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# ELEMENTARY CONSTRUCTION OF RESIDUE CURRENTS ASSOCIATED TO COHEN–MACAULAY IDEALS

by Richard LÄRKÄNG & Emmanuel MAZZILLI (\*)

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ABSTRACT. — For a Cohen–Macaulay ideal of holomorphic functions, we construct by elementary means residue currents whose annihilator is precisely the given ideal. We give two proofs that the currents have the prescribed annihilator, one using the theory of linkage, and another using an explicit division formula involving these residue currents to express the ideal membership.

RÉSUMÉ. — Pour un idéal Cohen–Macaulay de fonctions holomorphes, nous construisons de manière élémentaire des courants résiduels qui s’annulent précisément sur cet idéal. Nous donnons deux constructions, l’une utilisant la théorie des idéaux en algèbre commutative, et l’autre utilisant des représentations intégrales qui donnent une décomposition dans l’idéal modulo ces courants résiduels.

## 1. Introduction

Let  $\mathcal{O} := \mathcal{O}_{\mathbb{C}^n, 0}$  be the ring of germs of holomorphic functions at  $0 \in \mathbb{C}^n$ . If  $f \in \mathcal{O}$ , and  $U$  is a  $(0, 0)$ -current such that  $fU = 1$ , then it follows easily by regularity for the  $\bar{\partial}$ -operator on  $(0, 0)$ -currents that

$$(1.1) \quad g\bar{\partial}U = 0 \text{ if and only if } g \in J(f),$$

where  $J(f)$  is the principal ideal generated by  $f$ . For a current  $T$ , we let  $\text{ann } T$  denote the annihilator of  $T$ , i.e. all holomorphic functions  $g$  such that  $gT = 0$ . Thus, if  $fU = 1$ , we get that

$$\text{ann } \bar{\partial}U = J(f).$$

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One natural choice of such a current  $U$  is the so-called principal value current  $[1/f]$ , which is defined as

$$\left\langle \left[ \frac{1}{f} \right], \phi \right\rangle := \lim_{\epsilon \rightarrow 0^+} \int \frac{\chi(|f|^2/\epsilon)}{f} \wedge \phi,$$

where  $\phi$  is a test form and  $\chi$  is the cut-off function which is the characteristic function of the interval  $[1, \infty)$ , or a smooth regularization of this function. The existence of this current was proven by Dolbeault, [10], and Herrera–Lieberman, [13]. That this limit exists relies on Hironaka’s theorem about resolution of singularities, and is thus far from elementary. Anyhow, any such choice of a current  $U$  gives rise to a description of a principal ideal  $J(f)$ . A construction of such a current by elementary means, which in general is different from the principal value current was done by the second author in [18].

Consider now a complete intersection ideal  $J$  of codimension  $p$ , i.e.  $J = J(f_1, \dots, f_p)$  can be generated by exactly  $p$  holomorphic functions,  $f_1, \dots, f_p$ . Coleff and Herrera showed in [5] that one can give a reasonable meaning to  $\bar{\partial}[1/f_p] \wedge \dots \wedge \bar{\partial}[1/f_1]$  in a similar way as for the principal value current. Again, for all the different ways of regularizing the current, the existence of the limit relies on Hironaka’s theorem. In [16] it is described various ways that this product can be defined through some regularization procedure. It was proven independently by Passare, [20] and Dickenstein–Sessa, [9], that this so-called Coleff–Herrera product satisfies the duality principle,

$$(1.2) \quad \text{ann } \bar{\partial} \left[ \frac{1}{f_1} \right] \wedge \dots \wedge \bar{\partial} \left[ \frac{1}{f_p} \right] = J(f_1, \dots, f_p).$$

The proof of Passare relied on constructing an explicit division formula involving the Coleff–Herrera product in order to obtain the ideal membership, while the proof in [9] essentially reduced to solving a series of  $\bar{\partial}$ -equations.

Especially in relation to extension problems of holomorphic functions, it has turned out to be useful to consider other currents for describing complete intersection ideals similar to (1.2). It turns out that, generalizing the case of principal ideals in the beginning, if  $J = J(f_1, \dots, f_p)$  is a complete intersection ideal of codimension  $p$ , and if  $X_k$  are  $(0, k-1)$ -currents for  $k = 1, \dots, p$  such that

$$(1.3) \quad \begin{aligned} f_1 X_1 &= 1, \\ f_j X_k &= 0 \text{ for } 1 \leq j < k \leq p \text{ and } f_k X_k = \bar{\partial} X_{k-1} \text{ for } 2 \leq k \leq p, \end{aligned}$$

then

$$(1.4) \quad \text{ann } \bar{\partial} X_p = J(f_1, \dots, f_p).$$

In [19], the second author gave an elementary construction of such currents for any complete intersection ideal, using only the much more elementary Weierstrass preparation theorem, and not relying on Hironaka’s theorem.

Consider now a more general ideal  $J = J(f_1, \dots, f_m)$ , which is not necessarily a complete intersection ideal. In [2], Andersson and Wulcan constructed, given a free resolution  $(E, \varphi)$  of  $\mathcal{O}/J$ , a  $(\text{Hom}(E_0, E))$ -valued current  $R^E$  such that

$$\text{ann } R^E = J,$$

and two proofs of this description of the annihilator were given, one essentially reducing ideal membership to solving a series of  $\bar{\partial}$ -equations, and the second by constructing an explicit division formula. If  $J = J(f_1, \dots, f_p)$  is a complete intersection ideal, and one takes the Koszul complex of  $f$  as a free resolution of  $\mathcal{O}/J$ , then  $R^E$  equals the Coleff–Herrera product of  $f$ . In general, although the current  $R^E$  is explicitly expressed in terms of the free resolution  $(E, \varphi)$ , it is in general quite difficult to understand, and the proof of existence of this current again relies on Hironaka’s theorem.

In [14], the first author described a way of relating the currents  $R^E$  of Andersson and Wulcan, related to different free resolutions, of possibly different ideals. We consider the particular case when  $J$  is a Cohen–Macaulay ideal of codimension  $p$ , i.e.  $\mathcal{O}/J$  has a free resolution  $(E, \varphi)$  of length  $p$ . We also assume that  $\text{rank } E_0 = 1$ , which is always possible to choose. One can always find a complete intersection ideal  $I = J(f_1, \dots, f_p)$  of codimension  $p$  contained in  $J$ , for example by taking  $p$  generic linear combinations of a set of generators of  $J$ , cf. for example [14, Example 2]. If one lets  $(K, \psi)$  be the Koszul complex of  $f$ , then it is quite elementary homological algebra that one can construct a morphism of complexes  $a : (K, \psi) \rightarrow (E, \varphi)$  which extends the natural surjection  $\pi : \mathcal{O}/I \rightarrow \mathcal{O}/J$ , i.e. which is such that the following diagram is commutative:

$$(1.5) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & E_p & \xrightarrow{\varphi_p} & E_{p-1} & \longrightarrow & \cdots & \xrightarrow{\varphi_1} & E_0 & \longrightarrow & \mathcal{O}/J & \longrightarrow & 0 \\ & & \uparrow a_p & & \uparrow a_{p-1} & & & & \uparrow a_0 & & \uparrow \pi & & \\ 0 & \longrightarrow & K_p & \xrightarrow{\psi_p} & K_{p-1} & \longrightarrow & \cdots & \xrightarrow{\psi_1} & K_0 & \longrightarrow & \mathcal{O}/I & \longrightarrow & 0, \end{array}$$

cf. Proposition 2.2 below. By [14, Example 3], the current  $R^E$  can then be described as

$$(1.6) \quad R^E = a_p(e) \bar{\partial} \begin{bmatrix} 1 \\ f_1 \end{bmatrix} \wedge \cdots \wedge \bar{\partial} \begin{bmatrix} 1 \\ f_p \end{bmatrix},$$

where  $e_1, \dots, e_p$  is a frame for  $K_1$  such that  $\psi_1 = f_1 e_1^* + \cdots + f_p e_p^*$ , and  $e := e_p \wedge \cdots \wedge e_1$  is the induced frame for  $K_p \cong \wedge^p K_1$ . Hence, the current

$R^E$  can be described as an explicit tuple of holomorphic functions times a Coleff–Herrera product. If  $J = J(g_1, \dots, g_p)$  is also a complete intersection ideal of codimension  $p$ , and  $(E, \varphi)$  is the Koszul complex of  $g$ , then  $R^E$  is also a Coleff–Herrera product, and (1.6) then becomes the transformation law for Coleff–Herrera products, see [14, Remark 2].

Our main result is the following combination of (1.4) and (1.6), which thus with the help of the construction from [19] allows for constructing currents representing Cohen–Macaulay ideals by elementary means, in particular not relying on Hironaka’s theorem about resolution of singularities.

**THEOREM 1.1.** — *Let  $J$  be a Cohen–Macaulay ideal of codimension  $p$ ,  $(E, \varphi)$  be a free resolution of  $\mathcal{O}/J$  such that  $\text{rank } E_0 = 1$ ,  $I = J(f_1, \dots, f_p)$  a complete intersection ideal of codimension  $p$  contained in  $I$ ,  $(K, \psi)$  the Koszul complex of  $f$ , and let  $a : (K, \psi) \rightarrow (E, \varphi)$  be a morphism of complexes extending the natural surjection  $\pi : \mathcal{O}/I \rightarrow \mathcal{O}/J$  as in (1.5). If  $X_1, \dots, X_p$  are currents satisfying (1.3), then*

$$\text{ann } a_p(e)\bar{\partial}X_p = J.$$

The requirement that  $\text{rank } E_0 = 1$  implies that the entries of  $\varphi_1$  generate  $J$ , and one can always find a free resolution such that this is the case.

We give two different proofs of this result, one in Section 2, which with the help of the theory of linkage reduces the problem to the complete intersection case and (1.4), and as well a more direct proof in Section 3 by means of an explicit division formula for expressing the ideal membership.

In [17], Lundqvist defined by elementary means cohomological residues for a Cohen–Macaulay  $J$ , which act on test forms which are  $\bar{\partial}$ -closed in a neighborhood of  $\text{supp } J$ . By the construction in [17], it follows easily that the action of the current  $R^E$  on such test forms equals the residues by Lundqvist. These residues and its relation to other residues is elaborated a bit in [15, Section 7]. Since these cohomological residues are only defined acting on a restricted class of test forms, the construction can be done by elementary means, depending only on finding a free resolution, and in particular avoiding resolutions of singularities. This is at the cost of not showing that these residues can act on arbitrary test forms. However, the main result in [17] is that even by only acting on this restricted class of test forms, one still obtains a duality theorem.

## 2. Proof by the theory of linkage

In this section, we give the first proof of Theorem 1.1, which is based on the theory of linkage. In a somewhat different setting, similar methods

were used in [15]. We recall that if  $I$  and  $J$  are two ideals in a ring  $R$ , then  $I : J$  is the ideal  $I : J = \{r \in R \mid rJ \subseteq I\}$ . The key result in proving Theorem 1.1 is the following result, which can be found in (the proof of) [21, Proposition 3.41].

**THEOREM 2.1.** — *Let  $J \subseteq \mathcal{O}$  be an ideal of pure codimension  $p$ , and let  $I = J(f_1, \dots, f_p)$  be a complete intersection ideal of codimension  $p$  contained in  $J$ . If  $K := I : J$ , then  $J = I : K$ .*

We can describe the ideal  $K$  appearing in Theorem 2.1 in a different way, when  $\mathcal{O}/J$  is Cohen–Macaulay. In order to do this, we use the following standard fact from homological algebra, see [11], Proposition A3.13.

**PROPOSITION 2.2.** — *Let  $\alpha : F \rightarrow G$  be a homomorphism of  $\mathcal{O}$ -modules, and let  $(K, \psi)$  and  $(E, \varphi)$  be free resolutions of  $F$  and  $G$ . Then, there exists a morphism  $a : (K, \psi) \rightarrow (E, \varphi)$  of complexes which extends  $\alpha$ .*

We will apply this in the case when  $F = \mathcal{O}/I$ ,  $G = \mathcal{O}/J$ ,  $I \subseteq J$  and  $\alpha$  is the natural surjection  $\pi : \mathcal{O}/I \rightarrow \mathcal{O}/J$ , as in (1.5). We remind for the following lemma, that for any ideal  $J \subseteq \mathcal{O}$  of codimension  $p$ , there always exists a complete intersection ideal  $I \subseteq J$  of codimension  $p$ . The following follows from Lemma 3.2 in [12].

**LEMMA 2.3.** — *Let  $J \subseteq \mathcal{O}$  be a Cohen–Macaulay ideal of codimension  $p$ , and assume that  $I = J(f_1, \dots, f_p) \subseteq J$  is a complete intersection ideal of codimension  $p$ . Let  $(E, \varphi)$  be a free resolution of  $\mathcal{O}/J$  such that  $\text{rank } E_0 = 1$ , and let  $(K, \psi)$  be the Koszul complex of  $f$ , which is a free resolution of  $\mathcal{O}/I$ . Let  $a : (K, \psi) \rightarrow (E, \varphi)$  be the morphism induced by the natural surjection  $\mathcal{O}/I \rightarrow \mathcal{O}/J$  as in Proposition 2.2. Let  $L$  be the ideal generated by the entries of  $a_p$ . Then,*

$$I : J = I + L.$$

**Remark 2.4.** — By reformulating this result, one can in fact drop the Cohen–Macaulay assumption, see [15, Lemma 4.6], but for simplicity, we stick to this case here.

In [8], a topic is treated which is related to this article, namely, given an analytic functional annihilated by some Cohen–Macaulay ideal, to express this functional in terms of residue currents, or more precisely Coleff–Herrera products. In order to do this, Lemma 2.3 plays an important role, see the proof of [8, Theorem 4.1]. In this article, we construct currents with a prescribed Cohen–Macaulay ideal  $J$  as its annihilator. From the currents we construct, one could construct an analytic functional annihilated by  $J$  and

by our construction, this functional could be directly expressed in terms of a current  $\bar{\partial}X_p$  whose annihilator is some complete intersection ideal contained in  $J$ . The current  $\bar{\partial}X_p$  could either be the current constructed in [19] or a Coleff–Herrera product, and in this latter case, one would obtain an expression for the functional like in [8].

*Proof of Theorem 1.1.* — Since  $J = I : (I : J)$  by Theorem 2.1, and  $I : J = I + L$  by Lemma 2.3,  $J = I : L$ . We thus get that  $g \in J$  if and only if all the entries of  $ga_p(e)$  are in  $I$ . By (1.4), this holds if and only if  $ga_p(e)\bar{\partial}X_p = 0$ . □

*Example 2.5.* — We consider now the most basic case, namely when  $I = J(f_1, \dots, f_p)$  and  $J = J(g_1, \dots, g_p)$  are both complete intersection ideals of codimension  $p$ . Then  $I \subseteq J$  is equivalent to that that  $f = gA$  for some holomorphic  $p \times p$ -matrix  $A$ . In this case, when  $(E, \varphi)$  and  $(K, \psi)$  are the Koszul complexes of  $g$  and  $f$  respectively, being free resolutions of  $\mathcal{O}/J$  and  $\mathcal{O}/I$  respectively, then the morphism  $a : (E, \varphi) \rightarrow (K, \psi)$  extending the natural surjection  $\pi : \mathcal{O}/I \rightarrow \mathcal{O}/J$  is given by  $a_k : \bigwedge^k \mathcal{O}^{\oplus p} \rightarrow \bigwedge^k \mathcal{O}^{\oplus p}$ ,  $a_k = \bigwedge^k A$ . In particular,  $a_p = \det A$ . Thus, reasoning as above, we get that

$$(2.1) \quad g \in J \text{ if and only if } (\det A)g \in I.$$

This was an important part of the construction in [19], since (2.1) allowed to reduce the problem to constructing such currents for just for certain special “adapted” complete intersections.

*Example 2.6.* — Let  $\pi : \mathbb{C} \rightarrow \mathbb{C}^3$ ,  $\pi(t) = (t^3, t^4, t^5)$ , and let  $Z$  be the germ at 0 of  $\pi(\mathbb{C})$ . One can show that the ideal of holomorphic functions vanishing at  $Z$  equals  $J = J(y^2 - xz, x^3 - yz, x^2y - z^2)$ . The module  $\mathcal{O}/J$  has a minimal free resolution  $(E, \varphi)$  of the form

$$0 \rightarrow \mathcal{O}^{\oplus 2} \xrightarrow{\varphi_2} \mathcal{O}^{\oplus 3} \xrightarrow{\varphi_1} \mathcal{O} \rightarrow \mathcal{O}/J,$$

where

$$\varphi_2 = \begin{bmatrix} -z & -x^2 \\ -y & -z \\ x & y \end{bmatrix} \text{ and } \varphi_1 = [ \ y^2 - xz \quad x^3 - yz \quad x^2y - z^2 \ ].$$

In particular, since  $\mathcal{O}/J$  has a minimal free resolution of length 2, with rank  $E_2 = 2$ ,  $\mathcal{O}/J$  is Cohen–Macaulay but  $J$  is not a complete intersection. However,  $Z$  is in fact a set-theoretic complete intersection, which one can see by verifying that indeed, if  $f = (z^2 - x^2y, x^4 + y^3 - 2xyz)$ , and  $I = J(f)$ , then  $Z(I) = Z$ .



Let  $(F, \psi)$  be the Koszul complex of  $f$ , which is a free resolution of  $\mathcal{O}/I$  since  $f$  is a complete intersection. One verifies that  $a : (F, \psi) \rightarrow (E, \varphi)$  given by,

$$a_2 = \begin{bmatrix} x^3 - yz \\ y^2 - xz \end{bmatrix}, a_1 = \begin{bmatrix} 0 & y \\ 0 & x \\ -1 & 0 \end{bmatrix} \text{ and } a_0 = [ 1 ],$$

is a morphism of complexes extending the natural surjection  $\pi : \mathcal{O}/I \rightarrow \mathcal{O}/J$ . In the appendix of [14], we give an example of how such a morphism can be computed with the help of the computer algebra system Macaulay2.

By Theorem 1.1, we then get that

$$g \in J \text{ if and only if } (x^3 - yz)g \in I \text{ and } (y^2 - xz)g \in I.$$

For general Cohen–Macaulay ideals, one cannot expect that  $Z(I) = Z(J)$  as in this example, since it by definition only is possible for set-theoretic complete intersections.

### 2.1. Construction of the currents from [18] and [19]

In order to calculate the currents satisfying (1.3) as constructed in [18] and [19] for the complete intersection in the example above, we will first recall briefly the construction in general for a complete intersection ideal of codimension 2. (The case of codimension  $> 2$  is similar, but a bit more technically involved.)

We thus consider a tuple  $(f_1, f_2)$  of germs of holomorphic functions in  $\mathbb{C}^n$  defining a complete intersection of codimension 2. By a linear change of coordinates, we can assume we have coordinates  $z$  on  $\mathbb{C}^n$  such that  $f_1$  and  $f_2$  are of the form  $f_i = v_i Q_i$ , where  $v_i$  are invertible, and  $Q_i$  are Weierstrass polynomials in  $z_1$  for  $i = 1, 2$ . Then, the resultant  $r_2$  of  $Q_1$  and  $Q_2$  (where  $Q_1$  and  $Q_2$  are considered as polynomials in  $z_1$  for the calculation of the resultant) is a holomorphic function independent of  $z_1$ . By a linear change of variables only in  $(z_2, \dots, z_n)$ , we can assume that  $r_2(z) = u_2(z)P_2(z_2, \dots, z_n)$ , where  $P_2$  is a Weierstrass polynomial in  $z_2$  independent of  $z_1$  and  $u_2$  is invertible.

If we let  $g_1 := f_1$  and  $g_2 := r_2$ , then  $(g_1, g_2)$  is a complete intersection which satisfies that one can write  $g_i(z) = u_i(z)P_i(z)$ , where  $u_i(z)$  is a unit, and  $P_i(z)$  is a Weierstrass polynomials in  $z_i$  of degree  $N_i$ , and in addition,  $P_2(z)$  is independent of  $z_1$ . The construction of the currents  $X_1$  and  $X_2$

satisfying (1.3) is based on first constructing currents  $Y_1, Y_2$  satisfying the corresponding conditions for  $(g_1, g_2)$ , i.e.

$$g_1 Y_1 = 1, g_1 Y_2 = 0 \text{ and } g_2 Y_2 = \bar{\partial} Y_1.$$

To do this, one defines  $Y_1$  by

$$(2.2) \quad \langle Y_1, \phi \wedge dz_I \wedge d\bar{z}_J \rangle := C_1 \int \frac{\bar{P}_1^\gamma}{g_1} \partial_{z_1}^{N_1 \gamma}(\phi) dz_I \wedge d\bar{z}_J,$$

and  $Y_2$  by

$$(2.3) \quad \langle Y_2, \phi \wedge dz_I \wedge d\bar{z}_J \rangle := C_2 \int \frac{\bar{P}_2}{g_2} \partial_{z_2}^{N_2} \left( \frac{\bar{\partial} \bar{P}_1^\gamma}{g_1} \partial_{z_1}^{N_1 \gamma}(\phi) \right) dz_I \wedge d\bar{z}_J,$$

where  $\gamma$  is an integer chosen so the integrand in the definition of  $Y_2$  becomes integrable, which indeed holds for  $\gamma$  larger than  $N_2$ . The constant  $C_1$  is chosen so that  $f_1 X_1 = 1$  and the constant  $C_2$  is then chosen such that  $g_2 Y_2 = \bar{\partial} Y_1$ . Through integration by parts one can calculate that  $C_1 = (-1)^{N_1 \gamma} (N_1 \gamma)!$  and  $C_2 = -(-1)^{N_2} C_1 / (N_2!)$ .

In order to construct the currents  $X_1, X_2$  for  $(f_1, f_2)$ , one then first writes  $r_2 = a f_1 + b f_2$  for some holomorphic functions  $a, b$ , which indeed is possible, since by construction,  $r_2 = a' Q_1 + b' Q_2$ . Then, one defines  $X_1 := Y_1$  and  $X_2 := b Y_2$ , which one can verify satisfies the properties (1.3).

Note that the greatest common divisor, i.e. the last remainder term in the Euclidean algorithm for  $Q_1$  and  $Q_2$  considered as polynomials in  $z_1$  gives the resultant  $r_2$  up to a constant which depends on the degrees of the polynomials appearing when running the algorithm, see for example [6, Exercise 3.6.10-11]. For our purposes it does not matter if we take a constant multiple of  $r_2$  as  $g_2$ , and we will thus below take the greatest common divisor instead of the resultant. This is advantageous since the Euclidean algorithm is convenient for computation, and additionally, by going backwards in the algorithm, with the help of the terms that appear, one obtains a decomposition  $r_2 = a' Q_1 + b' Q_2$ .

*Example 2.7.* — We now consider the construction as described above for  $f = (f_1, f_2) := (z^2 - x^2 y, x^4 - 2xyz + y^3)$ , as in Example 2.6. In order to make  $f_1$  and  $f_2$  Weierstrass polynomials in  $x$  around zero (times units), we do the change of coordinates:

$$x = X, \quad y = Y, \quad z = X + Z.$$

In these new coordinates,  $(f_1, f_2)$  becomes:

$$f = ((1 - Y)X^2 + 2XZ + Z^2, X^4 - 2X^2Y - 2XYZ + Y^3).$$

If we let  $g = 1 - Y$ , the Weierstrass polynomial in  $X$  associated to  $f_1$  is equal to  $P_1 := X^2 + \frac{2XZ}{g} + \frac{Z^2}{g}$ , which has degree  $N_1 = 2$ .

To calculate the resultant  $r_2$  of  $P_1$  and  $f_2$ , we use the Euclidean algorithm as mentioned above, and after an elementary but tedious calculations, we obtain that

$$r_2 = \frac{Z^2 F^2}{g} - \frac{2ZFG}{g} + G^2,$$

where  $F$  and  $G$  are Weierstrass polynomials with respect to  $Z$  defined by:

$$F(Y, Z) = \frac{4}{g^2} \left(1 - \frac{2}{g}\right) Z^3 + 2Y \left(\frac{2}{g} - 1\right) Z,$$

$$G(Y, Z) = \frac{1}{g^2} \left(1 - \frac{4}{g}\right) Z^4 + \frac{2Y}{g} Z^2 + Y^3.$$

Since  $g(0) = 1$ , it is easy to see that  $r_2 = UP_2$  where  $U$  is a unit near zero and  $P_2$  is a Weierstrass polynomial in  $Z$  of degree  $N_2 = 8$ . We finally do the linear coordinate change  $(z_1, z_2, z_3) = (X, Z, Y)$  keeping the first variable fixed so that  $P_1$  is a Weierstrass polynomial in  $z_1$  and  $P_2$  is a Weierstrass polynomial in  $z_2$ , and is independent of  $z_1$ .

We thus let  $(g_1, g_2) = (f_1, r_2)$  and define  $Y_1, Y_2$  by (2.2) and (2.3). By going backwards in the Euclidean algorithm, one finds that  $r_2 = af_1 + bf_2$ , where  $b = -(x + \frac{2z}{g})F + G$ . Thus, we get that

$$X_1 := Y_1 \quad \text{and} \quad X_2 := bY_2$$

satisfies all the conditions (1.3).

### 3. Proof by explicit division formulas

In this section, we give an explicit division formula which proves Theorem 1.1. The proof relies on the following two lemmas. On  $\mathbb{C}^n$  with coordinates  $\zeta$ , and for  $z \in \mathbb{C}^n$  fixed,  $\delta_\eta$  denotes contraction with the vector field  $(\zeta_1 - z_1)\frac{\partial}{\partial \zeta_1} + \dots + (\zeta_n - z_n)\frac{\partial}{\partial \zeta_n}$ .

LEMMA 3.1. — *Let  $Q$  be a  $(1, 0)$ -form on  $\mathbb{C}^n$ , and  $H$  a holomorphic  $(k + 1)$ -form. Then*

$$(n - k)\bar{\partial}(\delta_\eta Q) \wedge (\bar{\partial}Q)^{n-k-1} \wedge H = (\bar{\partial}Q)^{n-k} \wedge \delta_\eta H.$$

LEMMA 3.2. — *Let  $I = J(f_1, \dots, f_p), J, (E, \varphi), (K, \psi)$  and  $a : (K, \psi) \rightarrow (E, \varphi)$  be as in Theorem 1.1, and let  $X_k$  be  $(0, k - 1)$ -currents for  $k = 1, \dots, p$ , which satisfy (1.3). Let  $Y_k$  be defined as*

$$Y_k := a_k(e_1 \wedge \dots \wedge e_k \wedge X_k).$$

Then  $Y$  satisfies

$$(3.1) \quad \nabla Y := 1 - \bar{\partial} Y_p,$$

where  $Y = Y_1 + \dots + Y_p$  and  $\nabla = \varphi - \bar{\partial}$ .

More explicitly, the equation (3.1) means that

$$(3.2) \quad \varphi_1 Y_1 = 1 \text{ and } \varphi_k Y_k = \bar{\partial} Y_{k-1} \text{ for } 2 \leq k \leq p.$$

We let  $e_1, \dots, e_p$  be the standard basis of  $K_1 \cong \mathcal{O}^p$  such that the morphism in  $(K, \psi)$  is contraction with  $\sum f_i e_i^*$ , and in particular,  $K_k$  has as a basis  $e_{I_1} \wedge \dots \wedge e_{I_k}$  for  $1 \leq I_1 < \dots < I_k \leq p$ .

*Remark 3.3.* — To be precise,  $\varphi_1 Y_1$  is a  $E_0$ -valued  $(0, 0)$ -current. However, since we assume that  $\text{rank } E_0 = 1$ , we have that  $E_0 \cong \mathcal{O} \cong K_0$ . Note that  $K_0 \cong \bigwedge^0 K_1$  has a canonical frame,  $e_\emptyset$ . In addition,  $a_0 : K_0 \rightarrow E_0$  is an isomorphism, so  $a_0$  induces a frame  $a_0(e_\emptyset)$  of  $E_0$ . In order to simplify the notation, we have identified  $\mathcal{O} \xrightarrow{\cong} E_0$  through the map,  $f \mapsto f a_0(e_\emptyset)$ , so that we write  $\varphi_1 Y_1 = 1$  instead of  $\varphi_1 Y_1 = a_0(e_\emptyset)$ .

In order to prove the division formula, we will also use the so-called *generalized Hefer forms* associated to a free resolution  $(E, \varphi)$  as introduced by Andersson in [1]. They consist of  $(k - \ell, 0)$ -form valued holomorphic morphisms  $H_k^\ell : E_k \rightarrow E_\ell$ , satisfying  $H_k^\ell = 0$  if  $k < \ell$ ,  $H_\ell^\ell = I_{E_\ell}$  and

$$(3.3) \quad \delta_\eta H_{k+1}^\ell = H_k^\ell \varphi_{k+1}(\zeta) - \varphi_{\ell+1}(z) H_{k+1}^{\ell+1}$$

for  $k > \ell$ .

In a similar way to in [18] and [19], we will show that (3.1) and the following integral representation formula lead to our sought after division formulas. Although similar formulas exist also when  $D$  is strongly pseudoconvex, [4, 7], for simplicity of the presentation, we stick to the case when  $D$  is convex, see for example [3, Chapter 4].

**THEOREM 3.4.** — *Let  $D \subseteq \mathbb{C}^n$  be a smooth convex domain with defining function  $\rho$ , and let  $Q := \partial \log(1/(-\rho))$  and*

$$P^{N,\ell}(\zeta, z) := \frac{1}{(\delta_\eta Q + 1)^{N+\ell}} (\bar{\partial} Q)^\ell,$$

which is holomorphic in  $z \in D$ , and for  $N \gg 1$ , it is smooth in  $\zeta \in \bar{D}$  and vanishes to arbitrarily high order (depending on  $N$ ) on  $\partial D$ . If  $h \in \mathcal{O}(\bar{D})$ , then

$$(3.4) \quad h(z) = c_{N,n} \int_D P^{N,n}(\zeta, z) h(\zeta), \quad \text{for } z \in D,$$

where  $c_{N,n} := \binom{N+n-1}{n}$ .

We now give the second proof of Theorem 1.1. More precisely, we have the following result, which implies Theorem 1.1.

**THEOREM 3.5.** — *Let  $D \subseteq \mathbb{C}^n$  be a smooth convex domain,  $(E, \varphi)$  a free resolution of  $\mathcal{O}/J$  for some ideal  $J$ , and assume that  $\text{rank } E_0 = 1$ . Let  $Y_1, \dots, Y_p$  be currents satisfying (3.1), and take  $P^{N,n}$  as in Theorem 3.4, where  $N \gg 1$  is such that  $P^{N,n}$  vanishes on  $\partial D$  to order higher than the order of  $Y_1, \dots, Y_p$  on  $\bar{D}$ , and let  $H$  be a generalized Hefer form for  $(E, \varphi)$ . If  $h \in \mathcal{O}(\bar{D})$ , then*

$$(3.5) \quad h(z) = \varphi_1(z)P(h)(z) + R(h)(z), \quad \text{for } z \in D,$$

where  $R(h)(z) =: R(z)$  is a holomorphic function given by

$$(3.6) \quad R(z) = c_p \int_D h(\zeta) P^{N,n-p}(\zeta, z) H_p^0 \bar{\partial} Y_p,$$

and  $P(h)(z) =: P(z)$  is given as

$$P(z) = P_0(z) + \dots + P_{p-1}(z),$$

where  $P_k(z)$  is a vector of holomorphic functions given by

$$(3.7) \quad P_k(z) = c_k \int_D h(\zeta) P^{N,n-k}(\zeta, z) H_{k+1}^1 Y_{k+1},$$

for suitably chosen constants  $c_0, \dots, c_p$ .

We recall that if  $(E, \varphi)$  is a free resolution of an ideal  $\mathcal{O}/J$  and  $\text{rank } E_0 = 1$ , then the entries of  $\varphi_1$  are generators of  $J$ , so the first term in the right-hand side of (3.5) belongs to  $J$ .

*Proof of Theorem 1.1.* — Since the kernel defining  $R(h)(z)$  is smooth in  $\zeta$  except for the term  $Y_p$ , and if we thus as in Lemma 3.2 take  $Y_p = a_p(e)X_p$ , we get by the division formula that  $\text{ann } a_p(e)\bar{\partial}X_p \subseteq J$ . Conversely, by the inclusion  $J \subseteq I : (I : J)$ , and Lemma 2.3, if  $h \in J$ , then  $a_p(e)h \in I$ , so  $a_p(e)h \in \text{ann } \bar{\partial}X_p$  by (1.4).  $\square$

*Remark 3.6.* — It might seem like we also for this proof use the theory of linkage, using Lemma 2.3, and the inclusion  $J \subseteq I : (I : J)$ . However, Lemma 2.3 is rather straight-forward homological algebra, and the inclusion  $J \subseteq I : (I : J)$  is trivial, while the real use of the theory of linkage in the previous section was the non-trivial inclusion  $I : (I : J) \subseteq J$ . In this section, we prove Theorem 1.1 with the help of integral formulas, instead of using this inclusion. We then note that Theorem 1.1 indeed implies this inclusion. By (1.4) and Lemma 2.3, we get that  $ha_p(e)\bar{\partial}X_p = 0$  is equivalent to that  $h \in I : (I : J)$ . By Theorem 1.1, we thus get that  $h \in J$ , proving the desired inclusion.

*Proof of Theorem 3.5.* — We define for  $0 \leq k \leq p - 1$ ,

$$(3.8) \quad R_k(z) := c_k \int h(\zeta) P^{N,n-k}(\zeta, z) H_k^0 \varphi_{k+1}(\zeta) Y_{k+1},$$

where the constants  $c_k$  are the same as the constants  $c_k$  in (3.7), and these constants will be determined below. For  $k = p$ , we let  $R_p(z) := R(z)$ , where  $R(z)$  is given by (3.6).

We start by defining  $c_0 := C_{N,n}$ , where  $C_{N,n}$  is the constant in (3.4). Since  $H_0^0 = \text{Id}_{E_0}$ , and  $\varphi_1 Y_1 = 1$ , we then get by (3.4) that

$$(3.9) \quad h(z) = R_0(z) = c_0 \int h(\zeta) P^{N,n}(\zeta, z) \varphi_1(\zeta) Y_1.$$

Having chosen  $c_0$ , the proof then proceeds by induction, by proving that for  $0 \leq k < p$  we can choose  $c_{k+1}$  such that

$$(3.10) \quad R_k(z) = \varphi_1(z) P_k(z) + R_{k+1}(z),$$

where  $P_k$  is given by (3.7). To see this, we note first that by using (3.3) on the term  $H_k^0 \varphi_{k+1}(\zeta)$  in (3.8), we get that

$$(3.11) \quad R_k(z) = \varphi_1(z) P_k(z) + c_k \int h(\zeta) P^{N,n-k}(\zeta, z) \delta_\eta H_{k+1}^0 Y_{k+1}.$$

By Lemma 3.1, we then obtain since  $H_{k+1}^0$  is a row of holomorphic  $(k + 1)$ -forms that

$$\begin{aligned} \bar{\partial} P^{N,n-k-1}(\zeta, z) \wedge H_{k+1}^0 &= -\frac{(N + n - k - 1)}{(\delta_\eta Q + 1)^{N+n-k}} (\bar{\partial} \delta_\eta Q) \wedge (\bar{\partial} Q)^{n-k-1} \wedge H_{k+1}^0 \\ &= \frac{C_k}{(\delta_\eta Q + 1)^{N+n-k}} (\bar{\partial} Q)^{n-k} \wedge \delta_\eta H_{k+1}^0 \\ &= C_k P^{N,n-k}(\zeta, z) \wedge \delta_\eta H_{k+1}^0, \end{aligned}$$

where  $C_k = \frac{-(N+n-k-1)}{(n-k)}$ . Inserting this in (3.11), we get that

$$(3.12) \quad R_k(z) = \varphi_1(z) P_k(z) + (c_k/C_k) \int h(\zeta) \bar{\partial} P^{N,n-k-1}(\zeta, z) \wedge H_{k+1}^0 Y_{k+1}.$$

If  $\Psi$  is a smooth  $(k + 1, 0)$  form on  $\bar{D}$ , then by extending  $\Psi \wedge P^{N,n-k-1}$  by 0 outside of  $\bar{D}$ , by the choice of  $N$ , this extension is a form which is differentiable to a higher order than the order of  $Y_{k+1}$  for  $0 \leq k < p$ . Thus, we can consider the extension of  $\Psi \wedge P^{N,n-k-1}$  as a test form of bidegree  $(n, n - k - 1)$ , and we thus get by definition of  $\bar{\partial} Y_{k+1}$  that

$$(3.13) \quad \int_{\bar{D}} \Psi \wedge P^{N,n-k-1} \wedge \bar{\partial} Y_{k+1} = \pm \int_{\bar{D}} \bar{\partial}(\Psi \wedge P^{N,n-k-1}) \wedge Y_{k+1}.$$

Since  $h$  and  $H_{k+1}^0$  are holomorphic, we get by applying (3.13) to the last term in (3.12) that

$$(3.14) \quad R_k(z) = \varphi_1(z)P_k(z) \pm (c_k/C_k) \int h(\zeta)P^{N,n-k-1}(\zeta, z) \wedge H_{k+1}^0 \bar{\partial}Y_{k+1},$$

for  $0 \leq k < p$ . If we then let  $c_{k+1} := \pm c_k/C_k$ , and for  $0 \leq k \leq p - 2$  use that  $\bar{\partial}Y_{k+1} = \varphi_{k+2}(\zeta)Y_{k+2}$  by (3.2), we obtain that the right-most term in (3.14) equals  $R_{k+1}(z)$ . For  $k = p - 1$ , we see directly from (3.6) that the right-most term equals  $R(z) = R_p(z)$ . We have thus proven that (3.10) holds for  $k = 0, \dots, p - 1$ .

To conclude, starting with (3.9), then using (3.10) repeatedly for  $k = 0, \dots, p - 1$ , and finally that  $R_p(z) = R(z)$ , we obtain (3.5).  $\square$

We finally also remark that indeed, using the framework of integral formulas of Andersson, as in [1], it follows from (3.1) that one has a division formula

$$h(z) = \varphi_1(z) \sum_k \int H_k^1 Y_k(\zeta) h(\zeta) \wedge g_{n-k} + \int H_p^0 \bar{\partial}Y_p(\zeta) h(\zeta) \wedge g_{n-p},$$

where  $g$  is a weight with compact support as in [2, Section 5]. Here we have preferred to give a more direct proof based on the basic Theorem 3.4, avoiding the need to use this full machinery.

*Proof of Lemma 3.1.* — We note first that since  $\delta_\eta$  is an anti-derivation,  $\delta_\eta \bar{\partial} = -\bar{\partial} \delta_\eta$ . In addition, for degree-reasons,  $Q \wedge (\bar{\partial}Q)^{n-k-1} \wedge H = 0$ . Thus,

$$0 = \delta_\eta \bar{\partial}(Q \wedge (\bar{\partial}Q)^{n-k-1} \wedge H) = -\bar{\partial} \delta_\eta(Q \wedge (\bar{\partial}Q)^{n-k-1} \wedge H).$$

Hence, since  $\delta_\eta$  is an anti-derivation, and  $Q$  and  $\bar{\partial}Q$  are of odd and even degree respectively, and  $(\bar{\partial}Q)^{n-k-1} \wedge H$  is  $\bar{\partial}$ -closed,

$$(\bar{\partial} \delta_\eta Q) \wedge (\bar{\partial}Q)^{n-k-1} \wedge H - \bar{\partial}(Q \wedge \delta_\eta(\bar{\partial}Q)^{n-k-1} \wedge H) = (\bar{\partial}Q)^{n-k} \wedge \delta_\eta H.$$

It then only remains to see that

$$(3.15) \quad -\bar{\partial}(Q \wedge \delta_\eta(\bar{\partial}Q)^{n-k-1} \wedge H) = (n - k - 1) \bar{\partial} \delta_\eta Q \wedge (\bar{\partial}Q)^{n-k-1} \wedge H.$$

To see this, we first note that since  $\bar{\partial}Q$  has even degree,

$$\bar{\partial}(Q \wedge \delta_\eta(\bar{\partial}Q)^{n-k-1} \wedge H) = (n - k - 1) \bar{\partial}(Q \wedge (\delta_\eta \bar{\partial}Q) \wedge (\bar{\partial}Q)^{n-k-2} \wedge H)$$

In addition,  $\delta_\eta \bar{\partial}Q = -\bar{\partial} \delta_\eta Q$ , which is  $\bar{\partial}$ -closed, so

$$-\bar{\partial}(Q \wedge (\delta_\eta \bar{\partial}Q) \wedge (\bar{\partial}Q)^{n-k-2} \wedge H) = \bar{\partial}Q \wedge \bar{\partial} \delta_\eta Q \wedge (\bar{\partial}Q)^{n-k-2} \wedge H,$$

which gives (3.15).  $\square$

*Proof of Lemma 3.2.* — To prove that  $Y$  satisfies (3.2), we note first that since  $\varphi_1 a_1 = \psi_1$  (where we identify  $E_0$  and  $K_0$  with  $\mathcal{O}$  as in Remark 3.3),

$$\varphi_1 Y_1 = \varphi_1 a_1 (e_1 \wedge X_1) = \psi_1 e_1 X_1 = f_1 X_1 = 1.$$

For  $2 \leq k \leq p$ , we get that

$$\begin{aligned} \varphi_k Y_k &= a_{k-1}(\psi_k(e_1 \wedge \cdots \wedge e_k)) \wedge X_k \\ &= \sum_{j=1}^k (-1)^{j-1} f_j a_{k-1}(e_1 \wedge \cdots \wedge \widehat{e}_j \wedge \cdots \wedge e_k) \wedge X_k \\ &= (-1)^{k-1} a_{k-1}(e_1 \wedge \cdots \wedge e_{k-1}) \wedge f_k X_k \\ &= (-1)^{k-1} a_{k-1}(e_1 \wedge \cdots \wedge e_{k-1}) \wedge \bar{\partial} X_{k-1} = \bar{\partial} Y_{k-1}, \end{aligned}$$

where all the other terms in the sum vanish since  $f_j X_k = 0$  for  $j < k$ , and the sign in the last equality is due to the superstructure, cf. for example [14, Section 2.1], since  $a_{k-1}(e_1 \wedge \cdots \wedge e_{k-1})$  has degree  $k - 1$ .  $\square$

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