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DIRECT IMAGES OF SEMI-MEROMORPHIC CURRENTS

by Mats ANDERSSON & Elizabeth WULCAN (*)

Abstract. — We introduce a calculus for the class $ASM(X)$ of direct images of semi-meromorphic currents on a reduded analytic space X, that extends the classical calculus due to Coleff, Herrera and Passare. Our main result is that each element in this class acts as a kind of multiplication on the sheaf $\mathcal{PM}_X$ of pseudomeromorphic currents on X. We also prove that $ASM(X)$ as well as $\mathcal{PM}_X$ and certain subsheaves are closed under the action of holomorphic differential operators and interior multiplication by holomorphic vector fields.

Résumé. — Nous introduisons un calcul pour la classe $ASM(X)$ d'images directes de courants semi-méromorphes sur un espace analytique reduit X, qui étend le calcul classique de Coleff, Herrera et Passare. Notre résultat principal montre que chaque élément de cette classe agit de manière analogue à une multiplication sur le faisceau $\mathcal{PM}_X$ de courants pseudoméromorphes sur X. Nous prouvons également que $ASM(X)$ ainsi que $\mathcal{PM}_X$ et certains sous-faisceaux sont fermés sous l'action des opérateurs différentiels holomorphes et la multiplication intérieure par des champs vectoriels holomorphes.

1. Introduction

Let f be a generically nonvanishing holomorphic function on a reduced analytic space X of pure dimension n. It was proved by Herrera and Lieberman, [14], that one can define the principal value current

$$\left[\frac{1}{f}\right] \cdot \xi := \lim_{\epsilon \to 0} \int_{|f|^2 > \epsilon} \frac{\xi}{f}, \tag{1.1}$$

for test forms ξ. It follows that $\bar\partial[1/f]$ is a current with support on the zero set $Z(f)$ of f; such a current is called a residue current. Coleff and Herrera,

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[13], introduced products of principal value and residue currents, like

$$[1/f_1]\dots[1/f_r]\bar\partial[1/f_{r+1}]\wedge\dots\wedge\bar\partial[1/f_m]. \tag{1.2}$$

The product of principal value currents is commutative, but when there are residue factors, like $\bar\partial[1/f_j]$, present these products are not (anti-)commutative in general.

In the literature there are various generalizations and related currents, for instance the abstract so-called Coleff–Herrera currents introduced by Björk, see [12], the Bochner–Martinelli type residue currents introduced in [21], and generalizations in, e.g., [3, 5, 9].

In order to obtain a coherent approach to questions about residue and principal value currents the sheaf $\mathcal{PM}_X$ of *pseudomeromorphic currents* on X was introduced in [10], and further developed in [7]; this sheaf consists of direct images under holomorphic mappings of products of test forms and currents like (1.2). See Section 2 below for the precise definition. This sheaf is closed under $\bar\partial$ and under multiplication by smooth forms. Pseudomeromorphic currents have a geometric nature, similar to positive closed (or normal) currents. For example, the *dimension principle* states that if the pseudomeromorphic current μ has bidegree $(*,p)$ and support on a variety of codimension larger than p, then μ must vanish. Moreover one can form restrictions $\mathbb{1}_W\mu$ of the pseudomeromorphic current μ to analytic (or constructible) subsets $W\subset X$, such that

$$\mathbb{1}_V\mathbb{1}_W\mu=\mathbb{1}_{V\cap W}\mu, \tag{1.3}$$

see Section 2.2. The notion of pseudomeromorphic currents plays a decisive role in, for instance, [7, 8, 10, 11, 15, 16, 18, 22, 23, 24, 25].

It is well-known that one cannot multiply currents in general. Several attempts to find a working calculus for principal value and residue currents have been made. A famous result by Coleff and Herrera, [13], see also Passare, [20], asserts that (1.2) has all expected (anti-)commutativity properties as long as the common zero set of $f_1,\dots,f_m$ has codimension m. Various extension are introduced in the references above. In [10] we proved that one can give a reasonable meaning to a product $[1/f]\mu$ for any holomorphic function f and pseudomeromorphic current μ; more precisely one should consider this as an operator

$$\mu\mapsto[1/f]\mu \tag{1.4}$$

on the sheaf $\mathcal{PM}_X$.

We have not found a way to define a reasonable product of general pseudomeromorphic currents. Our first objective in this paper is to study a generalization of principal value currents leading to an extension of (1.4). Following [7] we say that a current a is *almost semi-meromorphic*, $a \in ASM(X)$, if it is the direct image under a modification of a semi-meromorphic current, i.e., a current of the form $\omega[1/f]$, where f is a holomorphic section of a line bundle and ω is a smooth form with values in the same bundle. Almost semi-meromorphic currents are pseudomeromorphic and in many ways they generalize principal value currents. For example, it turns out that they form an (anti-)commutative algebra, see Section 4. Moreover $ASM(X)$ is closed under ∂, see Proposition 4.16. Taking $\bar\partial$ of $a \in ASM(X)$, however, yields an almost semi-meromorphic current plus a residue current supported on the *Zariski singular support*, $ZSS(a)$, of a, which is the smallest analytic set where a is not smooth. Many of the currents in the references above can be considered as (products of) the residues of almost semi-meromorphic currents. Theorem 4.8 states that the mapping (1.4) holds for any almost semi-meromorphic current a instead of $[1/f]$. More precisely, there is a unique extension to X of the current $a\wedge\mu$, defined in the obvious way in $X \setminus ZSS(a)$, such that its restriction to $ZSS(a)$ is zero.

A second objective is to prove that $\mathcal{PM}_X$ and $ASM(X)$ are closed under interior multiplication by a holomorphic vector field ξ and under the Lie derivative with respect to ξ; see Sections 3 and 4.5.

In Section 2 we recall basic known properties of the sheaf $\mathcal{PM}_X$ and provide some new results, e.g., Theorem 2.15 gives a new quite natural characterization of pseudomeromorphicity. Section 4 is devoted to the study of $ASM(X)$.

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2. Pseudomomeromorphic currents

In one complex variable s one can define the principal value current $[1/s^m]$ for instance as the value

$$\left[\frac{1}{s^m}\right] = \frac{|s|^{2\lambda}}{s^m}\Big|_{\lambda=0}$$

of the current-valued analytic continuation of $\lambda \mapsto |s|^{2\lambda}/s^m$, a priori defined for $\operatorname{Re}\lambda \gg 0$, see, e.g., [3, Lemma 2.1]. We have the relations

$$\frac{\partial}{\partial s}\left[\frac{1}{s^m}\right] = -m\left[\frac{1}{s^{m+1}}\right], \quad s\left[\frac{1}{s^{m+1}}\right] = \left[\frac{1}{s^m}\right]. \tag{2.1}$$

It is also well-known that

$$\bar\partial\left[\frac{1}{s^m}\right]\cdot\xi\,\mathrm{d}s=\frac{2\pi i}{(m-1)!}\frac{\partial^{m-1}}{\partial s^{m-1}}\xi(0) \tag{2.2}$$

for test functions ξ and $m\geqslant 1$; in particular, $\bar\partial[1/s^m]$ has support at $\{s=0\}$. Thus

$$\bar s\bar\partial\left[\frac{1}{s^m}\right]=0,\quad \mathrm{d}\bar s\wedge\bar\partial\left[\frac{1}{s^m}\right]=0. \tag{2.3}$$

We say that a function χ on the real line is a *smooth approximand of the characteristic function* $\chi_{[1,\infty)}$ of the interval $[1,\infty)$, and write

$$\chi\sim\chi_{[1,\infty)},$$

if χ is smooth, equal to 0 in a neighborhood of 0 and 1 in a neighborhood of ∞. It is well-known that $[1/s^m]=\lim_{\epsilon\to 0}\chi(|s|^2/\epsilon)(1/s^m)$.

Let t_j be coordinates in an open set $\mathcal{U}\subset\mathbb{C}^N$ and let α be a smooth form with compact support in $\mathcal{U}$. Then

$$\tau=\alpha\wedge\left[\frac{1}{t_1^{m_1}}\right]\cdots\left[\frac{1}{t_k^{m_k}}\right]\bar\partial\left[\frac{1}{t_{k+1}^{m_{k+1}}}\right]\wedge\ldots\wedge\bar\partial\left[\frac{1}{t_r^{m_r}}\right], \tag{2.4}$$

where $m_1,\ldots,m_r\geqslant 1$, is a well-defined current, since it is the tensor product of one-variable currents (times α). We say that τ is an *elementary (pseudomeromorphic) current*, and we refer to $[1/t_j^{m_j}]$ and $\bar\partial[1/t_\ell^{m_\ell}]$ as its *principal value factors* and *residue factors*, respectively. It is clear that (2.4) is commuting in the principal value factors and anti-commuting in the residue factors. We say the intersection of $\mathcal{U}$ and the coordinate plane $\{t_{k+1}=\cdots=t_r=0\}$ is the *elementary support* of τ. Clearly the support of τ is contained in the intersection of the elementary support of τ and the support of α.

Remark 2.1. — Since ∂ does not introduce new residue factors, $\partial\tau$ is an elementary current, cf. (2.1), whose elementary support either equals the elementary support H of τ or is empty. Moreover $\bar\partial\tau$ is a finite sum of elementary currents, whose elementary supports are either equal to H or coordinate planes of codimension 1 in H, cf. (2.2).

2.1. Definition and basic properties

Let X be a reduced complex space of pure dimension n. Fix a point $x\in X$. We say that a germ μ of a current at x is *pseudomeromorphic* at

x, $\mu \in \mathcal{PM}_x$, if it is a finite sum of currents of the form

$$\pi_*\tau = \pi^1_* \dots \pi^m_* \tau, \tag{2.5}$$

where $\mathcal{U} \subset X$ is a neighborhood of x,

$$\mathcal{U}_m \xrightarrow{\pi^m} \dots \xrightarrow{\pi^2} \mathcal{U}_1 \xrightarrow{\pi^1} \mathcal{U}_0 = \mathcal{U}, \tag{2.6}$$

each $\pi^j : \mathcal{U}_j \to \mathcal{U}_{j-1}$ is either a modification, a simple projection $\mathcal{U}_{j-1} \times Z \to \mathcal{U}_{j-1}$, or an open inclusion (i.e., $\mathcal{U}_j$ is an open subset of $\mathcal{U}_{j-1}$), and τ is elementary on $\mathcal{U}_m \subset \mathbb{C}^N$.

By definition the union $\mathcal{PM} = \mathcal{PM}_X = \cup_x \mathcal{PM}_x$ is an open subset (of the étalé space) of the sheaf $\mathcal{C} = \mathcal{C}_X$ of currents, and hence it is a subsheaf, which we call the sheaf of *pseudomeromorphic* currents[(1)]. A section μ of $\mathcal{PM}$ over an open set $\mathcal{V} \subset X$, $\mu \in \mathcal{PM}(\mathcal{V})$, is then a locally finite sum

$$\mu = \sum (\pi_\ell)_* \tau_\ell, \tag{2.7}$$

where each π_ℓ is a composition of mappings as in (2.6) (with $\mathcal{U} \subset \mathcal{V}$) and τ_ℓ is elementary. For simplicity we will always suppress the subscript ℓ in π_ℓ. If ξ is a smooth form, then

$$\xi \wedge \pi_* \tau = \pi_* \left(\pi^* \xi \wedge \tau\right). \tag{2.8}$$

Thus $\mathcal{PM}$ is closed under exterior multiplication by smooth forms. Since $\bar\partial$ and ∂ commute with push-forwards it follows that $\mathcal{PM}$ is closed under $\bar\partial$ and ∂, cf. Remark 2.1.

Remark 2.2. — Let τ be an elementary current with elementary support H. Since H is the intersection of an open set $\mathcal{U}$ and a linear subspace, each of its components is irreducible, and it follows that, in fact, τ is a finite sum of currents τ_ℓ such that the support of τ_ℓ is contained in an irreducible component of H. We may therefore assume that each τ_ℓ in (2.7) has irreducible elementary support.

Remark 2.3. — One may assume that each τ_ℓ in (2.7) has at most one residue factor. Indeed, in [21], see also [4, Corollary 3.5], it is shown that the Coleff–Herrera product

$$\bar\partial[1/t_{k+1}^{m_{k+1}}] \wedge \dots \wedge \bar\partial[1/t_r^{m_r}]$$

equals the Bochner–Martinelli residue current of $t_{k+1}^{m_{k+1}}, \dots, t_r^{m_r}$, which, see, e.g., [3], is the direct image under a modification of a current of the form $\alpha \wedge \bar\partial[1/f]$, cf. Example 4.18 below. It follows, cf. [6, Lemma 3.2], that (2.4)

(1) The definition here is from [7]; in the original definition in [10] simple projections were not included.

is the direct image under another modification of a finite sum of elementary currents with at most one residue factor.

Proposition 2.4. — *Assume that $\mu \in \mathcal{PM}$ has support on the subvariety $V \subset X$.*

(1) *If the holomorphic function h vanishes on V, then $\bar{h}\mu = 0$ and $\mathrm{d}\bar{h}\wedge\mu = 0$.*
(2) *If μ has bidegree $(*, p)$ and $\operatorname{codim} V > p$, then $\mu = 0$.*

This proposition is from [10]; for the adaption to nonsmooth X, see [7, Proposition 2.3]. Part (1) means that the action of the current μ only involves holomorphic derivatives of test forms. We refer to part (2) as the *dimension principle*. We will also need, [6, Proposition 1.2]:

Proposition 2.5. — *If $\pi\colon X' \to X$ is a modification, then $\pi_*\colon \mathcal{PM}(X') \to \mathcal{PM}(X)$ is surjective.*

2.2. Basic operations on pseudomeromorphic currents

Assume that μ is pseudomeromorphic on X and that $V \subset X$ is a subvariety. It was proved in [10], see also [7], that the restriction of μ to the open set $X \setminus V$ has a natural pseudomeromorphic extension $\mathbb{1}_{X\setminus V}\mu$ to X. In [10] it was obtained as the value

$$\mathbb{1}_{X\setminus V}\mu := |f|^{2\lambda}\mu|_{\lambda=0} \tag{2.9}$$

at $\lambda = 0$ of the analytic continuation of the current valued function $\lambda \mapsto |f|^{2\lambda}\mu$, where f is any tuple of holomorphic functions such that $Z(f) = V$. It follows that

$$\mathbb{1}_V\mu := \mu - \mathbb{1}_{X\setminus V}\mu$$

has support on V. It is proved in [10] that this operation extends to all constructible sets and that (1.3) holds. If α is a smooth form, then

$$\mathbb{1}_V(\alpha\wedge\mu) = \alpha\wedge\mathbb{1}_V\mu. \tag{2.10}$$

Moreover, if $\pi\colon X' \to X$ is a modification, a simple projection or an open inclusion and $\mu = \pi_*\mu'$, then

$$\mathbb{1}_V\mu = \pi_*\left(\mathbb{1}_{\pi^{-1}V}\mu'\right). \tag{2.11}$$

In this paper it is convenient to express $\mathbb{1}_{X\setminus V}\mu$ as a limit of currents that are pseudomeromorphic themselves.

Lemma 2.6. — *Let V be a germ of a subvariety at $x \in X$, let f be a tuple of holomorphic functions whose common zero set is precisely V, let v be a positive and smooth function, and let $\chi \sim \chi_{[1,\infty)}$. For each germ of a pseudomeromorphic current μ at x we have*

$$\mathbb{1}_{X\setminus V}\mu = \lim_{\epsilon\to 0} \chi(|f|^2 v/\epsilon)\mu. \tag{2.12}$$

Because of the factor v, the lemma holds just as well for a holomorphic section f of a Hermitian vector bundle.

In case V is a hypersurface and f is one single holomorphic function, or section of a line bundle, the lemma follows directly from Lemma 6 in [17] by just taking $T = f\mu$. We will reduce the general case to this lemma. The proof of this lemma relies on the proof of Theorem 1.1 in [17], which is quite involved. For a more direct proof of Lemma 2.6, see the proof of Proposition 3.4 in [1, Chapter 2].

Proof. — Let $\pi\colon X' \to X$ be a smooth modification such that $\pi^* f = f^0 f'$, where f^0 is a holomorphic section of a Hermitian line bundle $L \to X'$ and f' is a nonvanishing tuple of holomorphic sections of L^{-1}. In view of Proposition 2.5 we can assume that $\mu = \pi_* \mu'$, where μ' is pseudomeromorphic on X'. Then

$$|\pi^* f|^2 \pi^* v = |f^0|^2 |f'|^2 \pi^* v,$$

and from [17, Lemma 6] we thus have that

$$\lim_{\epsilon\to 0} \chi(|\pi^* f|^2 \pi^* v/\epsilon)\mu' = \mathbb{1}_{X'\setminus \pi^{-1}V}\mu'.$$

In view of (2.11) we get (2.12). □

Remark 2.7. — Lemma 2.6 holds even if $\chi = \chi_{[1,\infty)}$. However, in general it is not obvious what $\chi(|f|^2 v/\epsilon)\mu$ means. Let χ^δ be smooth approximands such that $\chi^\delta \to \chi_{[1,\infty)}$. It follows from the proof of Lemma 6 in [17] that for small enough ϵ, depending on μ, f, and v, the limit $\lim_{\delta\to 0} \chi^\delta(|f|^2 v/\epsilon)\mu$ exists and is independent of the choice of χ^δ; thus we can take it as the definition of $\chi(|f|^2 v/\epsilon)\mu$. In fact, it turns out that after a suitable change of real coordinates one can realize $\chi(|f|^2 v/\epsilon)\mu$ as a tensor product of two currents. In particular we get

$$\chi(|f|^2/\epsilon)\frac{1}{f} \cdot \xi = \int_{|f|^2>\epsilon} \frac{\xi}{f},$$

cf. (1.1).

We will need the following observation.

Lemma 2.8. — *If μ has the form (2.7), then*

$$\mathbb{1}_V \mu = \sum_{\operatorname{supp} \tau_\ell \subset \pi^{-1} V} \pi_* \tau_\ell.$$

It follows from the proof below that we just as well can take the sum over all ℓ such that the elementary supports of τ_ℓ are contained in $\pi^{-1}V$.

Proof. — In view of (2.11) we have that

$$\mathbb{1}_V \mu = \sum_\ell \pi_* \left(\mathbb{1}_{\pi^{-1}V} \tau_\ell \right).$$

If $\operatorname{supp} \tau_\ell \subset \pi^{-1}V$, then clearly $\mathbb{1}_{\pi^{-1}V}\tau_\ell = \tau_\ell$. We now claim that if $\operatorname{supp} \tau_\ell$ is not contained in $\pi^{-1}V$, then $\mathbb{1}_{\pi^{-1}V}\tau_\ell = 0$. If $\operatorname{supp} \tau_\ell \not\subset \pi^{-1}V$, the elementary support H of τ_ℓ is not contained in $\pi^{-1}V$. Assume that H has codimension q. Then τ_ℓ is of the form $\tau_\ell = \alpha \wedge \tau'$, where α is smooth and τ' is elementary of bidegree $(0, q)$. It follows from (2.10) that

$$\mathbb{1}_{\pi^{-1}V} \tau_\ell = \alpha \wedge \mathbb{1}_{\pi^{-1}V} \tau'.$$

By Remark 2.2 we may assume that H is irreducible, and therefore $\pi^{-1}V \cap H$ has codimension at least $q+1$ in $\mathcal{U}$. Since $\mathbb{1}_{\pi^{-1}V}\tau'$ has support on $\pi^{-1}V \cap H$ it must vanish in view of the dimension principle. Thus the lemma follows. □

We now consider another fundamental operation on $\mathcal{PM}$ introduced in [10].

Proposition 2.9 ([10]). — *Given a holomorphic function h and a pseudomeromorphic current μ there is a pseudomeromorphic current T such that $T = (1/h)\mu$ in the open set where $h \neq 0$ and $\mathbb{1}_{\{h=0\}}T = 0$.*

Here h may just as well be a holomorphic section of a line bundle. Clearly this current T must be unique and we denote it by $[1/h]\mu$. In [10] the current $[1/h]\mu$ was defined as $(|h|^{2\lambda}\mu/h)|_{\lambda=0}$.

Remark 2.10. — Notice that[2] $h[1/h]\mu = \mathbb{1}_{\{h\neq 0\}}\mu$; in particular, $h[1/h]\mu \neq \mu$ in general. For example, $z[1/z]\bar\partial[1/z] = 0$.

Since $[1/h]\mu = (1/h)\mu$ in $\{h \neq 0\}$ and $[1/h]\mu = \mathbb{1}_{\{h\neq 0\}}[1/h]\mu$, it follows from (2.12) that

$$\left[\frac{1}{h}\right]\mu = \lim_{\epsilon\to 0} \chi(|h|^2 v/\epsilon)\frac{1}{h}\mu. \tag{2.13}$$

[2] We have not exluded the possibility that h vanishes identically on some (or all) irreducible components of X.

One can also define

$$\bar\partial\left[\frac{1}{h}\right]\wedge\mu := \bar\partial\left(\left[\frac{1}{h}\right]\mu\right) - \left[\frac{1}{h}\right]\bar\partial\mu, \tag{2.14}$$

i.e., so that "Leibniz's rule" holds. Notice that if $\pi\colon X'\to X$ is a modification and $\mu=\pi_*\mu'$, then

$$\left[\frac{1}{h}\right]\mu = \pi_*\left(\left[\frac{1}{\pi^*h}\right]\mu'\right), \quad \bar\partial\left[\frac{1}{h}\right]\wedge\mu = \pi_*\left(\bar\partial\left[\frac{1}{\pi^*h}\right]\wedge\mu'\right). \tag{2.15}$$

This follows, e.g., from (2.8) and (2.13). It is also readily checked that

$$\bar\partial\left(\bar\partial\left[\frac{1}{h}\right]\wedge\mu\right) = -\bar\partial\left[\frac{1}{h}\right]\wedge\bar\partial\mu. \tag{2.16}$$

Remark 2.11. — Since $[1/f][1/g]=[1/(fg)]=[1/g][1/f]$ it follows from (2.14) that

$$\bar\partial\left[\frac{1}{f}\right]\cdot\left[\frac{1}{g}\right] + \left[\frac{1}{f}\right]\bar\partial\left[\frac{1}{g}\right] = \bar\partial\left[\frac{1}{g}\right]\cdot\left[\frac{1}{f}\right] + \left[\frac{1}{g}\right]\bar\partial\left[\frac{1}{f}\right].$$

However, it is not true in general that $[1/g]\bar\partial[1/f]=\bar\partial[1/f]\cdot[1/g]$. For instance, $[1/z]\bar\partial[1/z]=0$, whereas $\bar\partial[1/z]\cdot[1/z]=\bar\partial[1/z^2]$.

We now consider tensor products and direct images under simple projections.

LEMMA 2.12. — *If $\mu\in\mathcal{PM}_X$ and $\mu'\in\mathcal{PM}_{X'}$, then $\mu\otimes\mu'\in\mathcal{PM}_{X\times X'}$.*

This is precisely [6, Lemma 3.3]. It is easy to verify that

$$\mathbb{1}_{V\times V'}\mu\otimes\mu' = \mathbb{1}_V\mu\otimes\mathbb{1}_{V'}\mu'. \tag{2.17}$$

LEMMA 2.13. — *Assume that $p\colon Z\times W\to Z$ is a simple projection. If μ is in $\mathcal{PM}_{Z\times W}$ and $p^{-1}K\cap\operatorname{supp}\mu$ is compact for each compact set $K\subset Z$, then $p_*\mu$ is in $\mathcal{PM}_Z$.*

Proof. — Since pseudomeromorphicity is a local property, after multiplying μ if necessary by a suitable cutoff function we can assume that μ has compact support. By compactness and a partition of unity we then have a finite representation $\mu=\sum_\ell\pi_*\tau_\ell$. Now the lemma follows from the very definition of $\mathcal{PM}$. □

Example 2.14. — Assume that τ is an elementary current on X, p is a simple projection $X\times X'\to X$, and χ is any test form in X' with total integral 1. Then the tensor product $\tau\otimes\chi$ is an elementary current in $X\times X'$ such that $p_*(\tau\otimes\chi)=\tau$.

The following result provides a new, quite natural definition of pseudomeromorphicity.

THEOREM 2.15.

(1) *Assume that X is smooth. Then a germ of a current μ at $x \in X$ is pseudomeromorphic if and only if it is a finite sum*

$$\mu = \sum_{\ell} (f_\ell)_* \tau_\ell, \tag{2.18}$$

where $f_\ell \colon \mathcal{U}_\ell \to X$ are holomorphic mappings and τ_ℓ are elementary.

(2) *If X is a reduced space of pure dimension and $\pi \colon X' \to X$ is a smooth modification, then a current μ on X is pseudomeromorphic if and only if there is a pseudomeromorphic current μ' on X' such that $\mu = \pi_* \mu'$.*

Proof. — By definition a germ of a pseudomeromorphic current is of the form (2.18). Now assume that $f \colon \mathcal{U} \to X$ is any holomorphic mapping and τ is elementary in $\mathcal{U} \subset \mathbb{C}^N$. Let $F \colon \mathcal{U} \to \mathcal{U} \times X$ be the mapping $F(s) = (s, f(s))$. Let $\widetilde{F}$ be F considered as a biholomorphism onto the graph $\Gamma \subset \mathcal{U} \times X$ and let $i \colon \Gamma \to \mathcal{U} \times X$ be the natural injection. Then clearly $\widetilde{F}_* \tau$ is pseudomeromorphic on Γ and in view of [6, Theorem 1.1 (i)], $F_* \tau = i_* \widetilde{F}_* \tau$ is pseudomeromorphic in $\mathcal{U} \times X$. Clearly, it has compact support in $\mathcal{U} \times X$. If p is the projection $\mathcal{U} \times X \to X$, we can therefore apply Lemma 2.13, and conclude that $f_* \tau = p_* F_* \tau$ is pseudomeromorphic in X. Thus part (1) is proved. Part (2) is just Proposition 2.5. □

COROLLARY 2.16. — *Assume that $f \colon W \to X$ is a holomorphic mapping and X is smooth. If μ is pseudomeromorphic on W with compact support, then $f_* \mu$ is pseudomeromorphic on X.*

Proof. — We may assume that $\mu = \pi_* \tau$, where $\pi \colon \mathcal{U} \to W$ is a mapping as in the definition of pseudomeromorphicity and τ is elementary in $\mathcal{U}$. Then we can apply Theorem 2.15 (1) to the mapping $f \circ \pi \colon \mathcal{U} \to X$. It follows that $f_* \mu = f_* \pi_* \tau = (f \circ \pi)_* \tau$ is pseudomeromorphic in X. □

Remark 2.17. — Notice that in the proof of Theorem 2.15 we only used [6, Theorem 1.1 (i)], which asserts that i_* maps $\mathcal{PM}_W$ into $\mathcal{PM}_X$ if $i : W \to X$ is an embedding of a reduced pure-dimensional space W into a manifold X, in the relatively simple case when W is a smooth submanifold. The general case now follows from Corollary 2.16. Part (ii) of [6, Theorem 1.1] is a partial converse: *If $\mu = i_* \nu$ is pseudomeromorphic in X and $\mathbb{1}_{W_{\text{sing}}} \mu = 0$, then ν is pseudomeromorphic on W.* The proof of this fact relies on the possibility to make a so-called strong resolution. This means that there is a resolution $X' \to X$ that is a biholomorphism outside W, and such that the strict transform of W is a smooth resolution of W.

3. Action of holomorphic differential operators and vector fields

Let X be a reduced analytic space of pure dimension. We already know that ∂ maps $\mathcal{PM}_X$ into itself. We shall now consider a more general statement, and to this end we need the following result that is interesting in itself.

Proposition 3.1. — *Assume that $\mu \in \mathcal{PM}_x$ where $x \in X$. If $h \in \mathcal{O}_x$ is not identically zero on any irreducible component of X at x, then there is $\mu' \in \mathcal{PM}_x$ such that $h\mu' = \mu$.*

Remark 3.2. — By a partition of unity we can get a global such μ' if μ and h are global. If μ has compact support in $\mathcal{U} \subset X$ we can choose μ' with compact support in $\mathcal{U}$.

Remark 3.3. — If μ has support on V we may assume as well that μ' has. Indeed, $\mu = \mathbb{1}_V \mu = \mathbb{1}_V h\mu' = h\mathbb{1}_V \mu'$, so we can replace a given solution μ' by $\mathbb{1}_V \mu'$.

Example 3.4. — Proposition 3.1 is not true if h is anti-holomorphic. In fact, if $\bar{z}\mu' = 1$, then $[1/z]\mu'$ is equal to $1/|z|^2$ outside 0. Thus $\lim_{\epsilon\to 0} \chi(|z|^2/\epsilon)\mu'/z$ does not exist, and hence μ' cannot be pseudomeromorphic, cf. Proposition 2.9 and (2.13).

Proof of Proposition 3.1. — First assume that τ is an elementary pseudomeromorphic current in $\mathbb{C}^N_t$ and h is a monomial. By induction it is enough to assume that $h = t_1$. If t_1 is a residue factor in τ, then we just raise the power of t_1 in that factor one unit. Otherwise we take $\tau' = (1/t_1)\tau$. Then $h\tau' = \tau$.

We may assume that $\mu = \pi_*\tau$, where $\pi : \mathcal{U} \to X$ and τ is elementary of the form (2.4). By Hironaka's theorem we can find a modification $\nu : \mathcal{U}' \to \mathcal{U}$ such that, locally in $\mathcal{U}'$, $\nu^*\pi^*h$ is a monomial and ν^*t_j are monomials (times nonvanishing functions). By a partition of unity in $\mathcal{U}'$ and repeated use of (2.15) it follows that τ is a finite sum of currents $\nu_*\tau'$, where

$$\tau' := \nu^*\alpha\wedge \left[\frac{1}{\nu^*t_1^{m_1}}\right]\cdots\left[\frac{1}{\nu^*t_k^{m_k}}\right]\bar\partial\left[\frac{1}{\nu^*t_{k+1}^{m_{k+1}}}\right]\wedge\ldots\wedge\bar\partial\left[\frac{1}{\nu^*t_r^{m_r}}\right].$$

Each such term is a sum of elementary currents τ_ℓ in view of (2.14). By the first part of the proof there are elementary currents τ'_ℓ in $\mathcal{U}'$ such that $\nu^*\pi^*h\ \tau'_\ell = \tau_\ell$. Now the proposition follows in view of (2.8). $\square$

Theorem 3.5. — *Assume that X is smooth at $x \in X$.*

(1) *If z is a local holomorphic coordinate system at x and*

$$\mu = \sum_{|I|=p}' \mu_I \wedge \mathrm{d}z_I \tag{3.1}$$

is a germ in $\mathcal{PM}_x$, then each μ_I is in $\mathcal{PM}_x$.

(2) *If ξ is a germ of a holomorphic vector field, then the contraction $\xi \neg \mu$ and the Lie derivative $L_\xi \mu$ are in $\mathcal{PM}_x$.*

Notice that (2) is not true for anti-holomorphic vector fields. For example, $\mu = (\partial/\partial\bar{z})\neg\bar{\partial}(1/z)$ is a nonzero current of degree 0 with support at 0. In view of the dimension principle, it cannot be pseudomeromorphic.

Proof. — We will first assume that μ has bidegree $(n, *)$ so that $\mu = \hat{\mu}\wedge\mathrm{d}z$, where $\hat{\mu}$ has bidegree $(0, *)$, and show that $\hat{\mu}$ is pseudomeromorphic. We may assume that $\mu = \pi_*(\tau\wedge\mathrm{d}s)$, where $\pi : \mathcal{U} \to X$ is a mapping as in the definition of pseudomeromorphicity, s are local coordinates in $\mathcal{U} \subset \mathbb{C}^m$, and τ is elementary. Since π has generically surjective differential, we can write $s = (s', s'') = (s'_1, \dots, s'_n, s''_{n+1}, \dots, s''_m)$ so that $h := \det(\partial\pi/\partial s') = \det(\partial z/\partial s')$ is generically nonvanishing in $\mathcal{U}$. By Proposition 3.1 and Remark 3.2 there is a pseudomeromorphic τ' with compact support in $\mathcal{U}$ such that $h\tau' = \tau$ in $\mathcal{U}$. Now

$$\begin{aligned}\hat{\mu}\wedge\mathrm{d}z &= \pi_*(\tau\wedge\mathrm{d}s) = \pi_*(\tau'\wedge h\mathrm{d}s'\wedge\mathrm{d}s'') = \pi_*(\tau'\wedge\pi^*\mathrm{d}z\wedge\mathrm{d}s'') \\ &= \pm\pi_*(\tau'\wedge\mathrm{d}s'')\wedge\mathrm{d}z.\end{aligned}$$

Thus $\hat{\mu} = \pm\pi_*(\tau'\wedge\mathrm{d}s'')$ is pseudomeromorphic. In general, $\mu_I\wedge\mathrm{d}z = \pm\mu\wedge\mathrm{d}z_{I^c}$, where I^c is the complementary multiindex of I. It follows from above that μ_I is pseudomeromorphic. Thus (1) follows.

The first statement of (2) follows immediately from (1), and the second one follows since $L_\xi\mu = \partial(\xi\neg\mu) + \xi\neg(\partial\mu)$. □

3.1. The sheaves $\mathcal{PM}_X^Z$ and $\mathcal{W}_X^Z$

Let X be a reduced analytic space, let $Z \subset X$ be a (reduced) subspace of pure dimension, and denote by $\mathcal{PM}_X^Z$ the subsheaf of $\mathcal{PM}_X$ of currents that have support on Z. We say that $\mu \in \mathcal{PM}_X^Z$ has the *standard extension property, SEP, on Z* if $\mathbb{1}_W\mu = 0$ in $\mathcal{U}$ for each subvariety $W \subset \mathcal{U} \cap Z$ of positive codimension, where $\mathcal{U}$ is any open set in X. Let $\mathcal{W}_X^Z$ be the subsheaf of $\mathcal{PM}_X^Z$ of currents with the SEP on Z. In case $Z = X$ we usually write $\mathcal{W}_X$ rather than $\mathcal{W}_X^X$.

Example 3.6. — Note that an elementary current in $\mathcal{U}$ with elementary support H is in $\mathcal{W}_{\mathcal{U}}^{H}$.

It is easy to see that Theorem 3.5 holds for $\mathcal{PM}_X^Z$ as well, since neither ∂ nor contraction can increase support. Somewhat less obvious is that also the SEP is preserved.

THEOREM 3.7. — *The sheaf $\mathcal{W}_X^Z$ is invariant under ∂, and the statements in Theorem 3.5 hold for $\mathcal{W}_X^Z$ instead of $\mathcal{PM}$.*

This theorem is a consequence of the following general equalities.

PROPOSITION 3.8. — *Assume that μ is a pseudomeromorphic current on X. If $V \subset X$ is any analytic subset, then*

$$\mathbb{1}_V \partial\mu = \partial \mathbb{1}_V \mu. \tag{3.2}$$

If ξ is a holomorphic vector field, then

$$\mathbb{1}_V \xi \neg \mu = \xi \neg \mathbb{1}_V \mu. \tag{3.3}$$

Proof. — Note that (3.3) follows in view of (2.12). Let us therefore focus on (3.2). By (1.3) it is enough to consider $V = Z(h)$, where h is a nontrivial holomorphic function. Take $\chi \sim \chi_{[1,\infty)}$ and let $\chi_\epsilon = \chi(|h|^2/\epsilon)$. Now

$$\chi_\epsilon \partial\mu = \partial(\chi_\epsilon \mu) - \partial\chi_\epsilon \wedge \mu. \tag{3.4}$$

If the last term tends to 0 when $\epsilon \to 0$, after taking limits we get that $\mathbb{1}_{h\neq 0}\partial\mu = \partial(\mathbb{1}_{h\neq 0}\mu)$, which is equivalent to (3.2). Let $\hat{\chi}(t) = t\chi'(t) + \chi(t)$, and notice that also $\hat{\chi} \sim \chi_{[1,\infty)}$. According to Proposition 3.1 there is a pseudomeromorphic μ' such that $\mu = h\mu'$. The last term in (3.4) is therefore

$$\chi'(|h|^2/\epsilon)\bar{h}\partial h \wedge \mu/\epsilon = \chi'(|h|^2/\epsilon)|h|^2 \partial h \wedge \mu'/\epsilon = \hat{\chi}(|h|^2/\epsilon)\partial h \wedge \mu' - \chi_\epsilon \partial h \wedge \mu',$$

which tends to $\mathbb{1}_{h\neq 0}\partial h \wedge \mu' - \mathbb{1}_{h\neq 0}\partial h \wedge \mu' = 0$. □

4. Almost semi-meromorphic currents

We say that a current on X is *semi-meromorphic* if it is of the form $\omega[1/f]$, where f is a generically nonvanishing holomorphic section of a line bundle $L \to X$ and ω is a smooth form with values in L. For simplicity we will often omit the brackets [ ] indicating principal value in the sequel. Since furthermore $\omega[1/f] = [1/f]\omega$ when ω is smooth we can write just ω/f.

4.1. The algebra $ASM(X)$

Let X be a pure-dimensional reduced analytic space. We say that a current a is *almost semi-meromorphic* in X, $a \in ASM(X)$, if there is a modification $\pi\colon X' \to X$ such that

$$a = \pi_*(\omega/f), \tag{4.1}$$

where ω/f is semi-meromorphic in X'. We say that a is *almost smooth* in X if one can choose f to be nonvanishing. We can assume that X' is smooth because otherwise we take a smooth modification $\pi'\colon X'' \to X'$ and consider the pullbacks of f and ω to X'', cf. (2.15). If nothing else is said we tacitly assume that X' is smooth.

Notice that if $\mathcal{U} \subset X$ is an open subset, then the restriction $a_{\mathcal{U}}$ of $a \in ASM(X)$ to $\mathcal{U}$ is in $ASM(\mathcal{U})$. In fact, if (4.1) holds, then $\mathcal{U}' := \pi^{-1}\mathcal{U} \to \mathcal{U}$ is a modification of $\mathcal{U}$, and $a_{\mathcal{U}}$ is the direct image of the restriction of ω/f to $\mathcal{U}'$.

If V has positive codimension in $\mathcal{U} \subset X$, then $\pi^{-1}V$ has positive codimension in $\mathcal{U}'$ and $\mathbb{1}_V a = \pi_*(\mathbb{1}_{\pi^{-1}V}(\omega/f)) = \pi_*(\omega\mathbb{1}_{\pi^{-1}V}(1/f)) = 0$ in $\mathcal{U}$, cf. (2.11), (2.10), and the dimension principle. Thus $ASM(X)$ is contained in $\mathcal{W}(X)$.

Remark 4.1. — One can introduce a notion "locally almost semi-meromorphic current" and consider the associated sheaf. However, for the moment we have no need for such a concept.

Example 4.2. — Assume that $X = \{zw = 0\} \subset \mathbb{C}^2$. Let $a\colon X \to \mathbb{C}$ be 1 and 0 on the z-axis and the w-axis, respectively, except at the origin. Then a is almost smooth. Indeed the normalization $\nu\colon \widetilde{X} \to X$ consists of two disjoint components and $a = \nu_*\tilde{a}$, where $\tilde{a}$ is 0 and 1, respectively, on these components.

Given a modification $\pi\colon X' \to X$, let $\operatorname{sing}(\pi) \subset X'$ be the (analytic) set where π is not a biholomorphism. By the definition of a modification it has positive codimension. Let a be given by (4.1) and let $Z \subset X'$ be the zero set of f. By assumption also Z has positive codimension. Notice that $a \in ASM(X)$ is smooth outside $\pi(Z \cup \operatorname{sing}(\pi))$ which has positive codimension in X. We let $ZSS(a)$, the *Zariski-singular support* of a, be the smallest Zariski-closed set $V \subset X$ such that a is smooth outside V.

Example 4.3. — Assume that $a \in ASM(X)$ is almost smooth. Then $a = \pi_*\omega$, where ω is smooth, and thus $ZSS(a) \subset \pi(\operatorname{sing}(\pi))$. This inclusion may be strict. For example if a is smooth, then $ZSS(a)$ is empty. In this case

$\omega = \pi^* a$ outside $\mathrm{sing}(\pi)$ and since both sides are smooth across $\mathrm{sing}(\pi)$, by continuity, then $\omega = \pi^* a$ everywhere in X'.

Given two modifications $X_1 \to X$ and $X_2 \to X$, there is a modification $\pi\colon X' \to X$ that factorizes over both X_1 and X_2, i.e., we have $X' \to X_j \to X$ for $j = 1, 2$. Therefore, given $a_1, a_2 \in ASM(X)$ we can assume that $a_j = \pi_*(\omega_j/f_j)$, $j = 1, 2$. It follows that

$$a_1 + a_2 = \pi_*\left(\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2}\right) = \pi_* \frac{f_2\omega_1 + f_1\omega_2}{f_1 f_2},$$

so that $a_1 + a_2$ is in $ASM(X)$ as well. Moreover, $A := \pi_*(\omega_1 \wedge \omega_2 / f_1 f_2)$ is an almost semi-meromorphic current that coincides with $a_1 \wedge a_2$ outside the set $\pi\left(\mathrm{sing}(\pi) \cup V(f_1) \cup V(f_2)\right)$. If we had other representations $a_j = \pi'_*(\omega'_j/f'_j)$, $j = 1, 2$, we would get an almost semi-meromorphic A' that coincides generically with $a_1 \wedge a_2$ on X. Since almost semi-meromorphic have the SEP, thus $A = A'$. Hence we can define $a_1 \wedge a_2$ as A. Similarly, since

$$a_2 \wedge a_1 = (-1)^{\deg a_1 \deg a_2} a_1 \wedge a_2, \quad a_1 \wedge (a_2 + a_3) = a_1 \wedge a_2 + a_1 \wedge a_3$$

and

$$a_1 \wedge (a_2 \wedge a_3) = (a_1 \wedge a_2) \wedge a_3$$

hold generically on X and because of the SEP they hold on X. Thus $ASM(X)$ is an algebra.

Remark 4.4. — Notice that the almost smooth currents form a subalgebra of $ASM(X)$.

Example 4.5. — Clearly $ZSS(a_1 \wedge a_2) \subset ZSS(a_1) \cup ZSS(a_2)$ but the inclusion may be strict. Take for instance z_1/z_2 and z_2/z_3.

Example 4.6. — The most basic example of an (almost semi-)meromorphic current is the principal value current associated with a meromorphic form. Let f a be meromorphic k-form on X, i.e., locally $f = g/h$ where h is a holomorphic function that is generically nonvanishing and g is a holomorphic $(k, 0)$-form. By definition $g/h = g'/h'$ if and only if $g'h - gh'$ vanishes outside a set of positive codimension. In that case

$$g\left[\frac{1}{h}\right] = g'\left[\frac{1}{h'}\right] \tag{4.2}$$

outside a set of positive codimension. By the dimension principle therefore (4.2) holds everywhere. Thus there is a well-defined almost semi-meromorphic current $[f]$ associated with f. Notice that $ZSS([f])$ is contained in the pole set of the meromorphic form f, so unless X is smooth it may have codimension larger than 1. Actually, $ZSS([f])$ is equal to the pole set of f. In fact, by continuity $\bar\partial f = 0$ where f is smooth, and by a classical result proved by Malgrange (at least for functions), [19], then f is holomorphic there.

The following lemma will be crucial in what follows.

Lemma 4.7. — *If a is almost semi-meromorphic in X, then there is a representation* (4.1) *such that f is nonvanishing in $X' \setminus \pi^{-1}ZSS(a)$.*

Proof. — Let $V = ZSS(a)$ and assume that we have a representation (4.1) and that X' is smooth. Let Z be the union of the irreducible components of the divisor defined by f that are not fully contained in $\pi^{-1}V$. Since X' is smooth, Z is a Cartier divisor and thus the divisor of a section f' of some line bundle $L' \to X'$. It follows that $g := f/f'$ is a holomorphic section of $L \otimes (L')^{-1}$ in X' that is nonvanishing in $X' \setminus \pi^{-1}V$. Outside $\operatorname{sing}(\pi) \cup Z \cup \pi^{-1}V$ we have that

$$\omega = f\pi^* a = f' g \pi^* a. \tag{4.3}$$

By continuity, (4.3) must hold in $X' \setminus \pi^{-1}V$ since both sides are smooth there.

We claim that $\widetilde\omega := \omega/f'$ is smooth in X'. Taking this for granted, then

$$\pi_* \frac{\widetilde\omega}{g} \tag{4.4}$$

is in $ASM(X)$ and the zero set of g is contained in $\pi^{-1}V$. Since (4.4) coincides with a outside $V \cup \pi(\operatorname{sing}(\pi))$ it follows by the SEP that (4.4) indeed is equal to a in X. Thus the lemma follows.

The claim is a local statement in X' so given a point in X' we can choose local coordinates t in a neighborhood $\mathcal{U}$ of that point and consider each coefficient of the form ω with respect to these coordinates. Thus we may assume that ω is a function and that $\omega = f'\gamma$ where $\gamma = g\pi^* a$ is smooth in $\mathcal{U} \setminus \pi^{-1}V$, cf. (4.3) and the comment thereafter. For all multiindices α thus

$$\frac{\partial^\alpha \omega}{\partial \bar t^\alpha} \bar\partial \frac{1}{f'} = 0 \tag{4.5}$$

in $\mathcal{U} \setminus \pi^{-1}V$, since $f'\bar\partial(1/f') = 0$. By assumption $Z \cap \pi^{-1}V$ has positive codimension in Z. By the dimension principle it follows that (4.5) holds in $\mathcal{U}$ for all α, since $\bar\partial(1/f')$ has support on Z. From [2, Theorem 1.2] we conclude that $\widetilde\omega$ is smooth in $\mathcal{U}$. It follows that $\widetilde\omega$ is smooth in X'. □

4.2. Action of $ASM(X)$ on $\mathcal{PM}_X$

We will now extend Proposition 2.9 to general almost semi-meromorphic currents.

THEOREM 4.8. — *Assume that $a \in ASM(X)$. For each $\mu \in \mathcal{PM}(X)$ there is a unique pseudomeromorphic current T in X that coincides with $a \wedge \mu$ in $X \setminus ZSS(a)$ and such that $\mathbb{1}_{ZSS(a)}T = 0$.*

Let $V = ZSS(a)$. If such an extension T exists then $T = \mathbb{1}_{X \setminus V}T = \mathbb{1}_{X \setminus V} a \wedge \mu$ and so T is unique. Moreover, if h is a holomorphic tuple such that $Z(h) = V$, then

$$T = \lim_{\epsilon \to 0} \chi(|h|^2 v/\epsilon) a \wedge \mu \tag{4.6}$$

in view of Lemma 2.6. We will denote the extension T by $a \wedge \mu$ as well.

Proof. — As observed above, if the extension T exists, then (4.6) holds. Conversely, if the limit in (4.6) exists as a pseudomeromorphic current T on X, then it must coincide with $a \wedge \mu$ in $X \setminus V$. In particular, $\chi(|h|^2 v/\epsilon)T = \chi(|h|^2 v/\epsilon) a \wedge \mu$ for each $\epsilon > 0$ and hence, taking limits and using Lemma 2.6, we get $\mathbb{1}_{X \setminus V}T = T$, i.e., $\mathbb{1}_{ZSS(a)}T = 0$. To prove the theorem it is thus enough to verify that the limit in (4.6) exists as a pseudomeromorphic current.

In view of Lemma 4.7 we may assume that a has the form (4.1), where $Z = Z(f)$ is contained in $\pi^{-1}V$ and $\omega/f = \pi^* a$ in $X' \setminus \pi^{-1}V$. Let $\chi_\epsilon = \chi(|h|^2 v/\epsilon)$, so that $\pi^*\chi_\epsilon = \chi(|\pi^* h|\pi^* v/\epsilon)$. By Proposition 2.5 there is $\mu' \in \mathcal{PM}(X')$ such that $\pi_* \mu' = \mu$. Thus

$$\chi_\epsilon a \wedge \mu = \chi_\epsilon a \wedge \pi_* \mu' = \pi_* \left(\pi^* \chi_\epsilon \pi^* a \wedge \mu'\right) = \pi_* \left(\pi^* \chi_\epsilon \frac{\omega}{f} \wedge \mu'\right).$$

In view of Proposition 2.9 and Lemma 2.6,

$$\pi^* \chi_\epsilon \frac{\omega}{f} \wedge \mu' \to \mathbb{1}_{X' \setminus \pi^{-1}V} \frac{\omega}{f} \wedge \mu'$$

when $\epsilon \to 0$. In particular, the limit is a pseudomeromorphic current. Thus the limit in (4.6) exists and is pseudomeromorphic. □

Notice that the definition of $a \wedge \mu$ is local, so that it commutes with restrictions to open subsets of X. Thus for each $a \in ASM(X)$ we get a linear sheaf mapping

$$\mathcal{PM}_X \to \mathcal{PM}_X, \quad \mu \mapsto a \wedge \mu. \tag{4.7}$$

Proposition 4.9. — *Assume that $a \in ASM(X)$. If W is an analytic subset of $\mathcal{U} \subset X$ and $\mu \in \mathcal{PM}(\mathcal{U})$, then*

$$\mathbb{1}_W(a \wedge \mu) = a \wedge \mathbb{1}_W \mu. \tag{4.8}$$

Proof. — On the one hand (4.8) holds in the open set $\mathcal{U} \setminus ZSS(a)$ by (2.10) since a is smooth there. On the other hand both sides vanish on $ZSS(a)$, so (4.8) holds in all of $\mathcal{U}$; indeed $\mathbb{1}_{ZSS(a)}(a \wedge \mathbb{1}_W \mu) = 0$ by definition, cf. Theorem 4.8, and $\mathbb{1}_{ZSS(a)} \mathbb{1}_W(a \wedge \mu) = \mathbb{1}_W \mathbb{1}_{ZSS(a)}(a \wedge \mu) = 0$ in view of (1.3). □

Proposition 4.10. — *Each $a \in ASM(X)$ induces a linear mapping*

$$\mathcal{W}_X^Z \to \mathcal{W}_X^Z, \quad \mu \mapsto a \wedge \mu. \tag{4.9}$$

Proof. — To begin with, certainly $a \wedge \mu$ has support on Z if μ has. Let $\mathcal{U}$ be an open subset of X and assume that $W \subset \mathcal{U} \cap Z$ has positive codimension in $\mathcal{U} \cap Z$. Then $\mathbb{1}_W(a \wedge \mu) = a \wedge \mathbb{1}_W \mu = 0$ if $\mathbb{1}_W \mu = 0$, cf. (4.8). □

Example 4.11. — Assume that μ is in $\mathcal{W}_X$. Then $\mu' := [1/h]\mu$ is in $\mathcal{W}$ as well and if h is generically nonvanishing, then $h\mu' = h[1/h]\mu = \mathbb{1}_{\{h \neq 0\}}\mu = \mu$, cf. Remark 2.10.

Proposition 4.12. — *Assume that $a_1, a_2 \in ASM(X)$ and $\mu \in \mathcal{PM}_X$. Then*

$$a_1 \wedge a_2 \wedge \mu = (-1)^{\deg a_1 \deg a_2} a_2 \wedge a_1 \wedge \mu. \tag{4.10}$$

Proof. — Notice that both sides of (4.10) coincide outside $ZSS(a_1) \cup ZSS(a_2)$ and the restictions to $ZSS(a_1) \cup ZSS(a_2)$ vanish. □

In particular, one of the a_j may be a smooth form. We conclude that both (4.7) and (4.9) are $\mathcal{E}$-linear.

Proposition 4.13. — *If $a_1, a_2 \in ASM(X)$ and $\mu \in \mathcal{W}_X$, then*

$$a_1 \wedge a_2 \wedge \mu = (a_1 \wedge a_2) \wedge \mu, \quad (a_1 + a_2) \wedge \mu = a_1 \wedge \mu + a_2 \wedge \mu. \tag{4.11}$$

In fact, (4.11) holds outside $V := ZSS(a_1) \cup ZSS(a_2)$ and since $\mathbb{1}_V \mu = 0$ the equalities follow from (4.8).

Example 4.14. — Both equalities in (4.11) may fail for a general $\mu \in \mathcal{PM}_X$. Let $a_1 = 1/z_1$, $a_2 = z_1/z_2$, $a_3 = 1/z_2$, and $\mu = \bar\partial(1/z_1)$. Then $(a_1 a_2)\mu = (1/z_2)\bar\partial(1/z_1)$, but $a_2\mu = 0$, and so $a_1 a_2 \mu = 0$. Moreover

$$(a_1 + a_3)\mu = \frac{z_2 + z_1}{z_1 z_2} \bar\partial \frac{1}{z_1} = 0$$

but

$$a_1\mu + a_3\mu = \frac{1}{z_1}\bar\partial\frac{1}{z_1} + \frac{1}{z_2}\bar\partial\frac{1}{z_1} = \frac{1}{z_2}\bar\partial\frac{1}{z_1}.$$

4.3. Vector-valued almost semi-meromorphic currents

We will need to consider almost semi-meromorphic currents that take values in a holomorphic vector bundle $E \to X$. We say that $a \in ASM(X,E)$ if there is a representation (4.1), where as before f is a holomorphic section of $L \to X'$ and now ω takes values in $L \otimes \pi^* E$. Clearly then a is a current with values in E. If η is a test form with values in the dual bundle E^*, then $a.\eta = \pi_*((\omega/f).\pi^*\eta)$. Let e_j be a local frame for E in $\mathcal{U}$ and let ξ be a test function with support in $\mathcal{U}$. If $\xi' = \pi^*\xi$, $e'_j = e_j \circ \pi$ and $\omega = \omega_1 e'_1 + \omega_2 e'_2 + \dots$, then

$$\xi a = \sum_j \pi_*(\xi' \omega_j / f) e_j. \tag{4.12}$$

Proposition 4.15. — *Assume that X is smooth. There are natural isomorphisms*

$$ASM^{p,*}(X,E) \simeq ASM^{0,*}(X, \Lambda^p T^*_{1,0}(X) \otimes E). \tag{4.13}$$

Proof. — First notice that if F, G are vector bundles of the same rank over X' and h is a holomorphic section of $\mathrm{Hom}(F,G)$ that is generically invertible, then there is a holomorphic section g of $\mathrm{Hom}(G,F) \otimes \det G \otimes (\det F)^{-1}$ such that $hg = s \cdot I_G$, where s is a generically nonvanishing section of $\det G \otimes (\det F)^{-1}$.

For simplicity we assume that E is a trivial line bundle; the general case is proved in the same way. Now, let $F = \pi^* \Lambda^p T^*_{1,0}(X)$ and $G = \Lambda^p T^*_{1,0}(X')$. Then we have a natural mapping $h\colon F \to G$ as above, defined by just mapping the frame element $\mathrm{d}z_I$ to its pullback $\pi^* \mathrm{d}z_I$. Clearly h is an isomorphism where $\pi\colon X' \to X$ is biholomorphic.

Now, if $a \in ASM^{0,*}(X, \Lambda^p T^*_{1,0}(X))$, then we have the representation $a = \pi_*(\omega/f)$, where ω takes values in $F \otimes L$. Then $h\omega$ is a $(p,*)$-form in X' with values in L. It follows that $a' := \pi_*(h\omega/f)$ is an element in $ASM^{p,*}(X)$. We claim that $a' = a$. By the SEP it is enough to verify the identity where π is a biholomorphism. Let z be coordinates in an open subset $\mathcal{U} \subset X \setminus \pi(\operatorname{sing} \pi)$, and let ξ be a test function with support in $\mathcal{U}$. Then, cf. (4.12),

$$\begin{aligned}\xi a = \sum_{|I|=p}{}' \pi_*(\xi' \omega_I / f) \wedge \mathrm{d}z_I = \pi_* \left(\xi' \sum_{|I|=p}{}' \omega_I / f \wedge \pi^* \mathrm{d}z_I \right) = \pi_*(\xi' h\omega/f) \\ = \xi \pi_*(h\omega/f) = \xi a'.\end{aligned}$$

Conversely, since $h^{-1} = g/s$, if $a' \in ASM^{p,*}(X)$, then $a' = \pi_*(\tilde\omega/f)$, where $\tilde\omega$ is a $(p,*)$-form with values in L, then $g\tilde\omega$ takes values in

$F\otimes \det G\otimes (\det F)^{-1}\otimes L$ and sf takes values in $\det G\otimes (\det F)^{-1}\otimes L$, so that $a=\pi_*(g\tilde{\omega}/sf)$ is an element in $ASM^{0,*}(X, \Lambda^p T^*_{0,1}(X))$. Again one verifies that they coincide in $X\setminus \pi(\operatorname{sing}\pi)$. □

Notice that if $p=1$, then s is a section of the relative canonical bundle $K_{X'/X}=K_{X'}\otimes \pi^* K_X^{-1}$.

4.4. Residues of almost semi-meromorphic currents

We shall now study the effect of ∂ and $\bar\partial$ on almost semi-meromorphic currents.

Proposition 4.16. — *If $a\in ASM(X)$, then $\partial a\in ASM(X)$ and $b:=\mathbb{1}_{X\setminus ZSS(a)}\bar\partial a\in ASM(X)$.*

Thus we have the decomposition

$$\bar\partial a = b + r, \tag{4.14}$$

where $r:=\mathbb{1}_{ZSS(a)}\bar\partial a$ has support on $ZSS(a)$.

Proof. — Assume that $a=\pi_*(\omega/f)$ and let $D=D'+\bar\partial$ be a Chern connection on $L\to X'$. Then

$$\partial a = \pi_*\left(\partial\frac{\omega}{f}\right) = \pi_*\frac{f\cdot D'\omega - D'f\wedge\omega}{f^2},$$

which is in $ASM(X)$.

In view of Lemma 4.7 we may assume that $Z(f)\subset \pi^{-1}V$, where $V=ZSS(a)$. Now

$$\bar\partial a = \pi_*\frac{\bar\partial\omega}{f} + \pi_*\bar\partial\frac{1}{f}\wedge\omega. \tag{4.15}$$

By (2.11),

$$\begin{aligned}\mathbb{1}_{X\setminus V}\ \bar\partial a &= \pi_*\left(\mathbb{1}_{\pi^{-1}(X\setminus V)}\frac{\bar\partial\omega}{f}\right) + \pi_*\left(\mathbb{1}_{\pi^{-1}(X\setminus V)}\ \bar\partial\frac{1}{f}\wedge\omega\right)\\ &= \pi_*\left(\frac{\bar\partial\omega}{f}\right);\end{aligned} \tag{4.16}$$

thus $\mathbb{1}_{X\setminus V}\ \bar\partial a\in ASM(X)$. For the last equality we have used Proposition 2.9 and the fact that $\bar\partial(1/f)$ has support on $\pi^{-1}V$. □

In the same way we have: *If $a\in ASM(X,E)$ then* (4.14) *holds, where $b=\mathbb{1}_{X\setminus ZSS(a)}\bar\partial a$ is in $ASM(X,E)$ and $r=\mathbb{1}_{ZSS(a)}\bar\partial a$ is a pseudomeromorphic current with support on $ZSS(a)$ that takes values in E.*

Clearly the decomposition (4.14) is unique. We call $r = r(a)$ the *residue (current)* of a. Notice that if a is almost smooth, then $r(a) = 0$.

Remark 4.17. — If $a = \pi_*(\omega/f)$ is any representation of a, then still (4.15) holds, and since the first term is in $ASM(X)$ we conclude that

$$r(a) = \pi_* \left(\bar\partial \frac{1}{f} \wedge \omega \right).$$

Notice that the current $\bar\partial(1/f)$ is the residue of the principal value current $1/f$. Similarly, the residue currents introduced, e.g., in [3, 9, 21] can be considered as residues of certain almost semi-meromorphic currents, generalizing $1/f$.

Example 4.18. — Let us describe the construction of the residue currents in [3]. Let f be a holomorphic section of a Hermitian vector bundle $E \to X$, and let σ be the section over $X \setminus Z(f)$ of the dual bundle E^* with minimal norm such that $f\sigma = 1$. We can find a modification $\pi\colon X' \to X$ that is a biholomorphism $X' \setminus \pi^{-1}Z(f) \simeq X \setminus Z(f)$ such that $\pi^* f = f^0 f'$, where f^0 is a holomorphic section of a line bundle $L \to X'$, $\operatorname{div} f^0$ is contained in $\pi^{-1}Z(f)$, and f' is a nonvanishing section of $\pi^*E \otimes L^{-1}$. Then

$$\pi^*\sigma = \sigma'/f^0,$$

where σ' is a smooth section of $\pi^*E^* \otimes L$. Thus

$$\pi^*\left(\sigma\wedge(\bar\partial\sigma)^{k-1}\right) = \frac{\sigma'\wedge(\bar\partial\sigma')^{k-1}}{(f^0)^k}$$

is a section of $\Lambda^k(\pi^*E \oplus T^*_{0,1}(X'))$ in $X' \setminus \pi^{-1}Z(f)$; for the reader's convenience note that $\bar\partial\sigma$ has even degree in $\Lambda^k(\pi^*E \oplus T^*_{0,1}(X'))$. It follows that

$$U_k := \sigma\wedge(\bar\partial\sigma)^{k-1}$$

has an extension to an almost semi-meromorphic section of $\Lambda^k(E\oplus T^*_{0,1}(X))$, as the push-forward of $\sigma'\wedge(\bar\partial\sigma')^{k-1}/(f^0)^k$. Clearly $ZSS(U_k) \subset Z(f)$. Now the residue current R in [3] is the residue of the almost semi-meromorphic current $U = \sum_k U_k$. More precisely, if δ_f denotes interior multiplication by f, then $(\delta_f - \bar\partial)U = 1 - R$, i.e., $\bar\partial U = R + \delta_f U - 1$, where R is the residue and $\delta_f U - 1$ is almost semi-meromorphic. If E is trivial with trivial metric, the coefficients of R are the Bochner–Martinelli residue currents introduced in [21].

Clearly Theorem 4.8 extends to vector-valued currents. As a consequence of this theorem we can define products of residues of almost semi-meromorphic currents and pseudomeromorphic currents:

Definition 4.19. — *For $a \in ASM(X,E)$ and $\mu \in \mathcal{PM}_X$ we define*

$$\bar\partial a \wedge \mu := \bar\partial(a\wedge\mu) - (-1)^{\deg a} a\wedge\bar\partial\mu, \tag{4.17}$$

where $a\wedge\mu$ and $a\wedge\bar\partial\mu$ are defined as in Theorem 4.8. Moreover we define

$$r(a)\wedge\mu := \mathbb{1}_{ZSS(a)}\bar\partial a\wedge\mu.$$

Thus $\bar\partial a\wedge\mu$ is defined so that the Leibniz rule holds. It is easily checked that

$$r(a)\wedge\mu = \lim_{\epsilon\to 0}\bar\partial\chi(|h|^2 v/\epsilon)a\wedge\mu, \tag{4.18}$$

if $Z(h) = ZSS(a)$. In particular this gives a way of defining products of $\bar\partial$ and residues of almost semi-meromorphic currents. For example, the Coleff–Herrera product $\bar\partial(1/f_1)\wedge\ldots\wedge\bar\partial(1/f_p)$ can be defined by inductively applying (4.17). In [5] the first author defined products of more general residue currents in this way.

Notice that in general $a_1\wedge\bar\partial a_2$ is *not* equal to $\pm\bar\partial a_2\wedge a_1$, cf. Remark 2.11, and neither is

$$\bar\partial a_1\wedge\bar\partial a_2 = \pm\bar\partial a_2\wedge\bar\partial a_1 \tag{4.19}$$

in general; take, e.g., $a_1 = 1/z$ and $a_2 = 1/zw$.

Theorem 4.20. — *Assume that $a_1,\ldots,a_p$ are almost semi-meromorphic currents of degree $(*,k_1-1),\ldots,(*,k_p-1)$, respectively, and that*

$$\operatorname{codim}(ZSS(a_{i_1})\cap\cdots\cap ZSS(a_{i_r})) \geqslant k_{i_1}+\cdots+k_{i_r} \tag{4.20}$$

for all $\{i_1,\ldots,i_r\}\subset\{1,\ldots,p\}$. Then

$$\begin{aligned}\bar\partial a_1\wedge\ldots\wedge\bar\partial a_j\wedge\bar\partial a_{j+1}\wedge\ldots\wedge\bar\partial a_p \\ = (-1)^{(\deg a_j+1)(\deg a_{j+1}+1)}\bar\partial a_1\wedge\ldots\wedge\bar\partial a_{j+1}\wedge\bar\partial a_j\wedge\ldots\wedge\bar\partial a_p.\end{aligned} \tag{4.21}$$

Remark 4.21. — In fact, one can modify the proof below so that one can replace any factor $\bar\partial a_i$ in (4.21) by a_i. More precisely, let b_i be either a_i or $\bar\partial a_i$ for $i = 1,\ldots,p$. Then

$$\begin{aligned}b_1\wedge\ldots\wedge b_j\wedge b_{j+1}\wedge\ldots\wedge b_p \\ = (-1)^{\deg b_j\cdot\deg b_{j+1}} b_1\wedge\ldots\wedge b_{j+1}\wedge b_j\wedge\ldots\wedge b_p.\end{aligned} \tag{4.22}$$

Remark 4.22. — If the almost semimeromorphic parts of $\bar\partial a_i$ vanish, then it is enough to assume

$$\operatorname{codim}(ZSS(a_1)\cap\cdots\cap ZSS(a_p)) \geqslant k_1+\cdots+k_p. \tag{4.23}$$

Indeed, note that in this case the currents in (4.21) have support on $V := ZSS(a_1) \cap \cdots \cap ZSS(a_p)$. Thus it is enough to prove (4.21) in a neighborhood of $x \in V$, and there (4.23) implies (4.20).

In particular, the Coleff–Herrera product $\bar\partial(1/f_1)\wedge\ldots\wedge\bar\partial(1/f_p)$ is (anti-)commutative in its factors if the codimension of $\{f_1 = \cdots = f_p = 0\}$ is at least p.

Proof. — Let $V_j = ZSS(a_j)$. Moreover, let b_i be either an almost semi-meromorphic current or $\bar\partial$ of an semi-meromorphic current for $i = 1, \ldots, r$, cf. Remark 4.21, and assume that α is smooth. Then note that

$$(4.24)\quad b_1\wedge\ldots\wedge b_\ell\wedge\alpha\wedge b_{\ell+1}\wedge\ldots\wedge b_r = (-1)^{\deg\alpha(\deg b_1+\cdots+\deg b_\ell)}\alpha\wedge b_1\wedge\ldots\wedge b_r.$$

Assume that

$$(4.25)\quad \bar\partial a_1\wedge\ldots\wedge\bar\partial a_{j-1}\wedge a_j\wedge\bar\partial a_{j+1}\wedge\ldots\wedge\bar\partial a_p = (-1)^{\deg a_j(\deg a_{j+1}+1)}\bar\partial a_1\wedge\ldots\wedge\bar\partial a_{j-1}\wedge\bar\partial a_{j+1}\wedge a_j\wedge\bar\partial a_{j+2}\wedge\ldots\wedge\bar\partial a_p.$$

Applying $\bar\partial$ to (4.25) yields (4.21) in view of (4.17).

To prove (4.25) we will proceed by induction. First assume that $p = 2$. Then in view of (4.24),

$$(4.26)\qquad a_1\wedge\bar\partial a_2 = (-1)^{\deg a_1(\deg a_2+1)}\bar\partial a_2 \wedge a_1,$$

where a_1 or a_2 is smooth, i.e., outside $V_1 \cap V_2$. Because of the assumption (4.20), (4.26) holds in all of X by the dimension principle. Next, assume that (4.25) holds for $p = \ell$. In view of (4.24), (4.25) holds for $p = \ell+1$, where a_j or a_{j+1} is smooth. Moreover, by (4.24) and the assumption that (4.25) holds for $p = \ell$, (4.25) holds for $p = \ell + 1$, where (at least) one of $a_1, \ldots, a_{j-1}, a_{j+2}, \ldots, a_{\ell+1}$ is smooth. Thus (4.25) holds for $p = \ell+1$ outside $V_1 \cap \cdots \cap V_{\ell+1}$, and thus by (4.20) and the dimension principle it holds in all of X. Hence (4.25) and thus (4.21) hold for all p. □

The following example shows that $r(a) = 0$ does not imply that $r(a)\wedge\mu = 0$. This points out the importance of keeping in mind that $\mu \mapsto r(a)\wedge\mu$ is an operator on $\mathcal{PM}_X$ rather than a "product".

Example 4.23. — Let us consider the setting in Example 4.18. Assume in addition that $Z(f)$ has codimension at least 2. Note that then $r(\sigma) = 0$ by the dimension principle, since it has bidegree $(0, 1)$ and support on $Z(f)$, which has codimension $\geqslant 2$. However, if τ is the almost semi-meromorphic part of $\bar\partial U$, then $r(\sigma)\wedge\tau$ is the residue current R from [3] which is nonzero, cf. Example 4.18.

Remark 4.24. — There are other (weighted) approaches to products of residue currents, see, e.g. [20, 26], which coincide with the products above under suitable conditions.

4.5. Action of holomorphic differential operators and vector fields

Finally we prove that $ASM(X)$ is preserved under the action of holomorphic vector fields.

THEOREM 4.25. — *Let ξ be a holomorphic vector field on a smooth manifold X. If $a \in ASM(X)$, then the contraction $\xi \neg a$ and the Lie derivative $L_\xi a$, a priori defined on $X \setminus ZSS(a)$, have extensions as elements in $ASM(X)$.*

Since the extensions, if they exist, must be unique, we can simply say that $\xi \neg a$ and $L_\xi a$ are in $ASM(X)$.

Proof. — Let $\pi\colon X' \to X$ be a modification so that a has the form (4.1). Then $\xi' := \pi^*\xi$ is a global section of $\pi^*T(X)$, that is the natural lifting of ξ to $T(X')$ over $X' \setminus \text{sing}(\pi)$. By duality the mapping $\pi^*T^*_{1,0}(X) \to T^*_{1,0}(X')$ from the proof of Proposition 4.15 induces a holomorphic mapping $T(X') \to \pi^*T(X)$ that is the identity outside $\text{sing}(\pi)$. If h denotes this dual map, by the first part of the same proof there is a holomorphic mapping $g\colon \pi^*T(X) \to T(X') \otimes K_{X'/X}$ such that $hg = sI_{\pi^*T(X)}$, where s is a holomorphic section of $K_{X'/X}$. Thus $g\xi'/s$ is a semi-meromorphic vector field on X' that coincides with ξ' on $X' \setminus \text{sing}(\pi)$. Moreover, $b := s\xi'$ is smooth. Outside $\pi(\text{sing}(\pi)) \cup ZSS(a)$ we now have that

$$\xi \neg a = \pi_*\left(\frac{\xi' \neg \omega}{f}\right) = \pi_*\left(\frac{b \neg \omega}{sf}\right)$$

and it is clear that the right hand side defines an almost semi-meromorphic current in X. Finally, $L_\xi a = \xi \neg (\partial a) + \partial(\xi \neg a)$ is in $ASM(X)$ in view of Proposition 4.16. □

By similar arguments one can prove that $\mathcal{L}a$ is in $ASM(X)$ if a is an almost semi-meromorphic $(0,q)$-current and $\mathcal{L}$ is any (global) holomorphic differential operator. More precisely, one can show that $\mathcal{L}a = \pi_*(s^{-N}\mathcal{L}'(\omega/f))$ for some N, where s is the section of $K_{X'/X}$ in the proof above and $\mathcal{L}'$ is a holomorphic differential operator (with values in $K^N_{X'/X}$).

Corollary 4.26. — *Let X be an open subset of $\mathbb{C}^n_z$. If*

$$a = \sum_{|I|=p}' a_I \wedge \mathrm{d}z_I \tag{4.27}$$

is in $ASM(X)$, then each a_I is in $ASM(X)$. If $a \in ASM(X)$ has bidegree $(0,*)$, then $\partial a/\partial z_j$ is in $ASM(X)$ for each j.

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