From Monge–Ampère equations to envelopes and geodesic rays in the zero temperature limit

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Robert J. Berman[1]

Abstract Let (X, θ) be a compact complex manifold X equipped with a smooth (but not necessarily positive) closed $(1, 1)$-form θ. By a well-known envelope construction this data determines, in the case when the cohomology class $[\theta]$ is pseudoeffective, a canonical θ-psh function u_θ. When the class $[\theta]$ is Kähler we introduce a family u_β of regularizations of u_θ, parametrized by a large positive number β, where u_β is defined as the unique smooth solution of a complex Monge–Ampère equation of Aubin–Yau type. It is shown that, as $\beta \to \infty$, the functions u_β converge to the envelope u_θ uniformly on X in the Hölder space $C^{1,\alpha}(X)$ for any $\alpha \in\,]0, 1[$ (which is optimal in terms of Hölder exponents). A generalization of this result to the case of a nef and big cohomology class is also obtained and a weaker version of the result is obtained for big cohomology classes. The proofs of the convergence results do not assume any a priori regularity of u_θ. Applications to the regularization of ω-psh functions and geodesic rays in the closure of the space of Kähler metrics are given. As briefly explained there is a statistical mechanical motivation for this regularization procedure, where β appears as the inverse temperature. This point of view also leads to an interpretation of u_β as a "transcendental" Bergman metric.

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1 Introduction

Let X be a compact complex manifold equipped with a smooth closed $(1, 1)$-form θ on X and denote by $[\theta]$ the corresponding class in the Bott–Chern cohomology group $H^{1,1}(X, \mathbb{R})$. There is a range of positivity notions for such cohomology classes, generalizing the classical positivity notions in algebraic geometry. The algebro-geometric situation concerns the special case when X is projective variety and the cohomology class in question has integral periods, which equivalently means that the class may be realized as the first Chern class $c_1(L)$ of a line bundle L over X [25–27]. Accordingly, general cohomology classes in $H^{1,1}(X, \mathbb{R})$ are some times referred to as *transcendental* classes and the corresponding notions of positivity may be formulated in terms of the convex subspace of positive currents in the cohomology class—the strongest notion of positivity is that of a *Kähler class,* which means that the class contains a Kähler metric, i.e. a smooth positive form (see [27] for equivalent numerical characterizations of positivity). In general, once the reference element θ in the cohomology class in question has been fixed the subspace of positive forms may be identified (mod $\mathbb{R}$) with the space $PSH(X, \theta)$ of all θ-*plurisubharmonic* function (θ-psh, for short), i.e. all integrable strongly upper semi-continuous functions u on X such that

$$\theta + dd^c u \geq 0, \quad dd^c := i\partial\bar{\partial}$$

holds in the sense of currents (in the integral case the space $PSH(X, \theta)$ may be identified with the space of all singular positively curved metrics on the corresponding line bundle L). When the class $[\theta]$ is pseudo-effective, i.e. it contains a positive current, there is a canonical element in $PSH(X, \theta)$ defined as the following envelope*:*

$$u_\theta(x) := \sup\{u(x) : \ u \leq 0, \ u \in PSH(X, \theta)\},$$

defining a θ-plurisubharmonic function with minimal singularities in the sense of Demailly [18,25].

In this paper we introduce a natural family of regularizations u_β of the envelope u_θ, indexed by a positive real parameter β, where u_β is determined by an auxiliary choice of volume form dV; the functions u_β will be defined as solutions to certain complex Monge–Ampère equations, parametrized by β. Several motivations for studying the functions u_β and their asymptotics as $\beta \to \infty$, will be given below. For the moment we just mention that u_β can, in a certain sense, be considered as a "transcendental" analog of the Bergman metric for a high power of a line bundle L over X and moreover from a statistical mechanical point of view the limit $\beta \to 0$ appears as a zero-temperature limit.

In order to introduce the precise setting and the main results we start with the simplest case of a *Kähler class* $[\theta]$. First note that the envelope construction above can be seen as a generalization of the process of replacing the graph of a given smooth functions with its convex hull. By this analogy it is already clear from the one-dimensional case that u_θ will almost never by C^2-smooth even if the class $[\theta]$ is Kähler (unless θ is semi-positive, so that $u_\theta = 0$).

Fixing a volume form dV we consider, for β a fixed positive number, the following complex Monge–Ampère equations for a smooth function u_β :

$$(\theta + dd^c u_\beta)^n = e^{\beta u_\beta} dV \tag{1.1}$$

By the seminal results of Aubin [1] and Yau [64] there exists indeed a unique smooth solution u_β to the previous equation. In fact, any smooth solution is automatically θ-psh and the form $\omega_\beta := \theta + dd^c u_\beta$ defines a Kähler metric in $[\theta]$.

Theorem 1.1 *Let θ be a smooth $(1, 1)$-form on a compact complex manifold X such that $[\theta]$ is a Kähler class. Denote by u_θ the corresponding θ-psh envelope and by u_β the unique smooth solution of the complex Monge–Ampère equations* 1.1 *determined by θ and a fixed volume form dV on X. Then, as $\beta \to \infty$, the functions u_β converge to u_θ in $\mathcal{C}^{1,\alpha}(X)$ for any $\alpha \in]0, 1[$, with a uniform bound on $dd^c u_\beta$.*

In particular, the previous theorem yields a new direct PDE proof of the Laplacian bound on u_θ in [14] in the case of a Kähler class, with a rather explicit geometrical control on the bound. More generally, the proof reveals that the result remains valid if dV is replaced by any family dV_β of volume forms such that $dd^c(\log(dV_\beta/dV_1) = o(\beta)$. As a consequence the convergence result above admits the following geometric formulation: let ω_β be a family of Kähler metrics in $[\theta]$ satisfying the following twisted Kähler–Einstein equation:

$$\text{Ric } \omega_\beta = -\beta\omega_\beta + \beta\theta + o(\beta),$$

where Ric ω_β denotes the form representing the Ricci curvature of the Kähler metric ω_β and $o(\beta)$ denotes a family of forms on X such that $o(\beta)/\beta \to 0$ in the L^∞-sense as $\beta \to \infty$. Then the previous theorem says that ω_β is uniformly bounded and converges to $\theta + dd^c u_\theta$ in the sense of currents and the normalized potentials of ω_β converge in $\mathcal{C}^{1,\alpha}(X)$ to u_θ.

More generally, we will consider the case when the cohomology class $[\theta]$ is merely assumed to be *big;* this is the most general setting where complex Monge–Ampère equations of the form make sense [18]. The main new feature in this general setting is the presence of $-\infty$-singularities of all θ-psh functions on X. Such singularities are, in general, inevitable for cohomological reasons. Still, by the results in [18], the corresponding complex Monge–Ampère equations admit a unique θ-psh function u_β with minimal singularities; in particular its singularities can only appear along a certain complex subvariety of X, determined by the class $[\theta]$, whose complement is called the *Kähler locus* Ω of $[\theta]$ (or the *ample locus)* introduced in [17] (which in the algebro-geometric setting corresponds to the complement of the *augmented base locus* of the corresponding line bundle). Moreover, in the case when the class $[\theta]$ is also assumed to be *nef* the solution u_β is known to be smooth on Ω, as follows from the results in [18]. In this general setting our main result may be formulated as follows:

Theorem 1.2 *Let θ be a smooth $(1, 1)$-form on a compact complex manifold X such that $[\theta]$ is a big class. Then, as $\beta \to \infty$, the functions u_β converge to u_θ uniformly, in the sense that $\left\| u_\beta - u_\theta \right\|_{L^\infty Z(X)} \to 0$. Moreover, if the class $[\theta]$ is also assumed to be nef, then the convergence holds in $\mathcal{C}^{1,\alpha}_{loc}(\Omega)$ on the Kähler locus Ω of X.*

In particular, in the general setting of a big class the proof of the previous theorem yields an a priori L^∞-bound on the Monge–Ampere measure of u_θ :

$$(\theta + dd^c u_\theta)^n \leq 1_D \theta^n, \quad D = \{x \in X : u_\theta(x) = 0\} \tag{1.2}$$

Some further remarks are in order. First of all, as pointed out above, it was previously known that the norm $\left\| u_\beta - u_\theta \right\|_{L^\infty(X)}$ is finite for any fixed β (since u_β and the envelope u_θ both

have minimal singularities) and the thrust of the first statement in the previous theorem is thus that the norm in fact tends to zero. This global uniform convergence is considerably stronger than a a local uniform convergence on Ω. Secondly, it seems natural to expect that the local convergence on Ω in the previous theorem always holds in the $\mathcal{C}^{1,\alpha}_{loc}(\Omega)$-topology, regardless of the nef assumption. However, already the smoothness on Ω of solutions of complex Monge–Ampère equations of the form 1.1 is an open problem; in fact, it even seems to be unknown whether there always exists a θ-psh functions with minimal singularities, which is smooth on Ω. On the other hand, for special big classes $[\theta]$, namely those which admit an appropriate Zariski decomposition on some resolution of X, the regularity and convergence problem can be reduced to the nef case (in the line bundle case this situation appears if the corresponding section ring is finitely generated).

1.1 Degenerations induced by a divisor and applications to geodesic rays

In the case of a Kähler class and when θ is positive, i.e. θ is Kähler form, it follows immediately from the definition that $u_\theta = 0$ and in this case the convergence in Theorem 1.1 holds in the $\mathcal{C}^\infty$-sense, as recently shown in [36] using a completely different proof. However, as shown in [45,47] in the integral case $[\omega] = c_1(L)$, a non-trivial variant of the previous envelopes naturally appear in the geometric context of *test configurations* for the polarized manifold (X, L), i.e. $\mathbb{C}^*$-equivariant polarized deformations $(\mathcal{X}, \mathcal{L})$ of (X, L) and they can be used to construct (weak) *geodesic rays* in the space of all Kähler metrics in $[\omega]$. Such test configurations were introduced by Donaldson in his algebro-geometric definition of K-stability of a polarized manifold (X, L), which according to the the Yau–Tian–Donaldson is equivalent to the existence of a Kähler metric in the class $c_1(L)$ with constant scalar curvature. Briefly, K-stability of (X, L) amounts to the positivity of the Donaldson–Futaki invariants for all test configurations, which in turn is closely related to the large time asymptotics of Mabuchi's K-energy functional along the corresponding geodesic rays (see [42] and references therein).

Let us briefly explain how this fits into the present setup in the special case of the test configurations defined by the deformation to the normal cone of a divisor Z in X (e.g. a smooth complex hypersurface in X). First we consider the following complex Monge–Ampère equations degenerating along the divisor Z,

$$(\omega - \lambda\theta_L + dd^c u)^n = e^{\beta u} \|s\|^{2\lambda\beta} dV,$$

where we have realized Z as the zero-locus of a holomorphic section s of a line bundle L over X equipped with a fixed Hermitian metric $\|\cdot\|$ with curvature form θ_L and where $\lambda \in [0, \infty[$ is an additional fixed parameter. As is well-known, for λ sufficiently small ($\lambda \leq \epsilon$) there is, for any $\beta > 0$, a unique continuous $\omega - \lambda\theta_L$-psh solution $u_{\beta,\lambda}$ to the previous equation, which is smooth on $X - Z$. We will show that, when $\beta \to \infty$, the solutions $u_{\beta,\lambda}$ converge in $\mathcal{C}^{1,\alpha}(X)$ to a variant of the envelope u_θ, that we will (abusing notation slightly) denote by u_λ:

$$u_\lambda(x) := \sup\{u(x) : \ u \leq -\lambda \log \|s\|^2 \ \ u \in PSH(X, \omega - \lambda\theta_L)\}$$

(see Sect. 4). It may identified with the envelopes with prescribed singularities introduced in [2] in the context of Bergman kernel asymptotics for holomorphic sections vanishing to high order along a given divisor (see [45] for detailed regularity results for such envelopes and the relations to Hele–Shaw type flows and [56] for related asymptotic results in the toric case).

Remarkably, as shown in [45,47] (in the line bundle case) taking the Legendre transform of the envelopes $u_\lambda + \lambda \log \|s\|^2$ with respect to λ produces a geodesic ray in the closure of the space of Kähler potentials in $[\omega]$, which coincides with the $C^{1,\alpha}$-geodesic constructed

by Phong-Sturm [40,41] (in general, the geodesics are not C^2-smooth). Here, building on [45,47], we show that the logarithm of the Laplace transform, with respect to λ, of the Monge–Ampère measures of the envelopes u_λ defines a family of subgeodesics in the space of Kähler potentials converging to the corresponding geodesic ray (see Corollary 5.3). In geometric terms the result may be formulated as follows

Corollary 1.3 *Let ω be a Kähler form, and fix a constant c such that $[\omega] - c[Z]$ is a Kähler class. Let $\omega_{\beta,\lambda}$ be a family of currents in $[\omega] - \lambda[Z]$, defining smooth Kähler metrics away from the support of Z and satisfying*

$$\operatorname{Ric} \omega_{\beta,\lambda} = -\beta\omega_{\beta,\lambda} + \beta(\omega - \lambda[Z]) + o(\beta)$$

Then

$$\varphi_\beta^t := \frac{1}{\beta} \log \int_{[0,c]} d\lambda e^{\beta(\lambda-c)t} \frac{\omega_{\beta,\lambda}^n}{\omega^n}$$

defines a family of subgeodesics converging in $C^0(X \times [0, T])$, for any fixed $T > 0$, to a geodesic ray φ^t associated to the test configuration $(\mathcal{X}, \mathcal{L}_c)$ defined by the deformation to the normal cone of Z. Moreover, in the case when $[\omega] \in H^2(X, \mathbb{Q})$ the convergence holds in $C^0(X \times [0, \infty])$.

This can be seen as a "transcendental" analogue of the approximation result of Phong-Sturm [44], which uses Bergman geodesic rays. However, while the latter convergence result holds point-wise almost everywhere and for t fixed, an important feature of the convergence in the previous corollary is that it is *uniform,* even when t ranges in all of $[0, \infty[$. More generally, we will establish an extension of the previous result to the case when $[\omega] - c[Z]$ (or equivalently $\mathcal{L}_c$) is merely assumed big.

The motivation for considering this "transcendental" approximation scheme for geodesic rays is two-fold. First, as is well-known, recent examples indicate that a more "transcenden-tal" notion of K-stability is needed for the validity of the Yau–Tian–Donaldson conjecture, obtained by relaxing the notion of a test configuration. One such notion, called *analytic test configurations,* was introduced in [47] and as shown in op. cit. any such test configuration determines a weak geodesic ray, which a priori has very low regularity. However, the approx-imation scheme above could be used to regularize the latter weak geodesic rays, which opens the door for defining a notion of generalized Donaldson–Futaki invariant by studying the large time asymptotics of the K-energy functional along the corresponding regularizations (as in the Bergman metrics approach in [44]). In another direction, the approximation scheme above should be useful when considering the analog of K-stability for a non-integral Kähler class $[\omega]$ (compare Sect. 5). The previous corollary is just a first illustration of this approxi-mation scheme and we leave the development of more general approximation results for the future.

On the proofs

Next, let us briefly discuss the proofs of the previous theorems, starting with the case of a Kähler class. First, the weak convergence of u_β towards u_θ (i.e. convergence in $L^1(X)$) is proved using variational arguments (building on [12]). In fact, we will give two different proofs of this convergence, where the first one is variational and has two merits: (1) it generalizes directly to the case of a big class and (2) it applies when dV is replaced with a

quite singular measure μ_0 (satifying a Bernstein–Markov property). The second proof uses a direct simple maximum principle argument.

In either way, to conclude the proof of Theorem 1.1 we just have to provide a priori estimates on u_β, which are uniform in β and which we deduce from Siu's variant of the Aubin–Yau Laplacian estimates. In particular, this implies convergence in $L^\infty(X)$. However, in the case of a general big class, in order to establish the global L^∞-convergence, we need to take full advantage of the variational argument, namely that the argument shows that u_β converges to u_θ *in energy* and not only in $L^1(X)$. This allows us to invoke the L^∞-stability results in [32]. Briefly, the point is that convergence in energy implies convergence in capacity, which together with an L^p-control on the corresponding Monge–Ampère measures opens the door for Kolodziej type L^∞-estimates. Moreover, a variant of the maximum principle argument used in the case of the Kähler class, based on the theory of viscosity subsolutions developed in [29], yields the bound 1.2 (only the local case of the results in [29] is needed).

1.2 Further background and motivation

Before turning to the proofs of the results introduced above it may be illuminating to place the result into a geometric and probabilistic context (see also Sect. 3.1 for the relation to Bergman kernel asymptotics).

Kähler–Einstein metrics and the continuity method

First of all we recall that the main geometric motivation for studying complex Monge–Ampère equations of the form 1.1 comes from *Kähler–Einstein geometry* and goes back to the seminal works of Aubin [1] and Yau [64] in setting when X is a canonically polarized projective algebraic variety, i.e. the canonical line bundle $K_X := \Lambda^n T^*X$ of X is ample. If the form θ is taken as a Kähler metric ω on X in the first Chern class $c_1(K_X)$ of K_X and dV is chosen to be depend on ω in a suitable sense (i.e. $dV = e^{h_\omega}\omega^n$, where h_ω is the Ricci potential of ω), then the corresponding solution u_β of the Eq. 1.1 for $\beta = 1$ is the Kähler potential of a *Kähler–Einstein metric* ω_{KE} on X with negative Ricci curvature. Similarly, in the case of $\beta = -1$ the Eq. 1.1 corresponds to the Kähler–Einstein equation for a *positively* curved Kähler–Einstein equation in $c_1(-K_X)$ on a Fano manifold X. For a general value on the parameter β the equation appears in the continuity method for the Kähler–Einstein equation. Indeed, for $L = \pm K_X$ the Eq. 1.1 is equivalent to the following equation for ω_β in $c_1(L)$

$$\text{Ric } \omega_\beta = -\beta\omega_\beta + (\beta - \pm 1)\theta, \tag{1.3}$$

which, for β negative, is precisely Aubin's continuity equation for the Kähler–Einstein problem on a Fano manifold (when θ is taken as Kähler form in $c_1(\pm K_X)$). In the present setting, where $c_1(\pm K_X)$ is replaced by a general Kähler (or big) cohomology class $[\theta]$ there is no canonical volume form dV attached to θ and we thus need to work with a general volume form dV, but this only changes the previous equation with a term which is independent of β and which, as we show, becomes negligible as $\beta \to \infty$.

Interestingly, as observed in [50] the equation 1.3 can also be obtained from the *Ricci flow* via a backwards Euler discretization. Accordingly, the corresponding continuity path is called the Ricci continuity path in the recent paper [36], where it (or rather its "conical" generalization) plays a crucial role in the construction of Kähler–Einstein metrics with edge/cone singularities, by deforming the "trivial" solution $\omega_\beta = \theta$ at $\beta = \infty$ to a Kähler–Einstein metric at $\beta = \pm 1$. It should however be stressed that the main point of the present paper

is to study the case of a *non-positive* form θ which is thus different from the usual settings appearing in the context of Kähler–Einstein geometry and where, as we show, the limit as $\beta \to \infty$ is a canonical positive current associated to θ.

Cooling down: the zero temperature limit

In [5,8] a probabilistic approach to the construction of Kähler–Einstein metrics, was introduced, using certain β-deformations of determinantal point processes on X (which may be described in terms of "free fermions" [5]). The point is that if θ is the curvature form of a given Hermitian metric $\|\cdot\|$ on a, say ample, line bundle $L \to X$, then

$$\mu^{(N_k, \beta)} := \frac{\left\| (\det S^{(k)})(x_1, x_2, \ldots x_{N_k}) \right\|^{2\beta/k} dV^{\otimes N_k}}{Z_{k,\beta}} \tag{1.4}$$

defines a random point process on X, i.e. symmetric probability measure on the space X^{N_k} (modulo the permutation group) of configurations of N_k points on X, where N_k is dimension of the vector space $H^0(X, L^{\otimes k})$ of global holomorphic sections of $L^{\otimes k}$ and $\det S^{(k)}$ is any fixed generator in the top exterior power $\Lambda^{N_k} H^0(X, L^{\otimes k})$, identified with a holomorphic section of $(L^{\otimes k})^{\boxtimes N_k} \to X^{N_k}$.

From a statistical mechanical point of view the parameter β appears as the "thermodynamical β", i.e. $\beta = 1/T$ is the *inverse temperature* of the underlying statistical mechanical system and the complex Monge–Ampère equations above appear as the mean field type equations describing the macroscopic equilibrium state of the system at inverse temperature β. More precisely $\mu_\beta := MA(u_\beta)$ describes the expected macroscopic distribution of a single particle when k and (hence also the number of particles N_k) tends to infinity,

$$\int_{X^{N_{k-1}}} \mu^{(N_k, \beta)} \to \mu_\beta$$

A formal proof of this convergence was first outlined in [5] and then a rigorous proof was obtained in [8] (in fact, a much stronger convergence result holds, saying that the convergence towards μ_β holds exponentially in probability in the sense of large deviations with a rate functional which may be identified with the twisted K-energy functional). Anyway, here we only want to provide a statistical motivation for the large β-limit, which thus corresponds to the zero-temperature limit, where the system is slowly cooled down. From this point of view the convergence result in Theorem 1.1 can then be interpreted as a *second order phase transition* for the corresponding equilibrium measures μ_β. Briefly, the point is that while the support of μ_β is equal to all of X for any *finite* β the limiting measure $\mu_\infty (= MA(u_\theta))$ is supported on a proper subset S of X as soon as θ is not globally positive. The formation of a limiting ordered structure (here $MA(u_\theta)$ and its support S) in the zero-temperature limit is typical for second order phase transitions in the study of disordered systems. In fact, in many concrete examples the limiting support S is a domain with piece-wise smooth boundary, but it should be stressed that there are almost no general regularity results for the boundary of S (when $n > 1$). In the one-dimensional case of the Riemann sphere the support set S appears as the "droplet" familiar from the study of Coulomb gases and normal random matrices (see [34,55] and references therein).

Added in proof

It has been pointed out by experts that the proof of the main result in [14], saying that the Laplacian of u_θ is in L^∞_{loc} on the Kähler locus of X, is incomplete (further details need to

be added about how to obtain the estimate 1.8 in [14]). The authors intend to complement the proof given in [14] in the future, but in the case of a nef and big class the present paper provides a direct PDE proof of the regularity in question. In the case of a big, but non-nef class, the bound 1.2 is weaker than the bound in [14], but it appears to be adequate for all current complex geometric applications of envelopes as above, such as the recent proof of the duality between the pseudoeffective and the movable cone on a projective manifold in [62].

Since the first preprint version of the present paper appeared on ArXiv there has been a number of interesting developments that we briefly describe. In [24] it was shown that u_θ is Lipschitz continuous as soon as θ has a Lipschitz potential, using the regularizations u_β above and Blocki's gradient estimate (as a replacement of the Aubin–Yau–Siu inequality used in Proposition 2.6). Moreover, very recently the convergence result for u_β in the present paper was used in [22,54] to prove the $C^{1,1}$-regularity of u_θ (in the case of a Kähler class), by using the recent $C^{1,1}$-estimates in [21] as a replacement of the Aubin–Yau–Siu inequality. In another direction it was shown in [38] how to extend the C^0-convergence implicit in Theorem 1.1 to the setting of Hessian equations on Kähler manifolds, leading to a new global regularization result for (ω, m)-subharmonic functions (see Remark 3.5). Furthermore, very recently it was shown in [51] and [28], independently, that a transcendtal Kähler class containing a constant scalar curvature metric is K-semistable, in general, and K-stable [28] if the automorphism group is discrete, which thus establishes one direction of the generalized Yau–Tian–Donaldson conjecture discussed in Sect. 5.0.2. Moreover, solutions u_β of global complex Monge–Ampère equations as above and their relative positivity properties were used in [20] to give an alternative proof of Chen's conjecture concerning the convexity of the K-energy (recently established in [9]) with u_β replacing the local Bergman metric approximations used in [9], which thus reinforces the intepretation of u_β as a transcendtal Bergman metric discussed in Sect. 3.1. See also the very recent work [33] for applications to viscosity theory. Finally, a dynamical analog of Theorem 1.1, formulated in terms of the zero-temperature limit of the twisted Kähler–Ricci flow, is obtained in [15].

1.2.1 Organization

After having setup the general framework in Sect. 2 we go on to first prove the main result (Theorem 1.1) in the case of Kähler class (by two different proofs) and then its generalization to big classes (Theorem 1.2). The interpretation in terms of transcendental Bergman metrics is discussed in Sect. 3, together with applications to regularization of ω-psh functions. Then in Sect. 4 we consider the singular version of the previous setup which appears in the presence of a divisor Z on X. Finally, the results in the latter section are applied in Sect. 5 to the construction and regularization of geodesic rays and relations to the transcendtal generalization of the Yau–Tian–Donaldson conjecture are discussed.

2 From Monge–Ampère equations to θ-psh envelopes

We start with the general setup. Let X be a compact complex manifold equipped with a smooth closed $(1, 1)$-form θ and denote by $[\theta]$ the corresponding (Bott–Chern) cohomology class of currents:

$$[\theta] := \left\{ \theta + dd^c u : \ u \in L^1(X) \right\} \quad \left(dd^c := \frac{i}{2\pi} \partial \bar{\partial} \right)$$

We will denote by $PSH(X, \theta)$ the space of all θ-*plurisubharmonic* functions, which may be defined as the space of all functions u on X taking values in $]-\infty, \infty[$ such that $u \in L^1(X)$ and $\theta + dd^c u \geq 0$ holds in the sense of currents and such that u is strongly upper semi-continuous in the following sense: u is upper semi-continuous (usc) and for any $x \in X$ and null set N for the Lesbesgue measure there exists a sequence $x_j \in X - N$ such that $x_j \rightarrow x$ and $u(x_j) \rightarrow u(x)$ (the point is that the identity principle holds for strongly usc functions: if they coincide a.e. on X then they coincide everywhere). The previous definition of $PSH(X, \theta)$ is equivalent to the following more standard one: $u \in PSH(X, \theta)$ iff $u(z) + \phi_0(z)$ is plurisubharmonic, i.e. subharmonic along complex lines in $\mathbb{C}^n$, where z denotes given local holomorphic coordinates on X and ϕ_0 is a local potential for θ (i.e. $\theta = dd^c \phi_0$ locally).

We equip, as usual, the space $PSH(X, \theta)$ with its L^1-topology. The class $[\theta]$ is said to be *pseudo-effective* if $PSH(X, \theta)$ is non-empty. There is then a canonical element u_θ in the space $PSH(X, \theta)$ defined as the following envelope:

$$u_\theta(x) := \sup\{u(x) : \ u \leq 0, \ u \in PSH(X, \theta)\}, \tag{2.1}$$

Given a smooth θ-psh function u we will write

$$MA_\theta(u) := (\theta + dd^c u)^n$$

for the corresponding Monge–Ampère measure (often dropping the subindex θ from the notation). In the case when the class $[\theta]$ is big (see Sect. 2.2 below) the Monge–Ampère measure $MA_\theta(u)$ is defined for any $u \in PSH(X, \theta)$ as the non-pluripolar Monge–Ampère operator [18].

Given a volume form dV on X we will denote by u_β the unique θ-psh function with minimial singularities solving the following complex Monge–Ampère equation:

$$MA(u_\beta) = e^{\beta u_\beta} dV \tag{2.2}$$

(the existence and uniqueness of a solution is shown in [18]). In the case when $[\theta]$ is a *Kähler class,* i.e. $[\theta]$ contains a smooth and strictly positive form ω (i.e. a Kähler form) the solution u_β is smooth, by the Aubin–Yau theorem.[1]

2.0.2 An alternative formulation in the Kähler case

It may be worth pointing out that, in the Kähler case, the following equivalent formulation of the previous setup may be given, where the role of smooth form θ is played by a smooth function f. We start by fixing a Kähler form ω on X and consider the corresponding Kähler class $[\omega]$. We can then define a projection operator P_ω from $\mathcal{C}^\infty(X)$ to $PSH(X, \omega)$ by setting

$$(P_\omega f)(x) := \sup\{\varphi(x) : \ \varphi \leq f, \ \varphi \in PSH(X, \omega)\} \tag{2.3}$$

Setting $\theta := \omega + dd^c f$ we see that $u_\theta = P_\omega f - f$. Similarly, given a volume form dV on X we denote by $\varphi_\beta (:= P_\beta(f))$ the unique smooth solution to

$$(\omega + dd^c \varphi_\beta)^n = e^{\beta(\varphi_\beta - f)} dV \tag{2.4}$$

so that $u_\beta = \varphi_\beta - f$. One advantage of this new formulation is that it allows one to consider case where f has $+\infty$-singularities, leading to degeneracies in the rhs of the previous Monge–Ampère equation. In particular, this will allow us to consider a framework of complex Monge–Ampère equations degenerating along a fixed divisor Z in X. Interestingly, this

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[1] In the case of a Kähler class any solution u_β is automatically θ-psh, since by the minimum principle there exists some point where $\theta + dd^c u_\beta > 0$ and hence $\theta + dd^c u_\beta > 0$ on all of X, since $(\theta + dd^c u_\beta)^n > 0$ on all of X.

latter framework can, from the analytic point view, be seen as a variant of the setting of a big class within a Kähler framework.

We will be interested in the limit when $\beta \to \infty$. In order to separate the different kind of analytical difficulties which appear in the case when $[\theta]$ is Kähler from those which appear in the general case when $[\theta]$ is big, we will start with the Kähler case, even though it can be seen as a special case of the latter.

2.1 The case of a Kähler class (Proof of Theorem 1.1)

In this section we will assume that $[\theta]$ is a Kähler class, i.e. there exists some smooth function $v \in PSH(X, \theta)$ such that $\omega := \theta + dd^c v > 0$, i.e. ω is a Kähler form.

2.1.1 Convergence in energy

For a given smooth function u we will write

$$\mathcal{E}(u) := \frac{1}{n+1} \int_X \sum_{j=0}^n u(\theta + dd^c u)^j \wedge \theta^{n-j} \tag{2.5}$$

More generally, the functional $\mathcal{E}(u)$ extends uniquely to the space $PSH(X, \theta)$, by demanding that it be increasing and (strongly) usc [12]. Following [12] we will say that a sequence u_j in $PSH(X, \theta)$ converges to u *in energy* if $u_j \to u$ in $L^1(X)$ and $\mathcal{E}(u_j) \to \mathcal{E}(u)$.

We recall that the functional $\mathcal{E}$ restricted to the convex space $PSH(X, \theta) \cap L^\infty(X)$ (or more generally, to the *finite energy space* $\{\mathcal{E}^1(X, \theta) > -\infty\}$) may be equivalently defined as a primitive for the Monge–Ampère operator, viewed as a one-form on the latter space, in the sense that

$$d\mathcal{E}_{|u} = MA(u) \tag{2.6}$$

(i.e. $d\mathcal{E}(u + tv)/dt = \int MA(u)v$ at $t = 0$).

The next theorem concerns the following general setting: given a finite measure μ_0 on X we denote by u_β the solution to the Eq. 2.2 obtained by replacing dV with μ_0 (the existence of a solution with full Monge–Ampère mass is equivalent to μ_0 not charging pluripolar subsets of X). Following [11] the measure μ_0 is said to have the *Bernstein–Markov property wrt* $PSH(X, \theta)$ if for any positive constant ϵ there exists a constant C such that for any $u \in PSH(X, \theta)$

$$\sup_X e^{\beta u} \le e^C e^{\epsilon\beta} \int_X e^{\beta u} \mu_0 \tag{2.7}$$

In particular, any volume form dV has the Bernstein–Markov property wrt $PSH(X, \theta)$ (as follows from the local submean property of psh functions).

Theorem 2.1 *Let μ_0 be a finite measure on X not charging pluripolar subsets. Denote by u_β the solution to the complex Monge–Ampère equation determined by the data (θ, μ_0, β). If μ_0 has the Bernstein–Markov property wrt $PSH(X, \theta)$, then u_β converges to u_θ in energy.*

Proof Without loss of generality we may assume that the volume V of the class $[\theta]$ is equal to one (by a trivial scaling). Consider the following functional:

$$\mathcal{G}_\beta(u) := \mathcal{E}(u) - \mathcal{L}_\beta(u), \quad \mathcal{L}_\beta(u) := \frac{1}{\beta} \log \int_X e^{\beta u} \mu_0,$$

which is invariant under the additive action of $\mathbb{R}$. Its critical point equation is the "normalized" equation $MA(u) = e^{\beta u} \mu_0 / \int_X e^{\beta u} \mu_0$, whose unique sup-normalized solution is given by

$U_\beta := u_\beta - \sup_X u_\beta$, where, as before, u_β denotes the unique solution of the corresponding "non-normalized" equation. We will use that U_β is a maximizer of $\mathcal{G}_\beta$, as follows from a concavity argument [6,12].

Step 1 *Any L^1-limit point of the family U_β is a maximizer of the following functional on $PSH(X, \theta)$:*

$$\mathcal{G}_\infty(u) := \mathcal{E}(u) - \sup_X u$$

First observe that after a harmless normalization we may as well assume that μ_0 is a probability measure. Then $\mathcal{L}_\beta(u) \leq \sup_X u$, which means that $\mathcal{G}_\beta \geq \mathcal{G}_\infty$. Hence, for any fixed $v \in PSH(X, \theta)$ we have

$$\mathcal{G}_\beta(U_\beta) \geq \mathcal{G}_\beta(v) \geq \mathcal{G}_\infty(v). \tag{2.8}$$

By the compactness of $PSH(X, \theta) \cap \{\sup_X u = 0\}$ in $L^1(X)$ [31] the family U_β has a limit point $U_\infty \in PSH(X, \theta)$, where $U_\infty := \lim_{j \to \infty} U_{\beta_j}$ in the L^1-topology. Now fix $\epsilon > 0$. By the Bernstein–Markov property of μ_0 there exists a constant C such that

$$\mathcal{L}_\beta(U_\beta) \geq \sup_X U_\beta - C/\beta - \epsilon$$

and hence

$$\mathcal{G}_\beta(U_\beta) \leq \mathcal{G}_\infty(U_\beta) + C/\beta + \epsilon.$$

Finally, using that the functional $\mathcal{E}$ is usc on $PSH(X, \theta)$ and $\sup_X(\cdot)$ is continuous (see [11, Corollary 1.16] for a more general continuity result) it follows that

$$\limsup_{j \to \infty} \mathcal{G}_\beta(U_\beta) \leq \mathcal{G}_\infty(U_\infty) + \epsilon$$

which combined with the inequality 2.8 concludes the proof of the first step.

Step two: u_θ is the unique sup-normalized maximizer of $\mathcal{G}_\infty$

First note that u_θ maximizes $\mathcal{G}_\infty$ on $PSH(X, \theta)$. To see this first observe that u_θ is sup-normalized, i.e. $\sup_X u_\theta = 0$. Indeed, if $\sup_X u_\theta \leq -\delta \leq 0$ then $u_\theta + \delta \leq 0$ and hence $u_\theta \geq u_\theta + \delta$ (from the very definition of u_θ) forcing $\delta = 0$. But if $U \in PSH(X, \theta)$ is also sup normalized, then $u_\theta \geq U$ and hence $\mathcal{E}(u_\theta) \geq \mathcal{E}(U)$, since $\mathcal{E}$ is increasing on $PSH(X, \theta)$, showing that u_θ is a maximizer of $\mathcal{G}_\infty$. The proof of Step two is then concluded by using the following fact: if u and v are two elements in $PSH(X, \theta)$ of finite energy such that $\mathcal{E}(v) = \mathcal{E}(u)$ and $v \leq u$, then $v = u$. Indeed, by the standard cocycle property of $\mathcal{E}$,

$$\mathcal{E}(v) - \mathcal{E}(u) = \frac{1}{n+1} \sum_{j=0}^{n} \int (v - u)(\theta + dd^c v)^{n-j} \wedge (\theta + dd^c v)^j.$$

As a consequence, if $v \leq u$ then all terms above have to vanish. In particular the term with $j = n$ vanishes, which means that $u \leq v$ a.e. wrt $MA(v)$. But, then $v \leq u$ on all of X, by the domination principle in the class $\mathcal{E}^1(X, \theta)$ (the domination principle in the finite energy space $\mathcal{E}^1(X, \theta)$ holds in the general setting of a big class; see [4, Remark 3.14]).

Finally, by the Bernstein–Markov property we have that $\lim_{\beta \to \infty} \mathcal{L}_\beta(U_\beta) = \lim_{\beta \to \infty} \sup(U_\beta) = 0$ and hence u_β also converges to u_θ in $L^1(X)$. Moreover, by Step one, we have $\mathcal{E}(u_\beta) \to \mathcal{E}(u_\theta)$, which concludes the proof of the theorem. $\qquad\square$

Remark 2.2 The present definition of the Bernstein–Markov property is the natural "transcendental" generalization of the definition used in [11, Definition 1.9], which concerns the case when $[\theta] = c_1(L)$ for a big line bundle L. More generally, as in [11, Definition 1.9]

one can consider the setting where a compact subset K of X has been fixed and say that a measure μ_0 supported on K has the *Bernstein–Markov property wrt $PSH(X, \theta)$ for K* if the inequality 2.7 holds when X has been replaced with K. Repeating the proof in the previous theorem then shows that if the latter Bernstein–Markov property holds, then u_β converges to $u_{\theta, K}$ defined as in formula 2.1 (with X replaced by K) under the condition that $u_{\theta, K}$ be continuous (i.e. (K, θ) is regular in the sense of [11]).

In the case when $[\theta]$ is a Kähler class we will only need the L^1-convergence implicit in the previous theorem. But it should be stressed that when we move on to the case of a big class the convergence in energy will be crucial in order to establish the convergence in L^∞-norms.

2.1.2 A direct proof using the maximum principle when μ_0 is a volume form

Next we show how to give an alternative direct proof of the convergence of u_β towards u_θ (in the case of a given volume form dV) by exploting that u_β is smooth, by the Aubin–Yau theorem. It gives a quantitative L^∞-convergence.

Proposition 2.3 *Let $[\theta]$ be a Kähler class and dV a volume form on X. Then the correspondng smooth solution u_β of Eq. 2.2 satisfies*

$$\sup_X |u_\beta - u_\theta| \leq \frac{A \log \beta}{\beta},$$

where the constant A only depends on an upper bound on $|\theta^n / \omega^n|$.

Proof Since the solution u_β is smooth and $dd^c u_\beta \geq 0$ at a point x_0 where the maximum of u_β is attained, Eq. 2.2 implies the uniform a priori estimate

$$u_\beta \leq C/\beta, \quad C := \sup_X \log\left(\left(\frac{\theta^n}{\omega^n}\right)_+\right), \quad a_+ := \max\{0, a\}.$$

Hence, $u_\beta - C/\beta \leq u'_\theta$ where u'_θ is defined as u_θ, but with the sup taken over the subspace of all θ-psh functions $u \leq 0$ which are *smooth*. Conversely, fixing a smooth and strictly θ-psh function v and positive numbers ϵ and δ we consider a candidate u for the sup defining u'_θ and set $u_{\epsilon, \delta} := (1 - \epsilon)u + \epsilon v - \delta$. Then

$$(\theta + dd^c u_{\epsilon, \delta})^n \geq e^{\beta u_{\epsilon, \delta}} dV, \tag{2.9}$$

as long as $e^{-\delta\beta} \leq C\epsilon^n$, for a constant C only depending on the volume form dV (and the fixed element v). In particular, the previous inequality holds for $\epsilon = 1/\beta$ and $\delta = \frac{C'}{\beta} \log \beta$ for C' suffcently large. But then, comparing the inequality 2.9 and the defining Eq. 2.2, it follows from the maximum principle that $u_{\epsilon, \delta} \leq u_\beta$ (see Lemma 2.4). All in all this means that

$$u_\beta - C/\beta \leq u'_\theta \leq \frac{1}{(1 - 1/\beta)} u_\beta + \frac{C'}{\beta} \log \beta,$$

and hence the proof is concluded by the observation that $u'_\theta = u_\theta$, which is an immediate consequence of Demailly's regularization theorem. In fact, it is not necessary to invoke the latter regularization result as the argument above leads to a new PDE proof of it, as explained in Sect. 3. $\qquad\square$

2.1.3 L^∞-estimates

We start with the following well-known

Lemma 2.4 *Assume that u and v are (say, bounded) θ-psh functions such that $MA(v) \geq e^{\beta v}dV$ and $MA(u) \leq e^{\beta u}dV$. Then $v \leq u$.*

Proof In the smooth case this follows immediately from the maximum principle and in the general case we can apply the comparison principle (which, by [18, Corollary 2.3], holds in the general setting of a big class considered below). Indeed, according to the comparison principle $\int_{\{u \leq v\}} MA(v) \leq \int_{\{u \leq v\}} MA(u)$ and hence $\int_{\{u \leq v\}} e^{\beta v}dV \leq \int_{\{u \leq v\}} e^{\beta u}dV$. But then it must be that $v \leq u$ a.e. on X and hence everywhere.   □

The previous lemma allows us to construct "barriers" to show that u_β is uniformly bounded:

Lemma 2.5 *Given $\beta_0 > 0$ there exists a constant C such that $\sup_X |u_\beta| \leq C$ when $\beta \geq \beta_0$.*

Proof Take β such that $\beta \geq \beta_0$ (the given positive number). Let us start with the proof of the lower bound on u_β. Since $[\theta]$ is a Kähler class there is a smooth θ-psh function v such that $MA(v) \geq e^{-A}dV$ for some constant A. After shifting v by a constant we may assume that $v \leq -A/\beta_0 \leq -A/\beta$. But then $MA(v) \geq e^{-A}dV \geq e^{\beta v}$ and hence by the previous lemma $v \leq u_\beta$ which concludes the proof of the lower bound. Similarly, taking v to be a smooth θ-psh function v such that $MA(v) \leq e^{A}dV$ and shifting v so that $A/\beta_0 \leq v$ proves that $u_\beta \leq v$, which concludes the proof of the lemma.   □

2.1.4 The Laplacian estimate

Next we will establish the following key Laplacian estimate:

Proposition 2.6 *Fix a Kähler form ω in $[\theta]$. Then there exists a constant C such that, for $\beta \geq \beta_0$,*

$$-C \leq \Delta_\omega u_\beta \leq C$$

Proof The lower bound follows immediately from $\theta + dd^c u_\beta \geq 0$. To prove the upper bound we first recall the following variant of the Aubin–Yau Laplacian estimate in this context due to Siu (compare [57, p. 99] and [19, Lemma 2.2]): given two Kähler forms ω' and ω such that $\omega'^n = e^f \omega^n$ we have that

$$\Delta_{\omega'} \log tr_\omega \omega' \geq \frac{\Delta_\omega f}{tr_\omega \omega'} - Btr_{\omega'}\omega,$$

where the constant B is proportional to the infimum of the holomorphic bisectional curvatures of ω. Fixing $\beta > 0$ and setting $\omega' := \theta + dd^c u$ for $u := u_\beta$ we have, by the MA-equation for u_β, that $f = \beta u + \log(dV/\omega^n)$ and hence

$$Btr_{\omega'}\omega + \Delta_{\omega'} \log tr_\omega \omega' \geq \beta \frac{\Delta_\omega(u + \beta^{-1}\log(dV/\omega^n))}{tr_\omega \omega'}$$

Next, we note that $\Delta_\omega u = tr_\omega \omega' - tr_\omega \theta$. Moreover, writing $\omega = \omega' - dd^c(u - v)$, where v is a smooth function such that

$$\omega = \theta + dd^c v, \tag{2.10}$$

also gives $tr_{\omega'}\omega = n - \Delta_{\omega'}(u-v)$. Accordingly, the previous inequality may be reformulated as follows:

$$nB + \Delta_{\omega'}(\log tr_\omega\omega' - B(u-v)) \geq \beta\frac{tr_\omega\omega' - tr_\omega\theta_\beta}{tr_\omega\omega'}, \quad \theta_\beta := \theta - \beta^{-1}dd^c\log(dV/\omega^n)$$

and hence, letting C be the sup of $tr_\omega\theta_\beta$,

$$(C\beta + nBtr_\omega\omega')e^{-B(u-v)} + \Delta_{\omega'}\log(tr_\omega\omega' - B(u-v))tr_\omega\omega'e^{-B(u-v)} \geq \beta tr_\omega\omega'e^{-B(u-v)} \tag{2.11}$$

Thus, setting $s := \sup_X e^{-B(u-v)}tr_\omega\omega'$ and taking the maximum over X in the previous inequality gives

$$\beta s \leq 0 + nBs + \beta\sup_X Ce^{-B(u-v)}$$

Since $u(:= u_\beta)$ is uniformly bounded in x (by the previous lemma) and and since v is bounded, it follows that $tr_\omega\omega'$ is uniformly bounded from above, as desired. More precisely, the previous argument gives the estimate

$$tr_\omega(\theta + dd^cu_\beta) \leq \frac{\sup_X(tr_\omega\theta_\beta)}{1 - nB/\beta}e^{B(u_\beta - v)}e^{-\inf_X B(u_\beta - v)} \tag{2.12}$$

when $\beta > nB$ $\square$

Remark 2.7 Note that, in general, the Ricci curvature of the Kähler forms $\omega_\beta := \theta + dd^cu_\beta$ is unbounded, both from above and below, as $\beta \to \infty$. Still, by the previous estimate, the Kähler forms ω_β are uniformly bounded from above. However it should be stressed that, unless $\theta > 0$, there is no uniform bound of the form $\omega_\beta \geq \delta\omega > 0$ as it will follow from Theorem 1.1 that $\omega_\beta^n \to 0$ on large portions of X (indeed, for β large, $\omega_\beta^n \leq Ce^{-\beta\epsilon}dV$ on the open set where $u_\theta < -2\epsilon$).

2.1.5 Proof of Theorem 1.1 using the variational approach

By Lemma 2.5 u_β is uniformly bounded and by the Laplacian estimate in Proposition 2.6 combined with Green's formula the gradients of u_β are uniformly bounded. Hence, it follows from basic compactness results that, after perhaps passing to a subsequence, u_β converges to a function u in $\mathcal{C}^{1,\alpha}(X)$ for any fixed $\alpha \in]0, 1[$. It will thus be enough to show that $u = u_\theta$ (since this will show that any limit point of $\{u_\beta\}$ is uniquely determined and coincides with u_θ). But this follows from either Theorem 2.1 or Proposition 2.3.

2.2 The case of a big class (proof of Theorem 1.2)

A (Bott–Chern) cohomology class $[\theta]$ in $H^{1,1}(X)$ is said to be *big,* if $[\theta]$ contains a Kähler current ω, i.e. a positive current ω such that that $\omega \geq \epsilon\omega_0$ for some positive number ϵ, where ω_0 is a fixed strictly positive form ω_0 on X. We also recall that a class $[\theta]$ is said to be *nef* if, for any $\epsilon > 0$, there exists a smooth form $\omega_\epsilon \in T$ such that $\omega_\epsilon \geq -\epsilon\omega_0$. To simplify the exposition we will assume that X is a Kähler manifold so that the form ω_0 may be chosen to be closed. Then the cone of all big classes in the cohomology group $H^{1,1}(X)$ may be defined as the interior of the cone of pseudo-effective classes and the cone of Kähler classes may be defined as the interior of the cone of nef classes.

We also recall that a function u in $PSH(X, \theta)$ is said to have *minimal singularities,* if for any $v \in PSH(X, \theta)$ the function $u - v$ is bounded from below on X. In particular, the

envelope u_θ has (by its very definition) minimal singularities (and this is in fact the standard construction of a function with minimal singularities). In the case when $[\theta]$ is big any function with minimal singularities is locally bounded on a Zariski open subset Ω, as a well-known consequence of Demailly's approximation results [26]. In fact, the subset Ω can be taken as the *Kähler (ample) locus* of $[\theta]$ defined in [17].

Example 2.8 Let Y be a singular algebraic variety in complex projective space $\P^N$ and ω a Kähler form on $\P^n$ (for example, ω could be taken as the Fubini-Study metric so that $[\omega_{|Y}]$ is the first Chern class of $\mathcal{O}_X(1)$). If now $X \to Y$ is a smooth resolution of Y, which can be taken to invertible over the regular locus of Y; then the pull-back of ω to X defines a class which is nef and big and such that its Kähler locus corresponds to the regular part of Y.

We will denote by MA the Monge–Ampère operator on $PSH(X, \theta)$ defined by replacing wedge products of smooth forms with the *non-pluripolar produc*t of positive currents introduced in [18]. The corresponding operator MA is usually referred to as the *non-pluripolar Monge–Ampère operator.* For example, if u has minimal singularities, then $MA(u) = 1_\Omega MA(u_{|\Omega})$ on the Kähler locus Ω, where $MA(u_{|\Omega})$ may be computed locally using the classical definition of Bedford–Taylor. We let V stand for the volume of the class $[\theta]$, which may be defined as the total mass of $MA(u)$ for any function u in $PSH(X, \theta)$ with minimal singularities. By [18] there exists a unique solution u_β to the Eq. 2.2 in $PSH(X, \theta)$ with minimal singularities. Moreover, by [18] the solution is smooth on the Kähler locus in the case when $[\theta]$ is nef and big (which is expected to be true also without the nef assumption; compare the discussion in [18]).

2.2.1 Convergence in energy

In the case of a big class one first defines, following [12], the following functional on the space of all functions in n $PSH(X, \theta)$ with minimal singularities:

$$\mathcal{E}(u) := \frac{1}{n+1} \int_X \sum_{j=0}^n (u - u_\theta)(\theta + dd^c u)^j \wedge (\theta + dd^c u_\theta)^{n-j} \qquad (2.13)$$

(the point is that we needs to subtract u_θ to make sure that the integral is finite). Equivalently, $\mathcal{E}$ may be defined as the primitive of the Monge–Ampère operator on the the space of all finite energy functions in $PSH(X, \theta)$, normalized so that $\mathcal{E}(u_\theta) = 0$. We then define convergence in energy as before.

Remark 2.9 Strictly speaking, in the case of a Kähler class the definition 2.13 of $\mathcal{E}$ only coincides with the previous one (formula 2.5) in the case when θ is semi-positive (since the definition in formula 2.5 corresponds to the normalization condition $\mathcal{E}(0) = 0$). But the point is that, in the Kähler case, different normalizations gives rise to functionals which only differ up to an overall additive constant and hence the choice of normalization does not effect the notion of convergence in energy.

The proof of Theorem 1.1 can now be repeated word for word to give the following

Proposition 2.10 *Suppose that θ is a smooth form such that the class $[\theta]$ is big. Then u_β converges to u_θ in energy.*

2.2.2 L^∞-estimates

We will also need the following upper bound on u_β :

Lemma 2.11 *There exists a constant C such that*

$$u_\beta \leq u_\theta + C/\beta$$

(the constant C may be taken as $\log\left((\theta^n/dV)_+\right)$, *where* $a_+ := \max\{0, a\}$*).*

Proof We recall that if u_β is smooth (as in the case of a Kähler class) then the inequality follows directly from the maximum principle. In the general case the inequality follows from the fact that u_β is a viscosity subsolution of the Eq. 2.2, as follows from the results in the first section of [29]. To explain this first assume that the maximum of u_β on X is achieved at a point x_0 in the Zariski open subset Ω (defined as the Kähler locus of the class $[\theta]$). Then we can introduce local holomorphic coordinates centered at x_0 and locally write $\theta = dd^c f$ for f smooth and set $\phi := u_\beta + f$, which defines a locally bounded psh function ϕ. The defining equation for u_β implies the following local inequality, say on a neighourhood of the the ball $B \subset \mathbb{C}^n$:

$$(dd^c\phi)^n \geq e^{\beta(\phi-f)}dV$$

in the pluripotential sense of Bedford–Taylor (in fact, equality holds, but we will only need the inequality above). Moreover, by assumption $\phi - f$ has a local maximum at 0. But then it follows from local considerations (based on the Bedford–Taylor comparison principle for bounded psh functions and a regularization argumetn using convolutions) that

$$e^{\beta(\phi-f)}dV \leq (dd^c f)^n \ \text{ at } z = 0,$$

(see Proposition 1.11 in Sect. 1 on [29] or more precisely the implication in Proposition 1.1. saying that pluripotential subsolutions are viscosity subsolutions).[2] In other words,

$$u_\beta \leq C_0/\beta, \ \ C_0 = \log(\theta^n/dV)_+.$$

which proves the lemma in this case. In the general case we fix a sup-normalized function $v \in PSH(X, \theta)$ wich is smooth on Ω and such that $v - u_\theta \to -\infty$ along the analytic subvariety $X - \Omega$. Given $\epsilon > 0$ we set $u_{\beta,\epsilon} := (1 - \epsilon)u_\beta + \epsilon v \in PSH(X, \theta)$ which is locally bounded on Ω and satisfies the following inequality in the sense of Bedford–Taylor on Ω

$$MA_\theta(u_{\beta,\epsilon}) \geq (1 - \epsilon)^n e^{\beta u_\beta}dV \geq (1 - \epsilon)^n e^{\beta_\epsilon u_{\beta,\epsilon}}dV, \ \ \beta_\epsilon := \beta(1 - \epsilon)^{-1}$$

using that $v \leq 0$ in the last inequality. By assumption there exists a point x_ϵ in Ω where $u_{\beta,\epsilon}$ achieves its maximum. Hence, we can apply the previous argument to $\phi := u_{\beta,\epsilon} + f$ with parameter β_ϵ to get an inequality of the form $u_{\beta,\epsilon} \leq C_\epsilon/\beta_\epsilon$, where $C_\epsilon \to C_0$ as $\epsilon \to 0$. Letting ϵ tend to zero thus concludes the proof of the lemma. $\square$

We recall that in the case of a Kähler class the estimate in the previous lemma was obtained as consequence of the maximum principle in the proof of Proposition 2.3. Next, we generalize the L^∞-convergence in Proposition 2.3 to a general big class, using the convergence in energy in Proposition 2.10.

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[2] There is an erratum [30] to [29], but it only concerns the proof of the global comparison principle for viscosity solutions in Sect. 2 of [29] and not the local results in Sect. 1 of [29].

Proposition 2.12 *Suppose that θ is a smooth form such that the class $[\theta]$ is big. Then u_β converges uniformly to u_θ on X, i.e.*

$$\lim_{\beta \to 0} \|u_\beta - u_\theta\|_{L^\infty(X)} = 0$$

Proof According to the previous lemma we have that $u_\beta \leq u_\theta + C/\beta$ and hence $MA(u_\beta)/dV \leq e^C$. Moreover, by Proposition 2.10 u_β converges to u_θ in energy. As will be next explained these properties are enough to conclude that u_β converges uniformly to u. Indeed, it is well-known that if u_j is a sequence in $PSH(X, \theta)$ converging in capacity to u_∞ with a uniform bound L^p-bound on $MA(u_j)/dV$, then $\|u_j - u_\infty\|_{L^\infty(X)} \to 0$, as follows from a generalization of Kolodziej's L^∞-estimates to the setting of a big class (see [18,32] and references therein). Finally, as shown in [12], convergence in energy implies convergence in capacity, which thus concludes the proof of the previous proposition. In fact, using the stability results in [32] a more quantitative convergence result can be given. Indeed, according to Proposition 4.2 in [32] the following holds: assume that φ and ψ are functions in $PSH(X, \theta)$ normalized so that $\sup \varphi = \sup \psi = 0$ and such that $MA(\varphi) \leq f dV$, where $f \in L^p(X, dV)$. Then, for any sufficiently small positive number γ (see [32] for the precise condition) there exists a constant M, only depending on γ and an upper bound on $\|f\|_{L^p(dV)}$, such that

$$\sup_X (\psi - \varphi)^+ \leq M \|(\psi - \varphi)^+\|^\gamma_{L^1(X, MA(\varphi))}$$

Setting $\varphi := u_\beta - \epsilon_\beta$, where $\epsilon_\beta = \sup u_\beta$ and $\psi := u_\theta$ thus gives, for γ, fixed

$$\sup_X (u_\theta - u_\beta - \epsilon_\beta)^+ \leq M \left(\int |u_\theta - u_\beta - \epsilon_\beta| MA(u_\beta) \right)^\gamma$$

Now, by the convergence in energy and the L^1-convergence in Proposition 2.10 we have

$$\int (u_\beta - u_\theta) MA(u_\beta) \to 0$$

and since $\left| \int u_\theta - u_\beta - \epsilon_\beta \right| MA(u_\beta) \leq \int (u_\theta - u_\beta - C/\beta) MA(u_\beta) + C/\beta + \epsilon_\beta$ we deduce that $\sup_X (u_\theta - u_\beta - \epsilon_\beta)^+ \to 0$, i.e. $u_\theta \leq u_\beta + \epsilon'_\beta$, which concludes the proof.   □

2.2.3 Bound on the Monge–Ampère measure of u_θ

As shown above u_β converges to u_θ in energy (and even uniformly). In particular, the convergence holds weakly for the corresponding Monge–Ampère measures. The bound in Lemma 2.11 thus implies that

$$MA(u_\theta) \leq \sup_X \left(\frac{(\theta^n)_+}{dV} \right) dV$$

for any given volume form dV on X. Taking a sequence of volume forms dV_ϵ approximating the measure $(\theta^n)_+$ thus gives $MA(u_\theta) \leq (\theta^n)_+$ on X. Since $MA(u_\theta)$ is supported on the coincidence set D (which is contained in the set where $\theta \geq 0$) this proves the inequality 1.2.

2.2.4 Laplacian estimates

For the Laplacian estimate we will have to assume that the big class $[\theta]$ is nef.

Proposition 2.13 *Suppose that the class* $[\theta]$ *is nef and big. Then the Laplacian of* u_β *is locally bounded wrt* β *on the Zariski open set* $\Omega \subset X$ *defined as the Kähler locus of* X.

Proof We will assume that X is a Kähler manifold, i.e. X admits some Kähler form ω_0 (not necessarily cohomologous to θ). Then θ is nef precisely when the class $[\theta] + \epsilon[\omega_0]$ is Kähler for any $\epsilon > 0$. Setting $\theta_\epsilon := \theta + \epsilon \omega_0$ and fixing $\epsilon > 0$ and $\beta > 0$ we denote by $u_{\beta,\epsilon}$ the solutions of the Monge–Ampère equations obtained by replacing θ with θ_ϵ. Then it follows from well-known results [18] that, as $\epsilon \to 0$,

$$u_{\beta,\epsilon} \to u_\beta \text{ in } \mathcal{C}^\infty_{loc}(\Omega).$$

Moreover, since $[\theta]$ is assumed big there exists a positive current ω in $[\theta]$ such that the restriction of ω to Ω coincides with the restriction of a Kähler form on X. More precisely, we can take ω to be a Kähler current on X such that $\omega = dd^c v + \theta$ for a function v on X such that v is smooth on Ω and $u - v \to -\infty$ at the "boundary" of Ω (using that u has minimal singularities; compare [18]). Setting $u := u_{\beta,\epsilon}$ the inequality 2.11 still applies on Ω. Moreover, since $u - v \to -\infty$ at the boundary of Ω the sup s defined above is attained at some point of Ω and $\sup_X Ce^{-B(u-v)} \leq C'$. Accordingly, we deduce that

$$s := \sup_X e^{-B(u-v)} tr_\omega \omega' \leq C''$$

precisely as before, which in particular implies that $tr_\omega(\theta + dd^c u_{\beta,\epsilon})$ is locally bounded from above (wrt β and ϵ). Finally, letting $\epsilon \to 0$ concludes the proof. $\square$

In the special case when θ is semi-positive and big (the latter condition then simply means that $V > 0$) it follows from the results in [29] that u_β is continuous on all of X and hence Proposition 2.12 then says that $u_\beta \to u_\theta$ in $C^0(X)$.

Remark 2.14 The precise Laplacian estimate obtained in the previous proof may, for v and ω as in the proof above may be formulated as

$$tr_\omega \omega_{u_\beta} \leq \frac{1}{1 - nB/\beta} e^{B(u_\beta - v)} \sup_X \left(tr_\omega \theta_\beta \right) e^{-\inf_X B(u_\beta - v)} \tag{2.14}$$

In particular, normalizing v so that $\sup_X v = 0$ gives

$$tr_\omega \omega_{u_\beta} \leq \frac{e^{\sup u_\beta - \inf u_\beta}}{1 - nB/\beta} e^{-Bv} \sup_X \left(tr_\omega \theta_\beta \right)$$

By the L^∞-estimates above $\sup_X u_\beta - \inf_X u_\beta$ is uniformly bounded in terms of $\sup_X |\theta^n / dV|$. In particular, letting $\beta \to \infty$ gives the following a priori estimate for the Laplacian of the envelope u_θ :

$$tr_\omega \omega_{u_\theta} \leq Ce^{-Bv}, \tag{2.15}$$

where the constant C only depends on an upper bound on $|\theta|_\omega$. Interestingly, the estimate 2.15 is of the same form as the one obtained in [14], in the more general setting of a big class, by a completely different method where the constant B (i.e. the lower bound on the bisectional curvature) arises in the initial step of the proof where the envelope is regularized by the global convolution type operator associated to the exponential flow determined by the Chern connection.

2.2.5 End of the proof of Theorem 1.2 in the big case

This is proved exactly as in the case of a Kähler class, given the convergence results established above.

3 Transcendental Bergman metric asymptotics and Applications to regularization of ω-psh functions

3.1 Transcendental Bergman kernels

Consider an ample line bundle $L \to X$ and a pair $(\|\cdot\|, dV)$ consisting of an Hermitian metric $\|\cdot\|$ on L and a volume form dV on X. We denote by θ the normalized curvature form of $\|\cdot\|$, which represents the first Chern class $c_1(L)$ in $H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$. The corresponding *Bergman function* ρ_k (also called the *density of states function*), at level k, may be defined

$$\rho_k(x) = \sum_{i=1}^{N_k} \left\| s_i^{(k)}(x) \right\|^2,$$

in terms of any fixed basis $s_i^{(k)}$ in $H^0(X, L^{\otimes k})$ which is orthonormal wrt the corresponding L^2-norm determined by the pair $(\|\cdot\|, dV)$. In other words, $\rho_k(x)$ is the restriction to the diagonal of the squared point-wise norm of the Bergman kernel of $H^0(X, L^{\otimes k})$ (see [2] and references therein). The function $v_k := \frac{1}{k} \log \rho_k$ is often referred to as the *Bergman metric (potential)* at level k, determined by $(\|\cdot\|, dV)$ (geometrically, $\|\cdot\| e^{-k v_k}$ is the pull-back of the Fubini-Study metric on the projective space $\mathbb{P} H^0(X, L^{\otimes k})$ under the corresponding Kodaira embedding). As shown in [2] the corresponding *Bergman measures*

$$\nu_k := \frac{1}{N_k} \rho_k(x) dV$$

converge weakly to $MA_\theta(u_\theta)$ and v_k converges uniformly to u_θ. In particular,

$$MA_\theta(v_k) \approx e^{k v_k} dV$$

in the sense that both measures have the same weak limit (namely $MA_\theta(u_\theta)$). We can thus view the Bergman metric v_k as an approximate solution to the Eq. 1.1, for $\beta = k$. This motivates thinking of the family u_β of *exact* solutions, defined with respect to a general smooth closed $(1, 1)$-form θ (not necessarily corresponding to a line bundle) as a *transcendental Bergman metric,* in the sense that it behaves (at least asymptotically as $\beta \to \infty$) as a Bergman metric associated to an Hermitian line bundle. Similarly, $e^{k u_\beta} dV (= MA_\theta(u_\beta))$ can be thought of as a transcendental Bergman measure.

The main virtue of the family u_β is that it is canonically determined by the pair (θ, dV) and exists also in the general transcendental setting of a Kähler class $[\theta]$ which can not be realized as the first Chern class $c_1(L)$ of a line bundle. Accordingly, it seems natural to expect that it can be used as a substitute for the timehonoured technique in complex geometry of using Bergman kernels as an approximation tool. In Sects. 3.2 and 5 we will give two such applications to the regularization problem of ω-psh functions and weak geodesic rays, respectively.

In the following it will be convenient to use the equivalent formulation of envelopes of the form $P_\omega(f)$ in Sect. 2.0.2 (occasionally dropping the subscript ω). In other words, we start with a reference Kähler form ω on X. Given a smooth function f we denote by $P_\beta(f)$ the solution φ_β of the corresponding Monge–Ampère equation 2.4. In the line bundle setting above this corresponds to fixing a reference metric $\|\cdot\|_0$ on L and writing $\|\cdot\|^2 = \|\cdot\|_0 e^{-f}$ wich has curvature form $\theta = \omega + dd^c f$.

Lemma 3.1 *The operator* P_β : $C^\infty(X) \to SPSH(X, \omega) \cap C^\infty(X$ *is decreasing, i.e. if* $f \leq g$, *then* $P_\beta f \leq P_\beta g$. *Moreover,* $P_\beta(f + c) = P_\beta(f) + c$ *for any* $c \in \mathbb{R}$ *and hence*

$$\left\| P_\beta f - P_\beta g \right\|_{L^\infty(X)} \leq \| f - g \|_{L^\infty(X)} . \tag{3.1}$$

Proof The decreasing property follows directly from the comparison principle (Lemma 2.4) and the scaling property from the very definitions of P_β. □

By Proposition 2.3 P_β converges to the projection operator P :

$$\left\| P_\beta f - P f \right\|_{L^\infty(X)} \leq \frac{A \log \beta}{\beta}, . \tag{3.2}$$

where the constant A only depends on an upper bound on $(\omega + dd^c f)^n$. In particular, by a simple approximation argument (using 3.1) $P_\beta f$ converges to f uniformly, for any continuous function f on X. These convergence results can be viewed as transcental analogs of the Bergman metric asymptotics in [2] (which has the corresponding rate with $\beta = k$). Moreover, for f continuous the corresponding weak convergence of the transcendental Bergman measures:

$$\lim_{\beta \to \infty} e^{k(P_\beta f - f)} dV = (\omega + dd^c P f)^n$$

(resulting from the convergence of Monge–Ampère measues) is the analog of the convergence of Bergman measures towards equilibrium measures in [2] (first shown by Bouche and Tian, independently, in the case of a smooth and metrics with strictly positive curvature form θ).

Remark 3.2 Let us briefly explain how the setting above fits into the statistical mechanical setup recalled in Sect. 1.2. The point is that one can let the inverse temperature β, defining the probability measures 1.4, depend on k. In particular, for $\beta = k$ one obtains a *determinantal* random point process. A direct calculation (compare [4]) reveals that the corresponding one point correlation measure $\int_{X^{N_{k-1}}} \mu_{k,\beta}$ then coincides with the Bergman measure ν_k defined above. This means that the limit $k \to \infty$ which appears in the "Bergman setting" can—from a statistical mechanical point of view—be seen as a limit where the number N_k of particles *and* the inverse temperature β jointly tend to infinity.

3.2 Regularization of ω-psh functions

In this section we consider the case of a Kähler class $[\omega]$. We show how to give a simple global PDE proof of the following special case of the general regularization results of Demailly [26]:

Theorem 3.3 *Let* $[\omega]$ *be a Kähler class. Then any function* $\psi \in PSH(X, \omega)$ *can be written as a decreasing limit of functions* ψ_j *which are smooth and strictly* ω-*psh.*

Proof Since ψ is usc we can write it as a decreasing limit of smooth functions f_j. Setting

$$(P'_\omega f)(x) := \sup \left\{ \varphi(x) : \ \varphi \leq f : \ \varphi \in PSH(X, \omega) \cap C^\infty \right\} \tag{3.3}$$

we note that the sequence $\varphi_j := P'_\omega f_j$ decreases to ψ. Indeed, since the operator P'_ω is decreasing the sequence φ_j is decreasing and $\varphi_j \geq \psi$. Moreover, fixing a point x and $\epsilon > 0$ we have that $\varphi_j(x) \leq f_j(x) \leq \psi(x) + \epsilon$ for $j \geq j_\epsilon$ showing that $\varphi_j(x)$ decreases to $\psi(x)$ for any x, as claimed. Next, fixing $\beta > 0$ we set $\varphi_{j,\beta} := P_\beta f_j$ converging uniformly to $\varphi_{j,\beta}$ as $\beta \to \infty$ (by Proposition 2.3; compare formula 3.2). Hence, for appropriate choices of sequence $\epsilon_j \to 0$ and $\beta_j \to \infty$ the sequence $\psi_j := \varphi_{j,\beta_j} + \epsilon_j$ has the desired property (and as a consequence we actually have $P'_\omega f = P_\omega f$, by approximation). □

It should be pointed out that by a local gluing argument of Richberg [43] the regularization result above can be reduced to the case of a *continuous* ω-psh function ψ (using the usual local regularizations involving convolutions). In turn, it was shown in [16] that the continuity assumption can be replaced by the assumption of vanishing Lelong numbers and hence, as explained in [16], approximating a general element $\psi \in PSH(X, \omega)$ with the decreasing sequence $\psi_l := \max\{\psi, l\}$ in $PSH(X, \omega) \cap L^\infty$ gives a simple elemenary proof of the previous theorem. In the light of the discussion in the previous section the present global regularization scheme can be seen as a transcental analog of the well-known Bergman kernel approach to regularization used in the line bundle setting (see [26,31]). The present approach has the virtue of preserving higher order regularity properties of ψ as summarized in the following.

Theorem 3.4 *Let (X, ω) be a compact Kähler manifold φ an ω-psh function such that its Monge–Ampère measure $(\omega + dd^c\varphi)^n$ has an L^∞-density. Then $\varphi_\beta := P_\beta(\varphi)$ is in $PSH(X, \omega) \cap C^{2,\alpha}$ for some $\alpha > 0$ and satisfies*

$$\sup_X |\varphi_\beta - \varphi| \le C \frac{\log \beta}{\beta}, \quad (\omega + dd^c\varphi_\beta)^n \le C\omega^n$$

where the constant C only depends on an upper bound on the density $(\omega + dd^c\varphi)^n/\omega^n$. Moreover, if the positive current $(\omega + dd^c\varphi)$ has coefficents in L^∞ then $\omega + dd^c\varphi_\beta \le C'\omega$ and φ_β is in $PSH(X, \omega) \cap C^{3,\alpha}$ for any $\alpha < 1$.

Proof Since $(\omega + dd^c f)^n$ has an L^∞-density [37] gives that φ is in $C^\alpha(X)$ for some Hölder exponent $\alpha' > 0$. By the complex generalization of Evans-Krylov theory in [60] it then follows that φ_β is in $C^{2,\alpha}(X)$ for some $\alpha > 0$. Moreover, if $(\omega + dd^c\varphi)$ has coefficents in L^∞ then elliptic boot strapping gives that φ_β is in $C^{3,\alpha}$ for any $\alpha < 1$ and Proposition 2.6 shows that $\omega + dd^c\varphi_\beta \le C'\omega$. □

In particular, the transcendtal Bergman measure $e^{k(P_\beta\varphi-\varphi)}dV$ is uniformly bounded from above as long as $(\omega + dd^c\varphi)^n$ has an L^∞-density. For the ordinary Bergman measure the corresponding uniform bound was recently established in [9], under the stronger assumption that $(\omega + dd^c\varphi)$ has coefficents in L^∞. The latter result was used in the proof, involving local Bergman metric approximations, of Chen's conjecture concerning the convexity of the K-energy along weak geodesics in the closure of the space of Kähler metrics.

Remark 3.5 Inspired by the first preprint version of the present paper on ArXiv it was shown in [38] how to use a genaralization of the transcental Bergman kernels introduced here, using Hessian equations as a substitute for Monge–Ampère equations, in order to establish the corresponding conjectural global regularization result for (ω, m)-subharmonic functions (i.e. usc functions u such that $(\omega + dd^c u)^p \wedge \omega^{n-p} \ge 0$ for $p = 1, 2\ldots, m$; the case $m = n$ corresonds to the present setting). The elegant argument in [38] uses the notion of viscosity solutions of Hessian equations based on the technique introduced in [29].

4 Degenerations induced by a divisor

Let now (X, ω) be a compact Kähler manifold with a fixed divisor Z, i.e. Z is cut out by a holomorphic section s of a line bundle $L \to X$. We identify the divisor Z with the corresponding current of integration $[Z] := [s = 0]$. Let us also fix a smooth Hermitian

metric $\|\cdot\|$ on L and denote by θ_L its normalized curvature form. Fixing a parameter $\lambda \in [0, 1[$ we set

$$\varphi_\lambda := \sup\{\varphi \ \ \varphi \leq 0, \ \ \varphi \leq \lambda \log \|s\|^2 + O(1)\} \tag{4.1}$$

The upper bound on φ is equivalent to demanding that $\nu_Z(\varphi) \geq \lambda$, where $\nu_Z(\varphi)$ denotes the Lelong number of φ along Z. To the pair $([\omega], Z)$ we associate the following two constants:

$$\epsilon := \sup\{\lambda : [\omega] - \lambda[Z] \text{ is Kähler }\}$$

and

$$\epsilon' := \sup\{\lambda : [\omega] - \lambda[Z] \text{ is big}\},$$

so that $\epsilon \leq \epsilon'$ (the constants ϵ and ϵ' appears as nef and psef thresholds, respectively, in the algebraic geometry litterature). In the following we will always assume that $\lambda \in [0, \epsilon'[$, which ensures that φ_λ is not identically equal to $-\infty$.

Set $u_\lambda := \varphi_\lambda - \lambda \log \|s\|^2$, defining a function in $PSH(X, \theta)$, where $\theta := \omega - \lambda\theta_L$. Equivalently,

$$u_\lambda := P_\theta(-\lambda \log \|s\|^2) \tag{4.2}$$

in the sense of formula 2.3. This is equivalent to the construction of envelopes of metrics with prescribed singularities out-lined in the introduction of [2] (see also [45] where it is shown that u_λ is in $\mathcal{C}_{loc}^{1,1}(X - Z)$) in the case of an integral class).

Note that it follows immediately from the definition that u_λ has minimal singularities. In particular, if $\lambda < \epsilon$, then u_λ is bounded. In fact, u_λ is even continuous. The point is that, as long as the function φ_0 is lower semi-continuous the corresponding envelope $P_\theta(\varphi_0)$ will also be continuous. Indeed, it follows immediately that $P_\theta(\varphi_0)^* \leq \varphi_0$ and hence $P_\theta(\varphi_0)^* = P_\theta(\varphi_0)$, showing upper-semi continuity. The lower semi-continuity is then a standard consequence of Demailly's approximation theorem applied to the Kähler class $[\theta]$ (Theorem 3.3).

Theorem 4.1 *Let (X, ω) be a Kähler manifold and Z a divisor on X and fix a positive number $\lambda < \epsilon'$. Setting $\theta := \omega - \lambda\theta_L$, let $u_{\beta,\lambda}$ be the unique θ-psh function with minimal singularities solving*

$$(\theta + dd^c u)^n = e^{\beta u} \|s\|^{2\lambda\beta} \, dV$$

Then $u_{\beta,\lambda}$ converges uniformly, as $\beta \to \infty$, to the envelope u_λ. More precisely,

$$\sup_X |u_{\beta,\lambda} - u_\lambda| \leq \delta_\beta$$

for some family of positive numbers δ_β (independent of λ) tending to 0 as $\beta \to \infty$. Moreover, if $\lambda < \epsilon'$, then $\theta + dd^c u_{\beta,\lambda} \leq C\omega$ and hence the convergence holds in $C^{1,\alpha}(X)$ for any $\alpha < 1$.

Proof Set $f := -\|s\|^2$, which is a lsc function $X \to]-\infty, \infty]$ such that $dd^c f \leq C\omega$. The convergence in energy and hence the uniforme convergence then follows as before. Finally, the uniform bound on $dd^c u_{\beta,\lambda}$ is obtained by writing f is a decreasing limit of smooth function f_j such that $dd^c f_j \leq C'\omega$, applying Proposition cr for a fixed j and finally letting $j \to \infty$. $\qquad\square$

Note that $\varphi_{\lambda,\beta} := u_\lambda + \lambda \log \|s\|^2 \in PSH(X, \omega)$ is uniquely determined by the following equation on $X - Z$:

$$(\omega + dd^c \varphi_{\lambda,\beta})^n = e^{\beta\varphi_{\lambda,\beta}} dV \tag{4.3}$$

together with the asymptotics $\varphi_{\lambda,\beta} = \lambda \log \|s\|^2 + O(1)$ close to Z.

Remark 4.2 More generally, it is enough to assume that ω is semi-positive and big; then the uniform bound on $dd^c u_{\beta,\lambda}$ in the previous theorem holds on any compact subset of the Kähler locus of X (by Proposition 2.13). For example, this situation appears naturally when Z is the expectional divisor in the blow-up of a point on a Kähler manifold (M, ω_M) and ω is the pull-back of M. Then the corresponding constant ϵ is the Seshadri constant of p wrt $[\omega_M]$.

5 Applications to geodesic rays and test configurations

Let us start by briefly recalling the notions of geodesic rays and test configurations in Kähler geometry (see [42,47] and references therein). Given an n-dimensional Kähler manifold (X, ω) we denote by $\mathcal{K}_\omega$ the space of all ω-Kähler potentials φ, i.e. φ is smooth and $\omega + dd^c\varphi > 0$ (which equivalently means that φ is in the interior of the space $PSH(X, \omega) \cap C^\infty(X)$). The infinite dimensional space $\mathcal{K}_\omega$ comes with a canonical Riemannian metric, the Mabuchi-Semmes-Donaldson metric. The corresponding geodesics rays $\varphi^t(x)$ satisfy a PDE on $X \times [0, \infty[$ which, upon complexification of t (where $t := -\log|\tau|^2$) is equivalent to an S^1-invariant smooth solution to the Dirichlet problem for the Monge–Ampère equation on the product $X \times \Delta^*$ of X with the punctured unit-disc in the one-dimensional complex torus $\mathbb{C}^*$. In other words, $\varphi(x, \tau) := \varphi^t(x)$ satisfies

$$(dd^c\varphi + \pi^*\omega)^{n+1} = 0, \quad \text{on } X \times \Delta^*$$

and φ^t is called a *subgeodesic* if $dd^c\varphi + \pi^*\omega \geq 0$. In the case of an integral class $[\omega]$, i.e. when the class is equal to the first Chern class $c_1(L)$ of a line bundle L, there is a particularly important class of (weak) geodesics which are associated to so called *test configurations* for (X, L). This is an algebro-geometric gadget which gives an appropriate $\mathbb{C}^*$-equivariant polarized closure $\mathcal{X}$ of $X \times \mathbb{C}^*$ over $\mathbb{C}$. More precisely, the data defining a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) consists of

- A normal variety $\mathcal{X}$ with a $\mathbb{C}^*$-action and flat equivariant map $\pi : \mathcal{X} \to \mathbb{C}$
- A relatively ample $\mathbb{Q}$-line bundle $\mathcal{L}$ over $\mathcal{X}$ equipped with an equivariant lift ρ of the $\mathbb{C}^*$-action on X
- An isomorphism of (X, L) with $(\mathcal{X}, \mathcal{L})$ over $1 \in \mathbb{C}$

Here, we note that a "transcendental" analog of a test configuration can be defined in the setting of non-integer classes.

Definition 1 Let $(X, [\omega])$ be a complex manifold equipped with a Kähler class $[\omega]$. A test configuration for $(X, [\omega])$ consists of the following data:

- A normal Kähler space $\mathcal{X}$ equipped with a holomorphic S^1-action and a flat holomorphic map $\pi : \mathcal{X} \to \mathbb{C}$.
- An S^1-equivariant embedding of $X \times \mathbb{C}^*$ in $\mathcal{X}$ such that π commutes with projection onto the second factor of $X \times \mathbb{C}^*$.
- A $(1, 1)$-cohomology Kähler class $[\Omega]$ on $\mathcal{X}$ whose restriction to $X \times \{1\}$ may be identified with $[\omega]$ under the previous embedding.

In particular, a test configuration $(\mathcal{X}, \mathcal{L})$ for a polarized variety (X, L) induces a test configuration for $(X, c_1(L))$. The point is that the $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ induces the required isomorphism between $\mathcal{X}$ and $X \times \mathbb{C}^*$ over $\mathbb{C}^*$.

Next, we explain how to obtain geodesic rays from a test configuration. Given a test configuration $(\mathcal{X}, [\Omega])$ for $(X, [\omega])$ we fix a smooth representative form Ω which is S^1-invariant. For the sake of notational simplicity we also assume that Ω coincides with ω on $X \times \{1\}$. First we let Φ be the unique bounded Ω-psh function on $\mathcal{M} := \pi^{-1}(\Delta) \subset \mathcal{X}$ satisfying the Dirichlet problem

$$(dd^c \Phi + \Omega)^{n+1} = 0, \quad \text{on int}(\mathcal{M}) \tag{5.1}$$

with vanishing boundary values (in the sense that $\Phi(p) \to 0$ as p approaches a point in $\partial \mathcal{M}$). In fact, it can be shown, that Φ is automatically continuous up to the boundary (see below). Next, we fix an S^1-invariant function F on $X \times \mathbb{C}^*$ such that

$$\Omega = \pi^* \omega + dd^c F$$

and set $\varphi := \Phi + F$, which gives a correspondence

$$PSH(X \times \mathbb{C}^*, \Omega) \longleftrightarrow PSH(X \times \mathbb{C}^*, \pi^* \omega), \quad \Phi \leftrightarrow \varphi \tag{5.2}$$

Setting $\varphi^t(x) := \varphi(x, \tau)$ for φ corresponding to the solution Φ of the Dirichlet problem 5.1 then defines the geodesic ray in question.

Let us also recall that the solution Φ of the Dirichlet problem 5.1 may alternatively be defined as the following envelope:

$$\Phi(x) = \sup \{\Psi(x) : \quad \Psi \in PSH(\mathcal{M}, \Omega) : \quad \Psi_{\partial \mathcal{M}} \leq 0\} \tag{5.3}$$

As shown in [47], in the line bundle case, the geodesic ray φ^t may be realized as a Legendre transform of certain envelopes determined by the test configuration. Here we note that the latter result may be generalized to the "transcendental" setting. To this end first observe that a test configuration $(\mathcal{X}, [\Omega])$ for $(X, [\omega])$ determines a concave decreasing family

$$\mathcal{F}^\mu(X, \omega) \subset PSH(X, \omega)$$

of convex subspaces indexed by $\mu \in \mathbb{R}$, defined as follows: the subspace $\mathcal{F}^\mu(X, \omega)$ consists of all φ in $PSH(X, \omega)$ such that, setting $\bar{\varphi}(x, t) := \varphi(x)$, the current

$$dd^c(\bar{\varphi} - \mu \log |\tau|^2) + \pi^* \omega$$

on $X \times \mathbb{C}^*$ extends to a positive current on $\mathcal{X}$ in $[\Omega]$. In other words, we demand that the current $dd^c \bar{\varphi} + \pi^* \omega$ extends to current on $\mathcal{X}$ in $[\Omega]$ with Lelong number at least μ along the central fiber of $\mathcal{X}$ (in a generalized sense, as we are allowing negative Lelong numbers). The family $\mathcal{F}^\mu(X, \omega)$, thus defined, is clearly a concave decreasing family of convex subspaces (it is the "psh analogue" of the filtrations of $H^0(X, kL)$ defined in [47,61]). Next, to the family $\mathcal{F}^\mu(X, \omega)$ we associate the following family of envelopes ψ_μ in $PSH(X, \omega)$:

$$\psi_\mu(x) := \sup_{\psi \in \mathcal{F}^\mu(X, \omega)} \{\psi(x), \quad \psi \leq 0\}, \tag{5.4}$$

Proposition 5.1 *Let $(\mathcal{X}, [\Omega])$ be a test configuration for $(X, [\omega])$. Then the corresponding geodesic ray φ^t in $PSH(X, \omega)$ may be realized as the Legendre transform (wrt t) of the envelopes ψ_μ, i.e.*

$$\varphi^t(x) = \sup_{\mu \in \mathbb{R}} \{\psi_\mu(x) + \mu t\}$$

Proof By the definition of the envelopes it is equivalent to prove that

$$\varphi^t(x) = \sup_{\psi_\mu} \left\{ \psi_\mu(x) + \mu t \right\}$$

where the sup ranges over *all* $\psi_\mu \in \mathcal{F}^\mu(X, \omega)$ with $\psi_\mu \leq 0$ on X. Using the correspondence 5.2 we may identify $\psi_\mu(x) + \mu t$ with a function Φ_μ in $PSH(X \times \mathbb{C}^*, \Omega)$, which, by the extension assumption for the elements in the subspace $\mathcal{F}^\mu(X, \omega)$, extends uniquely to define an element in $PSH(\mathcal{X}, \Omega)$ (which by construction vanishes on the boundary of $\mathcal{M}$). But then $\Phi_\mu \leq \Phi$, the envelope defining the geodesic ray φ^t. This proves the lower bound on $\varphi^t(x)$. To prove the upper bound we note that, by the convexity in t, we may write

$$\varphi^t(x) = \sup_{\mu \in \mathbb{R}} \left\{ \phi_\mu^*(x) + \mu t \right\},$$

where ϕ_μ^* is the Legendre transform, wrt t, of φ^t (with our sign conventions ϕ_μ^* is thus concave wrt μ) :

$$\phi_\mu^*(x) = \inf_t \left\{ \mu t + \varphi^t(x) \right\}$$

In particular, $\phi_\mu^*(x) + \mu t \leq \varphi^t$ and moreover, by Kiselman's minimum principle, $\phi_\mu^*(x)$ is ω-psh on X. Identifying $\phi_\mu^*(x) + \mu t$ with a function Φ_μ in $PSH(X \times \mathbb{C}, \Omega)$, as before, it thus follows that $\Phi_\mu \leq \Phi$. In particular, Φ_μ is bounded from above and thus extends to define an element in $PSH(\mathcal{X}, \Omega)$, i.e. the corresponding curvature current is positive. But this means that $\phi_\mu^*(x) \in \mathcal{F}^\mu(X, \omega)$ which concludes the proof of the upper bound. $\square$

Example 5.2 (deformation to the normal cone; compare [48,49]). Any given (say reduced) divisor Z in X determines a special test configuration whose total space $\mathcal{X}$ is the deformation to the normal cone of Z. In other words, $\mathcal{X}$ is the blow-up of $X \times \mathbb{C}$ along the subscheme $Z \times \{0\}$. Denote by π the corresponding flat morphism $\mathcal{X} \to \mathbb{C}$ which factors through the blow-down map p from $\mathcal{X}$ to $X \times \mathbb{C}$. This construction also induces a natural embedding of $X \times \mathbb{C}^*$ in $\mathcal{X}$. Given a Kähler class $[\omega]$ on X, which we may identify with a class on $X \times \mathbb{C}$ and a positive number c we denote by $[\Omega_c]$ the corresponding class $[p^*\omega] - c[E]$ on $\mathcal{X}$, where E is the exceptional divisor and we are assuming that $c < \epsilon$, where ϵ is defined as the sup over all positive numbers c such that the class $[\Omega_c]$ is Kähler (i.e. ϵ is the Seshadri constant of Z wrt $[\omega]$). In this setting it is not hard to check that $\varphi \in \mathcal{F}^\mu(X, \omega)$ iff $\nu_Z(\varphi) \geq \mu + c$, where $\nu_Z(\varphi)$ denotes the Lelong number of φ along the divisor Z in X. The point is that $[p^*\omega] - cE$ may be identified with the subspace of currents in $[p^*\omega]$ with Lelong number at least c along the divisor E in $\mathcal{X}$ which in this case is equivalent to having Lelong number at least c along the central fiber $[\mathcal{X}_0]$, which in turn is equivalent to φ having Lelong number at least c along Z in X. In particular, setting $\mu = \lambda - c$ we have $\varphi_\lambda = \psi_\mu$, where φ_λ is the envelope defined by formula 4.1, i.e. $u_\lambda = \psi_\mu - \lambda \log \|s\|^2$, where u_λ is defined by 4.2.

Now we observe that one obtains a family of subgeodesics, approximating the weak geodesic φ^t in the closure of $\mathcal{K}_\omega$, associated to a divisor Z and a number $c \in [0, \epsilon[$, as in the previous example, by setting

$$\varphi_\beta^t := \frac{1}{\beta} \log \int_{[0,c]} d\lambda e^{\beta((\lambda-c)t + \varphi_{\lambda,\beta})},$$

where $\varphi_{\lambda,\beta}$ is the regularization of φ_λ introduced in Sect. 4, solving the Monge–Ampère equation 4.3 (which is indeed a subgeodesic as it is a superposition of the subgeodesics $(\lambda - c)t + \varphi_{\lambda,\beta}$). Combining Theorem 4.1 with the previous proposition we arrive at the following

Theorem 5.3 *Let* $[\omega]$ *be a Kähler class on* X *and* Z *a divisor in* X *and fix a positive number* $c \in [0, \epsilon[$. *Then the corresponding subgeodesics* φ_β^t *converge, as* $\beta \to \infty$, *to the weak geodesic* φ^t, *uniformly on* $X \times [0, T[$ *for any fixed* $T < \infty$ *(and for* $T = \infty$ *in the case when* $[\omega] \in H^2(X, \mathbb{Q})$). *Moreover, the first order space-time derivatives of* φ_β^t *are uniformly bounded on* $X \times [0, \infty]$.

Proof By Theorem 4.1

$$\varphi_\beta^t = \frac{1}{\beta} \log \int_{[0,\Lambda]} d\lambda e^{\beta((\lambda-c)t+\varphi_\lambda)} + o(1), \quad \varphi_\lambda := u_{\theta,\lambda} + \lambda \log \|s\|^2,$$

where the $o(1)$-term is independent of t and converges uniformly to 0 on $X \times [0, c]$ as $\beta \to \infty$. As a consequence, for $t \in [0, T]$ we clearly have

$$\varphi_\beta^t = \sup_{\mu \in [-c,0]} \left(\mu t + \psi_\mu \right) + o(1)$$

(where, as explained in the previous example, $\psi_\mu = \varphi_\lambda$ for $\mu = \lambda - c$) and by Proposition 5.1 the first term above defines the desired geodesic ray φ^t. Finally, we need to show that the error term above is uniform at $T \to \infty$ in the case when $[\omega] \in H^2(X, \mathbb{Q})$). To this end we will use a compactification argument. Set, as before $t = -\log |\tau|^2$, where $\tau \in \mathbb{C}^*$. By the definition of the deformation to the normal cone $\mathcal{X}$ (see the previous example) the function Φ_μ defined in the proof of Proposition 5.1 defines an Ω-psh function on $\mathcal{X}$. We thus a get a family of functions on $\mathcal{X}$ defined by

$$\Psi_\beta := \frac{1}{\beta} \log \int_{[-c,0]} d\mu e^{\beta \Phi_\mu}$$

and such that Ψ_β increases (by Hölder's inequality) to the function $\Psi_\infty := \sup_\mu \Phi_\mu$, which, according to the proof of Proposition 5.1, coincides with the envelope Φ defined by formula 5.3. But the latter envelope is continuous (up to the boundary) on $\mathcal{M}$ and hence it follows from Dini's lemma that Ψ_β converges to Ψ uniformly, as desired. The continuity of the envelope Φ follows from standard arguments in the case when $\mathcal{M}$ is smooth and the background form η is Kähler. We recall that the argument just uses that any sequence of η-psh functions may be approximated by a decreasing sequence of continuous η-psh functions, as follows from the approximation results in [26] (see for example [14] for a similar situation). The latter approximation property has been generalized, in the case of rational classes, to the case when η is merely assumed to be semi-positive (and big) [23] and hence the proof of the continuity still applies in the present situation (strictly speaking the results in op. cit. apply to *compac*t complex manifolds, but we can simply pass to a resolution of the the $\mathbb{C}^*$-equivariant compactification of $\mathcal{X}$ fibered over the standard $\mathbb{P}^1$-compactification of $\mathbb{C}$ and adopt the argument using barriers in [7]).

Finally, to prove the last statement we observe that, fixing a first order differential operator D_x on X, we have

$$\frac{d}{dt} \varphi_\beta^t(x) := \int_{[0,c]} (\lambda - c) v_{(x,t)}^{(\beta)}(\lambda), \quad D_x \varphi_\beta^t(x) = \int_{[0,c]} D_x \varphi_{\beta,\lambda}(x) v_{(x,t)}^{(\beta)}(\lambda),$$

in terms of the following probability measure $v_{(x,t)}^{(\beta)}$ on $[0, c]$:

$$v_{(x,t)}^{(\beta)}(\lambda) := e^{\beta((\lambda-c)t+\varphi_{\lambda,\beta})} / \int_{[0,c]} d\lambda e^{\beta((\lambda-c)t+\varphi_{\lambda,\beta})}$$

But then the estimate on the time derivative follows immediately from the uniform bound $|\lambda| \le c$ and the estimate on the space derivative form the uniform bound on $D_x\varphi_{\beta,\lambda}$ (Theorem 4.1). □

Remark 5.4 In the case when $[\omega] = c_1(L)$ it was shown in [44] how to approximate (in a point-wise almost everywhere sense) a weak geodesic φ_t associated to a test configuration by smooth Bergman geodesics associated to higher powers of the line bundle L (see also [47] for an alternative proof). Accordingly, it seems natural to view φ_β^t as a transcendtal analog of the Phong-Sturm Bergman geodesics. One advantage of φ_β^t is that the convergence is uniform (even when t is not constrained to be in a bounded interval in the case of a rational class). Assuming the conjectural validity of the appoximation result in [23] for general transcendental classes, the uniformity in the previous theorem holds for $T = \infty$, in general. It is also interesting to compare the bound on the first derivatives above with the case of toric Bergman geodesics studied in [58], where uniform C^1-convergence is established. It seems likely that a similar C^1-convergence holds in the present setting (even in the general non-toric setting), but we will not go further into this here. It would also be interesting to see if there is a uniform bound on the space Laplacians of φ_β^t (say on any fixed time inverval).

5.0.1. General (analytic) test configurations

Of course, the test configurations defined by the deformation to the normal cone of a divisor are very special ones. But the convergence result in Corollary 5.3 can be extended to general test configurations for a polarized manifold (X, L) (by replacing $MA(u_{\beta,\lambda})$ with $MA(\varphi_{\beta,\mu})$ where $\varphi_{\beta,\mu} \in \mathcal{F}^\lambda(X, \omega)$ satisfies the Eq. 4.3). The argument uses Odaka's generalization of the Ross-Thomas slope theory [39] defined in terms of a flag of ideals on X. The point is that by blowing up the corresponding ideals one sees that the pullback of the corresponding envelopes ψ^μ have divisorial singularities (compare Proposition 3.22 in [35]) so that the previous convergence argument can be repeated (as they apply also when L is merely semi-ample and big, which is the case on the blow-up).

More generally, an analytic generalization of test configurations for a polarization (X, L) was introduced in [47]. Similarly, an *analytic test configuration* for a Kähler manifold (X, ω) may be defined as a concave family $[\psi^\mu]$ of singularity classes in $PSH(X, \omega)$. The corresponding space $\mathcal{F}^\mu(X, \omega)$ may then be defined as all elements ψ in such that $[\psi] = [\psi^\mu]$. To any such family one associates a family of envelopes ψ_μ defined by formula 5.4. As shown in [47] taking the Legendre transform of ψ_μ wrt μ gives a curve φ^t in $PSH(X, \omega)$ which is a weak geodesic. The regularization scheme introduced in this paper could be adapted to this general framework by first introducing suitable algebraic regularizations of the singularity classes and using blow-ups (as in [39]). But we leave these developments and their relation to K-stability and the Yau–Tian–Donaldson conjecture for the future. For the moment we just observe that the latter conjecture admits a natural generalization to transcendental classes.

Example 5.5 Continuing with the previous example of deformation to the normal cone, we observe that one obtains a (transcendtal) analytic test configuration, which is not a bona fide test configuration, when $c \in]\epsilon, \epsilon'[$. In geometric terms this corresponds to allowing the line bundle $\mathcal{L}$ (or the corresponding Kähler class on the total space) to be merely big. In this setting the C^0-convergence in Theorem 5.3 still holds (with the same proof) as long as t is restricted to a bounded interval.

5.0.2. A generalization of the Yau–Tian–Donaldson conjecture to transcendetal classes

Using Wang's intersection formula [59] there is a natural generalization of the notion of K-stability of a polarization (X, L): by definition, a Kähler class $[\omega]$ on X is *K-stable* if, for any test configuration $(\mathcal{X}, [\Omega])$ for $(X, [\omega])$ the corresponding Donaldson–Futaki invariant satisfies $DF(\mathcal{X}, [\Omega]) \geq 0$ with equality iff $\mathcal{X}$ is equivariantly isomorphic to a product. Similarly, K-polystability is defined by not requiring that the isomorphism be equivariant. Here $DF(\mathcal{X}, [\Omega])$ is defined as the following sum of intersection numbers

$$DF(\mathcal{X}, [\Omega]) := a[\Omega]^{n+1} + (n+1)K_{\mathcal{X}/\P^1} \cdot [\Omega]^n, \quad a := n(-K_X) \cdot [\omega]^{n-1}/[\omega]^n$$

where we have replaced $\mathcal{X}$ with its equivariant compactification over $\P^1$ and $[\Omega]$ with the corresponding class on the compactification and the intersection numbers are computed on the compactification. The transcendental version of the Yau–Tian–Donaldson conjecture may then be formulated as the conjecture that $[\omega]$ admits a constant scalar curvature metric iff $(X, [\omega])$ is K-polystable. It is interesting to compare this generalization with Demailly–Paun's generalization of the Nakai-Moishezon criterium for ample line bundles [27], which in the case when X is a projective manifold says that if a $(1, 1)$-class $[\theta]$ has positive intersections with all p-dimensional subvarieties of X then $[\theta]$ contains a Kähler form ω. The difference is thus that in order to draw the considerably stronger conclusion that ω can be chosen to have constant scalar curvature one needs to impose conditions on "secondary" intersection numbers as well, i.e. intersection numbers defined over all suitable degenerations of $(X, [\theta])$. Finally, it should be pointed out that it may very well be that the notion of (transcendental) test configuration above has to be generalized a bit further in order for the previous conjecture to stand a chance of being true (compare the discussion in the introduction of the paper).

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