, $\mathcal{S}_\epsilon$, the graphs of the functions ϕ, ϕ_ϵ over $\mathcal{D}$, by the kernel $\varphi(x)\mathcal{K}(X; Z)\psi(Y)$, as in (2.7). In what follows, we need to compare the eigenvalues of these two operators. However, these operators act on different surfaces, this means, in different spaces. Therefore, to make them comparable, acting in the same space, we consider operators, unitary equivalent to $\varphi\mathfrak{K}\psi$ and $\varphi\mathfrak{K}_\epsilon\psi$, but written in the coordinates in $\mathcal{D}$, as in (2.10), with

$$(\varphi\mathbf{K}\psi u)(x) = \int\limits_{\mathcal{D}} \varphi(x)\psi(y)\sigma(x)\mathcal{K}(x, \phi(x); x - y, \phi(x) - \phi(y))\sigma(y)u(y)dy,$$

with $u(y)$ denoting $U(y, \varphi(y))$, $y \in \mathcal{D}$, and

$$(\varphi\mathbf{K}_\epsilon\psi u)(x) = \int\limits_{\mathcal{D}} \varphi(x)\psi(y)\sigma_\epsilon(x)\mathcal{K}(x, \phi_\epsilon(x); x - y, \phi_\epsilon(x) - \phi_\epsilon(y))\sigma_\epsilon(y)u(y)dy;$$

this time, with $u(y)$ denoting $U(y, \phi_\epsilon(y))$, $y \in \mathcal{D}$. Here $\sigma(x), \sigma_\epsilon(x)$ are defined in (2.6), (2.12). These two operators are unitary equivalent to $\varphi\mathfrak{K}\psi$ resp., $\varphi\mathfrak{K}_\epsilon\psi$, but they act now in one and the same space, just what we need. For a fixed $h > 0$, let $\rho_h(x, x-y) = \rho(x, x-y)$ (the subscript h will be omitted here, provided this does not cause a misunderstanding) be a bounded function, $|\rho(x, x-y)| \le 1$, vanishing for $|x-y| > h$. We define the kernel

$$\mathcal{K}^h = \varphi(x)\mathcal{K}(X; X-Y)\rho(x, x-y)\psi(y).$$

The corresponding operators in $L_2(\mathcal{D})$ are

$$\mathbf{K}^h u(x) = \int_{\mathcal{D}} \sigma(x)\mathcal{K}^h(x, \phi(x); x-y, \phi(x)-\phi(y))\sigma(y)u(y)dy \tag{4.1}$$

and

$$\mathbf{K}^h_\epsilon u(x) = \int_{\mathcal{D}} \sigma_\epsilon(x)\mathcal{K}^h(x, \phi_\epsilon(x); x-y, \phi_\epsilon(x)-\phi_\epsilon(y))\sigma_\epsilon(x)u(y)dy. \tag{4.2}$$

In this section we are interested in estimating the singular numbers (s-numbers) of the difference $\mathbf{K}^h - \mathbf{K}^h_\epsilon$. We recall that a compact operator $\mathbf{T}$ with s-numbers $s_n(\mathbf{T})$ belongs to the Schatten ideal $\mathfrak{S}_q$, $0 < q < \infty$, if the sum in $\|\mathbf{T}\|_q = (\sum s_n(\mathbf{T})^q)^{1/q}$ converges. For $q \ge 1$ the latter expression defines the norm in $\mathfrak{S}_q$; and $\|\mathbf{T}\|_q^q$ is a metric in $\mathfrak{S}_q, q < 1$. For $p < 2$ a convenient sufficient condition for the integral operator $\mathbf{T}$ with kernel $\mathcal{T}(x,y)$ to belong to $\mathfrak{S}_{p'}$ (as usual, $p' = \frac{p}{p-1}$) was found by G. Karadzhov [15], see also [20].

Lemma 4.1. *Let $\mathcal{T}(x,y)$ be the kernel of the operator $\mathbf{T}$, $\mathcal{T}^*(x,y) = \overline{\mathcal{T}(y,x)}$, and $\|\mathcal{T}\|_{p,p'}$ denote the expression*

$$\|\mathcal{T}\|_{p,p'} = \left(\left(\int |\mathcal{T}(x,y)|^p dx\right)^{p'/p} dy\right)^{1/p'}.$$

Suppose that both $\|\mathcal{T}\|_{p,p'}$ and $\|\mathcal{T}^*\|_{p,p'}$ are finite for some $p \in (1,2)$. Then $\mathbf{T} \in \mathfrak{S}_{p'}$ and

$$\|\mathbf{T}\|_{\mathfrak{S}_{p'}} \le C(\|\mathcal{T}\|_{p,p'}\|\mathcal{T}^*\|_{p,p'})^{1/2}. \tag{4.3}$$

Lemma 4.1 was established in [15] with *some* value of the constant C in (4.3). In [20] it was found that, actually, it is possible to take $C = 1$ as this constant, but the additional condition $\mathcal{T}(x,y) \in L_2$ was imposed; one can easily dispose of it (see, e.g., [11]).

Proposition 4.2. *For $pm < d$, $1 < p < 2$, the operator $\mathbf{K}_\epsilon - \mathbf{K}^h_\epsilon$ belongs to $\mathfrak{S}_{p'}$, and*

$$\|\mathbf{K}_\epsilon - \mathbf{K}^h_\epsilon\|_{\mathfrak{S}_{p'}} \le Ch^{-m+\frac{d}{p}}(\min(1, \|\nabla\phi - \nabla\phi_\epsilon\|_{L_{p'}}) + \|\phi - \phi_\epsilon\|_{L_\infty}).$$

Proof. In a rather lengthy proof, there is just one difference compared with Proposition 3.2 in [18]: due to the change of the codimension, the factors $\langle \operatorname{grad}\phi(x)\rangle$, $\langle \operatorname{grad}\phi_\epsilon(x)\rangle$ should pe replaced by $\sigma(x)$, resp., $\sigma_\epsilon(x)$ in all formulas. However, by our Proposition 2.4, the functions σ_ϵ converge to σ in any L_q, $q < \infty$, as $\epsilon \to 0$, and it is this convergence that is, actually, used in the proof of the estimate in [18]. □

As it is explained in Section 2, we must take care of one more case, the log-homogeneous kernels having the form

$$\mathcal{K}(X; X-Y) = \mathcal{Q}(X; X-Y)\log|X-Y|, \tag{4.4}$$

where $\mathcal{Q}(X;Z)$ a homogeneous polynomial in Z of degree m, for an nonnegative integer m. For a fixed h and some $\tau > 0$ to be chosen later, we introduce the kernel

$$\mathcal{K}_h(X; X-Y) = \rho(x, |x-y|)\mathcal{Q}(X; X-Y)(\log|X-Y| - \log(\tau h)) \tag{4.5}$$

and two operators $\mathbf{K}^h$, $\mathbf{K}^h_\epsilon$ defined by the formulas (4.1) and (4.2), with the kernels $\mathcal{K}^h$, given now by the expression (4.5).

Proposition 4.3. *Let $\nu < d/p, p \in (1,2)$. Then*

$$\|\mathbf{K}^h_\epsilon - \mathbf{K}^h\|_{\mathfrak{S}_{p'}} \leq Ch^{-\nu+\frac{d}{p}}(\min(1, \|\sigma-\sigma_\epsilon\|_{L_{p'}})(1+|\log\tau|) + \|\phi-\psi\|_{L_\infty}). \tag{4.6}$$

The proof, again, follows the proof of Proposition 3.3 in [18], with the remark concerning the change of the form of density, made in the previous proof. In [18], a special attention was given to the extra logarithmical term, which is treated now in the same way.

5. OPERATOR APPROXIMATION

The considerations to follow deal with the approximation of the kernels under study by degenerate kernels outside the $h-$ neighborhood of the diagonal $X = Y$ (note that h is still to be determined). We will modify somewhat the construction of finite rank approximating kernels, used in Theorems 3.1, 3.5 in [18], improving the estimates obtained there. We will apply the estimates obtained above in Section 3, however, now, to estimate the spectrum of the difference $\mathbf{K} - \mathbf{K}_h$, i.e., of operators on the projection of $\mathcal{S}, \mathcal{S}_\epsilon$ to $\mathbb{R}^d$, having kernels vanishing near the diagonal $x = y$. We need also to introduce a scaling parameter κ; further on, optimizing in κ, h will produce the eigenvalue estimates we need.

Let again $\mathcal{S} \equiv \mathcal{S}_\phi \subset \mathbb{R}^{d+\mathfrak{d}}$ be a piece of a Lipschitz surface, defined over the bounded domain $\mathcal{D} \subset \mathbb{R}^d$ as the graph of a Lipschitz function ϕ. Let $\mathcal{D}$ lie in the unit cube $Q_0 = [0,1]^d \subset \mathbb{R}^d$ having edges parallel to the co-ordinate axes in $\mathbb{R}^d$. We cover $\mathcal{D}$ by N^d small cubes

$$Q^\alpha = [(\alpha_1 - 1)b, \alpha_1 b) \times \cdots \times [(\alpha_d - 1)b, \alpha_d b)$$

with edge $b = 1/N$, $\alpha = (\alpha_1, \ldots, \alpha_d)$. We denote by $\daleth_N$ the set of cubes Q^α intersecting $\mathcal{D}$.

For any cube $Q = Q^\alpha \in \daleth_N$ we consider the "tower" of cubes of the form $\mathbf{Q} = Q \times (lb, (l+1)b)$, $l \in \mathbb{Z}^{\mathfrak{d}}$, in $\mathbb{R}^{d+\mathfrak{d}}$, over Q. Due to the Lipschitz property, for some constant C_0, there are no more than C_0 cubes in this tower intersecting the surface $\mathcal{S}$. Also, for any $\epsilon > 0$, there are no more than C_0 cubes in this tower intersecting

$\mathcal{S}_\epsilon \equiv \mathcal{S}_{\phi_\epsilon}$. We denote by Θ^Q the set of these two types of $d+\mathfrak{d}$-dimensional cubes, $|\Theta^Q| \le 2C_0$, and by Θ the union $\Theta = \bigcup_Q \Theta^Q$. Note that even for a fixed N, the set Θ may depend on ϵ, however the size of this set satisfies always the estimate

$$|\Theta| \le 2C_0 N^d.$$

We will denote the $d+\mathfrak{d}$-dimensional cubes in Θ by boldface, $\mathbf{Q} \in \Theta$, marking, if this is necessary, the dependence on N.

Again, due to the Lipschitz property, in each cube $\mathbf{Q} \in \Theta$ there exists a point lying controllably far away from $\mathcal{S}$, at least on the distance $c_0 N^{-1}$, with some constant c_0 depending only on the Lipschitz constant for $\mathcal{S}$. We denote this point $X_{\mathbf{Q}}$, and by $x_{\mathbf{Q}}$ we denote its projection on $\mathbb{R}^d$.

Let $\boldsymbol{\mathcal{K}}(X,Y) = \mathcal{K}(X; X-Y)$ be a homogeneous kernel of degree $m > -d$. For $\mathbf{Q} \in \Theta$ we define the polynomial in X variable:

$$\mathcal{P}_{\mathbf{Q}}(X,Y) = \sum_{|\beta| \le r} k_{\mathbf{Q},\beta}(Y)(X - X_{\mathbf{Q}})^\beta, \tag{5.1}$$

where the sum is the starting segment of the Taylor expansion of $\boldsymbol{\mathcal{K}}(X,Y)$ in X variable at the point $(X_{\mathbf{Q}}, Y)$

$$k_{\mathbf{Q},\beta}(Y) = \beta!^{-1} \partial_X^\beta \boldsymbol{\mathcal{K}}(X;Y) \restriction_{X = X_{\mathbf{Q}}} . \tag{5.2}$$

For any $Q \in \beth_N$ we denote by $\tilde{Q}$ the concentric cube with edge κb, where $\kappa > 1$ is a number to be specified later. Now we define the approximating degenerate kernel on $\mathcal{S}$

$$\mathcal{P}^N(X,Y) = \begin{cases} \mathcal{P}_{\mathbf{Q}}(x,\phi(x),y,\phi(y)), & x \in Q, y \notin \tilde{Q} \\ 0, & x \in Q, y \in \tilde{Q} \end{cases}, \ Q \in \Upsilon_N, \tag{5.3}$$

$X = (x,\phi(x)), Y = (y,\phi(y))$, Q is the projection of $\mathbf{Q}$. The kernel $\mathcal{P}^N(X,Y)$ is a polynomial in X variable in each set of the form $Q \times Q^0$ therefore it is degenerate, its rank is not greater that CN^d. This kernel vanishes for (x,y) in the set $\Sigma_N = \cup(Q \times \tilde{Q})$, which contains the $\kappa b(d+\mathfrak{d})^{-\frac{1}{2}}$-neighborhood of the diagonal $x = y$ (of course, for small N and large κ this set may cover the whole of $\mathcal{D} \times \mathcal{D}$, and then $\mathcal{P}^N$ vanishes everywhere). Similar to $\mathcal{P}^N$, we denote by $\boldsymbol{\mathcal{K}}^N$ the restricted kernel

$$\boldsymbol{\mathcal{K}}^N(X,Y) = \begin{cases} \boldsymbol{\mathcal{K}}(x,\phi(x),y,\phi(y)), & (x,y) \notin \Sigma_N, \\ 0, & (x,y) \in \Sigma_N. \end{cases} \tag{5.4}$$

Proposition 5.1. *Denote by* $\mathfrak{P}^N, \mathfrak{K}^N$ *the operators in* $L_2(\mathcal{S})$ *with kernels, resp.,* $\varphi\mathcal{P}^N\psi, \varphi\boldsymbol{\mathcal{K}}^N\psi$. *Then for* $r > d+m$,

$$\|\mathfrak{P}^N - \mathfrak{K}^N\| \le C\kappa^{m-r+d} N^{-m-d}. \tag{5.5}$$

(*Recall that* r *is the order of the approximating Taylor polynomial in* (5.2).)

Proof. Our construction differs only slightly from the one in [3], and the reasoning is similar to the proof of Proposition 2.5 there; we need however to trace down the dependence on the parameter κ. We pass to operators $\mathbf{P}^N$, $\mathbf{K}^N$ on $\mathcal{D}$. The difference of the kernels $\mathcal{P}^N(x,\phi(x),y,\phi(y)) - \boldsymbol{\mathcal{K}}^N(x,\phi(x);y-x,\phi(y)-\phi(x))$ is zero in Σ_N, and outside Σ_N it is the remainder in the Taylor expansion (5.1) of $\boldsymbol{\mathcal{K}}$, so, by homogeneity, we have

$$\left|\mathcal{P}^N(X,Y) - \boldsymbol{\mathcal{K}}^N(X,Y)\right| \le C|X-Y|^{m-r}N^{-r}.$$

Thus, the norm of the difference of the operators $\mathbf{P}^N - \mathbf{K}^N$ is estimated as

$$\|\mathbf{P}^N - \mathbf{K}^N\| \le CN^{-r} \sup_{y\in Q} \int\limits_{\mathcal{D}\cap\{|x-y|>\kappa b\}} |x-y|^{m-r}dx. \tag{5.6}$$

The latter integral is bounded by

$$CN^{-l}(\kappa b)^{m-r+d} \asymp C\kappa^{m-r+d}N^{m-d}, \tag{5.7}$$

which proves (5.5). □

We can also apply this estimate to the kernels $\boldsymbol{\mathcal{K}}_\epsilon^N$ and $\mathcal{P}_\epsilon^N$ defined similarly to $\boldsymbol{\mathcal{K}}^N$ and $\mathcal{P}^N$, with the function ϕ replaced by ϕ_ϵ.

Now we are ready for the main result on eigenvalue estimates for localized operators.

Proposition 5.2. *Let ϕ be a Lipschitz function and ϕ_ϵ be its smooth approximation on $\mathcal{D}$, defining surfaces $\mathcal{S}, \mathcal{S}_\epsilon$ so that*

$$\|\phi - \phi_\epsilon\|_{L_\infty} \le \epsilon, \quad \|\nabla\phi - \nabla\phi_\epsilon\|_{L_{p'}} \le \epsilon, \quad \|\sigma - \sigma_\epsilon\|_{p'} < \epsilon, \tag{5.8}$$

for some $p' > 2$, $pm > -d$. The kernel $\varphi\boldsymbol{\mathcal{K}}\psi$ defines operators $\mathfrak{K}$ on $\mathcal{S}$ and $\mathfrak{K}_\epsilon$ on $\mathcal{S}_\epsilon$. Let $\mathbf{K}$ and $\mathbf{K}_\epsilon$ be the corresponding operators in $L_2(\mathcal{D})$, under the unitary equivalence induced by projecting of $\mathcal{S}, \mathcal{S}_\epsilon$ to $\mathcal{D} \subset \mathbb{R}^d$. Then for the s-numbers of the difference $\mathbf{K} - \mathbf{K}_\epsilon$, for the following estimate holds

$$s_n(\mathbf{K} - \mathbf{K}_\epsilon) \le C\epsilon^\lambda n^{m/d-1}, \lambda = -m + d/p, \tag{5.9}$$

with constant,probably, depending on p but independent on ϵ, n.

Proof. We start by choosing an integer $r \ge 0$ such that $r - m - d$ is positive. With some κ, to be chosen later, we fix N and construct the approximating degenerate kernels $\mathcal{P}^N$ as in (5.3) and $\mathcal{P}_\epsilon^N$ in the same way, with ϕ replaced by ϕ_ϵ. These kernels, restricted to the surfaces $\mathcal{S}$ and $\mathcal{S}_\epsilon$, define finite rank operators $\mathbf{P}^N$ and $\mathbf{P}_\epsilon^N$ in $L_2(\mathcal{D})$. Consider the cut-off operators $\mathbf{K}^N$ and $\mathbf{K}_\epsilon^N$, the kernel of the latter defined as in the formula (5.4), with ϕ replaced by ϕ_ϵ. This construction generates the following decomposition of the difference $\mathbf{K} - \mathbf{K}_\epsilon$:

$$\begin{aligned}\mathbf{K} - \mathbf{K}_\epsilon = [(\mathbf{K}^N - \mathbf{P}^N) - (\mathbf{K}_\epsilon^N - \mathbf{P}_\epsilon^N)] + [\mathbf{P}^N - \mathbf{P}_\epsilon^N] \\ + [(\mathbf{K} - \mathbf{K}^N) - (\mathbf{K}_\epsilon - \mathbf{K}_\epsilon^N)] = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3.\end{aligned} \tag{5.10}$$

For both terms in $\mathbf{T}_1$, Proposition 5.1 can be applied, so the norm of $\mathbf{T}_1$ is estimated by

$$\|\mathbf{T}_1\| \leq C\kappa^{-m-r+d}N^{-m-d}. \tag{5.11}$$

The operator $\mathbf{T}_2$ has finite rank,

$$\operatorname{rank}\mathbf{T}_2 \leq CN^d. \tag{5.12}$$

The last term, the operator $\mathbf{T}_3$ has kernel vanishing for (x, y) outside Σ_N, i.e., outside some neighborhood of the diagonal, and we can apply Proposition 4.2 with the function $\rho(x, x-y)$ being the characteristic function of Σ_N and with $h = \kappa N^{-1}$. As a result, $\mathbf{T}_3$ belongs to the Shatten class $\mathfrak{S}_{p'}$ and $\|\mathbf{T}_3\|_{\mathfrak{S}_{p'}} \leq C\epsilon(\kappa N^{-1})^{m+d/p}$. This inequality implies

$$s_n(\mathbf{T}_3) \leq C\epsilon(\kappa N^{-1})^{m+d/p}n^{-1/p'}. \tag{5.13}$$

By Weyl's inequality (see, e.g., [12]), we have the following estimate for the s-numbers:

$$s_{n+k+1}(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3) \leq \|\mathbf{T}_1\| + s_{k+1}(\mathbf{T}_2) + s_{n+1}(\mathbf{T}_3), \tag{5.14}$$

for any $k, n \geq 0$. If we set here $k = CN^d$, where C is the constant in (5.12), then $s_{k+1}(\mathbf{T}_2)$ becomes zero. After substituting (5.11) and (5.13) into (5.14) we arrive at

$$s_{n+CN^d+1}(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3) \leq C\kappa^{m-k+d}N^{m-d} + C\epsilon(\kappa N^{-1})^{m+d/p}n^{-1/p'}. \tag{5.15}$$

We set now $n = N^d$. Then (5.15) gives

$$\begin{aligned} s_{(C+1)N^d+1}(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3) &\leq C\kappa^{m-k+d}N^{-m-d} + C\epsilon(\kappa N^{-1})^{m+d/p}N^{-d/p'} \\ &\leq C(\kappa^{m-k+d} + \epsilon\kappa^{m+d/p})N^{-m-d}. \end{aligned}$$

Now we can adapt the value of κ so that the coefficient in front of N^{-m-d} minimizes; we set $\kappa = \epsilon^{(r+d/p')^{-1}}$, and then this coefficient becomes $C\epsilon^{m+d/p}$. The resulting inequality is equivalent to (5.9). $\square$

A slight modification of the reasoning in the last two propositions takes care of the log-homogeneous kernels considered in Proposition 4.3.

Proposition 5.3. *Let ϕ be a Lipschitz function and ϕ_ϵ be its smooth approximation, so that* (5.8) *holds for some $p' > 2$, $pm > -d$ and the surfaces $\mathcal{S}$ and $\mathcal{S}_\epsilon$ in $\mathbb{R}^{d+\mathfrak{d}}$ be the graphs of ϕ and ϕ_ϵ over $\mathcal{D} \in \mathbb{R}^d$. Let the operators $\mathfrak{K}$ and $\mathfrak{K}_\epsilon$ be defined by a* log-*homogeneous kernel $\varphi(X)\mathcal{K}(X; X-Y)\psi(Y)$ described in* (4.4)*, on the surfaces $\mathcal{S}, \mathcal{S}_\epsilon$, moreover $\mathcal{Q}(X; X-Y)$ in* (4.4) *is a homogeneous polynomial of degree m. Let $\mathbf{K}$, $\mathbf{K}_\epsilon$ be the corresponding operators in $L_2(\mathcal{D})$. Fix some integer $r > m + d$. Then for the s-numbers of the difference $\mathbf{K} - \mathbf{K}_\epsilon$ the following estimate holds*

$$s_n(\mathbf{K} - \mathbf{K}_\epsilon) \leq C(\epsilon^\lambda(1 + |\log\epsilon|)n^{m/d-1} + n^{-r/d}), \quad \lambda = m + d/p. \tag{5.16}$$

Proof. The proof follows the lines of the reasoning in Propositions 5.1 and 5.2, however the construction of the approximating kernels $\boldsymbol{\mathcal{K}}^N$, $\boldsymbol{\mathcal{K}}_\epsilon^N$ is slightly different. As it was proposed in [3], we set, similar to (5.1), (5.3),

$$\mathcal{P}^N(X,Y) = \mathcal{Q}(X;X-Y)T_{l-1}^Q(X,Y), \quad X \in \mathbf{Q}, Y \notin \tilde{\mathbf{Q}}, \, \mathbf{Q} \in \Theta_N,$$

where $T_{l-1}^{\mathbf{Q}}(X, X-Y)$ is the Taylor polynomial of degree $r-1$ with respect to X for the function $\log|X-Y|$ with center point at $X_{\mathbf{Q}}$. For (X,Y) close to the diagonal, $(x,y) \in \Sigma_N$, we set

$$\mathcal{P}^N(X,Y) = -\mathcal{Q}(X, X-Y)\log N, \quad X \in \mathbf{Q}, Y \in \tilde{\mathbf{Q}}, \, \mathbf{Q} \notin \Theta_N.$$

Since $\mathcal{Q}(X;X-Y)$ is a polynomial in the second variable, the kernel $\mathcal{P}^N(X,Y)$ is degenerate, with rank not greater than CN^d. The same construction, with ϕ_ϵ replacing ϕ, defines the degenerate kernel $\mathcal{P}_\epsilon^N$. Let the kernel $\boldsymbol{\mathcal{K}}^N$ coincide with $\boldsymbol{\mathcal{K}}$ for x outside the set Σ_N and be equal to $\mathcal{P}^N$ in Σ_N, i.e. in the neighborhood of the diagonal. Similarly constructed, the kernel $\boldsymbol{\mathcal{K}}_\epsilon^N$ coincides with $\boldsymbol{\mathcal{K}}_\epsilon$ outside Σ_N and with $\mathcal{P}_\epsilon^N$ in Σ_N. We denote by $\mathbf{K}^N, \mathbf{K}_\epsilon^N$ the operators defined on $\mathcal{D}$ by the kernels $\boldsymbol{\mathcal{K}}^N, \boldsymbol{\mathcal{K}}_\epsilon^N$ restricted to $\mathcal{S}, \mathcal{S}_\epsilon$ and by $\mathbf{P}^N$, $\mathbf{P}_\epsilon^N$, respectively, the operators defined by the approximating kernels $\mathcal{P}^N, \mathcal{P}_\epsilon^N$ restricted to these surfaces. Again, as in Proposition 5.2, we estimate the norm of $\mathbf{K}^N - \mathbf{P}^N$. The kernel of this operator vanishes for (x,y) outside Σ_N. In the set Σ_N, so for $|X-Y| > \kappa N^{-1}$, we can use the estimate found in [3], see the formula (2.30) there,

$$|\mathcal{K}(X;X-Y) - \mathcal{P}^N(X,Y)| \le C\big(N^{-r}\log N + |X-Y|^{m-r}N^{-r}\big)$$

with r large enough, so that $-r+m+d<0$. Then the norm of $\mathbf{K}^N - \mathbf{P}^N$ can be estimated as

$$\|\mathbf{K}^N - \mathbf{P}^N\| \le C\left(N^{-l}\log N + N^{-l}\sup_{y\in\mathcal{D}} \int\limits_{\{|x-y|>\kappa b\}} |x-y|^{m-l}dx\right).$$

We evaluate the last integral as in (5.6), (5.7) and arrive at

$$\|\mathbf{P}^N - \mathbf{K}^N\| \le C(N^{-l}\log N + \kappa^{m-r+d}N^{-m-d}).$$

The same estimate holds for the difference $\mathbf{K}_\epsilon^N - \mathbf{P}_\epsilon^N$.

We pass now to the study of the operator $\mathbf{K} - \mathbf{K}_\epsilon$. We represent it as

$$\begin{aligned} \mathbf{K} - \mathbf{K}_\epsilon &= [(\mathbf{K}^N - \mathbf{P}^N) - (\mathbf{K}_\epsilon^N - \mathbf{P}_\epsilon^N)] + [\mathbf{P}^N - \mathbf{P}_\epsilon^N)] \\ &\quad + [(\mathbf{K} - \mathbf{K}^N) - (\mathbf{K}_\epsilon - \mathbf{K}_\epsilon^N)] = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 \end{aligned} \tag{5.17}$$

For the operator $\mathbf{T}_1$ we have just found the norm estimate,

$$\|\mathbf{T}_1\| \le C(N^{-r}\log N + \kappa^{-m-r+d}N^{-m-d}). \tag{5.18}$$

The operator $\mathbf{T}_2$ has finite rank, rank $\mathbf{T}_2 \le CN^d$. The operator $\mathbf{T}_3$ has exactly the form required by Proposition 4.3, see (4.5), with the function ρ being the characteristic function of Σ_N, $h = \sqrt{d+\mathfrak{d}}\kappa N^{-1}$, and $\tau = (\sqrt{d+\mathfrak{d}}\kappa)^{-1}$. Applying (4.6), we obtain

$$\|T_3\|_{\mathfrak{S}_{p'}} \le C\epsilon(1+|\log\kappa|)(\kappa N^{-1})^{m+d/p},$$

that implies

$$s_n(\mathbf{T}_3) \le C\epsilon(1+|\log\kappa|)(\kappa N^{-1})^{m+d/p} n^{-1/p'}. \tag{5.19}$$

Now the reasoning goes exactly as in the proof of Proposition 5.2, we only have to take care of the extra logarithm factor that spoils our estimates just slightly. We set again $j = CN^d$ in the Weyl inequality (5.14) written for the decomposition (5.17), and the term $s_{k+1}(\mathbf{T}_2)$ vanishes. After this, we substitute there the estimates (5.18), (5.19) and set $\kappa = \epsilon^{(-r+d/p)^{-1}}$, arriving at (5.16). □

6. EIGENVALUE ASYMPTOTICS

In this final section we will establish the asymptotic formula for eigenvalues of potential type operators on Lipschitz surfaces. Such formulas are well-known in the smooth case, see, e.g., [2] and references therein. The proof goes mostly in the same way as in the codimension 1 case in [18]. Therefore we just indicate the main steps, showing, of course, technical differences, as soon as they appear.

The starting point will be the known asymptotic formula for eigenvalues of a weakly polar integral operator on a smooth manifold. We will rewrite it in a form that does not involve a smooth structure on the manifold. Further on, we apply the asymptotic perturbation theorem by Birman-Solomyak, which enables one to pass to the limit in an asymptotic eigenvalue formula as soon as sufficiently general eigenvalue estimates are obtained. We show that under the approximation procedure described in this paper, the convergence of operators localized to a co-ordinate patch satisfies the conditions of this theorem, while it is also possible to pass to the limit in the expressions for the asymptotic coefficient. Finally, we show that it is possible to delocalize the problem, gluing together operators and asymptotic formulas in different co-ordinate patches to arrive at the final result, the asymptotic eigenvalue formula on the Lipschitz surface of an arbitrary codimension.

If $\mathfrak{a}_0(x,\xi)$, $(x,\xi) \in T^*\mathcal{S}$, is the principal symbol of the classical self-adjoint pseudodifferential operator $\mathfrak{K}$ of order $l = -(m+d) < 0$ on a smooth d-dimensional surface $\mathcal{S}$ and f is a real bounded measurable function on $\mathcal{S}$, then the positive (negative) eigenvalues $\lambda_n^\pm$ of $f\mathfrak{K}f$ satisfy

$$\lim_{n\to\infty} \lambda_n^\pm n^{1+\frac{m}{d}} = \mathbf{C}^\pm, \tag{6.1}$$

with the coefficients $\mathbf{C}^\pm$ expressed in the terms of the function f and the principal symbol of the operator $\mathfrak{K}$.

There are different ways to write down the expression for the coefficients $\mathbf{C}^{\pm}$ in (6.1). We need a form which would make sense also for a Lipschitz surface $\mathcal{S}$. Our starting point will be the standard expression,

$$(\mathbf{C}^{\pm})^{\frac{d}{m+d}} = (2\pi)^{-d}\mathrm{vol}\,\{(x,\xi) \in T^*\mathcal{S} : \pm|f(x)|^2\mathfrak{a}_0(x,\xi) > 1\}. \tag{6.2}$$

Here vol denotes the invariant measure on the cotangent bundle $T^*\mathcal{S}$ which is a smooth manifold. The eigenvalue asymptotics with coefficient (6.2) was proved in [6], for operators in a domain in the Euclidean space; the possibility of carrying over to manifolds was mentioned there as well. A detailed presentation for the case of manifolds appeared first in [13], and an alternative proof was given in [9] (without the weight function f).

The expression (6.2) for the coefficient in the asymptotics makes no sense for a Lipschitz surface since here the cotangent bundle does not possess a manifold structure and no reasonable measure on $T^*\mathcal{S}$ is defined. Therefore we need a modification of the formula (6.2) that would involve only the measure μ on $\mathcal{S} \subset \mathbb{R}^{d+\mathfrak{d}}$ induced by the Lebesgue measure in $\mathbb{R}^{d+\mathfrak{d}}$; this measure is discussed in Section 2. For a point $X_0 \in \mathbb{R}^{d+\mathfrak{d}}$, $X_0 \in \mathcal{S}$, we denote by $\mathbf{N}_{X_0}$ the normal plane to $\mathcal{S}$ at X_0, i.e., the subspace of $\zeta \in \mathbb{R}^{d+\mathfrak{d}}$ orthogonal to the tangent space to $\mathcal{S}$ at X_0.

Let $\mathcal{S}$ be locally the graph of a (still smooth) function ϕ. Then the tangent space $T_{X_0}(\mathcal{S})$, can be identified with the cotangent space $T^*_{X_0}(\mathcal{S})$ via the standard Euclidean structure in $\mathbb{R}^{d+\mathfrak{d}}$. We consider the Euclidean structure on $T_{X_0}(\mathcal{S})$ together with the measure $d\xi$, induced on $T_{X_0}(\mathcal{S})$ by the Lebesgue measure in $\mathbb{R}^{d+\mathfrak{d}}$. Then in the expression for the coefficient in (6.2) we can perform the integration in ξ variable for X fixed, in polar co-ordinates in each cotangent space $T^*_X(\mathcal{S})$, and then in X over $\mathcal{S}$ which gives

$$\begin{aligned}(\mathbf{C}^{\pm})^{d/(d+m)} &= (2\pi)^{-d}\int_{\mathcal{S}} d\mu(X) \int_{T_X(\mathcal{S})\cap\{\pm|f(x)|^2\mathfrak{a}_0(x,\xi)>1\}} 1 d\xi \\ &= (2\pi)^{-d}d^{-1}\int_{\mathcal{S}} \mathbf{w}^{\pm}(X)d\mu(X),\end{aligned} \tag{6.3}$$

with

$$\mathbf{w}^{\pm}(X) = \int_{S_{X_0}(\mathcal{S})} (|f(x)|^2(\mathfrak{a}_0(X,\omega))_{\pm})^{d/(m+d)}d\omega, \tag{6.4}$$

where the integration is performed over the ($d-1$-dimensional) tangent sphere to the surface $\mathcal{S}$ at the point X_0. We need another expression for the latter integral. To derive it, we represent the above tangent sphere as $S^{d+\mathfrak{d}-1} \cap (\mathbf{N}_{X_0}(\mathcal{S}))^{\perp}$, i.e., as the set of unit tangent vectors which are orthogonal to the normal plane at X_0; this gives

$$\mathbf{w}^{\pm}(X) = \int_{S^{d+\mathfrak{d}-1}\cap\mathbf{N}_{X_0}(\mathcal{S})^{\perp}} (|f(x)|^2(\mathfrak{a}_0(X,\omega))_{\pm})^{d/(m+d)}d\omega, \tag{6.5}$$

where $d\omega$ is the measure on $S^{d+\mathfrak{d}-1} \cap (\mathbf{N}_{X_0}(\mathcal{S}))^{\perp}$ induced from $\mathbb{R}^{d+\mathfrak{d}}$.

The expression (6.3), (6.5) makes now sense for any Lipschitz surface, because the normal plane $\mathbf{N}_{X_0}(\mathcal{S})$ and the tangent space $T_{X_0}(\mathcal{S})$ exist almost everywhere on $\mathcal{S}$ and the symbol $\mathfrak{a}_0(X,\omega)$ makes also sense almost everywhere, being not related to any pseudodifferential operator on $\mathcal{S}$ any more. The direction of the normal plane $\mathbf{N}_{X_0}$ is involved into (6.5) because it defines the integration set. Additionally, as it was explained in Section 2, this normal plane determines the symbol $\mathfrak{a}_0(X,\omega)$ by (2.8). We emphasize here that the expression in (6.5) depends on $\mathbf{N}_{X_0}$ and thus on $\nabla\phi(x_0)$ continuously.

As soon as the eigenvalue estimates for the approximation of the potential type operators are obtained, proving the eigenvalue asymptotic formulas is now a routine matter. The general approach to this kind of problems originated in [5] and since then it has been the main (probably, the only) way for proving eigenvalue asymptotic formulas for non-smooth problems. The approach is based upon the following perturbation lemma, see [5].

Lemma 6.1. *Let the self-adjoint compact operator* $\mathbf{T}$ *admit a decomposition for every* ϵ*:*

$$\mathbf{T} = \mathbf{T}_\epsilon + \mathbf{T}'_\epsilon,$$

so that for some $q > 0$*, the positive and negative eigenvalues* $\lambda_n^\pm(\mathbf{T}_\epsilon)$ *of the operator* $\mathbf{T}_\epsilon$ *follow the asymptotical law*

$$\lambda_n^\pm(\mathbf{T}_\epsilon) \sim \mathbf{C}_\epsilon^\pm n^{-\gamma}, \quad n \to \infty, \tag{6.6}$$

and the term $\mathbf{T}'_\epsilon$ *is small in the sense*

$$\limsup_{\nu\to\infty} s_n(\mathbf{T}'_\epsilon)n^\gamma \le \varphi(\epsilon); \quad \varphi(\epsilon) \to 0 \text{ as } \epsilon \to 0. \tag{6.7}$$

Then the coefficients $\mathbf{C}_\epsilon^\pm$ *have limits* $\mathbf{C}^\pm$ *as* $\epsilon \to 0$ *and*

$$\lambda_n^\pm(\mathbf{T}) \sim \mathbf{C}^\pm n^{-\gamma}, \quad n \to \infty. \tag{6.8}$$

We will apply Lemma 6.1 twice, first to justify the eigenvalue asymptotics for localized operators and then to pass from localized to global operators on the surface.

Proposition 6.2. *Let* $\mathcal{S}$ *be a Lipschitz surface in* $\mathbb{R}^{d+\mathfrak{d}}$*,* $\mathcal{S} \equiv \mathcal{S}_\phi = \{(x,\phi(x)) : x \in \mathcal{D}\}$*, where* $\mathcal{D}$ *is a bounded domain in* $\mathbb{R}^d$ *and* ϕ *is a Lipschitz function* $\phi : \mathcal{D} \to \mathbb{R}^{\mathfrak{d}}$*. Let* A *be a classical self-adjoint pseudodifferential operator of order* $l - \mathfrak{d}$*,* $l < 0$ *in* $\mathbb{R}^{d+\mathfrak{d}}$ *with principal symbol* $a_0(X,\Xi)$*. For a bounded measurable function* f *on* $\mathcal{D}$*, supported in a compact in* $\mathcal{D}$*, we consider the potential type operator* $f\mathbf{K}f$ *in* $L_2(\mathcal{D})$*, where* $\mathbf{K}$ *is the operator generated by* A *on* $\mathcal{S}_\phi$*. Then the eigenvalues* $\lambda_n^\pm(f\mathbf{K}f)$ *satisfy the asymptotic formula*

$$\lim_{n\to\infty} \lambda_n^\pm(f\mathbf{K}f)n^\gamma = \mathbf{C}^\pm, \tag{6.9}$$

where $\mathbf{C}^\pm$ *is defined in* (6.3)*, the density* $\mathbf{w}^\pm(X)$ *is given by* (6.5)*, and the symbol* $\mathfrak{a}_0(X,\omega)$ *is expressed via* $A_0(X,\Xi)$ *as*

$$\mathfrak{a}_0(X,\omega) = (2\pi)^{-\mathfrak{d}} \int\limits_{\mathbf{N}_X(\mathcal{S})} a_0(X,\omega+\zeta)d\zeta. \tag{6.10}$$

Proof. We are going to apply Lemma 6.1. In our case the role of the operator $\mathbf{T}$ will be played by $f\mathbf{K}f$ and the approximations $\mathbf{T}_\epsilon$ will be the operators $f\mathbf{K}_\epsilon f$ in $L_2(\mathcal{D})$ corresponding to the smooth functions ϕ_ϵ approximating ϕ, $\|\phi - \phi_\epsilon\|_{L_\infty} < \epsilon$, $\|\nabla\phi - \nabla\phi_\epsilon\|_{L_{p'}} < \epsilon$, $1 < p < 2, p < d/(d+m)$, $\|\sigma - \sigma_\epsilon\|_{L_{p'}} < \epsilon$. For the operators $f\mathbf{K}_\epsilon f$ the spectral asymptotics of the form (6.6) is known from [6,13]. The coefficients $\mathbf{C}_\epsilon^\pm$ in (6.6) are given by

$$(\mathbf{C}_\epsilon^\pm)^{-\frac{1}{\gamma}} = (2\pi)^{-d} d^{-1} \int\limits_{\mathcal{S}_\epsilon} \mathbf{w}_\epsilon^\pm(X) d\mu_\epsilon(X), \tag{6.11}$$

where the integration is performed over the surface $\mathcal{S}_\epsilon$, the graph of ϕ_ϵ over $\mathcal{D}$, with the corresponding measure $d\mu_\epsilon$. The density $\mathbf{w}_\epsilon^\pm$ is defined as in (6.5) with ϕ replaced by ϕ_ϵ and the normal plane $\mathbf{N}_X(\mathcal{S})$ replaced by $\mathbf{N}_X(\mathcal{S}_\epsilon)$.

We consider the difference $\mathbf{T}'_\epsilon = \mathbf{T} - \mathbf{T}_\epsilon$. Fix some M, large enough. Due to the representation (2.4), we split the integral kernel $\mathcal{H}(X,Y) = \mathcal{K}(X; X-Y)$ of the operator $\mathbf{A}$ as

$$\mathcal{H}(X,Y) = \mathcal{H}^0(X,Y) + \mathcal{H}^1(X,Y) + \mathcal{R}^M(X,Y), \tag{6.12}$$

where $\mathcal{H}^0(X,Y) = \mathcal{K}_0(X; X-Y)$ (or $\mathcal{K}_{0,\log}(X; X-Y)\log|X-Y|$) is the leading term in (2.4), $\mathcal{H}^1(X,Y)$ is the sum of the terms in (2.4) with $d+m-1 \le \nu \le M$ and $\mathcal{R}^M(X,Y)$ is the remainder term in (2.4). With accordance to this decomposition, operators $\mathbf{T}$ and $\mathbf{T}_\epsilon$ also split into three terms each, which produces the representation of $\mathbf{T}'_\epsilon$,

$$\mathbf{T}'_\epsilon = \mathbf{S}^0 + \mathbf{S}^1 + \mathbf{R}. \tag{6.13}$$

The operator $\mathbf{R}$ is the difference of operators generated by C^M - continuous kernels on $\mathcal{S}$, $\mathcal{S}_\epsilon$. As explained in Section 3, operators with such kernels have s-numbers decaying faster than $n^{-M/d}$, so, faster than $n^{-\gamma}$, if M is chosen large enough. The second term in (6.13) is the sum of a finite number of operators with kernels having the order of homogeneity higher than $m+d$. By Theorems 3.1, 3.3, the s-numbers of these operators decay, again, faster than $n^{-\gamma}$. Finally, the main term $\mathbf{S}^0$ in (6.13), is the difference of operators generated by the kernel with homogeneity (or log-homogeneity) of order m on surfaces $\mathcal{S}$, $\mathcal{S}_\epsilon$. By Propositions 5.2, 5.3, the s-numbers of this difference are estimated by $n^{-\gamma}$ with a small constant, tending to zero together with ϵ. By Lemma 6.1, these estimates imply the asymptotics (6.8), with coefficients $\mathbf{C}^\pm$ which are limits of $\mathbf{C}_\epsilon^\pm$ as $\epsilon \to 0$. It remains to notice that the densities $\mathbf{w}_\epsilon(X) = \mathbf{w}(x, \phi_\epsilon(x))$ entering in the formula (6.11) as well as the densities of the measures $d\mu_\epsilon = \sigma_\epsilon(x)dx$ converge respectively to the densities $\mathbf{w}(x, \phi(x))$ and $\sigma(x)$ at all points $x \in \mathcal{D}$ where $\nabla\psi_\epsilon(x)$ converges to $\nabla\phi(x)$, moreover, converge in any $L_q, q < \infty$, i.e., almost everywhere in $\mathcal{D}$. Since the gradients of ϕ_ϵ are uniformly bounded, the dominated convergence theorem implies that one can pass to the limit under the integral in (6.11), obtaining the expression (6.3), (6.5) for the asymptotical coefficients in (6.9). □

In order to prove the global asymptotic formula, we need one more, rather simple, spectral estimate showing a sort of locality property for potential type operators.

Proposition 6.3. *Let $\mathcal{S}$ be a compact Lipschitz surface in $\mathbb{R}^{d+\mathfrak{d}}$, $\mathfrak{K}$ a potential type operator on $\mathcal{S}$ and φ, ψ be bounded measurable functions on $\mathcal{S}$ such that the distance δ between supports of φ and ψ is positive. Then the s-numbers of the operator $\varphi\mathfrak{K}\psi$ satisfy*

$$s_n(\varphi\mathfrak{K}\psi) = o(n^{-k}) \text{ for any } k. \tag{6.14}$$

The property is established very similar to the one in [3] or [18], where the case of codimension 1 was considered. In the same way as in Theorems 3.1, 3.3, a piecewise polynomial finite rank approximation to the operator $\varphi\mathfrak{K}\psi$ is constructed, however, this time, since the distance between the supports of φ, ψ is positive, the support of the integral kernel of the operator lies on the positive distance from the diagonal, and therefore, the blow-up of the derivatives at the diagonal does not happen.

Now we can establish our global theorem on the spectral asymptotics.

Theorem 6.4. *Let A be a pseudodifferential operator of order $l - \mathfrak{d}$, $l < 0$, with principal symbol $a_0(X, \Xi)$ and $\mathcal{S}$ be a compact Lipschitz surface in $\mathbb{R}^{d+\mathfrak{d}}$. For a bounded measurable function f defined on $\mathcal{S}$ we consider the potential type operator $f\mathfrak{K}f$. Then the eigenvalues of $f\mathfrak{K}f$ satisfy the asymptotic formula*

$$\lim_{n\to\infty} \lambda_n^{\pm} n^{-\gamma} = \mathbf{C}^{\pm}, \gamma = \frac{d+m}{d} = -\frac{l}{d} \tag{6.15}$$

with coefficients $\mathbf{C}^{\pm}$ given by (6.3), (6.5), (6.10).

Proof. As soon as the local version of the Theorem and the locality property are established, the proof is rather standard. Consider a covering of $\mathcal{S}$ by co-ordinate patches Ω_j so that each part $\mathcal{S}_j = \mathcal{S} \cap \Omega_j$ is the graph of a Lipschitz function ϕ_j in proper local co-ordinates in Ω_j. For any fixed ϵ we can find closed sets $\overline{\Omega_j^{\epsilon}} \subset \Omega_j$ so that $\overline{\Omega_j^{\epsilon}}$ are disjoint, mutual distances are all positive and the measure of $\mathcal{S} \setminus \cup\Omega_j$ is smaller than ϵ. Let φ_j be the characteristic function of $\overline{\Omega_j^{\epsilon}}$, $F = \sum \varphi_j$, $G = 1 - F$. Consider the operator

$$\mathfrak{T}_\epsilon = \sum_j f\varphi_j\mathfrak{K}\varphi_j f = \sum \mathfrak{T}^{(j)}. \tag{6.16}$$

Since the functions φ_j have disjoint supports, the operator $\mathfrak{T}_\epsilon$ in representation (6.16) is the direct sum of operators $\mathfrak{T}^{(j)}$. We can consider each $\mathfrak{T}^{(j)}$ as the local operator on the piece $\mathcal{S}_j$ of the surface and apply Proposition 6.2, thus obtaining an asymptotic formula for the eigenvalues of each $\mathfrak{T}^{(j)}$. Since the spectrum of the direct sum of operators is the union of the spectra of summands, this gives the asymptotic formula for the eigenvalues of $\mathfrak{T}_\epsilon$. Next we consider the difference, the operator

$$\mathfrak{T}'_\epsilon = f\mathfrak{K}f - \mathfrak{T}_\epsilon.$$

We represent it as

$$\mathfrak{T}'_\epsilon = \sum_{i\neq j} f_i f\mathfrak{K}f_j f + Gf\mathfrak{K}fF + Ff\mathfrak{K}fG + Gf\mathfrak{K}fG. \tag{6.17}$$

The s-numbers of the first term in (6.17) decay fast by Proposition 6.3. For three remaining terms the s-numbers satisfy the condition of the form (6.7), as was established in [3], see Lemma 2.3 there. These two facts, via Lemma 6.1, produce our result. □

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