MITTAG-LEFFLER EULER INTEGRATOR FOR A STOCHASTIC FRACTIONAL ORDER EQUATION WITH ADDITIVE NOISE

MIHÁLY KOVÁCS, STIG LARSSON, AND FARDIN SAEDPANAH

Abstract. A semilinear stochastic fractional order equation, and its deterministic counterpart, is considered. Full discretization of the model problem is carried out and optimal strong rate of convergence is proved, which is (almost) twice the rate of the rate for the implicit Euler method. A generalised exponential Euler method, named here as the Mittag-Leffler Euler integrator, is used for the temporal discretization. Spatial discretization by the spectral Galerkin method is then performed. The framework allows for nonlinearities from a general class of Nemytskij operators. Multiple spatial dimension is allowed when the noise is of trace class. Numerical experiments are presented to validate the theory.

1. Introduction

We study numerical approximation of a class of semilinear stochastic fractional order equations. Our main example is the fractional order partial differential equation

$$u_t(x,t) - (J_0^\alpha \Delta u(x,\cdot))(t) = f(u(x,t)) + \dot{\xi}(x,t), \quad (x,t) \in \mathcal{D} \times (0,T], \tag{1.1}$$

with a bounded domain $\mathcal{D} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, together with initial condition $u(\cdot,0) = u_0$ in $\mathcal{D}$, and boundary condition $u = 0$ on $\partial\mathcal{D}$. Further, Δ is the Laplacian, f is a smooth real-valued function, $\dot{\xi}$ is zero-mean, real-valued, Gaussian noise, and J_0^α is the fractional integral of order α, [12],

$$(J_0^\alpha g)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\,\mathrm{d}s, \quad 0 < \alpha < 1.$$

The present work applies also to the deterministic equation (1.1) with $\dot{\xi} = 0$.

By considering the solution $u(t) = u(\cdot,t)$ as a Hilbert space-valued stochastic process, equation (1.1) can be regarded in an abstract setting as a semilinear stochastic Volterra type evolution equation

$$\mathrm{d}u(t) + \int_0^t b(t-s)Au(s)\,\mathrm{d}s\,\mathrm{d}t = F(u(t))\,\mathrm{d}t + \mathrm{d}W(t), \quad t \in (0,T], \tag{1.2}$$

where A is a self-adjoint, positive definite, not necessarily bounded, operator on the Hilbert space H, W is an H-valued Wiener process with covariance operator

---

Date: March 8, 2018.

1991 *Mathematics Subject Classification.* 34A08, 45D05, 45K05, 60H15, 60H35, 65M12, 65M60.

Key words and phrases. Euler integrator, fractional equations, Riesz kernel, strong convergence, integro-differential equations, stochastic differential equations.

Research supported in part by Nordforsk, project number 74756.

Q, $F\colon H \to H$ is a nonlinear operator, and with the Riesz kernel

$$b(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \alpha < 1. \tag{1.3}$$

In particular, the kernel has the important property of being positive, that is, for any continuous function g on $[0, T]$,

$$\int_0^T \int_0^t b(t-s) g(s) g(t) \, \mathrm{d}s \, \mathrm{d}t \geq 0. \tag{1.4}$$

The framework of this work applies also to slightly more general kernels, which have similar smoothing effects, e.g., the tempered Riesz kernel

$$b(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-\eta t}, \quad 0 < \alpha < 1, \ \eta \geq 0, \tag{1.5}$$

see Remark 1 for further discussion.

We note that (1.2) is an integro-differential equation. When the kernel b in (1.2) is smooth, e.g., exponential kernels, these equations reveal a hyperbolic behaviour, whereas for weakly singular kernels, e.g., the Riesz kernel (1.3), they exhibit certain parabolic features.

Equation (1.2) can be seen either as an abstract stochastic fractional order ordinary differential equation (ODE) on a suitable Hilbert space H, or as a system of stochastic fractional order ODEs derived from a suitable spatial discretization, e.g., spectral Galerkin method or finite element method.

The literature on numerical methods for stochastic PDEs, such as stochastic parabolic and hyperbolic PDEs, is mature. In some works, by using exponential integrators [6], the strong rate of convergence has been improved for the stochastic heat equation, see, e.g., [4, 17], and for the stochastic wave equation, see, e.g., [2], and the references therein. The drawback of the exponential integrators for stochastic PDEs is that, the eigenfunctions of the operator A and of the covariance operator Q of the noise must coincide and must be known explicitly, so that the scheme can be implemented.

However, the literature on numerical analysis of stochastic fractional order equations is more scarce, containing only [1], [8], [9], to the best of our knowledge. Here, we study full discretization of the model problem (1.1), with strong form (1.2), and its deterministic counterpart, i.e., when $\dot{\xi} = 0$. We use a generalised exponential Euler method, named here as the Mittag-Leffler Euler integrator, for the temporal discretization. Full discretization is then formulated by the spectral Galerkin method for spatial discretization. We prove optimal strong rate of convergence, which the temporal rate is (almost) twice the rate of the rate for the implicit Euler method. The framework presented here allows for a general class of Nemytskij operators. Multiple spatial dimension variable is allowed, for the trace class noise. The obtained rate in this work is the best possible and optimal, for both the stochastic equation and its deterministic counterpart, see Remark 3–Remark 6.

Applying implicit integrators, such as the implicit Euler method, to stochastic PDEs do not yield the expected rate of strong convergence compared to the regularity of the solution, see, e.g., [1, 8], and the references therein. This is the case also for explicit integrators. Besides, these integrators suffer from relatively small stability domain, that require unrealistic small time step sizes for integrating stiff problems. Though implicit integrators, for deterministic PDEs, reach the expected rate of convergence, however, the exponential integrators, which are explicit are exact for the linear parts and compute the non-linear part by simple quadratures. Therefore they are computationally more efficient, [6].

The solution operator (resolvent) contains full information on regularity of the solution. Exponential integrators, based on using an explicit representation of the (mild) solution in terms of the variation of constant formula, take advantage of this information. It solves the linear part exactly, using linear functionals of the noise, together with approximating the semilinear part using the fact that the integral of the resolvent and its square power is computable exactly. We note that including more information about the noise, and the smoothing effect of the resolvent, generated by the linear operator A in (1.1), has an important role in proving higher order rate of strong convergence.

The outline of the paper is as follows. In §2, we present some preliminaries. Then we formulate the full discretization in §3 and we prove the strong order of convergence. In §4 we discuss numerical implementation and present some experimental results to illustrate the theory.

2. **Preliminaries**

Let H be a separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and A be a self-adjoint, positive definite, not necessarily bounded operator in H with compact inverse. An important example is $H = L_2(\mathcal{D})$ and $A = -\Delta$ with homogeneous Dirichlet boundary conditions. Let $\{(\lambda_k,\varphi_k)\}_{k=1}^{\infty}$ be the eigenpairs of A, i.e.,

$$A\varphi_k = \lambda_k\varphi_k, \quad k \in \mathbb{N}. \tag{2.1}$$

It is known that

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the eigenvectors $\{\varphi_k\}_{k=1}^{\infty}$ form an orthonormal basis for H. We introduce the Sobolev spaces

$$\dot{H}^{\nu} := \mathrm{dom}(A^{\frac{\nu}{2}}), \quad \|v\|_{\nu}^2 := \|A^{\frac{\nu}{2}}v\|^2 = \sum_{k=1}^{\infty} \lambda_k^{\nu}(v,\varphi_k)^2, \quad \nu \in \mathbb{R},\ v \in \dot{H}^{\nu}.$$

Let $\mathcal{L} = \mathcal{L}(H)$ denote the space of all bounded linear operators on H. We also consider the space of Hilbert–Schmidt operators, that is, the space of all operators $T \in \mathcal{L}$ for which

$$\|T\|_{\mathrm{HS}} = \sum_{k=1}^{\infty} \|T\varphi_k\|^2 < \infty.$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, with Bochner spaces $L_p(\Omega; H) = L_p\big((\Omega,\mathcal{F},\mathbb{P}); H\big)$, $p \ge 2$. We let $Q \in \mathcal{L}$ be a self-adjoint, positive semidefinite operator and $H_0 = Q^{\frac{1}{2}}(H)$ be the Hilbert space with the inner product $\langle u, v\rangle_{H_0} = \langle Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v\rangle$, where $Q^{-\frac{1}{2}}$ denotes the pseudoinverse of $Q^{\frac{1}{2}}$, when it is not injective, and $Q^{\frac{1}{2}}$ is the unique positive semidefinite square root of Q. By $\mathcal{L}_2^0 = \mathcal{L}_2^0(H)$ we denote the space of Hilbert–Schmidt operators $H_0 \to H$. Thus, $\|T\|_{\mathcal{L}_2^0} = \|TQ^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$, for $T \in \mathcal{L}_2^0$. Then we let W be Q-Wiener process in H

with respect to $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$. We recall the Itô isometry,

$$\Big\|\int_0^t \phi(s)\,\mathrm{d}W(s)\Big\|_{L_2(\Omega;H)} = \Big\|\Big(\int_0^t \|\phi(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s\Big)^{\frac12}\Big\|_{L_2(\Omega;\mathbb{R})}, \tag{2.2}$$

and the Burkholder–Davis–Gundy inequality, for $p\ge 2$,

$$\Big\|\int_0^t \phi(s)\,\mathrm{d}W(s)\Big\|_{L_p(\Omega;H)} \le C_p\Big\|\Big(\int_0^t \|\phi(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s\Big)^{\frac12}\Big\|_{L_p(\Omega;\mathbb{R})}, \tag{2.3}$$

for strongly measurable functions $\phi\colon [0,T]\to\mathcal{L}_2^0$, [5].

We denote, henceforth,

$$\rho=\alpha+1,\quad \rho\in(1,2),$$

recalling that $\alpha\in(0,1)$ in (1.3). We quantify the the regularity of the noise by $\beta\in(0,\frac1\rho]$ through the assumption that there is a constant B such that

$$\big\|A^{\frac{\beta-\frac1\rho}{2}}\big\|_{\mathcal{L}_2^0} = \big\|A^{\frac{\beta-\frac1\rho}{2}}Q^{\frac12}\big\|_{\mathrm{HS}}\le B. \tag{2.4}$$

Trace class noise, $\mathrm{Tr}(Q)=\|Q^{\frac12}\|^2_{\mathrm{HS}}<\infty$, corresponds to $\beta=\frac1\rho$.

Let us consider the linear non-homogeneous deterministic problem

$$u_t(t)+\int_0^t b(t-s)Au(s)\,\mathrm{d}s = f(t),\ t\in(0,T];\quad u(0)=u_0. \tag{2.5}$$

There exists a resolvent family $\{S(t)\}_{t\ge0}$ of bounded linear operators on H, which is strongly continuous for $t\ge0$ and differentiable for $t>0$, such that the unique mild solution of (2.5) is given by, [14],

$$u(t)=S(t)u_0+\int_0^t S(t-s)f(s)\,\mathrm{d}s,$$

where $u_0\in H,\ f\in L_2([0,T];H)$. We note that the resolvent family does not enjoy the semigroup property due to the nonlocality of the memory term in (2.5). However, an explicit representation is given by the spectral decomposition

$$S(t)v=\sum_{k=1}^\infty s_k(t)(v,\varphi_k)\varphi_k, \tag{2.6}$$

where the functions $s_k(t)$ are the solutions of

$$\dot s_k(t)+\lambda_k\int_0^t b(t-s)s_k(s)\,\mathrm{d}s=0,\ t>0;\quad s_k(0)=1.$$

Next, we collect the relevant properties of the resolvent family $\{S(t)\}_{t\ge0}$. A simple energy argument shows that the family $\{S(t)\}_{t\ge0}$ is contractive, i.e.,

$$\|S(t)\|_{\mathcal{L}}\le 1,\quad t\ge0. \tag{2.7}$$

Furthermore, we assume that it enjoys the following smoothing properties: There is M such that for $t>0$, we have

$$\|A^sS(t)\|_{\mathcal{L}}\le Mt^{-s\rho},\qquad s\in\Big[0,\frac1\rho\Big], \tag{2.8}$$

$$\|A^s\dot S(t)\|_{\mathcal{L}}\le Mt^{-s\rho-1},\qquad s\in\Big[0,\frac1\rho\Big], \tag{2.9}$$

$$\|A^{-s}\dot S(t)\|_{\mathcal{L}}\le Mt^{s\rho-1},\qquad s\in[0,1]. \tag{2.10}$$

Remark 1. These are verified in [11, Theorem 5.5] for the Riesz kernel and in [3, Lemma A.4] for more general kernels, We note that for the Riesz kernel (1.3), which is our main example, estimates (2.8) and (2.9) hold also for $s \in [0,1]$, see [11, Theorem 5.5], but we do not need this extended range of s for the present analysis. A more general class of kernels b for which (2.8)–(2.10) are satisfied is the the class of 4-monotone kernels such that $0 \neq b \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, $\lim_{t\to\infty} b(t) = 0$, with

$$\rho := 1 + \frac{2}{\pi} \sup\{|\arg \widehat{b}(\lambda)|,\ \mathrm{Re}\,\lambda > 0\} \in (1,2),$$

and $\widehat{b}(\lambda) \leq C\lambda^{1-\rho},\ \lambda > 1$, where this latter condition may be substituted by the condition $\|b\|_{L^1(0,t)} \leq Ct^{\rho-1}$, $t \in (0,1)$, see [3, Remark 3.8 and Lemma A.4]. In particular, b does not have to be analytic.

We now recall the semilinear stochastic equation (1.2), which is the abstract form of the model problem (1.1). Its mild solution is an adapted H-valued stochastic process, $u(t)$, such that, for $t \in [0,T]$,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\,\mathrm{d}s + \int_0^t S(t-s)\,\mathrm{d}W(s),\ \mathbb{P}\text{-a.s.} \tag{2.11}$$

In addition to the singularity exponent $\rho = \alpha + 1 \in (1,2)$ from (1.3) and the regularity parameter $\beta \in (0, \frac{1}{\rho}]$ in (2.4), we assume that there are $\delta \in [1, \frac{2}{\rho})$, $\gamma \in [0,\beta)$, $\eta \in [1, \frac{2}{\rho})$, and a constant L, such that

$$\|F(u)\| \leq L(1 + \|u\|), \quad \|F'(u)v\| \leq L\|v\|, \qquad u, v \in H, \tag{2.12}$$

$$\|F'(u)v\|_{-\delta} \leq L(1 + \|u\|_\gamma)\|v\|_{-\gamma}, \qquad u \in \dot{H}^\gamma,\ v \in \dot{H}^{-\gamma}, \tag{2.13}$$

$$\|F''(u)(v_1, v_2)\|_{-\eta} \leq L\|v_1\|\|v_2\|, \qquad v_1, v_2 \in H. \tag{2.14}$$

Our main example is $H = L_2(\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^d$, and $A = -\Delta$, the negative of the Dirichlet Laplacian. Here F can be taken to be a Nemytskij operator defined by $F(u)(x) = f(u(x))$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function with bounded derivatives of orders 1 and 2. Then (2.12) clearly holds and (2.14) is satisfied with $\eta > d/2$ because of Sobolev's inequality. The additional assumption $\eta \leq \frac{2}{\rho}$ puts a restriction on ρ, namely, $1 < \rho < 4/d$. For (2.13) we refer to Lemma 4.4 in [16], which can be extended from $d = 1$ to $d \leq 3$.

Lemma 1. *Under the above assumptions, let $p \geq 2$, and assume $\|u_0\|_{L_p(\Omega;\dot{H}^\gamma)} \leq K$. Then, there is a unique mild solution $u \in C([0,T]; L_p(\Omega;H))$ of (2.12). Furthermore, for a constant $C = C(B,K,L,M,T,\beta,\gamma,\delta,\rho,p)$,*

$$\sup_{t\in[0,T]} \|u(t)\|_{L_p(\Omega;\dot{H}^\gamma)} \leq C. \tag{2.15}$$

Proof. The existence and uniqueness of a mild solution $u \in C([0,T]; L_p(\Omega;H))$ of (2.12) can be proved, even only under Assumption (2.12), via a standard Banach fixed point argument using (2.4) and (2.8)–(2.10), see, for example the proof of [3, Theorem 3.3]. Therefore,

$$\|u(t)\|_{L_p(\Omega;H)} \leq C, \quad t \in [0,T], \tag{2.16}$$

which is (2.15) with $\gamma = 0$. For $\gamma \in (0, \beta)$, using (2.11), we have

$$\begin{aligned}\|u(t)\|_{L_p(\Omega;\dot H^\gamma)} \le \|S(t)\|_{\mathcal{L}}\|u_0\|_{L_p(\Omega;\dot H^\gamma)} &\\ + \int_0^t \|A^{\frac{\gamma}{2}} S(t-s)\|_{\mathcal{L}} \|F(u(s))\|_{L_p(\Omega;H)}\,\mathrm{d}s &\\ + \Big\| \int_0^t A^{\frac{\gamma}{2}} S(t-s)\,\mathrm{d}W(s)\Big\|_{L_p(\Omega;H)}. &\end{aligned}$$

By using (2.7), (2.8), (2.12), (2.3), and (2.16), we obtain

$$\begin{aligned}\|u(t)\|_{L_p(\Omega;\dot H^\gamma)} &\le \|u_0\|_{L_p(\Omega;\dot H^\gamma)} + L\int_0^t \|A^{\frac{\gamma}{2}} S(t-s)\|_{\mathcal{L}}\big(1 + \|u(s)\|_{L_p(\Omega;H)}\big)\,\mathrm{d}s\\ &\quad + C_p \Big\| \Big(\int_0^t \|A^{\frac{\gamma}{2}} S(t-s) Q^{\frac12}\|^2_{\mathrm{HS}}\,\mathrm{d}s\Big)^{\frac12}\Big\|_{L_p(\Omega;\mathbb{R})}\\ &\le C + C\int_0^t (t-s)^{-\frac{\gamma\rho}{2}}\,\mathrm{d}s\\ &\quad + C_p \big\|A^{\frac{\beta-\frac{1}{\rho}}{2}} Q^{\frac12}\big\|_{\mathrm{HS}} \Big(\int_0^t \Big\|A^{\frac{(\gamma-\beta)+\frac{1}{\rho}}{2}} S(t-s)\Big\|^2 \mathrm{d}s\Big)^{\frac12}.\end{aligned}$$

By using (2.4) and (2.8) again, we have

$$\|u(t)\|_{L_p(\Omega;\dot H^\gamma)} \le C + BC\Big(\int_0^t (t-s)^{-1+(\beta-\gamma)\rho}\,\mathrm{d}s\Big)^{\frac12},$$

where the integral is finite, since $(\beta-\gamma)\rho \in (0,1)$. This completes the proof. □

Remark 2. In the deterministic case, i.e., when $\mathrm{d}W = 0$, by following the proof of Lemma 1, it is straightforward to prove that, assuming $u_0 \in \dot H^{2\gamma}$ for some $\gamma \in [0, \frac{1}{\rho})$, we have the regularity estimate

$$\sup_{t\in[0,T]} \|u(t)\|_{\dot H^{2\gamma}} \le C. \tag{2.17}$$

3. Full discretization

In this section we formulate a fully discrete method for approximation of (1.2), the abstract form of (1.1). We use the spectral Galerkin method for spatial discretization in combination with time discretization based on an exponential Euler type method. We refer to the proposed time discretization method as the Mittag-Leffler Euler integrator (MLEI), since the solution operator can be represented in the form of a generalized Mittag-Leffler operator. We give more details in §4, where numerical examples are presented.

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a uniform partition of the time interval $[0,T]$, with time step $\Delta t = t_{m+1} - t_m,\ m = 0, 1, \cdots, M-1$. Then, for $m = 0, 1, \cdots, M$, by using the variation of constant formula (2.11), we have

(3.1)

$$\begin{aligned}u(t_m) &= S(t_m)u_0 + \int_0^{t_m} S(t_m-\sigma)F(u(\sigma))\,\mathrm{d}\sigma + \int_0^{t_m} S(t_m-\sigma)\,\mathrm{d}W(\sigma)\\ &= S(t_m)u_0 + \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} S(t_m-\sigma)F(u(\sigma))\,\mathrm{d}\sigma + \int_0^{t_m} S(t_m-\sigma)\,\mathrm{d}W(\sigma),\end{aligned}$$

Following the idea of exponential integrators, we formulate the MLEI

$$U_m = S(t_m)u_0 + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma)\,\mathrm{d}\sigma\, F(U_j) + \int_0^{t_m} S(t_m - \sigma)\,\mathrm{d}W(\sigma), \tag{3.2}$$

where the convolution containing the nonlinear term is approximated but the linear terms, including the stochastic convolution integral, are computed exactly, see §4 for details.

For spatial discretization, we define finite-dimensional subspaces H_N of H by $H_N = \operatorname{span}\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$, where $\{\varphi_k\}_{k=1}^{\infty}$ are the eigenvectors of A, (2.1). Then we define the projector

$$\mathcal{P}_N \colon H \to H_N, \quad \mathcal{P}_N v = \sum_{k=1}^{N} (v, \varphi_k)\varphi_k, \quad v \in H. \tag{3.3}$$

We also define the operator

$$A_N \colon H_N \to H_N, \quad A_N = A\mathcal{P}_N, \tag{3.4}$$

which generates a family of resolvent operators $\{S_N(t)\}_{t\geq 0}$ in H_N. It is known that

$$S_N(t)\mathcal{P}_N = S(t)\mathcal{P}_N, \tag{3.5}$$

$$\|A^{-\nu}(I - \mathcal{P}_N)\| = \sup_{k\geq N+1} \lambda_k^{-\nu} = \lambda_{N+1}^{-\nu}, \quad \nu \geq 0. \tag{3.6}$$

The representation of S_N, similar to (2.6), is given by

$$S_N(t)v = \sum_{k=1}^{N} s_k(t)(v, \varphi_k)\varphi_k.$$

Therefore, the smoothing properties (2.7) and (2.8)–(2.10) also hold for S_N with constants independent of N.

Hence, the fully discrete approximation of (1.1), based on the temporal approximation (3.2), is given by

$$\begin{aligned} U_m^N = S_N(t_m)\mathcal{P}_N u_0 &+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S_N(t_m - \sigma)\,\mathrm{d}\sigma\, \mathcal{P}_N F(U_j^N) \\ &+ \int_0^{t_m} S_N(t_m - \sigma)\mathcal{P}_N\,\mathrm{d}W(\sigma), \end{aligned} \tag{3.7}$$

with initial value $U_0^N = \mathcal{P}_N u_0$. Now we state and prove the main theorem, that shows the strong rate of convergence.

Theorem 1. *Under the above assumptions, with $\gamma \in [0, \beta)$, we assume in addition that $\|u_0\|_{L_4(\Omega;\dot{H}^\gamma)} \leq K$. Then, for a constant $C = C(B, K, L, T, \beta, \rho, \gamma)$, we have*

$$\sup_{t_m\in[0,T]} \|u(t_m) - U_m^N\|_{L_2(\Omega;H)} \leq C\big(\lambda_N^{-\frac{\gamma}{2}} + \Delta t^{\gamma\rho}\big).$$

Proof. By subtracting (3.7) from (3.1), we get

$$\begin{aligned} u(t_m) - U_m^N &= S(t_m)u_0 - S_N(t_m)\mathcal{P}_N u_0 \\ &\quad + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \Big\{ S(t_m-\sigma)F(u(\sigma)) - S_N(t_m-\sigma)\mathcal{P}_N F(U_j^N) \Big\}\,\mathrm{d}\sigma \\ &\quad + \int_0^{t_m} \big\{ S(t_m-\sigma) - S_N(t_m-\sigma)\mathcal{P}_N \big\}\,\mathrm{d}W(\sigma). \end{aligned}$$

By recalling (3.5) and taking norms, we obtain

$$\begin{aligned} \|u(t_m) - U_m^N\|_{L_2(\Omega;H)} &\le \|S(t_m)(I-\mathcal{P}_N)u_0\|_{L_2(\Omega;H)} \\ &\quad + \Big\| \int_0^{t_m} S(t_m-\sigma)(I-\mathcal{P}_N)F(u(\sigma))\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)} \\ &\quad + \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m-\sigma)\mathcal{P}_N\big(F(u(\sigma)) - F(U_j^N)\big)\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)} \\ &\quad + \Big\| \int_0^{t_m} S(t_m-\sigma)(I-\mathcal{P}_N)\,\mathrm{d}W(\sigma) \Big\|_{L_2(\Omega;H)} = \sum_{l=1}^4 I_l. \end{aligned} \tag{3.8}$$

We note that I_1, I_2, and I_4 correspond to the *spatial discretization error*, while I_3 corresponds to the *temporal error*.

1. **Spatial error**: The estimate of I_1 is a consequence of (2.7) and (3.6), as

$$\begin{aligned} I_1 &\le \|S(t_m)\|_{\mathcal{L}} \|A^{-\frac{\gamma}{2}}(I-\mathcal{P}_N)A^{\frac{\gamma}{2}}u_0\|_{L_2(\Omega;H)} \\ &\le C\lambda_{N+1}^{-\frac{\gamma}{2}} \|u_0\|_{L_2(\Omega;\dot H^\gamma)} \le C\lambda_{N+1}^{-\frac{\gamma}{2}}. \end{aligned} \tag{3.9}$$

For I_2, by using (2.8) and (3.6), we have

$$\begin{aligned} I_2 &\le \int_0^{t_m} \|A^\gamma S(t_m-\sigma)\|_{\mathcal{L}} \\ &\qquad \times \|A^{-\gamma}(I-\mathcal{P}_N)\|_{\mathcal{L}} \|F(u(\sigma))\|_{L_2(\Omega;H)}\,\mathrm{d}\sigma \\ &\le C\int_0^{t_m} (t_m-\sigma)^{-\gamma\rho} \lambda_{N+1}^{-\gamma} \|F(u(\sigma))\|_{L_2(\Omega;H)}\,\mathrm{d}\sigma \\ &\le C\lambda_{N+1}^{-\gamma}\big(1 + \|u_0\|_{L_2(\Omega;H)}\big) \le C\lambda_{N+1}^{-\gamma}, \end{aligned} \tag{3.10}$$

where we recall that $\gamma\rho < 1$ and use (2.12) and (2.15) with $p = 2, \gamma = 0$.

Now we estimate I_4. Using the Itô isometry (2.2), we have

$$\begin{aligned} I_4 &\le \Big\| \Big(\int_0^{t_m} \big\|S(t_m-\sigma)(I-\mathcal{P}_N)Q^{\frac12}\big\|_{\mathrm{HS}}^2\,\mathrm{d}\sigma \Big)^{\frac12} \Big\| \\ &\le \big\|A^{\frac{\beta-\frac{1}{\rho}}{2}} Q^{\frac12}\big\|_{\mathrm{HS}} \|A^{-\frac{\gamma}{2}}(I-\mathcal{P}_N)\|_{\mathcal{L}} \Big(\int_0^{t_m} \big\|A^{\frac{(\gamma-\beta)+\frac{1}{\rho}}{2}} S(t_m-\sigma)\big\|_{\mathrm{HS}}^2\,\mathrm{d}\sigma \Big)^{\frac12}, \end{aligned}$$

which, by (2.8), (3.6), and since $(\beta-\gamma)\rho \in (0,1)$, implies

$$\begin{aligned} I_4 &\le C\lambda_{N+1}^{-\frac{\gamma}{2}} \big\|A^{\frac{\beta-\frac{1}{\rho}}{2}} Q^{\frac12}\big\|_{\mathrm{HS}} \Big(\int_0^{t_m} (t_m-\sigma)^{-1+(\beta-\gamma)\rho}\,\mathrm{d}\sigma \Big)^{\frac12} \\ &\le C\lambda_{N+1}^{-\frac{\gamma}{2}}. \end{aligned} \tag{3.11}$$

2. **Temporal error**: Here we estimate I_3, i.e.,

$$I_3 = \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma)\mathcal{P}_N\big(F(u(\sigma)) - F(U_j^N)\big)\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)}.$$

We use the Taylor expansion

$$F(u(\sigma)) = F(u(t_j)) + F'(u(t_j))\big(u(\sigma) - u(t_j)\big) + R_{F,j}(\sigma)$$

where the remainder is

$$R_{F,j}(\sigma) = \int_0^1 F''\Big(u(t_j) + \gamma\big(u(\sigma) - u(t_j)\big)\Big)\big(u(\sigma) - u(t_j), u(\sigma) - u(t_j)\big)(1-\gamma)\,\mathrm{d}\gamma,$$

to get

$$\begin{aligned} I_3 \le{}& \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma)\mathcal{P}_N\big(F(u(t_j)) - F(U_j^N)\big)\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)} \\ &+ \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma)\mathcal{P}_N F'(u(t_j))\big(u(\sigma) - u(t_j)\big)\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)} \\ &+ \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma)\mathcal{P}_N R_{F,j}(\sigma)\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)}. \end{aligned}$$

By substituting $u(\sigma)$ and $u(t_j)$ from the variation of constant formula (2.11) in the second term, we have

$$I_3 \le \sum_{l=1}^{7} I_{3,l}, \tag{3.12}$$

where

$$I_{3,1} = \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma)\mathcal{P}_N\big(F(u(t_j)) - F(U_j^N)\big)\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)},$$

$$I_{3,2} = \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma)\mathcal{P}_N F'(u(t_j))\big(S(\sigma) - S(t_j)\big)u_0\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)},$$

$$I_{3,3} = \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma)\mathcal{P}_N F'(u(t_j)) \int_{t_j}^{\sigma} S(\sigma - \tau)F(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)},$$

$$\begin{aligned} I_{3,4} = \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} & S(t_m - \sigma)\mathcal{P}_N F'(u(t_j)) \\ &\times \int_0^{t_j} \big(S(\sigma - \tau) - S(t_j - \tau)\big)F(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)}, \end{aligned}$$

$$I_{3,5} = \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma)\mathcal{P}_N F'(u(t_j)) \int_{t_j}^{\sigma} S(\sigma - \tau)\,\mathrm{d}W(\tau)\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)},$$

$$I_{3,6} = \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m-\sigma)\mathcal{P}_N F'(u(t_j)) \\ \times \int_0^{t_j} \big(S(\sigma-\tau) - S(t_j-\tau)\big)\,\mathrm{d}W(\tau)\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)},$$

and

$$I_{3,7} = \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m-\sigma)\mathcal{P}_N R_{F,j}(\sigma)\,\mathrm{d}\sigma \Big\|_{L_2(\Omega;H)}.$$

First, using (2.12) and (2.7), we have

$$I_{3,1} \le L \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|u(t_j) - U_j\|_{L_2(\Omega;H)}. \tag{3.13}$$

To estimate $I_{3,2}$, we have

$$\begin{aligned} I_{3,2} &\le \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|A^{\frac{\delta}{2}} S(t_m-\sigma)\|_{\mathcal{L}} \|A^{-\frac{\delta}{2}} F'(u(t_j))\big(S(\sigma)-S(t_j)\big)u_0\|_{L_2(\Omega;H)}\,\mathrm{d}\sigma \\ &= \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|A^{\frac{\delta}{2}} S(t_m-\sigma)\|_{\mathcal{L}} \Big\|A^{-\frac{\delta}{2}} F'(u(t_j)) \int_{t_j}^{\sigma} \dot{S}(\tau)u_0\,\mathrm{d}\tau \Big\|_{L_2(\Omega;H)}\,\mathrm{d}\sigma, \end{aligned}$$

so that, using (2.13), (2.8), and (2.15), we obtain

$$\begin{aligned} I_{3,2} &\le C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \big(1 + \|u(t_j)\|_{L_4(\Omega;\dot{H}^{\gamma})}\big) \\ &\quad \times \Big\| \Big\| \int_{t_j}^{\sigma} \dot{S}(\tau)u_0\,\mathrm{d}\tau \Big\|_{-\gamma} \Big\|_{L_4(\Omega;\mathbb{R})}\,\mathrm{d}\sigma \\ &\le C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \Big\| \int_{t_j}^{\sigma} \|A^{-\gamma}\dot{S}(\tau)A^{\frac{\gamma}{2}}u_0\,\mathrm{d}\tau\| \Big\|_{L_4(\Omega;\mathbb{R})}\,\mathrm{d}\sigma. \end{aligned}$$

Now, by (2.10), we have

$$\begin{aligned} I_{3,2} &\le C\|u_0\|_{L_4(\Omega;\dot{H}^{\gamma})} \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \int_{t_j}^{\sigma} \tau^{\gamma\rho-1}\,\mathrm{d}\tau\,\mathrm{d}\sigma \\ &\le \frac{C}{\gamma\rho}(t_{j+1}^{\gamma\rho} - t_j^{\gamma\rho}) \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}}\,\mathrm{d}\sigma \le C(t_{j+1}^{\gamma\rho} - t_j^{\gamma\rho}), \end{aligned}$$

and, since $\gamma\rho \in (0,1)$, we consequently have

$$I_{3,2} \le C\Delta t^{\gamma\rho}. \tag{3.14}$$

Now we estimate $I_{3,3}$ in (3.12). Using (2.12) and (2.7), we have

$$\begin{aligned}
I_{3,3} &\le \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|S(t_m-\sigma)\|_{\mathcal{L}} \Big\| F'(u(t_j)) \int_{t_j}^{\sigma} S(\sigma-\tau)F(u(\tau))\,\mathrm{d}\tau \ \Big\|_{L_2(\Omega;H)} \mathrm{d}\sigma \\
&\le L \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|S(t_m-\sigma)\|_{\mathcal{L}} \int_{t_j}^{\sigma} \|S(\sigma-\tau)\|_{\mathcal{L}} \big\|F(u(\tau))\big\|_{L_2(\Omega;H)} \,\mathrm{d}\tau\,\mathrm{d}\sigma \\
&\le C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{\sigma} \big(1+\|u(\tau)\|_{L_2(\Omega;H)}\big)\,\mathrm{d}\tau\,\mathrm{d}\sigma,
\end{aligned}$$

that, by (2.15) with $p=2, \gamma=0$, implies

$$I_{3,3} \le C\Delta t. \tag{3.15}$$

To estimate $I_{3,4}$ in (3.12), we have

$$\begin{aligned}
I_{3,4} &\le \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|A^{\frac{\delta}{2}} S(t_m-\sigma)\|_{\mathcal{L}} \\
&\quad \times \Big\| A^{-\frac{\delta}{2}} F'(u(t_j)) \int_0^{t_j} \big(S(\sigma-\tau)-S(t_j-\tau)\big) F(u(\tau))\,\mathrm{d}\tau \ \Big\|_{L_2(\Omega;H)} \mathrm{d}\sigma \\
&= \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|A^{\frac{\delta}{2}} S(t_m-\sigma)\|_{\mathcal{L}} \\
&\quad \times \Big\|\Big\| A^{-\frac{\delta}{2}} F'(u(t_j)) \int_0^{t_j} \int_{t_j}^{\sigma} \dot{S}(\theta-\tau)\,\mathrm{d}\theta\ F(u(\tau))\,\mathrm{d}\tau \ \Big\|\Big\|_{L_2(\Omega;\mathbb{R})} \mathrm{d}\sigma,
\end{aligned}$$

which, in view of (2.8), (2.13), and (2.15), implies

$$\begin{aligned}
I_{3,4} &\le C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \\
&\quad \times \Big\|\Big\| \int_0^{t_j} \int_{t_j}^{\sigma} A^{-\frac{\gamma}{2}} \dot{S}(\theta-\tau)\,\mathrm{d}\theta\ F(u(\tau))\,\mathrm{d}\tau \ \Big\|\Big\|_{L_4(\Omega;\mathbb{R})} \mathrm{d}\sigma, \\
&\le C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \\
&\quad \times \int_0^{t_j} \int_{t_j}^{\sigma} \big\|A^{-\frac{\gamma}{2}} \dot{S}(\theta-\tau)\big\|_{\mathcal{L}}\,\mathrm{d}\theta\ \big\|F(u(\tau))\big\|_{L_4(\Omega;H)}\,\mathrm{d}\tau\,\mathrm{d}\sigma.
\end{aligned}$$

Now, by (2.12) and (2.14), we have

$$\begin{aligned}
I_{3,4} &\le C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \\
&\quad \times \int_0^{t_j} \int_{t_j}^{\sigma} (\theta-\tau)^{\frac{\gamma\rho}{2}-1}\,\mathrm{d}\theta\ \big(1+\|u(\tau)\|_{L_4(\Omega;H)}\big)\,\mathrm{d}\tau\,\mathrm{d}\sigma,
\end{aligned}$$

which, together with (2.15) with $p = 4$, implies

$$I_{3,4} \leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m - \sigma)^{-\frac{\delta\rho}{2}} \int_0^{t_j} \int_{t_j}^{\sigma} (\theta - \tau)^{\frac{\gamma\rho}{2}-1} \, \mathrm{d}\theta \;\, \mathrm{d}\tau \, \mathrm{d}\sigma.$$

Then, computing the double integral as

$$\begin{aligned} \int_0^{t_j} \int_{t_j}^{\sigma} (\theta - \tau)^{\frac{\gamma\rho}{2}-1} \, \mathrm{d}\theta \, \mathrm{d}\tau &= \int_{t_j}^{\sigma} \int_0^{t_j} (\theta - \tau)^{\frac{\gamma\rho}{2}-1} \, \mathrm{d}\tau \, \mathrm{d}\theta \\ &= \frac{2}{\gamma\rho} \int_{t_j}^{\sigma} \big(\theta^{\frac{\gamma\rho}{2}} - (\theta - t_j)^{\frac{\gamma\rho}{2}} \big) \, \mathrm{d}\theta \leq \frac{2}{\gamma\rho} t_j^{\frac{\gamma\rho}{2}} \Delta t, \end{aligned}$$

we conclude the estimate

$$I_{3,4} \leq C\Delta t. \tag{3.16}$$

We now estimate the terms in (3.12), which are affected by the noise. For $I_{3,5}$, using the fact that the expected value of independent processes is zero, and then the Cauchy–Schwarz inequality, we have

$$\begin{aligned} I_{3,5}^2 &= \mathbb{E} \Big\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - \sigma) \mathcal{P}_N F'(u(t_j)) \int_{t_j}^{\sigma} S(\sigma - \tau) \, \mathrm{d}W(\tau) \, \mathrm{d}\sigma \Big\|^2 \\ &= \sum_{j=0}^{m-1} \mathbb{E} \Big\| \int_{t_j}^{t_{j+1}} \int_{t_j}^{\sigma} S(t_m - \sigma) \mathcal{P}_N F'(u(t_j)) S(\sigma - \tau) \, \mathrm{d}W(\tau) \, \mathrm{d}\sigma \Big\|^2 \\ &\leq \Delta t \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \mathbb{E} \Big\| \int_{t_j}^{\sigma} S(t_m - \sigma) \mathcal{P}_N F'(u(t_j)) S(\sigma - \tau) \, \mathrm{d}W(\tau) \, \mathrm{d}\sigma \Big\|^2 . \end{aligned}$$

Then, by the Itô isometry (2.2) and (2.12), we have

$$\begin{aligned} I_{3,5}^2 &\leq \Delta t \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{\sigma} \Big\| S(t_m - \sigma) \mathcal{P}_N F'(u(t_j)) S(\sigma - \tau) Q^{\frac{1}{2}} \Big\|_{\mathrm{HS}}^2 \, \mathrm{d}\tau \, \mathrm{d}\sigma \\ &\leq \Delta t \big\| A^{\frac{\beta - \frac{1}{\rho}}{2}} Q^{\frac{1}{2}} \big\|_{\mathrm{HS}}^2 \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{\sigma} \Big\| S(t_m - \sigma) \Big\|_{\mathcal{L}}^2 \Big\| A^{\frac{-\beta + \frac{1}{\rho}}{2}} S(\sigma - \tau) \Big\|_{\mathcal{L}}^2 \, \mathrm{d}\tau \, \mathrm{d}\sigma. \end{aligned}$$

Now, using (2.4), (2.7), and (2.8), we obtain

$$I_{3,5}^2 \leq \Delta t \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{\sigma} (\sigma - \tau)^{\beta\rho - 1} \, \mathrm{d}\tau \, \mathrm{d}\sigma \leq t_N \Delta t^{1+\beta\rho},$$

and therefore, we conclude the estimate

$$I_{3,5} \leq C\Delta t^{\frac{1+\beta\rho}{2}}. \tag{3.17}$$

Now we estimate $I_{3,6}$. To this end, having

$$
\begin{aligned}
I_{3,6} &\le \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|A^{\frac{\delta}{2}} S(t_m-\sigma)\|_{\mathcal{L}} \\
&\quad \times \Big\| A^{-\frac{\delta}{2}} F'(u(t_j)) \int_0^{t_j} \big(S(\sigma-\tau) - S(t_j-\tau)\big)\,\mathrm{d}W(\tau)\Big\|_{L_2(\Omega;H)} \mathrm{d}\sigma \\
&= \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|A^{\frac{\delta}{2}} S(t_m-\sigma)\|_{\mathcal{L}} \\
&\quad \times \Big\| \|A^{-\frac{\delta}{2}} F'(u(t_j)) \int_0^{t_j} \int_{t_j}^{\sigma} \dot{S}(\theta-\tau)\,\mathrm{d}\theta\,\mathrm{d}W(\tau)\|\Big\|_{L_2(\Omega;H)} \mathrm{d}\sigma,
\end{aligned}
$$

and using (2.8) and (2.13), we obtain

$$
\begin{aligned}
I_{3,6} &\le C\big(1 + \sup_{t\in[0,T]} \|u(t)\|_{L_4(\Omega;\dot{H}^{\gamma})}\big) \\
&\quad \times \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \Big\| \int_0^{t_j} \int_{t_j}^{\sigma} A^{-\frac{\gamma}{2}} \dot{S}(\theta-\tau)\,\mathrm{d}\theta\,\mathrm{d}W(\tau)\Big\|_{L_4(\Omega;H)} \mathrm{d}\sigma,
\end{aligned}
$$

Then, by (2.15) and the Burkholder–Davis–Gundy inequality (2.3),

$$
\begin{aligned}
I_{3,6} &\le C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \\
&\quad \times \Big\| \Big(\int_0^{t_j} \| \int_{t_j}^{\sigma} A^{-\frac{\gamma}{2}} \dot{S}(\theta-\tau)\,\mathrm{d}\theta\, Q^{\frac{1}{2}} \|_{\mathrm{HS}}^2 \,\mathrm{d}\tau\Big)^{\frac{1}{2}} \Big\|_{L_4(\Omega;\mathbb{R})} \mathrm{d}\sigma \\
&\le C \big\| A^{\frac{\beta-\frac{1}{\rho}}{2}} Q^{\frac{1}{2}} \big\|_{\mathrm{HS}} \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \\
&\quad \times \Big\| \Big(\int_0^{t_j} \Big(\int_{t_j}^{\sigma} \|A^{\frac{-(\gamma+\beta)+\frac{1}{\rho}}{2}} \dot{S}(\theta-\tau)\|_{\mathcal{L}}\,\mathrm{d}\theta\ \Big)^2 \mathrm{d}\tau\Big)^{\frac{1}{2}} \Big\|,
\end{aligned}
$$

which, (2.4), (2.10), and (2.15), implies

$$
I_{3,6} \le C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\delta\rho}{2}} \Big\| \Big(\int_0^{t_j} \Big(\int_{t_j}^{\sigma} (\theta-\tau)^{\frac{(\gamma+\beta)\rho}{2} - \frac{3}{2}}\,\mathrm{d}\theta\ \Big)^2 \mathrm{d}\tau\Big)^{\frac{1}{2}} \Big\|.
$$

From this and

$$
\begin{aligned}
&\int_0^{t_j} \Big(\int_{t_j}^{\sigma} (\theta-\tau)^{\frac{(\gamma+\beta)\rho}{2}-\frac{3}{2}} \,\mathrm{d}\theta \Big)^2 \,\mathrm{d}\tau \\
&\quad = C \int_0^{t_j} \Big((\sigma-\tau)^{\gamma\rho-\frac{1}{2}+\frac{(\beta-\gamma)\rho}{2}} - (t_j-\tau)^{\gamma\rho-\frac{1}{2}+\frac{(\beta-\gamma)\rho}{2}} \Big)^2 \,\mathrm{d}\tau \\
&\quad = C \int_0^{t_j} \Big((\sigma-\tau)^{\gamma\rho}(\sigma-\tau)^{-\frac{1}{2}+\frac{(\beta-\gamma)\rho}{2}} - (t_j-\tau)^{\gamma\rho}(t_j-\tau)^{-\frac{1}{2}+\frac{(\beta-\gamma)\rho}{2}} \Big)^2 \,\mathrm{d}\tau \\
&\quad \le C \int_0^{t_j} \Big((\sigma-\tau)^{\gamma\rho}(t_j-\tau)^{-\frac{1}{2}+\frac{(\beta-\gamma)\rho}{2}} - (t_j-\tau)^{\gamma\rho}(t_j-\tau)^{-\frac{1}{2}+\frac{(\beta-\gamma)\rho}{2}} \Big)^2 \,\mathrm{d}\tau \\
&\quad = C \int_0^{t_j} (t_j-\tau)^{-1+(\beta-\gamma)\rho} \Big((\sigma-\tau)^{\gamma\rho} - (t_j-\tau)^{\gamma\rho} \Big)^2 \,\mathrm{d}\tau \\
&\quad \le C \Delta t^{2\gamma\rho} \int_0^{t_j} (t_j-\tau)^{-1+(\beta-\gamma)\rho} \,\mathrm{d}\tau \\
&\quad = C t_j^{(\beta-\gamma)\rho} \Delta t^{2\gamma\rho},
\end{aligned}
$$

we conclude the estimate

$$
I_{3,6} \le C\Delta t^{\gamma\rho}. \tag{3.18}
$$

To estimate $I_{3,7}$, the last term in (3.12), we have

$$
I_{3,7} \le \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|A^{\frac{\eta}{2}} S(t_m-\sigma)\|_{\mathcal{L}} \Big\| \|A^{-\frac{\eta}{2}} R_{F,j}(\sigma)\| \Big\|_{L_2(\Omega;H)} \,\mathrm{d}\sigma.
$$

By (2.8) and (2.14), this implies

$$
I_{3,7} \le C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (t_m-\sigma)^{-\frac{\eta\rho}{2}} \|u(\sigma)-u(t_j)\|_{L_4(\Omega;H)}^2 \,\mathrm{d}\sigma,
$$

which, considering the fact that, [1, Proposition 3.2],

$$
\|u(\sigma)-u(t_j)\|_{L_4(\Omega;H)} \le C(\sigma-t_j)^{\frac{\gamma\rho}{2}},
$$

we conclude the estimate

$$
I_{3,7} \le C\Delta t^{\gamma\rho}. \tag{3.19}
$$

Finally, inserting (3.9)–(3.11) and (3.13)–(3.19) into (3.12), we have

$$
\|u(t_m) - U_m^N\|_{L_2(\Omega;H)} \le C\big(\Delta t^{\gamma\rho} + \lambda_{N+1}^{-\frac{\gamma}{2}}\big) + C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \|u(t_j) - U_j\|_{L_2(\Omega;H)},
$$

which, by Gronwall's inequality, completes the proof.

□

Remark 3. We note that, the temporal strong rate of convergence is (almost) twice the rate of the backward Euler method, [1, 8]. For instance, in case of the space-time white noise ($Q = I$) in one dimension, for which $\beta < \frac{1}{2}$, we recover the rate almost $\frac{1}{2}$ for the exponential Euler method for the heat equation ($\rho = 1$), see e.g., [17].

Remark 4. For the deterministic form of the model problem (1.2), i.e., with $\mathrm{d}W = 0$, the rate is therefore $O\big(\Delta t + \lambda_{N+1}^{-\gamma})\big)$, as expected. Indeed, recalling (3.9) and Remark 2, we have

$$I_1 \leq \|S(t_m)\|_{\mathcal{L}} \|A^{-\gamma}(I - \mathcal{P}_N)A^{\gamma} u_0\| \leq C\lambda_{N+1}^{-\gamma} \|u_0\|_{2\gamma}.$$

We also recall (3.10), for which we have in this case

$$I_2 \leq C\lambda_{N+1}^{-\gamma}(1 + \|u_0\|).$$

Remark 5. Due to the Riesz kernel (1.3), the regularity of the solution of (1.1) and its deterministic counterpart is limited, see [11, Theorem 5.5]. Therefore, the obtained rate in this work is the best possible and optimal, for both the stochastic equation and its deterministic counterpart.

Remark 6. The temporal rate $O\big(\Delta t\big)$ for the deterministic problem, coincides with first order explicit and implicit methods, see, e.g., [11]. However, the proposed MLEI in this work does not suffer from limited stability domain, or the need for solving non-linear system of equations, and it is computationally more efficient.

4. **Numerical implementation**

In this section we present the explicit form of the exact solution of (1.1), in terms of the Mittag-Leffler functions. Then, we illustrate the temporal strong order of convergence, to confirm the proposed rate in Theorem 1.

4.1. **Explicit representation of the solution.**

First, we derive an explicit representation of the resolvent family in terms of the Mittag-Leffler functions when b is the Riesz kernel.

Recall that the one parameter Mittag-Leffler function $E_a(z)$ and its two parameters form $E_{a,b}(z)$, $a, b > 0$, are defined as

$$E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+1)}, \quad E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}, \quad \forall z \in \mathbb{C},$$

and obviously $E_a(z) = E_{a,1}(z)$.

Taking the Laplace transform of (2.11), when $b(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, we have

$$\mathcal{L}(s_k) = (s^{\rho} + \lambda_k)^{-1} s^{\rho-1},$$

which implies

$$s_k(t) = E_{\rho}(-t^{\rho}\lambda_k).$$

Then the resolvent family is given by

$$S(t)v = \sum_{k=1}^{\infty} E_{\rho}(-t^{\rho}\lambda_k)(v, \varphi_k)\varphi_k.$$

To explain the computer implementation of the full-discrete method (3.7), we note that

$$S_N(t_m) = \sum_{k=1}^{N} E_{\rho}(-t_m^{\rho}\lambda_k)(v, \varphi_k)\varphi_k,$$

Suppose that Q has the same eigenfunctions as A, so that $Qv = \sum_{k=1}^{\infty} \mu_k (v, \varphi_k)\varphi_k$. Then, for each time step $m = 1, \dots, M$, the approximation U_m^N defined by (3.7) is given by $U_m^N = \sum_{k=1}^{N} U_{m,k}^N \varphi_k$, where for $k = 1, \cdots, N$,

$$\begin{aligned} U_{m,k}^N = E_\rho(-\lambda_k t_m^\rho) u_{0,k} &+ \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} E_\rho(-\lambda_k (t_m - \sigma)^\rho) \,\mathrm{d}\sigma \; F_k(U_j^N) \\ &+ \sum_{j=0}^{m-1} \left(\int_{t_j}^{t_{j+1}} (E_\rho(-\lambda_k (t_m - \sigma)^\rho))^2 \,\mathrm{d}\sigma \right)^{\frac{1}{2}} \mu_k^{\frac{1}{2}} \xi_{m,k}, \end{aligned}$$

and where $u_{0,k} = (u_0, \varphi_k)$, $F_k(\cdot) = (F(\cdot), \varphi_k)$, and $\xi_{m,k}$ for $m = 1, \cdots, M$ and $k = 1, \cdots, N$ are independent standard normally distributed random variables.

We note that the integrals of the Mittag-Leffler functions are computable, e.g., by means of a simple quadrature, say the trapezoidal method. The first integral can be even computed exactly as

$$\begin{aligned} \int_{t_j}^{t_{j+1}} E_\rho(-\lambda_k (t_m - \sigma)^\rho)) \,\mathrm{d}\sigma &= \int_{t_{m-j-1}}^{t_{m-j}} E_\rho(-\lambda_k \sigma^\rho) \,\mathrm{d}\sigma \\ &= t_{m-j} E_{\rho,2}(-\lambda_k t_{m-j}^\rho) - t_{m-j-1} E_{\rho,2}(-\lambda_k t_{m-j-1}^\rho). \end{aligned}$$

For evaluating the Mittag-Leffler function we use `mlf.m` from [13].

4.2. **Numerical experiments.** Since the major contribution of this paper is the temporal approximation, we only present a simplified numerical experiment with uncoupled eigenmodes. More precisely, let $u = \sum_{k=1}^{\infty} u_k \varphi_k$ and define the nonlinear operator

$$F(u) = \sum_{k=1}^{\infty} \sin(u_k) \varphi_k.$$

We illustrate the temporal convergence by computing two uncoupled eigenmodes, one with small λ_k and one with large λ_k. We use space-time white noise $Q = I$ in one dimension, for which $\beta < \frac{1}{2}$, together with the Riesz kernel with two different values of ρ. The theoretical rate of convergence is then $\gamma\rho$, which is almost $\frac{1}{2}\rho$. We use 10000 sample paths in all experiments. See Figure 1 for the rate of convergence and Figures 2 and 3 for the behaviour of the solution.

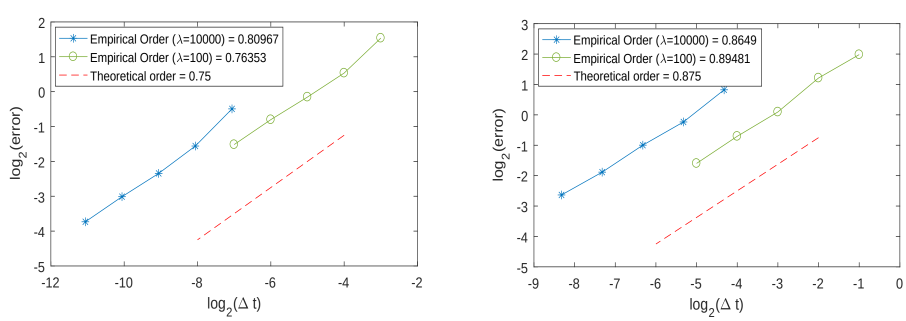

FIGURE 1. Temporal rate of convergence: (left) with $\rho = 1.5$, (right) with $\rho = 1.75$.

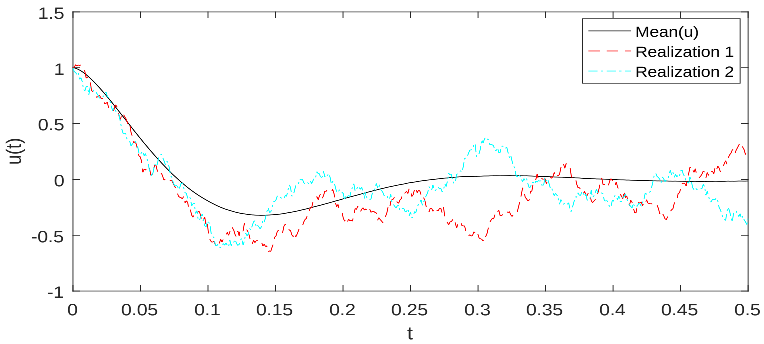

FIGURE 2. Solution behaviour with: $\rho = 1.5$ and $\lambda_k = 100$.

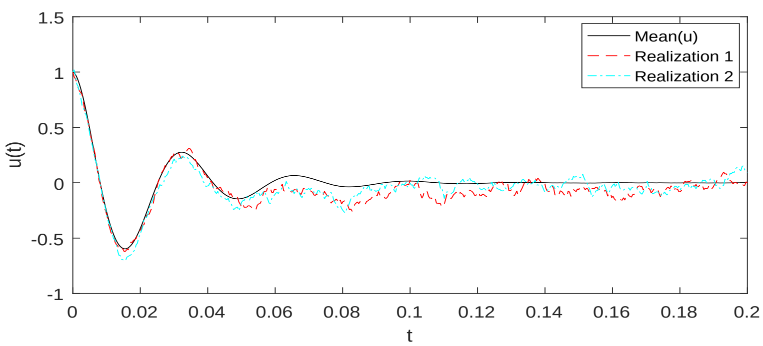

FIGURE 3. Solution behaviour with: $\rho = 1.75$ and $\lambda_k = 10000$.

REFERENCES

[1] A. Andersson, M. Kovács and S. Larsson, Weak error analysis for semilinear stochastic Volterra equations with additive noise. J. Math. Anal. Appl. 437 (2016), 1283–1304.

[2] R. Anton, D. Cohen, S. Larsson and X. Wang, Full discretization of semi-linear stochastic wave equations driven by multiplicative noise. SIAM J. Numer. Anal. 54 (2016), 1093–1119.

[3] B. Baeumer, M. Geissert and M. Kovács, Existence, uniqueness and regularity for a class of semilinear stochastic Volterra equations with multiplicative noise. J. Differential Equations 258 (2015), 535–554.

[4] S. Becker, A. Jentzen and P. Kloeden, An exponential Wagner-Platen type scheme for SPDES. SIAM J. Numer. Anal. 54 (2016), 2389–2426.

[5] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia Math. Appl., vol. 44, Cambridge University Press, Cambridge, 1992.

[6] M. Hochbruck and A. Ostermann, Exponential integrators. Acta Numerica (2010), 209–286.

[7] M. Kovács, S. Larsson and F. Lindgren, Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise II: Fully discrete schemes. BIT Numer. Math. 53 (2013), 497–525.

[8] M. Kovács and J. Printems, Strong order convergence of a fully discrete approximation of a linear stochastic Volterra equation. Math. Comp. 83 (2014), 2325–2346.

[9] M. Kovács and J. Printems, Weak convergence of a fully discrete approximation of a linear stochastic evolution equation with a positive-type memory term. J. Math. Anal. Appl. 413 (2014), 939–952.
[10] S. Larsson and M. Molteni, Numerical solution of parabolic problems based on a weak space-time formulation. Comput. Methods Appl. Math. 17 (2016), doi:10.1515/cmam-2016-0027.
[11] W. McLean and V. Thomée, Numerical solution of an evolution equation with positive memory term. J. Austral. Math. Soc. Ser. B 35 (1993), 23–70.
[12] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[13] I. Podlubny and M. Kacenak, The Matlab mlf code. MATLAB central File Exchange (2001-2009). File ID: 8738.
[14] J. Prüss, Evolutionary Integral Equations and Applications. Monographs in Mathematics, vol. 87, Birkhäuser, Basel, 1993.
[15] X. Wang, Strong convergence rates of the linear implicit Euler method for the finite element discretization of SPDEs with additive noise. IMA J. Numer. Anal. 37 (2017), 956–984.
[16] X. Wang, S. Gan, and J. Tang, Higher order strong approximations of semilinear stochastic wave equation with additive space-time white noise, arXiv:1308.4529 [math.NA].
[17] X. Wang and R. Qi, A note on an accelerated exponential Euler method for parabolic SPDEs with additive noise. Appl. Math. Letters 46 (2015), 31–37.

Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE–412 96 Gothenburg, Sweden
E-mail address: `mihaly@chalmers.se`

Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE–412 96 Gothenburg, Sweden
E-mail address: `stig@chalmers.se`

Department of Mathematics, University of Kurdistan, P.O. Box 416, Sanandaj, Iran
E-mail address: `f.saedpanah@uok.ac.ir, fardin.saedpanah@gmail.com`