



Direct images of semi-meromorphic currents

Downloaded from: <https://research.chalmers.se>, 2024-10-12 09:08 UTC

Citation for the original published paper (version of record):

Andersson, M., Wulcan, E. (2018). Direct images of semi-meromorphic currents. *Annales de l'Institut Fourier*, 68(2): 875-890. <http://dx.doi.org/10.5802/aif.3180>

N.B. When citing this work, cite the original published paper.



ANNALES

DE

L'INSTITUT FOURIER

Mats ANDERSSON & Elizabeth WULCAN

Direct images of semi-meromorphic currents

Tome 68, n° 2 (2018), p. 875-900.

http://aif.cedram.org/item?id=AIF_2018__68_2_875_0



© Association des Annales de l'institut Fourier, 2018,
Certains droits réservés.



Cet article est mis à disposition selon les termes de la licence
CREATIVE COMMONS ATTRIBUTION – PAS DE MODIFICATION 3.0 FRANCE.
<http://creativecommons.org/licenses/by-nd/3.0/fr/>

L'accès aux articles de la revue « Annales de l'institut Fourier »
(<http://aif.cedram.org/>), implique l'accord avec les conditions générales
d'utilisation (<http://aif.cedram.org/legal/>).

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>

DIRECT IMAGES OF SEMI-MEROMORPHIC CURRENTS

by Mats ANDERSSON & Elizabeth WULCAN (*)

ABSTRACT. — We introduce a calculus for the class $ASM(X)$ of direct images of semi-meromorphic currents on a reduced analytic space X , that extends the classical calculus due to Coleff, Herrera and Passare. Our main result is that each element in this class acts as a kind of multiplication on the sheaf \mathcal{PM}_X of pseudomeromorphic currents on X . We also prove that $ASM(X)$ as well as \mathcal{PM}_X and certain subsheaves are closed under the action of holomorphic differential operators and interior multiplication by holomorphic vector fields.

RÉSUMÉ. — Nous introduisons un calcul pour la classe $ASM(X)$ d'images directes de courants semi-méromorphes sur un espace analytique réduit X , qui étend le calcul classique de Coleff, Herrera et Passare. Notre résultat principal montre que chaque élément de cette classe agit de manière analogue à une multiplication sur le faisceau \mathcal{PM}_X de courants pseudoméromorphes sur X . Nous prouvons également que $ASM(X)$ ainsi que \mathcal{PM}_X et certains sous-faisceaux sont fermés sous l'action des opérateurs différentiels holomorphes et la multiplication intérieure par des champs vectoriels holomorphes.

1. Introduction

Let f be a generically nonvanishing holomorphic function on a reduced analytic space X of pure dimension n . It was proved by Herrera and Lieberman, [14], that one can define the principal value current

$$(1.1) \quad \left[\frac{1}{f} \right] \cdot \xi := \lim_{\epsilon \rightarrow 0} \int_{|f|^2 > \epsilon} \frac{\xi}{f},$$

for test forms ξ . It follows that $\bar{\partial}[1/f]$ is a current with support on the zero set $Z(f)$ of f ; such a current is called a residue current. Coleff and Herrera,

Keywords: residue current, semi-meromorphic current, analytic space, pseudomeromorphic current.

2010 *Mathematics Subject Classification:* 32A26, 32A27, 32B15, 32C30.

(*) The authors were partially supported by the Swedish Research Council.

[13], introduced products of principal value and residue currents, like

$$(1.2) \quad [1/f_1] \dots [1/f_r] \bar{\partial}[1/f_{r+1}] \wedge \dots \wedge \bar{\partial}[1/f_m].$$

The product of principal value currents is commutative, but when there are residue factors, like $\bar{\partial}[1/f_j]$, present these products are not (anti-)commutative in general.

In the literature there are various generalizations and related currents, for instance the abstract so-called Coleff–Herrera currents introduced by Björk, see [12], the Bochner–Martinelli type residue currents introduced in [21], and generalizations in, e.g., [3, 5, 9].

In order to obtain a coherent approach to questions about residue and principal value currents the sheaf \mathcal{PM}_X of *pseudomeromorphic currents* on X was introduced in [10], and further developed in [7]; this sheaf consists of direct images under holomorphic mappings of products of test forms and currents like (1.2). See Section 2 below for the precise definition. This sheaf is closed under $\bar{\partial}$ and under multiplication by smooth forms. Pseudomeromorphic currents have a geometric nature, similar to positive closed (or normal) currents. For example, the *dimension principle* states that if the pseudomeromorphic current μ has bidegree $(*, p)$ and support on a variety of codimension larger than p , then μ must vanish. Moreover one can form restrictions $\mathbb{1}_W \mu$ of the pseudomeromorphic current μ to analytic (or constructible) subsets $W \subset X$, such that

$$(1.3) \quad \mathbb{1}_V \mathbb{1}_W \mu = \mathbb{1}_{V \cap W} \mu,$$

see Section 2.2. The notion of pseudomeromorphic currents plays a decisive role in, for instance, [7, 8, 10, 11, 15, 16, 18, 22, 23, 24, 25].

It is well-known that one cannot multiply currents in general. Several attempts to find a working calculus for principal value and residue currents have been made. A famous result by Coleff and Herrera, [13], see also Passare, [20], asserts that (1.2) has all expected (anti-)commutativity properties as long as the common zero set of f_1, \dots, f_m has codimension m . Various extensions are introduced in the references above. In [10] we proved that one can give a reasonable meaning to a product $[1/f]\mu$ for any holomorphic function f and pseudomeromorphic current μ ; more precisely one should consider this as an operator

$$(1.4) \quad \mu \mapsto [1/f]\mu$$

on the sheaf \mathcal{PM}_X .

We have not found a way to define a reasonable product of general pseudomeromorphic currents. Our first objective in this paper is to study a

generalization of principal value currents leading to an extension of (1.4). Following [7] we say that a current a is *almost semi-meromorphic*, $a \in ASM(X)$, if it is the direct image under a modification of a semi-meromorphic current, i.e., a current of the form $\omega[1/f]$, where f is a holomorphic section of a line bundle and ω is a smooth form with values in the same bundle. Almost semi-meromorphic currents are pseudomeromorphic and in many ways they generalize principal value currents. For example, it turns out that they form an (anti-)commutative algebra, see Section 4. Moreover $ASM(X)$ is closed under ∂ , see Proposition 4.16. Taking $\bar{\partial}$ of $a \in ASM(X)$, however, yields an almost semi-meromorphic current plus a residue current supported on the *Zariski singular support*, $ZSS(a)$, of a , which is the smallest analytic set where a is not smooth. Many of the currents in the references above can be considered as (products of) the residues of almost semi-meromorphic currents. Theorem 4.8 states that the mapping (1.4) holds for any almost semi-meromorphic current a instead of $[1/f]$. More precisely, there is a unique extension to X of the current $a \wedge \mu$, defined in the obvious way in $X \setminus ZSS(a)$, such that its restriction to $ZSS(a)$ is zero.

A second objective is to prove that \mathcal{PM}_X and $ASM(X)$ are closed under interior multiplication by a holomorphic vector field ξ and under the Lie derivative with respect to ξ ; see Sections 3 and 4.5.

In Section 2 we recall basic known properties of the sheaf \mathcal{PM}_X and provide some new results, e.g., Theorem 2.15 gives a new quite natural characterization of pseudomeromorphicity. Section 4 is devoted to the study of $ASM(X)$.

Acknowledgment. We are grateful to the referee for carefully reading and pointing out unclarities and misprints.

2. Pseudomeromorphic currents

In one complex variable s one can define the principal value current $[1/s^m]$ for instance as the value

$$\left[\frac{1}{s^m} \right] = \left. \frac{|s|^{2\lambda}}{s^m} \right|_{\lambda=0}$$

of the current-valued analytic continuation of $\lambda \mapsto |s|^{2\lambda}/s^m$, a priori defined for $\text{Re } \lambda \gg 0$, see, e.g., [3, Lemma 2.1]. We have the relations

$$(2.1) \quad \frac{\partial}{\partial s} \left[\frac{1}{s^m} \right] = -m \left[\frac{1}{s^{m+1}} \right], \quad s \left[\frac{1}{s^{m+1}} \right] = \left[\frac{1}{s^m} \right].$$

It is also well-known that

$$(2.2) \quad \bar{\partial} \left[\frac{1}{s^m} \right] \cdot \xi \, ds = \frac{2\pi i}{(m-1)!} \frac{\partial^{m-1}}{\partial s^{m-1}} \xi(0)$$

for test functions ξ and $m \geq 1$; in particular, $\bar{\partial}[1/s^m]$ has support at $\{s = 0\}$. Thus

$$(2.3) \quad \bar{s} \bar{\partial} \left[\frac{1}{s^m} \right] = 0, \quad d\bar{s} \wedge \bar{\partial} \left[\frac{1}{s^m} \right] = 0.$$

We say that a function χ on the real line is a *smooth approximand of the characteristic function* $\chi_{[1,\infty)}$ of the interval $[1, \infty)$, and write

$$\chi \sim \chi_{[1,\infty)},$$

if χ is smooth, equal to 0 in a neighborhood of 0 and 1 in a neighborhood of ∞ . It is well-known that $[1/s^m] = \lim_{\epsilon \rightarrow 0} \chi(|s|^2/\epsilon)(1/s^m)$.

Let t_j be coordinates in an open set $\mathcal{U} \subset \mathbb{C}^N$ and let α be a smooth form with compact support in \mathcal{U} . Then

$$(2.4) \quad \tau = \alpha \wedge \left[\frac{1}{t_1^{m_1}} \right] \dots \left[\frac{1}{t_k^{m_k}} \right] \bar{\partial} \left[\frac{1}{t_{k+1}^{m_{k+1}}} \right] \wedge \dots \wedge \bar{\partial} \left[\frac{1}{t_r^{m_r}} \right],$$

where $m_1, \dots, m_r \geq 1$, is a well-defined current, since it is the tensor product of one-variable currents (times α). We say that τ is an *elementary (pseudomeromorphic) current*, and we refer to $[1/t_j^{m_j}]$ and $\bar{\partial}[1/t_\ell^{m_\ell}]$ as its *principal value factors* and *residue factors*, respectively. It is clear that (2.4) is commuting in the principal value factors and anti-commuting in the residue factors. We say the intersection of \mathcal{U} and the coordinate plane $\{t_{k+1} = \dots = t_r = 0\}$ is the *elementary support* of τ . Clearly the support of τ is contained in the intersection of the elementary support of τ and the support of α .

Remark 2.1. — Since $\bar{\partial}$ does not introduce new residue factors, $\bar{\partial}\tau$ is an elementary current, cf. (2.1), whose elementary support either equals the elementary support H of τ or is empty. Moreover $\bar{\partial}\tau$ is a finite sum of elementary currents, whose elementary supports are either equal to H or coordinate planes of codimension 1 in H , cf. (2.2).

2.1. Definition and basic properties

Let X be a reduced complex space of pure dimension n . Fix a point $x \in X$. We say that a germ μ of a current at x is *pseudomeromorphic* at

$x, \mu \in \mathcal{PM}_x$, if it is a finite sum of currents of the form

$$(2.5) \quad \pi_*\tau = \pi_*^1 \dots \pi_*^m \tau,$$

where $\mathcal{U} \subset X$ is a neighborhood of x ,

$$(2.6) \quad \mathcal{U}_m \xrightarrow{\pi^m} \dots \xrightarrow{\pi^2} \mathcal{U}_1 \xrightarrow{\pi^1} \mathcal{U}_0 = \mathcal{U},$$

each $\pi^j : \mathcal{U}_j \rightarrow \mathcal{U}_{j-1}$ is either a modification, a simple projection $\mathcal{U}_{j-1} \times Z \rightarrow \mathcal{U}_{j-1}$, or an open inclusion (i.e., \mathcal{U}_j is an open subset of \mathcal{U}_{j-1}), and τ is elementary on $\mathcal{U}_m \subset \mathbb{C}^N$.

By definition the union $\mathcal{PM} = \mathcal{PM}_X = \cup_x \mathcal{PM}_x$ is an open subset (of the étalé space) of the sheaf $\mathcal{C} = \mathcal{C}_X$ of currents, and hence it is a subsheaf, which we call the sheaf of *pseudomeromorphic* currents⁽¹⁾. A section μ of \mathcal{PM} over an open set $\mathcal{V} \subset X, \mu \in \mathcal{PM}(\mathcal{V})$, is then a locally finite sum

$$(2.7) \quad \mu = \sum (\pi_\ell)_* \tau_\ell,$$

where each π_ℓ is a composition of mappings as in (2.6) (with $\mathcal{U} \subset \mathcal{V}$) and τ_ℓ is elementary. For simplicity we will always suppress the subscript ℓ in π_ℓ . If ξ is a smooth form, then

$$(2.8) \quad \xi \wedge \pi_*\tau = \pi_* (\pi^* \xi \wedge \tau).$$

Thus \mathcal{PM} is closed under exterior multiplication by smooth forms. Since $\bar{\partial}$ and ∂ commute with push-forwards it follows that \mathcal{PM} is closed under $\bar{\partial}$ and ∂ , cf. Remark 2.1.

Remark 2.2. — Let τ be an elementary current with elementary support H . Since H is the intersection of an open set \mathcal{U} and a linear subspace, each of its components is irreducible, and it follows that, in fact, τ is a finite sum of currents τ_ℓ such that the support of τ_ℓ is contained in an irreducible component of H . We may therefore assume that each τ_ℓ in (2.7) has irreducible elementary support.

Remark 2.3. — One may assume that each τ_ℓ in (2.7) has at most one residue factor. Indeed, in [21], see also [4, Corollary 3.5], it is shown that the Coleff–Herrera product

$$\bar{\partial}[1/t_{k+1}^{m_{k+1}}] \wedge \dots \wedge \bar{\partial}[1/t_r^{m_r}]$$

equals the Bochner–Martinelli residue current of $t_{k+1}^{m_{k+1}}, \dots, t_r^{m_r}$, which, see, e.g., [3], is the direct image under a modification of a current of the form $\alpha \wedge \bar{\partial}[1/f]$, cf. Example 4.18 below. It follows, cf. [6, Lemma 3.2], that (2.4)

(1) The definition here is from [7]; in the original definition in [10] simple projections were not included.

is the direct image under another modification of a finite sum of elementary currents with at most one residue factor.

PROPOSITION 2.4. — Assume that $\mu \in \mathcal{PM}$ has support on the subvariety $V \subset X$.

- (1) If the holomorphic function h vanishes on V , then $\bar{h}\mu = 0$ and $d\bar{h} \wedge \mu = 0$.
- (2) If μ has bidegree $(*, p)$ and $\text{codim } V > p$, then $\mu = 0$.

This proposition is from [10]; for the adaption to nonsmooth X , see [7, Proposition 2.3]. Part (1) means that the action of the current μ only involves holomorphic derivatives of test forms. We refer to part (2) as the *dimension principle*. We will also need, [6, Proposition 1.2]:

PROPOSITION 2.5. — If $\pi: X' \rightarrow X$ is a modification, then $\pi_*: \mathcal{PM}(X') \rightarrow \mathcal{PM}(X)$ is surjective.

2.2. Basic operations on pseudomeromorphic currents

Assume that μ is pseudomeromorphic on X and that $V \subset X$ is a subvariety. It was proved in [10], see also [7], that the restriction of μ to the open set $X \setminus V$ has a natural pseudomeromorphic extension $\mathbb{1}_{X \setminus V} \mu$ to X . In [10] it was obtained as the value

$$(2.9) \quad \mathbb{1}_{X \setminus V} \mu := |f|^{2\lambda} \mu|_{\lambda=0}$$

at $\lambda = 0$ of the analytic continuation of the current valued function $\lambda \mapsto |f|^{2\lambda} \mu$, where f is any tuple of holomorphic functions such that $Z(f) = V$. It follows that

$$\mathbb{1}_V \mu := \mu - \mathbb{1}_{X \setminus V} \mu$$

has support on V . It is proved in [10] that this operation extends to all constructible sets and that (1.3) holds. If α is a smooth form, then

$$(2.10) \quad \mathbb{1}_V(\alpha \wedge \mu) = \alpha \wedge \mathbb{1}_V \mu.$$

Moreover, if $\pi: X' \rightarrow X$ is a modification, a simple projection or an open inclusion and $\mu = \pi_* \mu'$, then

$$(2.11) \quad \mathbb{1}_V \mu = \pi_* (\mathbb{1}_{\pi^{-1}V} \mu').$$

In this paper it is convenient to express $\mathbb{1}_{X \setminus V} \mu$ as a limit of currents that are pseudomeromorphic themselves.

LEMMA 2.6. — Let V be a germ of a subvariety at $x \in X$, let f be a tuple of holomorphic functions whose common zero set is precisely V , let v be a positive and smooth function, and let $\chi \sim \chi_{[1,\infty)}$. For each germ of a pseudomeromorphic current μ at x we have

$$(2.12) \quad \mathbf{1}_{X \setminus V} \mu = \lim_{\epsilon \rightarrow 0} \chi(|f|^2 v / \epsilon) \mu.$$

Because of the factor v , the lemma holds just as well for a holomorphic section f of a Hermitian vector bundle.

In case V is a hypersurface and f is one single holomorphic function, or section of a line bundle, the lemma follows directly from Lemma 6 in [17] by just taking $T = f\mu$. We will reduce the general case to this lemma. The proof of this lemma relies on the proof of Theorem 1.1 in [17], which is quite involved. For a more direct proof of Lemma 2.6, see the proof of Proposition 3.4 in [1, Chapter 2].

Proof. — Let $\pi: X' \rightarrow X$ be a smooth modification such that $\pi^* f = f^0 f'$, where f^0 is a holomorphic section of a Hermitian line bundle $L \rightarrow X'$ and f' is a nonvanishing tuple of holomorphic sections of L^{-1} . In view of Proposition 2.5 we can assume that $\mu = \pi_* \mu'$, where μ' is pseudomeromorphic on X' . Then

$$|\pi^* f|^2 \pi^* v = |f^0|^2 |f'|^2 \pi^* v,$$

and from [17, Lemma 6] we thus have that

$$\lim_{\epsilon \rightarrow 0} \chi(|\pi^* f|^2 \pi^* v / \epsilon) \mu' = \mathbf{1}_{X' \setminus \pi^{-1} V} \mu'.$$

In view of (2.11) we get (2.12). □

Remark 2.7. — Lemma 2.6 holds even if $\chi = \chi_{[1,\infty)}$. However, in general it is not obvious what $\chi(|f|^2 v / \epsilon) \mu$ means. Let χ^δ be smooth approximands such that $\chi^\delta \rightarrow \chi_{[1,\infty)}$. It follows from the proof of Lemma 6 in [17] that for small enough ϵ , depending on μ , f , and v , the limit $\lim_{\delta \rightarrow 0} \chi^\delta(|f|^2 v / \epsilon) \mu$ exists and is independent of the choice of χ^δ ; thus we can take it as the definition of $\chi(|f|^2 v / \epsilon) \mu$. In fact, it turns out that after a suitable change of real coordinates one can realize $\chi(|f|^2 v / \epsilon) \mu$ as a tensor product of two currents. In particular we get

$$\chi(|f|^2 / \epsilon) \frac{1}{f} \cdot \xi = \int_{|f|^2 > \epsilon} \frac{\xi}{f},$$

cf. (1.1).

We will need the following observation.

LEMMA 2.8. — *If μ has the form (2.7), then*

$$\mathbb{1}_V \mu = \sum_{\text{supp } \tau_\ell \subset \pi^{-1}V} \pi_* \tau_\ell.$$

It follows from the proof below that we just as well can take the sum over all ℓ such that the elementary supports of τ_ℓ are contained in $\pi^{-1}V$.

Proof. — In view of (2.11) we have that

$$\mathbb{1}_V \mu = \sum_{\ell} \pi_* (\mathbb{1}_{\pi^{-1}V} \tau_\ell).$$

If $\text{supp } \tau_\ell \subset \pi^{-1}V$, then clearly $\mathbb{1}_{\pi^{-1}V} \tau_\ell = \tau_\ell$. We now claim that if $\text{supp } \tau_\ell$ is not contained in $\pi^{-1}V$, then $\mathbb{1}_{\pi^{-1}V} \tau_\ell = 0$. If $\text{supp } \tau_\ell \not\subset \pi^{-1}V$, the elementary support H of τ_ℓ is not contained in $\pi^{-1}V$. Assume that H has codimension q . Then τ_ℓ is of the form $\tau_\ell = \alpha \wedge \tau'$, where α is smooth and τ' is elementary of bidegree $(0, q)$. It follows from (2.10) that

$$\mathbb{1}_{\pi^{-1}V} \tau_\ell = \alpha \wedge \mathbb{1}_{\pi^{-1}V} \tau'.$$

By Remark 2.2 we may assume that H is irreducible, and therefore $\pi^{-1}V \cap H$ has codimension at least $q + 1$ in \mathcal{U} . Since $\mathbb{1}_{\pi^{-1}V} \tau'$ has support on $\pi^{-1}V \cap H$ it must vanish in view of the dimension principle. Thus the lemma follows. □

We now consider another fundamental operation on \mathcal{PM} introduced in [10].

PROPOSITION 2.9 ([10]). — *Given a holomorphic function h and a pseudomeromorphic current μ there is a pseudomeromorphic current T such that $T = (1/h)\mu$ in the open set where $h \neq 0$ and $\mathbb{1}_{\{h=0\}}T = 0$.*

Here h may just as well be a holomorphic section of a line bundle. Clearly this current T must be unique and we denote it by $[1/h]\mu$. In [10] the current $[1/h]\mu$ was defined as $(|h|^{2\lambda}\mu/h)|_{\lambda=0}$.

Remark 2.10. — Notice that⁽²⁾ $h[1/h]\mu = \mathbb{1}_{\{h \neq 0\}}\mu$; in particular, $h[1/h]\mu \neq \mu$ in general. For example, $z[1/z]\bar{\partial}[1/z] = 0$.

Since $[1/h]\mu = (1/h)\mu$ in $\{h \neq 0\}$ and $[1/h]\mu = \mathbb{1}_{\{h \neq 0\}}[1/h]\mu$, it follows from (2.12) that

$$(2.13) \quad \left[\frac{1}{h} \right] \mu = \lim_{\epsilon \rightarrow 0} \chi(|h|^2 v / \epsilon) \frac{1}{h} \mu.$$

⁽²⁾We have not excluded the possibility that h vanishes identically on some (or all) irreducible components of X .

One can also define

$$(2.14) \quad \bar{\partial} \left[\frac{1}{h} \right] \wedge \mu := \bar{\partial} \left(\left[\frac{1}{h} \right] \mu \right) - \left[\frac{1}{h} \right] \bar{\partial} \mu,$$

i.e., so that “Leibniz’s rule” holds. Notice that if $\pi: X' \rightarrow X$ is a modification and $\mu = \pi_* \mu'$, then

$$(2.15) \quad \left[\frac{1}{h} \right] \mu = \pi_* \left(\left[\frac{1}{\pi^* h} \right] \mu' \right), \quad \bar{\partial} \left[\frac{1}{h} \right] \wedge \mu = \pi_* \left(\bar{\partial} \left[\frac{1}{\pi^* h} \right] \wedge \mu' \right).$$

This follows, e.g., from (2.8) and (2.13). It is also readily checked that

$$(2.16) \quad \bar{\partial} \left(\bar{\partial} \left[\frac{1}{h} \right] \wedge \mu \right) = -\bar{\partial} \left[\frac{1}{h} \right] \wedge \bar{\partial} \mu.$$

Remark 2.11. — Since $[1/f][1/g] = [1/(fg)] = [1/g][1/f]$ it follows from (2.14) that

$$\bar{\partial} \left[\frac{1}{f} \right] \cdot \left[\frac{1}{g} \right] + \left[\frac{1}{f} \right] \bar{\partial} \left[\frac{1}{g} \right] = \bar{\partial} \left[\frac{1}{g} \right] \cdot \left[\frac{1}{f} \right] + \left[\frac{1}{g} \right] \bar{\partial} \left[\frac{1}{f} \right].$$

However, it is not true in general that $[1/g]\bar{\partial}[1/f] = \bar{\partial}[1/f] \cdot [1/g]$. For instance, $[1/z]\bar{\partial}[1/z] = 0$, whereas $\bar{\partial}[1/z] \cdot [1/z] = \bar{\partial}[1/z^2]$.

We now consider tensor products and direct images under simple projections.

LEMMA 2.12. — *If $\mu \in \mathcal{PM}_X$ and $\mu' \in \mathcal{PM}_{X'}$, then $\mu \otimes \mu' \in \mathcal{PM}_{X \times X'}$.*

This is precisely [6, Lemma 3.3]. It is easy to verify that

$$(2.17) \quad \mathbf{1}_{V \times V'} \mu \otimes \mu' = \mathbf{1}_V \mu \otimes \mathbf{1}_{V'} \mu'.$$

LEMMA 2.13. — *Assume that $p: Z \times W \rightarrow Z$ is a simple projection. If μ is in $\mathcal{PM}_{Z \times W}$ and $p^{-1}K \cap \text{supp } \mu$ is compact for each compact set $K \subset Z$, then $p_* \mu$ is in \mathcal{PM}_Z .*

Proof. — Since pseudomeromorphicity is a local property, after multiplying μ if necessary by a suitable cutoff function we can assume that μ has compact support. By compactness and a partition of unity we then have a finite representation $\mu = \sum_{\ell} \pi_* \tau_{\ell}$. Now the lemma follows from the very definition of \mathcal{PM} . □

Example 2.14. — Assume that τ is an elementary current on X , p is a simple projection $X \times X' \rightarrow X$, and χ is any test form in X' with total integral 1. Then the tensor product $\tau \otimes \chi$ is an elementary current in $X \times X'$ such that $p_*(\tau \otimes \chi) = \tau$.

The following result provides a new, quite natural definition of pseudomeromorphicity.

THEOREM 2.15.

- (1) Assume that X is smooth. Then a germ of a current μ at $x \in X$ is pseudomeromorphic if and only if it is a finite sum

$$(2.18) \quad \mu = \sum_{\ell} (f_{\ell})_* \tau_{\ell},$$

where $f_{\ell}: \mathcal{U}_{\ell} \rightarrow X$ are holomorphic mappings and τ_{ℓ} are elementary.

- (2) If X is a reduced space of pure dimension and $\pi: X' \rightarrow X$ is a smooth modification, then a current μ on X is pseudomeromorphic if and only if there is a pseudomeromorphic current μ' on X' such that $\mu = \pi_* \mu'$.

Proof. — By definition a germ of a pseudomeromorphic current is of the form (2.18). Now assume that $f: \mathcal{U} \rightarrow X$ is any holomorphic mapping and τ is elementary in $\mathcal{U} \subset \mathbb{C}^N$. Let $F: \mathcal{U} \rightarrow \mathcal{U} \times X$ be the mapping $F(s) = (s, f(s))$. Let \tilde{F} be F considered as a biholomorphism onto the graph $\Gamma \subset \mathcal{U} \times X$ and let $i: \Gamma \rightarrow \mathcal{U} \times X$ be the natural injection. Then clearly $\tilde{F}_* \tau$ is pseudomeromorphic on Γ and in view of [6, Theorem 1.1 (i)], $F_* \tau = i_* \tilde{F}_* \tau$ is pseudomeromorphic in $\mathcal{U} \times X$. Clearly, it has compact support in $\mathcal{U} \times X$. If p is the projection $\mathcal{U} \times X \rightarrow X$, we can therefore apply Lemma 2.13, and conclude that $f_* \tau = p_* F_* \tau$ is pseudomeromorphic in X . Thus part (1) is proved. Part (2) is just Proposition 2.5. \square

COROLLARY 2.16. — Assume that $f: W \rightarrow X$ is a holomorphic mapping and X is smooth. If μ is pseudomeromorphic on W with compact support, then $f_* \mu$ is pseudomeromorphic on X .

Proof. — We may assume that $\mu = \pi_* \tau$, where $\pi: \mathcal{U} \rightarrow W$ is a mapping as in the definition of pseudomeromorphicity and τ is elementary in \mathcal{U} . Then we can apply Theorem 2.15(1) to the mapping $f \circ \pi: \mathcal{U} \rightarrow X$. It follows that $f_* \mu = f_* \pi_* \tau = (f \circ \pi)_* \tau$ is pseudomeromorphic in X . \square

Remark 2.17. — Notice that in the proof of Theorem 2.15 we only used [6, Theorem 1.1 (i)], which asserts that i_* maps \mathcal{PM}_W into \mathcal{PM}_X if $i: W \rightarrow X$ is an embedding of a reduced pure-dimensional space W into a manifold X , in the relatively simple case when W is a smooth submanifold. The general case now follows from Corollary 2.16. Part (ii) of [6, Theorem 1.1] is a partial converse: If $\mu = i_* \nu$ is pseudomeromorphic in X and $\mathbb{1}_{W_{\text{sing}}} \mu = 0$, then ν is pseudomeromorphic on W . The proof of this fact relies on the possibility to make a so-called strong resolution. This means that there is a resolution $X' \rightarrow X$ that is a biholomorphism outside W , and such that the strict transform of W is a smooth resolution of W .

3. Action of holomorphic differential operators and vector fields

Let X be a reduced analytic space of pure dimension. We already know that ∂ maps \mathcal{PM}_X into itself. We shall now consider a more general statement, and to this end we need the following result that is interesting in itself.

PROPOSITION 3.1. — *Assume that $\mu \in \mathcal{PM}_x$ where $x \in X$. If $h \in \mathcal{O}_x$ is not identically zero on any irreducible component of X at x , then there is $\mu' \in \mathcal{PM}_x$ such that $h\mu' = \mu$.*

Remark 3.2. — By a partition of unity we can get a global such μ' if μ and h are global. If μ has compact support in $\mathcal{U} \subset X$ we can choose μ' with compact support in \mathcal{U} .

Remark 3.3. — If μ has support on V we may assume as well that μ' has. Indeed, $\mu = \mathbb{1}_V \mu = \mathbb{1}_V h \mu' = h \mathbb{1}_V \mu'$, so we can replace a given solution μ' by $\mathbb{1}_V \mu'$.

Example 3.4. — Proposition 3.1 is not true if h is anti-holomorphic. In fact, if $\bar{z}\mu' = 1$, then $[1/z]\mu'$ is equal to $1/|z|^2$ outside 0. Thus $\lim_{\epsilon \rightarrow 0} \chi(|z|^2/\epsilon)\mu'/z$ does not exist, and hence μ' cannot be pseudomero-morphic, cf. Proposition 2.9 and (2.13).

Proof of Proposition 3.1. — First assume that τ is an elementary pseudomero-morphic current in \mathbb{C}_t^N and h is a monomial. By induction it is enough to assume that $h = t_1$. If t_1 is a residue factor in τ , then we just raise the power of t_1 in that factor one unit. Otherwise we take $\tau' = (1/t_1)\tau$. Then $h\tau' = \tau$.

We may assume that $\mu = \pi_*\tau$, where $\pi : \mathcal{U} \rightarrow X$ and τ is elementary of the form (2.4). By Hironaka’s theorem we can find a modification $\nu : \mathcal{U}' \rightarrow \mathcal{U}$ such that, locally in \mathcal{U}' , $\nu^*\pi^*h$ is a monomial and ν^*t_j are monomials (times nonvanishing functions). By a partition of unity in \mathcal{U}' and repeated use of (2.15) it follows that τ is a finite sum of currents $\nu_*\tau'$, where

$$\tau' := \nu^* \alpha \wedge \left[\frac{1}{\nu^* t_1^{m_1}} \right] \cdots \left[\frac{1}{\nu^* t_k^{m_k}} \right] \bar{\partial} \left[\frac{1}{\nu^* t_{k+1}^{m_{k+1}}} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{\nu^* t_r^{m_r}} \right].$$

Each such term is a sum of elementary currents τ_ℓ in view of (2.14). By the first part of the proof there are elementary currents τ'_ℓ in \mathcal{U}' such that $\nu^*\pi^*h \tau'_\ell = \tau_\ell$. Now the proposition follows in view of (2.8). □

THEOREM 3.5. — Assume that X is smooth at $x \in X$.

(1) If z is a local holomorphic coordinate system at x and

$$(3.1) \quad \mu = \sum_{|I|=p}^I \mu_I \wedge dz_I$$

is a germ in \mathcal{PM}_x , then each μ_I is in \mathcal{PM}_x .

(2) If ξ is a germ of a holomorphic vector field, then the contraction $\xi \lrcorner \mu$ and the Lie derivative $L_\xi \mu$ are in \mathcal{PM}_x .

Notice that (2) is not true for anti-holomorphic vector fields. For example, $\mu = (\partial/\partial\bar{z}) \lrcorner \bar{\partial}(1/z)$ is a nonzero current of degree 0 with support at 0. In view of the dimension principle, it cannot be pseudomeromorphic.

Proof. — We will first assume that μ has bidegree $(n, *)$ so that $\mu = \hat{\mu} \wedge dz$, where $\hat{\mu}$ has bidegree $(0, *)$, and show that $\hat{\mu}$ is pseudomeromorphic. We may assume that $\mu = \pi_*(\tau \wedge ds)$, where $\pi : \mathcal{U} \rightarrow X$ is a mapping as in the definition of pseudomeromorphicity, s are local coordinates in $\mathcal{U} \subset \mathbb{C}^m$, and τ is elementary. Since π has generically surjective differential, we can write $s = (s', s'') = (s'_1, \dots, s'_n, s''_{n+1}, \dots, s''_m)$ so that $h := \det(\partial\pi/\partial s') = \det(\partial z/\partial s')$ is generically nonvanishing in \mathcal{U} . By Proposition 3.1 and Remark 3.2 there is a pseudomeromorphic τ' with compact support in \mathcal{U} such that $h\tau' = \tau$ in \mathcal{U} . Now

$$\begin{aligned} \hat{\mu} \wedge dz &= \pi_*(\tau \wedge ds) = \pi_*(\tau' \wedge h ds' \wedge ds'') = \pi_*(\tau' \wedge \pi^* dz \wedge ds'') \\ &= \pm \pi_*(\tau' \wedge ds'') \wedge dz. \end{aligned}$$

Thus $\hat{\mu} = \pm \pi_*(\tau' \wedge ds'')$ is pseudomeromorphic. In general, $\mu_I \wedge dz = \pm \mu \wedge dz_{I^c}$, where I^c is the complementary multiindex of I . It follows from above that μ_I is pseudomeromorphic. Thus (1) follows.

The first statement of (2) follows immediately from (1), and the second one follows since $L_\xi \mu = \partial(\xi \lrcorner \mu) + \xi \lrcorner (\partial\mu)$. □

3.1. The sheaves \mathcal{PM}_X^Z and \mathcal{W}_X^Z

Let X be a reduced analytic space, let $Z \subset X$ be a (reduced) subspace of pure dimension, and denote by \mathcal{PM}_X^Z the subsheaf of \mathcal{PM}_X of currents that have support on Z . We say that $\mu \in \mathcal{PM}_X^Z$ has the *standard extension property, SEP*, on Z if $\mathbf{1}_W \mu = 0$ in \mathcal{U} for each subvariety $W \subset \mathcal{U} \cap Z$ of positive codimension, where \mathcal{U} is any open set in X . Let \mathcal{W}_X^Z be the subsheaf of \mathcal{PM}_X^Z of currents with the SEP on Z . In case $Z = X$ we usually write \mathcal{W}_X rather than \mathcal{W}_X^X .

Example 3.6. — Note that an elementary current in \mathcal{U} with elementary support H is in $\mathcal{W}_{\mathcal{U}}^H$.

It is easy to see that Theorem 3.5 holds for \mathcal{PM}_X^Z as well, since neither ∂ nor contraction can increase support. Somewhat less obvious is that also the SEP is preserved.

THEOREM 3.7. — *The sheaf \mathcal{W}_X^Z is invariant under ∂ , and the statements in Theorem 3.5 hold for \mathcal{W}_X^Z instead of \mathcal{PM} .*

This theorem is a consequence of the following general equalities.

PROPOSITION 3.8. — *Assume that μ is a pseudomeromorphic current on X . If $V \subset X$ is any analytic subset, then*

$$(3.2) \quad \mathbb{1}_V \partial \mu = \partial \mathbb{1}_V \mu.$$

If ξ is a holomorphic vector field, then

$$(3.3) \quad \mathbb{1}_V \xi \lrcorner \mu = \xi \lrcorner \mathbb{1}_V \mu.$$

Proof. — Note that (3.3) follows in view of (2.12). Let us therefore focus on (3.2). By (1.3) it is enough to consider $V = Z(h)$, where h is a nontrivial holomorphic function. Take $\chi \sim \chi_{[1, \infty)}$ and let $\chi_\epsilon = \chi(|h|^2/\epsilon)$. Now

$$(3.4) \quad \chi_\epsilon \partial \mu = \partial(\chi_\epsilon \mu) - \partial \chi_\epsilon \wedge \mu.$$

If the last term tends to 0 when $\epsilon \rightarrow 0$, after taking limits we get that $\mathbb{1}_{h \neq 0} \partial \mu = \partial(\mathbb{1}_{h \neq 0} \mu)$, which is equivalent to (3.2). Let $\hat{\chi}(t) = t\chi'(t) + \chi(t)$, and notice that also $\hat{\chi} \sim \chi_{[1, \infty)}$. According to Proposition 3.1 there is a pseudomeromorphic μ' such that $\mu = h\mu'$. The last term in (3.4) is therefore

$$\chi'(|h|^2/\epsilon) \bar{h} \partial h \wedge \mu / \epsilon = \chi'(|h|^2/\epsilon) |h|^2 \partial h \wedge \mu' / \epsilon = \hat{\chi}(|h|^2/\epsilon) \partial h \wedge \mu' - \chi_\epsilon \partial h \wedge \mu',$$

which tends to $\mathbb{1}_{h \neq 0} \partial h \wedge \mu' - \mathbb{1}_{h \neq 0} \partial h \wedge \mu' = 0$. □

4. Almost semi-meromorphic currents

We say that a current on X is *semi-meromorphic* if it is of the form $\omega[1/f]$, where f is a generically nonvanishing holomorphic section of a line bundle $L \rightarrow X$ and ω is a smooth form with values in L . For simplicity we will often omit the brackets $[]$ indicating principal value in the sequel. Since furthermore $\omega[1/f] = [1/f]\omega$ when ω is smooth we can write just ω/f .

4.1. The algebra $ASM(X)$

Let X be a pure-dimensional reduced analytic space. We say that a current a is *almost semi-meromorphic* in X , $a \in ASM(X)$, if there is a modification $\pi: X' \rightarrow X$ such that

$$(4.1) \quad a = \pi_*(\omega/f),$$

where ω/f is semi-meromorphic in X' . We say that a is *almost smooth* in X if one can choose f to be nonvanishing. We can assume that X' is smooth because otherwise we take a smooth modification $\pi': X'' \rightarrow X'$ and consider the pullbacks of f and ω to X'' , cf. (2.15). If nothing else is said we tacitly assume that X' is smooth.

Notice that if $\mathcal{U} \subset X$ is an open subset, then the restriction $a_{\mathcal{U}}$ of $a \in ASM(X)$ to \mathcal{U} is in $ASM(\mathcal{U})$. In fact, if (4.1) holds, then $\mathcal{U}' := \pi^{-1}\mathcal{U} \rightarrow \mathcal{U}$ is a modification of \mathcal{U} , and $a_{\mathcal{U}}$ is the direct image of the restriction of ω/f to \mathcal{U}' .

If V has positive codimension in $\mathcal{U} \subset X$, then $\pi^{-1}V$ has positive codimension in \mathcal{U}' and $\mathbb{1}_V a = \pi_*(\mathbb{1}_{\pi^{-1}V}(\omega/f)) = \pi_*(\omega \mathbb{1}_{\pi^{-1}V}(1/f)) = 0$ in \mathcal{U} , cf. (2.11), (2.10), and the dimension principle. Thus $ASM(X)$ is contained in $\mathcal{W}(X)$.

Remark 4.1. — One can introduce a notion “locally almost semi-meromorphic current” and consider the associated sheaf. However, for the moment we have no need for such a concept.

Example 4.2. — Assume that $X = \{zw = 0\} \subset \mathbb{C}^2$. Let $a: X \rightarrow \mathbb{C}$ be 1 and 0 on the z -axis and the w -axis, respectively, except at the origin. Then a is almost smooth. Indeed the normalization $\nu: \tilde{X} \rightarrow X$ consists of two disjoint components and $a = \nu_*\tilde{a}$, where \tilde{a} is 0 and 1, respectively, on these components.

Given a modification $\pi: X' \rightarrow X$, let $\text{sing}(\pi) \subset X'$ be the (analytic) set where π is not a biholomorphism. By the definition of a modification it has positive codimension. Let a be given by (4.1) and let $Z \subset X'$ be the zero set of f . By assumption also Z has positive codimension. Notice that $a \in ASM(X)$ is smooth outside $\pi(Z \cup \text{sing}(\pi))$ which has positive codimension in X . We let $ZSS(a)$, the *Zariski-singular support* of a , be the smallest Zariski-closed set $V \subset X$ such that a is smooth outside V .

Example 4.3. — Assume that $a \in ASM(X)$ is almost smooth. Then $a = \pi_*\omega$, where ω is smooth, and thus $ZSS(a) \subset \pi(\text{sing}(\pi))$. This inclusion may be strict. For example if a is smooth, then $ZSS(a)$ is empty. In this case

$\omega = \pi^*a$ outside $\text{sing}(\pi)$ and since both sides are smooth across $\text{sing}(\pi)$, by continuity, then $\omega = \pi^*a$ everywhere in X' .

Given two modifications $X_1 \rightarrow X$ and $X_2 \rightarrow X$, there is a modification $\pi: X' \rightarrow X$ that factorizes over both X_1 and X_2 , i.e., we have $X' \rightarrow X_j \rightarrow X$ for $j = 1, 2$. Therefore, given $a_1, a_2 \in ASM(X)$ we can assume that $a_j = \pi_*(\omega_j/f_j)$, $j = 1, 2$. It follows that

$$a_1 + a_2 = \pi_* \left(\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} \right) = \pi_* \frac{f_2\omega_1 + f_1\omega_2}{f_1f_2},$$

so that $a_1 + a_2$ is in $ASM(X)$ as well. Moreover, $A := \pi_*(\omega_1 \wedge \omega_2 / f_1 f_2)$ is an almost semi-meromorphic current that coincides with $a_1 \wedge a_2$ outside the set $\pi(\text{sing}(\pi) \cup V(f_1) \cup V(f_2))$. If we had other representations $a_j = \pi'_*(\omega'_j/f'_j)$, $j = 1, 2$, we would get an almost semi-meromorphic A' that coincides generically with $a_1 \wedge a_2$ on X . Since almost semi-meromorphic have the SEP, thus $A = A'$. Hence we can define $a_1 \wedge a_2$ as A . Similarly, since

$$a_2 \wedge a_1 = (-1)^{\text{deg } a_1 \text{ deg } a_2} a_1 \wedge a_2, \quad a_1 \wedge (a_2 + a_3) = a_1 \wedge a_2 + a_1 \wedge a_3$$

and

$$a_1 \wedge (a_2 \wedge a_3) = (a_1 \wedge a_2) \wedge a_3$$

hold generically on X and because of the SEP they hold on X . Thus $ASM(X)$ is an algebra.

Remark 4.4. — Notice that the almost smooth currents form a subalgebra of $ASM(X)$.

Example 4.5. — Clearly $ZSS(a_1 \wedge a_2) \subset ZSS(a_1) \cup ZSS(a_2)$ but the inclusion may be strict. Take for instance z_1/z_2 and z_2/z_3 .

Example 4.6. — The most basic example of an (almost semi-)meromorphic current is the principal value current associated with a meromorphic form. Let f be meromorphic k -form on X , i.e., locally $f = g/h$ where h is a holomorphic function that is generically nonvanishing and g is a holomorphic $(k, 0)$ -form. By definition $g/h = g'/h'$ if and only if $g'h - gh'$ vanishes outside a set of positive codimension. In that case

$$(4.2) \quad g \left[\frac{1}{h} \right] = g' \left[\frac{1}{h'} \right]$$

outside a set of positive codimension. By the dimension principle therefore (4.2) holds everywhere. Thus there is a well-defined almost semi-meromorphic current $[f]$ associated with f . Notice that $ZSS([f])$ is contained in the pole set of the meromorphic form f , so unless X is smooth

it may have codimension larger than 1. Actually, $ZSS([f])$ is equal to the pole set of f . In fact, by continuity $\bar{\partial}f = 0$ where f is smooth, and by a classical result proved by Malgrange (at least for functions), [19], then f is holomorphic there.

The following lemma will be crucial in what follows.

LEMMA 4.7. — *If a is almost semi-meromorphic in X , then there is a representation (4.1) such that f is nonvanishing in $X' \setminus \pi^{-1}ZSS(a)$.*

Proof. — Let $V = ZSS(a)$ and assume that we have a representation (4.1) and that X' is smooth. Let Z be the union of the irreducible components of the divisor defined by f that are not fully contained in $\pi^{-1}V$. Since X' is smooth, Z is a Cartier divisor and thus the divisor of a section f' of some line bundle $L' \rightarrow X'$. It follows that $g := f/f'$ is a holomorphic section of $L \otimes (L')^{-1}$ in X' that is nonvanishing in $X' \setminus \pi^{-1}V$. Outside $\text{sing}(\pi) \cup Z \cup \pi^{-1}V$ we have that

$$(4.3) \quad \omega = f\pi^*a = f'g\pi^*a.$$

By continuity, (4.3) must hold in $X' \setminus \pi^{-1}V$ since both sides are smooth there.

We claim that $\tilde{\omega} := \omega/f'$ is smooth in X' . Taking this for granted, then

$$(4.4) \quad \pi_* \frac{\tilde{\omega}}{g}$$

is in $ASM(X)$ and the zero set of g is contained in $\pi^{-1}V$. Since (4.4) coincides with a outside $V \cup \pi(\text{sing}(\pi))$ it follows by the SEP that (4.4) indeed is equal to a in X . Thus the lemma follows.

The claim is a local statement in X' so given a point in X' we can choose local coordinates t in a neighborhood \mathcal{U} of that point and consider each coefficient of the form ω with respect to these coordinates. Thus we may assume that ω is a function and that $\omega = f'\gamma$ where $\gamma = g\pi^*a$ is smooth in $\mathcal{U} \setminus \pi^{-1}V$, cf. (4.3) and the comment thereafter. For all multiindices α thus

$$(4.5) \quad \frac{\partial^\alpha \omega}{\partial t^\alpha} \bar{\partial} \frac{1}{f'} = 0$$

in $\mathcal{U} \setminus \pi^{-1}V$, since $f'\bar{\partial}(1/f') = 0$. By assumption $Z \cap \pi^{-1}V$ has positive codimension in Z . By the dimension principle it follows that (4.5) holds in \mathcal{U} for all α , since $\bar{\partial}(1/f')$ has support on Z . From [2, Theorem 1.2] we conclude that $\tilde{\omega}$ is smooth in \mathcal{U} . It follows that $\tilde{\omega}$ is smooth in X' . \square

4.2. Action of $ASM(X)$ on \mathcal{PM}_X

We will now extend Proposition 2.9 to general almost semi-meromorphic currents.

THEOREM 4.8. — *Assume that $a \in ASM(X)$. For each $\mu \in \mathcal{PM}(X)$ there is a unique pseudomeromorphic current T in X that coincides with $a \wedge \mu$ in $X \setminus ZSS(a)$ and such that $\mathbb{1}_{ZSS(a)}T = 0$.*

Let $V = ZSS(a)$. If such an extension T exists then $T = \mathbb{1}_{X \setminus V}T = \mathbb{1}_{X \setminus V}a \wedge \mu$ and so T is unique. Moreover, if h is a holomorphic tuple such that $Z(h) = V$, then

$$(4.6) \quad T = \lim_{\epsilon \rightarrow 0} \chi(|h|^2 v / \epsilon) a \wedge \mu$$

in view of Lemma 2.6. We will denote the extension T by $a \wedge \mu$ as well.

Proof. — As observed above, if the extension T exists, then (4.6) holds. Conversely, if the limit in (4.6) exists as a pseudomeromorphic current T on X , then it must coincide with $a \wedge \mu$ in $X \setminus V$. In particular, $\chi(|h|^2 v / \epsilon)T = \chi(|h|^2 v / \epsilon)a \wedge \mu$ for each $\epsilon > 0$ and hence, taking limits and using Lemma 2.6, we get $\mathbb{1}_{X \setminus V}T = T$, i.e., $\mathbb{1}_{ZSS(a)}T = 0$. To prove the theorem it is thus enough to verify that the limit in (4.6) exists as a pseudomeromorphic current.

In view of Lemma 4.7 we may assume that a has the form (4.1), where $Z = Z(f)$ is contained in $\pi^{-1}V$ and $\omega/f = \pi^*a$ in $X' \setminus \pi^{-1}V$. Let $\chi_\epsilon = \chi(|h|^2 v / \epsilon)$, so that $\pi^*\chi_\epsilon = \chi(|\pi^*h|^2 \pi^*v / \epsilon)$. By Proposition 2.5 there is $\mu' \in \mathcal{PM}(X')$ such that $\pi_*\mu' = \mu$. Thus

$$\chi_\epsilon a \wedge \mu = \chi_\epsilon a \wedge \pi_*\mu' = \pi_* (\pi^*\chi_\epsilon \pi^*a \wedge \mu') = \pi_* \left(\pi^*\chi_\epsilon \frac{\omega}{f} \wedge \mu' \right).$$

In view of Proposition 2.9 and Lemma 2.6,

$$\pi^*\chi_\epsilon \frac{\omega}{f} \wedge \mu' \rightarrow \mathbb{1}_{X' \setminus \pi^{-1}V} \frac{\omega}{f} \wedge \mu'$$

when $\epsilon \rightarrow 0$. In particular, the limit is a pseudomeromorphic current. Thus the limit in (4.6) exists and is pseudomeromorphic. □

Notice that the definition of $a \wedge \mu$ is local, so that it commutes with restrictions to open subsets of X . Thus for each $a \in ASM(X)$ we get a linear sheaf mapping

$$(4.7) \quad \mathcal{PM}_X \rightarrow \mathcal{PM}_X, \quad \mu \mapsto a \wedge \mu.$$

PROPOSITION 4.9. — Assume that $a \in ASM(X)$. If W is an analytic subset of $\mathcal{U} \subset X$ and $\mu \in \mathcal{PM}(\mathcal{U})$, then

$$(4.8) \quad \mathbb{1}_W(a \wedge \mu) = a \wedge \mathbb{1}_W \mu.$$

Proof. — On the one hand (4.8) holds in the open set $\mathcal{U} \setminus ZSS(a)$ by (2.10) since a is smooth there. On the other hand both sides vanish on $ZSS(a)$, so (4.8) holds in all of \mathcal{U} ; indeed $\mathbb{1}_{ZSS(a)}(a \wedge \mathbb{1}_W \mu) = 0$ by definition, cf. Theorem 4.8, and $\mathbb{1}_{ZSS(a)} \mathbb{1}_W(a \wedge \mu) = \mathbb{1}_W \mathbb{1}_{ZSS(a)}(a \wedge \mu) = 0$ in view of (1.3). □

PROPOSITION 4.10. — Each $a \in ASM(X)$ induces a linear mapping

$$(4.9) \quad \mathcal{W}_X^Z \rightarrow \mathcal{W}_X^Z, \quad \mu \mapsto a \wedge \mu.$$

Proof. — To begin with, certainly $a \wedge \mu$ has support on Z if μ has. Let \mathcal{U} be an open subset of X and assume that $W \subset \mathcal{U} \cap Z$ has positive codimension in $\mathcal{U} \cap Z$. Then $\mathbb{1}_W(a \wedge \mu) = a \wedge \mathbb{1}_W \mu = 0$ if $\mathbb{1}_W \mu = 0$, cf. (4.8). □

Example 4.11. — Assume that μ is in \mathcal{W}_X . Then $\mu' := [1/h]\mu$ is in \mathcal{W} as well and if h is generically nonvanishing, then $h\mu' = h[1/h]\mu = \mathbb{1}_{\{h \neq 0\}}\mu = \mu$, cf. Remark 2.10.

PROPOSITION 4.12. — Assume that $a_1, a_2 \in ASM(X)$ and $\mu \in \mathcal{PM}_X$. Then

$$(4.10) \quad a_1 \wedge a_2 \wedge \mu = (-1)^{\deg a_1 \deg a_2} a_2 \wedge a_1 \wedge \mu.$$

Proof. — Notice that both sides of (4.10) coincide outside $ZSS(a_1) \cup ZSS(a_2)$ and the restrictions to $ZSS(a_1) \cup ZSS(a_2)$ vanish. □

In particular, one of the a_j may be a smooth form. We conclude that both (4.7) and (4.9) are \mathcal{E} -linear.

PROPOSITION 4.13. — If $a_1, a_2 \in ASM(X)$ and $\mu \in \mathcal{W}_X$, then

$$(4.11) \quad a_1 \wedge a_2 \wedge \mu = (a_1 \wedge a_2) \wedge \mu, \quad (a_1 + a_2) \wedge \mu = a_1 \wedge \mu + a_2 \wedge \mu.$$

In fact, (4.11) holds outside $V := ZSS(a_1) \cup ZSS(a_2)$ and since $\mathbb{1}_V \mu = 0$ the equalities follow from (4.8).

Example 4.14. — Both equalities in (4.11) may fail for a general $\mu \in \mathcal{PM}_X$. Let $a_1 = 1/z_1$, $a_2 = z_1/z_2$, $a_3 = 1/z_2$, and $\mu = \bar{\partial}(1/z_1)$. Then $(a_1 a_2) \mu = (1/z_2) \bar{\partial}(1/z_1)$, but $a_2 \mu = 0$, and so $a_1 a_2 \mu = 0$. Moreover

$$(a_1 + a_3) \mu = \frac{z_2 + z_1}{z_1 z_2} \bar{\partial} \frac{1}{z_1} = 0$$

but

$$a_1 \mu + a_3 \mu = \frac{1}{z_1} \bar{\partial} \frac{1}{z_1} + \frac{1}{z_2} \bar{\partial} \frac{1}{z_1} = \frac{1}{z_2} \bar{\partial} \frac{1}{z_1}.$$

4.3. Vector-valued almost semi-meromorphic currents

We will need to consider almost semi-meromorphic currents that take values in a holomorphic vector bundle $E \rightarrow X$. We say that $a \in ASM(X, E)$ if there is a representation (4.1), where as before f is a holomorphic section of $L \rightarrow X'$ and now ω takes values in $L \otimes \pi^*E$. Clearly then a is a current with values in E . If η is a test form with values in the dual bundle E^* , then $a.\eta = \pi_*((\omega/f).\pi^*\eta)$. Let e_j be a local frame for E in \mathcal{U} and let ξ be a test function with support in \mathcal{U} . If $\xi' = \pi^*\xi$, $e'_j = e_j \circ \pi$ and $\omega = \omega_1 e'_1 + \omega_2 e'_2 + \dots$, then

$$(4.12) \quad \xi a = \sum_j \pi_*(\xi' \omega_j / f) e_j.$$

PROPOSITION 4.15. — *Assume that X is smooth. There are natural isomorphisms*

$$(4.13) \quad ASM^{p,*}(X, E) \simeq ASM^{0,*}(X, \Lambda^p T^*_{1,0}(X) \otimes E).$$

Proof. — First notice that if F, G are vector bundles of the same rank over X' and h is a holomorphic section of $\text{Hom}(F, G)$ that is generically invertible, then there is a holomorphic section g of $\text{Hom}(G, F) \otimes \det G \otimes (\det F)^{-1}$ such that $hg = s \cdot I_G$, where s is a generically nonvanishing section of $\det G \otimes (\det F)^{-1}$.

For simplicity we assume that E is a trivial line bundle; the general case is proved in the same way. Now, let $F = \pi^* \Lambda^p T^*_{1,0}(X)$ and $G = \Lambda^p T^*_{1,0}(X')$. Then we have a natural mapping $h: F \rightarrow G$ as above, defined by just mapping the frame element dz_I to its pullback $\pi^* dz_I$. Clearly h is an isomorphism where $\pi: X' \rightarrow X$ is biholomorphic.

Now, if $a \in ASM^{0,*}(X, \Lambda^p T^*_{1,0}(X))$, then we have the representation $a = \pi_*(\omega/f)$, where ω takes values in $F \otimes L$. Then $h\omega$ is a $(p, *)$ -form in X' with values in L . It follows that $a' := \pi_*(h\omega/f)$ is an element in $ASM^{p,*}(X)$. We claim that $a' = a$. By the SEP it is enough to verify the identity where π is a biholomorphism. Let z be coordinates in an open subset $\mathcal{U} \subset X \setminus \pi(\text{sing } \pi)$, and let ξ be a test function with support in \mathcal{U} . Then, cf. (4.12),

$$\begin{aligned} \xi a &= \sum_{|I|=p} \pi_*(\xi' \omega_I / f) \wedge dz_I = \pi_* \left(\xi' \sum_{|I|=p} \omega_I / f \wedge \pi^* dz_I \right) = \pi_*(\xi' h\omega / f) \\ &= \xi \pi_*(h\omega / f) = \xi a'. \end{aligned}$$

Conversely, since $h^{-1} = g/s$, if $a' \in ASM^{p,*}(X)$, then $a' = \pi_*(\tilde{\omega}/f)$, where $\tilde{\omega}$ is a $(p, *)$ -form with values in L , then $g\tilde{\omega}$ takes values in

$F \otimes \det G \otimes (\det F)^{-1} \otimes L$ and sf takes values in $\det G \otimes (\det F)^{-1} \otimes L$, so that $a = \pi_*(g\tilde{\omega}/sf)$ is an element in $ASM^{0,*}(X, \Lambda^p T_{0,1}^*(X))$. Again one verifies that they coincide in $X \setminus \pi(\text{sing } \pi)$. \square

Notice that if $p = 1$, then s is a section of the relative canonical bundle $K_{X'/X} = K_{X'} \otimes \pi^* K_X^{-1}$.

4.4. Residues of almost semi-meromorphic currents

We shall now study the effect of ∂ and $\bar{\partial}$ on almost semi-meromorphic currents.

PROPOSITION 4.16. — *If $a \in ASM(X)$, then $\partial a \in ASM(X)$ and $b := \mathbb{1}_{X \setminus ZSS(a)} \bar{\partial} a \in ASM(X)$.*

Thus we have the decomposition

$$(4.14) \quad \bar{\partial} a = b + r,$$

where $r := \mathbb{1}_{ZSS(a)} \bar{\partial} a$ has support on $ZSS(a)$.

Proof. — Assume that $a = \pi_*(\omega/f)$ and let $D = D' + \bar{\partial}$ be a Chern connection on $L \rightarrow X'$. Then

$$\partial a = \pi_* \left(\partial \frac{\omega}{f} \right) = \pi_* \frac{f \cdot D' \omega - D' f \wedge \omega}{f^2},$$

which is in $ASM(X)$.

In view of Lemma 4.7 we may assume that $Z(f) \subset \pi^{-1}V$, where $V = ZSS(a)$. Now

$$(4.15) \quad \bar{\partial} a = \pi_* \frac{\bar{\partial} \omega}{f} + \pi_* \bar{\partial} \frac{1}{f} \wedge \omega.$$

By (2.11),

$$(4.16) \quad \begin{aligned} \mathbb{1}_{X \setminus V} \bar{\partial} a &= \pi_* \left(\mathbb{1}_{\pi^{-1}(X \setminus V)} \frac{\bar{\partial} \omega}{f} \right) + \pi_* \left(\mathbb{1}_{\pi^{-1}(X \setminus V)} \bar{\partial} \frac{1}{f} \wedge \omega \right) \\ &= \pi_* \left(\frac{\bar{\partial} \omega}{f} \right); \end{aligned}$$

thus $\mathbb{1}_{X \setminus V} \bar{\partial} a \in ASM(X)$. For the last equality we have used Proposition 2.9 and the fact that $\bar{\partial}(1/f)$ has support on $\pi^{-1}V$. \square

In the same way we have: *If $a \in ASM(X, E)$ then (4.14) holds, where $b = \mathbb{1}_{X \setminus ZSS(a)} \bar{\partial} a$ is in $ASM(X, E)$ and $r = \mathbb{1}_{ZSS(a)} \bar{\partial} a$ is a pseudomeromorphic current with support on $ZSS(a)$ that takes values in E .*

Clearly the decomposition (4.14) is unique. We call $r = r(a)$ the *residue (current)* of a . Notice that if a is almost smooth, then $r(a) = 0$.

Remark 4.17. — If $a = \pi_*(\omega/f)$ is any representation of a , then still (4.15) holds, and since the first term is in $ASM(X)$ we conclude that

$$r(a) = \pi_* \left(\bar{\partial} \frac{1}{f} \wedge \omega \right).$$

Notice that the current $\bar{\partial}(1/f)$ is the residue of the principal value current $1/f$. Similarly, the residue currents introduced, e.g., in [3, 9, 21] can be considered as residues of certain almost semi-meromorphic currents, generalizing $1/f$.

Example 4.18. — Let us describe the construction of the residue currents in [3]. Let f be a holomorphic section of a Hermitian vector bundle $E \rightarrow X$, and let σ be the section over $X \setminus Z(f)$ of the dual bundle E^* with minimal norm such that $f\sigma = 1$. We can find a modification $\pi: X' \rightarrow X$ that is a biholomorphism $X' \setminus \pi^{-1}Z(f) \simeq X \setminus Z(f)$ such that $\pi^*f = f^0f'$, where f^0 is a holomorphic section of a line bundle $L \rightarrow X'$, $\text{div } f^0$ is contained in $\pi^{-1}Z(f)$, and f' is a nonvanishing section of $\pi^*E \otimes L^{-1}$. Then

$$\pi^*\sigma = \sigma'/f^0,$$

where σ' is a smooth section of $\pi^*E^* \otimes L$. Thus

$$\pi^* (\sigma \wedge (\bar{\partial}\sigma)^{k-1}) = \frac{\sigma' \wedge (\bar{\partial}\sigma')^{k-1}}{(f^0)^k}$$

is a section of $\Lambda^k(\pi^*E \oplus T_{0,1}^*(X'))$ in $X' \setminus \pi^{-1}Z(f)$; for the reader's convenience note that $\bar{\partial}\sigma$ has even degree in $\Lambda^k(\pi^*E \oplus T_{0,1}^*(X'))$. It follows that

$$U_k := \sigma \wedge (\bar{\partial}\sigma)^{k-1}$$

has an extension to an almost semi-meromorphic section of $\Lambda^k(E \oplus T_{0,1}^*(X))$, as the push-forward of $\sigma' \wedge (\bar{\partial}\sigma')^{k-1}/(f^0)^k$. Clearly $ZSS(U_k) \subset Z(f)$. Now the residue current R in [3] is the residue of the almost semi-meromorphic current $U = \sum_k U_k$. More precisely, if δ_f denotes interior multiplication by f , then $(\delta_f - \bar{\partial})U = 1 - R$, i.e., $\bar{\partial}U = R + \delta_f U - 1$, where R is the residue and $\delta_f U - 1$ is almost semi-meromorphic. If E is trivial with trivial metric, the coefficients of R are the Bochner–Martinelli residue currents introduced in [21].

Clearly Theorem 4.8 extends to vector-valued currents. As a consequence of this theorem we can define products of residues of almost semi-meromorphic currents and pseudomeromorphic currents:

DEFINITION 4.19. — For $a \in ASM(X, E)$ and $\mu \in \mathcal{PM}_X$ we define

$$(4.17) \quad \bar{\partial}a \wedge \mu := \bar{\partial}(a \wedge \mu) - (-1)^{\deg a} a \wedge \bar{\partial}\mu,$$

where $a \wedge \mu$ and $a \wedge \bar{\partial}\mu$ are defined as in Theorem 4.8. Moreover we define

$$r(a) \wedge \mu := \mathbb{1}_{ZSS(a)} \bar{\partial}a \wedge \mu.$$

Thus $\bar{\partial}a \wedge \mu$ is defined so that the Leibniz rule holds. It is easily checked that

$$(4.18) \quad r(a) \wedge \mu = \lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|^2 v/\epsilon) a \wedge \mu,$$

if $Z(h) = ZSS(a)$. In particular this gives a way of defining products of $\bar{\partial}$ and residues of almost semi-meromorphic currents. For example, the Coleff–Herrera product $\bar{\partial}(1/f_1) \wedge \dots \wedge \bar{\partial}(1/f_p)$ can be defined by inductively applying (4.17). In [5] the first author defined products of more general residue currents in this way.

Notice that in general $a_1 \wedge \bar{\partial}a_2$ is *not* equal to $\pm \bar{\partial}a_2 \wedge a_1$, cf. Remark 2.11, and neither is

$$(4.19) \quad \bar{\partial}a_1 \wedge \bar{\partial}a_2 = \pm \bar{\partial}a_2 \wedge \bar{\partial}a_1$$

in general; take, e.g., $a_1 = 1/z$ and $a_2 = 1/zw$.

THEOREM 4.20. — Assume that a_1, \dots, a_p are almost semi-meromorphic currents of degree $(*, k_1 - 1), \dots, (*, k_p - 1)$, respectively, and that

$$(4.20) \quad \text{codim}(ZSS(a_{i_1}) \cap \dots \cap ZSS(a_{i_r})) \geq k_{i_1} + \dots + k_{i_r}$$

for all $\{i_1, \dots, i_r\} \subset \{1, \dots, p\}$. Then

$$(4.21) \quad \begin{aligned} \bar{\partial}a_1 \wedge \dots \wedge \bar{\partial}a_j \wedge \bar{\partial}a_{j+1} \wedge \dots \wedge \bar{\partial}a_p \\ = (-1)^{(\deg a_j + 1)(\deg a_{j+1} + 1)} \bar{\partial}a_1 \wedge \dots \wedge \bar{\partial}a_{j+1} \wedge \bar{\partial}a_j \wedge \dots \wedge \bar{\partial}a_p. \end{aligned}$$

Remark 4.21. — In fact, one can modify the proof below so that one can replace any factor $\bar{\partial}a_i$ in (4.21) by a_i . More precisely, let b_i be either a_i or $\bar{\partial}a_i$ for $i = 1, \dots, p$. Then

$$(4.22) \quad \begin{aligned} b_1 \wedge \dots \wedge b_j \wedge b_{j+1} \wedge \dots \wedge b_p \\ = (-1)^{\deg b_j \cdot \deg b_{j+1}} b_1 \wedge \dots \wedge b_{j+1} \wedge b_j \wedge \dots \wedge b_p. \end{aligned}$$

Remark 4.22. — If the almost semimeromorphic parts of $\bar{\partial}a_i$ vanish, then it is enough to assume

$$(4.23) \quad \text{codim}(ZSS(a_1) \cap \dots \cap ZSS(a_p)) \geq k_1 + \dots + k_p.$$

Indeed, note that in this case the currents in (4.21) have support on $V := ZSS(a_1) \cap \dots \cap ZSS(a_p)$. Thus it is enough to prove (4.21) in a neighborhood of $x \in V$, and there (4.23) implies (4.20).

In particular, the Coleff–Herrera product $\bar{\partial}(1/f_1) \wedge \dots \wedge \bar{\partial}(1/f_p)$ is (anti-)commutative in its factors if the codimension of $\{f_1 = \dots = f_p = 0\}$ is at least p .

Proof. — Let $V_j = ZSS(a_j)$. Moreover, let b_i be either an almost semi-meromorphic current or $\bar{\partial}$ of an semi-meromorphic current for $i = 1, \dots, r$, cf. Remark 4.21, and assume that α is smooth. Then note that

$$(4.24) \quad b_1 \wedge \dots \wedge b_\ell \wedge \alpha \wedge b_{\ell+1} \wedge \dots \wedge b_r \\ = (-1)^{\deg \alpha (\deg b_1 + \dots + \deg b_\ell)} \alpha \wedge b_1 \wedge \dots \wedge b_r.$$

Assume that

$$(4.25) \quad \bar{\partial} a_1 \wedge \dots \wedge \bar{\partial} a_{j-1} \wedge a_j \wedge \bar{\partial} a_{j+1} \wedge \dots \wedge \bar{\partial} a_p \\ = (-1)^{\deg a_j (\deg a_{j+1} + 1)} \bar{\partial} a_1 \wedge \dots \wedge \bar{\partial} a_{j-1} \wedge \bar{\partial} a_{j+1} \wedge a_j \wedge \bar{\partial} a_{j+2} \wedge \dots \wedge \bar{\partial} a_p.$$

Applying $\bar{\partial}$ to (4.25) yields (4.21) in view of (4.17).

To prove (4.25) we will proceed by induction. First assume that $p = 2$. Then in view of (4.24),

$$(4.26) \quad a_1 \wedge \bar{\partial} a_2 = (-1)^{\deg a_1 (\deg a_2 + 1)} \bar{\partial} a_2 \wedge a_1,$$

where a_1 or a_2 is smooth, i.e., outside $V_1 \cap V_2$. Because of the assumption (4.20), (4.26) holds in all of X by the dimension principle. Next, assume that (4.25) holds for $p = \ell$. In view of (4.24), (4.25) holds for $p = \ell + 1$, where a_j or a_{j+1} is smooth. Moreover, by (4.24) and the assumption that (4.25) holds for $p = \ell$, (4.25) holds for $p = \ell + 1$, where (at least) one of $a_1, \dots, a_{j-1}, a_{j+2}, \dots, a_{\ell+1}$ is smooth. Thus (4.25) holds for $p = \ell + 1$ outside $V_1 \cap \dots \cap V_{\ell+1}$, and thus by (4.20) and the dimension principle it holds in all of X . Hence (4.25) and thus (4.21) hold for all p . \square

The following example shows that $r(a) = 0$ does not imply that $r(a) \wedge \mu = 0$. This points out the importance of keeping in mind that $\mu \mapsto r(a) \wedge \mu$ is an operator on \mathcal{PM}_X rather than a “product”.

Example 4.23. — Let us consider the setting in Example 4.18. Assume in addition that $Z(f)$ has codimension at least 2. Note that then $r(\sigma) = 0$ by the dimension principle, since it has bidegree $(0, 1)$ and support on $Z(f)$, which has codimension ≥ 2 . However, if τ is the almost semi-meromorphic part of $\bar{\partial}U$, then $r(\sigma) \wedge \tau$ is the residue current R from [3] which is nonzero, cf. Example 4.18.

Remark 4.24. — There are other (weighted) approaches to products of residue currents, see, e.g. [20, 26], which coincide with the products above under suitable conditions.

4.5. Action of holomorphic differential operators and vector fields

Finally we prove that $ASM(X)$ is preserved under the action of holomorphic vector fields.

THEOREM 4.25. — *Let ξ be a holomorphic vector field on a smooth manifold X . If $a \in ASM(X)$, then the contraction $\xi \lrcorner a$ and the Lie derivative $L_\xi a$, a priori defined on $X \setminus ZSS(a)$, have extensions as elements in $ASM(X)$.*

Since the extensions, if they exist, must be unique, we can simply say that $\xi \lrcorner a$ and $L_\xi a$ are in $ASM(X)$.

Proof. — Let $\pi: X' \rightarrow X$ be a modification so that a has the form (4.1). Then $\xi' := \pi^* \xi$ is a global section of $\pi^* T(X)$, that is the natural lifting of ξ to $T(X')$ over $X' \setminus \text{sing}(\pi)$. By duality the mapping $\pi^* T_{1,0}^*(X) \rightarrow T_{1,0}^*(X')$ from the proof of Proposition 4.15 induces a holomorphic mapping $T(X') \rightarrow \pi^* T(X)$ that is the identity outside $\text{sing}(\pi)$. If h denotes this dual map, by the first part of the same proof there is a holomorphic mapping $g: \pi^* T(X) \rightarrow T(X') \otimes K_{X'/X}$ such that $hg = sI_{\pi^* T(X)}$, where s is a holomorphic section of $K_{X'/X}$. Thus $g\xi'/s$ is a semi-meromorphic vector field on X' that coincides with ξ' on $X' \setminus \text{sing}(\pi)$. Moreover, $b := s\xi'$ is smooth. Outside $\pi(\text{sing}(\pi)) \cup ZSS(a)$ we now have that

$$\xi \lrcorner a = \pi_* \left(\frac{\xi' \lrcorner \omega}{f} \right) = \pi_* \left(\frac{b \lrcorner \omega}{sf} \right)$$

and it is clear that the right hand side defines an almost semi-meromorphic current in X . Finally, $L_\xi a = \xi \lrcorner (\partial a) + \partial(\xi \lrcorner a)$ is in $ASM(X)$ in view of Proposition 4.16. □

By similar arguments one can prove that $\mathcal{L}a$ is in $ASM(X)$ if a is an almost semi-meromorphic $(0, q)$ -current and \mathcal{L} is any (global) holomorphic differential operator. More precisely, one can show that $\mathcal{L}a = \pi_*(s^{-N} \mathcal{L}'(\omega/f))$ for some N , where s is the section of $K_{X'/X}$ in the proof above and \mathcal{L}' is a holomorphic differential operator (with values in $K_{X'/X}^N$).

COROLLARY 4.26. — *Let X be an open subset of \mathbb{C}_z^n . If*

$$(4.27) \quad a = \sum_{|I|=p}^{\prime} a_I \wedge dz_I$$

*is in $ASM(X)$, then each a_I is in $ASM(X)$. If $a \in ASM(X)$ has bidegree $(0, *)$, then $\partial a / \partial z_j$ is in $ASM(X)$ for each j .*

BIBLIOGRAPHY

- [1] M. ANDERSSON, “Multivariable residue calculus and integral formulas”, <http://www.math.chalmers.se/~matsa/3introresidy.pdf>.
- [2] ———, “Ideals of smooth functions and residue currents”, *J. Funct. Anal.* **212** (2004), no. 1, p. 76-88.
- [3] ———, “Residue currents and ideals of holomorphic functions”, *Bull. Sci. Math.* **128** (2004), no. 6, p. 481-512.
- [4] ———, “Uniqueness and factorization of Coleff-Herrera currents”, *Ann. Fac. Sci. Toulouse, Math.* **18** (2009), no. 4, p. 651-661.
- [5] ———, “A residue criterion for strong holomorphicity”, *Ark. Mat.* **48** (2010), no. 1, p. 1-15.
- [6] ———, “Pseudomeromorphic currents on subvarieties”, *Complex Var. Elliptic Equ.* **61** (2016), no. 11, p. 1533-1540.
- [7] M. ANDERSSON & H. SAMUELSSON, “A Dolbeault-Grothendieck lemma on complex spaces via Koppelman formulas”, *Invent. Math.* **190** (2012), no. 2, p. 261-297.
- [8] M. ANDERSSON, H. SAMUELSSON & J. SZNAJDMAN, “On the Briançon-Skoda theorem on a singular variety”, *Ann. Inst. Fourier* **60** (2010), no. 2, p. 417-432.
- [9] M. ANDERSSON & E. WULCAN, “Residue currents with prescribed annihilator ideals”, *Ann. Sci. Éc. Norm. Supér.* **40** (2007), no. 6, p. 985-1007.
- [10] ———, “Decomposition of residue currents”, *J. Reine Angew. Math.* **638** (2010), p. 103-118.
- [11] ———, “Global effective versions of the Briançon-Skoda-Huneke theorem”, *Invent. Math.* **200** (2015), no. 2, p. 607-651.
- [12] J.-E. BJÖRK, “Residues and \mathcal{D} -modules”, in *The legacy of Niels Henrik Abel*, Springer, 2004, p. 605-651.
- [13] N. R. COLEFF & M. E. HERRERA, *Les courants résiduels associés à une forme méromorphe*, Lecture Notes in Mathematics, vol. 633, Springer, 1978, x+211 pages.
- [14] M. E. HERRERA & D. LIEBERMAN, “Residues and principal values on complex spaces”, *Math. Ann.* **194** (1971), p. 259-294.
- [15] R. LÄRKÄNG, “Residue currents associated with weakly holomorphic functions”, *Ark. Mat.* **50** (2012), no. 1, p. 135-164.
- [16] ———, “On the duality theorem on an analytic variety”, *Math. Ann.* **355** (2013), no. 1, p. 215-234.
- [17] R. LÄRKÄNG & H. SAMUELSSON KALM, “Various approaches to products of residue currents”, *J. Funct. Anal.* **264** (2013), no. 1, p. 118-138.
- [18] J. LUNDQVIST, “An effective uniform Artin-Rees lemma”, in *Analysis Meets Geometry*, Trends in Mathematics, Birkhäuser/Springer, 2017, p. 335-348.
- [19] B. MALGRANGE, “Sur les fonctions différentiables et les ensembles analytiques”, *Bull. Soc. Math. Fr.* **91** (1963), p. 113-127.

- [20] M. PASSARE, “A calculus for meromorphic currents”, *J. Reine Angew. Math.* **392** (1988), p. 37-56.
- [21] M. PASSARE, A. TSIKH & A. YGER, “Residue currents of the Bochner-Martinelli type”, *Publ. Mat.* **44** (2000), no. 1, p. 85-117.
- [22] J. RUPPENTHAL, H. SAMUELSSON KALM & E. WULCAN, “Explicit Serre duality on complex spaces”, *Adv. Math.* **305** (2017), p. 1320-1355.
- [23] H. SAMUELSSON KALM, “Holomorphic forms, the $\bar{\partial}$ -equation, and duality on a reduced complex space”, <http://arxiv.org/abs/1506.07842>.
- [24] J. SZNAJDMAN, “A Brianchon-Skoda-type result for a non-reduced analytic space”, to appear in *J. Reine Angew. Math.*
- [25] ———, “A residue calculus approach to the uniform Artin-Rees lemma”, *Isr. J. Math.* **196** (2013), no. 1, p. 33-50.
- [26] E. WULCAN, “Products of residue currents of Cauchy-Fantappiè-Leray type”, *Ark. Mat.* **45** (2007), no. 1, p. 157-178.

Manuscrit reçu le 7 décembre 2016,
révisé le 11 septembre 2017,
accepté le 14 septembre 2017.

Mats ANDERSSON
University of Gothenburg
and Chalmers University of Technology,
Department of Mathematical Sciences, Division of
Mathematics,
SE-412 96 Göteborg (Sweden)
matsa@chalmers.se

Elizabeth WULCAN
University of Gothenburg
and Chalmers University of Technology,
Department of Mathematical Sciences, Division of
Mathematics,
SE-412 96 Göteborg (Sweden)
wulcan@chalmers.se