Evolution of Time-Harmonic Electromagnetic and Acoustic Waves Along Waveguides

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Integral Equations and Operator Theory

Evolution of Time-Harmonic Electromagnetic and Acoustic Waves Along Waveguides

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Abstract. We study time-harmonic electromagnetic and acoustic waveguides, modeled by an infinite cylinder with a non-smooth cross section. We introduce an infinitesimal generator for the wave evolution along the cylinder and prove estimates of the functional calculi of these first order non-self adjoint differential operators with non-smooth coefficients. Applying our new functional calculus, we obtain a one-to-one correspondence between polynomially bounded time-harmonic waves and functions in appropriate spectral subspaces.

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1. Introduction

A linear partial differential equation, PDE, or a system of PDEs, is often analyzed by studying the evolution of solutions u with respect to one of the variables, say t. Recall that if the PDE is of second or higher order, then we can rewrite it as a system of first order equations, so without loss of generality we can assume that the PDE only contains first order derivatives in t. In this way the PDE becomes a vector-valued ordinary differential equation, ODE, like

$$\partial_t u(t,x) + Tu(t,x) = 0 \tag{1.1}$$

in the homogeneous case. Here T, an infinitesimal generator, is a differential operator acting in the remaining variables x only, for each fixed t.

Formally solutions to (1.1) are given by

$$u(t,x) = (\exp(-tT)u(0,\cdot))(x). \tag{1.2}$$

Andreas Rosén was formerly named Andreas Axelsson.

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However, since T is an unbounded operator, we need to be careful in the definition and analysis of such a solution operator $\exp(-tT)$. The heuristics are as follows. For a parabolic equation, say the heat equation, T is the positive Laplace operator, and $\exp(-tT)$ is a well defined bounded operator for any $t \geq 0$ and any initial function. For a hyperbolic equation, say the wave equation as a first order system, T is skew symmetric and $\exp(-tT)$ is unitary and well defined for any $-\infty < t < \infty$ and any initial function. For an elliptic equation, say the Cauchy–Riemann system, T is symmetric but with spectrum running from $-\infty$ to $+\infty$. In this case we need to split the function space for initial data as a direct sum of two Hardy subspaces. Then $\exp(-tT)$ is well defined and bounded for $t > 0$ when the initial data is in one of the Hardy subspaces, and for $t < 0$ when the initial data is in the other Hardy subspace.

The aim of the present paper is to study infinitesimal generators T arising as above in the elliptic case. Our motivation comes from the theory for waveguides, and our results yield a powerful mathematical representation of time-harmonic waves propagating along waveguides with general non-smooth materials. The waveguide is modeled by the unbounded region $\mathrm{R}\times\Omega$, where Ω is a bounded domain in R^2, or more generally in R^n. Note that we study time-harmonic waves. Therefore the PDE is elliptic rather than hyperbolic, and t is not time but rather the spatial variable along the waveguide. For an acoustic waveguide, the PDE is of Helmholtz type, as in Sect. 2.1, with coefficients which we allow to vary non-smoothly over the cross section Ω, but they are homogeneous along the waveguide. For an electromagnetic waveguide, the system of PDEs is Maxwell's equations as we describe in Sect. 2.2.

We show in Sect. 2 that the infinitesimal generators T arising in this way when studying waveguide propagation are of the form

$$T = (D_1 + D_0)B, \tag{1.3}$$

where D_1 is a self-adjoint first-order differential operator, D_0 is a normal bounded multiplication operator, and B is a bounded accretive operator depending on the material properties of the cross section of the waveguide. With such variable coefficients, the operator T will not be self-adjoint. Even in the static case $D_0 = 0$, T is only a bi-sectoral operator (see [3]), and $L^2(\Omega)$ bounds of $\exp(-tT)$ and more general functions $f(T)$ of T, are non-trivial matters. However, in the general non-smooth case, this is well understood from the works of Axelsson et al. [5] and Auscher et al. [4]. In the present paper we extend these results to the case $D_0 \neq 0$ which occurs in general time-harmonic, but non-static, wave propagation in waveguides.

In Sect. 3 we study functional calculi of operators of the form (1.3), which we show have $L^2(\Omega)$ spectra contained in regions

$$S_{\omega,\tau} := \{x + iy \in \mathrm{C} : |y| < |x| \tan\omega + \tau\}.$$

To have a theory for general frequencies of oscillation, encoded by the zero-order term D_0, it is essential to require the cross section Ω to be bounded, which ensures that the spectrum is discrete. However, the compactness of resolvents and the discreteness of spectrum only holds for T in the range of

D_1+D_0, which is invariant under T. Building on fundamental quadratic estimates (see [1]) for operators T in the static case, we are able to construct and prove $L^2(\Omega)$ estimates of a generalised Riesz–Dunford functional calculus of T. To yield a well defined and bounded operator $f(T)$, the symbol $f(z)$ is required to be uniformly bounded and holomorphic on an open neighbourhood of the spectrum of T except at ∞, where it is only required to be bounded and holomorphic on a bi-sector $|y| \le \tan\omega|x|$, $\omega < \pi/2$, in a neighbourhood of ∞. Due to the deep quadratic estimates from harmonic analysis used in Proposition 3.16, this suffices to bound $f(T)$ at ∞.

Another novelty in estimating $f(T)$, due to the non-self adjointness of T, is that $\|f(T)\|$ may depend not only on $|f(\lambda)|$, but also on a finite number of derivatives $f^{(k)}(\lambda)$ at a given eigenvalue λ of T. In particular, an eigenvalue of T on the imaginary axis with index/algebraic multiplicity greater than 1, will result in propagating waves $u_t = \exp(-tT)u_0$ which grow polynomially.

Note that since the spectrum is discrete, a symbol like

$$f(z) = \begin{cases} e^{-tz}, & \text{if } \mathrm{Re} z > a, \\ 0, & \text{if } \mathrm{Re} z \le a, \end{cases}$$

for $t > 0$, is admissible provided no eigenvalue lies on $\mathrm{Re} z = a$, and will yield an operator bounded on $L^2(\Omega)$. In this sense the functional calculus that we here construct is more general than that considered by Morris in [11].

In the final Sect. 4, we apply our new functional calculus for operators T to show how all polynomially bounded time-harmonic waves in the semi- or bi-infinite waveguide can be represented like (1.2), with u_0 in appropriate spectral subspace for T.

2. Partial Differential Equations Expressed as Vector-Valued Ordinary Differential Equations

In this section we consider the Helmholtz and Maxwell's equations and express them as vector-valued ordinary differential equations in terms of operator DB, which is introduced later.

Throughout this paper $\Omega = \Omega^+ \subset \mathrm{R}^n$ denotes a bounded open set, separated from the exterior domain, $\Omega^- = \mathrm{R}^n \setminus \Omega$, by a weakly Lipschitz interface $\Gamma = \partial\Omega$, defined as follows.

Definition 2.1. The interface Γ is weakly Lipschitz if, for all $y \in \Gamma$, there exists a neighbourhood $V_y \ni y$ and a global bilipschitz map $\rho_y : \mathrm{R}^n \to \mathrm{R}^n$ such that

$$\Omega^\pm \cap V_y = \rho_y\left(\mathrm{R}^n_\pm\right) \cap V_y,$$
$$\Gamma \cap V_y = \rho_y\left(\mathrm{R}^{n-1}\right) \cap V_y,$$

where $\mathrm{R}^n_+ = \mathrm{R}^{n-1} \times (0, +\infty)$ and $\mathrm{R}^n_- = \mathrm{R}^{n-1} \times (-\infty, 0)$. In this case Ω is called a weakly Lipschitz domain.

We will use the symbols $\mathbf{D}(\cdot)$, $\mathbf{N}(\cdot)$, and $\mathbf{R}(\cdot)$ to denote the domain, null space, and range of an operator, respectively.

2.1. The Helmholtz Equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded weakly Lipschitz domain and $A \in L_\infty\left(\Omega; \mathcal{L}\left(\mathbb{C}^{n+2}\right)\right)$ be t-independent and pointwise strictly accretive in the sense that there exists $\alpha > 0$ such that

$$\mathrm{Re}(A(x)v, v) \geq \alpha \|v\|^2 \tag{2.1}$$

for all $x \in \mathbb{R}^n$ and $v \in \mathbb{C}^{n+2}$. For a complex number $k \neq 0$, we consider the equation

$$\begin{bmatrix}\mathrm{div}_{(t,x)} & k\end{bmatrix} A \begin{bmatrix}\nabla_{(t,x)} \\ k\end{bmatrix} u = 0 \tag{2.2}$$

in $\Omega \times \mathbb{R}$ with $u \in H_0^1(\Omega)$ for all $t \in \mathbb{R}$.

Let us set

$$H_{\mathrm{div}}(\Omega; \mathbb{C}^n) := \{f \in L_2(\Omega; \mathbb{C}^n) : \ \mathrm{div} f \in L_2(\Omega)\}.$$

By div and ∇_0, we denote the divergence and gradient operators on $H_{\mathrm{div}}(\Omega)$ and $H_0^1(\Omega)$ respectively.

Splitting $\mathbb{C}^{n+2}$ into $\mathbb{C}$ and $\mathbb{C}^{n+1}$, we decompose the matrix $A(x)$ in the following way

$$A(x) = \begin{bmatrix} A_{\perp\perp}(x) & A_{\perp\|}(x) \\ A_{\|\perp}(x) & A_{\|\|}(x)\end{bmatrix}.$$

Then we can write Eq. (2.2) in the form

$$\begin{bmatrix}\partial_t & \begin{bmatrix}\mathrm{div} & k\end{bmatrix}\end{bmatrix} \begin{bmatrix} A_{\perp\perp}(x) & A_{\perp\|}(x) \\ A_{\|\perp}(x) & A_{\|\|}(x)\end{bmatrix} \begin{bmatrix} \partial_t u \\ \begin{bmatrix}\nabla_0 u \\ ku\end{bmatrix}\end{bmatrix} = 0.$$

Hence

$$\begin{bmatrix}\partial_t & \begin{bmatrix}\mathrm{div} & k\end{bmatrix}\end{bmatrix} \begin{bmatrix} A_{\perp\perp}\partial_t u + A_{\perp\|}\begin{bmatrix}\nabla_0 u \\ ku\end{bmatrix} \\ A_{\|\perp}\partial_t u + A_{\|\|}\begin{bmatrix}\nabla_0 u \\ ku\end{bmatrix}\end{bmatrix} = 0. \tag{2.3}$$

Next, we define f as

$$f = \begin{bmatrix} f_\perp \\ f_\| \end{bmatrix} := \begin{bmatrix} A_{\perp\perp}\partial_t u + A_{\perp\|}\begin{bmatrix}\nabla_0 u \\ ku\end{bmatrix} \\ \begin{bmatrix}\nabla_0 u \\ ku\end{bmatrix}\end{bmatrix}. \tag{2.4}$$

Since A is pointwise strictly accretive, all diagonal blocks are pointwise strictly accretive, and consequently invertible. In particular, $A_{\perp\perp}$ is invertible. Hence, due to (2.4), we obtain $\partial_t u = A_{\perp\perp}^{-1}(f_\perp - A_{\perp\|} f_\|)$. Therefore we can write Eq. (2.3) in terms of f

$$\begin{bmatrix}\partial_t & \begin{bmatrix}\mathrm{div} & k\end{bmatrix}\end{bmatrix} \begin{bmatrix} A_{\perp\perp}A_{\perp\perp}^{-1}(f_\perp - A_{\perp\|} f_\|) + A_{\perp\|} f_\| \\ A_{\|\perp}A_{\perp\perp}^{-1}(f_\perp - A_{\perp\|} f_\|) + A_{\|\|} f_\| \end{bmatrix} = 0,$$

hence

$$\begin{bmatrix}\partial_t & \begin{bmatrix}\mathrm{div} & k\end{bmatrix}\end{bmatrix} \begin{bmatrix} f_\perp \\ A_{\|\perp}A_{\perp\perp}^{-1}(f_\perp - A_{\perp\|} f_\|) + A_{\|\|} f_\| \end{bmatrix} = 0. \tag{2.5}$$

On the other hand, from definition of $f_\|$, we obtain

$$\partial_t f_\| = \begin{bmatrix} \nabla_0 \partial_t u \\ k\partial_t u \end{bmatrix} = \begin{bmatrix} \nabla_0 \\ k \end{bmatrix} (A_{\perp\perp}^{-1}(f_\perp - A_{\perp\|} f_\|)),$$

which, together with (2.5), gives us the system of equations

$$\begin{cases} \partial_t f_\perp + \begin{bmatrix} \operatorname{div} k \end{bmatrix} (A_{\|\perp} A_{\perp\perp}^{-1}(f_\perp - A_{\perp\|} f_\|) + A_{\|\|} f_\|) = 0, \\ \partial_t f_\| - \begin{bmatrix} \nabla_0 \\ k \end{bmatrix} A_{\perp\perp}^{-1}(f_\perp - A_{\perp\|} f_\|) = 0. \end{cases}$$

In vector notation, we equivalently have

$$\partial_t \begin{bmatrix} f_\perp \\ f_\| \end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} \operatorname{div} k \end{bmatrix} \\ -\begin{bmatrix} \nabla_0 \\ k \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1} A_{\perp\|} \\ A_{\|\perp} A_{\perp\perp}^{-1} & A_{\|\|} - A_{\|\perp} A_{\perp\perp}^{-1} A_{\perp\|} \end{bmatrix} \begin{bmatrix} f_\perp \\ f_\| \end{bmatrix} = 0.$$

Define

$$B := \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1} A_{\perp\|} \\ A_{\|\perp} A_{\perp\perp}^{-1} & A_{\|\|} - A_{\|\perp} A_{\perp\perp}^{-1} A_{\perp\|} \end{bmatrix}$$

and

$$D := \begin{bmatrix} 0 & \begin{bmatrix} \operatorname{div} k \end{bmatrix} \\ -\begin{bmatrix} \nabla_0 \\ k \end{bmatrix} & 0 \end{bmatrix}$$

with domains $\mathbf{D}(B) = L_2(\Omega; \mathrm{C}^{n+2})$ and

$$\mathbf{D}(D) = \Big\{ f = (f_1, f_2, f_3) \in L_2(\Omega; \mathrm{C}^{2+n}) : \; f_1 \in H_0^1(\Omega), \\ f_2 \in H_{\operatorname{div}}(\Omega; \mathrm{C}^n), \; f_3 \in L_2(\Omega) \Big\},$$

respectively. Then the equation becomes

$$\partial_t f + DBf = 0, \tag{2.6}$$

together with the constraint that $f \in \mathbf{R}(D)$ for each fixed $t \in \mathrm{R}$.

Since A is a pointwise strictly accretive operator, B is a strictly accretive multiplication operator just like A, see [4, Proposition 3.2]. By the above arguments, equation (2.2) for u implies that f, defined above, solves (2.6). Moreover, the converse is also true, that is the following proposition holds.

Proposition 2.2. *If $(f, \nabla_0 g, kg) \in \mathbf{R}(D)$ solves Eq. (2.6), then g solves Eq. (2.2).*

Proof. Let $(f, \nabla_0 g, kg) \in \mathbf{R}(D)$ be a solution of Eq. (2.6), then

$$\begin{cases} \partial_t f + \begin{bmatrix} \operatorname{div} k \end{bmatrix} \left(A_{\|\perp} A_{\perp\perp}^{-1} \left(f - A_{\perp\|} \begin{bmatrix} \nabla_0 g \\ kg \end{bmatrix} \right) + A_{\|\|} \begin{bmatrix} \nabla_0 g \\ kg \end{bmatrix} \right) = 0, \\ \partial_t \begin{bmatrix} \nabla_0 g \\ kg \end{bmatrix} - \begin{bmatrix} \nabla_0 \\ k \end{bmatrix} A_{\perp\perp}^{-1} \left(f - A_{\perp\|} \begin{bmatrix} \nabla_0 g \\ kg \end{bmatrix} \right) = 0. \end{cases} \tag{2.7}$$

The first equation of (2.7) can be written in the form

$$\begin{bmatrix}\partial_t & \begin{bmatrix}\mathrm{div} & k\end{bmatrix}\end{bmatrix}\begin{bmatrix} f \\ A_{\parallel\perp}A_{\perp\perp}^{-1}\left(f - A_{\perp\parallel}\begin{bmatrix}\nabla_0 g \\ kg\end{bmatrix}\right) + A_{\parallel\parallel}\begin{bmatrix}\nabla_0 g \\ kg\end{bmatrix}\end{bmatrix} = 0. \tag{2.8}$$

From the second equation of the system (2.7), we see

$$\partial_t g = A_{\perp\perp}^{-1}\left(f - A_{\perp\parallel}\begin{bmatrix}\nabla_0 g \\ kg\end{bmatrix}\right), \tag{2.9}$$

thus

$$f = A_{\perp\perp}\partial_t g + A_{\perp\parallel}\begin{bmatrix}\nabla_0 g \\ kg\end{bmatrix}. \tag{2.10}$$

Setting (2.9) and (2.10) into the formula (2.8), we get

$$\begin{bmatrix}\partial_t & \begin{bmatrix}\mathrm{div} & k\end{bmatrix}\end{bmatrix}\begin{bmatrix} A_{\perp\perp}\partial_t g + A_{\perp\parallel}\begin{bmatrix}\nabla_0 g \\ kg\end{bmatrix} \\ A_{\parallel\perp}\partial_t g + A_{\parallel\parallel}\begin{bmatrix}\nabla_0 g \\ kg\end{bmatrix}\end{bmatrix} = 0.$$

This shows that g solves Eq. (2.2). □

Let us define operators

$$D_1 := \begin{bmatrix} 0 & \mathrm{div} & 0 \\ -\nabla_0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}, \qquad D_0 := \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ -k & 0 & 0\end{bmatrix}$$

with domains $\mathbf{D}(D_1) = \mathbf{D}(D)$ and $\mathbf{D}(D_0) = L_2(\Omega; \mathrm{C}^{n+2})$. Then

$$D = D_1 + D_0.$$

Remark 2.3. Note that D_1 is a self-adjoint operator, see [9, Theorem 6.2], and D_0 is a bounded operator. Therefore D is a closed operator and

$$D^* = D_1^* + D_0^* = \begin{bmatrix} 0 & \mathrm{div} & -\overline{k} \\ -\nabla_0 & 0 & 0 \\ \overline{k} & 0 & 0\end{bmatrix}.$$

2.2. Maxwell's Equation

Let $\Omega \subset \mathrm{R}^2$ be a bounded weakly Lipschitz domain. By Rademacher's Theorem the surface $\partial\Omega$ has a tangent plane and an outward pointing unit normal $n(x)$ at almost every $x \in \partial\Omega$. We introduce the Sobolev spaces

$$\begin{aligned}
H_{\mathrm{div}}(\Omega; \mathrm{C}^2) &:= \{f \in L_2(\Omega; \mathrm{C}^2) :\ \mathrm{div} f \in L_2(\Omega)\},\\
H_{\mathrm{curl}}(\Omega; \mathrm{C}^2) &:= \{f \in L_2(\Omega; \mathrm{C}^2) :\ \mathrm{curl} f \in L_2(\Omega)\},\\
H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2) &:= \{f \in H_{\mathrm{div}}(\Omega; \mathrm{C}^2) :\ \mathrm{div}(\tilde{f}) \in L_2(\mathrm{R}^2)\},\\
H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2) &:= \{f \in H_{\mathrm{curl}}(\Omega; \mathrm{C}^2) :\ \mathrm{curl}(\tilde{f}) \in L_2(\mathrm{R}^2)\},
\end{aligned}$$

where $\tilde{f}$ denotes the zero-extension of f to R^2.

The last two spaces have the following geometric meaning. Assume that $f \in H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2)$, then there exists a sequence $\{\psi_k\}_{k=1}^{\infty} \subset C_0^\infty(\Omega; \mathrm{C}^2)$ such that $\psi_k \to f$ and $\mathrm{div}\psi_k \to \mathrm{div} f$. Hence, for $\phi \in C^\infty(\mathrm{R}^2)$, we obtain

$$\int_\Omega (\mathrm{div} f, \phi) - \int_\Omega (f, -\nabla\phi) = \lim_{k\to\infty} \left(\int_\Omega (\mathrm{div}\psi_k, \phi) - \int_\Omega (\psi_k, -\nabla\phi) \right) = 0.$$

Hence the Stokes' theorem implies formally

$$\int_{\partial\Omega} (f \cdot n, \phi) = \int_\Omega (\mathrm{div} f, \phi) - \int_\Omega (f, -\nabla\phi) = 0.$$

Therefore we interpret $f \in H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2)$ to mean that $\mathrm{div} f \in L_2(\Omega)$, and that f is tangential on the boundary in a weak sense. Similarly, the condition $f \in H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$ means that $\mathrm{curl} f \in L_2(\Omega)$, and f is normal on the boundary in a weak sense.

By ∇, ∇_0, div and div_0, we define the gradient and divergence operators on $H^1(\Omega)$, $H^1_0(\Omega)$, $H_{\mathrm{div}}(\Omega; \mathrm{C}^2)$ and $H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2)$ respectively.

Remark 2.4. For a bounded weakly Lipschitz domain $\Omega \subset \mathrm{R}^2$ and function $f \in H_{\mathrm{div}}(\Omega; \mathrm{C}^2)$, we see

$$\mathrm{curl} Jf = \mathrm{div} f, \qquad f \cdot n = Jf \times n$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This gives

$$JH_{\mathrm{div}}(\Omega; \mathrm{C}^2) = H_{\mathrm{curl}}(\Omega; \mathrm{C}^2), \qquad JH^0_{\mathrm{div}}(\Omega; \mathrm{C}^2) = H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2).$$

Let $\mu, \varepsilon \in L_\infty\left(\mathrm{R}^2; \mathcal{L}\left(\mathrm{C}^3\right)\right)$ be pointwise strictly accretive matrices, see (2.1). For a complex number $\omega \neq 0$, we consider Maxwell's system of equations

$$\begin{cases} \mathrm{div}_{(t,x)} \mu H = 0, \\ i\omega\mu H + \mathrm{curl}_{(t,x)} E = 0, \\ i\omega\varepsilon E - \mathrm{curl}_{(t,x)} H = 0, \\ \mathrm{div}_{(t,x)} \varepsilon E = 0 \end{cases} \tag{2.11}$$

in $\mathrm{R} \times \Omega$ with

$$\begin{aligned} \mu H &\in L_2(\Omega) \times H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2), \\ E &\in L_2(\Omega) \times H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2) \end{aligned}$$

for any fixed $t \in \mathrm{R}$.

According to the splitting of C^3 into C and C^2, we write

$$H = \begin{bmatrix} H_\perp \\ H_\parallel \end{bmatrix}, \quad E = \begin{bmatrix} E_\perp \\ E_\parallel \end{bmatrix},$$

$$\mu = \begin{bmatrix} \mu_{\perp\perp} & \mu_{\perp\parallel} \\ \mu_{\parallel\perp} & \mu_{\parallel\parallel} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{\perp\perp} & \varepsilon_{\perp\parallel} \\ \varepsilon_{\parallel\perp} & \varepsilon_{\parallel\parallel} \end{bmatrix}$$

and define auxiliary matrices

$$\overline{\mu} := \begin{bmatrix} \mu_{\perp\perp} & \mu_{\perp\|} \\ 0 & I \end{bmatrix}, \quad \underline{\mu} := \begin{bmatrix} 1 & 0 \\ \mu_{\|\perp} & \mu_{\|\|} \end{bmatrix},$$

$$\overline{\varepsilon} := \begin{bmatrix} \varepsilon_{\perp\perp} & \varepsilon_{\perp\|} \\ 0 & I \end{bmatrix}, \quad \underline{\varepsilon} := \begin{bmatrix} 1 & 0 \\ \varepsilon_{\|\perp} & \varepsilon_{\|\|} \end{bmatrix},$$

$$A = \begin{bmatrix} \mu & 0 \\ 0 & \varepsilon \end{bmatrix}, \quad \overline{A} := \begin{bmatrix} \overline{\mu} & 0 \\ 0 & \overline{\varepsilon} \end{bmatrix}, \qquad \underline{A} := \begin{bmatrix} \underline{\mu} & 0 \\ 0 & \underline{\varepsilon} \end{bmatrix}.$$

Since μ, ε are pointwise strictly accretive, we conclude that $\mu_{\perp\perp}$, $\varepsilon_{\perp\perp}$ are pointwise strictly accretive, and consequently $\overline{\mu}$, $\overline{\varepsilon}$, and $\overline{A}$ are invertible.

Let $I_\perp = \{I_\perp^{i,j}\}_{i,j=1}^6$ be a 6 by 6 matrix such that $I_\perp^{1,1} = I_\perp^{4,4} = 1$, and all other elements are zero. We set $I_\| = I - I_\perp$. From the first and forth equations of (2.11), we get

$$\partial_t I_\perp AG + \begin{bmatrix} 0 & \mathrm{div}_0 & 0 & 0 \\ -\nabla & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{div} \\ 0 & 0 & -\nabla_0 & 0 \end{bmatrix} I_\| AG = 0, \quad \text{where} \quad G := \begin{bmatrix} H \\ E \end{bmatrix}. \tag{2.12}$$

From the second and third equations of (2.11), we obtain

$$\partial_t I_\| G + \begin{bmatrix} 0 & \mathrm{div}_0 & 0 & 0 \\ -\nabla & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{div} \\ 0 & 0 & -\nabla_0 & 0 \end{bmatrix} I_\perp G - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\omega J \\ 0 & 0 & 0 & 0 \\ 0 & -i\omega J & 0 & 0 \end{bmatrix} I_\| AG = 0. \tag{2.13}$$

Since $I_\perp AG = I_\perp \overline{A}G$, $I_\| AG = I_\| \underline{A}G$, $G_\| = I_\| \overline{A}G$, and $I_\perp G = I_\perp \underline{A}G$, we can combine Eqs. (2.12) and (2.13) in the following way

$$\partial_t \overline{A}G + \begin{bmatrix} 0 & \mathrm{div}_0 & 0 & 0 \\ -\nabla & 0 & 0 & i\omega J \\ 0 & 0 & 0 & \mathrm{div} \\ 0 & -i\omega J & -\nabla_0 & 0 \end{bmatrix} \underline{A}G = 0. \tag{2.14}$$

Define

$$D := \begin{bmatrix} 0 & \mathrm{div}_0 & 0 & 0 \\ -\nabla & 0 & 0 & i\omega J \\ 0 & 0 & 0 & \mathrm{div} \\ 0 & -i\omega J & -\nabla_0 & 0 \end{bmatrix}$$

with domain

$$\mathbf{D}(D) = \{f = (f_1, f_2, f_3, f_4) \in L_2(\Omega) : \ f_1 \in H^1(\Omega), \ f_2 \in H^0_{\mathrm{div}}(\Omega; \mathbb{C}^2), \\ f_3 \in H^1_0(\Omega), \ f_4 \in H_{\mathrm{div}}(\Omega; \mathbb{C}^2)\}.$$

Let $B := \underline{A}\,\overline{A}^{-1}$, $F := \overline{A}G$, so that Eq. (2.14) becomes

$$\partial_t F + DBF = 0 \tag{2.15}$$

together with the constraint that $F \in \mathbf{R}(D)$ for each fixed $t \in \mathbb{R}$.

To see that (2.11) and (2.15) are equivalent, we prove an analogue of Proposition 2.2.

Proposition 2.5. *Let $f(t,x)$ and $g(t,x)$ be three dimensional vector-valued functions such that (f,g) solves Eq. (2.15), and $(f,g) \in \mathbf{R}(D) \cap \mathbf{D}(DB)$ for each fixed $t \in \mathrm{R}$. Then the vector-valued functions*

$$H = \overline{\mu}^{-1} f, \qquad E = \overline{\varepsilon}^{-1} g \tag{2.16}$$

solve the system of equations (2.11), and for any fixed $t \in \mathrm{R}$,

$$\mu H \in L_2(\Omega) \times H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2),$$
$$E \in L_2(\Omega) \times H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2).$$

Proof. Splitting C^3 into C and C^2, we write

$$f = \begin{bmatrix} f_\perp \\ f_\parallel \end{bmatrix}, \quad g = \begin{bmatrix} g_\perp \\ g_\parallel \end{bmatrix}, \quad H = \begin{bmatrix} H_\perp \\ H_\parallel \end{bmatrix}, \quad E = \begin{bmatrix} E_\perp \\ E_\parallel \end{bmatrix}.$$

Since (f,g) is a solution for (2.15), we see

$$\partial_t \overline{A} \begin{bmatrix} H \\ E \end{bmatrix} + DB\overline{A} \begin{bmatrix} H \\ E \end{bmatrix} = \partial_t \overline{A} \begin{bmatrix} H \\ E \end{bmatrix} + D\underline{A} \begin{bmatrix} H \\ E \end{bmatrix} = 0.$$

Thus

$$\begin{cases} \partial_t \left(\mu_{\perp\perp} H_\perp + \mu_{\perp\parallel} H_\parallel\right) + \mathrm{div}_0 \left(\mu_{\parallel\perp} H_\perp + \mu_{\parallel\parallel} H_\parallel\right) = 0, \\ \partial_t H_\parallel - \nabla H_\perp + i\omega J \left(\varepsilon_{\parallel\perp} E_\perp + \varepsilon_{\parallel\parallel} E_\parallel\right) = 0, \\ \partial_t \left(\varepsilon_{\perp\perp} E_\perp + \varepsilon_{\perp\parallel} E_\parallel\right) + \mathrm{div} \left(\varepsilon_{\parallel\perp} E_\perp + \varepsilon_{\parallel\parallel} E_\parallel\right) = 0, \\ \partial_t E_\parallel - \nabla_0 E_\perp - i\omega J \left(\mu_{\parallel\perp} H_\perp + \mu_{\parallel\parallel} H_\parallel\right) = 0. \end{cases} \tag{2.17}$$

By the assumption, $(f,g) \in \mathbf{R}(D)$ for fixed $t \in \mathrm{R}$, and hence Proposition 2.11 implies

$$\begin{cases} \mathrm{curl} f_\parallel - i\omega g_\perp = 0, \\ \mathrm{curl} g_\parallel + i\omega f_\perp = 0. \end{cases}$$

Therefore, in terms of H and E, we can write

$$\begin{cases} \mathrm{curl} H_\parallel - i\omega \left(\varepsilon_{\perp\perp} E_\perp + \varepsilon_{\perp\parallel} E_\parallel\right) = 0, \\ \mathrm{curl} E_\parallel + i\omega \left(\mu_{\perp\perp} H_\perp + \mu_{\perp\parallel} H_\parallel\right) = 0. \end{cases} \tag{2.18}$$

Combining (2.17) and (2.18), we conclude that H, E solve the system of equations (2.11).

Since $\overline{\mu} H = f$ and $f \in \mathbf{D}(DB)$ for each fixed $t \in \mathrm{R}$, it follows that

$$\underline{\mu} H \in H^1(\Omega) \times H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2).$$

Hence

$$\mu H \in L_2(\Omega) \times H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2).$$

Proposition 2.11 and (2.16) lead to $E_\parallel \in H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$. Therefore, for any fixed $t \in \mathrm{R}$,

$$E \in L_2(\Omega) \times H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2). \qquad \square$$

Let us define operators

$$D_1 := \begin{bmatrix} 0 & \mathrm{div}_0 & 0 & 0 \\ -\nabla & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{div} \\ 0 & 0 & -\nabla_0 & 0 \end{bmatrix}, \qquad D_0 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\omega J \\ 0 & 0 & 0 & 0 \\ 0 & -i\omega J & 0 & 0 \end{bmatrix}$$

with domains $\mathbf{D}(D_1) = \mathbf{D}(D)$ and $\mathbf{D}(D_0) = L_2(\Omega; \mathrm{C}^6)$. Then

$$D = D_1 + D_0.$$

Remark 2.6. Note that D_1 is a self-adjoint operator, see [9, Theorem 6.2], and D_0 is a bounded operator. Therefore D is a closed operator and

$$D^* = D_1^* + D_0^* = \begin{bmatrix} 0 & \mathrm{div}_0 & 0 & 0 \\ -\nabla & 0 & 0 & \overline{i\omega} J \\ 0 & 0 & 0 & \mathrm{div} \\ 0 & -\overline{i\omega} J & -\nabla_0 & 0 \end{bmatrix}.$$

2.3. Properties of D

Here we prove that the operators defined in Sects. 2.1 and 2.2 have closed range and compact resolvents. We will use the symbols $\sigma(\cdot)$ and $\rho(\cdot)$ to denote the spectrum and resolvent sets of an operator, respectively.

Let us start by considering the operator D defined in Sect. 2.1. First, we prove that $\mathbf{R}(D)$ is closed.

Proposition 2.7. *Let $\Omega \subset \mathrm{R}^n$ be a bounded, weakly Lipschitz domain, and D be the operator defined in Sect. 2.1. Then $\mathbf{R}(D)$ is a closed subspace of $L_2\left(\Omega; \mathrm{C}^{n+2}\right)$.*

Proof. According to [8, Theorem 5.2], it suffices to prove that $\gamma(D) > 0$, where $\gamma(D)$ is the reduced minimum modulus of D, that is the greatest number γ such that

$$\|Du\| \geq \gamma \inf_{v \in \mathbf{N}(D)} \|u - v\| \quad \text{for all} \quad u \in \mathbf{D}(D).$$

Let $h = (h_1, h_2, h_3) \in \mathbf{D}(D)$, then $g = \left(0, h_2, -\frac{1}{k}\mathrm{div} h_2\right) \in \mathbf{N}(D)$, and therefore

$$\inf_{v \in \mathbf{N}(D)} \|h - v\| \leq \|h - g\| = \frac{1}{|k|}\|kh_1\| + \frac{1}{|k|}\|kh_3 + \mathrm{div} h_2\| \leq \frac{1}{|k|}\|Dh\|.$$

This implies that $\gamma(D) \geq |k| > 0$, and consequently that $\mathbf{R}(D)$ is closed. □

To prove Proposition 2.7 we used that $k \neq 0$. However, by applying the Poincaré inequality, one can prove that Proposition 2.7 also holds for $k = 0$.

Next, we find the exact expression for $\mathbf{R}(D)$.

Proposition 2.8. *Let $\Omega \subset \mathrm{R}^n$ be a bounded, weakly Lipschitz domain, and D be the operator defined in Sect. 2.1. Then $\mathbf{R}(D) = \mathcal{H}$, where*

$$\mathcal{H} := \left\{ f = (f_1, f_2, f_3) \in L_2(\Omega; \mathrm{C}^{2+n}) : \ f_3 \in H_0^1(\Omega), \ f_2 = \frac{1}{k}\nabla_0 f_3 \right\}.$$

Proof. By definition of operator D, we obtain $\mathbf{R}(D) \subset \mathcal{H}$. Conversely, assume that $f = (f_1, f_2, f_3) \in \mathcal{H}$. Since

$$L_2(\Omega) = \mathbf{N}(\nabla_0) \oplus \overline{\mathbf{R}(\mathrm{div})},$$

there exists a function $h \in \mathbf{N}(\nabla_0)$ and sequence $\{g^l\}_{l=1}^{\infty} \subset H_{\mathrm{div}}(\Omega; \mathrm{C}^n)$ such that $h + \mathrm{div} g^l \to f_1$ in the L_2 norm. Therefore

$$D \begin{bmatrix} -\frac{1}{k} f_3 \\ g^l \\ h \end{bmatrix} \to \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

in the L_2 norm. This, by Proposition 2.7, implies that $f \in \mathbf{R}(D)$. □

Finally, we prove that the resolvent operators are compact. This implies that the spectrum $\sigma(D|_{\mathbf{R}(D)})$ contains only the eigenvalues of $D|_{\mathbf{R}(D)}$, and each eigenvalue has finite geometric multiplicity. In fact, we prove in Proposition 3.14 that the indexes/algebraic multiplicities are finite.

Proposition 2.9. *Let $\Omega \subset \mathrm{R}^n$ be a bounded, weakly Lipschitz domain, and D be the operator defined in Sect. 2.1. Assume $\lambda \in \rho(D|_{\mathbf{R}(D)})$, then*

$$(\lambda - D|_{\mathbf{R}(D)})^{-1} : \mathbf{R}(D) \to \mathbf{R}(D)$$

is a compact operator.

Proof. Since

$$D|_{\mathbf{R}(D)} (\lambda - D|_{\mathbf{R}(D)})^{-1} : \mathbf{R}(D) \to \mathbf{R}(D)$$

is a bounded operator, it suffices to show that the embedding

$$\big(\mathbf{D}(D) \cap \mathbf{R}(D), \| \cdot \|_{\mathbf{D}(D)\cap\mathbf{R}(D)}\big) \hookrightarrow (\mathbf{R}(D), \| \cdot \|_{L_2})$$

is compact, where

$$\|f\|_{\mathbf{D}(D)\cap\mathbf{R}(D)} = \|Df\| + \|f\|.$$

Let $\{(f^l, \nabla g^l, kg^l)\}_{l=1}^{+\infty}$ be a sequence in $\big(\mathbf{D}(D) \cap \mathbf{R}(D), \| \cdot \|_{\mathbf{D}(D)\cap\mathbf{R}(D)}\big)$ such that

$$\left\| \begin{bmatrix} f^l \\ \nabla_0 g^l \\ kg^l \end{bmatrix} \right\| + \left\| D \begin{bmatrix} f^l \\ \nabla_0 g^l \\ kg^l \end{bmatrix} \right\| < C \tag{2.19}$$

for some $C > 0$. In particular, we get

$$\|f^l\| + \|\nabla_0 f^l\| \leq C.$$

Therefore, the sequence $\{f^l\}_{l=1}^{\infty}$ is bounded in $H^1(\Omega)$. Since $\Omega \subset \mathrm{R}^n$ is bounded, the Sobolev Embedding Theorem gives that $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact. Hence, the sequence $\{f^l\}_{l=1}^{\infty}$ contains a Cauchy subsequence in $L_2(\Omega)$. The same conclusion can be drawn for $\{g^l\}_{l=1}^{\infty}$.

From estimate (2.19), we obtain

$$\|\mathrm{div}\nabla_0 g^l + k^2 g^l\| + \|g^l\| \leq C,$$

and hence $\|\mathrm{div}\nabla_0 g^l\| \leq C$. Next, since $\{g^l\}_{l=1}^{\infty} \subset H_0^1(\Omega)$ and $\mathrm{curl}\nabla_0 g^l = 0$, we see that $\{\nabla_0 g^l\}_{l=1}^{\infty}$ is a bounded sequence in $H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^n) \cap H_{\mathrm{div}}(\Omega; \mathrm{C}^n)$.

Consequently, the sequence $\{\nabla_0 g^l\}_{l=1}^{\infty}$ contains a Cauchy subsequence in $L_2(\Omega; \mathrm{C}^n)$, because the embedding

$$H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^n) \cap H_{\mathrm{div}}(\Omega; \mathrm{C}^n) \hookrightarrow L_2(\Omega; \mathrm{C}^n)$$

is compact, see [6] or [12].

Finally, after passing to subsequences three times, we conclude that $\{(f^l, \nabla_0 g^l, kg^l)\}_{l=1}^{\infty}$ contains a Cauchy subsequence in $(\mathbf{R}(D), \|\cdot\|_{L_2})$. $\square$

We next derive similar results for the operator D defined in Sect. 2.2.

Proposition 2.10. *Let $\Omega \subset \mathrm{R}^2$ be a bounded, weakly Lipschitz domain, and D be the operator defined in Sect. 2.2. Then $\mathbf{R}(D)$ is a closed subspace of $L_2(\Omega; \mathrm{C}^6)$.*

Proof. As in Proposition 2.7, it suffices to show that $\gamma(D) > 0$. Let us choose any $h = (h_1, h_2, h_3, h_4) \in \mathbf{D}(D)$. In particular, $h_1 \in H^1(\Omega)$, $h_3 \in H^1_0(\Omega)$, and hence $\nabla h_1 \in H_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$ and $\nabla_0 h_3 \in H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$. By Remark 2.4,

$$\frac{1}{i\omega} J^{-1} \nabla h_1 \in H_{\mathrm{div}}(\Omega; \mathrm{C}^2), \quad \frac{1}{i\omega} J^{-1} \nabla_0 h_3 \in H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2).$$

Hence

$$g = \left(h_1, \frac{1}{i\omega} J^{-1} \nabla_0 h_1, h_3, \frac{1}{i\omega} J^{-1} \nabla h_1 \right) \in \mathbf{D}(D).$$

Moreover, straightforward calculations show that $g \in \mathbf{N}(D)$. Therefore

$$\begin{aligned} \inf_{v \in \mathbf{N}(D)} \|h - v\| \leq \|h - g\| &= \|h_2 + \frac{1}{i\omega} J^{-1} \nabla_0 h_3\| + \|h_4 - \frac{1}{i\omega} J^{-1} \nabla h_1\| \\ &= \frac{1}{|\omega|} \|i\omega J h_2 + \nabla_0 h_3\| + \frac{1}{|\omega|} \|i\omega J h_4 - \nabla h_1\| \\ &\leq \frac{1}{|\omega|} \|Dh\|. \end{aligned}$$

This implies that $\gamma(D) \geq |\omega| > 0$, and consequently that $\mathbf{R}(D)$ is closed. $\square$

The following proposition gives the exact expression for $\mathbf{R}(D)$.

Proposition 2.11. *Let $\Omega \subset \mathrm{R}^2$ be a bounded, weakly Lipschitz domain, and D be the operator defined in Sect. 2.2. Then $\mathbf{R}(D) = \mathcal{H}$, where*

$$\begin{aligned} \mathcal{H} := \big\{ (f_\perp, f_\|, g_\perp, g_\|) \in L_2(\Omega; \mathrm{C}^6) :\ & f_\| \in H_{\mathrm{curl}}(\Omega; \mathrm{C}^2),\ g_\| \in H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2) \\ & \text{and } \mathrm{curl} f_\| - i\omega g_\perp = 0,\ \mathrm{curl} g_\| + i\omega f_\perp = 0 \big\}. \end{aligned}$$

Proof. Assume $(f, g) \in \mathbf{R}(D)$. Then there exists $(F, G) \in \mathbf{D}(D)$ such that

$$\begin{bmatrix} f_\perp \\ f_\| \\ g_\perp \\ g_\| \end{bmatrix} = \begin{bmatrix} \mathrm{div}_0 F_\| \\ -\nabla F_\perp + i\omega J G_\| \\ \mathrm{div} G_\| \\ -\nabla_0 G_\perp - i\omega J F_\| \end{bmatrix}.$$

Since $F_\| \in H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2)$, $G_\| \in H_{\mathrm{div}}(\Omega; \mathrm{C}^2)$, we see that $f_\perp, g_\perp \in L_2(\Omega)$. From Remark 2.4, we conclude that $JF_\| \in H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$ and $JG_\| \in H_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$. Therefore, since $F_\perp \in H^1(\Omega)$ and $G_\perp \in H^1_0(\Omega)$, we obtain $f_\| \in H_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$ and $g \in H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$.

Next, we compute

$$\mathrm{curl} f_{\|} = -\mathrm{curl}\nabla F_{\perp} + i\omega \mathrm{curl} J G_{\|} = i\omega \mathrm{div} G_{\|} = i\omega g_{\perp}$$

and similarly

$$\mathrm{curl} g_{\|} = -i\omega f_{\perp}.$$

From the arguments above, we can assert that $\mathbf{R}(D) \subset \mathcal{H}$.

Conversely, assume $(f, g) \in \mathcal{H}$. Let us set

$$F_{\|} = \frac{1}{i\omega} J g_{\|}, \qquad G_{\|} = \frac{1}{i\omega} J f_{\|}.$$

Then, from Remark 2.4, we obtain

$$F_{\|} \in H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2), \qquad G_{\|} \in H_{\mathrm{div}}(\Omega; \mathrm{C}^2),$$

and

$$f_{\perp} = \mathrm{div}_0 F_{\|}, \qquad g_{\perp} = \mathrm{div} G_{\|}.$$

Next, since

$$\mathrm{curl}\left(f_{\|} - i\omega J G_{\|}\right) = \mathrm{curl} f_{\|} - i\omega \mathrm{div} G_{\|} = \mathrm{curl} f_{\|} - i\omega g_{\perp} = 0$$

and $f_{\|} - i\omega J G_{\|} \in H_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$, there exists a function $F_{\perp} \in H^1(\Omega)$ such that $-\nabla F_{\perp} = f_{\|} - i\omega J G_{\|}$.

Likewise, since $\mathrm{curl}\left(g_{\|} + i\omega J F_{\|}\right) = 0$ and $g_{\|} + i\omega J F_{\|} \in H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2)$, there exists a function $G_{\perp} \in H^1_0(\Omega)$ such that $-\nabla_0 G_{\perp} = g_{\|} + i\omega J F_{\|}$.

Combining all relations between (f, g) and (F, G), we conclude that $(F, G) \in \mathbf{D}(D)$, and

$$D \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}.$$

This implies that $\mathcal{H} \subset \mathbf{R}(D)$, hence that $\mathcal{H} = \mathbf{R}(D)$. □

There is also the following analogue of Proposition 2.9.

Proposition 2.12. *Let $\Omega \subset \mathrm{R}^2$ be a bounded, weakly Lipschitz domain, and D be the operator defined in Sect. 2.2. Assume that $\lambda \in \rho(D|_{\mathbf{R}(D)})$, then*

$$(\lambda - D|_{\mathbf{R}(D)})^{-1} : \mathbf{R}(D) \to \mathbf{R}(D)$$

is a compact operator.

Proof. Since

$$D|_{\mathbf{R}(D)} (\lambda - D|_{\mathbf{R}(D)})^{-1} : \mathbf{R}(D) \to \mathbf{R}(D)$$

is a bounded operator, it remains to verify that the embedding

$$\left(\mathbf{D}(D) \cap \mathbf{R}(D), \|\cdot\|_{\mathbf{R}(D)}\right) \hookrightarrow (\mathbf{R}(D), \|\cdot\|_{L_2})$$

is compact.

Let $\{h^l\}_{l=1}^{\infty} \subset \mathbf{D}(D)$ be a sequence such that $\{Dh^l\}_{l=1}^{\infty} \subset \mathbf{D}(D) \cap \mathbf{R}(D)$ and

$$\|Dh^l\| + \|DDh^l\| < C \tag{2.20}$$

for some constant $C > 0$. In particular,

$$\|\mathrm{div}_0 h_2^l\| + \| - \nabla \mathrm{div}_0 h_2^l + i\omega J(-\nabla_0 h_3^l - i\omega J h_2^l)\| < C,$$
$$\| - \nabla_0 h_3^l - i\omega J h_2^l\| < C.$$

Therefore

$$\|\nabla \mathrm{div}_0 h_2^l\| + \|\mathrm{div}_0 h_2^l\| < C. \tag{2.21}$$

As in Proposition 2.9, (2.21) implies that $\{\mathrm{div}_0 h_2^l\}_{l=1}^\infty$ contains a Cauchy subsequence in $L_2(\Omega)$. Similarly, this statement holds for $\{\mathrm{div} h_4^l\}_{l=1}^\infty$.

Since $\|DDh^l\| \le C$, we obtain

$$\|\mathrm{div}(-\nabla_0 h_3^l - i\omega J h_2^l)\| \le C$$

and

$$\| - \mathrm{curl}\nabla_0 h_3^l - i\omega \mathrm{curl} J h_2^l)\| = \|i\omega \mathrm{curl} J h_2^l)\| = \|i\omega \mathrm{div} h_2^l\| \le C.$$

Therefore $\{-\nabla_0 h_3^l - i\omega J h_2^l\}_{l=1}^\infty$ is bounded in $H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2) \cap H_{\mathrm{div}}(\Omega; \mathrm{C}^2)$. From the compact embedding (see [6] or [12])

$$H^0_{\mathrm{curl}}(\Omega; \mathrm{C}^2) \cap H_{\mathrm{div}}(\Omega; \mathrm{C}^2) \hookrightarrow L_2(\Omega; \mathrm{C}^2),$$

we conclude that $\{-\nabla_0 h_3^l - i\omega J h_2^l\}_{l=1}^\infty$ contains a Cauchy subsequence in $L_2(\Omega; \mathrm{C}^2)$.

Likewise, $\{-\nabla h_1^l + i\omega J h_4^l\}_{l=1}^\infty$ is bounded in $H_{\mathrm{curl}}(\Omega; \mathrm{C}^2) \cap H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2)$. Since $H_{\mathrm{curl}}(\Omega; \mathrm{C}^2) \cap H^0_{\mathrm{div}}(\Omega; \mathrm{C}^2)$ is also compactly embedded into $L_2(\Omega; \mathrm{C}^2)$, $\{-\nabla h_1^l + i\omega J h_4^l\}_{l=1}^\infty$ contains a convergent subsequence in $L_2(\Omega; \mathrm{C}^2)$.

From the arguments above, we conclude that $\{Dh^l\}_{l=1}^\infty$ contains a Cauchy subsequence in $L_2(\Omega; \mathrm{C}^6)$. □

3. Spectral Projections and Functional Calculus for DB

In this section we modify the functional calculus designed by McIntosh in [10], for the operators described below.

Let $\Omega \subset \mathrm{R}^n$ be a bounded, weakly Lipschitz domain. From now on we consider a pointwise accretive multiplication operator $B \in L_\infty(\Omega; \mathrm{C}^M \times \mathrm{C}^M)$ on $L_2(\Omega; \mathrm{C}^M)$ and a closed range operator

$$D : L_2(\Omega; \mathrm{C}^M) \to L_2(\Omega; \mathrm{C}^M)$$

satisfying the following conditions

1. There exists a bounded operator D_0 and a self-adjoint homogeneous first order differential operator D_1 with constant coefficients and local boundary conditions so that
$$D = D_1 + D_0.$$
2. The operator $(\lambda - D|_{\mathbf{R}(D)})^{-1}$ is compact for some, and therefore for all λ belonging to the resolvent set $\rho(D|_{\mathbf{R}(D)})$.

Remark 3.1. In both the Helmholtz and the Maxwell's cases, the operators B and D satisfy the conditions above. Moreover, D_0 is a normal operator, and hence D is normal as well, in particular $\mathbf{D}(D) = \mathbf{D}(D^*)$.

3.1. Preliminary for Functional Calculus

Here we consider basic properties of the operator DB in order to construct a functional calculus in the next subsections. We begin with a well known result and give its proof for the sake of completeness.

Proposition 3.2. *We have topological splittings for $L_2(\Omega;\mathrm{C}^M)$,*

$$L_2(\Omega;\mathrm{C}^M) = \mathbf{N}(D^*B) \oplus \mathbf{R}(D),$$
$$L_2(\Omega;\mathrm{C}^M) = \mathbf{N}(D^*) \oplus B\mathbf{R}(D).$$

Proof. Since $\mathbf{N}(B^*D^*) = \mathbf{N}(D^*)$, $\mathbf{R}(DB) = \mathbf{R}(D)$, and $B^*D^* = (DB)^*$, we obtain the following orthogonal splitting

$$L_2(\Omega;\mathrm{C}^M) = \mathbf{R}(DB) \oplus \mathbf{N}(B^*D^*) = \mathbf{R}(D) \oplus \mathbf{N}(D^*).$$

For any non-zero $g \in \mathbf{N}(D^*)$, $(B^{-1}g, g) \neq 0$. Thus

$$\mathbf{R}(DB) \cap B\mathbf{N}(D^*) = \{0\}.$$

Since B^* is an accretive operator, for $g \in \mathbf{R}(D)$ and $h \in \mathbf{N}(D^*)$, we obtain

$$C^{-1}\|g\|^2 + 0 \leq \mathrm{Re}(B^*g, g) + \mathrm{Re}(g, h) = \mathrm{Re}(B^*g, g) + \mathrm{Re}(B^*g, B^{-1}h) \tag{3.1}$$

$$= \mathrm{Re}(B^*g, g + B^{-1}h) \leq C\|g\|\|g + B^{-1}h\|$$

for some constant $C > 0$. Similarly,

$$\begin{aligned} C^{-1}\|B^{-1}h\|^2 &\leq \mathrm{Re}(B^*B^{-1}h, B^{-1}h) = \mathrm{Re}(B^{-1}h, h) \\ &= \mathrm{Re}(B^{-1}h + g, h) \leq C\|B^{-1}h + g\|\|h\| \end{aligned} \tag{3.2}$$

for some constant $C > 0$. Therefore $B^{-1}\mathbf{N}(D^*) \oplus \mathbf{R}(D)$ is a Hilbert space. Assume that $f \in (B^{-1}\mathbf{N}(D^*) \oplus \mathbf{R}(D))^\perp$. In particular, $f \in \mathbf{N}(D^*)$ and $f \perp \mathbf{R}(D)$. Since B is an accretive operator, we see that $f = 0$. Therefore

$$L_2(\Omega;\mathrm{C}^M) = B^{-1}\mathbf{N}(D^*) \oplus \mathbf{R}(D) = \mathbf{N}(D^*B) \oplus \mathbf{R}(D).$$

One can prove the second splitting similarly. □

Proposition 3.3. *The operator*

$$DB|_{\mathbf{R}(D)} : \mathbf{R}(D) \to \mathbf{R}(D)$$

is a closed and densely defined operator.

Proof. Note that $\mathbf{N}(D^*B) \subset \mathbf{D}(DB)$. Therefore, from Proposition 3.2, we obtain

$$\mathbf{D}(DB) = [\mathbf{D}(DB) \cap \mathbf{R}(D)] \oplus \mathbf{N}(D^*B). \tag{3.3}$$

Let us fix $\varepsilon > 0$ and $f \in \mathbf{R}(D)$. Since B is an invertible bounded operator, and $\mathbf{D}(D)$ is a dense set in $L_2(\Omega;\mathrm{C}^M)$, we deduce that $\mathbf{D}(DB) = B^{-1}\mathbf{D}(D)$ is dense in $L_2(\Omega;\mathrm{C}^M)$. Therefore, from (3.3), we can find $g \in \mathbf{D}(DB) \cap \mathbf{R}(D)$ and $h \in \mathbf{N}(D^*B)$ such that $\|g+h-f\| \leq \varepsilon$. On the other hand, Proposition 3.2 gives

$$\|g + h - f\| \geq C(\|g - f\| + \|h\|).$$

Hence $\|g - f\| \leq \frac{\varepsilon}{C}$, and consequently $\mathbf{D}(DB) \cap \mathbf{R}(D)$ is dense in $\mathbf{R}(D)$.

The operator

$$DB : L_2(\Omega; \mathbb{C}^M) \to L_2(\Omega; \mathbb{C}^M)$$

is closed, and $\mathbf{R}(D)$ is closed in $L_2(\Omega; \mathbb{C}^M)$. Hence, the operator $DB|_{\mathbf{R}(D)}$ is closed. $\square$

To state the next proposition let us set

$$S_{\alpha,\tau} := \{x + iy \in \mathbb{C} : |y| < |x| \tan \alpha + \tau\}$$

for $\alpha \in [0, \frac{\pi}{2})$ and $\tau \geq 0$. Define the angle and constant of accretivity of B to be

$$\omega := \sup_{v \in \mathbb{C}^M} |\arg(Bv, v)| < \frac{\pi}{2}, \quad \beta := \inf_{v \in \mathbb{C}^M} \frac{\mathrm{Re}(Bv, v)}{\|v\|^2},$$

respectively.

Proposition 3.4. *There exist constants $\tau, C > 0$, depending only on $\|D_0 B\|$, $\|B\|$, and β such that $\sigma(DB) \subset S_{\omega,\tau}$ and*

$$\|(\lambda - DB)^{-1}\| \leq \frac{C}{dist(\lambda, S_{\omega,0})} \tag{3.4}$$

for any $\lambda \notin S_{\omega,\tau}$.

Proof. Since D_1 is self-adjoint, $D_1 B$ is bisectorial, see [4, Proposition 3.3]. Therefore, for any $\lambda \notin S_{\omega,0}$ and $u \in \mathbf{D}(DB)$,

$$\|(\lambda - DB)u\| \geq \|(\lambda - D_1 B)u\| - \|D_0 Bu\| \geq C\mathrm{dist}(\lambda, S_{\omega,0})\|u\| - \|D_0 B\|\|u\|.$$

Thus, for sufficiently large $\tau > 0$ and any $\lambda \notin S_{\omega,\tau}$,

$$\frac{C}{2}\mathrm{dist}(\lambda, S_{\omega,0})\|u\| \geq \|D_0 B\|\|u\|,$$

and therefore

$$\|(\lambda - DB)u\| \geq \frac{C}{2}\mathrm{dist}(\lambda, S_{\omega,0})\|u\|. \tag{3.5}$$

Hence $\lambda - DB$ is an injective operator with closed range. Next, let us consider the adjoint operator

$$(\lambda - DB)^* = \overline{\lambda} - B^* D^* = B^*(\overline{\lambda} - D^* B^*)B^{*-1}.$$

Similarly, we see that $\overline{\lambda} - D^* B^*$ is injective. Consequently, $(\lambda - DB)^*$ is also injective. Hence $\lambda - DB$ is a surjective operator. Thus, $\lambda \notin S_{\omega,\tau}$ is contained in the resolvent set, and (3.5) implies (3.4). $\square$

Let $P_{\mathbf{R}(D)}$ and $P_{\mathbf{N}(D^*)}$ be the orthogonal projections to $\mathbf{R}(D)$ and $\mathbf{N}(D^*)$ corresponding to the splitting

$$L_2(\Omega; \mathbb{C}^M) = \mathbf{R}(D) \oplus \mathbf{N}(D^*). \tag{3.6}$$

Lemma 3.5. *The operator*

$$P_{\mathbf{R}(D)}\big|_{B\mathbf{R}(D)} : B\mathbf{R}(D) \to \mathbf{R}(D)$$

is bounded and invertible.

Proof. If $P_{\mathbf{R}(D)}BDf = 0$, then $(BDf, Df) = 0$. This implies that $Df = 0$, and hence that $P_{\mathbf{R}(D)}\big|_{B\mathbf{R}(D)}$ is an injective operator. The second splitting in Proposition 3.2 implies that $P_{\mathbf{R}(D)}\big|_{B\mathbf{R}(D)}$ is surjective. Thus, by the bounded inverse theorem, we get the statement of the lemma. □

Proposition 3.6. *Let $\lambda \in \rho(DB|_{\mathbf{R}(D)})$, then*

$$(\lambda - DB|_{\mathbf{R}(D)})^{-1} : \mathbf{R}(D) \to \mathbf{R}(D)$$

is a compact operator.

Proof. As in Propositions 2.9 and 2.12, it suffices to prove that the embedding

$$\big(\mathbf{D}(DB) \cap \mathbf{R}(D), \|\cdot\|_{\mathbf{D}(DB)\cap\mathbf{R}(DB)}\big) \hookrightarrow (\mathbf{R}(D), \|\cdot\|_{L_2})$$

is compact.

Let $\{f^l\}_{l=1}^{\infty} \subset \big(\mathbf{D}(DB) \cap \mathbf{R}(D), \|\cdot\|_{\mathbf{D}(DB)\cap\mathbf{R}(DB)}\big)$ be a sequence such that

$$\|f^l\| + \|DBf^l\| \le C$$

for some $C > 0$. Since $D|_{\mathbf{N}(D^*)}$ is bounded, splitting (3.6) implies

$$\|f^l\| + \|DP_{\mathbf{R}(D)}Bf^l\| \le C,$$

and therefore

$$\|P_{\mathbf{R}(D)}Bf^l\| + \|DP_{\mathbf{R}(D)}Bf^l\| \le C$$

for some $C > 0$. Since $(\lambda - D|_{\mathbf{R}(D)})^{-1}$ is a compact operator, we see that

$$\big(\mathbf{D}(D) \cap \mathbf{R}(D), \|\cdot\|_{\mathbf{D}(D)\cap\mathbf{R}(D)}\big) \hookrightarrow (\mathbf{R}(D), \|\cdot\|_{L_2})$$

is a compact embedding. Hence the sequence $\{P_{\mathbf{R}(D)}Bf^l\}_{l=1}^{\infty}$ contains a Cauchy subsequence, and therefore Lemma 3.5 implies that the sequence $\{f^l\}_{l=1}^{\infty}$ contains a Cauchy subsequence in $L_2(\Omega; \mathrm{C}^M)$ as well. □

We conclude this preliminary subsection by introducing the following setup. We fix a constant $\tau > 0$ from Proposition 3.4 and define

$$\mathcal{H} := \mathbf{R}(D), \qquad T := DB|_{\mathcal{H}}, \qquad T_1 := D_1B|_{\mathcal{H}}, \qquad T_0 := D_0B|_{\mathcal{H}}.$$

By summarizing Propositions 3.3, 3.4, and 3.6, we conclude that T is a closed densely defined operator with $\sigma(T) \subset S_{\omega,\tau}$. Moreover, for each $\lambda \notin S_{\omega,\tau}$, the operator $(\lambda - T)^{-1}$ is compact, and hence there may be only a finite number of eigenvalues of T on the imaginary axis. We denote them by $\{\lambda_i^0\}_{i=1}^N$. We fix positive constants a and R such that $R < a$ and

$$\sigma(T) \cap \{\zeta \in \mathrm{C} : \ |\mathrm{Re}\zeta| \le a\} = \{\lambda_i^0\}_{i=1}^N, \tag{3.7}$$

$$\{\zeta \in \mathrm{C} : |\zeta - \lambda_i^0| \le R\} \cap \{\zeta \in \mathrm{C} : |\zeta - \lambda_j^0| \le R\} = \emptyset, \tag{3.8}$$

$$\{\zeta \in \mathrm{C} : |\zeta - \lambda_i^0| \le R\} \subset S_{\omega,\tau}$$

for $1 \le i < j \le N$.

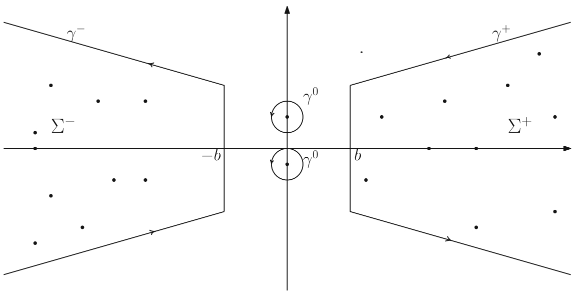

FIGURE 1. N=2

For $\mu \in (\omega, \frac{\pi}{2})$, we fix the open set

$$\Sigma := \Sigma^- \cup \Sigma^+ \cup \Sigma^0,$$

where

$$\Sigma^\pm := \{\zeta \in \mathrm{C} :\ \pm\mathrm{Re}\zeta > a,\ |\mathrm{Im}\zeta| < \tau + |\mathrm{Re}\zeta| \tan\mu\}$$

and

$$\Sigma^0 := \cup_{i=1}^N \{\zeta \in \mathrm{C} : |\zeta - \lambda_i^0| < R\}.$$

Due to (3.7) and (3.8), Σ is a disjoint union of $N+2$ open, connected sets, and $\sigma(T) \subset \Sigma$.

Next, we define

$$H^\infty(\Sigma) := \{h : \Sigma \to \mathrm{C} \text{ holomorphic}, \sup_{z\in\Sigma} |h(z)| < \infty\},$$

$$\Theta(\Sigma) := \{\psi \in H^\infty(\Sigma) : |\psi(z)| \le \frac{C}{|z|^\alpha}, \text{ for some } \alpha, C > 0 \text{ and all } z \in \Sigma\}.$$

For $b > a$ such that

$$\sigma(T) \cap \{\zeta \in \mathrm{C} :\ a \le |\mathrm{Re}\zeta| \le b\} = \emptyset \tag{3.9}$$

and $\nu \in (\omega, \mu)$, $r < R$, we define anti-clockwise oriented curves

$$\gamma^\pm := \{\zeta \in \mathrm{C} :\ \pm\mathrm{Re}\zeta = b,\ |\mathrm{Im}\zeta| \le \tau + |\mathrm{Re}\zeta| \tan\nu\}$$

$$\bigcup \{\zeta \in \mathrm{C} :\ \pm\mathrm{Re}\zeta > b,\ \mathrm{Im}\zeta = \tau + |\mathrm{Re}\zeta| \tan\nu\}$$

$$\bigcup \{\zeta \in \mathrm{C} :\ \pm\mathrm{Re}\zeta > b,\ \mathrm{Im}\zeta = -(\tau + |\mathrm{Re}\zeta| \tan\nu)\},$$

$$\gamma^0 := \bigcup_{i=1}^N \{\zeta \in \mathrm{C} :\ |\zeta - \lambda_i^0| = r\} \tag{3.10}$$

and

$$\gamma := \gamma^- \cup \gamma^+ \cup \gamma^0. \tag{3.11}$$

See Fig. 1.

3.2. The $\Theta(\Sigma)$ Functional Calculus

Here we introduce the following preliminary functional calculus.

Definition 3.7. Let $r < R$, $0 < \nu < \mu$, and $b > a$ such that (3.9) holds. For $\psi \in \Theta(\Sigma)$, we define $\psi(T)$ by

$$\psi(T) = \frac{1}{2\pi i} \int_\gamma \frac{\psi(\zeta)}{\zeta - T} d\zeta, \tag{3.12}$$

where γ is the curve defined in (3.11).

A justification of this definition follows from the next proposition.

Proposition 3.8. *For $\psi \in \Theta(\Sigma)$, the integral*

$$\frac{1}{2\pi i} \int_\gamma \frac{\psi(\zeta)}{\zeta - T} d\zeta$$

converges absolutely. Moreover, the integral is independent of the choice of $\gamma = \gamma^{\pm}(r, \nu, b)$, where $0 < r < R$, $b > a$, and $\nu \in (\omega, \mu)$ such that (3.9) holds.

Proof. We give only the main ideas of the proof. For $\psi \in \Theta(\Sigma)$, Proposition 3.4 implies

$$|\psi(\zeta)| \|(\zeta - T)^{-1}\| \le C \frac{1}{|\zeta|} \frac{1}{|\zeta|^\alpha}.$$

Therefore the first statement follows from the convergence

$$\int_\varepsilon^{+\infty} \frac{1}{x^{\alpha+1}} dx < \infty$$

for $\varepsilon > 0$, since $\alpha > 0$.

Next, let us prove that the integral is independent of the choice of ν. Assume $\omega < \nu_1 < \nu_2 < \mu$. For $P > 0$, we set

$$\delta_P^{\pm}(t) := b \pm i(b \tan \nu + \tau) + Pe^{\pm i(t\nu_2 + (1-t)\nu_1)}.$$

Then

$$\left\| \int_{\delta_P^\pm} \frac{\psi(\zeta)}{\zeta - T} d\zeta \right\| \le C l(\delta_P^\pm) \frac{1}{P^\alpha} \frac{1}{P} \le C \frac{1}{P^\alpha},$$

where $l(\delta_P^\pm)$ is the length of $\delta_P^\pm$. Letting $P \to \infty$, we obtain the desired independence of the choice of ν.

Finally, suppose b_1 and b_2 satisfy the assumptions of the proposition, and $b_1 < b_2$. Then, there is no spectral point inside the region $b_1 \le \mathrm{Re}\lambda \le b_2$. This shows that the integral is independent of the choice of b. □

The proofs of the next three propositions are standard and based on proofs for bisectorial operators, see for instance [1,2]. First we prove that the map given by (3.12) is an algebra homomorphism.

Proposition 3.9. *If $\psi_1, \psi_2 \in \Theta(\Sigma)$, then*

$$\psi_1(T) + \psi_2(T) = (\psi_1 + \psi_2)(T)$$

and

$$\psi_1(T)\psi_2(T) = (\psi_1\psi_2)(T).$$

Proof. For $0 < r_1 < r_2 < R$, $0 < \nu_1 < \nu_2 < \mu$, and $b_1 > b_2 > a$ such that

$$\sigma(T) \cup \{\zeta \in \mathbb{C} :\ a < |\mathrm{Re}\zeta| < b_1\} = \emptyset,$$

we define two curves γ_1 and γ_2 as in (3.11). Note that γ_1 belongs to the interior of γ_2. Then

$$\begin{aligned}(2\pi i)^2 \psi_1(T)\psi_2(T) &= \left(\int_{\gamma_1} \frac{\psi_1(\lambda)}{\lambda - T} d\lambda\right)\left(\int_{\gamma_2} \frac{\psi_2(\zeta)}{\zeta - T} d\zeta\right)\\ &= \int_{\gamma_1}\int_{\gamma_2} \psi_1(\lambda)\psi_2(\zeta)\frac{1}{\zeta-\lambda}\left(\frac{1}{\lambda - T} - \frac{1}{\zeta - T}\right) d\zeta d\lambda\\ &= \int_{\gamma_1} \frac{\psi_1(\lambda)}{\lambda - T}\left(\int_{\gamma_2} \frac{\psi_2(\zeta)}{\zeta - \lambda} d\zeta\right) d\lambda\\ &\quad - \int_{\gamma_2}\left(\int_{\gamma_1} \frac{\psi_1(\lambda)}{\zeta - \lambda} d\lambda\right) \frac{\psi_2(\zeta)}{\zeta - T} d\zeta.\end{aligned}$$

Using the Cauchy formula, we see that the second term vanishes. Therefore

$$(2\pi i)^2 \psi_1(T)\psi_2(T) = 2\pi i \int_{\gamma_1} \frac{\psi_1(\lambda)}{\lambda - T}\psi_2(\lambda) d\lambda = (2\pi i)^2 (\psi_1\psi_2)(T). \qquad \square$$

Next we prove the convergence lemma for the $\Theta(\Sigma)$ functional calculus.

Proposition 3.10. *Let $\psi_n, \psi \in \Theta(\Sigma)$ for $n \in \mathrm{N}$. Assume that $\psi_n \to \psi$ uniformly on compact subsets of Σ, and there exist n-independent constants $\alpha > 0$, $C > 0$ such that*

$$|\psi_n(\zeta)| < \frac{C}{|\zeta|^\alpha}$$

for $\zeta \in \Sigma$. Then $\psi_n(T) \to \psi(T)$ in the operator norm.

Proof. Let us fix $\varepsilon > 0$. One can find an integer $m_1 \in \mathrm{N}$ such that for any $n > m_1$,

$$\left\|\int_{\gamma^0} \frac{\psi_n(\zeta) - \psi(\zeta)}{\zeta - T} d\zeta\right\| \le C\|\psi_n - \psi\|_{L^\infty(\gamma^0)} \left\|\int_{\gamma^0} \frac{1}{\zeta - T} d\zeta\right\| \le \frac{2\pi\varepsilon}{3}.$$

Let $\gamma_{p,q} := \{\zeta \in \gamma :\ p \le |\zeta| < q\}$, then we can fix $M > 0$ such that

$$\left\|\int_{\gamma_{M,\infty}} \frac{\psi_n(\zeta) - \psi(\zeta)}{\zeta - T} d\zeta\right\| \le C\int_M^{+\infty} \frac{1}{r^{\alpha+1}} dr < \frac{2\pi\varepsilon}{3}.$$

Moreover, since $a > 0$, there exists $m_2 \in \mathrm{N}$ such that for any $n > m_2$,

$$\left\|\int_{\gamma_{b,M}} \frac{\psi_n(\zeta) - \psi(\zeta)}{\zeta - T} d\zeta\right\| \le C\|\psi_n - \psi\|_{L^\infty(\gamma_{b,M})} \left\|\int_{\gamma_{a,M}} \frac{1}{\zeta - T} d\zeta\right\| < \frac{2\pi\varepsilon}{3}.$$

By choosing $n > \max(m_1, m_2)$, we obtain $\|\psi_n(T) - \psi(T)\| < \varepsilon$. $\square$

The following proposition, together with Proposition 3.16, allows us to derive an $H^\infty(\Sigma)$ functional calculus from the $\Theta(\Sigma)$ functional calculus.

Proposition 3.11. *Let $\{f_j\}_{j=1}^{\infty} \subset \Theta(\Sigma)$ be a sequence such that $\|f_j\|_{L^\infty(\Sigma)} < C$ and $\|f_j(T)\| < C$ for all $j \in \mathrm{N}$ and some $C > 0$. Assume $f \in H^\infty(\Sigma)$ and $f_j \to f$ uniformly on compact subsets of Σ. Then, for any $u \in \mathcal{H}$, the sequence $\{f_j(T)u\}_{j=1}^{\infty}$ is convergent in $\mathcal{H}$. Moreover, if $f(z) = 1$ on Σ, then $f_j(T)u \to u$ in $\mathcal{H}$.*

Proof. Let $\tau_1 > \tau$ and $u \in \mathbf{D}(T)$. Since $i\tau_1 \notin S_{\omega,\tau}$, there exists $v \in \mathcal{H}$ such that

$$u = (i\tau_1 - T)^{-1}v.$$

Let $\psi(z) = f(z)\frac{1}{i\tau_1 - z}$ and $\psi_j(z) = f_j(z)\frac{1}{i\tau_1 - z}$ on Σ. By Proposition 3.9, we see that $f_j(T)u = \psi_j(T)v$, and therefore Proposition 3.10 implies that $\{f_j(T)u\}_{j=1}^{\infty}$ converges to $\psi(T)v$ in $\mathcal{H}$.

Next, let $u \in \mathcal{H}$. Since $\mathbf{D}(T)$ is a dense set in $\mathcal{H}$, there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset \mathbf{D}(T)$ converging to u in $\mathcal{H}$. Thus

$$\begin{aligned}\|f_m(T)u - f_n(T)u\| &\le \|(f_m(T) - f_n(T))(u - u_k)\| + \|f_m(T)u_k - f_n(T)u_k\| \\ &\le 2C\|u - u_k\| + \|(f_m(T) - f_n(T))u_k\|.\end{aligned}$$

By choosing k large enough and then letting $m, n \to \infty$, we conclude that $\{f_j(T)u\}_{j=1}^{\infty}$ is a Cauchy sequence.

Finally, if $f(z) = 1$ on Σ and $u \in \mathbf{D}(T)$, then the arguments above imply that $f_j(T)u \to u$ in $\mathcal{H}$. For $u \in \mathcal{H}$, there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset \mathbf{D}(T)$ converging to u in $\mathcal{H}$. Thus

$$\|f_j(T)u - u\| = \|f_j(T)u - f_j(T)u_k\| + \|u_k - u\| + \|f_j(T)u_k - u_k\|.$$

By choosing k large enough and then letting $j \to \infty$, we get $f_j(T)u \to u$ in $\mathcal{H}$. $\square$

Remark 3.12. Note that we do not use the uniform boundedness of the sequence $\{f_k(T)\}_{k=1}^{\infty}$ to prove the second part of Proposition 3.11.

Definition 3.13. For an eigenvalue $\lambda \in \sigma(T)$, define the index of λ as the smallest non-negative integer m such that

$$\mathbf{N}((\lambda - T)^m) = \mathbf{N}((\lambda - T)^{m+1}).$$

Next, we prove that each imaginary eigenvalue of T has finite index.

Proposition 3.14. *The index m_i of λ_i^0 is a finite number for $i = 1, \ldots, N$.*

Proof. Let us set

$$p_i(z) = \begin{cases} 1, & \text{if } |z - \lambda_i^0| \le R, \\ 0, & \text{otherwise.} \end{cases}$$

Since $p_i \in \Theta(\Sigma)$, we can define $\Pi_i := p_i(T)$ for $i = 1, \ldots N$. Proposition 3.6 implies that $(\lambda - T)^{-1}$ is a compact operator for all $\lambda \in \rho(T)$. Hence Π_i is a compact operator as the Riemann sum of compact operators. Moreover, by Proposition 3.9, Π_i is a projection. Therefore Π_i is a finite rank operator, and

$$\mathcal{H} = \mathbf{N}(\Pi_i) \oplus \mathbf{R}(\Pi_i). \tag{3.13}$$

Finally, for any integer $m > 0$, we obtain $\mathbf{N}((\lambda_i^0 - T)^m) \subset \mathbf{R}(\Pi_i)$. Therefore the index of λ_i^0 is a finite number. $\square$

We conclude this subsection with the following inequality, which will be used in Sect. 4.

Proposition 3.15. *For fixed $i = 1, \dots, N$, there exists a constant $C > 0$ such that for all $h \in H^\infty(\Sigma)$ satisfying $h(z) = 0$ for $z \notin \{\zeta \in \mathrm{C} : |\lambda_i^0 - \zeta| < R\}$, the following estimate holds*

$$\|h(T)\| \leq C \max_{0 \leq j \leq m_i - 1} |h^{(j)}(\lambda_i^0)|.$$

Proof. From the assumption, $h(T)u = h(T)\Pi_i u = 0$ for $u \in \mathbf{N}(\Pi_i)$. Therefore, due to (3.13), it suffices to prove

$$\|h(T)v\| \leq C \max_{0 \leq j < m_i} |h^{(j)}(\lambda_i^0)| \|v\| \tag{3.14}$$

for all $v \in \mathbf{R}(\Pi_i)$ and some $C > 0$.

Since $T|_{\mathbf{R}(\Pi_i)}$ is bounded and $\mathbf{R}(\Pi_i) = \mathbf{N}((\lambda - T)^{m_i})$, we obtain

$$h(T)v = \sum_{k=0}^{m_i - 1} \frac{h^{(k)}(\lambda_i^0)}{k!} (\lambda_i^0 - T)^k v$$

for any $v \in \mathbf{R}(\Pi_i)$. This implies (3.14). $\square$

3.3. The $\mathbf{H}^\infty(\boldsymbol{\Sigma})$ Functional Calculus

Here we prove that T has a bounded $H^\infty(\Sigma)$ functional calculus. In order to do this, analogous to the functional calculus for bisectorial operators, we need the following quadratic estimate.

Proposition 3.16. *There exists a constant $C > 0$ such that*

$$\int_0^{\frac{1}{\tau}} \left\| \frac{tT}{1 + t^2 T^2} u \right\|^2 \frac{dt}{t} \leq C \|u\|^2 \tag{3.15}$$

for all $u \in \mathcal{H}$.

Proof. Note that $\pm \frac{i}{t} \notin S_{\omega,\tau}$ for $t \in (0, \frac{1}{\tau})$. Hence, by Proposition 3.4, we obtain

$$\|(1 + itT)^{-1} - (1 + itT_1)^{-1}\| = \|(1 + itT)^{-1}(tT_0)(1 + itT_1)^{-1}\| \leq C|t|.$$

Thus

$$\begin{aligned} &\|tT(1 + t^2T^2)^{-1} - tT_1(1 + t^2T_1^2)^{-1}\| \\ &\quad = \frac{1}{2i} \|(1 + itT)^{-1} - (1 - itT)^{-1} + (1 + itT_1)^{-1} - (1 - itT_1)^{-1}\| \leq C|t|. \end{aligned} \tag{3.16}$$

The quadratic estimate (3.15) for T_1 was proved in [5, Theorem 3.1]. Therefore (3.16) implies (3.15). $\square$

Next we prove the following auxiliary lemma.

Lemma 3.17. *Let P, Q be the operators defined by*

$$Pu = \frac{\tau^2}{\tau^2 + T^2} u \quad \text{and} \quad Qu = 2\int_0^{\frac{1}{\tau}} \left(sT\frac{1}{1+s^2T^2} \right)^2 u\frac{ds}{s}$$

for $u \in \mathcal{H}$. Then the following identity

$$(P+Q)u = u$$

holds for $u \in \mathcal{H}$.

Proof. Let us consider the functions

$$f_m(z) = \frac{\tau^2}{\tau^2+z^2} + 2\sum_{j=1}^{m} \frac{1}{j} \frac{\left(\frac{j}{\tau m}z\right)^2}{\left(1+\left(\frac{j}{\tau m}z\right)^2\right)^2}.$$

Observe that $f_m \to 1$ pointwise on Σ. Actually, $\{f_m\}_{m=1}^{\infty}$ converges uniformly on compact subsets of Σ. Indeed, assume there exist a compact subset $K \subset \Sigma$ and $\{x_k\}_{k=1}^{\infty} \subset K$ such that

$$|f_m(x_m) - 1| > c$$

for some $c > 0$. Since K is compact, without lost of generality we assume that $x_m \to x$ for some $x \in K$. Then

$$c < |f_m(x_m) - 1| < |f_m(x) - 1| + |f_m(x_m) - f_m(x)|.$$

The first term tends to zero because of pointwise convergence. To estimate the second term, let us note that $\operatorname{dist}(i\tau, \Sigma) > 0$, and hence there exists $C > 0$ such that

$$\left| \frac{1}{1+(\alpha z)^2} \right| < C$$

for any $\alpha \in [0, \frac{1}{\tau}]$, $z \in \Sigma$. Therefore, straightforward calculations give

$$|f_m(x_m) - f_m(x)| \leq \sum_{j=1}^{m} \frac{1}{j} \left(\frac{j}{\tau m} \right)^2 C|x - x_m| \leq C|x - x_m|.$$

This contradicts our assumption $c > 0$. Thus $f_m \to 1$ uniformly on compact subsets of Σ.

Therefore Proposition 3.11 and Remark 3.12 imply that

$$f_m(T)u \to u \tag{3.17}$$

for all $u \in \mathcal{H}$.

On the other hand, Proposition 3.9 yields

$$f_m(T)u = \frac{\tau^2}{\tau^2+T^2} + 2\sum_{j=1}^{m} \frac{1}{j} \left(\frac{j}{\tau m} T \left(1 + \left(\frac{j}{\tau m} T \right)^2 \right)^{-1} \right)^2 u$$

for each $u \in \mathcal{H}$, and therefore

$$f_m(T)u \to Pu + Qu.$$

Hence, due to (3.17), we derive $Pu + Qu = u$. □

Now we prove that T has a bounded $H^\infty(\Sigma)$ functional calculus . The main idea is contained in [2], [7].

Theorem 3.18. *There exists a constant $C > 0$ such that the following estimate*

$$\|f(T)\| \leq C\|f\|_\infty$$

holds for all $f \in \Theta(\Sigma)$.

Proof. Let P, Q be the operators defined in Lemma 3.17. Then, for $v, u \in \mathcal{H}$,

$$\begin{aligned}
|(v, f(T)u)| &= |(v, (P+Q)f(T)(P+Q)u)| \\
&\leq |(v, Pf(T)Pu)| + |(v, (I-P)f(T)Pu)| \\
&\quad + |(v, Pf(T)(I-P)u)| + |(v, Qf(T)Qu)| \\
&\leq 3|(v, Pf(T)Pu)| + 2|(v, Pf(T)u)| + |(v, Qf(T)Qu)|.
\end{aligned}$$

We estimate each summand separately. For the first two terms, by using Proposition 3.9, we obtain

$$\begin{aligned}
|(v, Pf(T)Pu)| &\leq \|v\|\|u\| \left\| \int_\gamma \frac{\tau^4 f(z)}{(\tau^2+z^2)^2}(z-T)^{-1}dz \right\| \\
&\leq C\|v\|\|u\|\|f\|_\infty
\end{aligned}$$

and

$$\begin{aligned}
|(v, Pf(T)u)| &\leq \|v\|\|u\| \left\| \int_\gamma \frac{\tau^2 f(z)}{\tau^2+z^2}(z-T)^{-1}dz \right\| \\
&\leq C\|v\|\|u\|\|f\|_\infty.
\end{aligned}$$

To estimate the last term, let us set $\psi_t(z) = \frac{tz}{1+t^2z^2} \in \Theta(\Sigma)$, and note that

$$\begin{aligned}
\|\psi_s(T)f(T)\psi_t(T)\| &\leq \|f\|_\infty \int_0^{+\infty} \frac{stx^2}{(1+s^2x^2)(1+t^2x^2)}\frac{dx}{x} \\
&\leq \|f\|_\infty \min\left(\left(\frac{t}{s}\right)^\alpha, \left(\frac{s}{t}\right)^\alpha\right)\left(1+\left|\log\left(\frac{t}{s}\right)\right|\right)
\end{aligned}$$

for $t, s \in (0, \frac{1}{\tau})$ and some $\alpha > 0$. Denote $\eta(x) = \min(x^\alpha, x^{-\alpha})(1+|\log x|)$. Then

$$\begin{aligned}
|(v, Qf(T)Qu)| &\leq C\int_0^{\frac{1}{\tau}}\int_0^{\frac{1}{\tau}} \|\psi_s^*(T)v\|\|\psi_s(T)f(T)\psi_t(T)\|\|\psi_t(T)u\|\frac{dt}{t}\frac{ds}{s} \\
&\leq C\|f\|_\infty \int_0^{\frac{1}{\tau}}\int_0^{\frac{1}{\tau}} \|\psi_s^*(T)v\|\|\psi_t(T)u\|\eta\left(\frac{t}{s}\right)\frac{dt}{t}\frac{ds}{s}.
\end{aligned}$$

The Cauchy-Schwartz inequality yields

$$\begin{aligned}
|(v, Qf(T)Qu)|^2 \leq C\|f\|_\infty^2 &\left(\int_0^{\frac{1}{\tau}} \|\psi_s^*(T)v\|^2 \left(\int_0^{\frac{1}{\tau}} \eta\left(\frac{t}{s}\right)\frac{dt}{t}\right)\frac{ds}{s}\right) \\
&\times \left(\int_0^{\frac{1}{\tau}} \|\psi_t(T)u\|^2 \left(\int_0^{\frac{1}{\tau}} \eta\left(\frac{t}{s}\right)\frac{ds}{s}\right)\frac{dt}{t}\right).
\end{aligned}$$

Finally, using the quadratic estimate from Proposition 3.16, we get

$$|(v, Qf(T)Qu)| \leq C\|f\|_\infty \|u\|\|v\|. \qquad \square$$

Now we are in a position to introduce the following $H^\infty(\Sigma)$ functional calculus for the operator T.

Definition 3.19. Let $f \in H^\infty(\Sigma)$ and $\{\psi_i\}_{i=1}^\infty \subset \Theta(\Sigma)$ be a uniformly bounded sequence such that $\psi_i \to f$ uniformly on compact subsets of Σ. We define

$$f(T)u = \lim_{i\to\infty} \psi_i(T)u$$

for $u \in \mathcal{H}$.

By Proposition 3.11, the definition of $f(T)$ is independent of the choice of sequence $\{\psi_i\}_{i=1}^\infty$. Also observe that the sequence $\{\frac{im}{im+z} f(z)\}_{m=1}^\infty \subset \Theta(\Sigma)$ converges to f uniformly on compact subsets of Σ for $f \in H^\infty(\Sigma)$. Therefore Proposition 3.18 implies that we have a well defined bounded operator $f(T)$ on $\mathcal{H}$ for any $f \in H^\infty(\Sigma)$.

Proposition 3.11 also shows that Definition 3.19 agrees with Definition 3.7 for functions in $\Theta(\Sigma)$.

Let us consider the basic properties of the $H^\infty(\Sigma)$ functional calculus. First we prove that the map given by Definition 3.19 is an algebra homomorphism.

Proposition 3.20. *Let f, $g \in H^\infty(\Sigma)$. Then*

$$f(T) + g(T) = (f + g)(T),$$

and

$$f(T)g(T) = (fg)(T).$$

Proof. Let f, $g \in H^\infty(\Sigma)$ and $\{f_j\}_{j=1}^\infty$, $\{g_j\}_{j=1}^\infty \subset \Theta(\Sigma)$ be the corresponding sequences, see Definition 3.19. Then $\{fg_j\}_{j=1}^\infty \subset \Theta(\Sigma)$ is uniformly bounded and $fg_j \to fg$ on compact subsets of Σ. Therefore

$$(fg)(T)u = \lim_{j\to\infty} (fg_j)(T)u \tag{3.18}$$

for each $u \in \mathcal{H}$. Similarly, for a fixed j, we see that $f_i g_j \to fg_j$ on compact subset of Σ, so that

$$(fg_j)(T)u = \lim_{i\to\infty} (f_i g_j)(T)u \tag{3.19}$$

for any $u \in \mathcal{H}$. Finally, Proposition 3.9 together with (3.18) and (3.19) give

$$\begin{aligned}(fg)(T)u &= \lim_{j\to\infty} \left(\lim_{i\to\infty} (f_i g_j)(T)u\right) = \lim_{j\to\infty} \left(\lim_{i\to\infty} (f_i(T)g_j(T)u)\right) \\ &= \lim_{j\to\infty} (f(T)g_j(T)u) = f(T) \lim_{j\to\infty} (g_j(T)u) = f(T)g(T)u\end{aligned}$$

for each $u \in \mathcal{H}$. $\square$

Next we show the convergence lemma for the $H^\infty(\Sigma)$ functional calculus.

Proposition 3.21. *Let* $\{f_n\}_{n=1}^\infty \subset H^\infty(\Sigma)$ *be a uniformly bounded sequence. Assume* $f \in H^\infty(\Sigma)$ *and* $f_n \to f$ *uniformly on compact subsets of* Σ*. Then* $f_n(T)u \to f(T)u$ *for any* $u \in \mathcal{H}$*.*

Proof. Fix $u \in \mathcal{H}$. By Proposition 3.11, there exists a sequence $\{m_n\}_{n=1}^\infty \subset \mathrm{N}$ such that $m_n > n$ and

$$\left\| \frac{im_n}{im_n - T} f_n(T)u - f_n(T)u \right\| \to 0 \tag{3.20}$$

as $n \to \infty$. On the other hand, the sequence $\left\{ \frac{im_n}{im_n - z} f_n(z) \right\}_{n=1}^\infty \subset \Theta(\Sigma)$ is uniformly bounded and converges to f on compact subsets of Σ. Therefore

$$\left\| \frac{im_n}{im_n - T} f_n(T)u - f(T)u \right\| \to 0 \tag{3.21}$$

as $n \to \infty$. The triangle inequality together with (3.20) and (3.21) imply that

$$\|f_n(T)u - f(T)u\| \to 0.$$

□

3.4. Important Examples of the Functional Calculus

We conclude this section by considering several important examples.

Let us define the following functions on Σ

$$\pi_\pm(z) = \begin{cases} 1, & \text{if } z \in \Sigma^\pm \\ 0, & \text{if } z \in \Sigma \backslash \Sigma^\pm \end{cases}, \quad \pi_0(z) = \begin{cases} 1, & \text{if } z \in \Sigma^0 \\ 0, & \text{if } z \in \Sigma \backslash \Sigma^0 \end{cases}$$

and the corresponding operators $\Pi_\pm := \pi_\pm(T)$, $\Pi_0 := \pi_0(T)$.

Proposition 3.22. *The operators* $\Pi_\pm$ *and* Π_0 *are bounded complementary projections.*

Proof. By Proposition 3.20, we see that

$$\Pi_\pm \Pi_\pm u = \pi_\pm(T)\pi_\pm(T)u = (\pi_\pm\pi_\pm)(T)u = \pi_\pm(T)u = \Pi_\pm u$$

for any $u \in \mathcal{H}$. Similarly, we obtain

$$\Pi_0\Pi_0 u = \Pi_0 u, \quad \Pi_0\Pi_\pm u = 0, \quad \Pi_\pm\Pi_\mp u = 0.$$

Since $(\pi_- + \pi_0 + \pi_+)(z) = 1$ for $z \in \Sigma$, Propositions 3.11 and 3.20 give

$$\Pi_- u + \Pi_0 u + \Pi_+ u = u$$

for any $u \in \mathcal{H}$. □

According to the above proposition, we have a topological splitting

$$\mathcal{H} = \mathbf{R}(\Pi_-) \oplus \mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_+).$$

For a given $u \in \mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_\pm)$, we define

$$u_t := \left(e^{-tT}\right) u$$

for $\pm t > 0$, where e^{-tT} is the operator obtained from the function

$$h_t^\pm(z) = \begin{cases} e^{-tz}, & \text{if } z \in \Sigma^0 \cup \Sigma^\pm, \\ 0, & \text{if } z \in \Sigma \backslash \left(\Sigma^0 \cup \Sigma^\pm\right), \end{cases}$$

by the functional calculus.

Proposition 3.23. *Let $u \in \mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_\pm)$. Then, in $\mathcal{H}$, we have*

$$\partial_t u_t + T u_t = 0 \tag{3.22}$$

for $\pm t > 0$. Moreover, $u_t \to u$ as $t \to 0$.

Proof. Let us fix $u \in \mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_\pm)$. Note that $\partial_t h_t^\pm(z) \in \Theta(\Sigma)$, and

$$\frac{h_{t+\delta}^\pm(z) - h_t^\pm(z)}{\delta} \to \partial_t h_t^\pm(z)$$

uniformly on compact subsets of Σ as $\delta \to 0$. Therefore Proposition 3.11 yields

$$\partial_t h_t^\pm(T)u = \left(\partial_t h_t^\pm\right)(T)u = -T h_t^\pm(T)u.$$

This implies (3.22).

Next, for any compact subset of Σ, we have the uniform convergence of $h_t^\pm \in \Theta(\Sigma)$ to $\pi_0 + \pi_\pm$ as $t \to 0$. Therefore Proposition 3.11 gives

$$\lim_{t\to 0} u_t = \lim_{t \to 0} h_t^\pm(T)u = (\Pi_0 + \Pi_\pm)u = u. \qquad \square$$

4. Application to Waveguide Propagation

In this section, we return to the Helmholtz equation and Maxwell's system of equations and use our new functional calculus for the operator $T := DB|_{\mathbf{R}(D)}$ to investigate acoustic and electromagnetic waves along the waveguide. More precisely, in Theorems 4.1 and 4.2 we prove that all polynomially bounded time-harmonic waves in the semi- or bi-infinite waveguide have representations in $\mathbf{R}(\Pi_0)$ or $\mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_+)$, respectively.

4.1. The Bi-infinite Waveguide

We start by considering the bi-infinite waveguide, that is we consider the ordinary differential equation

$$(\partial_t + T)f = 0, \qquad (t, x) \in \mathrm{R} \times \Omega. \tag{4.1}$$

Theorem 4.1. $(\mathbf{A})$: *Let $f_0 \in \mathbf{R}(\Pi^0)$ and*

$$h_t(z) = \begin{cases} e^{-tz}, & \text{if } z \in \Sigma^0 \\ 0, & \text{if } z \in \Sigma \backslash \Sigma^0. \end{cases}$$

Then $f_t := h_t(T)f_0 \in C(\mathrm{R}; \mathbf{R}(\Pi_0))$ solves Eq. (4.1). Moreover, for any non-negative integer j, there exists a constant $C = C(j) > 0$, which is independent of the choice of f_0, such that

$$\|\partial_t^j f_t\| + \|T^j f_t\| < C(1 + |t|^l)\|f_0\| \tag{4.2}$$

with $l = \sup_i m_i - 1$, where m_i is the index of λ_i^0 for $i = 1, \dots, N$.

$(\mathbf{B})$: *Conversely, let $f_t \in C(\mathrm{R}; \mathcal{H})$ such that $f_t \in \mathbf{D}(T)$ for all $t \in \mathrm{R}$. Assume that f_t solves Eq. (4.1) and satisfies*

$$\|f_t\| < Ce^{\varepsilon|t|} \tag{4.3}$$

for all $t \in \mathrm{R}$ *and some* t*-independent constants* $C > 0$ *and* $\varepsilon \in (0, a)$*. Then* $f_0 \in \mathbf{R}(\Pi^0)$ *and*

$$f_t = h_t(T) f_0$$

for any $t \in \mathrm{R}$*.*

Proof. $(\mathbf{A})$: Note that $h_t(z) \in \Theta(\Sigma)$ for any $t \in \mathrm{R}$. Therefore Theorems 3.10 and 3.18 imply

$$\|h_t(T) - h_{t+\delta}(T)\| \leq C\|e^{-tz} - e^{-(t+\delta)z}\|_{L^\infty(\Sigma^0)} \to 0$$

as $\delta \to 0$, so that $f_t \in C(\mathbf{R}; \mathcal{H})$. By Proposition 3.23, f_t solves Eq. (4.1). The boundedness of $T|_{\mathbf{R}(\Pi_0)}$ and Proposition 3.15 together imply

$$\|\partial_t^j f_t\| + \|T^j f_t\| \leq C \sum_{i=1}^{N} \max_{0 \leq k \leq m_i - 1} |h^{(k)}(\lambda_i^0)| \|f_0\|,$$

which shows (4.2).

$(\mathbf{B})$: Let us set

$$g_t^+(z) = \begin{cases} e^{-tz}, & \text{if } z \in \Sigma^+, \\ 0, & \text{if } z \in \Sigma \setminus \Sigma^+ \end{cases}$$

for $t > 0$. By assumption, f_t solves (4.1). Therefore, for $t_0 \in \mathrm{R}$ and $t < t_0$, we obtain

$$\partial_t \left(g_{t_0-t}^+(T) \Pi_+ f_t \right) = g_{t_0-t}^+(T) \left(\partial_t + T \right) \Pi_+ f_t = g_{t_0-t}^+(T) \Pi_+ \left(\partial_t + T \right) f_t = 0.$$

Integrating over (P, t_0), for some $P < t_0$, gives

$$\Pi_+ f_{t_0} - g_{t_0-P}^+(T) \Pi^+ f_P = 0.$$

By Theorem 3.18 and estimate (4.3), we obtain

$$\|g_{t_0-P}^+(T) \Pi_+ f_P\| \leq C \sup_{z \in \Sigma^+} \left| e^{-(t_0-P)z} \right| e^{\varepsilon|P|} \leq C e^{-(t_0-P)a} e^{\varepsilon|P|}.$$

Letting $P \to -\infty$, we conclude that $\Pi_+ f_{t_0} = 0$ for $t_0 \in \mathrm{R}$.

Similarly, let

$$g_t^-(z) = \begin{cases} e^{tz}, & \text{if } z \in \Sigma^-, \\ 0, & \text{if } z \in \Sigma \setminus \Sigma^- \end{cases}$$

for $t > 0$. Then, for $t_0 \in \mathrm{R}$ and $t > t_0$, we derive

$$\partial_t \left(g_{t-t_0}^-(T) \Pi_- f_t \right) = 0.$$

By integrating over (t_0, P) and letting $P \to +\infty$, we conclude $\Pi_- f_{t_0} = 0$ for $t_0 \in \mathrm{R}$, and hence $f_0 \in \mathbf{R}(\Pi^0)$. Then the first part of this theorem implies that $\widetilde{f}_t = h_t(T) f_0$ solves Eq. (4.1), and hence

$$\partial_t (h_{s-t}(T)(f_t - \widetilde{f}_t)) = 0$$

for $t < s$. By integrating over (P, s) and letting $P \to 0$, one can prove $f_s = \widetilde{f}_s$, so that $f_t = h_t(T) f_0$. $\square$

4.2. The Semi-infinite Waveguide

Next to obtain a similar result for the semi-infinite waveguide we consider the ordinary differential equation

$$(\partial_t + T)f = 0, \quad (t, x) \in \mathrm{R}^+ \times \Omega, \tag{4.4}$$

where $\mathrm{R}^+ := (0, +\infty)$.

Theorem 4.2. $(\mathbf{A})$: *Let* $f_0 \in \mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_+)$ *and*

$$h_t(z) = \begin{cases} e^{-tz}, & \text{if } z \in \Sigma^0 \cup \Sigma^+, \\ 0, & \text{if } z \in \Sigma^- \end{cases}$$

for $t > 0$. *Then* $f_t := h_t(T)f_0 \in C(\mathrm{R}^+; \mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_+))$ *solves* (4.4). *Moreover, for any nonnegative integer* j, *there exists a constant* $C = C(j) > 0$, *which is independent of the choice of* f_0, *such that*

$$\|\partial_t^j f_t\| + \|T^j f_t\| < C(t^l + t^{-j})\|f_0\| \tag{4.5}$$

with $l = \sup_i m_i - 1$, *where* m_i *is the index of* λ_i^0 *for* $i = 1, \ldots, N$. *Furthermore,* $\lim_{t\to 0} f_t = f_0$ *in* $\mathcal{H}$.

$(\mathbf{B})$: *Conversely, let* $f_t \in C(\mathrm{R}^+; \mathcal{H})$ *such that* $f_t \in \mathbf{D}(T)$ *for all* $t \in \mathrm{R}^+$. *Assume that* f_t *solves* (4.4) *and satisfies*

$$\|f_t\| < Ce^{\varepsilon|t|} \tag{4.6}$$

for all $t \in \mathrm{R}^+$ *and some* t*-independent constants* $C > 0$ *and* $\varepsilon \in (0, a)$. *Then there exists* $f_0 \in \mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_+)$ *such that*

$$f_t = h_t(T)f_0$$

for $t \in \mathrm{R}^+$. *Moreover,* $f_0 \in \mathbf{R}(\Pi_+)$ *if and only if* $\|f_t\| \to 0$ *as* $t \to \infty$.

Proof. $(\mathbf{A})$: By Proposition 3.23, f_t solves Eq. (4.4). From Theorem 4.1, we see that

$$\|\partial_t^j h_t(T)\Pi_0 f_0\| + \|T^j h_t(T)\Pi_0 f_0\| \le C(1 + |t|^l)\|\Pi_0 f_0\|. \tag{4.7}$$

Theorem 3.18 implies now that

$$\|\partial_t^j h_t(T)\Pi_+ f_0\| + \|T^j h_t(T)\Pi_+ f_0\| \le \sup_{z\in\Sigma^+} \left|(1 + z^j)e^{-tz}\right| \|\Pi_+ f_0\|.$$

This gives

$$\|\partial_t^j h_t(T)\Pi_+ f_0\| + \|T^j h_t(T)\Pi_+ f_0\| \le Ct^{-j}\|\Pi_+ f_0\| \tag{4.8}$$

as $t \to 0$, and

$$\|\partial_t^j h_t(T)\Pi_+ f_0\| + \|T^j h_t(T)\Pi_+ f_0\| \le Ce^{-ta}\|\Pi_+ f_0\| \tag{4.9}$$

as $t \to \infty$. Combining (4.7)–(4.9), we obtain estimate (4.5).

Since $h_t \to \pi_0 + \pi_+$ uniformly on compact subsets of Σ, Proposition 3.11 implies

$$\lim_{t\to 0} h_t(T)f_0 = (\pi_0(T) + \pi_+(T))f_0 = (\Pi_0 + \Pi_+)f_0 = f_0.$$

(**B**) : Let us set

$$g_t(z) = \begin{cases} e^{tz}, & \text{if } z \in \Sigma^-, \\ 0, & \text{if } z \in \Sigma \backslash \Sigma^- \end{cases}$$

for $t > 0$. By our assumption, f_t solves (4.4). Therefore, for $t > s > 0$, we obtain

$$\partial_t \left(g_{t-s}(T)\Pi_- f_t\right) = g_{t-s}(T)(\partial_t + T)\Pi_- f_t = g_{t-s}(T)\Pi_-(\partial_t + T)f_t = 0.$$

Integrating over (s, P), for some $P > s$, gives

$$g_{P-s}(T)\Pi_- f_P - \Pi_- f_s = 0.$$

Theorem 3.18 and estimate (4.6) imply now that

$$\|g_{P-s}(T)\Pi_- f_P\| \leq Ce^{-aP}e^{\varepsilon P}.$$

Letting $P \to \infty$, we get $\Pi_- f_s = 0$, so that $f_s \in \mathbf{R}(\Pi_0) \oplus \mathbf{R}(\Pi_+)$ for all $s \in \mathrm{R}^+$.

Fix $s > 0$. The first part of this theorem implies that $f_{s+t} - h_t(T)f_s$ solves (4.4), and hence

$$\partial_t(h_{r-t}(T)(f_{s+t} - h_t(T)f_s)) = 0$$

for $0 < t < r$. Let $\varepsilon \in (0, r)$, then integration over $(P, r - \varepsilon)$ gives

$$h_{r-(r-\varepsilon)}(T)(f_{s+(r-\varepsilon)} - h_{r-\varepsilon}(T)f_s) - h_{r-P}(T)(f_{s+P} - h_P(T)f_s) = 0.$$

Letting P, $\varepsilon \to 0$, we obtain $f_{s+r} - h_r(T)f_s = 0$ for $r > 0$, or equivalently

$$f_t = h_{t-s}(T)f_s \tag{4.10}$$

for $0 < s < t$.

Since f_t is uniformly bounded as $t \to 0$, one can find a decreasing sequence $\{s_k\}_{k=1}^{\infty} \subset \mathrm{R}^+$ such that $s_k \to 0$ and $f_{s_k} \to f_0$ weakly in $\mathcal{H}$. Let ϕ be a test function. Then, due to (4.10),

$$\begin{aligned}(f_t, \phi) &= (h_{t-s_k}(T)f_{s_k}, \phi) = (f_{s_k}, h_{t-s_k}(T)^*\phi) \\ &= (f_{s_k}, h_{t-s_k}(T)^*\phi - h_t(T)^*\phi) + (f_{s_k}, h_t(T)^*\phi)\end{aligned}$$

for $t > s_k$. Therefore

$$|(f_t, \phi) - (f_{s_k}, h_t(T)^*\phi)| \leq \|f_{s_k}\| \, \|h_{t-s_k}(T)^*\phi - h_t(T)^*\phi\| \, .$$

Letting $k \to \infty$, we obtain

$$|(f_t, \phi) - (f_0, h_t(T)^*\phi)| \leq 0.$$

Hence $f_t = h_t(T)f_0$. Since $h_t \to \pi_0 + \pi_+$ uniformly on compact subsets of Σ, we conclude that $f_t \to f_0$ strongly in $\mathcal{H}$.

Finally, if $f_0 \in \mathbf{R}(\Pi_+)$, then

$$\|f_t\| \leq Ce^{-at}\|f_0\|$$

for $t > 0$. Hence $\|f_t\| \to 0$ as $t \to \infty$.

Conversely, assume that $\|f_t\| \to 0$ as $t \to \infty$. Then $\|h_t(T)\Pi_0 f_0\| \to 0$ as $t \to \infty$, and therefore

$$\left\| \sum_{i=1}^{N} \sum_{k=0}^{m_i-1} \frac{(-t)^k e^{-t\lambda_i^0}}{k!} (\lambda_i^0 - T)^k \Pi_i \Pi_0 f_0 \right\| \to 0$$

as $t \to \infty$. Since $\mathbf{R}(\Pi_0) = \bigoplus_{j=1}^{N} \mathbf{R}(\Pi_j)$, and λ_i^0 is purely imaginary, we obtain

$$\left\| \sum_{k=0}^{m_i-1} \frac{(-t)^k}{k!} (\lambda_i^0 - T)^k \Pi_i f_0 \right\| \to 0 \tag{4.11}$$

as $t \to \infty$ for $i = 1, \dots, N$. Let us define

$$l_i := \sup\{k = 1, \dots, m_i - 1 : \ (\lambda_i^0 - T)^k \Pi_i f_0 \neq 0\}.$$

If $l_i > 0$, the identity

$$t^{l_i} = (t-1)(t^{l_i-1} + t^{l_i-2} + \cdots + 1) + 1$$

implies

$$\left\| \sum_{k=0}^{m_i-1} \frac{(-t)^k}{k!} (\lambda_i^0 - T)^k \Pi_i f_0 \right\| \geq \frac{1}{2} \left\| \frac{(-t)^{l_i}}{l_i!} (\lambda_i^0 - T)^{l_i} \Pi_i f_0 \right\|$$

for sufficiently large $t > 0$. By (4.11), the left hand side tends to 0 as $t \to \infty$, while the right hand side tends to ∞. This contradiction shows that $l_i = 0$ for $i = 1, \dots, N$. Hence (4.11) implies $\Pi_i f_0 = 0$ for $i = 1, \dots, N$. Therefore $f_0 \in \mathbf{R}(\Pi_+)$. $\square$

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