

THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

**Vector-valued Eisenstein series of
congruence types and their products**

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Gothenburg, Sweden 2019

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Printed in Gothenburg, Sweden 2019

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Abstract:

Historically, Kohnen and Zagier connected modular forms with period polynomials, and as a consequence of this association concluded that the products of at most two Eisenstein series span all spaces of classical modular forms of level 1. Later Borisov and Gunnells among other authors extended the result to higher levels. We consider this problem for vector-valued modular forms, establish the framework of congruence types and obtain the structure of the space of vector-valued Eisenstein series using tools from representation theory. Based on this development and historic results, we show that the space of vector-valued modular forms of certain weights and any congruence type can be spanned by the invariant vectors of that type tensor at most two Eisenstein series.

Keywords : vector-valued modular forms, congruence type, Hecke operator, products of Eisenstein series, Fourier expansion of modular forms ■

Preface

This monograph constitutes my thesis for a Licentiate degree at the department of Mathematical Sciences at the Chalmers University of Technology and the University of Gothenburg. The work in this thesis is based on material gathered during my study as a Ph.D. candidate employed by the Chalmers University of Technology, from August 2016 to April 2019.

Acknowledgement

It is my great pleasure to thank my supervisor Martin Raum for motivating me to study the Eisenstein series, sharing with me deep insights and higher points of view about mathematics, providing technical supports, from explaining important scientific notions to details about writing techniques, and for encouraging me to learn how to find a suitable way for myself to study and work in a progressive manner. Meanwhile, I would especially like to express my sincere gratitude to my co-supervisor Dennis Eriksson for encouraging me to study literature about another aspect of Eisenstein series, the Kronecker limit formula, sharing with me broad knowledge and deep insights in many aspects of mathematics, and very positive attitude about the process of study and growth.

I would like to warmly thank all the people who have helped me, made useful comments and suggestions, taught me important things, and discussed interesting mathematics with me. I would particularly like to thank Christian Johansson for the great patience and help of answering my beginner-level questions and providing excellent advice on representation theory and its literature, thank Per Salberger for useful comments on geometric and number theoretic questions and great suggestion on publication, thank Julia Brandes for critical comments and valuable help on circle method, as well as for concrete suggestions on how to prepare for a good math talk, thank Anders Södergren for sharing both great passion and precision on dealing with non-trivial summation and series, thank Genkai Zhang for sharing broad knowledge on Lie groups and algebraic groups, especially on the notion of circle bundles over locally symmetric space, as well as how to do math with the help of geometric intuition, thank Julie Rowlett for discussing about the notions of Fourier dimension vs. Hausdorff dimension, Salem set, and introducing very interesting open problem about Siegel disc, thank Håkan Samuelsson for great advice and deep insight on career planning and the math community, thank Elizabeth Wulcan for discussing a complete proof of the Oka's coherence theorem, thank Michael Björklund for giving wonderfully structured and crystal clean lectures on harmonic analysis, and introducing the spectral perspective of Eisenstein series, thank Peter Hegarty for sharing wonderful problems, ideas, and experience on how to solve problems, as well as useful suggestion on the thesis writing, thank Jan Stevens for teaching me wonderful tricks for problem solving, and for sharing with all the students a physical model of 27 lines on a cubic surface, thank Magnus Goffeng for showing the topological origin of algebraic K-theory, thank Lyudmila Turowska for discussing possible generalization of the Gelfand representation and many useful concrete examples, thank Sebastian Herrero for sharing the idea of using Hecke operators to compare Fourier coefficients of all the Eisenstein series of a given level, thank Jeff Steif for discussing Brownian motions with an outsider, and thank Maria Roginskaya for sharing insights about measure theory, Polish spaces and their Borel sets, and for useful suggestions on a possible undergraduate project.

Outside of our math department, I would like to particularly thank the Mittag-Leffler Institute for hosting a wonderful conference “N-cube week VIII” in 2018 and generous support for young researchers to attend the conference, and all the organizers of conference,

who have made it such a memorable experience for the attenders. I would like to thank the Max Plank Institute for Mathematics and all the organizers of conference for hosting the very successful and high-level summer school-conference “Modular forms are everywhere” in 2017, and generous support for young researchers. I would also like to thank Lund University and all the organizers of conference for hosting “N-cube days IX” in 2018, and generous support for young researchers. I am also grateful to all the online math communities from which I learned much about mathematics in general, especially the Mathoverflow and the Mathematics Stack Exchange.

During my study and work, I have constantly received invaluable support and help from people all around the department. I would especially like to thank Aila Särkkä and Bernt Wennberg for caring about the well-being of Ph.D. students in general, and for supporting me when I faced enormous difficulty. I would also like to express my deepest gratitude to my manager, Håkan Samuelsson, for constant and caring support for my every opportunity to study and to broaden my scientific horizons. I am also grateful to the study advisor for Ph.D. students, Peter Hegarty, for many useful suggestions and strategies to choose a good path. I would like to thank all the administrators in our department, who make this place such an enjoyable place to stay, especially Marie Kühn, Lotta Fernström, and Fia Eriksson. All these messy things and troubles in my expectation are always magically reduced to triviality in your magical hands, and I simply could not imagine a world without you. Finally, I would like to express my respect and gratitude to our math library, joint with the main library of Chalmers, which provides literally speaking everything I need for my research and self-improvement. I would especially like to thank the librarians, both previous and current, Lars Almqvist and Sofia Arvidsson, for great understanding and support, patience and assistance in finding relevant literature. All these great people make my study at the department like home.

Last but not least, I would also like to thank my family, for the invaluable support as always, as well as all of my friends and colleagues, both old and new, for making my stay in town a really wonderful and enjoyable time. Without you, life would have been much drier, and winter much longer. Because of you, everything becomes more meaningful and colorful.

Jiacheng Xia
Göteborg, April 2019

To you

“Depuis quinze jours, je m’efforçais de démontrer qu’il ne pouvait exister aucune fonction analogue à ce que j’ai appelé depuis les fonctions fuchsiennes ; j’étais alors fort ignorant ; tous les jours, je m’asseyais à ma table de travail, j’y passais une heure ou deux, j’essayais un grand nombre de combinaisons et je n’arrivais à aucun résultat. Un soir, je pris du café noir contrairement à mon habitude ; je ne pus m’endormir ; les idées surgissaient en foule ; je les sentais comme se heurter, jusqu’à ce que deux d’entre elles s’accrochassent pour ainsi dire pour former une combinaison stable. Le matin, j’avais établi l’existence d’une classe de fonctions fuchsiennes, celles qui dérivent de la série hypergéométrique ; je n’eus plus qu’à rédiger les résultats, ce qui ne me prit que quelques heures.”

Henri Poincaré, *Science et méthode*, 1908

“For fifteen days I strove to prove that there could not be any functions like those I have since called Fuchsian functions. I was then very ignorant; every day I seated myself at my work table, stayed an hour or two, tried a great number of combinations and reached no results. One evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination. By the next morning I had established the existence of a class of Fuchsian functions, those which come from the hypergeometric series; I had only to write out the results, which took but a few hours.”

Henri Poincaré, *Science and Method*, 1913, translated by George Bruce Halsted

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1 Introduction

1.1 Introduction In the contents of a textbook on mathematical analysis [God03], the author Roger Godement simply nicknamed modular forms, a subject or rather a tool, as the “opium of mathematicians”. One could argue that the word “mathematician” for him might be exclusive to the Bourbaki, hence there is nothing really so fascinating if it is only about a small circle of people. However, such a unique feeling more or less exists among many other mathematicians as well. One of the most famous quotations on it, probably attributed to Martin Eichler, is “There are five fundamental operations in mathematics: addition, subtraction, multiplication, division, and modular forms”. Indeed, this saying might not be much exaggerated in terms of the Modularity Theorem, that all rational elliptic curves arise from modular forms. The Modularity Theorem was first hinted at by Taniyama in 1955, and later formulated by Shimura in such form. It was first proved for a large part, by Wiles and Taylor in [Wil95] and [TW95], and later completed by Breuil, Conrad, Diamond, and Taylor in [Dia96], [CDT99], and finally in [Bre+01]. More concretely, we can describe the theorem in a simple version as follows. Let $E : Y^2 = 4X^3 - g_2X - g_3$ for $g_2, g_3 \in \mathbb{Z}$ such that $g_2^3 - 27g_3^2 \neq 0$ be a cubic equation, which then define elliptic curves over \mathbb{Q} . For a given prime p , we define $a_p(E)$ to be $p - |(x, y) \in \mathbb{F}_p^2, (x, y) \text{ solves the equation } E \pmod{p}|$. Till now everything looks purely arithmetic, but the magic is that all the arithmetic information encoded in $a_p(E)$ can be *always* captured from some modular form, which arises from an analytic setting. More precisely, we can define an operator acting on the whole space of modular forms of a given weight and level, a Hecke operator T_p for each p , so that there is an eigenform $f = f_E$ of T_p , with the p -th Fourier coefficient $a_p(f)$ (which is also the eigenvalue of T_p), such that

$$a_p(E) = a_p(f).$$

One could go on and count many roles that modular forms play in connecting with other parts of mathematics, and more recently, string theory [Fle+18]. Such ubiquitous functions not surprisingly have a very simple definition. The word “modular” indicates that such functions should be defined on some moduli space of very fundamental objects. If we view the complex plane \mathbb{C} as a two dimensional real vector space, and consider the moduli space \mathcal{L} of all lattices of rank 2 in \mathbb{C} . We then find that two lattices correspond to the same point in \mathcal{L} if and only if they can be transformed via a homothety $\lambda \in \mathbb{C}^\times$. We thus define a function f on the set of all the lattices Λ to be a “modular form” if it satisfies some very nice analytic condition, and behaves in the simplest possible way under the action of homotheties $\lambda : \Lambda \mapsto \lambda\Lambda$ for all $\lambda \in \mathbb{C}^\times$. More precisely, if f furthermore satisfies

$$f \circ \lambda = \lambda^{-k} \circ f$$

for some number k , we call f a modular form of weight k . Such simplest type of modular forms is also called Elliptic Modular forms, in that a lattice Λ_z with a chosen base 1, z can be identified with an elliptic curve $E_z = \mathbb{C}/\Lambda_z$ up to an oriented basis. This point of view

will be later called the “lattice point of view” in this thesis, in contrast to the following group point of view.

We can also show that it is equivalent to define modular forms via the slash action, i.e. $f : \mathfrak{H} \rightarrow \mathbb{C}$ is called a modular function if it satisfies certain analytic condition and the modularity condition $f|_k\gamma = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where the slash action $\cdot|_k$ is defined by

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

Modular forms are always defined to be holomorphic on some geometric object, while modular functions are allowed to be meromorphic. This consideration is due to the requirement that, in order to connect with more interesting mathematics, the analytic condition is most of the time crucial. Slightly weaker than being holomorphic, real-analytic functions which satisfy modularity condition are also interesting objects, and we compute the Fourier expansions of one simple case in Appendix A.

One of the key features of modular forms, is that the space of these functions of a given weight (and other parameters later to be introduced in the thesis) is always finite dimensional, with very explicit bound. For example, in the simplest case of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight k , the dimension of the spaces of all such forms is less or equal than $\lfloor k/12 \rfloor + 1$. This fact in general allows us to determine all the information by just computing a few Fourier coefficients of modular forms, which guarantees the computational efficiency in this subject. Moreover, classical modular forms have Fourier coefficient in some cyclotomic field of bounded degree except the constant term, which makes the computation very ideal.

Modular forms can be defined on general discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$. By the superrigidity theorem of Margulis, for $\mathrm{SL}_n(\mathbb{R})$ where $n \geq 3$, lattice subgroups (i.e. discrete with finite covolume) and arithmetic subgroups coincide, and each complex finite dimensional representation for $\mathrm{SL}_n(\mathbb{Z})$ has canonical decomposition. By a theorem due to Bass–Lazard–Serre [BLS64] and independently Mennicke, every finite-index subgroup in $\mathrm{SL}_n(\mathbb{Z})$ is a congruence subgroup. However, for $\mathrm{SL}_2(\mathbb{R})$ situations are much more subtle, and there are many non-congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index. In this thesis, we focus on modular forms for congruence subgroups, while stating each result in its natural generality.

The first non-trivial question is on the existence of nonzero modular forms. Indeed this question also once bothered Henri Poincaré when he first believed that this kind of modularity would be too strong to have any solution. In fact, if we carry out the Reynolds operator with respect to the slash action, then it should be invariant at least formally under the slash action, so the only questions left are: do they converge nicely, and are they nonzero? For the first question, the assumption of absolute convergence of infinite series together with the Riemann rearrangement theorem would guarantee the modularity condition and certain analytic property of such infinite series, which we call Eisenstein series. To answer the second question, we have to at least compute something, and the simplest choice might be to compute its constant Fourier coefficient. In fact, an Eisenstein series is in one-to-one correspondence with its constant coefficient, which in some sense

suggests that Eisenstein series is the simplest object among modular forms. In this thesis, we carry out computation of Fourier coefficients of all classical Eisenstein series of weight $k > 2$ and level N for any positive integer N .

The cusp expansion of a modular form f of weight k , at some cusp $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ of a congruence subgroup, is defined by the Fourier expansion of $f|_k\gamma$, up to a cyclotomic unit. Its computation for Eisenstein series for $\Gamma_0(N)$ has been obtained in [Coh18]. We comment that it is also useful to consider this problem by collecting all the cusp expansion information of a classical modular form into a vector-valued modular form. This turns out to be particularly powerful when there is twist by a pair of Dirichlet characters, in fact vector-valued modular form allows us to focus mainly on the analytic part of the problem by encoding all the other algebraic information into a finite dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{Z}/N)$, which we call an irreducible congruence type. We find in Section 5 a complete description for the structure of spaces of vector-valued Eisenstein series using simple tools in representation theory.

We consider another important aspect of Eisenstein series in Section 7, that products of at most two Eisenstein series can span the whole space of modular forms of a given weight. The fundamental historic contribution on this aspect are due to Rankin [Ran52], Kohnen and Zagier [KZ84], and Borisov and Gunnells [BG01]. Based on these results, and the language of Hecke operator for congruence types developed in Raum [Wes17], we state in Theorem 7.5 for the first time a simplified formula to look at this aspect, and with all the knowledge about classical Eisenstein series, this allows us to compute general vector-valued modular forms of congruence types in a more efficient way. We notice the advantage of using the tool of representation theory here, for example in Lemma 7.3, which is rather difficult to prove (for a weaker version of it) by finding a purely combinatorial bijection, even only for the fact that the dimension of these isomorphic spaces are the same. Throughout this thesis we try to keep balance of using the conventional notations from classical modular forms and the standard language from representation theory. Along the way towards the main result, we also discover some results known in the literature from different point of view, which we summarize in Section 1.4.

1.2 Notation Throughout this note we use the following notations. We denote by \mathfrak{H} the upper half plane $\{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. We denote by Γ the full modular group $\mathrm{SL}_2(\mathbb{Z})$, and $\Gamma' \subseteq \Gamma$ a general congruence subgroup. Let Γ_∞ denote the subgroup of Γ generated by $-\mathrm{id}$ and $T := \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$. In other words, $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$. For each element $\gamma \in \Gamma$, entries of γ are denoted by $a(\gamma), b(\gamma), c(\gamma)$, and $d(\gamma)$. For a positive integer N , we define the congruence subgroups

$$\begin{aligned} \Gamma_0(N) &:= \{\gamma \in \Gamma : c(\gamma) \equiv 0 \pmod{N}\}, \\ \Gamma_1(N) &:= \{\gamma \in \Gamma : c(\gamma) \equiv 0, a(\gamma) \equiv d(\gamma) \equiv 1 \pmod{N}\}, \\ \Gamma(N) &:= \{\gamma \in \Gamma : b(\gamma) \equiv c(\gamma) \equiv 0, a(\gamma) \equiv d(\gamma) \equiv 1 \pmod{N}\}. \end{aligned}$$

Let $L_N := (\mathbb{Z}/N)^2$ and L_N^\times be the set of the primitive points of L_N , namely those of order N . Let λ denote an element in L_N (or L_N^\times).

We denote by $\mathcal{H}_k^{\text{md}}$ the space of holomorphic functions of moderate growth at all cusps, with respect to the weight k slash action. In other words, $f : \mathfrak{H} \rightarrow \mathbb{C} \in \mathcal{H}_k^{\text{md}}$ if and only if for each $\gamma \in \Gamma$,

$$(f|_k\gamma)(\tau) = O(1)$$

as $\tau \rightarrow i\infty$. For a finite dimensional complex representation ρ of Γ , we denote by $\mathcal{H}_k^{\text{md}}(\rho)$ the space of vector-valued holomorphic functions $f : \mathfrak{H} \rightarrow V(\rho)$ of moderate growth. Since $V(\rho)$ is finite-dimensional, we can make the identification

$$\mathcal{H}_k^{\text{md}}(\rho) := \mathcal{H}_k^{\text{md}} \otimes \rho.$$

For a congruence subgroup Γ' and a representation ρ of Γ' , we denote by $M_k(\Gamma', \rho) \subseteq \mathcal{H}_k^{\text{md}}(\rho)$ the space of holomorphic vector-valued modular forms of weight k and arithmetic type ρ . In other words, for a vector-valued function $f \in \mathcal{H}_k^{\text{md}}(\rho)$, $f \in M_k(\Gamma', \rho)$ if and only if

$$f(\gamma\tau) = (c\tau + d)^k \rho(\gamma) f(\tau)$$

holds for all $\gamma \in \Gamma$. The space $S_k(\Gamma', \rho) \subseteq M_k(\Gamma', \rho)$ of vector-valued cusp forms of weight k and arithmetic type ρ , is defined via

$$S_k(\Gamma', \rho) := \{f \in M_k(\Gamma', \rho) : \forall w \in V(\rho)^\vee : (w \circ f)(\tau) \rightarrow 0 \text{ as } \tau \rightarrow i\infty\}.$$

When $\Gamma' = \Gamma$, we often omit the group and use the notations $M_k(\rho)$ and $S_k(\rho)$, respectively. If classical modular forms are involved within the same context, we choose to use a different font, e.g. $\mathcal{M}_k, \mathcal{S}_k$, to further clarify them.

For a complex number z , we denote by $e(z) := \exp(2\pi iz)$.

Let \mathcal{R}_1 be a set of representatives for $\Gamma_1(N)\backslash\Gamma$, fixed once and for all. We also fix a set of representatives \mathcal{R}_0 for $\Gamma_0(N)\backslash\Gamma$ in Section 4, and do not require \mathcal{R}_1 and \mathcal{R}_0 to be compatible to each other in any sense.

1.3 Definitions

Definition 1.1 (Linear Permutation Representation). Let G be a discrete group. Let X be a finite G -set with the action $\pi : G \rightarrow \text{sym}(X)$. We define the permutation representation ρ of G corresponding to π , via $V(\rho) := \mathbb{C}^X$, and

$$\rho(g)\mathbf{e}_x := \mathbf{e}_{\pi(g)x},$$

for all $g \in G$ and $x \in X$.

We refer to Example 9.1 for a linear permutation of S_3 , presented via permutation matrices.

Definition 1.2 (Homomorphism of representations). Let G be a discrete group. Let ρ, σ be two representations of G , and V, W their representation spaces, respectively. We define a homomorphism of representations from ρ to σ , to be a \mathbb{C} -linear map $f : V \rightarrow W$, such that

$$\sigma(g) \circ f = f \circ \rho(g),$$

for all $g \in G$. We say ρ, σ are isomorphic, if there are homomorphisms f from ρ to σ , and g from σ to ρ , respectively, such that $f \circ g = \text{id}_V$ and $g \circ f = \text{id}_W$.

Definition 1.3. Given a congruence subgroup Γ' of Γ , let \mathcal{R} be a set of representatives for the cosets $\Gamma' \backslash \Gamma$. For $\beta \in \mathcal{R}$ and $\gamma \in \Gamma$, we define $I_\beta(\gamma) \in \Gamma'$ by the equation $\beta\gamma = I_\beta(\gamma)\overline{\beta\gamma}$, where $\overline{\beta\gamma} \in \mathcal{R}$ is the representative element of the class $[\beta\gamma]$.

We record that $I_\bullet(\bullet)$ a 1-cocycle¹.

Definition 1.4 (Induced Representation). Given a congruence subgroup Γ' of Γ and an arithmetic type ρ for Γ' , we fix a set of representatives \mathcal{R} for the cosets $\Gamma' \backslash \Gamma$. For such a fixed set \mathcal{R} of representatives, we define the induced representation $\text{Ind}_{\Gamma'}^\Gamma \rho$ by

$$\begin{aligned} V(\text{Ind}_{\Gamma'}^\Gamma \rho) &:= V(\rho) \otimes \mathbb{C}[\mathcal{R}] \text{ and} \\ \text{Ind}_{\Gamma'}^\Gamma \rho(\gamma)(\mathbf{e}_\beta) &:= \rho\left(\left(I_\beta(\gamma^{-1})\right)^{-1}\right)\mathbf{e}_{\overline{\beta\gamma^{-1}}}. \end{aligned}$$

Similar to the induced representation, we define the Hecke operators for representations and vector-valued modular forms. For a positive integer N , denote by

$$\Delta_N = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = N, 0 \leq b < d \right\}$$

a set of upper triangular matrices of determinant N that are inequivalent under the left Γ -action. For any 2 by 2 matrix β with integer coefficients and determinant N , there is a unique $\gamma \in \Gamma$ and $\delta \in \Delta_N$ such that $\beta = \gamma\delta$. We denote $\overline{\beta} := \delta$. Note that the right action π_N of Γ on Δ_N defined via

$$\pi_N(\gamma) : \beta \mapsto \overline{\beta\gamma}$$

yields a 1-cocycle defined by the equation $\beta\gamma = I_\beta(\gamma)\overline{\beta\gamma}$. Therefore, we can define vector-valued Hecke operators as follows.

Definition 1.5 (Hecke Operator for arithmetic types). Given an arithmetic type ρ for Γ , the representation $T_N(\rho)$ is defined via

$$V(T_N(\rho)) := V(\rho) \otimes \mathbb{C}[\Delta_N] \text{ and } (T_N(\rho))(\gamma)(v \otimes \mathbf{e}_\beta) := \rho\left(\left(I_\beta(\gamma^{-1})\right)^{-1}\right)(v) \otimes \mathbf{e}_{\overline{\beta\gamma^{-1}}}.$$

¹See Lemma 8.18.

For $\beta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$, and a complex function on the upper half plane $f : \mathfrak{H} \rightarrow \mathbb{C}$, we define the slash action

$$(f|_k\beta)(\tau) := \left(\frac{a}{d}\right)^{k/2} f(\beta\tau)$$

Definition 1.6 (Hecke Operator for vector-valued Modular Forms). Let N be a positive integer, ρ be an arithmetic type of finite kernel index for Γ . Given a vector-valued modular form $f \in M_k(\rho)$, $T_N(f)$ is defined via

$$(T_N(f))(\tau) := \sum_{\beta \in \Delta_N} (f|_k\beta) \otimes \mathbf{e}_\beta.$$

Definition 1.7 (Induced vector-valued Modular Forms). Given a congruence subgroup Γ' of Γ and a character χ of Γ' , we fix a set of representatives \mathcal{R} for the cosets $\Gamma' \backslash \Gamma$. Let $\rho := \mathrm{Ind}_{\Gamma'}^\Gamma \chi$ be the induced arithmetic type for Γ . We define the map

$$\mathrm{Ind} : M_k(\Gamma', \chi) \rightarrow M_k(\rho)$$

by sending a modular form $f \in M_k(\Gamma', \chi)$ to the vector-valued modular form

$$(\mathrm{Ind}(f))(\tau) := \sum_{\beta \in \mathcal{R}} (f|_k\beta) \otimes \mathbf{e}_\beta.$$

1.4 Structure of this thesis In Section 2, we start from some basic properties of congruence subgroups, that we often encounter in later sections. We then state and prove Lemma 3.5 in Section 3, a generalization of center split in representation theory, for explaining decomposition of modular forms later. At the end of this section, we show that vector-valued modular forms of a general congruence type can be identified as invariant vectors in holomorphic sections, and obtain the self-duality of the permutation representation ρ_N^\times , which is of central importance in the thesis.

In Section 4, we take an elementary number theoretic method to compute all the T -invariant vectors of $\mathrm{Ind}_{\Gamma_0(N)}^\Gamma \chi$, and obtain a characterization of orbits (double cosets) under the action of T to have a nonzero T -invariant vector. The central notion we define in this section is the “girth” of an orbit (which should correspond to the width of a cusp from a geometric point of view), and by using this notion, we obtain an arbitrary component of vector-valued Eisenstein series of this type.

In Section 5, we first prove the decomposition of modular forms into Eisenstein series part and cusp form part, both for vector-valued modular forms and for their components. We then use this fact to study the structure of spaces of vector-valued Eisenstein series of a congruence type, and reaches the goal at the end of this section.

We compute Fourier expansions for all the classical Eisenstein series of level N in Section 6, and emphasize on its mysterious constant term, which features the reflection phenomenon of special values of Hurwitz zeta functions.

And finally in Section, we state and prove the main result Theorem 7.5 based on the framework we adopt in this thesis and historic results on products of Eisenstein series.

The proofs we provide in Section 8 are from mostly our perspective suitable for this thesis. For auxiliary results that are very much in the standard literature, we cite their original proofs.

2 Congruence Subgroups

Throughout this section, we fix a set of representatives $[N] := \{0, 1, \dots, N-1\}$ for \mathbb{Z}/N . Let $[N]^\times$ denote the set of representatives for $(\mathbb{Z}/N)^\times$, chosen from $[N]$. Given $d \in [N]^\times$, we denote $d^{-1} \in [N]^\times$ to be the representative element of the inverse of d modulo N .

Lemma 2.1.

$$\Gamma_1(N) = \bigcup_{b \in [N]} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Gamma(N).$$

Proof. It suffices to show that for each $\gamma \in \Gamma_1(N)$,

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} \gamma \in \Gamma(N)$$

for some integer $b \in \{0, 1, \dots, N-1\}$. We choose such an integer b such that $b \equiv b(\gamma) \pmod{N}$, and the rest is clear from Lemma 8.31. ■

Lemma 2.2.

$$\Gamma_0(N) = \bigcup_{d \in [N]^\times} \begin{pmatrix} d^{-1} & * \\ N & d \end{pmatrix} \Gamma_1(N),$$

where for each matrix appeared in the formula, $*$ is determined by d .

Proof. It suffices to show that for each $\gamma \in \Gamma_0(N)$,

$$\begin{pmatrix} d^{-1} & * \\ N & d \end{pmatrix}^{-1} \gamma \in \Gamma_1(N)$$

for some integer $d \in [N]^\times$. We choose such a representative element d such that $d \equiv d(\gamma) \pmod{N}$, and the rest is then clear from Lemma 8.31. ■

Lemma 2.3. *We have the formula of index for the tower $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \Gamma$ as follows.*

$$\begin{aligned} [\Gamma : \Gamma_0(N)] &= \prod_{p|N} \left(1 + \frac{1}{p}\right) N, \\ [\Gamma_0(N) : \Gamma_1(N)] &= \prod_{p|N} \left(1 - \frac{1}{p}\right) N, \\ [\Gamma_1(N) : \Gamma(N)] &= N. \end{aligned}$$

Proof. First of all, we compute the size of the group $GL_2(\mathbb{F}_p)$ for a prime p . Since $|M_2(\mathbb{F}_p)| = p^4$, and the set of non-invertible matrices $M_2(\mathbb{F}_p) \setminus GL_2(\mathbb{F}_p)$ is a disjoint union of the following sets: matrices with the first row $(0, 0)$ and arbitrary second row, and matrices with the first row $\lambda \neq (0, 0)$ and the second row being one of $t\lambda$ for $t \in \mathbb{F}_p$, we have $|GL_2(\mathbb{F}_p)| = p^4 - (p^2 + (p^2 - 1)p) = (p - 1)^2 p(p + 1)$. The surjective group homomorphism $\det : GL_2(\mathbb{F}_p) \rightarrow \mathbb{F}_p$ thus yields the size of the kernel, i.e. $|SL_2(\mathbb{F}_p)| = \frac{|GL_2(\mathbb{F}_p)|}{|\mathbb{F}_p|} = (p - 1)p(p + 1) = p^3(1 - \frac{1}{p^2})$. Now we use induction to show that for each $e \in \mathbb{Z}_{\geq 1}$, we have $|SL_2(\mathbb{Z}/p^e)| = p^{3e}(1 - \frac{1}{p^2})$. For $e = 1$, it is showed to be true. Suppose this is true for $e \geq 1$, then we need to show that $|SL_2(\mathbb{Z}/p^{e+1})| = p^{3e+3}(1 - \frac{1}{p^2})$. Consider the group homomorphism

$$\begin{aligned} SL_2(\mathbb{Z}/p^{e+1}) &\longrightarrow SL_2(\mathbb{Z}/p^e) \\ \gamma &\longmapsto \gamma \pmod{p^e}, \end{aligned}$$

we find that it is surjective by solving linear equations over \mathbb{F}_p . Moreover, the kernel of the map is bijective with the set of elements $\begin{pmatrix} xp^{e+1} & yp^e \\ zp^e & wp^{e+1} \end{pmatrix}$ satisfying $x, y, z, w \in \{0, 1, \dots, p - 1\}$, and $x + w - y - z \equiv 0 \pmod{p}$. We then further identify this set with a subspace space in \mathbb{F}_p^4 of codimension 1 over \mathbb{F}_p , and therefore the kernel of the map has cardinality p^3 . By the induction assumption, we then find $|SL_2(\mathbb{Z}/p^{e+1})| = p^3 |SL_2(\mathbb{Z}/p^e)| = p^{3e+3}(1 - \frac{1}{p^2})$. Finally, by the Chinese remainder theorem and the fact that

$$SL_2(R \oplus S) \cong SL_2(R) \oplus SL_2(S)$$

for any commutative rings R and S , we conclude that $|SL_2(\mathbb{Z}/N)| = N^3 \prod_{p|N} (1 - \frac{1}{p^2})$. The canonical isomorphism $\Gamma/\Gamma(N) \cong SL_2(\mathbb{Z}/N)$ then gives the total index $[\Gamma : \Gamma(N)] = N^3 \prod_{p|N} (1 - \frac{1}{p^2})$. For the third identity $[\Gamma_1(N) : \Gamma(N)] = N$, it is clear from the surjective group homomorphism

$$\begin{aligned} \Gamma_1(N) &\longrightarrow \mathbb{Z}/N \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto b \pmod{N}, \end{aligned}$$

whose kernel is by definition $\Gamma(N)$. For the second identity $[\Gamma_0(N) : \Gamma_1(N)] = \prod_{p|N} (1 - \frac{1}{p})N$, it is clear from the surjective group homomorphism

$$\begin{aligned} \Gamma_0(N) &\longrightarrow (\mathbb{Z}/N)^\times \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto d \pmod{N}, \end{aligned}$$

whose kernel is by definition $\Gamma_1(N)$. Last, the first identity $[\Gamma : \Gamma_0(N)] = \prod_{p|N} (1 + \frac{1}{p})N$ is then clear from the total index and the rest two indices. \blacksquare

Lemma 2.4. $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$.

Proof. Consider the group homomorphism

$$\begin{aligned} \pi : \Gamma_0(N) &\longrightarrow (\mathbb{Z}/N)^\times, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto d \pmod{N}. \end{aligned}$$

It is clear that $\ker \rho = \Gamma_1(N)$, hence $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$. ■

Lemma 2.5.

$$\overline{\Gamma_0(N)} \cong (\mathbb{Z}/N)^\times \rtimes \overline{\Gamma_1(N)},$$

where $\overline{\Gamma_i(N)}$ denote the images of $\Gamma_i(N)$ in $\mathrm{SL}_2(\mathbb{Z}/N)$, for $i = 0$ and 1 , under the natural projection $\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/N)$.

Proof. Consider the group homomorphism

$$\begin{aligned} \pi : \overline{\Gamma_0(N)} &\longrightarrow (\mathbb{Z}/N)^\times, \\ \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} &\longmapsto \bar{d}. \end{aligned}$$

For any $\bar{d} \in (\mathbb{Z}/N)^\times$, we can lift it to an integer d coprime to N , and find integers a, b such that $ad - bN = 1$. So $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_0(N)$ yields a preimage of \bar{d} , and the map is surjective. It is clear that the kernel is equal to $\overline{\Gamma_1(N)}$. Therefore, $\overline{\Gamma_1(N)} \trianglelefteq \overline{\Gamma_0(N)}$ and $\overline{\Gamma_0(N)}/\overline{\Gamma_1(N)} \cong (\mathbb{Z}/N)^\times$. It then suffices to show that every element $\gamma_0 \in \overline{\Gamma_0(N)}$ can be uniquely written in the form $\gamma_0 = \begin{pmatrix} \bar{d}^{-1} & 0 \\ 0 & \bar{d} \end{pmatrix} \gamma_1$, where $\bar{d} \in (\mathbb{Z}/N)^\times$ and $\gamma_1 \in \overline{\Gamma_1(N)}$. Indeed, if such a form exists, it must be unique, as \bar{d} is equal to $d(\gamma_0)$. Conversely, if we let $\bar{d} := d(\gamma_0)$, we get $\gamma_1 := \begin{pmatrix} \bar{d}^{-1} & 0 \\ 0 & \bar{d} \end{pmatrix}^{-1} \gamma_0 \in \overline{\Gamma_1(N)}$. ■

Remark 2.6. Literally speaking we have just proved the inner semidirect product decomposition holds. Since each inner semidirect product can be naturally viewed as an outer semidirect product via the conjugation action, we will later view this lemma without specifying the word inner (resp. outer).

Corollary 2.7. $\overline{\Gamma_1(N)}$ is a normal subgroup of $\overline{\Gamma_0(N)}$.

3 Representation theory: vector-valued Modular Forms of congruence types

Throughout this section, we fix the following notations. Let \mathcal{R}_1 be a set of representatives for $\Gamma_1(N)\backslash\Gamma$, which is fixed throughout. Let $\mathbb{1} : \Gamma_1(N) \longrightarrow 1$ be the trivial character of $\Gamma_1(N)$. We define $\mathrm{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbb{1}$ based on \mathcal{R}_1 . Similarly, we fix a set of representatives \mathcal{R}_0 for $\Gamma_0(N)\backslash\Gamma$ and define $\mathrm{Ind}_{\Gamma_0(N)}^{\Gamma} \chi$ based on \mathcal{R}_0 , for a Dirichlet character $\chi \pmod{N}$.

Let $L_N := (\mathbb{Z}/N)^2$ and L_N^\times be the set of the primitive points of L_N . In other words, a point of L_N is in L_N^\times if and only if it has order N .

We identify L_N with row vectors of dimension 2, whose entries lie in \mathbb{Z}/N . Similarly, we identify L_N^\times with row vectors (c, d) , where $c, d \in \mathbb{Z}/N$ and $\gcd(c, d, N) = 1$. Then Γ naturally acts on both L_N and L_N^\times via the formula $\pi(\gamma)\lambda := \lambda\gamma^{-1}$ (so that they are still left actions) for $\lambda \in L_N$ and $\gamma \in \Gamma$. We denote these two actions by π_N and π_N^\times , respectively, and the corresponding permutation representations by ρ_N and ρ_N^\times .

Definition 3.1. We call ρ an arithmetic type for Γ if it is a finite dimensional complex representation of Γ . If furthermore there is a positive integer N , such that $\Gamma(N) \subseteq \ker \rho$, we call ρ a congruence type of Γ .

Lemma 3.2. Both ρ_N and ρ_N^\times are congruence types.

Proof. First of all, they are both finite dimensional since

$$\dim \rho_N^\times \leq \dim \rho_N = |(\mathbb{Z}/N)^2| = N^2.$$

By definition, $\gamma \in \ker \rho_N$ if and only if

$$\lambda\gamma^{-1} = \lambda$$

for all $\lambda \in (\mathbb{Z}/N)^2$. Clearly this holds for all $\gamma \in \Gamma(N)$, hence $\Gamma(N) \subseteq \ker \rho_N$. Similarly, we have $\Gamma(N) \subseteq \ker \rho_N^\times$. \blacksquare

Lemma 3.3. Let G be a group, such that it has finite center $Z := Z(G)$. Let \hat{Z} be the character group of Z , and ρ a finite dimensional complex representation of G . For each $\xi \in \hat{Z}$, we define a subspace of $V(\rho)$

$$V_\xi := \{v \in V(\rho) : \forall z \in Z : \rho(z)v = \xi(z)v\},$$

then V_ξ is a subrepresentation of ρ , which we also denote by ρ_ξ . Moreover, we have the decomposition

$$\rho = \bigoplus_{\xi \in \hat{Z}} \rho_\xi.$$

Proof. We first check that V_ξ is a subrepresentation of ρ . For any $v \in V_\xi$ and $g \in G$, we have that for all $z \in Z$,

$$\begin{aligned} \rho(z)(\rho(g)v) &= \rho(zg)v \\ &= \rho(gz)v = \rho(g)(\rho(z)v) \\ &= \rho(g)(\xi(z)v) = \xi(z)(\rho(g)v), \end{aligned} \tag{3.1}$$

where we used the fact that $zg = gz$, since Z is the center of G . Therefore $\rho(g)v \in V_\xi$. Next, we show the direct sum decomposition as vector spaces (hence also as representations by (3.1))

$$V(\rho) = \bigoplus_{\xi \in \hat{Z}} V_\xi. \tag{3.2}$$

Since Z is a finite abelian group, as a representation of Z , $\text{Res}_Z \rho$ can be decomposed into irreducible representations $\xi \in \hat{Z}$, which are all of degree 1 (Theorem 9 in [Ser77]). Let d_ξ be the multiplicity of ξ in the canonical decomposition of $\text{Res}_Z \rho$, we have

$$\text{Res}_Z \rho = \bigoplus_{\xi \in \hat{Z}} \xi^{\oplus d_\xi}. \quad (3.3)$$

Therefore, to prove Decomposition (3.2), it suffices to show that

$$V_\xi = V(\xi^{\oplus d_\xi})$$

for each $\xi \in \hat{Z}$. By definition of V_ξ , it is clear that $V(\xi^{\oplus d_\xi}) \subseteq V_\xi$. To see the other direction, let $v \in V_\xi$, hence

$$\rho(z)v = \xi(z)v \quad (3.4)$$

for all $z \in Z$. By the decomposition 3.3, we can write

$$v = \sum_{\xi' \in \hat{Z}} v_{\xi'},$$

where $v_{\xi'} \in V(\xi'^{\oplus d_{\xi'}})$ for each $\xi' \in \hat{Z}$. On the one hand, since $\rho(z)v_{\xi'} = \xi'(z)v_{\xi'}$, we have

$$\rho(z)v = \sum_{\xi' \in \hat{Z}} \rho(z)v_{\xi'} = \sum_{\xi' \in \hat{Z}} \xi'(z)v_{\xi'}.$$

On the other hand,

$$\xi(z)v = \sum_{\xi' \in \hat{Z}} \xi(z)v_{\xi'}.$$

Therefore, by the direct sum decomposition (3.3), Condition (3.4) implies that

$$\xi(z)v_{\xi'} = \xi'(z)v_{\xi'}$$

for all $\xi' \in \hat{Z}$ and $z \in Z$. For any $\xi' \neq \xi$, we can pick some element $z \in Z$ such that $\xi(z) \neq \xi'(z)$, therefore $v_{\xi'} = 0$ and $v = v_\xi \in V(\xi^{\oplus d_\xi})$. ■

Now, we apply Lemma 3.3 to our case, where $G = \Gamma$ and $Z = \langle \pm \text{id} \rangle$, which has two characters: ξ_+ is the one such that $\xi_+(-\text{id}) = 1$, and ξ_- is the other such that $\xi_+(-\text{id}) = -1$.

Corollary 3.4. *The representation ρ_N^\times can be decomposed as*

$$\rho_N^\times = \rho_{+,N}^\times \oplus \rho_{-,N}^\times,$$

where $V(\rho_{\pm,N}^\times)$ are the eigenspaces for the eigenvalues ± 1 of $\rho_N^\times(-\text{id})$, respectively.

Note that Lemma 3.3 is based on that the center of a group plays both rôles as an abelian group and a normal subgroup. We now give an alternative criterion for decomposition of representation, by viewing these two rôles separately.

Lemma 3.5. *Let B be a finite group, $A \subseteq B$ an abelian subgroup and $N \trianglelefteq B$ a normal subgroup, such that $B = AN$. Let ρ be a finite dimensional complex representation of B . For a character $\chi \in \hat{A}$, Let V_χ be the χ -isotypical component of ρ , that is,*

$$V_\chi := \{v \in V(\rho) : \forall a \in A, \rho(a)v = \chi(a)v\}.$$

Then the following statements are equivalent for each character $\chi \in \hat{A}$:

- i) V_χ is a subrepresentation of $\text{Res}_N \rho$.
- ii) $\rho(ana^{-1})v = \rho(n)v$ for all $a \in A$, $n \in N$ and $v \in V_\chi$.
- iii) V_χ is a subrepresentation of ρ .

We denote by ρ_χ the subrepresentation of ρ (for B) on V_χ , if Condition (iii) holds for some χ . If (iii) holds for all $\chi \in \hat{A}$, then we have

$$\rho = \bigoplus_{\chi \in \hat{A}} \rho_\chi. \tag{3.5}$$

In particular, this decomposition holds when $\text{Res}_N \rho$ is trivial.

Proof. Suppose Condition (iii) holds, then Condition (i) is clearly satisfied. Now assume Condition (i), then for all $a \in A$, $n \in N$ and $v \in V_\chi$, we have $\rho(n)v \in V_\chi$, hence

$$\rho(ana^{-1})v = \rho(a)\rho(n)(\rho(a^{-1})v) = \chi(a)^{-1}\rho(a)(\rho(n)v) = \chi(a)^{-1}\chi(a)\rho(n)v = \rho(n)v,$$

that is, Condition (ii) holds. Finally, assume Condition (ii), and we show that Condition (iii) holds, i.e. for any $v \in V_\chi$, $a \in A$ and $b \in B$, $\rho(a)(\rho(b)v) = \chi(a)\rho(b)v$. Since $B = AN$, we may write $b = \tilde{a}n$ for some $\tilde{a} \in A$ and $n \in N$. By Condition (ii), we have $\rho(n)v = \rho(a^{-1}na)v$. Together with the assumption that A is abelian, we get

$$\rho(a)(\rho(b)v) = \rho(a\tilde{a})\rho(n)v = \rho(\tilde{a}a)\rho(a^{-1}na)v = \chi(a)\rho(\tilde{a}n)v = \chi(a)\rho(b)v.$$

If Condition (iii) holds for all $\chi \in \hat{A}$, we simply repeat the second half of the proof of Lemma 3.3, with $Z \rightsquigarrow A$, and obtain the decomposition (3.5). ■

Remark 3.6. Note that this lemma does not assume $A \cap N = \{1_B\}$ as part of the condition, hence it includes the setting of semidirect product $B = A \ltimes N$ as a special case.

Lemma 3.7.

$$\rho_N \cong \bigoplus_{N' | N} \rho_{N'}^\times.$$

Proof. By Lemma 8.13, we have a bijection

$$u : L_N \longrightarrow \prod_{N'|N} L_{N'}^\times,$$

together with its inverse map

$$r : \prod_{N'|N} L_{N'}^\times \longrightarrow L_N.$$

Consider the linear map

$$\begin{aligned} f : V(\rho_N) &\longrightarrow V\left(\bigoplus_{N'|N} \rho_{N'}^\times\right), \\ \mathbf{e}_\lambda &\longmapsto \mathbf{e}_{u(\lambda)}. \end{aligned}$$

It is clear that the inverse is given by

$$\begin{aligned} g : V\left(\bigoplus_{N'|N} \rho_{N'}^\times\right) &\longrightarrow V(\rho_N), \\ \mathbf{e}_\lambda &\longmapsto \mathbf{e}_{r(\lambda)}. \end{aligned}$$

By Lemma 8.1, in order to see that f is an isomorphism of representations, it suffices to check the compatibility condition, i.e. the following identity

$$\left(\bigoplus_{N'|N} \rho_{N'}^\times(\gamma)\right) \circ f = f \circ \rho_N(\gamma)$$

holds for all $\gamma \in \Gamma$. In fact, for any $\lambda \in L_N^\times$, we have

$$\begin{aligned} &\left(\left(\bigoplus_{N'|N} \rho_{N'}^\times(\gamma)\right) \circ f\right)(\mathbf{e}_\lambda) \\ &= \left(\bigoplus_{N'|N} \rho_{N'}^\times(\gamma)\right)(\mathbf{e}_{u(\lambda)}). \end{aligned}$$

Let $N(\lambda)$ be the order of λ , that is, if $u(\lambda) \in L_{N(\lambda)}^\times$, we have then

$$\begin{aligned} &\left(\bigoplus_{N'|N} \rho_{N'}^\times(\gamma)\right)(\mathbf{e}_{u(\lambda)}) \\ &= \rho_{N(\lambda)}^\times(\gamma)(\mathbf{e}_{u(\lambda)}) \\ &= \mathbf{e}_{u(\lambda)\gamma^{-1}}. \end{aligned}$$

On the other hand, for any $\lambda \in L_N$, we have

$$\begin{aligned} &f \circ \rho_N(\gamma)(\mathbf{e}_\lambda) \\ &= \mathbf{e}_{u(\lambda\gamma^{-1})}. \end{aligned}$$

By Lemma 8.14, we conclude that $u(\lambda\gamma^{-1}) = u(\lambda)\gamma^{-1}$, which completes the proof. ■

Lemma 3.8.

$$\mathrm{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1} \cong \rho_N^{\times}.$$

Proof. By Lemma 8.15, we have a bijection $l : \mathcal{R}_1 \rightarrow L_N^{\times}$ and its inverse map $r : L_N^{\times} \rightarrow \mathcal{R}_1$. Consider the linear map

$$\begin{aligned} f : V(\mathrm{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}) &\longrightarrow V(\rho_N^{\times}), \\ \mathbf{e}_{\gamma} &\longmapsto \mathbf{e}_{l(\gamma)}. \end{aligned}$$

It is clear that the inverse is given by

$$\begin{aligned} g : V(\rho_N^{\times}) &\longrightarrow V(\mathrm{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}), \\ \mathbf{e}_{\lambda} &\longmapsto \mathbf{e}_{r(\lambda)}. \end{aligned}$$

By Lemma 8.1, in order to see that f is an isomorphism of representations, it suffices to check the compatibility condition, i.e. the following identity

$$\rho_N^{\times}(\gamma) \circ f = f \circ \mathrm{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}(\gamma)$$

holds for all $\gamma \in \Gamma$. Indeed, for any $\beta \in \mathcal{R}_1$, we have

$$(\rho_N^{\times}(\gamma) \circ f)(\mathbf{e}_{\beta}) = \mathbf{e}_{l(\beta)\gamma^{-1}} = \mathbf{e}_{l(\overline{\beta\gamma^{-1}})} = (f \circ \mathrm{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}(\gamma))(\mathbf{e}_{\beta}),$$

where the equality $l(\beta)\gamma^{-1} = l(\overline{\beta\gamma^{-1}})$ is proved in Lemma 8.17. ■

Lemma 3.9.

$$\mathrm{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1} \cong \bigoplus_{\chi' \pmod{N}} \mathrm{Ind}_{\Gamma_0(N)}^{\Gamma} \chi'.$$

Proof. Given a Dirichlet character $\chi \pmod{N}$, we define the following linear map:

$$\begin{aligned} p_{\chi} : V(\mathrm{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}) &\longrightarrow \bigoplus_{\chi' \pmod{N}} V(\mathrm{Ind}_{\Gamma_0(N)}^{\Gamma} \chi'), \\ \mathbf{e}_{\gamma} &\longmapsto \chi\left((I_{\mathrm{id}}(\gamma))^{-1}\right) \mathbf{e}_{\chi, \bar{\gamma}} \in V(\mathrm{Ind}_{\Gamma_0(N)}^{\Gamma} \chi), \end{aligned}$$

where for each $\gamma \in \mathcal{R}_1$, $\bar{\gamma} \in \mathcal{R}_0$ is the representative element for the class $[\gamma] \in \Gamma_0(N) \backslash \Gamma$. Moreover, Lemma 8.21 states that p_{χ} is a homomorphism of representations.

On the other hand, for each χ , we define a linear map

$$\begin{aligned} \iota_{\chi} : \bigoplus_{\chi' \pmod{N}} V(\mathrm{Ind}_{\Gamma_0(N)}^{\Gamma} \chi') &\longrightarrow V(\mathrm{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}), \\ \mathbf{e}_{\chi, \beta} &\longmapsto \sum_{\substack{\gamma' \in \mathcal{R}_1, \\ \bar{\gamma}' = \beta}} \chi(I_{\mathrm{id}}(\gamma')) \mathbf{e}_{\gamma'}, \\ \mathbf{e}_{\chi', \beta} &\longmapsto 0, \text{ for } \chi' \neq \chi. \end{aligned}$$

Now consider the homomorphism

$$p := \varphi(N)^{-\frac{1}{2}} \sum_{\chi \pmod{N}} p_\chi : V(\mathrm{Ind}_{\Gamma_1(N)}^\Gamma \mathbf{1}) \longrightarrow \bigoplus_{\chi \pmod{N}} V(\mathrm{Ind}_{\Gamma_0(N)}^\Gamma \chi),$$

together with the linear map

$$\iota := \varphi(N)^{-\frac{1}{2}} \sum_{\chi \pmod{N}} \iota_\chi : \bigoplus_{\chi \pmod{N}} V(\mathrm{Ind}_{\Gamma_0(N)}^\Gamma \chi) \longrightarrow V(\mathrm{Ind}_{\Gamma_1(N)}^\Gamma \mathbf{1}).$$

By Lemma 8.23, they are inverse maps to one another. Note that as a linear combination of homomorphisms, p is a homomorphism, hence an isomorphism between the two representations by Lemma 8.1. \blacksquare

For a Dirichlet character $\chi \pmod{N}$ and an integer $k > 2$ such that $\chi(-1) = (-1)^k$, let $\mathcal{M}_k(N, \chi)$ be the space of classical modular forms of weight k , level N , and character χ . That is,

$$\mathcal{M}_k(N, \chi) := \{f \in \mathcal{M}_k(\Gamma_1(N)) : \forall \gamma \in \Gamma_0(N) : f|_k \gamma = \chi(\gamma)f\}.$$

The following lemma can be found in Section 4.3 of [DS05], and here we first reproduce a proof in terms of representation theory. We record the Diamond operator $\langle d \rangle$ for $d \in (\mathbb{Z}/N)^\times$, defined via

$$\begin{aligned} \langle d \rangle : \mathcal{M}_k(\Gamma_1(N)) &\longrightarrow \mathcal{M}_k(\Gamma_1(N)), \\ f &\longmapsto f|_k \gamma, \end{aligned}$$

for any $\gamma = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \Gamma_0(N)$ such that $d_0 \equiv d \pmod{N}$. Note that the image $f|_k \gamma$ does not depend on the choice of γ , due to the fact that $\gamma_2 \in \Gamma_1(N)\gamma_1$, if both γ_1 and γ_2 have the same bottom row \pmod{N} , by Lemma 8.15. Furthermore, the Diamond operator also defines a representation, denoted by $\langle \cdot \rangle$, for the abelian group $(\mathbb{Z}/N)^\times$ on the space $\mathcal{M}_k(\Gamma_1(N))$, via

$$\langle \cdot \rangle : (\mathbb{Z}/N)^\times \ni d \longmapsto \langle d \rangle.$$

Since any finite dimensional representation of a finite abelian group splits into 1-dim irreducible representations, and all the 1-dim irreducible representations of $(\mathbb{Z}/N)^\times$ are just Dirichlet characters $\chi \pmod{N}$, we know that its isotypical components are the χ -eigenspaces.

Next, we apply representation theory to classical modular forms. It is clear that in the case of $\langle \cdot \rangle$, $\mathcal{M}_k(N, \chi)$ is the isotypical χ -component of the space $\mathcal{M}_k(\Gamma_1(N))$, i.e. we have

$$\mathcal{M}_k(N, \chi) = \left\{ f \in \mathcal{M}_k(\Gamma_1(N)) : \forall d \in (\mathbb{Z}/N)^\times : \langle d \rangle f = \chi(d)f \right\}. \quad (3.6)$$

Therefore, by the canonical decomposition of the representation $\langle \cdot \rangle$ into its isotypical components, we obtain Lemma 3.10 below.

Conceptually, we record another proof as an application of Lemma 3.5. Consider the linear action of $\Gamma_0(N)$ on $\mathcal{M}_k(\Gamma_1(N))$, via

$$\gamma f := f|_k \gamma. \quad (3.7)$$

Note that here we are literally defining the right action, since we follow the convention of defining $\mathcal{M}_k(N, \chi)$ as the χ -eigenspace, rather than the χ^{-1} -eigenspace, see for instance [DS05]. Fortunately, (3.7) is in fact also a left action. Indeed, by Lemma 2.4, $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$, and f is invariant by the action of $\Gamma_1(N)$, hence the linear action (3.7) factors through the action by the quotient $\Gamma_1(N)\backslash\Gamma_0(N) \cong (\mathbb{Z}/N)^\times$, which is abelian. It is well defined, in that for any $\gamma_1 \in \Gamma_1(N)$, $\gamma\gamma_1 = \gamma'_1\gamma$ for some $\gamma'_1 \in \Gamma_1(N)$, hence $(f|_k \gamma)|_k \gamma_1 = f|_k(\gamma\gamma_1) = f|_k(\gamma'_1\gamma) = f|_k \gamma$. Therefore, we have a complex representation ρ for $\Gamma_0(N)$ on $\mathcal{M}_k(\Gamma_1(N))$. Since $\Gamma(N) \subseteq \ker \rho$, it follows that ρ induces the homomorphism

$$\bar{\rho} : \overline{\Gamma_0(N)} \cong \Gamma_0(N)/\Gamma(N) \longrightarrow \mathrm{GL}(\mathcal{M}_k(\Gamma_1(N))).$$

By Lemma 2.5, we have

$$\overline{\Gamma_0(N)} \cong (\mathbb{Z}/N)^\times \times \overline{\Gamma_1(N)}.$$

Therefore, for the group $B := \overline{\Gamma_0(N)}$, there is a subgroup A isomorphic to $(\mathbb{Z}/N)^\times$. Moreover, the normal subgroup $N := \overline{\Gamma_1(N)}$ acts trivially on $\mathcal{M}_k(\Gamma_1(N))$, i.e. $\mathrm{Res}_N \bar{\rho}$ is trivial. We then apply Lemma 3.5 to $\bar{\rho}$, and get

$$V(\bar{\rho}) = \bigoplus_{\chi \in \hat{A}} V(\bar{\rho}_\chi).$$

By definition, $V(\bar{\rho}) = \mathcal{M}_k(\Gamma_1(N))$, and each $\chi \in \hat{A}$ is actually a Dirichlet character $(\mathrm{mod} N)$, so that

$$V(\bar{\rho}_\chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) : \forall \gamma \in A : f|_k \gamma = \chi(\gamma)f\} = \mathcal{M}_k(N, \chi),$$

which completes the proof.

Now we will also present a more explicit proof based on the spirit of Lemma 3.9. Instead of directly employing the extrinsic projection p_χ in the proof of Lemma 3.9, we need to find an intrinsic analogue of it. The key is to average the Diamond operators twisted by χ over $(\mathbb{Z}/N)^\times$ to obtain this projection map.

Lemma 3.10.

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi \pmod{N}} \mathcal{M}_k(N, \chi).$$

Proof. Consider the linear map

$$\begin{aligned} p_\chi : \mathcal{M}_k(\Gamma_1(N)) &\longrightarrow \mathcal{M}_k(N, \chi), \\ f &\longmapsto \frac{1}{\varphi(N)} \sum_{d \in (\mathbb{Z}/N)^\times} \chi(d)^{-1} \langle d \rangle f. \end{aligned}$$

The map is well-defined, since for $f \in \mathcal{M}_k(\Gamma_1(N))$ and any $d' \in (\mathbb{Z}/N)^\times$, we have

$$\begin{aligned} \langle d' \rangle p_\chi(f) &= \frac{1}{\varphi(N)} \sum_{d \in (\mathbb{Z}/N)^\times} \chi(d)^{-1} \langle d' d \rangle f \\ &= \chi(d') \frac{1}{\varphi(N)} \sum_{d' d \in (\mathbb{Z}/N)^\times} \chi(d' d)^{-1} \langle d' d \rangle f \\ &= \chi(d') p_\chi(f), \end{aligned}$$

so that by Equation (3.6), $p_\chi(f) \in \mathcal{M}_k(N, \chi)$. Then we define the linear map

$$\begin{aligned} p : \mathcal{M}_k(\Gamma_1(N)) &\longrightarrow \bigoplus_{\chi \pmod{N}} \mathcal{M}_k(N, \chi), \\ f &\longmapsto \left(p_\chi(f) \right)_\chi. \end{aligned}$$

We claim that p is an isomorphism, with the inverse map given by

$$\begin{aligned} \iota : \bigoplus_{\chi \pmod{N}} \mathcal{M}_k(N, \chi) &\longrightarrow \mathcal{M}_k(\Gamma_1(N)), \\ (f_\chi)_\chi &\longmapsto \sum_{\chi \pmod{N}} f_\chi. \end{aligned}$$

In fact, following the definitions above and applying the orthogonal relations for Dirichlet characters, we have

$$\begin{aligned} \iota \circ p(f) &= \sum_{\chi \pmod{N}} p_\chi(f) \\ &= \frac{1}{\varphi(N)} \sum_{\chi \pmod{N}} \sum_{d \in (\mathbb{Z}/N)^\times} \chi(d)^{-1} \langle d \rangle f \\ &= \sum_{d \in (\mathbb{Z}/N)^\times} \left(\frac{1}{\varphi(N)} \sum_{\chi \pmod{N}} \chi(d)^{-1} \right) \langle d \rangle f \\ &= \sum_{d \in (\mathbb{Z}/N)^\times} \delta_{d \equiv 1 \pmod{N}} \langle d \rangle f \\ &= \langle 1 \rangle f \\ &= f, \end{aligned}$$

where δ is the Kronecker delta. Similarly, we also have

$$\begin{aligned}
 p \circ \iota((f_\chi)_\chi) &= p\left(\sum_{\chi' \pmod{N}} f_{\chi'}\right) \\
 &= \left(\sum_{\chi' \pmod{N}} p_\chi(f_{\chi'})\right)_\chi \\
 &= \left(\sum_{\chi' \pmod{N}} \frac{1}{\varphi(N)} \sum_{d \in (\mathbb{Z}/N)^\times} \chi(d)^{-1} \langle d \rangle f_{\chi'}\right)_\chi \\
 &= \left(\sum_{\chi' \pmod{N}} \frac{1}{\varphi(N)} \sum_{d \in (\mathbb{Z}/N)^\times} \chi(d)^{-1} \chi'(d) f_{\chi'}\right)_\chi \\
 &= \left(\sum_{\chi' \pmod{N}} \left(\frac{1}{\varphi(N)} \sum_{d \in (\mathbb{Z}/N)^\times} \frac{\chi'(d)}{\chi(d)}\right) f_{\chi'}\right)_\chi \\
 &= \left(\sum_{\chi' \pmod{N}} \delta_{\chi', \chi} f_{\chi'}\right)_\chi \\
 &= (f_\chi)_\chi.
 \end{aligned}$$

Since $\mathcal{M}_k(N, \chi)$ are already subspaces of $\mathcal{M}_k(\Gamma_1(N))$, the isomorphism p gives the set-theoretic identity. \blacksquare

We denote the (classical) cusp forms of weight k , level N , and character $\chi \pmod{N}$ by $\mathcal{S}_k(N, \chi) := \mathcal{M}_k(N, \chi) \cap \mathcal{S}_k(\Gamma_1(N))$, and by $\mathcal{E}_k(\Gamma_1(N)) := \mathcal{M}_k(\Gamma_1(N))/\mathcal{S}_k(\Gamma_1(N))$ and $\mathcal{E}_k(N, \chi) := \mathcal{M}_k(N, \chi)/\mathcal{S}_k(N, \chi)$ the quotient space of modular forms by cusp forms (which should not be confused later with the notation $\mathcal{E}_{k,N}$ for the space of classical Eisenstein series), following the convention of this notation for Eisenstein series, adopted for instance in [DS05] (and in later chapters in [DS05] it can be shown that it is canonically isomorphic to a subspace of modular forms as the complement of the cusp forms). Then we have the same type of decomposition as in Lemma 3.10 for them as well.

Corollary 3.11.

$$\begin{aligned}
 \mathcal{S}_k(\Gamma_1(N)) &= \bigoplus_{\chi \pmod{N}} \mathcal{S}_k(N, \chi), \\
 \mathcal{E}_k(\Gamma_1(N)) &\cong \bigoplus_{\chi \pmod{N}} \mathcal{E}_k(N, \chi).
 \end{aligned}$$

Proof. For the first decomposition, we simply repeat the proof of Lemma 3.10, and check furthermore that the maps are well-defined also between cusp forms, which is clear by definition.

For the second decomposition, we consider the surjective linear map

$$\begin{aligned}
 \varphi: \bigoplus_{\chi \pmod{N}} \mathcal{M}_k(N, \chi) &\twoheadrightarrow \bigoplus_{\chi \pmod{N}} \mathcal{M}_k(N, \chi)/\mathcal{S}_k(N, \chi) \\
 \bigoplus_{\chi \pmod{N}} f_\chi &\longmapsto \bigoplus_{\chi \pmod{N}} (f_\chi \bmod \mathcal{S}_k(N, \chi)).
 \end{aligned}$$

It is clear that $\ker \varphi = \bigoplus_{\chi \pmod{N}} \mathcal{S}_k(N, \chi)$, hence we have

$$\begin{aligned} \mathcal{E}_k(\Gamma_1(N)) &= \mathcal{M}_k(\Gamma_1(N)) / \mathcal{S}_k(\Gamma_1(N)) = \frac{\bigoplus_{\chi \pmod{N}} \mathcal{M}_k(N, \chi)}{\bigoplus_{\chi \pmod{N}} \mathcal{S}_k(N, \chi)} \\ &\cong \bigoplus_{\chi \pmod{N}} \mathcal{M}_k(N, \chi) / \mathcal{S}_k(N, \chi) = \bigoplus_{\chi \pmod{N}} \mathcal{E}_k(N, \chi). \quad \blacksquare \end{aligned}$$

The examples for classical modular forms point to the decomposition of vector-valued modular forms, from their arithmetic types. To start the preparation, we view M_k as a functor from the category of arithmetic types for Γ to the category of \mathbb{C} -vector spaces of vector-valued modular forms of weight k and certain arithmetic type. In fact, given a morphism between two arithmetic types for Γ

$$\varphi : \rho_1 \longrightarrow \rho_2,$$

we can define the associated morphism

$$\begin{aligned} M_k(\varphi) : M_k(\rho_1) &\longrightarrow M_k(\rho_2), \\ f &\longmapsto \varphi \circ f. \end{aligned}$$

This is well-defined, since

$$(\varphi \circ f)|_{k, \rho_2 \gamma} = (\rho_2(\gamma^{-1}) \circ \varphi) f|_{k \gamma} = (\varphi \circ \rho_1(\gamma^{-1})) f|_{k \gamma} = \varphi \circ f|_{k, \rho_1 \gamma} = \varphi \circ f.$$

Moreover, if ϕ is a morphism between ρ_1 and ρ_2 , and ψ is a morphism between ρ_2 and ρ_3 , then we have that for all $f \in M_k(\rho)$,

$$M_k(\psi \circ \phi)(f) = (\psi \circ \phi) \circ f = \psi \circ (\phi \circ f) = (M_k(\psi) \circ M_k(\phi))(f),$$

hence $M_k(\psi \circ \phi) = M_k(\psi) \circ M_k(\phi)$. Further, we have that for all $f \in M_k(\rho)$,

$$M_k(\text{id}_\rho)(f) = \text{id}_\rho \circ f = f,$$

hence $M_k(\text{id}_\rho) = \text{id}_{M_k(\rho)}$. As a corollary to the functoriality of M_k , we have

Lemma 3.12. *Let ρ_1, ρ_2 be two arithmetic types for Γ . If $\rho_1 \cong \rho_2$, then $M_k(\rho_1) \cong M_k(\rho_2)$.*

We also remark that M_k keeps the direct sums. Here we give a concrete proof.

Lemma 3.13. *For each $i \in \{1, \dots, n\}$, let ρ_i be an arithmetic type for Γ , then we have*

$$M_k(\bigoplus_i \rho_i) \cong \bigoplus_i M_k(\rho_i).$$

Proof. If we fix a linear isomorphism

$$p : V(\oplus_i \rho_i) \longrightarrow \bigoplus_i V(\rho_i),$$

with the inverse map ι , then we can write down explicitly the corresponding isomorphism

$$\tilde{p} : M_k(\oplus_i \rho_i) \longrightarrow \bigoplus_i M_k(\rho_i),$$

via $\tilde{p}(f) := p \circ f$ for $f \in M_k(\oplus_i \rho_i)$. Its inverse $\tilde{\iota}$ is given by $\tilde{\iota}(h) := \iota \circ h$ for $h \in \bigoplus_i M_k(\rho_i)$. ■

Proposition 3.14.

$$M_k(\rho_N) \cong \bigoplus_{N'|N} M_k(\rho_{N'}^\times) \cong \bigoplus_{N'|N} M_k(\text{Ind}_{\Gamma_1(N')}^\Gamma \mathbb{1}).$$

Proof. The first isomorphism follows from Lemma 3.7, Lemma 3.12 and Lemma 3.13. And the second isomorphism follows from Lemma 3.8, Lemma 3.12 and Lemma 3.13. ■

Recall that $\mathcal{H}_k^{\text{md}}$ is the space of holomorphic functions of moderate growth with respect to weight k , at all cusps. Since for each $\gamma' \in \Gamma$,

$$f|_k \gamma'(\tau) = O(1)$$

as $\tau \rightarrow i\infty$, for all $\gamma \in \Gamma$, we have

$$(f|_k \gamma^{-1})|_k \gamma'(\tau) = (f|_k(\gamma^{-1} \gamma'))(\tau) = O(1)$$

as $\tau \rightarrow i\infty$ as well. Therefore, $\mathcal{H}_k^{\text{md}}$ is closed under the group action $\gamma f := f|_k \gamma^{-1}$, hence is naturally a (left) representation of Γ . Depending on the context, the slash action $\cdot|_k$ can be omitted, and the space $\mathcal{H}_k^{\text{md}}$ will then also denote the representations.

Proposition 3.15. *Let ρ be an arithmetic type for Γ , then we have*

$$\begin{aligned} M_k(\rho) &\cong \text{Hom}_\Gamma(\mathbb{1}, \mathcal{H}_k^{\text{md}}(\rho)) \\ &\cong \text{Hom}_\Gamma(\rho^\vee, \mathcal{H}_k^{\text{md}}), \end{aligned}$$

where we can also use other notations for the Hom set, namely

$$\text{Hom}_\Gamma(\mathbb{1}, \mathcal{H}_k^{\text{md}}(\rho)) = \text{Hom}_\Gamma(\mathbb{1}, \mathcal{H}_k^{\text{md}} \otimes \rho) \cong \text{H}^0(\Gamma, \mathcal{H}_k^{\text{md}} \otimes \rho) = \left(\mathcal{H}_k^{\text{md}} \otimes \rho \right)^\Gamma.$$

Proof. For the first isomorphism, consider the linear map

$$s : M_k(\rho) \longrightarrow \text{Hom}_\Gamma(\mathbb{1}, \mathcal{H}_k^{\text{md}}(\rho)),$$

which sends a vector-valued modular form $f \in M_k(\rho)$ to the homomorphism $s(f) := (1 \mapsto f)$. It is clear that s is injective, if well-defined. To see s is well-defined, i.e. $s(f) \in \text{Hom}(\mathbf{1}, \mathcal{H}_k^{\text{md}}(\rho))$, and that s is surjective, we claim that for any $h \in \mathcal{H}_k^{\text{md}}(\rho)$, $(1 \mapsto h) \in \text{Hom}(\mathbf{1}, \mathcal{H}_k^{\text{md}}(\rho))$ if and only if $h \in M_k(\rho)$. Indeed, by Definition 1.2, $(1 \mapsto h) \in \text{Hom}(\mathbf{1}, \mathcal{H}_k^{\text{md}}(\rho))$ is tantamount to that

$$h = \rho(\gamma)(h|_k\gamma^{-1}) = h|_{k,\rho}(\gamma^{-1})$$

holds for all $\gamma \in \Gamma$, i.e. $h \in M_k(\rho)$. This completes the proof for the first isomorphism. As for the second isomorphism, we apply Lemma 8.24 by inserting $\pi \rightsquigarrow \mathbf{1}$, $\rho \rightsquigarrow \mathcal{H}_k^{\text{md}}$ (with the slash action), and $\sigma \rightsquigarrow \rho$, with the fact that $\mathbf{1} \otimes \rho \cong \rho$ for any representation ρ . ■

Note that ρ_N^\times has self-duality (although the one we construct below is not canonical). In fact, since Γ is generated by $T = \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, a linear map $\varphi : V(\rho_N^\times) \rightarrow V(\rho_N^{\times\vee})$ intertwines with ρ_N^\times and $\rho_N^{\times\vee}$ if and only if it satisfies

$$\rho_N^{\times\vee}(\gamma) \circ \varphi = \varphi \circ \rho_N^\times(\gamma)$$

for $\gamma = T, S$. Note that since $\rho_N^{\times\vee}(\gamma)w = w \circ \rho_N^\times(\gamma^{-1})$ for $w \in V(\rho_N^{\times\vee})$, this condition in terms of the matrix form $\varphi(\mathbf{e}_\lambda) = \sum_{\mu \in L_N^\times} \kappa_{\lambda,\mu} \mathbf{e}_\mu^\vee$, with respect to a chosen basis \mathbf{e}_λ for $V(\rho_N^\times)$ and its dual basis \mathbf{e}_λ^\vee , can be written as

$$\kappa\left((c, d), (g, h)\right) = \kappa\left((c, c+d), (g, g+h)\right) = \kappa\left((d, -c), (h, -g)\right), \quad (3.8)$$

with the identification $(\kappa_{\lambda,\mu}) \rightsquigarrow \kappa : L_N^\times \times L_N^\times \rightarrow \mathbb{C}$. Note that for any function $\xi : \mathbb{Z}/N \rightarrow \mathbb{C}$, $\kappa\left((c, d), (g, h)\right) = \xi(ch - dg)$ is a natural construction satisfying Equation (3.8). In other words, let $\pi : L_N^\times \times L_N^\times \rightarrow \mathbb{Z}/N$ be the map sending $((c, d), (g, h)) \in L_N^\times \times L_N^\times$ to $ch - dg$, then for all functions $\xi : \mathbb{Z}/N \rightarrow \mathbb{C}$, we have $\kappa = \xi \circ \pi$ satisfies Equation (3.8). Therefore, we may construct self-dual maps in this way, and furthermore, we can find those with matrix forms over \mathbb{Q} .

Lemma 3.16. *There is an isomorphism*

$$\varphi : \rho_N^\times \longrightarrow \rho_N^{\times\vee}$$

whose matrix form $(\varphi_{\lambda,\mu})$ has entries $\varphi_{\lambda,\mu} \in \mathbb{Q}$ for all $\lambda, \mu \in L_N^\times$, with respect to the basis $\mathbf{e}_\lambda \in V(\rho)$ and $\mathbf{e}_\mu^\vee \in V(\rho)^\vee$.

Proof. Let $\pi : L_N^\times \times L_N^\times \rightarrow \mathbb{Z}/N$ be the map sending $((c, d), (g, h)) \in L_N^\times \times L_N^\times$ to $ch - dg \in \mathbb{Z}/N$. From the discussion above, we know that the composition $\xi \circ \pi$ always yields a homomorphism of representations for any function $\xi : \mathbb{Z}/N \rightarrow \mathbb{C}$. By Lemma 8.1 and the fact that both linear spaces have the same dimension, if φ is furthermore injective, then it is an isomorphism of representations. Therefore, we only need to look for some function ξ taking values in \mathbb{Q} , so that $\det(\kappa(\lambda, \mu)) \neq 0$ for $\kappa = \xi \circ \pi$.

Let K be a field, we view $\kappa : L_N^\times \times L_N^\times \longrightarrow K$ as a point in the affine space $\mathbb{A}_K^{L_N^\times \times L_N^\times}$ (resp. ξ as a point in $\mathbb{A}_K^{\mathbb{Z}/N}$), and the determinant map $\det_K : \mathbb{A}_K^{L_N^\times \times L_N^\times} \longrightarrow K$ as a morphism of K -affine varieties, which then induces the morphism

$$\begin{aligned} \widetilde{\det}_K : \mathbb{A}_K^{\mathbb{Z}/N} &\longrightarrow K, \\ \xi &\longmapsto \det_K(\xi \circ \pi). \end{aligned}$$

By definition, $\widetilde{\det}_K$ is always defined over \mathbb{Q} for any field K . First of all, we show that $\widetilde{\det}_K$ is a nonzero morphism for any field K . It suffices to show that the coefficient of the term $\xi(\bar{0})^{|L_N^\times|}$ that appears in the determinant has nonzero coefficient. In fact, when we expand the determinant formally, each monomial contributes a coefficient ± 1 to the sum, therefore it suffices to show that the number of all the monomials of the form $\xi(\bar{0})^{|L_N^\times|}$ is an odd integer. This is equivalent to showing that $|S| \equiv 1 \pmod{2}$ for the set

$$S := \{\sigma \in S(L_N^\times) : \forall \lambda \in L_N^\times : \pi(\lambda, \sigma(\lambda)) = \bar{0}\}.$$

Note that π is skew symmetric, i.e. $\pi(\lambda, \mu) = -\pi(\mu, \lambda)$ for all $\lambda, \mu \in L_N^\times$, we have

$$\forall \lambda \in L_N^\times : \pi(\lambda, \sigma(\lambda)) = \bar{0} \iff \forall \lambda \in L_N^\times : \pi(\sigma(\lambda), \lambda) = \bar{0} \iff \forall \lambda \in L_N^\times : \pi(\lambda, \sigma^{-1}\lambda) = \bar{0},$$

hence $\sigma \in S$ if and only if $\sigma^{-1} \in S$. Moreover, it is clear that $\text{id} \in S$, hence $|S| \equiv 1 \pmod{2}$.

Now we put $K = \mathbb{R}$, and let $\xi \in \widetilde{\mathbb{A}}_{\mathbb{R}}^{\mathbb{Z}/N}$ be a point with nonzero image, i.e. $\widetilde{\det}_{\mathbb{R}}(\xi) \neq 0$. Since $\mathbb{A}_{\mathbb{Q}}^{\mathbb{Z}/N}$ is dense in $\mathbb{A}_{\mathbb{R}}^{\mathbb{Z}/N}$ and $\widetilde{\det}_{\mathbb{R}}$ is continuous, with respect to the normal topology induced from the topology of the field \mathbb{R} , we find infinitely many (countable) points $\xi_n \in \mathbb{A}_{\mathbb{Q}}^{\mathbb{Z}/N}$ for $n \in \mathbb{N}$ with nonzero images, in some neighbourhood of ξ . Putting $\kappa_n := \xi_n \circ \pi$, we thus find infinitely many (countable) different isomorphisms $\varphi_n : \rho_N^\times \longrightarrow \rho_N^{\times \vee}$ defined over \mathbb{Q} with the matrix form κ_n . \blacksquare

Remark 3.17. In fact, if we consider the self-duality of ρ_N instead of ρ_N^\times , then the Fourier transform of vector-valued functions together with its inverse transform provides a concrete example of self-dual isomorphism. More explicitly, similar to the proof above, we can choose $\xi := e(\frac{\cdot}{N})$ and find that $\xi \circ \pi$ indeed defines an isomorphism between $V(\rho_N)$ and $V(\rho_N)^\vee$. Here $\pi : (\mathbb{Z}/N)^2 \times (\mathbb{Z}/N)^2 \longrightarrow \mathbb{Z}/N$ is the map sending $((c, d), (g, h)) \in (\mathbb{Z}/N)^2 \times (\mathbb{Z}/N)^2$ to $ch - dg \in \mathbb{Z}/N$. Further discussions can be found in Section 3.1 of [Car12].

As a corollary of Proposition 3.15 and Lemma 3.16, we have

Corollary 3.18.

$$M_k(\rho_N^\times) \cong \text{Hom}(\mathbf{1}, \mathcal{H}_k(\rho_N^\times)) \cong \text{Hom}(\rho_N^\times, \mathcal{H}_k).$$

We record here some basic properties of induced representations.

Lemma 3.19. *Let $G_1 \subseteq G_2 \subseteq G_3$ be finite groups, and ρ an arithmetic type for G_1 , then we have*

$$\mathrm{Ind}_{G_2}^{G_3}(\mathrm{Ind}_{G_1}^{G_2}\rho) \cong \mathrm{Ind}_{G_1}^{G_3}\rho.$$

Proof. See chapter 7 in [Ser77], for instance. ■

Lemma 3.20. *Let $G_1 \subseteq G_2 \subseteq G_3$ be finite groups, then we have*

$$\mathrm{Ind}_{G_2}^{G_3}\mathbb{1} \hookrightarrow \mathrm{Ind}_{G_1}^{G_3}\mathbb{1}.$$

Proof. By Lemma 3.19, we have

$$\mathrm{Ind}_{G_2}^{G_3}\mathbb{1} \hookrightarrow \mathrm{Ind}_{G_2}^{G_3}(\mathrm{Ind}_{G_1}^{G_2}\mathbb{1}) \cong \mathrm{Ind}_{G_1}^{G_3}\mathbb{1}. \quad \blacksquare$$

4 Vector-valued Eisenstein series and their components for $\Gamma_0(N)$

The goal of this section is to compute components of vector-valued Eisenstein series in a concrete case, from the perspective of arithmetic. From the results obtained in this section, we could also subsume classical Eisenstein series under this class.

Definition 4.1 (Slash action). Let ρ be an arithmetic type for Γ . For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and every vector-valued function $f : \mathfrak{H} \rightarrow V(\rho)$, the slash action of γ is defined via

$$(f|_{k,\rho}\gamma)(\tau) := (c\tau + d)^{-k}\rho(\gamma^{-1})f(\gamma\tau).^2$$

In particular, for a vector $v \in V(\rho)$, the slash action is given by

$$(v|_{k,\rho}\gamma)(\tau) := (c\tau + d)^{-k}\rho(\gamma^{-1})v.$$

Lemma 4.2. *The slash action is a group action of Γ .*

Proof. We need to follow a straightforward computation, and note that $\rho(\gamma^{-1})$ instead of $\rho(\gamma)$ is the correct version, in that it transforms the right action to the left one. ■

Throughout this paper, the space of (Γ_∞, ρ) -invariant vectors $v \in V(\rho)$ is denoted by $V(\rho)^{\Gamma_\infty}$. Namely, $v \in V(\rho)^{\Gamma_\infty}$ if and only if $v|_{k,\rho}\gamma = v$ for all $\gamma \in \Gamma_\infty$.

Definition 4.3 (Vector-Valued Eisenstein Series). Let ρ be an arithmetic type for Γ , and $k > 2$ an integer. Given a (Γ_∞, ρ) -invariant vector $v \in V(\rho)^{\Gamma_\infty}$, we define the vector-valued Eisenstein series of weight k , arithmetic type ρ , and constant term v via

$$E_{k,\rho,v}(\tau) := \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma} (v|_{k,\rho}\gamma)(\tau)$$

assuming that the right hand side converges absolutely at every point $\tau \in \mathfrak{H}$ ³. We denote by $E_k(\rho)$ the space linearly spanned by $E_{k,\rho,v}$ for $v \in V(\rho)^{\Gamma_\infty}$.

³If ρ is furthermore of finite kernel index, this condition is automatically satisfied.

Remark 4.4. Note that some authors define $E_{k,\rho,v}$ in a slightly different way than ours. They assume that $\ker \rho$ has finite index in Γ , and define

$$\tilde{E}_{k,\rho,v} := \frac{1}{[\Gamma_\infty : \Gamma_\infty(v)]} \sum_{[\gamma] \in \Gamma_\infty(v) \backslash \Gamma} (v|_{k,\rho} \gamma),$$

where $\Gamma_\infty(v) := \Gamma_\infty \cap \text{Stab}(v)$. The space $\tilde{E}_k(\rho)$ is then defined as the linear span of $\tilde{E}_{k,\rho,v}$ for $v \in V(\rho)$. In fact, under their assumption, these two versions agree, i.e., $\tilde{E}_{k,\rho,v} = E_{k,\rho,v}$ for all $v \in V(\rho)^{\Gamma_\infty}$, and $\tilde{E}_k(\rho) = E_k(\rho)$. See Lemma 8.25.

For the remainder of this section, let χ be a Dirichlet character mod N , and let $\rho := \text{Ind}_{\Gamma_0(N)}^\Gamma \chi$. In particular, ρ is a congruence type of Γ .

First we identify the cosets $\Gamma_0(N) \backslash \Gamma$ with the projective line over the ring \mathbb{Z}/N , by sending the bottom row of a matrix to the affine coordinates of the projective line⁴. Second it is worth mentioning that the Γ_∞ -action is so special that the ordering of natural numbers naturally helps with simplifying the computation. Bearing these points in mind, we make the following definition.

Definition 4.5 (The Standard Representatives of $\Gamma_0(N) \backslash \Gamma$). Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ be an arbitrary element. We define a nonnegative integer associated to the coset $[\gamma] \in \Gamma_0(N) \backslash \Gamma$, called $c([\gamma])$, to be $\gcd(c, N)$ if it is not equal to N . Otherwise we define $c([\gamma])$ to be 0.⁵ Among all the elements in the coset $[\gamma]$ which are in the shape of $\begin{pmatrix} * & * \\ c([\gamma]) & \tilde{d} \end{pmatrix}$,⁶ we pick one having the smallest nonnegative integer \tilde{d} entry, say d_0 . We call the pair $(c([\gamma]), d_0)$ the standard representative⁷ of the coset $[\gamma]$.

From now on we fix once and for all the representative set $\mathcal{R}_0 \subseteq \Gamma$ for $\Gamma_0(N) \backslash \Gamma$, so that each element δ in $\mathcal{R}_0 \subseteq \Gamma$ has its bottom row (c, d) being equal to the standard representative of its coset $[\delta]$. For convenience, we may and will assume $\text{id} \in \mathcal{R}_0$. We also fix π from now on to be the right action of Γ_∞ on the standard representatives \mathcal{R}_0 arising from the natural action on $\Gamma_0(N) \backslash \Gamma$.

Lemma 4.6. *Given $\delta \in \mathcal{R}_0$, let Δ be the orbit of π containing the element δ , then v_δ is a \mathbb{C} -linear combination of \mathbf{e}_β 's where $\beta \in \Delta$. More precisely, we have*

$$v_\delta = \frac{1}{N} \sum_{n=0}^{N-1} \chi((I_\delta(T^n))^{-1}) \mathbf{e}_{\delta T^n} \tag{4.1}$$

$$= \frac{1}{N} \sum_{[\gamma] \in \Gamma_\infty(N) \backslash \Gamma_\infty} \chi((I_\delta(\gamma))^{-1}) \mathbf{e}_{\delta \gamma}, \tag{4.2}$$

⁴See Lemma 8.35.

⁵We follow the convention that $\gcd(a, b)$ for any integer a and b not all zero should always be a positive integer. It is easy to check that $\gcd(c, N)$ does not depend on the choice of the element γ , see Lemma 8.37.

⁶it exists, see Lemma 8.37

⁷Strictly speaking it is not a representative element, but these pairs are in one-to-one correspondence to the elements we have picked as representative elements, which have exactly these standard representatives as their bottom row.

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. As a corollary of (4.2), we have the decomposition of the (Γ_∞, χ) -invariant vector space

$$V(\rho)^{\Gamma_\infty} = \bigoplus_{\Delta: \Gamma_\infty\text{-orbit}} V_\Delta, \quad (4.3)$$

where V_Δ is the subspace generated by v_β 's for $\beta \in \Delta$. Moreover, $\dim V_\Delta \leq 1$ for every orbit Δ . If $\dim V_\Delta = 1$, then v_β is a nonzero vector for all $\beta \in \Delta$.

Proof. Before presenting the proof, we remark that the proof itself does not depend on the choice of \mathcal{R}_0 . Recall that $v_\delta := \mathbf{e}_\delta^T$, which is then defined to be equal to

$$\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{e}_\delta \Big|_{k,\rho} T^n = \frac{1}{N} \sum_{[\gamma] \in \Gamma_\infty(N) \backslash \Gamma_\infty} \mathbf{e}_\delta \Big|_{k,\rho} \gamma. \quad (4.4)$$

By Definition 4.1, for each $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$,

$$\mathbf{e}_\delta \Big|_{k,\rho} \gamma := (0\tau + 1)^{-k} \rho(\gamma^{-1}) \mathbf{e}_\delta = \rho(\gamma^{-1}) \mathbf{e}_\delta,$$

and by Definition 1.4 with $\rho := \text{Ind}_{\Gamma_0(N)}^\Gamma \chi$,

$$\rho(\gamma^{-1}) \mathbf{e}_\delta := \chi \left(\left(I_\delta \left((\gamma^{-1})^{-1} \right) \right)^{-1} \right) \mathbf{e}_{\overline{\delta(\gamma^{-1})^{-1}}} = \chi \left((I_\delta(\gamma))^{-1} \right) \mathbf{e}_{\overline{\delta\gamma}}.$$

After inserting $\mathbf{e}_\delta \Big|_{k,\rho} \gamma = \chi \left((I_\delta(\gamma))^{-1} \right) \mathbf{e}_{\overline{\delta\gamma}}$ into the right hand side of (4.4), we get (4.2); and by replacing γ with T^n we also get (4.1).

Given a fixed orbit Δ , if $v_\beta = 0$ for all $\beta \in \Delta$, then V_Δ is of dimension 0. Otherwise there is an element $\delta \in \Delta$, such that v_δ is a nonzero vector. In this case, the dimension of V_Δ is 1. Indeed, for an arbitrary element β in Δ , since both β and δ are in the same orbit, we have $\beta = \gamma_0 \delta \alpha$ for some $\gamma_0 \in \Gamma_0(N)$ and $\alpha \in \Gamma_\infty$, which yields a scalar multiple relation $v_\beta = \chi(\gamma_0^{-1}) v_\delta$ by the computation in the next paragraph. Note that since $\chi(\gamma_0^{-1})$ is a nonzero scalar, v_β is also a nonzero vector.

Here is the computation to show $v_\beta = \chi(\gamma_0^{-1}) v_\delta$. Replacing δ by $\beta = \gamma_0 \delta \alpha$ in (4.1), we have

$$v_\beta = \frac{1}{N} \sum_{n=0}^{N-1} \chi \left((I_{\gamma_0 \delta \alpha}(T^n))^{-1} \right) \mathbf{e}_{\overline{\gamma_0 \delta \alpha T^n}}. \quad (4.5)$$

We then simplify $I_{\gamma_0 \delta \alpha}(T^n)$ and $\overline{\gamma_0 \delta \alpha T^n}$. Since $\gamma_0 \in \Gamma_0(N)$, $\overline{\gamma_0 \delta \alpha T^n} = \overline{\delta \alpha T^n}$. By Definition 1.3, we have $I_{\gamma_0 \delta \alpha}(T^n) = (\gamma_0 \delta \alpha T^n) (\overline{\gamma_0 \delta \alpha T^n})^{-1} = (\gamma_0 \delta \alpha T^n) (\overline{\delta \alpha T^n})^{-1}$. So (4.5) can be simplified as

$$v_\beta = \frac{1}{N} \sum_{n=0}^{N-1} \chi \left(\overline{\delta(\alpha T^n)} (\alpha T^n)^{-1} \delta^{-1} \gamma_0^{-1} \right) \mathbf{e}_{\overline{\delta \alpha T^n}}. \quad (4.6)$$

Note that in (4.6), both γ_0 and $\overline{\delta(\alpha T^n)}(\alpha T^n)^{-1}\delta^{-1}\gamma_0^{-1} = (I_{\gamma_0\delta\alpha}(T^n))^{-1}$ are elements in $\Gamma_0(N)$, hence $\chi(\overline{\delta(\alpha T^n)}(\alpha T^n)^{-1}\delta^{-1}\gamma_0^{-1}) = \chi(\gamma_0^{-1})\chi(\overline{\delta(\alpha T^n)}(\alpha T^n)^{-1})$. Also note that $\alpha \in \Gamma_\infty$, so we can write $\alpha = T^k$ for some integer k , hence $\alpha T^n = T^{n+k}$ and finally we get

$$v_\beta = \chi(\gamma_0^{-1})\frac{1}{N}\sum_{n=k}^{N+k-1}\chi(\overline{\delta T^n}(\delta T^n)^{-1})\mathbf{e}_{\overline{\delta T^n}}. \quad (4.7)$$

Since $T^N \in \Gamma(N)$, by Lemma 8.31, each summand depends only on $n \pmod{N}$. Therefore (4.7) can be simplified to

$$\begin{aligned} v_\beta &= \chi(\gamma_0^{-1})\frac{1}{N}\sum_{n=0}^{N-1}\chi(\overline{\delta T^n}(\delta T^n)^{-1})\mathbf{e}_{\overline{\delta T^n}} \\ &= \chi(\gamma_0^{-1})\frac{1}{N}\sum_{n=0}^{N-1}\chi((I_\delta(T^n))^{-1})\mathbf{e}_{\overline{\delta T^n}} \\ &= \chi(\gamma_0^{-1})v_\delta, \end{aligned}$$

as is desired to show. ■

Definition 4.7. Let χ be the Dirichlet character fixed throughout the note. We say a (Γ_∞, χ) -orbit Δ contributes to the (Γ_∞, χ) -invariant vectors if $\dim V_\Delta = 1$, where V_Δ is the space generated by v_β 's for $\beta \in \Delta$.

We denote by $\Delta \subseteq \mathcal{R}_0$ an arbitrary orbit of π . Motivated by the notion of $c([\gamma])$ in Definition 4.5, we first introduce the notation

$$c(\Delta). \quad (4.8)$$

Note that each element $\delta = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Delta$ has the same value of c ,⁸ we thus define $c(\Delta)$ to be this value.

After doing some experiment with concrete examples⁹, we are motivated to prove the following formula of orbit's size.

Proposition 4.8. *Let Δ be an arbitrary orbit of the right action $\pi : \mathcal{R}_0 \times \Gamma_\infty \rightarrow \mathcal{R}_0$ arising from the natural action $\Gamma_0(N)\backslash\Gamma \times \Gamma_\infty \rightarrow \Gamma_0(N)\backslash\Gamma$, with $c(\Delta) = c$, then the size of Δ agrees with the size of the ideal $c^2\mathbb{Z}/N$, which we denote by $M := \frac{N}{\gcd(c^2, N)}$. Furthermore, given an arbitrary element $\delta \in \Delta$, $\overline{\delta T^n}$ depends only on $n \pmod{M}$, and we have an explicit list of all the elements in Δ :*

$$\Delta = \left\{ \overline{\delta T^n} : 0 \leq n \leq M-1 \right\}. \quad (4.9)$$

⁸See Lemma 8.40.

⁹See Ex.9.2 and Ex.9.3.

Proof. To show (4.9), we start from the definition of Δ containing an arbitrary element δ , i.e. $\Delta = \{\overline{\delta T^n} : n \in \mathbb{Z}\}$, where $T = \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$. Since $\overline{\gamma_1 \gamma_2} = \overline{\gamma_1} \overline{\gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$, if we can show that $\overline{\beta T^M} = \beta$ for any $\beta \in \Delta$, then it follows that $\overline{\delta T^n}$ depends only on $n \pmod{M}$. Plus if we can also show that the $\overline{\delta T^n}$'s are all distinct if some integers n 's are distinct with one another \pmod{M} , then we conclude (4.9). In other words, it suffices to show that M is the smallest positive integer n such that

$$\overline{\beta T^n} = \beta \tag{4.10}$$

for an arbitrary element $\beta \in \Delta$, which we deduce in two parts as follows, where we let (c, d) be the bottom row of β , hence βT^n has the bottom row $(c, cn + d)$.

First we show that if (4.10) holds, then M divides n . By Lemma 8.35, (4.10) implies that there is an integer λ coprime to N , such that

$$\begin{aligned} c &\equiv \lambda c \pmod{N}, \text{ hence } c(\lambda - 1) \equiv 0 \pmod{N}, \\ cn + d &\equiv \lambda d \pmod{N}, \text{ hence } d(\lambda - 1) \equiv cn \pmod{N}. \end{aligned}$$

Therefore, we have that $c^2 n \equiv cd(\lambda - 1) \equiv 0 \pmod{N}$, which implies that $M \mid n$.

Second we show that (4.10) holds for $n = M$. By Lemma 8.34, it suffices to show that there is an integer λ coprime to N , such that

$$\begin{aligned} c &\equiv \lambda c \pmod{N}, \\ cM + d &\equiv \lambda d \pmod{N}, \end{aligned}$$

which is equivalent to the equations

$$(\lambda - 1)c \equiv 0 \pmod{N}, \tag{4.11}$$

$$(\lambda - 1)d \equiv cM \pmod{N}. \tag{4.12}$$

In order to find such a λ , first we observe that

$$c^2 M = \frac{c^2}{\gcd(c^2, N)} N$$

is divided by N . Set $\lambda - 1 = cM\mu$ for some integer μ , (4.11) is satisfied. Moreover, since any λ set in this way is coprime with cM , which is exactly equal to the *girth* $g(\Delta)$ in Definition 4.11, by Lemma 8.26, any λ set in this way is coprime to N . Therefore it suffices to find an integer μ such that Equation (4.12) is satisfied with $\lambda - 1 = cM\mu$, in which case (4.12) is equivalent to the equation

$$d\mu \equiv 1 \pmod{\tilde{c}},$$

where $\tilde{c} = \frac{\gcd(c^2, N)}{c}$. Since $\gcd(c, d) = 1$ and $\tilde{c} \mid c$, d is coprime with \tilde{c} , hence such μ exists. ■

Regarding the formula of $|\Delta|$, we have the following corollary.

Corollary 4.9. *For an arbitrary orbit Δ , $|\Delta| = 1$ if and only if $c(\Delta)^2 \equiv 0 \pmod{N}$. In particular, if N is square free, the only orbit of size 1 is $\{\text{id}\}$. Conversely, if N is not square free, then there is more than one orbit of size 1.*

Regarding the formula of an arbitrary invariant vector v_δ , it can be further simplified in the following corollary, from (4.2), by the claim in Proposition 4.8 that given an arbitrary element $\delta \in \Delta$, $\overline{\delta T^n}$ depends only on $n \pmod{M}$.

Corollary 4.10. *For an arbitrary orbit Δ , we denote by $c := c(\Delta)$ and by M the integer $\frac{N}{\gcd(c^2, N)}$. Let $\delta \in \Delta$ be an arbitrary element, we then have*

$$v_\delta = \frac{1}{N} \sum_{n=0}^{M-1} \left(\sum_{k=0}^{\frac{N}{M}-1} \overline{\chi} \left(I_\delta \left(T^{n+kM} \right) \right) \right) \mathbf{e}_{\overline{\delta T^n}}, \quad (4.13)$$

and by (4.9), $\mathbf{e}_{\overline{\delta T^n}}$'s are linearly independent vectors, with $n \in \{0, \dots, M-1\}$.

Motivated by some interesting examples¹⁰, a notion naturally arises to measure the intrinsic potential of an arbitrary orbit Δ to remain untwisted under the effect of a character \pmod{N} , which we call the *girth* of an orbit.

Definition 4.11. Let $\Delta \subseteq \mathcal{R}_0$ be an orbit of π . We define the girth of the orbit Δ to be

$$g(\Delta) := c(\Delta)|\Delta| = \frac{c(\Delta)N}{\gcd(c(\Delta)^2, N)}.$$

Remark 4.12. You may like to think of the orbit as a circle with each point of the orbit equally distributed on it. The distance between each pair of neighbour points are to be read as the c -value of the orbit, since it is exactly the leap on the d -value each time moved under the action of T . Therefore the circumference of the circle should be the c -value times the number of the orbit, hence the definition of the girth.

Proposition 4.13. *Let Δ be an orbit of the right action $\pi : \mathcal{R}_0 \times \Gamma_\infty \longrightarrow \mathcal{R}_0$ arising from the natural action $\Gamma_0(N) \backslash \Gamma \times \Gamma_\infty \longrightarrow \Gamma_0(N) \backslash \Gamma$, and $g = g(\Delta)$ be its girth. Let N^* be the conductor of χ . Then Δ contributes to the (Γ_∞, χ) -invariant vectors if and only if N^* divides g .¹¹*

Proof. Let $c := c(\Delta)$ and M be the integer $\frac{N}{\gcd(c^2, N)}$. We fix an arbitrary element $\delta \in \Delta$. By Definition 4.7 and Lemma 4.6, Δ contributes to the (Γ_∞, χ) -invariant vectors if and only if $v_\delta \neq 0$. By Corollary 4.10, $v_\delta \neq 0$ if and only if there is an integer $n \in \{0, \dots, M-1\}$, such that

$$\sum_{k=0}^{\frac{N}{M}-1} \overline{\chi} \left(I_\delta \left(T^{n+kM} \right) \right) \neq 0. \quad (4.14)$$

¹⁰See, for instance, Ex.9.4.

¹¹joke: so the girth is well-defined, which measures how fat an orbit is \pmod{N} .

Dropping all the conjugates in (4.14) and applying Lemma 8.18 to $\beta = \text{id}$, $\gamma_1 = \delta$ and $\gamma_2 = T^{n+kM}$, we have $I_\delta(T^{n+kM}) = I_{\text{id}}(\delta T^{n+kM})$ and simplify (4.14) to

$$\sum_{k=0}^{\frac{N}{M}-1} \chi\left(I_{\text{id}}(\delta T^{n+kM})\right) \neq 0,$$

which is furthermore, for all $n \in \{0, \dots, M-1\}$, equivalent to

$$\sum_{k=0}^{\frac{N}{M}-1} (\chi(1+g))^k \neq 0, \tag{4.15}$$

by Lemma 8.43. Moreover, since

$$(\chi(1+g))^{\frac{N}{M}} = \chi\left(1 + \frac{N}{M}g\right) = \chi(1+cN) = 1,$$

(4.15) holds if and only if

$$\chi(1+g) = 1. \tag{4.16}$$

To conclude, Δ contributes to the (Γ_∞, χ) -invariant vectors if and only if (4.16) holds, which by Lemma 8.27 is equivalent to that N^* divides g . \blacksquare

Corollary 4.14. *It depends only on N^* , N , and $c(\Delta)$ to determine whether Δ contributes to the (Γ_∞, χ) -invariant vectors. If moreover the character χ is primitive, then Δ contributes to the (Γ_∞, χ) -invariant vectors if and only if*

$$\gcd\left(c(\Delta), \frac{N}{c(\Delta)}\right) = 1.$$

In particular, if χ is primitive, all of the orbits contribute to the (Γ_∞, χ) -invariant vectors if and only if N is square free.

Proposition 4.15. *Let \mathcal{R}_0 be fixed as before in this section, and let Δ be an orbit which contributes to the (Γ_∞, χ) -invariant vectors. Let $\delta \in \Delta$ be an arbitrary element. Then we have an explicit formula*

$$v_\delta = \frac{1}{|\Delta|} \sum_{n=0}^{|\Delta|-1} \bar{\chi}\left(I_{\text{id}}(\delta T^n)\right) \mathbf{e}_{\overline{\delta T^n}}. \tag{4.17}$$

We denote by v_Δ to be the vector v_{δ_0} , for $\delta_0 \in \Delta$ with the smallest value of d among all the elements of Δ .

Proof. Let $c := c(\Delta)$ and $M := \frac{N}{\gcd(c^2, N)}$. By Lemma 4.10, we have

$$v_\delta = \frac{1}{N} \sum_{n=0}^{M-1} \left(\sum_{k=0}^{\frac{N}{M}-1} \bar{\chi} \left(I_\delta \left(T^{n+kM} \right) \right) \right) \mathbf{e}_{\delta T^n}.$$

By Lemma 8.43, we simplify each coefficient

$$\sum_{k=0}^{\frac{N}{M}-1} \bar{\chi} \left(I_\delta \left(T^{n+kM} \right) \right)$$

to

$$\bar{\chi}(\lambda_n) \sum_{k=0}^{\frac{N}{M}-1} \overline{\left(\chi(1+g) \right)^k},$$

where λ_n is an integer coprime to N , satisfying

$$\chi(\lambda_n) = \chi \left(I_{\text{id}} \left(\delta T^n \right) \right).$$

Since Δ contributes to the (Γ_∞, χ) -invariant vectors, by Proposition 4.13, we have $\chi(1+g) = 1$, hence $\sum_{k=0}^{\frac{N}{M}-1} \overline{\left(\chi(1+g) \right)^k} = \frac{N}{M}$. Therefore,

$$v_\delta = \frac{1}{M} \sum_{n=0}^{M-1} \bar{\chi}(\lambda_n) \mathbf{e}_{\delta T^n}, \quad (4.18)$$

to which we insert $M = |\Delta|$ and apply Proposition 4.8 to write the formula in a self-contained way:

$$v_\delta = \frac{1}{|\Delta|} \sum_{n=0}^{|\Delta|-1} \bar{\chi} \left(I_{\text{id}} \left(\delta T^n \right) \right) \mathbf{e}_{\delta T^n}. \quad (4.19) \quad \blacksquare$$

We also give below an algorithm to compute all the coefficients that appear in (4.17), namely $\bar{\chi} \left(I_{\text{id}} \left(\delta T^n \right) \right) = \bar{\chi}(\lambda_n)$, more explicitly for finding such a value.

Lemma 4.16. *The λ_n 's in (4.18) can be computed by the following steps. Let n be an integer, and (c, d) a pair of integers which are coprime with one another. Denote by $g_n := \gcd(cn + d, N)$, and $\mu_n := \frac{cn+d}{g_n}$. First we find the smallest non-negative integer λ_n , such that $\gcd(\lambda_n, \frac{N}{g_n}) = 1$ and $\lambda_n \equiv \mu_n \pmod{\left(\frac{N}{cg_n}\right)}$, and denote by $\bar{\lambda}_n := \lambda_n \pmod{\left(\frac{N}{cg_n}\right)} \in \mathbb{Z}/\left(\frac{N}{g_n}\right)$. Then we solve the following equations for λ , which has a solution, unique modulo N . The solution is then equal to λ_n modulo N .*

$$\begin{cases} \lambda & \equiv \bar{\lambda}_n^{-1} \mu_n \pmod{\left(\frac{N}{g_n}\right)} \\ \lambda & \equiv 1 \pmod{\left(\frac{N}{c}\right)}. \end{cases} \quad (4.20)$$

Now we are in position to find all the components of the vector-valued Eisenstein series $E_{k,\rho,v}$ for $\rho = \text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi$. Given an arbitrary $v \in V(\rho)^{\Gamma_\infty}$, with the set of representatives \mathcal{R}_0 fixed after Definition 4.5, Lemma 4.6 and Proposition 4.15 allow us to write $v = \sum_{\Delta} c_{\Delta} v_{\Delta}$, where $c_{\Delta} \in \mathbb{C}$ and the orbits Δ 's under the summation symbol run over those contributing to the (Γ_∞, χ) -invariant vectors. Therefore to determine a component of $E_{k,\rho,v} = \sum_{\Delta} c_{\Delta} E_{k,\rho,v_{\Delta}}$, it suffices to work with each $E_{k,\rho,v_{\Delta}}$.

Let Δ be an arbitrary orbit which contributes to the (Γ_∞, χ) -invariant vectors, and let $\delta \in \Delta$ be a fixed element throughout this section. Recall the definition $v_{\Delta} := v_{\delta} := \mathbf{e}_{\delta}^T$. By writing down explicitly the Reynolds operator acting on \mathbf{e}_{δ} and recalling Definition 4.3 for $E_{k,\rho,v}$, we have

$$E_{k,\rho,v_{\Delta}}(\tau) = \sum_{[\gamma_2] \in \Gamma_\infty \backslash \Gamma} \left(\frac{1}{N} \sum_{[\gamma_1] \in \Gamma_\infty(N) \backslash \Gamma_\infty} \mathbf{e}_{\delta} \Big|_{k,\rho} \gamma_1 \right) \Big|_{k,\rho} \gamma_2.$$

Note that the slash action is a group action on the right. We also observe that as $[\gamma_2]$ runs through $\Gamma_\infty \backslash \Gamma$ and $[\gamma_1]$ runs through $\Gamma_\infty(N) \backslash \Gamma_\infty$, $[\gamma]$ with $\gamma = \gamma_1 \gamma_2$ runs through $\Gamma_\infty(N) \backslash \Gamma$. Therefore we have

$$E_{k,\rho,v_{\Delta}}(\tau) = \frac{1}{N} \sum_{[\gamma] \in \Gamma_\infty(N) \backslash \Gamma} \mathbf{e}_{\delta} \Big|_{k,\rho} \gamma.$$

Writing down the slash action explicitly by Definition 4.1 and 1.4, we have

$$\begin{aligned} \mathbf{e}_{\delta} \Big|_{k,\rho} \gamma(\tau) &:= (c\tau + d)^{-k} \rho(\gamma^{-1}) \mathbf{e}_{\delta} \\ &:= (c\tau + d)^{-k} \chi \left(\left(I_{\delta} \left((\gamma^{-1})^{-1} \right) \right)^{-1} \right) \otimes \mathbf{e}_{\overline{\delta(\gamma^{-1})^{-1}}} \\ &= (c\tau + d)^{-k} \chi \left(\left(I_{\delta}(\gamma) \right)^{-1} \right) \otimes \mathbf{e}_{\delta\gamma}. \end{aligned}$$

Therefore for an arbitrary $\beta \in \mathcal{R}_0$, $\mathbf{e}_{\delta} \Big|_{k,\rho} \gamma$ contributes to the \mathbf{e}_{β} -component of $E_{k,\rho,v_{\Delta}}$, which we call the (Δ, β) -component, if and only if $\overline{\delta\gamma} = \beta$, which is equivalent to $\delta\gamma\beta^{-1} \in \Gamma_0(N)$. For these γ 's which contribute to the (Δ, β) -component, we have that $I_{\delta}(\gamma) = \delta\gamma\beta^{-1}$ by Definition 1.3. We now conclude with the following formula for the (Δ, β) -component:

$$\frac{1}{N} \sum_{\substack{[\gamma] \in \Gamma_\infty(N) \backslash \Gamma, \\ \delta\gamma\beta^{-1} \in \Gamma_0(N)}} (c\tau + d)^{-k} \chi(\beta\gamma^{-1}\delta^{-1}), \quad (4.21)$$

where the pair of integers (c, d) appeared in each summand is the bottom row of γ in that summand.

Combining the results in this section, we have the following summary, which describes $E_{k,\rho,v}$ in terms of $E_{k,\rho,v_{\Delta}}$'s.

Proposition 4.17. *Let χ be a Dirichlet character (mod N), and $\rho := \text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi$. Given a (Γ_∞, ρ) -invariant constant vector v , $E_{k,\rho,v}$ is a \mathbb{C} -linear combination of $E_{k,\rho,v_{\Delta}}$'s.*

Given $\beta \in \mathcal{R}_0$ and Δ , we fix an element $\delta \in \Delta$. Then, the \mathfrak{e}_β -component of $E_{k,\rho,v_\Delta}(\tau)$ is given by the following formula:

$$\frac{1}{|\Delta|} \sum_{[\gamma] \in \delta^{-1} \left(\frac{\delta \Gamma_\infty \delta^{-1}}{\Gamma_0(N)} \setminus \Gamma_0(N) \right)^\beta} (c\tau + d)^{-k} \chi(\beta \gamma^{-1} \delta^{-1})^{12}, \quad (4.22)$$

where we write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

5 Spaces of Eisenstein Series

Let ρ be an arithmetic type for Γ and k a positive integer. Recall the definition of the space of vector-valued Eisenstein series of weight k and type ρ , $E_k(\rho)$, in Section 4.

Lemma 5.1. *Let $k > 2$ and ρ be a congruence type for Γ , then we have*

$$M_k(\rho) = E_k(\rho) + S_k(\rho).$$

Proof. Given a vector-valued modular form $f \in M_k(\rho)$, we have to find $f_E \in E_k(\rho)$ and $f_S \in S_k(\rho)$, such that $f = f_E + f_S$. Let $v \in V(\rho)$ be the constant term of f , that is, $v = \lim_{\tau \rightarrow i\infty} f(\tau)$. We shall prove later that $v \in V(\rho)^{\Gamma_\infty}$, which together with the assumption about k and ρ implies that $f_E := E_{k,\rho,v}$ is well-defined. By Lemma 8.5, f_E has constant term v , so $f_S := f - f_E$ has constant term $0 \in V(\rho)$. Since $f \in M_k(\rho)$ and $f_E \in E_k(\rho) \subseteq M_k(\rho)$, we have $f_S \in M_k(\rho)$, and therefore $f_S \in S_k(\rho)$. To see the fact that $v \in V(\rho)^{\Gamma_\infty}$, we take the limit $\tau \rightarrow i\infty$ on both sides of $f|_{k,\rho}\gamma = f$ for all $\gamma \in \Gamma_\infty$. On the right hand side, $\lim_{\tau \rightarrow i\infty} f(\tau) = v$. On the left hand side, we find

$$\lim_{\tau \rightarrow i\infty} f|_{k,\rho}\gamma(\tau) = (c\tau + d)^{-k} \rho(\gamma^{-1}) \lim_{\tau \rightarrow i\infty} f(\gamma\tau) = (c\tau + d)^{-k} \rho(\gamma^{-1})v = v|_{k,\rho}\gamma,$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty$, and so we conclude that $v|_{k,\rho}\gamma = v$ for all $\gamma \in \Gamma_\infty$, that is, $v \in V(\rho)^{\Gamma_\infty}$. \blacksquare

In what follows, we define spaces of components of modular forms and work on the relations between spaces of vector-valued modular forms of certain types and spaces of components of these types. Let ρ be a congruence type for Γ and $k > 2$ an integer. We define

$$\begin{aligned} \mathcal{M}_k[\rho] &:= \text{span}\{w \circ f : w \in V(\rho)^\vee \text{ and } f \in M_k(\rho)\}, \\ \mathcal{E}_k[\rho] &:= \text{span}\{w \circ f : w \in V(\rho)^\vee \text{ and } f \in E_k(\rho)\}, \\ \mathcal{S}_k[\rho] &:= \text{span}\{w \circ f : w \in V(\rho)^\vee \text{ and } f \in S_k(\rho)\}. \end{aligned}$$

We view all these spaces as (left) representations of Γ , via the slash action $\cdot|_k(\gamma^{-1})$. In fact, if $f \in \mathcal{M}_k[\rho]$, by definition there are vector-valued modular forms $\tilde{f}_i \in M_k(\rho)$, and

¹²We will see in the proof that each summand in (4.22) does not depend on the choice of γ for each class $[\gamma]$.

linear functionals $w_i \in V(\rho)^\vee$, such that $f = \sum_i w_i \circ \tilde{f}_i$. By the linearity of the slash action $\cdot|_k$, and the assumption that $\tilde{f}_i|_{k,\rho}\gamma = \tilde{f}_i$ for all $\gamma \in \Gamma$, we have

$$\begin{aligned} f|_k\gamma &= \left(\sum_i w_i \circ \tilde{f}_i \right) |_k\gamma \\ &= \sum_i \left(w_i \circ \rho(\gamma) \right) \circ \left(\tilde{f}_i|_{k,\rho}\gamma \right) \\ &= \sum_i w'_i \circ \tilde{f}_i, \end{aligned}$$

where $w'_i := w_i \circ \rho(\gamma) \in V(\rho)^\vee$. Hence $\mathcal{M}_k[\rho]$ is closed under the slash action $f|_k(\gamma^{-1})$, and similarly, $\mathcal{E}_k[\rho]$ and $\mathcal{S}_k[\rho]$ are also closed under the slash action. Note that from the definition of spaces of components, we have surjections between representations

$$\begin{aligned} \rho^\vee \otimes M_k(\rho) &\twoheadrightarrow \mathcal{M}_k[\rho] \\ \rho^\vee \otimes E_k(\rho) &\twoheadrightarrow \mathcal{E}_k[\rho] \\ \rho^\vee \otimes S_k(\rho) &\twoheadrightarrow \mathcal{S}_k[\rho], \end{aligned}$$

where $M_k(\rho)$, $E_k(\rho)$, $S_k(\rho)$ are viewed as trivial representations. We then consider the space of all components of vector-valued modular forms (Eisenstein series and cusp forms, respectively) of the same weight k and any congruence type. One way to do so is to define each element simply as finite sums of components from different congruence types ρ . Another slightly more standard way is via irreducible representations, which is equivalent to the first for the following reasons. Take for example finite sums of $f_\rho \in \mathcal{M}_k[\rho]$ for all congruence types ρ . Recall from Section 2.6 of [Ser77] that for any finite dimensional complex representation of a finite group, the canonical decomposition exists and is unique. In particular, this applies to every arithmetic type for Γ of finite kernel index, hence to every congruence type ρ . By Lemma 3.13, the split of congruence types passes on to that of modular forms and their components. Therefore, each element from this total space can be written as a finite sum of $f_\rho \in \mathcal{M}_k[\rho^{\oplus n}]$ where ρ are irreducible congruence types. Moreover, we have $\mathcal{M}_k[\rho^{\oplus n}] = \mathcal{M}_k[\rho]$ for all positive integers n , since for any $w = \bigoplus_{i=1}^n w_i$ and $f = \bigoplus_{i=1}^n f_i$, where $w_i \in V(\rho)^\vee$ and $f_i \in M_k(\rho)$, we have $w \circ f = \sum_{i=1}^n w_i \circ f_i \in \mathcal{M}_k[\rho]$. Furthermore, for any two congruence types ρ_1, ρ_2 that are isomorphic via φ , we have $\mathcal{M}_k[\rho_1] = \mathcal{M}_k[\rho_2]$, since $w \circ f = (w \circ \varphi^{-1}) \circ (\varphi \circ f) \in \mathcal{M}_k[\rho_2]$ for any $w \in V(\rho_1)^\vee$ and $f \in M_k(\rho_1)$, and vice versa by switching ρ_1 and ρ_2 . By the same token, we can replace the modular forms by the Eisenstein series and cusp forms, respectively in this argument. Combining these facts it is clear that the following definition is equivalent to the one using finite sums of components from all different congruence types (not necessarily irreducible).

Notation 5.2.

$$\begin{aligned}\mathcal{M}_k &:= \bigoplus_{\rho \text{ irred}} \mathcal{M}_k[\rho], \\ \mathcal{E}_k &:= \bigoplus_{\rho \text{ irred}} \mathcal{E}_k[\rho], \text{ and} \\ \mathcal{S}_k &:= \bigoplus_{\rho \text{ irred}} \mathcal{S}_k[\rho],\end{aligned}$$

where the direct sums are over all non-isomorphic irreducible congruence types ρ for Γ .

Lemma 5.3. *Let $k > 2$ be an integer, then we have*

$$\mathcal{E}_k \cap \mathcal{S}_k = \{0\}.$$

Proof. Suppose $f \in \mathcal{E}_k \cap \mathcal{S}_k$, by definition there exist some irreducible congruence types ρ_i and ρ'_j for Γ , for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, $f_{E,i} \in \mathcal{E}_k[\rho_i]$, $f_{S,j} \in \mathcal{S}_k[\rho'_j]$, such that

$$f = \sum_{i=1}^n f_{E,i} = \sum_{j=1}^m f_{S,j}.$$

By definition, each $f_{E,i}$ (similarly $f_{S,j}$) is the sum of terms in the forms of $w_E \circ \tilde{f}_E$ (similarly $w_S \circ \tilde{f}_S$), with $\tilde{f}_E \in E_k(\rho_i)$ and $w_E \in V(\rho_i)^\vee$ (similarly $\tilde{f}_S \in S_k(\rho'_j)$ and $w_S \in V(\rho'_j)^\vee$). Therefore, we have

$$f = \sum_{i \in I} w_{E,i} \circ \tilde{f}_{E,i} = \sum_{j \in J} w_{S,j} \circ \tilde{f}_{S,j}$$

for some finite sets I and J , where $\tilde{f}_{E,i} \in E_k(\rho_i)$, $\tilde{f}_{S,j} \in S_k(\rho'_j)$ and $w_{E,i} \in V(\rho_i)^\vee$, $w_{S,j} \in V(\rho'_j)^\vee$ (ρ_i might be the same for some $i \in I$, and similarly also for ρ'_j).

We first compute the constant term of $f|_k \gamma$ for all $\gamma \in \Gamma$, namely $\lim_{\tau \rightarrow i\infty} (f|_k \gamma)(\tau)$. On the one hand, we express it with respect to the constant terms of Eisenstein series. Let v_i be the constant term of $\tilde{f}_{E,i}$, i.e., $v_i := \lim_{\tau \rightarrow i\infty} \tilde{f}_{E,i}(\tau)$. By the linearity of $w_{E,i}$ and modularity of $\tilde{f}_{E,i}$, for all $\gamma \in \Gamma$, we have

$$\begin{aligned}(f|_k \gamma)(\tau) &= \left(\sum_{i \in I} (w_{E,i} \circ \tilde{f}_{E,i})|_k \gamma \right)(\tau) \\ &= \sum_{i \in I} w_{E,i} \circ \left(\tilde{f}_{E,i}|_k \gamma(\tau) \right) \\ &= \sum_{i \in I} \left(w_{E,i} \circ \rho_i(\gamma) \right) \left(\tilde{f}_{E,i}|_{k, \rho_i} \gamma(\tau) \right) \\ &= \sum_{i \in I} \left(w_{E,i} \circ \rho_i(\gamma) \right) \tilde{f}_{E,i}(\tau).\end{aligned}$$

Therefore, the constant term of $f|_k\gamma$ is

$$\begin{aligned} \lim_{\tau \rightarrow i\infty} (f|_k\gamma)(\tau) &= \lim_{\tau \rightarrow i\infty} \sum_{i \in I} (w_{E,i} \circ \rho_i(\gamma)) \tilde{f}_{E,i}(\tau) \\ &= \sum_{i \in I} (w_{E,i} \circ \rho_i(\gamma)) \left(\lim_{\tau \rightarrow i\infty} \tilde{f}_{E,i}(\tau) \right) = \sum_{i \in I} w_{E,i}(\rho_i(\gamma)v_i). \end{aligned} \quad (5.1)$$

On the other hand, replacing the letter E by S , similarly we find

$$(f|_k\gamma)(\tau) = \sum_{j \in J} (w_{S,j} \circ \rho'_j(\gamma)) \tilde{f}_{S,j}(\tau) \text{ for all } \gamma \in \Gamma.$$

Since $\tilde{f}_{S,j} \in S_k(\rho'_j)$ for each j , by definition of vector-valued cusp forms with $w_j := w_{S,j} \circ \rho'_j(\gamma) \in V(\rho'_j)^\vee$, we have that

$$\lim_{\tau \rightarrow i\infty} (f|_k\gamma)(\tau) = \lim_{\tau \rightarrow i\infty} \sum_{j \in J} (w_j \circ \tilde{f}_{S,j}(\tau)) = 0 \text{ for all } \gamma \in \Gamma. \quad (5.2)$$

Combining Equation (5.1) and Equation (5.2), we get

$$\sum_{i \in I} w_{E,i}(\rho_i(\gamma)v_i) = 0 \text{ for all } \gamma \in \Gamma. \quad (5.3)$$

Then, we insert Equation (5.3) into the expression of Eisenstein series. Since $k > 2$ and ρ_i is a congruence type, the application of $w_{E,i}$ intertwines with taking the limit. By Lemma 8.5 with the constant term of $\tilde{f}_{E,i} \rightsquigarrow v_i$, we have

$$\begin{aligned} f &= \sum_{i \in I} w_{E,i} \circ \tilde{f}_{E,i} \\ &= \sum_{i \in I} w_{E,i} \circ \left(\sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma} v_i|_{k, \rho_i \gamma} \right) \\ &= \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma} \sum_{i \in I} w_{E,i} \circ (v_i|_{k, \rho_i \gamma}) \\ &= \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma} (c\tau + d)^{-k} \sum_{i \in I} w_{E,i}(\rho_i(\gamma^{-1})v_i) \\ &= 0, \end{aligned}$$

which completes the proof. ■

Combining Lemma 5.1 and Lemma 5.3, we have the following decompositions.

Corollary 5.4. *Let $k > 2$ be an integer and ρ be a congruence type for Γ , then we have the decompositions*

$$\begin{aligned} M_k(\rho) &= E_k(\rho) \oplus S_k(\rho), \\ \mathcal{M}_k[\rho] &= \mathcal{E}_k[\rho] \oplus \mathcal{S}_k[\rho], \text{ and} \\ \mathcal{M}_k &= \mathcal{E}_k \oplus \mathcal{S}_k. \end{aligned}$$

Proposition 5.5. *Let $k > 2$, ρ and ρ' be two arithmetic types for Γ of finite kernel index, such that $\rho \hookrightarrow \rho'$, then we have a linear isomorphism*

$$s : E_k(\rho) \cong \mathrm{Hom}_{\Gamma}(\mathbf{1}, \mathcal{E}_k[\rho'] \otimes \rho) \cong (\mathcal{E}_k[\rho'] \otimes \rho)^{\Gamma}$$

$$f \longmapsto (1 \longmapsto f).$$

Proof. It is clear that s is injective, if it is well-defined. Moreover, under a chosen basis \mathbf{e}_i of $V(\rho)$ and its dual basis \mathbf{e}_i^{\vee} , we have $f = \sum_i (\mathbf{e}_i^{\vee} \circ f) \otimes \mathbf{e}_i \in \mathcal{E}_k[\rho] \otimes V(\rho) \subseteq \mathcal{E}_k[\rho'] \otimes V(\rho)$. Similar to the proof of Proposition 3.15, after a simple calculation, we know that $s(f) \in \mathrm{Hom}_{\Gamma}(\mathbf{1}, \mathcal{E}_k[\rho'] \otimes \rho)$ if and only if

$$f|_{k, \rho \gamma^{-1}} = f \text{ for all } \gamma \in \Gamma, \quad (5.4)$$

which is satisfied by all $f \in E_k(\rho) \subseteq M_k(\rho)$. Therefore, the linear map s is well-defined.

We show next that s is surjective. If $(1 \longmapsto f) \in \mathrm{Hom}_{\Gamma}(\mathbf{1}, \mathcal{E}_k[\rho'] \otimes \rho)$ for some $f \in \mathcal{E}_k[\rho'] \otimes \rho$, then we need to show that $f \in E_k(\rho)$. First, the homomorphism condition of representations implies that Equation (5.4) is satisfied. Second, since $w \circ f \in \mathcal{E}_k[\rho'] \subseteq \mathcal{H}_k^{\mathrm{md}}$ for all $w \in V(\rho)^{\vee}$, $f \in \mathcal{H}_k^{\mathrm{md}}(\rho)$, hence $f \in M_k(\rho)$. Therefore, it suffices to prove that if $f \in M_k(\rho)$ so that $w \circ f \in \mathcal{E}_k[\rho']$ for all $w \in V(\rho)^{\vee}$, then $f \in E_k(\rho)$.

By Lemma 5.1, there exist $f_E \in E_k(\rho)$ and $f_S \in S_k(\rho)$, such that $f = f_E + f_S$. In order to conclude that $f \in E_k(\rho)$, it suffices to show $f_S = 0$. Equivalently, we show that $w \circ f_S = 0$ for all $w \in V(\rho)^{\vee}$. In fact, for such w , since $w \circ f_E \in \mathcal{E}_k[\rho]$ and $w \circ f \in \mathcal{E}_k[\rho']$ for all $w \in V(\rho)^{\vee}$, their linear combinations lie in \mathcal{E}_k . In particular, the relation $w \circ f = w \circ f_E + w \circ f_S$ forces the element $w \circ f - w \circ f_E = w \circ f_S$ to be in $\mathcal{E}_k \cap \mathcal{S}_k[\rho] \subseteq \mathcal{E}_k \cap \mathcal{S}_k$. By Lemma 5.3, the intersection is $\{0\}$, hence $w \circ f_S = 0$ for all $w \in V(\rho)^{\vee}$, as desired. ■

Remark 5.6. For instance, when $\rho = \rho_N^{\times}$ and $\rho' = \rho_{lN}^{\times}$ for a positive integer l , see Example 9.7. Furthermore, one can also easily loose the condition $\rho \hookrightarrow \rho'$ to that any irreducible component of ρ that admits a nonzero T -fixed vector also occurs in ρ' , and still have the same isomorphism.

Since an Eisenstein series is determined by a T -fixed vector (or a Γ_{∞} -invariant vector, depending on the slash action of $-\mathrm{id}$.), we first need to determine which irreducible congruence types of level N have a T -fixed vector.

Lemma 5.7. *Let ρ be an irreducible congruence type of level N . If ρ has a nonzero T -fixed vector, then ρ occurs in the decomposition of ρ_N^{\times} , i.e. $\rho \hookrightarrow \rho_N^{\times}$.*

Proof. We show the lemma in two steps. First, we fix a T -fixed vector $v \neq 0$ in the representation space $V(\rho)$. Since ρ is irreducible, the submodule generated by v must be the whole space $V(\rho)$. Since ρ is finite dimensional, we can choose some elements $\gamma_i \in \Gamma$ for $i = 1, 2, \dots, n$, such that $\rho(\gamma_i)v$ consist a basis of $V(\rho)$. Let Γ' denote the stabilizer of v in Γ . We claim that there is an inclusion of representations

$$\varphi : \rho \hookrightarrow \mathrm{Ind}_{\Gamma'}^{\Gamma} \mathbf{1}$$

$$\rho(\gamma_i)v \longmapsto \mathbf{e}_{\gamma_i^{-1}\Gamma}.$$

First of all, it is straightforward to check that φ is a homomorphism of representations. Then we compute $\ker \varphi$ as follows. If $v_0 := \sum_{i=1}^n c_i \rho(\gamma_i)v \in \ker \varphi$, after fixing a set of representatives \mathcal{R} for $\Gamma' \backslash \Gamma$, we have

$$0 = \sum_{i=1}^n c_i \mathfrak{e}_{\gamma_i^{-1}} = \sum_{\beta \in \mathcal{R}} \mathfrak{e}_{\beta} \sum_{i: \gamma_i^{-1} = \beta} c_i,$$

which implies that $c_\beta := \sum_{i: \gamma_i^{-1} = \beta} c_i = 0$ for all $\beta \in \mathcal{R}$. Note that for all i , $\rho(\gamma_i)v = \rho(\beta^{-1})v$ for $\beta = \overline{\gamma_i^{-1}}$, since v is fixed by Γ' . Therefore, we have

$$v_0 = \sum_{i=1}^n c_i \rho(\gamma_i)v = \sum_{\beta \in \mathcal{R}} c_\beta \rho(\beta^{-1})v = 0,$$

which implies $\ker \varphi = 0$.

Second, we complete the inclusion by showing that $\text{Ind}_{\Gamma'}^{\Gamma} \mathbf{1} \hookrightarrow \text{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}$. since v is fixed by T , and ρ is of level N , v is thus fixed by any element in $\Gamma_1(N)$, i.e. $\Gamma' \supseteq \Gamma_1(N)$. By Lemma 3.20 and Lemma 3.8, we have

$$\text{Ind}_{\Gamma'}^{\Gamma} \mathbf{1} \hookrightarrow \text{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1} \cong \rho_N^{\times},$$

which completes the proof by the composition with the first inclusion $\varphi : \rho \hookrightarrow \text{Ind}_{\Gamma'}^{\Gamma} \mathbf{1}$. ■

Lemma 5.8. *Let $k > 2$, ρ a congruence type for Γ of level N , then we have a linear isomorphism*

$$s : E_k(\rho) \cong \text{Hom}_{\Gamma}(\mathbf{1}, \mathcal{E}_k[\rho_N^{\times}] \otimes \rho) \cong (\mathcal{E}_k[\rho_N^{\times}] \otimes \rho)^{\Gamma}$$

$$f \longmapsto (1 \longmapsto f).$$

Proof. Since ρ factors through finite group $\text{SL}_2(\mathbb{Z}/N)$, it is completely reducible, hence it suffices to show the lemma for ρ irreducible. There are two cases: if ρ has a nonzero T -fixed vector, then by Lemma 5.7 we have an embedding $\rho \hookrightarrow \rho_N^{\times}$, hence by Proposition 5.5 we conclude that s is a linear isomorphism. If ρ does not have a nonzero T -fixed vector, then $V(\rho)^{\Gamma^{\infty}} = 0$ and therefore by definition we have $E_k(\rho) = 0$. On the other hand, let $(1 \longmapsto f) \in \text{Hom}_{\Gamma}(\mathbf{1}, \mathcal{E}_k[\rho_N^{\times}] \otimes \rho)$ be a homomorphism of representations, then we have $f \in M_k(\rho)$. By Corollary 5.4, we can write $f = f_E + f_S$ for $f_E \in E_k(\rho) = 0$ and $f_S \in S_k(\rho)$, which forces all the components of f to lie in the space $\mathcal{E}_k[\rho_N^{\times}] \cap \mathcal{S}_k[\rho] \subseteq \mathcal{E}_k \cap \mathcal{S}_k$. By Lemma 5.3, all the components must therefore be zero, hence $f = 0$ and $\text{Hom}_{\Gamma}(\mathbf{1}, \mathcal{E}_k[\rho_N^{\times}] \otimes \rho) = 0$. ■

Now we apply Proposition 5.5 with $\rho = \rho_N^{\times}$ to study the structure of $E_k(\rho_N^{\times})$. We start with moving ρ_N^{\times} from the right to the left by taking the duality (which is isomorphic to itself), to further simplify the Hom set.

Lemma 5.9.

$$\text{Hom}_{\Gamma}(\mathbf{1}, \mathcal{E}_k[\rho_N^{\times}] \otimes \rho_N^{\times}) \cong \text{Hom}_{\Gamma}(\rho_N^{\times \vee}, \mathcal{E}_k[\rho_N^{\times}]) \cong \text{Hom}_{\Gamma}(\rho_N^{\times}, \mathcal{E}_k[\rho_N^{\times}]).$$

Proof. The first isomorphism follows from Lemma 8.24, with $\pi \rightsquigarrow \mathbf{1}$, $\rho \rightsquigarrow \mathcal{E}_k[\rho_N^\times]$, and $\sigma \rightsquigarrow \rho_N^\times$. The second isomorphism follows from the self duality of ρ_N^\times , i.e., Lemma 3.16. ■

We then study the space of components $\mathcal{E}_k[\rho_N^\times]$, which can be identified with the space of classical Eisenstein series defined via double sums over a pair of integers (c, d) with the restriction $\gcd(c, d) = 1$. For $\lambda \in L_N^\times$, we define the classical Eisenstein series of weight k , level N , associated with λ via

$$E_{k,N,\lambda} := \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ (c,d) \equiv \lambda \pmod{N}, \\ \gcd(c,d)=1}} (c\tau + d)^{-k},$$

The \mathbb{C} -linear space spanned by $E_{k,N,\lambda}$, for all $\lambda \in L_N^\times$, is denoted by $\mathcal{E}_{k,N}$. We call it the space of (classical) Eisenstein series of weight k and level N . Note that $\mathcal{E}_{k,N}$ is closed under the slash action of Γ by Lemma 8.6, hence $\mathcal{E}_{k,N}$ is naturally a (left) representation of Γ via

$$\gamma E_{k,N,\lambda} := E_{k,N,\lambda}|_k \gamma^{-1}.$$

Depending on the context, $\mathcal{E}_{k,N}$ also denotes this representation. We then identify the space of components of vector-valued Eisenstein series $\mathcal{E}_k[\rho_N^\times]$, with the space of classical Eisenstein series $\mathcal{E}_{k,N}$, as representations of Γ .

Lemma 5.10. *Let $k > 2$ be an integer and N a positive integer, we have*

$$\mathcal{E}_k[\rho_N^\times] = \mathcal{E}_{k,N}.$$

Proof. For $N = 1$, both sides are equal to 0 if k is an odd integer, and agree with the 1-dim space generated by $E_k(\tau)$ if k is an even integer. For $N = 2$, both sides vanish if k is an odd integer.

For $N \geq 3$, or $N = 2$ and k even, to see $\mathcal{E}_{k,N} \subseteq \mathcal{E}_k[\rho_N^\times]$, it suffices to show that for each $\lambda \in L_N^\times$, $E_{k,N,\lambda} \in \mathcal{E}_k[\rho_N^\times]$. Let $v_0 := \mathbf{e}_{(\bar{0}, \bar{1})} + (-1)^k \mathbf{e}_{(\bar{0}, \bar{-1})} \in V(\rho_N^\times)$, it is clear that $v_0|_{k, \rho_N^\times} \gamma = v_0$ for $\gamma = -\text{id}$ and $\gamma = T$, hence $v_0 \in V(\rho_N^\times)^{\Gamma_\infty}$, and $E_{k, \rho_N^\times, v_0} \in E_k(\rho_N^\times)$ is well-defined. By definition, we find

$$E_{k, \rho_N^\times, v_0} = \sum_{\substack{(c,d) \in \mathbb{Z}^2: \\ \gcd(c,d)=1}} (c\tau + d)^{-k} \left(\mathbf{e}_{(\bar{c}, \bar{d})} + (-1)^k \mathbf{e}_{(\bar{-c}, \bar{-d})} \right). \quad (5.5)$$

Applying $\mathbf{e}_\lambda^\vee \in V(\rho_N^\times)^\vee$ to both sides of (5.5), and noting that $\lambda \neq -\lambda$ when $N \geq 3$ (Lemma 8.44), as well as $\lambda = -\lambda$ when $N = 2$, we find the λ -component

$$\mathbf{e}_\lambda^\vee \circ E_{k, \rho_N^\times, v_0} = E_{k,N,\lambda},$$

hence $E_{k,N,\lambda} \in \mathcal{E}_k[\rho_N^\times]$.

Conversely, to show $\mathcal{E}_k[\rho_N^\times] \subseteq \mathcal{E}_{k,N}$, it suffices to prove that for each $\lambda \in L_N^\times$ and $v \in V(\rho_N^\times)^{\Gamma_\infty}$, the component $\mathfrak{e}_\lambda^\vee(E_{k,\rho_N^\times,v}) \in \mathcal{E}_{k,N}$. Since $\mathfrak{e}_\lambda^\vee$ is a linear function continuous with respect to the norm (unique up to equivalence) on the finite dimensional space $V(\rho_N^\times)$, and the series $E_{k,\rho_N^\times,v}$ is convergent, we have

$$\mathfrak{e}_\lambda^\vee(E_{k,\rho_N^\times,v}) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} (c\tau + d)^{-k} \mathfrak{e}_\lambda^\vee(\rho_N^\times(\gamma^{-1})v) = \frac{1}{2} \sum_{\gamma \in \langle T \rangle \setminus \Gamma} (c\tau + d)^{-k} \mathfrak{e}_\lambda^\vee(\rho_N^\times(\gamma^{-1})v).$$

Note that any two elements $\gamma_1, \gamma_2 \in \Gamma$ have the same bottom row if and only if $\gamma_1\gamma_2^{-1} \in \langle T \rangle$. Fix a set of representatives for $\langle T \rangle \setminus \Gamma$, and let $\gamma_{(c,d)}$ denote the representative element with the bottom row (c, d) . Then we have

$$\mathfrak{e}_\lambda^\vee(E_{k,\rho_N^\times,v}) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ \gcd(c,d)=1}} (c\tau + d)^{-k} \mathfrak{e}_\lambda^\vee(\rho_N^\times(\gamma_{(c,d)}^{-1})v).$$

If we can show that $\mathfrak{e}_\lambda^\vee(\rho_N^\times(\gamma_{(c,d)}^{-1})v)$ depends only on $(\bar{c}, \bar{d}) \in L_N^\times$, say this value is $\epsilon_{\lambda'}$ for $\lambda' = (\bar{c}, \bar{d})$, then we have

$$\mathfrak{e}_\lambda^\vee(E_{k,\rho_N^\times,v}) = \frac{1}{2} \sum_{\lambda' \in L_N^\times} \epsilon_{\lambda'} E_{k,N,\lambda'} \in \mathcal{E}_{k,N}.$$

In fact, since ρ_N^\times factors through the representation $\overline{\rho_N^\times}$ for $\mathrm{SL}_2(\mathbb{Z}/N)$ (naturally induced from ρ_N^\times), and $v \in V(\rho_N^\times)^{\Gamma_\infty}$, we have $\rho_N^\times(\gamma)v = v$ for all $\gamma \in \Gamma_1(N)$, which implies that $\rho_N^\times(\gamma_1^{-1})v = \rho_N^\times(\gamma_2^{-1})v$ for any two elements γ_1, γ_2 in the same coset of $\Gamma_1(N) \setminus \Gamma$. This means that $\mathfrak{e}_\lambda^\vee(\rho_N^\times(\gamma_{(c,d)}^{-1})v)$ depends only on the coset of $\gamma_{(c,d)}$, hence only on $\lambda' = (\bar{c}, \bar{d})$ by Lemma 8.15. \blacksquare

To study the structure of $\mathcal{E}_{k,N}$, we start with the parity issue. The fact that

$$E_{k,N,\lambda}|_k(-\mathrm{id}) = E_{k,N,-\lambda} = (-1)^k E_{k,N,\lambda}$$

motivates us to consider it in terms of representations. The key fact is that for any group with a finite center, its representation always splits into isotypical components (not necessarily non-zero) of characters of its center. More precisely, as a corollary of Lemma 3.3 we have the following construction. Since the center of Γ is equal to $\pm \mathrm{id}$, we can split the space $V(\rho_N^\times)$ into two parts. Let $V(\rho_N^\times)_+$ denote the eigenspace of $+1$ and $V(\rho_N^\times)_-$ that of -1 , under the action of $\rho_N^\times(-\mathrm{id})$. Then by Lemma 3.3, they are subrepresentations of ρ_N^\times , which we denote by $\rho_{N,+}^\times$ and $\rho_{N,-}^\times$, respectively. Moreover, we have the decomposition of representations

$$\rho_N^\times = \rho_{N,+}^\times \oplus \rho_{N,-}^\times. \tag{5.6}$$

We then connect $\mathcal{E}_{k,N}$ with one of these components, by associating the parity of the weight k to such representations, and constructing an explicit basis for each eigenspace.

For $k \in \mathbb{Z}$ and $\lambda \in L_N^\times$, let

$$v_{k,\lambda} := \mathbf{e}_{-\lambda} + (-1)^k \mathbf{e}_\lambda,$$

and

$$\rho_{k,N}^\times := \langle v_{k,\lambda} : \lambda \in L_N^\times \rangle.$$

Since

$$\rho_N^\times(\gamma)v_{k,\lambda} = \rho_N^\times(\gamma)\mathbf{e}_{-\lambda} + (-1)^k \rho_N^\times(\gamma)\mathbf{e}_\lambda = \mathbf{e}_{-\lambda\gamma^{-1}} + (-1)^k \mathbf{e}_{\lambda\gamma^{-1}} = v_{k,\lambda\gamma^{-1}},$$

$\rho_{k,N}^\times$ is a subrepresentation of ρ_N^\times . Moreover, since

$$\rho_N^\times(-\text{id})v_{k,\lambda} = \rho_N^\times(-\text{id})\mathbf{e}_{-\lambda} + (-1)^k \rho_N^\times(-\text{id})\mathbf{e}_\lambda = \mathbf{e}_\lambda + (-1)^k \mathbf{e}_{-\lambda} = (-1)^k v_{k,\lambda},$$

we know $\rho_{k,N}^\times$ is contained in the eigenspace of $(-1)^k$, and $\rho_{k+1,N}^\times$ is contained in that of $(-1)^{k+1}$. In order to conclude that they actually coincide, by Decomposition 5.6, it suffices to show that their basis, $v_{k,\lambda}$ and $v_{k+1,\lambda}$ for all $\lambda \in L_N^\times$, generate the whole space $V(\rho_N^\times)$. But this is clear from the relation

$$\mathbf{e}_\lambda = \frac{1}{2} \left((-1)^k v_{k,\lambda} + (-1)^{k+1} v_{k+1,\lambda} \right).$$

To conclude, we have the decomposition for each $k \pmod{2}$

$$\rho_N^\times = \rho_{k,N}^\times \oplus \rho_{k+1,N}^\times,$$

which each component being eigenspace of $(-1)^k$ (resp. $(-1)^{k+1}$), for the centre action of ρ_N^\times .

Now the we are in position to state

Proposition 5.11. *Let $k \in \mathbb{Z}_{\geq 3}$, then we have an isomorphism of representations*

$$\begin{aligned} \mathcal{E}_{k,N} &\cong \rho_{k,N}^\times \\ E_{k,N,\lambda} &\longmapsto \mathbf{e}_\lambda. \end{aligned}$$

Proof. It follows from Lemma 8.6 and Lemma 8.8. ■

We start from the definition of group version Eisenstein series.

Lemma 5.12.

$$E_{k,N,\lambda} = \sum_{[\beta] \in \Gamma_\infty \backslash \Gamma_1(N)} 1|_k \beta \delta = \sum_{[\gamma] \in \Gamma_\infty(N) \backslash \Gamma(N)} 1|_k \gamma \delta,$$

where $\Gamma_\infty(N) := \Gamma_\infty \cap \Gamma(N)$, and $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ satisfies that $(c, d) \equiv \lambda \pmod{N}$.

Proof. The first equality follows directly from the definition of Eisenstein series, and the second equality follows from Lemma 2.1. \blacksquare

Similar to the Eisenstein series $E_{k,N,\lambda}$, We define a class of Eisenstein series by $G_{k,N,\lambda}$, via double sums without the restriction $\gcd(c, d) = 1$

$$G_{k,N,\lambda} := \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ (c,d) \equiv \lambda \pmod{N}}} (c\tau + d)^{-k},$$

for $\lambda \in L_N^\times$. Note that the difference is that the summation for $G_{k,N,\lambda}$ is via lattice (additive) structure, while that for $E_{k,N,\lambda}$ is via cosets of a group, or “primitive points” in the lattice of level N (multiplicative in nature).

Lemma 5.12 tells us that $\mathcal{E}_{k,N}$ is group theoretic in nature. On the other hand, we can define the lattice theoretic $\mathcal{G}_{k,N}$ in a similar way, namely the space spanned by all the $G_{k,N,\lambda}$, and view it as the representation ρ_N^\times via the slash action. Furthermore, Lemma 8.10 and Lemma 8.12 show that

$$\mathcal{E}_{k,N} = \mathcal{G}_{k,N}.$$

This tells us that in order to compute Fourier expansions, we can choose the more practical lattice version $G_{k,N,\lambda} \in \mathcal{G}_{k,N}$, which is carried out in Section 6.

Furthermore, as representations they are also isomorphic via the map

$$\begin{aligned} \varphi : \mathcal{E}_{k,N} &\cong \mathcal{G}_{k,N} \\ E_{k,N,\lambda} &\longmapsto G_{k,N,\lambda}. \end{aligned}$$

In fact, by Lemma 8.8 and Corollary 8.11, the map φ is a linear isomorphism. By lemma 8.1, it suffices to show that φ intertwines with the slash actions on both sides. By Lemma 8.6, we have that the slash action on $\mathcal{E}_{k,N}$ acts as

$$E_{k,N,\lambda}|_\gamma = E_{k,N,\lambda\gamma}.$$

On the other hand, the slash action on $\mathcal{G}_{k,N}$ can be computed via the slash action on $\mathcal{E}_{k,N}$, by Lemma 8.12. We have

$$\begin{aligned} G_{k,N,\lambda}|_\gamma &= \left(\sum_{a \in (\mathbb{Z}/N)^\times} \zeta_{a,N}(k) E_{k,N,a^{-1}\lambda} \right) |_\gamma \\ &= \sum_{a \in (\mathbb{Z}/N)^\times} \zeta_{a,N}(k) E_{k,N,a^{-1}\lambda} |_\gamma \\ &= \sum_{a \in (\mathbb{Z}/N)^\times} \zeta_{a,N}(k) E_{k,N,a^{-1}\lambda\gamma} \\ &= G_{k,N,\lambda\gamma}, \end{aligned}$$

hence the map $\varphi : E_{k,N,\lambda} \longmapsto G_{k,N,\lambda}$ intertwines with the slash actions. From now on, we will only use the notation $\mathcal{E}_{k,N}$ for the space of classical Eisenstein series of level N .

In order to connect classical Eisenstein series back to vector-valued Eisenstein series, we start from the natural identification of a family of classical modular forms with a vector-valued modular form. Given an arithmetic type ρ of Γ , Let $\{\mathbf{e}_\lambda\}$ be a basis of $V(\rho)$ and $\{\mathbf{e}_\lambda^\vee\}$ be the corresponding dual basis of $V(\rho)^\vee$. Then there is a natural identification of a function

$$f : \{\mathbf{e}_\lambda\} \times \mathfrak{H} \longrightarrow \mathbb{C}$$

(or equivalently, a family of functions $f_\lambda : \mathfrak{H} \longrightarrow \mathbb{C}$) with a vector-valued function $\tilde{f} : \mathfrak{h} \longrightarrow \mathbb{C} \otimes V(\rho)^\vee$, via

$$\tilde{f} := \sum_{\lambda} f(\lambda, \tau) \mathbf{e}_\lambda^\vee.$$

In particular, when $\rho = \rho_N^\times$ and $\lambda \in L_N^\times$ are points of order N , we naturally identify the set of $G_{k,N,\lambda}$ (and that of $E_{k,N,\lambda}$) and their linear combinations with a vector-valued Eisenstein series. Such point of view is useful for computing cusp expansions via vector-valued Eisenstein series, and in general for computational aspects of modular forms.

Lemma 5.13.

$$\mathrm{Hom}_{\Gamma}(\rho_N^\times, \rho_{k,N}^\times) \cong \mathrm{End}_{\Gamma}(\rho_{k,N}^\times).$$

Proof. It suffices to show that any element in $\mathrm{Hom}_{\Gamma}(\rho_N^\times, \rho_{k,N}^\times)$ factors through the projection p_k of $V(\rho_N^\times)$ onto $V(\rho_{k,N}^\times) \subseteq V(\rho_N^\times)$. In other words, for any $\varphi \in \mathrm{Hom}_{\Gamma}(\rho_N^\times, \rho_{k,N}^\times)$, there exists a unique element $\tilde{\varphi} \in \mathrm{End}(\rho_{k,N}^\times)$, such that

$$\varphi = \tilde{\varphi} \circ p_k. \tag{5.7}$$

In fact, if Factorization (5.7) holds, then the map $\varphi \longmapsto \tilde{\varphi}$ provides a linear isomorphism as desired. If $\tilde{\varphi} = 0$, then $\varphi = 0$, hence the injectivity. For surjectivity, given an arbitrary element $\tilde{\varphi} \in \mathrm{End}_{\Gamma}(\rho_{k,N}^\times)$, $\tilde{\varphi} \circ p_k$ is actually in its preimage, by the uniqueness of $\tilde{\varphi}$ in Factorization (5.7), and the fact that $p_k \in \mathrm{Hom}_{\Gamma}(\rho_N^\times, \rho_{k,N}^\times)$.

To see Factorization (5.7) holds, we look at the action of the center. Recall that

$$\rho_N^\times = \rho_{k,N}^\times \oplus \rho_{k+1,N}^\times.$$

In particular, $-\mathrm{id} \in \Gamma$ acts as scalar products $(-1)^k$ and $(-1)^{k+1}$ on $V(\rho_{k,N}^\times)$ and $V(\rho_{k+1,N}^\times)$, respectively. Therefore, for any element $\varphi \in \mathrm{Hom}_{\Gamma}(\rho_N^\times, \rho_{k,N}^\times)$, the restriction of φ on the subspace $V(\rho_{k+1,N}^\times)$ must be 0, which implies that the restriction of φ on the subspace $V(\rho_{k,N}^\times)$ uniquely satisfies (5.7). \blacksquare

Combining Proposition 5.5, Lemma 5.9, Lemma 5.10, Proposition 5.11, and Lemma 5.13, we have the main result of this Chapter.

Proposition 5.14. *There is a linear isomorphism*

$$E_k(\rho_N^\times) \cong \mathrm{End}_{\Gamma}(\rho_{k,N}^\times).$$

6 Fourier Expansions of Eisenstein series

Lemma 6.1. *Let $d_0 \in \{0, \dots, N-1\}$. Then, for any $k \in \mathbb{Z}_{\geq 2}$, we have*

$$\sum_{m \in d_0 + N\mathbb{Z} \setminus \{0\}} m^{-k} = \frac{1}{N^k} \left(\zeta(k, d_0/N) + (-1)^k \zeta(k, (N-d_0)/N) \right) \text{ if } d_0 \neq 0,$$

and

$$\sum_{m \in d_0 + N\mathbb{Z} \setminus \{0\}} m^{-k} = \frac{1}{N^k} \left(\zeta(k) + (-1)^k \zeta(k) \right) \text{ if } d_0 = 0.$$

Furthermore, this sum can be evaluated for all d_0 as

$$\begin{aligned} & \frac{(-2\pi i)^k}{(k-1)!N} \sum_{n=1}^N -\frac{B_k(n/N)}{k} e(d_0 n/N) \\ &= \frac{(-2\pi i)^k}{(k-1)!N} \sum_{j=0}^k -\frac{B_j}{j} \binom{k-1}{j-1} \sum_{n=1}^N (n/N)^{k-j} e(d_0 n/N). \end{aligned}$$

Proof. We insert the fact that

$$\cos\left(k\pi/2 - 2\pi dn/N\right) = \frac{i^k}{2} \left(e(-dn/N) + (-1)^k e(dn/N) \right)$$

into the functional equation in Lemma 8.47, and get

$$\begin{aligned} \zeta(1-k, n/N) &= \frac{(k-1)!}{(-2\pi i N)^k} \sum_{d=1}^N \zeta(k, d/N) \left(e(-dn/N) + (-1)^k e(dn/N) \right) \\ &= \frac{(k-1)!}{(-2\pi i N)^k} \left[\sum_{d=1}^{N-1} \left(\zeta(k, d/N) + (-1)^k \zeta(k, (N-d)/N) \right) e(-dn/N) \right. \\ &\quad \left. + \left(\zeta(k) + (-1)^k \zeta(k) \right) \right]. \end{aligned} \tag{6.1}$$

Now let $f(n) := \frac{1}{N} \frac{(-2\pi i N)^k}{(k-1)!} \zeta(1-k, n/N)$ for $n \in \{1, \dots, N\}$, and $g(d) := \zeta(k, d/N) + (-1)^k \zeta(k, (N-d)/N)$ for $d \in \{1, \dots, N-1\}$ and $g(d) := \zeta(k) + (-1)^k \zeta(k)$ for $d = N$. Then we can rewrite Equation 6.1 by

$$f(n) = \frac{1}{N} \sum_{d=1}^N g(d) e(-dn/N).$$

Fourier analysis over \mathbb{Z}/N then provides the inversion formula, in particular for $d = d_0$ we get

$$g(d_0) = \sum_{n=1}^N f(n) e(d_0 n/N). \tag{6.2}$$

Finally by Lemma 8.48, we have

$$\zeta(1 - k, n/N) = -\frac{B_k(n/N)}{k}$$

for $n \in \{1, \dots, N\}$, which we insert into Equation 6.2 to conclude the proof. \blacksquare

Remark 6.2. The reflection formula of Hurwitz function states that for all $x \in \mathbb{R}_{(0,1)}$

$$\begin{aligned} \zeta(k, 1 - x) + (-1)^k \zeta(k, x) &= -\frac{\pi}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \Big|_{t=x} \cot(\pi t) \\ &= -\frac{\pi^k}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \Big|_{t=\pi x} \cot(t), \end{aligned}$$

which makes the evaluation possible for real value x . However, when $x \in \mathbb{Q}_{(0,1)}$, from the perspective of computation, computing symbolic $(k-1)$ -th derivative of $\cot(t)$ is much less effective than computing \mathbb{Q} -linear combination of j -th Bernoulli numbers, for $j \leq k$ in a closed formula.

Remark 6.3. The formula in the lemma for $d_0 = 0$ also reproduces one relation between Bernoulli numbers and values of Bernoulli polynomials:

$$\sum_{n=1}^N B_k(n/N) = \frac{B_k}{N^{k-1}}, \quad (6.3)$$

by taking $d_0 = 0$ and comparing the case $N = 1$ and N being a general positive integer in the sum $\sum_{m \in d_0 + N\mathbb{Z} \setminus \{0\}} m^{-k}$. Note that Equation 6.3 is a special case (take $x = 0$) of Raabe's formula

$$\sum_{n=0}^{N-1} B_k(x + n/N) = \frac{B_k(Nx)}{N^{k-1}},$$

which can be found at 9.624 in [GR07].

For keeping our expression homogeneous in the sense that all Fourier coefficients are of “degree zero” in N , we define our divisor sum of level N as follows, where $(m/N)^{k-1}$ takes place of the usual m^{k-1} when $N = 1$.

$$\sigma_{k-1, N, \lambda}(n) := \sum_{\substack{m \in \mathbb{Z}, m \mid n, \\ n/m \equiv c_0 \pmod{N}}} \operatorname{sgn}(m) (m/N)^{k-1} e(d_0 m/N).$$

Now we are in position to state the Fourier expansion of $G_{k, N, \lambda}$.

Proposition 6.4. *Let $k \in \mathbb{Z}_{\geq 3}$ and $N \in \mathbb{Z}_{\geq 1}$. Let $\lambda := (c_0, d_0) \in L_N^\times$, then the Fourier expansion of $\frac{(k-1)!}{(-2\pi i)^k} G_{k,N,\lambda}$ is given by*

$$\begin{aligned} \frac{(k-1)!}{(-2\pi i)^k} G_{k,N,\lambda} &= \delta_{c_0, \bar{0}} \frac{1}{N} \sum_{n=1}^N -\frac{B_k(n/N)}{k} e(d_0 n/N) + \frac{1}{N} \sum_{n=1}^{\infty} \sigma_{k-1,N,\lambda}(n) e(n\tau/N) \quad (6.4) \\ &= \delta_{c_0, \bar{0}} \frac{1}{N} \sum_{j=0}^k -\frac{B_j}{j} \binom{k-1}{j-1} \sum_{n=1}^N (n/N)^{k-j} e(d_0 n/N) + \frac{1}{N} \sum_{n=1}^{\infty} \sigma_{k-1,N,\lambda}(n) e(n\tau/N). \end{aligned}$$

where δ is the Kronecker delta, $B_k(x) := \sum_{j=0}^k \binom{k}{j} B_j x^{k-j}$ is the k -th Bernoulli polynomial, and

Proof. We start by separating the sum

$$G_{k,N,\lambda} := \sum_{\substack{(c,d) \in \mathbb{Z}^2, \\ (c,d) \equiv \lambda \pmod{N}}} (c\tau + d)^{-k}$$

in terms of $\text{sgn}(c) \in \{-1, 0, 1\}$. Clearly the part of $c = 0$ contributes the sum

$$\delta_{c_0, \bar{0}} \sum_{d \equiv d_0 \pmod{N}} d^{-k},$$

which can be further simplified by Lemma 6.1, and then inserted to the left hand side of Formula 6.4.

For convenience, let d_0 also denote the representative element from $\{0, 1, \dots, N-1\}$ for $d_0 \in \mathbb{Z}/N$, then we have

$$\begin{aligned} & \sum_{\substack{c > 0 \\ (c,d) \equiv (c_0, d_0) \pmod{N}}} (c\tau + d)^{-k} \\ &= \frac{1}{N^k} \sum_{\substack{c > 0 \\ c \equiv c_0 \pmod{N}}} \sum_{l \in \mathbb{Z}} \left((c\tau + d_0)/N + l \right)^{-k} \\ &= \frac{C_k}{N^k} \sum_{\substack{c > 0 \\ c \equiv c_0 \pmod{N}}} \sum_{m \in \mathbb{Z}_{\geq 1}} m^{k-1} e(md_0/N) e(m\tau/N) \quad (6.5) \\ &= \frac{C_k}{N^k} \sum_{n \in \mathbb{Z}_{\geq 1}} \left(\sum_{\substack{m \in \mathbb{Z}_{\geq 1}, m|n, \\ n/m \equiv c_0 \pmod{N}}} m^{k-1} e(md_0/N) \right) e(n\tau/N), \end{aligned}$$

where $C_k := \frac{(-2\pi i)^k}{(k-1)!}$, and Equation 6.5 follows from application of Lemma 8.45 with $z := (c\tau + d_0)/N \in \mathfrak{H}$. We treat the part of $c < 0$ similarly, except that we set $z := -(c\tau + d_0)/N \in \mathfrak{H}$ in order to apply Lemma 8.45 again, and consequently in the last step we

consider the sum over $m' := -m$ instead of m . We thus obtain

$$\begin{aligned}
 & \sum_{\substack{c < 0 \\ (c,d) \equiv (c_0,d_0) \pmod{N}}} (c\tau + d)^{-k} \\
 &= \frac{1}{(-N)^k} \sum_{\substack{c < 0 \\ c \equiv c_0 \pmod{N}}} \sum_{l \in \mathbb{Z}} \left(-(c\tau + d_0)/N - l \right)^{-k} \\
 &= \frac{C_k}{(-N)^k} \sum_{\substack{c < 0 \\ c \equiv c_0 \pmod{N}}} \sum_{m \in \mathbb{Z}_{\geq 1}} m^{k-1} e(-md_0/N) e(-m\tau/N) \\
 &= \frac{C_k}{N^k} \sum_{n \in \mathbb{Z}_{\geq 1}} \left(\sum_{\substack{m' \in \mathbb{Z}_{\leq -1}, m' | n, \\ n/m' \equiv c_0 \pmod{N}}} -m'^{k-1} e(m'd_0/N) \right) e(n\tau/N),
 \end{aligned} \tag{6.6}$$

where Equation 6.6 follows from Lemma 8.45. Combining positive and negative parts of the divisor sum, we get the Fourier coefficient of $e(n\tau/N)$ for $n \geq 1$. \blacksquare

7 Products of Eisenstein series

Historically, Rankin [Ran52] was the first to associate periods with the scalar products of cuspidal Hecke eigenforms and products of two Eisenstein series. Later Kohnen and Zagier [KZ84] connected modular forms with period polynomials, and as a consequence of this association concluded that the products of at most two Eisenstein series span all spaces of modular forms for $\mathrm{SL}_2(\mathbb{Z})$. In higher level cases, Borisov and Gunnells [BG01] [BG03] solved the problem for $\Gamma_1(N)$ and weight $k \geq 3$, and Khuri-Makdisi [Khu12] considered it for $\Gamma(N)$ and weight 2. For the group $\Gamma_0(N)$, İmamoğlu and Kohnen [IK05] first considered the problem in the case of $\Gamma_0(2)$, and later Kohnen and Martin [KM08] generalized it to the prime level, for $\Gamma_0(p)$. Recently, Dickson and Neururer [DN18] showed that each cusp form of weight $k \geq 4$ for $\Gamma_0(N)$ can be spanned by products of explicit Eisenstein series for $\Gamma_1(N)$, under certain technical assumptions on the level N .

We continue to consider this problem for vector-valued modular forms, following the strategy and results in Raum [Wes17]. Recall Definition 1.6 of the vector-valued Hecke operator T_N for vector-valued modular forms, which yields a linear map

$$T_N : E_k(\mathbb{1}) \longrightarrow E_k(\rho_{T_N}),$$

where $\mathbb{1}$ is the trivial representation of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and $\rho_{T_N} := T_N(\mathbb{1})$ is defined in Definition 1.5. Hence we have the following inclusion.

Theorem 7.1. *Let $k, l \in \mathbb{Z}_{\geq 4}$ be even integers. Let ρ be a congruence type for Γ . Then we have*

$$M_{k+l}(\rho) = E_{k+l}(\rho) + \sum_{\substack{N, N' \in \mathbb{Z}_{\geq 1}, \\ \phi: \rho_{T_N} \otimes \rho_{T_{N'}} \rightarrow \rho}} \phi \left(T_N(E_k(\mathbb{1})) \otimes T_{N'}(E_l(\mathbb{1})) \right),$$

where the sum runs over all the homomorphisms of representations $\phi : \rho_{T_N} \otimes \rho_{T_{N'}} \longrightarrow \rho$ for all positive integers N, N' .

Proof. See [Wes17], and note that the weight k that appears in that paper corresponds to $k + l$ here. \blacksquare

Our goal is to formulate this result in terms of classical Eisenstein series. We need to use a basic fact for the representation ρ_N^\times .

Lemma 7.2. *Let $l > 2$ be an integer, then we have*

$$T_N(E_l(\mathbb{1})) \subseteq E_l(T_N(\mathbb{1})).$$

Proof. By Proposition 2.7 in [Wes17], we have

$$T_N(E_l(\mathbb{1})) \subseteq T_N(M_l(\mathbb{1})) \subseteq M_l(T_N(\mathbb{1})).$$

By Lemma 2.3 in [Wes17], we know $T_N(\mathbb{1})$ is a congruence type. From the proof of Proposition 5.5, for any $f \in E_l(\mathbb{1})$ we know that if we could show that all the components of $T_N(f)$ is in the space of all classical Eisenstein series \mathcal{E}_l , then we have $T_N(f) \in E_l(T_N(\mathbb{1}))$. But it is straightforward to compute each component of $T_N(f)$ as $f|_l\gamma$ for some $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta_N$, and it turns out they all belong to the classical Eisenstein series defined via congruence relations, and the level of the Eisenstein series $f|_l\gamma$ is at most $ad = N$. \blacksquare

Lemma 7.3.

$$T_N(\mathbb{1}) \cong \bigoplus_{d^2|N} \text{Ind}_{\Gamma_0(N/d^2)}^\Gamma \mathbb{1} \hookrightarrow \bigoplus_{d^2|N} \rho_{N/d^2}^\times \hookrightarrow \bigoplus_{d|N} \rho_d^\times \cong \rho_N.$$

Proof. We show the lemma in two steps. First, note that for any matrix with integer coefficients of determinant N , the action of Γ by the multiplication from left or right does not change the gcd of all the four matrix coefficients. This implies a decomposition $\Delta_N = \coprod_d \Delta_{N,d}$ and the corresponding decomposition of the action π_N by Γ , where $\Delta_{N,d}$ denotes the subset of Δ_N , whose elements have d as the gcd of all the coefficients. For any $\gamma \in \Delta_{N,d}$, we have $\gamma = d\gamma'$ for some $\gamma' \in \Delta_{N/d^2,1}$, since $\det\gamma = d^2\det\gamma'$. It is thus clear that we have the decomposition

$$T_N(\mathbb{1}) \cong \bigoplus_{d^2|N} T_{N/d^2}^\times(\mathbb{1}),$$

where $T_{N/d^2}^\times(\mathbb{1})$ is the permutation representation associated with the Γ -action on $\Delta_{N/d^2,1}$.

Second, we show that there is an isomorphism for every positive integer N'

$$\begin{aligned} \varphi : T_{N'}^\times(\mathbb{1}) &\cong \text{Ind}_{\Gamma_0(N')}^\Gamma \mathbb{1}, \\ \mathfrak{e} \begin{pmatrix} N' & 0 \\ 0 & 1 \end{pmatrix} &\longmapsto \mathfrak{e}_{\text{id}}. \end{aligned}$$

In fact, it is clear from the theory of Smith normal form applied to the principal ideal domain \mathbb{Z} that the action of Γ on $\Delta_{N',1}$ is transitive¹³, hence the vector $\mathfrak{e}_{\begin{pmatrix} N' & 0 \\ 0 & 1 \end{pmatrix}}$ generates the whole space $V\left(T_{N'}^{\times}(\mathbb{1})\right)$. Therefore, the homomorphism φ is well-defined via the association $\mathfrak{e}_{\begin{pmatrix} N' & 0 \\ 0 & 1 \end{pmatrix}} \mapsto \mathfrak{e}_{\text{id}}$ and the intertwining relation, given that the stabilizer of $\mathfrak{e}_{\begin{pmatrix} N' & 0 \\ 0 & 1 \end{pmatrix}}$, denoted by Γ_1 , is contained in that of \mathfrak{e}_{id} , denoted by Γ_0 . The surjectivity follows from the fact that the permutation representation of Γ on $\Gamma_0(N)\backslash\Gamma$ is transitive. On the other hand, it is also clear that the homomorphism φ is injective if $\Gamma_0 \subseteq \Gamma_1$ (similar to the injectivity part of the proof of Lemma 5.7). Now we have to show that $\Gamma_0 = \Gamma_1$. It follows directly from the definition of $\text{Ind}_{\Gamma_0(N')}^{\Gamma} \mathbb{1}$ that $\gamma \in \Gamma_0$ if and only if $\gamma \in \Gamma_0(N)$, and that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ if and only if

$$\begin{pmatrix} N' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} N' & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in \Gamma,$$

which is equivalent after direct computation that $c \equiv 0 \pmod{N}$, that is, $\gamma \in \Gamma_0(N')$. This shows that $\Gamma_0 = \Gamma_1 = \Gamma_0(N')$. Finally, the inclusion

$$\text{Ind}_{\Gamma_0(N')}^{\Gamma} \mathbb{1} \hookrightarrow \text{Ind}_{\Gamma_1(N')}^{\Gamma} \mathbb{1} \cong \rho_{N'}^{\times}$$

follows from Lemma 3.20 (or Lemma 3.9 as another way to view it). ■

Lemma 7.4. *Let N, M be positive integers such that $N \mid M$, then we have*

$$\rho_N^{\times} \hookrightarrow \rho_M^{\times}.$$

Proof. Since $N \mid M$, we have $\Gamma_1(N) \supseteq \Gamma_1(M)$. We now apply Lemma 3.20 with $G_1 := \Gamma_1(M)$, $G_2 := \Gamma_1(N)$, $G_3 := \Gamma$, and Lemma 3.8, to conclude the inclusion

$$\rho_N^{\times} \cong \text{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbb{1} \hookrightarrow \text{Ind}_{\Gamma_1(M)}^{\Gamma} \mathbb{1} \cong \rho_M^{\times}. \quad \blacksquare$$

Combining the above results, we have the following main theorem.

Theorem 7.5. *Let $k, l \in \mathbb{Z}_{\geq 4}$ be even integers. Let ρ be a congruence type for $\Gamma = \text{SL}_2(\mathbb{Z})$ of level N , i.e., N is the smallest positive integer such that $\Gamma(N) \subseteq \ker \rho$, then we have*

$$M_{k+l}(\rho) \cong (\mathcal{E}_{k+l,N} \otimes \rho)^{\Gamma} + \sum_{N'=1}^{\infty} \left(\mathcal{E}_{k,N'} \otimes \mathcal{E}_{l,N'} \otimes \rho \right)^{\Gamma}.$$

Proof. For any two congruence types ρ_1 and ρ_2 , and each homomorphism $\phi : \rho_1 \otimes \rho_2 \rightarrow \rho$, ϕ intertwines with the slash actions $(\cdot)|_{k,\rho_1} \otimes (\cdot)|_{l,\rho_2}$ and $(\cdot)|_{k+l,\rho}$, hence we have

$$M_{k+l}(\rho) \supseteq E_{k+l}(\rho) + \sum_{\substack{N' \in \mathbb{Z}_{\geq 1}, \\ \psi: \rho_{N'}^{\times} \otimes \rho_{N'}^{\times} \rightarrow \rho}} \psi \left(E_k(\rho_{N'}^{\times}) \otimes E_l(\rho_{N'}^{\times}) \right). \quad (7.1)$$

¹³for a typical example of transforming one diagonal matrix to a different diagonal matrix in $\Delta_{N',1}$, please refer to Example 9.8.

On the other hand, by Theorem 7.1, Lemma 7.2, Lemma 7.3, and Lemma 7.4, we have the inclusions

$$\begin{aligned}
 M_{k+l}(\rho) &= E_{k+l}(\rho) + \sum_{\substack{N_1, N_2 \in \mathbb{Z}_{\geq 1}, \\ \phi: T_{N_1}(\mathbf{1}) \otimes T_{N_2}(\mathbf{1}) \rightarrow \rho}} \phi \left(T_{N_1}(E_k(\mathbf{1})) \otimes T_{N_2}(E_l(\mathbf{1})) \right), \\
 &\subseteq E_{k+l}(\rho) + \sum_{\substack{N_1, N_2 \in \mathbb{Z}_{\geq 1}, \\ \phi: T_{N_1}(\mathbf{1}) \otimes T_{N_2}(\mathbf{1}) \rightarrow \rho}} \phi \left(E_k(T_{N_1}(\mathbf{1})) \otimes E_l(T_{N_2}(\mathbf{1})) \right), \\
 &\subseteq E_{k+l}(\rho) + \sum_{\substack{N_3, N_4 \in \mathbb{Z}_{\geq 1}, \\ \varphi: \rho_{N_3}^\times \otimes \rho_{N_4}^\times \rightarrow \rho}} \varphi \left(E_k(\rho_{N_3}^\times) \otimes E_l(\rho_{N_4}^\times) \right), \tag{7.2}
 \end{aligned}$$

$$\subseteq E_{k+l}(\rho) + \sum_{\substack{N' \in \mathbb{Z}_{\geq 1}, \\ \psi: \rho_{N'}^\times \otimes \rho_{N'}^\times \rightarrow \rho}} \psi \left(E_k(\rho_{N'}^\times) \otimes E_l(\rho_{N'}^\times) \right). \tag{7.3}$$

We argue for the last two inclusions a bit more: both are via composition of maps. For (7.2), note that for a fixed pair of positive integers (N_1, N_2) , Lemma 7.3 induces two projections

$$\pi_i : \bigoplus_{d_i | N_i} \rho_{d_i}^\times \longrightarrow T_{N_i}(\mathbf{1})$$

for $i = 1, 2$. We then consider for each homomorphism $\phi : T_{N_1}(\mathbf{1}) \otimes T_{N_2}(\mathbf{1}) \rightarrow \rho$ its composition with the projections, namely

$$\varphi := \phi \circ (\pi_1 \otimes \pi_2),$$

and all the components

$$\varphi_{d_1, d_2} := \varphi|_{\rho_{d_1}^\times \otimes \rho_{d_2}^\times} : \rho_{d_1}^\times \otimes \rho_{d_2}^\times \longrightarrow \rho.$$

We find that

$$\phi \left(E_k(T_{N_1}(\mathbf{1})) \otimes E_l(T_{N_2}(\mathbf{1})) \right) = \sum_{d_1 | N_1, d_2 | N_2} \varphi_{d_1, d_2} \left(E_k(\rho_{d_1}^\times) \otimes E_l(\rho_{d_2}^\times) \right),$$

which concludes the inclusion (7.2). For the last inclusion (7.3), we show it in a similar manner. For a fixed pair of positive integers (N_3, N_4) , Lemma 7.4 induces two projections

$$\varpi_i : \rho_{N'}^\times \longrightarrow \rho_{N_i}^\times$$

for $i = 3, 4$ and $N' := \text{lcm}(N_3, N_4)$. We find that for each homomorphism $\varphi : \rho_{N_3}^\times \otimes \rho_{N_4}^\times \rightarrow \rho$, we have

$$\varphi \left(E_k(\rho_{N_3}^\times) \otimes E_l(\rho_{N_4}^\times) \right) = \psi \left(E_k(\rho_{N'}^\times) \otimes E_l(\rho_{N'}^\times) \right)$$

for $\psi := \varphi \circ (\varpi_3 \otimes \varpi_4) : \rho_{N'}^\times \otimes \rho_{N'}^\times \rightarrow \rho$, which concludes (7.3). Therefore, combining (7.1) and (7.3), we obtain

$$M_{k+l}(\rho) = E_{k+l}(\rho) + \sum_{\substack{N' \in \mathbb{Z}_{\geq 1}, \\ \psi: \rho_{N'}^\times \otimes \rho_{N'}^\times \rightarrow \rho}} \psi \left(E_k(\rho_{N'}^\times) \otimes E_l(\rho_{N'}^\times) \right), \quad (7.4)$$

Finally, since ρ is a congruence type of level N , by Lemma 5.8 and Lemma 5.10, we obtain

$$E_{k+l}(\rho) \cong \mathrm{Hom}_{\Gamma} \left(\mathbf{1}, \mathcal{E}_{k+l}[\rho_N^\times] \otimes \rho \right) = \mathrm{Hom}_{\Gamma} \left(\mathbf{1}, \mathcal{E}_{k+l,N} \otimes \rho \right) \cong (\mathcal{E}_{k+l,N} \otimes \rho)^{\Gamma}.$$

Similarly, we can also simplify the second term of (7.4) by Lemma 5.8 and Lemma 5.10. For each homomorphism $\psi : \rho_{N'}^\times \otimes \rho_{N'}^\times \rightarrow \rho$, we have

$$\begin{aligned} \psi \left(E_k(\rho_{N'}^\times) \otimes E_l(\rho_{N'}^\times) \right) &\cong \psi \left((\mathcal{E}_{k,N'} \otimes \rho_{N'}^\times)^{\Gamma} \otimes (\mathcal{E}_{l,N'} \otimes \rho_{N'}^\times)^{\Gamma} \right) \\ &\subseteq \psi \left((\mathcal{E}_{k,N'} \otimes \rho_{N'}^\times \otimes \mathcal{E}_{l,N'} \otimes \rho_{N'}^\times)^{\Gamma} \right) \\ &\subseteq (\mathcal{E}_{k,N'} \otimes \mathcal{E}_{l,N'} \otimes \rho)^{\Gamma}, \end{aligned}$$

hence we conclude that

$$M_{k+l}(\rho) \hookrightarrow (\mathcal{E}_{k+l,N} \otimes \rho)^{\Gamma} + \sum_{N'=1}^{\infty} (\mathcal{E}_{k,N'} \otimes \mathcal{E}_{l,N'} \otimes \rho)^{\Gamma}. \quad (7.5)$$

On the other hand, since for each positive integer N'

$$\mathcal{E}_{k,N'} \otimes \mathcal{E}_{l,N'} \hookrightarrow \mathcal{M}_{k+l} \subseteq \mathcal{H}_{k+l}^{\mathrm{md}},$$

by Proposition 3.15 we have

$$(\mathcal{E}_{k+l,N} \otimes \rho)^{\Gamma} + \sum_{N'=1}^{\infty} (\mathcal{E}_{k,N'} \otimes \mathcal{E}_{l,N'} \otimes \rho)^{\Gamma} \hookrightarrow (\mathcal{H}_{k+l}^{\mathrm{md}} \otimes \rho)^{\Gamma} + (\mathcal{H}_{k+l}^{\mathrm{md}} \otimes \rho)^{\Gamma} \cong M_k(\rho),$$

which together with (7.5) completes the proof. \blacksquare

Remark 7.6. Note that the sum over N' in Theorem 7.5 is a finite sum, and the upper bound of N' can be found by means of Hecke theory.

8 Auxiliary Statements

Lemma 8.1. *Let φ, ρ be two representations of a group G . Let $f : \varphi \rightarrow \rho$ be a homomorphism of representations. If f is an isomorphism as a linear map, then it is an isomorphism of representations.*

Proof. Assume g is the inverse map of f , i.e.

$$f \circ g = \text{id}_{V(\rho)}, \text{ and } g \circ f = \text{id}_{V(\varphi)}. \quad (8.1)$$

Since f is a homomorphism, we have a commutative diagram, which can be extended from both sides by g , we obtain the following commutative diagram:

$$\begin{array}{ccccc} V(\rho) & \xrightarrow{g} & V(\varphi) & \xrightarrow{f} & V(\rho) \\ & & \varphi(\gamma) \downarrow & & \downarrow \rho(\gamma) \\ & & V(\varphi) & \xrightarrow{f} & V(\rho) & \xrightarrow{g} & V(\varphi) \end{array} \quad (8.2)$$

Insert Condition (8.1) into Diagram (8.2), we get the following commutative diagram, hence g is also a homomorphism of representations.

$$\begin{array}{ccc} V(\rho) & \xrightarrow{g} & V(\varphi) \\ \rho(\gamma) \downarrow & & \downarrow \varphi(\gamma) \\ V(\rho) & \xrightarrow{g} & V(\varphi) \end{array} \quad (8.3) \quad \blacksquare$$

Lemma 8.2. *Let $x \in \mathbb{R}_{(-1,1)}$ and $y \in \mathbb{R}_{>1}$. Let $\tau = x + iy$. Then we have*

$$|c\tau + d| \geq \frac{1}{\sqrt{5}}|ci + d|$$

for all real numbers c, d .

Proof. We show this inequality in two cases. If $|cx + d| > \frac{1}{2}|d|$, then we have

$$|c\tau + d|^2 = |cy|^2 + |cx + d|^2 > |c|^2 + \frac{1}{4}|d|^2 \geq \frac{1}{5}(|c|^2 + |d|^2) = \frac{1}{5}|ci + d|^2.$$

If $|cx + d| \leq \frac{1}{2}|d|$, then we have

$$|d| = |(cx + d) + (-cx)| \leq |cx + d| + |cx| \leq \frac{1}{2}|d| + |cx|,$$

hence $|cy| \geq |cx| \geq \frac{1}{2}|d|$ (where we use the assumption $|y| > 1 > |x|$). Therefore, we also conclude

$$|c\tau + d|^2 = |cy|^2 + |cx + d|^2 \geq |cy|^2 = \frac{1}{5}|cy|^2 + \frac{4}{5}|cy|^2 \geq \frac{1}{5}|c|^2 + \frac{1}{5}|d|^2 = \frac{1}{5}|ci + d|^2. \quad \blacksquare$$

Lemma 8.3. *Let $k > 2$ be a real number, then*

$$\sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |ci + d|^{-k} \leq 2\zeta(k) + 2^{2-k/2}(\zeta(k/2))^2.$$

Proof. Let $X := \{(0, d) : d \in \mathbb{Z} \setminus \{0\}\}$, $Y := \{(c, 0) : c \in \mathbb{Z} \setminus \{0\}\}$, and $Z := \{(c, d) \in \mathbb{Z}^2 : c \neq 0, d \neq 0\}$. Then it is clear that $\mathbb{Z}^2 \setminus \{(0, 0)\} = X \amalg Y \amalg Z$, hence

$$\sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |ci + d|^{-k} = \sum_{(c,d) \in X} |ci + d|^{-k} + \sum_{(c,d) \in Y} |ci + d|^{-k} + \sum_{(c,d) \in Z} |ci + d|^{-k}. \quad (8.4)$$

Since $k > 2$, $\sum_{(c,d) \in X} |ci + d|^{-k} = \sum_{(c,d) \in Y} |ci + d|^{-k}$ converge and are both equal to $2\zeta(k)$. For $\sum_{(c,d) \in Z} |ci + d|^{-k}$, we note that $|ci + d| = (c^2 + d^2)^{1/2} \geq (2|c||d|)^{1/2}$, so we have

$$\sum_{(c,d) \in Z} |ci + d|^{-k} \leq 2^{-k/2} \sum_{c \in \mathbb{Z} \setminus \{0\}} \sum_{d \in \mathbb{Z} \setminus \{0\}} |c|^{-k/2} |d|^{-k/2} = 2^{2-k/2} (\zeta(k/2))^2,$$

as $k/2 > 1$. Inserting these inequalities into (8.4), we finish the proof. \blacksquare

Lemma 8.4. *Let $k > 2$ be an integer and ρ an arithmetic type for Γ such that $\ker \rho$ is of finite index in Γ . Let v be a Γ_∞ -invariant vector. Then, the infinite series $E_{k,\rho,v}$ is dominated by some convergent series on some open neighbourhood (punctured) U of $i\infty$. In particular, $E_{k,\rho,v}$ converges uniformly on U . Moreover, $E_{k,\rho,v}$ converges absolutely at each point on \mathfrak{H} .*

Proof. Let U be the open subset $\{\tau \in \mathfrak{H} : \text{Im}(\tau) > 1, |\text{Re}(\tau)| < 1\}$. Since the norm on the finite dimensional vector space $V(\rho)$ is unique up to equivalence, we simply fix an arbitrary choice $\|\cdot\|$. We shall prove later that there is a constant $C_{\rho,v}$ depending only on (ρ, v) , and a positive real number $R(c, d)$ for each $(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, depending on k , such that

$$\|v|_{k,\rho}\gamma(\tau)\| \leq C_{\rho,v} R(c(\gamma), d(\gamma)) \quad (8.5)$$

for all $\gamma \in \Gamma$ and $\tau \in U$, and that $\sum_{(c,d) \neq (0,0)} R(c, d) < \infty$. Note that if we have this estimate (8.5), then the infinite series (of vector-valued functions) $E_{k,\rho,v}(\tau) = \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma} v|_{k,\rho}\gamma(\tau)$ is dominated on U by the infinite series (of non-negative real numbers)

$$C_{\rho,v} \sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma} R(c(\gamma), d(\gamma)) = \frac{1}{2} C_{\rho,v} \sum_{\gcd(c,d)=1} R(c, d) \leq \frac{1}{2} C_{\rho,v} \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} R(c, d) < \infty,$$

as desired. To show (8.5), we first recall that $\ker \rho$ is of finite index in Γ , hence there are only finitely many values of $\|\rho(\gamma)v\|$ for $\gamma \in \Gamma$. Let $C_{\rho,v}$ be the maximal value of $\|\rho(\gamma)v\|$, then we have $\|v|_{k,\rho}\gamma(\tau)\| = |c\tau + d|^{-k} \|\rho(\gamma)v\| \leq C_{\rho,v} |c\tau + d|^{-k}$. By Lemma 8.2, $|c\tau + d|^{-k} \leq 5^{k/2} |ci + d|^{-k}$ for all $(c, d) \neq (0, 0)$ and $\tau \in U$. Let $R(c, d) := 5^{k/2} |ci + d|^{-k}$, the first condition of (8.5) is satisfied. By Lemma 8.3, we have

$$\sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} R(c, d) = 5^{k/2} \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |ci + d|^{-k} < \infty,$$

hence the second condition of (8.5) is also satisfied. To see that $E_{k,\rho,v}$ is absolutely convergent at an arbitrary point $\tau = x + iy \in \mathfrak{H}$, we decompose the sum

$$\sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} |c\tau + d|^{-k} = \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left((cx + d)^2 + (cy)^2 \right)^{-k/2}$$

into three parts and estimate them separately. If $c = 0$, then the sum is bounded above by $2\zeta(k)$. If $c \neq 0$ and $|cx + d| < 1$, then for each fixed value of c , there are at most 2 corresponding values of d , hence the sum is bounded above by

$$\sum_{c \in \mathbb{Z} \setminus \{0\}} \sum_{d: |cx+d| < 1} (|c|y)^{-k} \leq 4y^{-k} \zeta(k).$$

If $c \neq 0$ and $|cx + d| \geq 1$, then the sum is bounded above by

$$\sum_{c \in \mathbb{Z} \setminus \{0\}} |2cy|^{-k/2} \sum_{d \in \mathbb{Z}: |cx+d| \geq 1} |cx + d|^{-k/2} \leq \sum_{c \in \mathbb{Z} \setminus \{0\}} |2cy|^{-k/2} 2\zeta(k/2) = 2^{2-k/2} y^{-k/2} (\zeta(k/2))^2,$$

Combining these inequalities and $\|v|_{k,\rho}\gamma(\tau)\| = |c\tau + d|^{-k} \|\rho(\gamma)v\| \leq C_{\rho,v} |c\tau + d|^{-k}$, we obtain the absolute convergence of $E_{k,\rho,v}$ at the point τ . \blacksquare

Lemma 8.5. *Let $k > 2$ be an integer. Let ρ be a congruence type for Γ and $v \in V(\rho)^{\Gamma_\infty}$, then the constant term of $E_{k,\rho,v}$ is equal to v . Conversely, if $f \in E_k(\rho)$ has the constant term $v \in V(\rho)^{\Gamma_\infty}$, then $f = E_{k,\rho,v}$.*

Proof. It is clear that the second claim follows from the first. For the first claim, the constant term of $E_{k,\rho,v}$ is by definition

$$\lim_{\tau \rightarrow i\infty} E_{k,\rho,v} = \lim_{\tau \rightarrow i\infty} \sum_{[\gamma] \in \Gamma_\infty \setminus \Gamma} v|_{k,\rho}\gamma,$$

where Lemma 8.4 allows us to interchange the limit with the sum here, so we find the constant term of $E_{k,\rho,v}$ as

$$\sum_{[\gamma] \in \Gamma_\infty \setminus \Gamma} \lim_{\tau \rightarrow i\infty} (c\tau + d)^{-k} \rho(\gamma^{-1})v.$$

Since $\lim_{\tau \rightarrow i\infty} (c\tau + d)^{-k} \rho(\gamma^{-1})v = 0$ if $c \neq 0$, and the only coset $[\gamma]$ corresponding to $c = 0$ is the trivial one, which contributes v to the sum, the constant term is therefore equal to v . \blacksquare

Lemma 8.6. *Let $k > 2$, $N \geq 1$ be integers. For all $\lambda \in L_N^\times$ and $\gamma \in \Gamma$, we have*

$$E_{k,N,\lambda}|_k \gamma = E_{k,N,\lambda\gamma} \tag{8.6}$$

Proof. For $N = 1$, both sides are always equal to E_k . For $N = 2$ and $k > 2$ an odd integer, both sides vanish. In the remaining cases, let $v_0 := \mathbf{e}_{(\bar{0}, \bar{1})} + (-1)^k \mathbf{e}_{(\bar{0}, \bar{-1})} \in V(\rho_N^\times)^{\Gamma_\infty}$, where the representation ρ_N^\times is defined in the beginning of Section 3. From the proof of Lemma 5.10, we know that

$$E_{k,N,\lambda} = \mathbf{e}_\lambda^\vee \circ E_{k,\rho_N^\times,v_0}.$$

Since $E_{k,\rho_N^\times,v_0} \in M_k(\rho_N^\times)$, we have $E_{k,\rho_N^\times,v_0}|_k\gamma = \rho_N^\times(\gamma) \circ E_{k,\rho_N^\times,v_0}|_k\rho_N^\times\gamma = \rho_N^\times(\gamma) \circ E_{k,\rho_N^\times,v_0}$, and therefore

$$E_{k,N,\lambda}|_k\gamma = \mathbf{e}_\lambda^\vee \circ E_{k,\rho_N^\times,v_0} = \left(\mathbf{e}_\lambda^\vee \circ \rho_N^\times(\gamma)\right) \circ E_{k,\rho_N^\times,v_0}.$$

Note that we also have $E_{k,N,\lambda\gamma} = \mathbf{e}_{\lambda\gamma}^\vee \circ E_{k,\rho_N^\times,v_0}$, to see (8.6), it suffices to show that $\mathbf{e}_{\lambda\gamma}^\vee = \mathbf{e}_\lambda^\vee \circ \rho_N^\times(\gamma)$. This is clear from the definition of ρ_N^\times and the dual basis. \blacksquare

Lemma 8.7. *Let $N \geq 3$ be an integer and $k > 2$ an integer, or $N = 2$ and $k > 2$ an even integer. Then, the constant Fourier coefficient of $E_{k,N,\lambda}$ is non-zero if and only if $\lambda = (\bar{0}, \bar{1})$ or $(\bar{0}, \bar{-1})$. Furthermore, when $N \geq 3$, if $\lambda = (\bar{0}, \bar{1})$, the constant Fourier coefficient of $E_{k,N,\lambda}$ is 1; if $\lambda = (\bar{0}, \bar{-1})$, it is $(-1)^k$. When $N = 2$ and $k > 2$ an even integer, the constant Fourier coefficient of $E_{k,N,\lambda}$ is 2 for $\lambda = (\bar{0}, \bar{1}) = (\bar{0}, \bar{-1})$.*

Proof. Let $v_0 := \mathbf{e}_{(\bar{0}, \bar{1})} + (-1)^k \mathbf{e}_{(\bar{0}, \bar{-1})} \in V(\rho_N^\times)^{\Gamma_\infty}$, where the representation ρ_N^\times is defined in the beginning of Section 3. From the proof of Lemma 5.10 we know that

$$E_{k,N,\lambda} = \mathbf{e}_\lambda^\vee \circ E_{k,\rho_N^\times,v_0}.$$

By the continuity of the application of \mathbf{e}_λ^\vee and Lemma 8.5, we find the constant Fourier coefficient of $E_{k,N,\lambda}$ is

$$\lim_{\tau \rightarrow i\infty} E_{k,N,\lambda}(\tau) = \mathbf{e}_\lambda^\vee \left(\lim_{\tau \rightarrow i\infty} E_{k,\rho_N^\times,v_0}(\tau) \right) = \mathbf{e}_\lambda^\vee(v_0),$$

and the rest is clear. \blacksquare

Lemma 8.8. *The Eisenstein series $E_{k,N,\lambda}$ are linearly independent over \mathbb{C} when λ runs over a set of representatives for $L_N^\times / \pm 1$ (which will be simply denoted by $\lambda \in L_N^\times / \pm 1$ later).*

Proof. Suppose there is a linear relation

$$\sum_{\lambda \in L_N^\times} \epsilon_\lambda E_{k,N,\lambda} = 0,$$

such that at least one of $\{\epsilon_\lambda, \epsilon_{-\lambda}\}$ equals 0 for all $\lambda \in L_N^\times$. Then, we need to show that $\epsilon_\lambda = 0$ for all element $\lambda \in L_N^\times$. We apply $\cdot|_k\gamma^{-1}$ on both sides, with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, such that $(c, d) \equiv \lambda \pmod{N}$. By Lemma 8.6, we get

$$\sum_{\lambda' \in L_N^\times} \epsilon_{\lambda'} E_{k,N,\lambda'\gamma^{-1}} = 0.$$

Now consider the constant Fourier coefficient on both sides. By Lemma 8.7, if $N \geq 3$, the constant Fourier coefficient on the left hand side is equal to

$$\sum_{\substack{\lambda' \in L_N^\times, \\ \lambda'\gamma^{-1} = (\bar{0}, \bar{1})}} \epsilon_{\lambda'} + \sum_{\substack{\lambda' \in L_N^\times, \\ \lambda'\gamma^{-1} = (\bar{0}, \bar{-1})}} (-1)^k \epsilon_{\lambda'} = 0,$$

Note that for any $\lambda' \in L_N^\times$, $\lambda'\gamma^{-1} = (\bar{0}, \bar{1})$ if and only if $\lambda' = (\bar{0}, \bar{1})\gamma = \lambda$, and $\lambda'\gamma^{-1} = (\bar{0}, \bar{-1})$ if and only if $\lambda' = -\lambda$, hence we get $\epsilon_\lambda + (-1)^k \epsilon_{-\lambda} = 0$. But at least one of $\{\epsilon_\lambda, \epsilon_{-\lambda}\}$ is 0 by assumption, so we must have $\epsilon_\lambda = 0$. If $N = 1$ or 2 , $(\bar{0}, \bar{1}) = (\bar{0}, \bar{-1})$ and the constant Fourier coefficient is equal to

$$\sum_{\substack{\lambda' \in L_N^\times \\ \lambda'\gamma^{-1} = (\bar{0}, \bar{1})}} \epsilon_{\lambda'} = 0,$$

whence we deduce $\epsilon_\lambda = 0$. ■

Remark 8.9. Conceptually speaking, this proof is from the cusp expansions of modular forms, namely each Eisenstein series $E_{k,N,\lambda}$ is characterized by a unique cusp of $\Gamma(N)$ which supports it. For a pair of coprime integers c, d such that $(c, d) \equiv \lambda \pmod{N}$, this cusp can be explicitly written as $\Gamma(N)(d : -c)$, which corresponds to the first column of $\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \Gamma$ in the proof.

Lemma 8.10. *For each $\lambda \in L_N^\times$, the Eisenstein series $E_{k,N,\lambda}$ is a linear combination of $G_{k,N,b\lambda}$ for all $b \in (\mathbb{Z}/N)^\times$.*

Proof. We start from the Möbius Transform of the one function

$$\sum_{m|n} \mu(d) = \delta_{n,1},$$

for all $n \in \mathbb{Z}_{\geq 1}$, where δ is the Kronecker delta. Then we apply it to the sum from the definition of $E_{k,N,\lambda}$ and obtain

$$\begin{aligned} & E_{k,N,\lambda} \\ := & \sum_{\substack{(c,d) \equiv \lambda \pmod{N}, \\ \gcd(c,d)=1}} (c\tau + d)^{-k} \\ = & \sum_{(c,d) \equiv \lambda \pmod{N}} \delta_{\gcd(c,d),1} (c\tau + d)^{-k} \\ = & \sum_{(c,d) \equiv \lambda \pmod{N}} \sum_{\substack{m \in \mathbb{Z}_{\geq 1}, \\ m | \gcd(c,d)}} \mu(m) (c\tau + d)^{-k} \\ = & \sum_{\substack{m \in \mathbb{Z}_{\geq 1}, \\ \gcd(m,N)=1}} \mu(m) \sum_{\substack{(c,d) \equiv \lambda \pmod{N}, \\ \gcd(c,d) \in m\mathbb{Z}}} (c\tau + d)^{-k} \end{aligned} \tag{8.7}$$

$$= \sum_{\substack{m \in \mathbb{Z}_{\geq 1}, \\ \gcd(m,N)=1}} \frac{\mu(m)}{m^k} \sum_{(c',d') \equiv \bar{m}^{-1}\lambda \pmod{N}} (c'\tau + d')^{-k} \tag{8.8}$$

$$= \sum_{a \in (\mathbb{Z}/N)^\times} \sum_{\substack{m \in \mathbb{Z}_{\geq 1}, \\ m \equiv a \pmod{N}}} \frac{\mu(m)}{m^k} G_{k,N,a^{-1}\lambda}$$

$$= \sum_{a \in (\mathbb{Z}/N)^\times} \zeta_{a,N,\mu}(k) G_{k,N,a^{-1}\lambda},$$

where $\zeta_{a,N,\mu}$ is our modified zeta function twisted by the Möbius function, given by

$$\zeta_{a,N,\mu}(k) := \sum_{\substack{m \in \mathbb{Z}_{\geq 1}, \\ m \equiv a \pmod{N}}} \frac{\mu(m)}{m^k},$$

which is not directly linked to the Hurwitz zeta function. Equation 8.7 follows from the fact that all the m appeared as factors of $\gcd(c, d)$ must be coprime to N . This is because $\gcd(c, d, N) = 1$ for all terms appeared in the previous sum, since $(c, d) \equiv \lambda \pmod{N}$ and $\lambda \in L_N^\times$. Equation 8.8 follows from the transform of variables $(c', d') = (\frac{c}{m}, \frac{d}{m})$, and that $\bar{m} \in (\mathbb{Z}/N)^\times$. ■

Corollary 8.11. *The Eisenstein series $G_{k,N,\lambda}$ are linearly independent over \mathbb{C} for $\lambda \in L_N^\times / \pm 1$.*

Proof. By Lemma 8.10 and the fact that $G_{k,N,-\lambda} = (-1)^k G_{k,N,\lambda}$, the set G of all the Eisenstein series $G_{k,N,\lambda}$ for $\lambda \in L_N^\times / \pm 1$, generates $\mathcal{E}_{k,N}$, so $\text{rk } G \geq \dim \mathcal{E}_{k,N}$. By Lemma 8.8, $\dim \mathcal{E}_{k,N} = |E|$, where E is the set of all the Eisenstein series $E_{k,N,\lambda}$ for $\lambda \in L_N^\times / \pm 1$. Since $|E| = |G|$, we obtain

$$|G| \geq \text{rk } G \geq \dim \mathcal{E}_{k,N} = |G|,$$

therefore we have $|G| = \text{rk } G$, i.e., all the Eisenstein series $G_{k,N,\lambda}$ are linearly independent over \mathbb{C} . ■

Lemma 8.12. *Given an element $\lambda \in L_N^\times$, the Eisenstein series $G_{k,N,\lambda}$ is a linear combination of $E_{k,N,b\lambda}$ for all $b \in (\mathbb{Z}/N)^\times$.*

Proof. We separate the sum over pairs (c, d) from the definition of $E_{k,N,\lambda}$, in terms of $n = \gcd(c, d)$, which are coprime to N since $\gcd(c, d, N) = \gcd(c_0, d_0, N) = 1$, and obtain

$$\begin{aligned} & G_{k,N,\lambda} \\ := & \sum_{(c,d) \equiv \lambda \pmod{N}} (c\tau + d)^{-k} \\ = & \sum_{\substack{n \in \mathbb{Z}_{\geq 1}, \\ \gcd(n,N)=1}} \sum_{\substack{(c,d) \equiv \lambda \pmod{N}, \\ \gcd(c,d)=n}} (c\tau + d)^{-k} \\ = & \sum_{\substack{n \in \mathbb{Z}_{\geq 1}, \\ \gcd(n,N)=1}} \frac{1}{n^k} \sum_{\substack{(c',d') \equiv \bar{n}^{-1}\lambda \pmod{N}, \\ \gcd(c',d')=1}} (c'\tau + d')^{-k} \end{aligned} \tag{8.9}$$

$$\begin{aligned} & = \sum_{a \in (\mathbb{Z}/N)^\times} \sum_{\substack{n \in \mathbb{Z}_{\geq 1}, \\ n \equiv a \pmod{N}}} \frac{1}{n^k} E_{k,N,a^{-1}\lambda} \\ & = \sum_{a \in (\mathbb{Z}/N)^\times} \zeta_{a,N}(k) E_{k,N,a^{-1}\lambda}, \end{aligned} \tag{8.10}$$

where Equation 8.9 follows from the transform of variables $(c', d') = (\frac{c}{n}, \frac{d}{n})$, and that $\bar{n} \in (\mathbb{Z}/N)^\times$. The modified zeta function $\zeta_{a,N}$ in Equation 8.10, given by

$$\zeta_{a,N}(k) := \sum_{\substack{n \in \mathbb{Z}_{\geq 1}, \\ n \equiv a \pmod{N}}} \frac{1}{n^k},$$

is a scalar multiple of a Hurwitz zeta function $\zeta(\cdot, \frac{a}{N})$, namely

$$\zeta_{a,N}(k) = \frac{1}{N^k} \zeta(k, \frac{a}{N}). \quad \blacksquare$$

Lemma 8.13. *We have a disjoint union decomposition of L_N , namely there is a bijection*

$$\begin{aligned} u : L_N &\longrightarrow \coprod_{N' | N} L_{N'}^\times, \\ \lambda &\longmapsto \frac{\text{ord}(\lambda)}{N} \lambda \pmod{\text{ord}(\lambda)}, \end{aligned}$$

where $\text{ord}(\lambda)$ denotes the order of λ in the group L_N .

Proof. For each $\lambda \in L_N$, we have $N\lambda = 0$, hence $\text{ord}(\lambda) | N$. In particular, we have

$$L_N = \coprod_{N' | N} L_{N,N'},$$

where $L_{N,N'} := \{\lambda \in L_N : \text{ord}(\lambda) = N'\}$. Therefore, it suffices to show that, for each $N' | N$, there is a bijection

$$\begin{aligned} u_{N'} : L_{N,N'} &\longrightarrow L_{N'}^\times, \\ \lambda &\longmapsto \frac{N'}{N} \lambda \pmod{N'}. \end{aligned}$$

To see this is well-defined, for a given $\lambda \in L_{N,N'}$, we can lift it to a pair of integers (c, d) . Since $N'(c, d) \in (N\mathbb{Z})^2$, we get $\frac{N'}{N}(c, d) \in \mathbb{Z}^2$. Furthermore, it is clear that the map does not depend on the choice of the lift, so $u_{N'}(L_{N,N'}) \subseteq L_{N'}^\times$. To see that the image is in $L_{N'}^\times$, it suffices to show that for any integer l , if $\frac{lN'}{N}\lambda = 0 \in L_{N'}$, then $N' | l$. Indeed, by considering the lift to integers again, this would imply that $l\lambda = 0 \in L_N$. But $\lambda \in L_{N,N'}$, which means as the order of λ , N' must divide l . Finally, following a similar procedure, we can check that the inverse of $u_{N'}$ is

$$\begin{aligned} r_{N'} : L_{N'}^\times &\longrightarrow L_{N,N'}, \\ \lambda &\longmapsto \frac{N}{N'} \lambda \pmod{N}. \end{aligned} \quad \blacksquare$$

Lemma 8.14. *For any $\lambda \in L_N$ and $\gamma \in \Gamma$, we have that*

$$u(\lambda\gamma) = u(\lambda)\gamma,$$

where u is the map defined in Lemma 8.13.

Proof. First we show that λ and $\lambda\gamma$ have the same order in the group L_N . In fact, let N_1 be the order of λ , and N_2 that of $\lambda\gamma$, then N_1 also annihilates $\lambda\gamma$, hence a multiple of N_2 . Similarly since $\lambda = (\lambda\gamma)\gamma^{-1}$, N_2 annihilates λ , hence a multiple of N_1 . The rest is clear from the definition of the map u . ■

Lemma 8.15. *We have a bijection*

$$\begin{aligned} l : \Gamma_1(N)\backslash\Gamma &\longrightarrow L_N^\times \\ [\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}] &\longmapsto (\bar{c}, \bar{d}) := (c, d) \pmod{N}. \end{aligned}$$

Remark 8.16. For convenience, the ensuing bijection between \mathcal{R}_1 and L_N^\times is also denoted by l . The inverse of l is denoted by r in both cases.

Proof. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have $\gcd(c, d) = 1$, hence $(\bar{c}, \bar{d}) \in L_N^\times$. For any $\gamma_1 \in \Gamma_1(N)$, by Lemma 8.31, we have

$$\gamma_1\gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \pmod{N},$$

hence the map is well-defined. For injectivity, if two matrices $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma$ for $i = 1, 2$ have the same bottom row mod N , then we apply Lemma 8.31 again to get

$$\begin{aligned} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}^{-1} &\equiv \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_1 & d_1 \end{pmatrix}^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} d_1 & -b_2 \\ -c_1 & a_2 \end{pmatrix} \\ &\equiv \begin{pmatrix} a_1d_1 - b_1c_1 & * \\ 0 & a_2d_2 - b_2c_2 \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}, \end{aligned}$$

hence these two matrices are in the same coset. For surjectivity, given a pair of integers (c, d) satisfying $\gcd(c, d, N) = 1$, we have to find a pair of coprime integers (c', d') , such that $(c', d') \equiv (c, d) \pmod{N}$. This is a direct application of Lemma 8.36 with $m := \gcd(d, N)$ and $n := d$. In fact, since $m \mid n$ and $\gcd(c, m) = 1$, Lemma 8.36 says that there is some integer k , such that $c + km$ is coprime with n , that is, $c + k \gcd(d, N)$ is coprime with d . Write $\gcd(d, N) = xd + yN$ for some integers x, y , then $c + kyN$ is coprime with d . It suffices to set $c' := c + kyN$ and $d' := d$. ■

Lemma 8.17. *For any $\gamma_1 \in \mathcal{R}_1, \gamma_2 \in \Gamma$, we have*

$$l(\overline{\gamma_1\gamma_2}) = l(\gamma_1)\gamma_2,$$

where l is the map defined in Lemma 8.15, and $\bar{\gamma} \in \mathcal{R}_1$ denotes the representative element of the coset $[\gamma] \in \Gamma_1(N)\backslash\Gamma$.

Proof. It suffices to show that the bottom row of $\gamma_1\gamma_2 \bmod N$ is equal to $l(\gamma_1)\gamma_2$. In fact, the bottom row of $\gamma_1\gamma_2 \bmod N$ is equal to the bottom row of $\gamma_1 \bmod N$ multiplied by γ_2 , hence equal to $l(\gamma_1)\gamma_2$. ■

Lemma 8.18. *Given a congruence subgroup Γ' of Γ , let \mathcal{R} be a set of representatives for the cosets $\Gamma'\backslash\Gamma$. Let I be the cocycle attached to \mathcal{R} , and for an arbitrary element $\alpha \in \Gamma$, let $\bar{\alpha} \in \mathcal{R}$ denote the representative element of the coset $[\alpha] \in \Gamma'\backslash\Gamma$. Then we have the 1-cocycle relation*

$$I_\beta(\gamma_1\gamma_2) = I_\beta(\gamma_1)I_{\overline{\beta\gamma_1}}(\gamma_2). \quad (8.11)$$

Proof. By the definition of cocycle, $\beta\gamma_1 = I_\beta(\gamma_1)\overline{\beta\gamma_1}$, so $\beta\gamma_1\gamma_2 = I_\beta(\gamma_1)(\overline{\beta\gamma_1}\gamma_2)$. Replacing $\overline{\beta\gamma_1}\gamma_2$ by $I_{\overline{\beta\gamma_1}}(\gamma_2)(\overline{\overline{\beta\gamma_1}\gamma_2})$, we get

$$\begin{aligned} & I_\beta(\gamma_1\gamma_2)\overline{\beta(\gamma_1\gamma_2)} \\ &= \beta\gamma_1\gamma_2 = I_\beta(\gamma_1)I_{\overline{\beta\gamma_1}}(\gamma_2)(\overline{\overline{\beta\gamma_1}\gamma_2}), \end{aligned}$$

where actually $\overline{\beta(\gamma_1\gamma_2)} = \overline{(\overline{\beta\gamma_1}\gamma_2)}$. In fact, since $\beta\gamma_1$ and $\overline{\beta\gamma_1}$ are in the same coset, so are $\beta(\gamma_1\gamma_2)$ and $\overline{\beta\gamma_1}\gamma_2$. Therefore the cancellation law of the group Γ yields (8.11). ■

Lemma 8.19. *If $\gamma, \gamma' \in \Gamma$ are in the same coset from $\Gamma_0(N)\backslash\Gamma$, and I is the cocycle attached to \mathcal{R}_0 . Then*

$$I_{\text{id}}(\gamma')(I_{\text{id}}(\gamma))^{-1} = \gamma'\gamma^{-1}.$$

Proof. Suppose $\overline{\gamma'} = \overline{\gamma} = \beta$, then we have

$$I_{\text{id}}(\gamma')(I_{\text{id}}(\gamma))^{-1} = (\gamma'\beta^{-1})(\gamma\beta^{-1})^{-1} = \gamma'\gamma^{-1}. \quad \blacksquare$$

Lemma 8.20. *For an arbitrary element $\alpha \in \Gamma$, let $\tilde{\alpha} \in \mathcal{R}_1$ denote the representative element for the coset of α in $\Gamma_1(N)\backslash\Gamma$, and $\bar{\alpha} \in \mathcal{R}_0$ that in $\Gamma_0(N)\backslash\Gamma$. Let I be the cocycle attached to \mathcal{R}_0 . Then we have the following two facts for all $\gamma, \delta \in \Gamma$:*

$$I_{\text{id}}(\gamma)I_{\overline{\gamma}}(\delta^{-1}) \text{ and } I_{\text{id}}(\widetilde{\gamma\delta^{-1}}) \text{ are in the same coset in } \Gamma_1(N)\backslash\Gamma, \quad (8.12)$$

and

$$\overline{\gamma}\delta^{-1} \text{ and } \widetilde{\gamma\delta^{-1}} \text{ are in the same coset in } \Gamma_0(N)\backslash\Gamma. \quad (8.13)$$

Proof. Apply Lemma 8.18 with $\beta = \text{id}$, $\gamma_1 = \gamma$, and $\gamma_2 = \delta^{-1}$, we get

$$I_{\text{id}}(\gamma)I_{\overline{\gamma}}(\delta^{-1}) = I_{\text{id}}(\gamma\delta^{-1}).$$

To see the rest of Fact 8.12, it suffices to check that for any $\alpha \in \Gamma$, $I_{\text{id}}(\alpha)$ and $I_{\text{id}}(\tilde{\alpha})$ are in the same coset in $\Gamma_1(N)\backslash\Gamma$. Since α and $\tilde{\alpha}$ are in the same coset in $\Gamma_1(N)\backslash\Gamma$, they are certainly in the same coset in $\Gamma_0(N)\backslash\Gamma$. Apply Lemma 8.19, we then get

$$I_{\text{id}}(\alpha)(I_{\text{id}}(\tilde{\alpha}))^{-1} = \alpha\tilde{\alpha}^{-1} \in \Gamma_1(N).$$

To see Fact 8.13, note that $\overline{\gamma}\delta^{-1}$ and $\gamma\delta^{-1}$ are in the same coset in $\Gamma_0(N)\backslash\Gamma$, and the latter is in the same coset with $\widetilde{\gamma\delta^{-1}}$, in $\Gamma_1(N)\backslash\Gamma$, hence certainly the same coset in $\Gamma_0(N)\backslash\Gamma$. ■

Lemma 8.21. *For any Dirichlet character $\chi \pmod{N}$, the following linear map is a homomorphism from $\text{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}$ to $\bigoplus_{\chi' \pmod{N}} \text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi'$:*

$$\begin{aligned} p_{\chi} : V(\text{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}) &\longrightarrow \bigoplus_{\chi' \pmod{N}} V(\text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi') \\ \mathbf{e}_{\gamma} &\longmapsto \chi\left((I_{\text{id}}(\gamma))^{-1}\right) \mathbf{e}_{\chi, \bar{\gamma}}, \end{aligned}$$

where $\bar{\gamma} \in \mathcal{R}_0$ is the representative element for the class $[\gamma] \in \Gamma_0(N) \backslash \Gamma$.

Proof. The natural injection

$$\begin{aligned} i_{\chi} : V(\text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi) &\longrightarrow \bigoplus_{\chi' \pmod{N}} V(\text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi') \\ \mathbf{e}_{\beta} &\longmapsto \mathbf{e}_{\chi, \beta} \end{aligned}$$

is a homomorphism from $\text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi$ to $\bigoplus_{\chi' \pmod{N}} \text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi'$. Plus, we have $p_{\chi} = i_{\chi} \circ \pi_{\chi}$, where

$$\begin{aligned} \pi_{\chi} : V(\text{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}) &\longrightarrow V(\text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi) \\ \mathbf{e}_{\gamma} &\longmapsto \chi\left((I_{\text{id}}(\gamma))^{-1}\right) \mathbf{e}_{\bar{\gamma}}. \end{aligned}$$

To show that p_{χ} is a homomorphism of representations, it suffices to show that π_{χ} is a homomorphism from $\text{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}$ to $\text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi$, i.e.

$$\text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi(\delta) \circ \pi_{\chi} = \pi_{\chi} \circ \text{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}(\delta)$$

holds for all $\delta \in \Gamma$. In fact, for an arbitrary $\gamma \in \mathcal{R}_1$, we have

$$\begin{aligned} &\text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi(\delta) \circ \pi_{\chi}(\mathbf{e}_{\gamma}) \\ &:= \chi\left((I_{\text{id}}(\gamma))^{-1}\right) \text{Ind}_{\Gamma_0(N)}^{\Gamma} \chi(\delta) \mathbf{e}_{\bar{\gamma}} \\ &:= \chi\left((I_{\text{id}}(\gamma))^{-1}\right) \chi\left((I_{\bar{\gamma}}(\delta^{-1}))^{-1}\right) \mathbf{e}_{\widetilde{\bar{\gamma}\delta^{-1}}} \\ &= \chi\left((I_{\text{id}}(\widetilde{\gamma\delta^{-1}}))^{-1}\right) \mathbf{e}_{\widetilde{\gamma\delta^{-1}}} \\ &=: \pi_{\chi} \mathbf{e}_{\widetilde{\gamma\delta^{-1}}} \\ &=: \pi_{\chi} \circ \text{Ind}_{\Gamma_1(N)}^{\Gamma} \mathbf{1}(\delta)(\mathbf{e}_{\gamma}), \end{aligned} \tag{8.14}$$

where $\tilde{\alpha} \in \mathcal{R}_1$ for $\alpha \in \Gamma$ denotes the representative element for the coset of α in $\Gamma_1(N) \backslash \Gamma$, and $\bar{\alpha} \in \mathcal{R}_0$ that in $\Gamma_0(N) \backslash \Gamma$. Equation 8.14 follows from Lemma 8.20. \blacksquare

Lemma 8.22. *Let \mathcal{R}_1 be a fixed set of representatives for $\Gamma_1(N) \backslash \Gamma$. For any fixed element $\beta \in \Gamma$, we have a bijection d_{β}*

$$\begin{aligned} d_{\beta} : \{\gamma \in \mathcal{R}_1 : \bar{\gamma} = \beta\} &\longrightarrow (\mathbb{Z}/N)^{\times} \\ \gamma &\longmapsto \bar{d}(I_{\text{id}}(\gamma)) = \bar{d}(\gamma\beta^{-1}), \end{aligned}$$

where $\bar{d}(\gamma') := d \pmod{N}$ for $\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proof. First we show the injectivity. For any two elements $\gamma_1, \gamma_2 \in \mathcal{R}_1$ such that $\overline{\gamma_1} = \overline{\gamma_2}$, if $\overline{d}(I_{\text{id}}(\gamma_1)) = \overline{d}(I_{\text{id}}(\gamma_2))$, then we need to show that $\gamma_1 = \gamma_2$. By Lemma 8.19, we have $\gamma_1 \gamma_2^{-1} = I_{\text{id}}(\gamma_1) I_{\text{id}}(\gamma_2)^{-1} \in \Gamma_0(N)$, hence

$$\overline{d}(\gamma_1 \gamma_2^{-1}) = \overline{d}(I_{\text{id}}(\gamma_1) (I_{\text{id}}(\gamma_2))^{-1}) = \overline{d}(I_{\text{id}}(\gamma_1)) \overline{d}((I_{\text{id}}(\gamma_2))^{-1}) = \overline{1},$$

which furthermore implies that $\gamma_1 \gamma_2^{-1} \in \Gamma_1(N)$. Since $\gamma_1, \gamma_2 \in \mathcal{R}_1$, this means that $\gamma_1 = \gamma_2$. Second we show surjectivity. Given $d_0 \in (\mathbb{Z}/N)^\times$, we find and fix an element $\gamma_0 \in \Gamma_0(N)$, such that $\overline{d}(\gamma_0) = d_0$. Let $\gamma := \widetilde{\gamma_0} \beta \in \mathcal{R}_1$, and we claim that γ is the pre-image of d_0 . In fact, from our construction, there is some $\gamma_1 \in \Gamma_1(N)$ such that $\gamma = \gamma_1 \gamma_0 \beta$, so we have

$$\overline{d}(\gamma \beta^{-1}) = \overline{d}(\gamma_1 \gamma_0) = \overline{d}(\gamma_0) = d_0. \quad \blacksquare$$

Lemma 8.23. *The maps p and ι constructed in the proof of Lemma 3.9 are inverse maps with one another.*

Proof. The result follows from a certain orthogonality relation of Dirichlet characters, applied to the following computation.

On the one side, we show that for each $\gamma \in \mathcal{R}_1$,

$$\iota \circ p(\mathbf{e}_\gamma) = \mathbf{e}_\gamma.$$

After summing up by definitions of the maps p and ι , we have

$$\begin{aligned} & \iota \circ p(\mathbf{e}_\gamma) \\ &= \frac{1}{\varphi(N)} \sum_{\substack{\gamma' \in \mathcal{R}_1, \\ \overline{\gamma'} = \overline{\gamma}}} \left(\sum_{\chi \pmod{N}} \chi((I_{\text{id}}(\gamma))^{-1}) \chi(I_{\text{id}}(\gamma')) \right) \mathbf{e}_{\gamma'} \\ &= \frac{1}{\varphi(N)} \sum_{\substack{\gamma' \in \mathcal{R}_1, \\ \overline{\gamma'} = \overline{\gamma}}} \left(\sum_{\chi \pmod{N}} \chi(\gamma' \gamma^{-1}) \right) \mathbf{e}_{\gamma'} \end{aligned} \quad (8.15)$$

$$\begin{aligned} &= \frac{1}{\varphi(N)} \sum_{\substack{\gamma' \in \mathcal{R}_1, \\ \overline{\gamma'} = \overline{\gamma}}} (\varphi(N) \delta_{\gamma', \gamma}) \mathbf{e}_{\gamma'} \\ &= \mathbf{e}_\gamma, \end{aligned} \quad (8.16)$$

where δ is the Kronecker delta. Equation 8.15 follows from Lemma 8.19, and Equation 8.16 follows from the orthogonality relation

$$\sum_{\chi \pmod{N}} \chi(d) = \varphi(N) \delta_{d \equiv 1 \pmod{N}}.$$

Here $\gamma' \gamma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and $d \equiv 1 \pmod{N}$ if and only if $\gamma' \gamma^{-1} \in \Gamma_1(N)$. Since both of them are in \mathcal{R}_1 , this actually means that $\gamma' = \gamma$.

On the other side, we show that for each $\chi \pmod{N}$ and each $\beta \in \mathcal{R}_0$,

$$p \circ \iota(\mathbf{e}_{\chi, \beta}) = \mathbf{e}_{\chi, \beta}.$$

After summing up by definitions of the maps ι and p , we have

$$\begin{aligned} & p \circ \iota(\mathbf{e}_{\chi, \beta}) \\ &= \frac{1}{\varphi(N)} \sum_{\chi' \pmod{N}} \sum_{\substack{\gamma' \in \mathcal{R}_1, \\ \bar{\gamma}' = \beta}} \chi(I_{\text{id}}(\gamma')) \chi'((I_{\text{id}}(\gamma'))^{-1}) \mathbf{e}_{\chi', \bar{\gamma}'} \\ &= \frac{1}{\varphi(N)} \sum_{\chi' \pmod{N}} \left(\sum_{\substack{\gamma' \in \mathcal{R}_1, \\ \bar{\gamma}' = \beta}} \frac{\chi}{\chi'}(I_{\text{id}}(\gamma')) \right) \mathbf{e}_{\chi', \beta} \\ &= \frac{1}{\varphi(N)} \sum_{\chi' \pmod{N}} \left(\sum_{d \in (\mathbb{Z}/N)^\times} \frac{\chi}{\chi'}(d) \right) \mathbf{e}_{\chi', \beta} \end{aligned} \tag{8.17}$$

$$\begin{aligned} &= \frac{1}{\varphi(N)} \sum_{\chi' \pmod{N}} \varphi(N) \delta_{\chi', \chi} \mathbf{e}_{\chi', \beta} \\ &= \mathbf{e}_{\chi, \beta}, \end{aligned} \tag{8.18}$$

where Equation 8.17 follows from Lemma 8.22, and Equation 8.18 follows from the following orthogonality relation

$$\sum_{d \in (\mathbb{Z}/N)^\times} \chi(d) = \varphi(N) \delta_{\chi, \chi_0}$$

for any character $\chi \pmod{N}$, where $\chi_0 \pmod{N}$ is the trivial character. ■

Lemma 8.24. *Let π , ρ , and σ be arithmetic types for a group Γ , then we have a linear isomorphism*

$$\text{Hom}_\Gamma(\pi, \rho \otimes \sigma) \cong \text{Hom}_\Gamma(\pi \otimes \sigma^\vee, \rho).$$

Proof. First of all, there is a natural isomorphism between linear spaces

$$\begin{aligned} F : \text{Hom}_{\mathbb{C}}(V(\pi), V(\rho) \otimes V(\sigma)) &\longrightarrow \text{Hom}_{\mathbb{C}}(V(\pi) \otimes V(\sigma)^\vee, V(\rho)), \\ \psi &\longmapsto F(\psi) := (v_\pi \otimes w_{\sigma^\vee} \longmapsto w_{\sigma^\vee}(\psi(v_\pi))), \end{aligned}$$

where $v_\pi \in V(\pi)$, and $w_{\sigma^\vee} \in V(\sigma)^\vee$ is linearly extended to $w_{\sigma^\vee} : V(\rho) \otimes V(\sigma) \longrightarrow V(\rho)$ via $w_{\sigma^\vee}(v_\rho \otimes v_\sigma) := (w_{\sigma^\vee}(v_\sigma))v_\rho$, for $v_\rho \in V(\rho)$, $v_\sigma \in V(\sigma)$, respectively. The inverse F^{-1} is given by

$$F^{-1}(\theta) := (v_\pi \longmapsto \theta(v_\pi \otimes \Omega_\sigma)),$$

where $\Omega_\sigma \in V(\sigma)^\vee \otimes V(\sigma)$ is the Casimir element, and $\theta \in \text{Hom}_{\mathbb{C}}(V(\pi) \otimes V(\sigma)^\vee, V(\rho))$ is linearly extended to $\theta : V(\pi) \otimes V(\sigma)^\vee \otimes V(\sigma) \longrightarrow V(\rho) \otimes V(\sigma)$ via $\theta(v_\pi \otimes w_\sigma^\vee \otimes v_\sigma) := \theta(v_\pi \otimes w_\sigma^\vee) \otimes v_\sigma$. It is straightforward to check that F and F^{-1} are indeed inverse of each another.

Next, we need to check that F and F^{-1} is compatible with group actions. In fact, to show that F restricts to a linear isomorphism $F : \text{Hom}_\Gamma(\pi, \rho \otimes \sigma) \longrightarrow \text{Hom}_\Gamma(\pi \otimes \sigma^\vee, \rho)$, it suffices to check $F(\text{Hom}_\Gamma(\pi, \rho \otimes \sigma)) \subseteq \text{Hom}_\Gamma(\pi \otimes \sigma^\vee, \rho)$ and $F^{-1}(\text{Hom}_\Gamma(\pi \otimes \sigma^\vee, \rho)) \subseteq \text{Hom}_\Gamma(\pi, \rho \otimes \sigma)$. To see the first inclusion, suppose we have $\varphi \in \text{Hom}_\Gamma(\pi, \rho \otimes \sigma)$, that is, for all $\gamma \in \Gamma$,

$$((\rho \otimes \sigma)(\gamma)) \circ \varphi = \varphi \circ (\pi(\gamma)). \quad (8.19)$$

Then we need to prove

$$(\rho(\gamma)) \circ (F(\varphi)) = (F(\varphi)) \circ ((\pi \otimes \sigma^\vee)(\gamma)) \quad (8.20)$$

for all $\gamma \in \Gamma$. By the linearity of the maps on both sides of (8.20), it suffices to check their evaluations at $v_\pi \otimes w_{\sigma^\vee}$ for any $v_\pi \in V(\pi)$ and $w_{\sigma^\vee} \in V(\sigma^\vee)$. By the definition of $F(\varphi)$, the evaluation of the left hand side of (8.20) is

$$\rho(\gamma)(w_{\sigma^\vee}(\varphi(v_\pi))). \quad (8.21)$$

For the evaluation of the right hand side, we first recall the definition of dual representation $\sigma^\vee(\gamma)w_{\sigma^\vee} := w_{\sigma^\vee} \circ \sigma(\gamma^{-1})$. By the definition of $F(\varphi)$, we then simplify the evaluation of the right hand side of (8.20) and get the expression

$$(w_{\sigma^\vee} \circ \sigma(\gamma^{-1}))(\varphi(\pi(\gamma)v_\pi)), \quad (8.22)$$

which is equal to

$$(w_{\sigma^\vee} \circ \sigma(\gamma^{-1}))((\rho \otimes \sigma)(\gamma)(\varphi(v_\pi))) \quad (8.23)$$

by (8.19). In order to show that the expressions in (8.21) and (8.23) agree, we may assume $\varphi(v_\pi) = v_\rho \otimes v_\sigma$ for some $v_\rho \in V(\rho)$ and $v_\sigma \in V(\sigma)$, since all the maps involved are linear. With further simplification under this assumption, we see that (8.21) and (8.23) agree, and both are equal to

$$(w_{\sigma^\vee}(v_\sigma))(\rho(\gamma)v_\rho).$$

To show the second inclusion $F^{-1}(\text{Hom}_\Gamma(\pi \otimes \sigma^\vee, \rho)) \subseteq \text{Hom}_\Gamma(\pi, \rho \otimes \sigma)$, we can choose a basis $\{\mathbf{e}_i\}$ of $V(\sigma)$ and its dual basis $\{\mathbf{e}_i^\vee\}$ of $V(\sigma^\vee)$, and write the Casimir element as $\Omega_\sigma = \sum_i \mathbf{e}_i^\vee \otimes \mathbf{e}_i$. If $\eta \in \text{Hom}_\Gamma(\pi \otimes \sigma^\vee, \rho)$, that is, for all $\gamma \in \Gamma$,

$$(\rho(\gamma)) \circ \eta = \eta \circ ((\pi \otimes \sigma^\vee)(\gamma)), \quad (8.24)$$

then we need to show

$$\left((\rho \otimes \sigma)(\gamma)\right) \circ \left(F^{-1}(\eta)\right) = \left(F^{-1}(\eta)\right) \circ \left(\pi(\gamma)\right) \quad (8.25)$$

for all $\gamma \in \Gamma$. Simplifying the evaluation at any $v_\pi \in V(\pi)$ for both sides of (8.25) by (8.24), in a similar way as for the first inclusion, we are reduced to show that

$$\sum_i \eta\left(\left(\pi(\gamma)v_\pi\right) \otimes \left(\mathbf{e}_i^\vee \circ \sigma(\gamma^{-1})\right)\right) \otimes \left(\sigma(\gamma)\mathbf{e}_i\right) = \sum_i \eta\left(\left(\pi(\gamma)v_\pi\right) \otimes \mathbf{e}_i^\vee\right) \otimes \mathbf{e}_i. \quad (8.26)$$

Let A be the matrix representing $\sigma(\gamma)$ under the basis $\{\mathbf{e}_i\}$ (\mathbf{e}_i as a column vector, and \mathbf{e}_i^\vee as a row vector), then the left hand side of (8.26) can be expressed as a linear combination

$$\sum_{j,k} c_{j,k} \eta\left(\left(\pi(\gamma)v_\pi\right) \otimes \mathbf{e}_j^\vee\right) \otimes \mathbf{e}_k,$$

with $c_{j,k} = \sum_i A_{k,i}(A^{-1})_{i,j} = (AA^{-1})_{k,j} = \delta_{k,j}$, where δ is the Kronecker delta. This expression completes the proof. \blacksquare

Lemma 8.25. *Let ρ be an arithmetic type for Γ , such that $\ker \rho$ has finite index in Γ . Then we have $[\Gamma_\infty : \Gamma_\infty(v)] < \infty$, where $\Gamma_\infty(v) := \Gamma_\infty \cap \text{Stab}(v)$. Moreover, for any $v \in V(\rho)$, we have*

$$\sum_{[\gamma] \in \Gamma_\infty(v) \backslash \Gamma} (v|_{k,\rho} \gamma) \in E_k(\rho).$$

Proof. Since $[\Gamma : \ker \rho] < \infty$, we have $[\Gamma_\infty : \Gamma_\infty \cap \ker \rho] \leq [\Gamma : \ker \rho] < \infty$. Furthermore, $\ker \rho \subseteq \text{Stab}(v)$, hence $\Gamma_\infty \cap \ker \rho \subseteq \Gamma_\infty \cap \text{Stab}(v) = \Gamma_\infty(v)$, and therefore $[\Gamma_\infty : \Gamma_\infty(v)] \leq [\Gamma_\infty : \Gamma_\infty \cap \ker \rho] < \infty$. For the second claim, it suffices to find a vector $w \in V(\rho)^{\Gamma_\infty}$, such that

$$\sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma} (w|_{k,\rho} \gamma) = \sum_{[\tilde{\gamma}] \in \Gamma_\infty(v) \backslash \Gamma} (v|_{k,\rho} \tilde{\gamma}). \quad (8.27)$$

We write the right hand side of Equation (8.27) as a double sum

$$\sum_{[\gamma] \in \Gamma_\infty \backslash \Gamma} \left(\sum_{[\gamma'] \in \Gamma_\infty(v) \backslash \Gamma_\infty} (v|_{k,\rho} \gamma') \right) \Big|_{k,\rho} \gamma,$$

and set

$$w := \sum_{[\gamma'] \in \Gamma_\infty(v) \backslash \Gamma_\infty} (v|_{k,\rho} \gamma').$$

It is then clear that $w \in V(\rho)^{\Gamma_\infty}$ and that Equation (8.27) is satisfied. \blacksquare

Lemma 8.26. *Let Δ be an orbit of π and $g = g(\Delta)$ its girth. Then we have that g divides N , and that $g^2 \equiv 0 \pmod{N}$. In particular, we have that any integer coprime to g is coprime to N as well, and the differential-like identity $(1+g)^n \equiv 1+ng \pmod{N}$ which yields that $\chi(1+ng) = \left(\chi(1+g)\right)^n$ for all integers n .*

Proof. Since $c(\Delta) \mid N$, we have $c(\Delta) \mid \gcd(c(\Delta)^2, N)$. The identity $g^{\frac{\gcd(c(\Delta)^2, N)}{c(\Delta)}} = N$ then implies that $g \mid N$. For seeing the second fact, write

$$g^2 = \frac{c(\Delta)^2}{\gcd(c(\Delta)^2, N)} \frac{N}{\gcd(c(\Delta)^2, N)} N.$$

The rest is clear. ■

Lemma 8.27. *Let Δ be an orbit of π and $g = g(\Delta)$ its girth. Then we have that*

$$\chi(1 + g) = 1$$

if and only if the conductor of χ , which we denote by N^ , divides g .*

Proof. Since for any integer k , $1 + kg$ is coprime to g , by Lemma 8.26, it is coprime to N as well. Let $\chi^* \pmod{N^*}$ be the primitive character that induces χ , we thus have that $\chi(1 + kg) = \chi^*(1 + kg)$ for all integers k . If N^* divides g , then $\chi(1 + g) = \chi^*(1 + g) = 1$. Conversely, if $\chi(1 + g) = 1$, then N^* must divide g . In fact, on the one hand, for any integer k , we have

$$\chi^*(1 + kg) = \chi(1 + kg) = \left(\chi(1 + g)\right)^k = 1,$$

hence χ^* is induced by some character $\chi' \pmod{\gcd(g, N^*)}$. On the other hand, χ^* is primitive, therefore $\gcd(g, N^*) = N^*$, which implies that $N^* \mid g$. ■

Definition 8.28. Let $G_1 \subseteq G_2$ be a subgroup of finite index. We fix \mathcal{R} to be a set of representatives of $G_1 \backslash G_2$ containing the identity element, and by $\bar{\gamma} \in \mathcal{R}$ the representative element of the coset containing an arbitrary element γ . For any $\beta \in \mathcal{R}$ and $\gamma \in G_2$, we define the cocycle I by the equation $\beta\gamma = I_\beta(\gamma)\bar{\beta}\bar{\gamma}$. Given a linear representation ρ over \mathbb{C} of G_1 , we define the induced representation $\text{Ind}_{G_1}^{G_2}(\rho)$ over \mathbb{C} of G_2 by

$$V(\text{Ind}_{G_1}^{G_2}(\rho)) := V(\rho) \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{R}] \text{ and } \text{Ind}_{G_1}^{G_2}(\rho)(\gamma)(v \otimes \mathbf{e}_\beta) := \left(\rho\left((I_\beta(\gamma^{-1}))^{-1}\right)v\right) \otimes \mathbf{e}_{\bar{\beta}\bar{\gamma}^{-1}}.$$

Let (\cdot, \cdot) be a scalar product on $V(\rho)$, we define the induced scalar product of (\cdot, \cdot) , denoted by $\langle \cdot, \cdot \rangle$, on $V(\text{Ind}_{G_1}^{G_2}(\rho))$ via the formula

$$\langle u \otimes \mathbf{e}_\alpha, v \otimes \mathbf{e}_\beta \rangle := \delta_{\alpha, \beta}(u, v),$$

where $\delta_{\cdot, \cdot}$ is the Kronecker delta.

Lemma 8.29. *Let $G_1 \subseteq G_2$ be a subgroup of finite index. We fix \mathcal{R} a set of representatives of $G_1 \backslash G_2$ containing the identity element. Recall Definition 8.28. If ρ is a unitary representation of G_1 with respect to a scalar product (\cdot, \cdot) on $V(\rho)$, then $\text{Ind}_{G_1}^{G_2}(\rho)$ is unitary with respect to the induced scalar product $\langle \cdot, \cdot \rangle$ of (\cdot, \cdot) .*

Proof. We have to show that $\langle \cdot, \cdot \rangle$ is invariant with respect to the action of G_2 via $\text{Ind}_{G_1}^{G_2}(\rho)$. Since the induced scalar product $\langle \cdot, \cdot \rangle$ is bilinear, and the \mathbf{e}_β 's with $\beta \in \mathcal{R}$ is a basis of $\mathbb{C}[\mathcal{R}]$, it suffices to check that for all $\alpha, \beta \in \mathcal{R}$, $u, v \in V(\rho)$, and $\gamma \in G_2$, the following identity holds:

$$\left\langle \text{Ind}_{G_1}^{G_2}(\rho)(\gamma)(u \otimes \mathbf{e}_\alpha), \text{Ind}_{G_1}^{G_2}(\rho)(\gamma)(v \otimes \mathbf{e}_\beta) \right\rangle = \langle u \otimes \mathbf{e}_\alpha, v \otimes \mathbf{e}_\beta \rangle. \quad (8.28)$$

The right hand side of (8.28) is $\delta_{\alpha, \beta}(u, v)$ by Definition 8.28, where $\delta_{\cdot, \cdot}$ is the Kronecker delta. The left hand side is, by Definition 8.28 again,

$$\delta_{\alpha\gamma^{-1}, \beta\gamma^{-1}} \left(\rho \left((I_\alpha(\gamma^{-1}))^{-1} \right) u, \rho \left((I_\beta(\gamma^{-1}))^{-1} \right) v \right).$$

Observe that $\delta_{\alpha\gamma^{-1}, \beta\gamma^{-1}} = \delta_{\alpha, \beta}$ for any $\gamma \in G_2$. Therefore, if $\alpha \neq \beta$, then (8.28) holds, since both the left and right hand side vanish. If $\alpha = \beta$, then (8.28) is equivalent to

$$\left(\rho \left((I_\alpha(\gamma^{-1}))^{-1} \right) u, \rho \left((I_\alpha(\gamma^{-1}))^{-1} \right) v \right) = (u, v),$$

which follows from the assumption that ρ is unitary. ■

Definition 8.30. Let N be the positive integer fixed through this note. Denote by $M_2(\mathbb{Z})$ the ring of 2×2 matrices with integer entries. Given two elements $\gamma_1, \gamma_2 \in M_2(\mathbb{Z})$, we say that $\gamma_1 \equiv \gamma_2 \pmod{N}$ if and only if $\gamma_1 - \gamma_2 \in NM_2(\mathbb{Z})$.

Lemma 8.31. *Recall Definition 8.30, we have the following common arithmetic properties modulo N . For any two elements $\gamma_1, \gamma_2 \in \Gamma$, if $\gamma_1 \equiv \gamma_2 \pmod{N}$, then $\gamma_1^{-1} \equiv \gamma_2^{-1} \pmod{N}$. For any four elements $\gamma_1, \gamma_2, \gamma'_1, \gamma'_2 \in M_2(\mathbb{Z})$, if $\gamma_1 \equiv \gamma_2 \pmod{N}$ and $\gamma'_1 \equiv \gamma'_2 \pmod{N}$, then $\gamma_1 + \gamma'_1 \equiv \gamma_2 + \gamma'_2 \pmod{N}$ and $\gamma_1 \gamma'_1 \equiv \gamma_2 \gamma'_2 \pmod{N}$. For any two elements $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \equiv \gamma_2 \pmod{N}$ if and only if $\gamma_1 \gamma_2^{-1} \in \Gamma(N)$, and if $\gamma_1 \equiv \gamma_2 \pmod{N}$, then $\overline{\gamma_1} = \overline{\gamma_2}$. If furthermore $\gamma_1, \gamma_2 \in \Gamma_0(N)$ and $\gamma_1 \equiv \gamma_2 \pmod{N}$, we have $\chi(\gamma_1) = \chi(\gamma_2)$.*

Proof. We prove the first property. If $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \gamma_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \pmod{N}$, then $a \equiv a' \pmod{N}$, $b \equiv b' \pmod{N}$, $c \equiv c' \pmod{N}$ and $d \equiv d' \pmod{N}$. Since $\gamma_1, \gamma_2 \in \Gamma$, we have that $\gamma_1^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and $\gamma_2^{-1} = \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix}$. Entry-wise congruence modulo N of these two matrices implies that $\gamma_1^{-1} \equiv \gamma_2^{-1} \pmod{N}$. The rest of the arithmetic properties is clear in that $NM_2(\mathbb{Z})$ is a two-sided ideal of the ring $M_2(\mathbb{Z})$. As a corollary, for any two elements $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \equiv \gamma_2 \pmod{N}$ if and only if $\gamma_1 \gamma_2^{-1} \equiv \text{id} \pmod{N}$, which is equivalent to that $\gamma_1 \gamma_2^{-1} \in \Gamma(N)$. If $\gamma_1 \equiv \gamma_2 \pmod{N}$, then $\gamma_1 \gamma_2^{-1} \in \Gamma(N) \subseteq \Gamma_0(N)$, hence $\overline{\gamma_1} = \overline{\gamma_2}$. If furthermore $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \gamma_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(N)$, since $\gamma_1 \equiv \gamma_2 \pmod{N}$, we have in particular $d \equiv d' \pmod{N}$, so $\chi(\gamma_1) = \chi(\gamma_2)$. ■

Lemma 8.32. *Let $\rho := \text{Ind}_{\Gamma_0(N)}^\Gamma(\chi)$ for a Dirichlet character χ of modulus N . Then we have*

$$\ker \rho = \left\{ \gamma \in \Gamma : \gamma \equiv \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \pmod{N} \text{ for some } e \in \mathbb{Z}, \text{ such that } \chi(e) = 1 \right\}.$$

In particular, $\Gamma(N) \subseteq \ker(\rho)$, where $\Gamma(N)$ is the principal congruence subgroup of level N .

Proof. For any $\gamma \in \Gamma$, we have that $\gamma \in \ker \rho$ if and only if

$$\overline{\beta\gamma^{-1}} = \beta \text{ and } \chi\left((I_\beta(\gamma^{-1}))^{-1}\right) = 1$$

for all $\beta \in \mathcal{R}_0$. By Definition 1.3 this can be simplified to the following conditions:

$$\beta\gamma\beta^{-1} \in \Gamma_0(N) \text{ and } \chi(\beta\gamma\beta^{-1}) = 1 \quad (8.29)$$

for all $\beta \in \Gamma$.

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ker \rho$, we have to show that there is an integer e satisfying $\chi(e) = 1$, such that $\gamma \equiv \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \pmod{N}$. Set β to be $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, respectively in (8.29), we get that $b \equiv c \equiv 0 \pmod{N}$ and $\chi(d) = 1$. Then we set $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in (8.29), and get further restriction that $a \equiv d \pmod{N}$. Let e be equal to the integer d , then we have the desired condition.

Conversely, for any $\gamma \in \Gamma$, if there is an integer e , such that $\gamma \equiv \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \pmod{N}$ and $\chi(e) = 1$, then by Lemma 8.31 we have that $\beta\gamma\beta^{-1} \equiv \beta \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \beta^{-1} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \pmod{N}$, hence $\beta\gamma\beta^{-1} \in \Gamma_0(N)$. Also by Lemma 8.31 we have that $\chi(\beta\gamma\beta^{-1}) = \chi\left(\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\right) = \chi(e) = 1$.

In particular, we have that

$$\begin{aligned} \Gamma(N) &:= \left\{ \gamma \in \Gamma : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} \subseteq \\ &\left\{ \gamma \in \Gamma : \gamma \equiv \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \pmod{N} \text{ for some } e \in \mathbb{Z}, \text{ such that } \chi(e) = 1 \right\} = \ker \rho. \quad \blacksquare \end{aligned}$$

Remark 8.33. It is worthwhile to note that even if χ is primitive, it still might not be the case that $\ker(\rho) = \Gamma(N)$ holds.¹⁴ Quadratic characters in general yield such examples.

Lemma 8.34. *Let $\gamma_1 = \begin{pmatrix} * & * \\ c_1 & d_1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} * & * \\ c_2 & d_2 \end{pmatrix}$ be two elements in Γ . If there is an integer λ coprime to N , such that $c_2 \equiv \lambda c_1 \pmod{N}$ and $d_2 \equiv \lambda d_1 \pmod{N}$, then $[\gamma_1] = [\gamma_2]$ in $\Gamma_0(N) \setminus \Gamma$.*

Proof. First we reduce the proof to the case $\lambda = 1$. In fact, given an integer λ coprime to N , we can always find two integers a and b , such that $a\lambda - bN = 1$, hence the element $\gamma_0 = \begin{pmatrix} a & b \\ N & \lambda \end{pmatrix}$ is in $\Gamma_0(N)$. Replacing γ_1 by $\gamma_0\gamma_1 \equiv \begin{pmatrix} * & * \\ c_2 & d_2 \end{pmatrix} \pmod{N}$, we can thus assume that $\lambda = 1$.

We treat the case $\lambda = 1$ straightforwardly, where $c_2 \equiv c_1 \pmod{N}$ and $d_2 \equiv d_1 \pmod{N}$, by the following multiplication formula:

$$\gamma_1\gamma_2^{-1} = \begin{pmatrix} * & * \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} d_2 & * \\ -c_2 & * \end{pmatrix} = \begin{pmatrix} * & * \\ c_1d_2 - c_2d_1 & * \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N}.$$

Therefore, $\gamma_1\gamma_2^{-1} \in \Gamma_0(N)$, i.e. $[\gamma_1] = [\gamma_2]$. ■

¹⁴See Ex. 9.5, for instance.

Lemma 8.35. *The map*

$$\begin{aligned}\phi : \Gamma_0(N) \backslash \Gamma &\longrightarrow \mathbb{P}^1(\mathbb{Z}/N) \\ [(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})] &\longmapsto [(\bar{c} : \bar{d})],\end{aligned}$$

where we denote the projection modulo N of an integer n by \bar{n} , is well-defined, and a bijection.

Proof. To see that the map is well-defined, we have to show that for any two elements $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ and $(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix})$ in Γ , if there is an element $\gamma_0 \in \Gamma_0(N)$, such that $(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}) = \gamma_0 (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, then $[(\bar{c}' : \bar{d}')] = [(\bar{c} : \bar{d})]$. In fact, since $\gamma_0 \in \Gamma_0(N)$, we have $\gamma_0 \equiv \begin{pmatrix} * & * \\ 0 & \lambda \end{pmatrix} \pmod{N}$ for some integer λ coprime to N . By Lemma 8.31, we then have

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \gamma_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ \lambda c & \lambda d \end{pmatrix} \pmod{N},$$

hence $(\bar{c}', \bar{d}') = (\lambda \bar{c}, \lambda \bar{d})$, with λ coprime to N . So $[(\bar{c}' : \bar{d}')] = [(\bar{c} : \bar{d})]$ in $\mathbb{P}^1(\mathbb{Z}/N)$ and the map is well-defined.

To show that ϕ is surjective, given an element $[(\bar{c} : \bar{d})] \in \mathbb{P}^1(\mathbb{Z}/N)$, we have to find an element $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma$, such that $\phi([\gamma]) = [(\bar{c} : \bar{d})]$. By the definition of ϕ , it suffices to find a pair of integers (c_1, d_1) with $\gcd(c_1, d_1) = 1$, such that $[(\bar{c}_1 : \bar{d}_1)] = [(\bar{c} : \bar{d})]$. First we find an arbitrary lift of (\bar{c}, \bar{d}) in \mathbb{Z}^2 , say (c_0, d_0) , and set $\lambda := \gcd(c_0, d_0)$. Since $(\bar{c} : \bar{d}) \in \mathbb{P}^1(\mathbb{Z}/N)$, we have $\gcd(\lambda, N) = 1$. Then we set (c_1, d_1) to be $(\lambda^{-1}c_0, \lambda^{-1}d_0)$, thus we have $\gcd(c_1, d_1) = 1$ and that

$$[(\bar{c}_1 : \bar{d}_1)] = [(\overline{\lambda^{-1}c_0} : \overline{\lambda^{-1}d_0})] = [(\overline{\lambda^{-1}c_0} : \overline{\lambda^{-1}d_0})] = [(\bar{c}_0 : \bar{d}_0)] = [(\bar{c} : \bar{d})].$$

To see that ϕ is injective, suppose two cosets $[\gamma_1]$ and $[\gamma_2]$ are sent to the same image by ϕ , we have to show that these two cosets are equal. Assume explicitly that $\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$, then by the definition of ϕ we have that $[(\bar{c}_1 : \bar{d}_1)] = [(\bar{c}_2 : \bar{d}_2)]$, which implies that there is an integer λ coprime to N , such that $c_2 \equiv \lambda c_1 \pmod{N}$ and $d_2 \equiv \lambda d_1 \pmod{N}$. Apply Lemma 8.34 to γ_1 and γ_2 , we conclude that $[\gamma_1] = [\gamma_2]$. ■

Lemma 8.36. *Let m, n be two positive integers such that m divides n . Then the natural map*

$$\begin{aligned}\varphi : \mathbb{Z}/n &\longrightarrow \mathbb{Z}/m \\ a &\longmapsto a \pmod{m}\end{aligned}$$

restricts to a surjective group homomorphism

$$\begin{aligned}\varphi^\times : (\mathbb{Z}/n)^\times &\longrightarrow (\mathbb{Z}/m)^\times \\ a &\longmapsto a \pmod{m}\end{aligned}$$

Proof. The group homomorphism is clearly well-defined. To see the surjectivity of φ^\times , we reduce it to the case where $n = p^e$ and $m = p^f$ for some prime number p and non-negative integers $f \leq e$. This is a trivial case, since for any integer l which is coprime with m , it is automatically coprime with n .

To see the general case, let $n = \prod_{i=1}^s p_i^{e_i}$ and $m = \prod_{i=1}^s p_i^{f_i}$ where $f_i \leq e_i$ be the prime-factorization of n and m , respectively. As a corollary of the Chinese Remainder Theorem, we have the decomposition

$$(\mathbb{Z}/n)^\times = \bigoplus_{i=1}^s (\mathbb{Z}/p_i^{e_i})^\times \quad \text{and} \quad (\mathbb{Z}/m)^\times = \bigoplus_{i=1}^s (\mathbb{Z}/p_i^{f_i})^\times.$$

For each component $(\mathbb{Z}/p_i^{e_i})^\times$ of $(\mathbb{Z}/n)^\times$, we define a homomorphism

$$\begin{aligned} \varphi_i^\times : (\mathbb{Z}/p_i^{e_i})^\times &\longrightarrow (\mathbb{Z}/p_i^{f_i})^\times \\ a &\longmapsto a \pmod{p_i^{f_i}}, \end{aligned}$$

and our task is to reduce the surjectivity of φ to that of each φ_i . To do this, first recall the general fact that the family of groups $\{\mathbb{Z}/n : n \in \mathbb{Z}_{\geq 1}\}$ indexed by the directed poset $\mathbb{Z}_{\geq 1}$ with respect to the division relation¹⁵, together with the family of group homomorphisms $f_{m,n} : \mathbb{Z}/n \rightarrow \mathbb{Z}/m$ for all $m|n$, defined by $f_{m,n}(a) := a \pmod{m}$, constitutes an inverse system over $\mathbb{Z}_{\geq 1}$. In particular, for any $a \in (\mathbb{Z}/n)^\times$, under the identification of $(\mathbb{Z}/n)^\times$ with $\bigoplus_{i=1}^s (\mathbb{Z}/p_i^{e_i})^\times$ and $(\mathbb{Z}/m)^\times$ with $\bigoplus_{i=1}^s (\mathbb{Z}/p_i^{f_i})^\times$, we have that $\text{pr}_i(\varphi^\times(a)) = \varphi_i^\times(\text{pr}_i(a))$ for each i , i.e. $\varphi^\times = \bigoplus_{i=1}^s \varphi_i^\times$. Therefore, if each φ_i^\times is surjective, so is φ^\times , thus the general case is reduced to the trivial case discussed at the beginning. ■

Lemma 8.37. *Every element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in a given coset in $\Gamma_0(N)\backslash\Gamma$ has the same value of $\text{gcd}(c, N)$. Moreover, this value is equal to \tilde{c} for some element $\tilde{\gamma} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ in this coset.*

Proof. For the first claim, we have to show that for any element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ and any element γ_0 in $\Gamma_0(N)$, if $\gamma_0\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, then $\text{gcd}(c', N) = \text{gcd}(c, N)$. Suppose γ_0 has the shape $\begin{pmatrix} * & * \\ * & \lambda \end{pmatrix}$ for some integer λ coprime to N . Then by Lemma 8.31,

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \gamma_0\gamma \equiv \begin{pmatrix} * & * \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ \lambda c & \lambda d \end{pmatrix} \pmod{N},$$

hence $c' \equiv \lambda c \pmod{N}$ and we have $\text{gcd}(c', N) = \text{gcd}(\lambda c, N) = \text{gcd}(c, N)$.

Set \tilde{c} to be $\text{gcd}(c, N)$. To show the second claim, we will construct an element $\tilde{\gamma} \in \Gamma$ in the shape of $\begin{pmatrix} * & * \\ \tilde{c} & * \end{pmatrix}$, such that $[\tilde{\gamma}] = [\gamma]$ in $\Gamma_0(N)\backslash\Gamma$. First we observe that it suffices to find an integer λ coprime to N , such that $\tilde{c} \equiv \lambda c \pmod{N}$. In fact, with such a λ , we can pass from γ to $\tilde{\gamma}$. Set $\tilde{d} := \lambda d$. Since \tilde{c} is a factor of N , it is coprime with λ ; and since \tilde{c} is also a factor of c , it is coprime with d . Therefore $\text{gcd}(\tilde{c}, \tilde{d}) = 1$, and we thus find an element $\tilde{\gamma} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \Gamma$. Apply Lemma 8.34 to $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\tilde{\gamma} \equiv \begin{pmatrix} * & * \\ \lambda c & \lambda d \end{pmatrix}$, we see that $[\tilde{\gamma}] = [\gamma]$.

¹⁵i.e. $m \leq n$ in the index poset if and only if $m | n$ as integers.

In order to find an integer λ coprime to N , such that $\tilde{c} \equiv \lambda c \pmod{N}$, we take the following two steps. First, since $\tilde{c} = \gcd(c, N)$, we have $\gcd(\frac{c}{\tilde{c}}, \frac{N}{\tilde{c}}) = 1$, hence there is an integer $\tilde{\lambda}$ coprime to $\frac{N}{\tilde{c}}$, such that $\tilde{\lambda} \frac{c}{\tilde{c}} \equiv 1 \pmod{\frac{N}{\tilde{c}}}$. Second, by applying Lemma 8.36 to the case $m = \frac{N}{\tilde{c}}$ and $n = N$, we find an integer λ coprime to N , such that $\lambda \equiv \tilde{\lambda} \pmod{\frac{N}{\tilde{c}}}$. So we have $\lambda \frac{c}{\tilde{c}} \equiv \tilde{\lambda} \frac{c}{\tilde{c}} \equiv 1 \pmod{\frac{N}{\tilde{c}}}$. Multiplying both sides by \tilde{c} , we get $\lambda c \equiv \tilde{c} \pmod{N}$. ■

Remark 8.38. To illustrate the process of finding such an element, we refer the reader to Ex. 9.6.

Lemma 8.39. *Let $\gamma \in \Gamma$ be an arbitrary element, then $\chi(I_{\text{id}}(\gamma))$ can be computed explicitly based on Definition 4.5 as follows. Let (c_0, d_0) be the standard representative of $[\gamma]$, i.e. the bottom row of $\bar{\gamma}$, and (c, d) the bottom row of γ , then there is an integer λ , unique modulo N , such that $c \equiv \lambda c_0 \pmod{N}$ and $d \equiv \lambda d_0 \pmod{N}$. Furthermore, it is coprime to N , and we have that $\chi(I_{\text{id}}(\gamma)) = \chi(\lambda)$. In particular, the map $\gamma \mapsto \chi(I_{\text{id}}(\gamma))$ defined on Γ factors through the projection onto the bottom row.*

Proof. By Lemma 8.34, the bottom row (c, d) of γ determines $[\gamma]$, hence its standard representative (c_0, d_0) , so the corollary is clear from the explicit computation.

To show the validity of such computation which only involves the bottom rows, first we recall that (c_0, d_0) is the bottom row of $\bar{\gamma}$. Apply the part of Lemma 8.35 that ϕ is well-defined, to two elements γ and $\bar{\gamma}$ in the same coset $[\gamma]$, we see that there is an integer λ coprime to N , such that $c \equiv \lambda c_0 \pmod{N}$ and $d \equiv \lambda d_0 \pmod{N}$. To see that this integer is actually unique modulo N , given another integer μ satisfying the same property, we have that $(\mu - \lambda)c_0 \equiv (\mu - \lambda)d_0 \equiv 0 \pmod{N}$. Since $\gcd(c_0, d_0) = 1$, we can find $x, y \in \mathbb{Z}$, such that $xc_0 + yd_0 = 1$. Therefore $\mu - \lambda = (\mu - \lambda)(xc_0 + yd_0) \equiv 0 \pmod{N}$.

With the bottom row being fixed as (c_0, d_0) , we write $\bar{\gamma}$ as $\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$, it then follows that

$$\begin{aligned} I_{\text{id}}(\gamma) &= \gamma \bar{\gamma}^{-1} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix} \\ &= \begin{pmatrix} * & * \\ * & a_0 d - b_0 c \end{pmatrix} \equiv \begin{pmatrix} * & * \\ * & \lambda a_0 d_0 - \lambda b_0 c_0 \end{pmatrix} = \begin{pmatrix} * & * \\ * & \lambda \end{pmatrix} \pmod{N}, \end{aligned}$$

hence we have that $\chi(I_{\text{id}}(\gamma)) = \chi(\lambda)$. ■

Lemma 8.40. *Let π be the action on the representative set \mathcal{R}_0 fixed after Definition 4.5, arising from the natural right action of Γ_∞ on $\Gamma_0(N) \backslash \Gamma$. Let Δ be an arbitrary orbit of π , then every element $\delta = \begin{pmatrix} * & * \\ c & * \end{pmatrix}$ in Δ has the same value of c .*

Proof. For an arbitrary element $\delta = \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \Delta \subseteq \mathcal{R}_0$, we need to show that if β is another element in Δ , that is, $\beta = \bar{\delta} \bar{\gamma}$ for some element $\gamma \in \Gamma_\infty$, then β is in the shape of $\begin{pmatrix} * & * \\ c & * \end{pmatrix}$.

On the one hand, viewing δ as an arbitrary element in Γ , by Definition 4.5, we have $c([\delta]) = \gcd(c, N)$. On the other hand, since $\delta \in \mathcal{R}_0$, from the exact way that \mathcal{R}_0 is fixed, we

know that the bottom row of δ is equal to the standard representative of $[\delta]$. In particular, their first entries are equal, i.e. $c = c([\delta])$. Therefore, we have that $\gcd(c, N) = c$.

Since $\gamma \in \Gamma_\infty$, $\delta\gamma$ is in the shape of $\begin{pmatrix} * & * \\ c & * \end{pmatrix}$, hence the standard representative of the coset $[\delta\gamma]$ is in the shape of $\left(\gcd(c, N), *\right) = (c, *)$. Since each element in \mathcal{R}_0 has its bottom row being equal to the standard representative of its coset, $\beta = \overline{\delta\gamma} \in \mathcal{R}_0$ has to be in the shape of $\begin{pmatrix} * & * \\ c & * \end{pmatrix}$. \blacksquare

Lemma 8.41. *Let Δ be an orbit of the action $\pi : \mathcal{R}_0 \times \Gamma_\infty \rightarrow \mathcal{R}_0$ arising from the natural action $\Gamma_0(N) \backslash \Gamma \times \Gamma_\infty \rightarrow \Gamma_0(N) \backslash \Gamma$, with $g = g(\Delta)$ its girth and $c := c(\Delta)$. Let $M := \frac{N}{\gcd(c^2, N)}$ and $\delta \in \Delta$ be an arbitrary element. Then for every integer n , there are two integers λ_n coprime to N and μ_n coprime to $\left(\frac{N}{g}\right)$, such that for any integer k ,*

$$\chi\left(I_{\text{id}}\left(\delta T^{n+kM}\right)\right) = \chi(\lambda_n + \mu_n kg). \quad (8.30)$$

Proof. By Lemma 8.40, $\delta \in \Delta$ has the bottom row in the shape of $(c, *)$, which we denote by (c, d) . For each integer n , we denote by (c, d_n) the bottom row of $\overline{\delta T^n}$. Apply Lemma 8.39 to $\gamma = \delta T^n$, whose bottom row is $(c, cn + d)$, we get an integer λ_n coprime to N , such that

$$\lambda_n c \equiv c \pmod{N}, \quad (8.31)$$

$$\lambda_n d_n \equiv cn + d \pmod{N}. \quad (8.32)$$

Recall from Proposition 4.8 that $|\Delta| = M$, and from Definition 4.11 that $g := c(\Delta)|\Delta|$, we get $g = cM$.

By the claim in Proposition 4.8 that $\overline{\delta T^n}$ depends only on $n \pmod{N}$, we have $\overline{\delta T^{n+kM}} = \overline{\delta T^n}$, whose bottom rows are equal to (c, d_n) for any integer k . We also have that the bottom row of δT^{n+kM} is $(c, c(n+kM) + d)$. Now apply Lemma 8.39 to $\gamma = \delta T^{n+kM}$. In order to find an integer μ_n coprime to $\frac{N}{g}$ such that (8.30) holds, it suffices to solve the equations determined by γ , in Lemma 8.39 for $\lambda = \lambda_n + \mu_n kg$, namely we have to solve the following equations for μ_n coprime to $\frac{N}{g}$:

$$(\lambda_n + \mu_n kg)c \equiv c \pmod{N}, \quad (8.33)$$

$$(\lambda_n + \mu_n kg)d_n \equiv c(n+kM) + d \pmod{N}. \quad (8.34)$$

Since $gc = \frac{c^2}{\gcd(c^2, N)}N$ is divided by N , and that λ_n satisfies (8.31), for any integers μ_n and k , $\lambda_n + \mu_n kg$ always satisfies (8.33). Since λ_n satisfies (8.32), to find a μ_n such that $\lambda_n + \mu_n kg$ solves (8.34), it suffices to solve the equation $(\mu_n kg)d_n \equiv ckM = kg \pmod{N}$ for μ_n , which is equivalent to $\mu_n d_n k \equiv k \pmod{\left(\frac{N}{g}\right)}$. Note that $\gcd(c, d_n) = 1$, we have thus $\gcd(d_n, \left(\frac{N}{g}\right)) = \gcd(d_n, \frac{\gcd(c^2, N)}{c}) = 1$, hence there is an integer μ_n coprime to $\left(\frac{N}{g}\right)$, such that $\mu_n d_n \equiv 1 \pmod{\left(\frac{N}{g}\right)}$, which implies $\mu_n d_n k \equiv k \pmod{\left(\frac{N}{g}\right)}$. For such μ_n , $\lambda_n + \mu_n kg$ satisfies both (8.33) and (8.34) for any integer k . \blacksquare

Lemma 8.42. *Let g be an integer which divides N . Let λ be an integer coprime to $\frac{N}{g}$, and L a multiple of $\left(\frac{N}{g}\right)$, then*

$$\sum_{k=0}^{L-1} \chi(1 + \lambda kg) = \sum_{k=0}^{L-1} \chi(1 + kg). \quad (8.35)$$

Proof. Since L is a multiple of $\left(\frac{N}{g}\right)$, we write $L = l\left(\frac{N}{g}\right)$ for some integer l . It is then clear that the multiset $\{k \pmod{\frac{N}{g}} : 0 \leq k \leq L - 1\}$ is equal to l copies of $\mathbb{Z}/\left(\frac{N}{g}\right)$, with each copy invariant under the multiplication by $\bar{\lambda} \in \left(\mathbb{Z}/\left(\frac{N}{g}\right)\right)^\times$. Therefore, the multiset $\{\lambda k \pmod{\frac{N}{g}} : 0 \leq k \leq L - 1\}$ is also l copies of $\mathbb{Z}/\left(\frac{N}{g}\right)$, hence equal to $\{k \pmod{\frac{N}{g}} : 0 \leq k \leq L - 1\}$. So we have an identity of multisets

$$\left\{1 + \lambda kg \pmod{N} : 0 \leq k \leq L - 1\right\} = \left\{1 + kg \pmod{N} : 0 \leq k \leq L - 1\right\},$$

which yields (8.35) as desired. ■

Lemma 8.43. *Let Δ be an orbit of the action $\pi : \mathcal{R}_0 \times \Gamma_\infty \longrightarrow \mathcal{R}_0$ arising from the natural action $\Gamma_0(N)\backslash\Gamma \times \Gamma_\infty \longrightarrow \Gamma_0(N)\backslash\Gamma$, with $g = g(\Delta)$ its girth and $c := c(\Delta)$. Let $M := \frac{N}{\gcd(c^2, N)}$ and $\delta \in \Delta$ be an arbitrary element. Then for each $n \in \{0, \dots, M - 1\}$, there is some integer λ_n coprime to N , such that*

$$\sum_{k=0}^{\frac{N}{M}-1} \chi\left(I_{\text{id}}(\delta T^{n+kM})\right) = \chi(\lambda_n) \sum_{k=0}^{\frac{N}{M}-1} (\chi(1 + g))^k. \quad (8.36)$$

In particular, for all $n \in \{0, \dots, M - 1\}$ we have that

$$\sum_{k=0}^{\frac{N}{M}-1} \chi\left(I_{\text{id}}(\delta T^{n+kM})\right) \neq 0,$$

if and only if

$$\sum_{k=0}^{\frac{N}{M}-1} (\chi(1 + g))^k \neq 0.$$

Proof. Given an integer $n \in \{0, \dots, M - 1\}$, to which we apply Lemma 8.41, there are two integers λ_n coprime to N and μ_n coprime to $\left(\frac{N}{g}\right)$, such that for any integer k ,

$$\chi\left(I_{\text{id}}(\delta T^{n+kM})\right) = \chi(\lambda_n + \mu_n kg). \quad (8.37)$$

Since $\gcd(\lambda_n, N) = 1$, there is an integer ν_n coprime to N , such that $\lambda_n \nu_n \equiv 1 \pmod{N}$. Therefore we have

$$\sum_{k=0}^{\frac{N}{M}-1} \chi(\lambda_n + \mu_n k g) = \chi(\lambda_n) \sum_{k=0}^{\frac{N}{M}-1} \chi(1 + \mu_n \nu_n k g), \quad (8.38)$$

Recall that μ_n is coprime with $\frac{N}{g}$ and ν_n is coprime to N , hence $\mu_n \nu_n$ is coprime to $\frac{N}{g}$. Since $\left(\frac{N}{M}\right)$ is a multiple of $\left(\frac{N}{g}\right)$, by Lemma 8.42 with $\lambda = \mu_n \nu_n$ and $L = \frac{N}{M}$, we have that

$$\sum_{k=0}^{\frac{N}{M}-1} \chi(1 + \mu_n \nu_n k g) = \sum_{k=0}^{\frac{N}{M}-1} \chi(1 + k g). \quad (8.39)$$

Furthermore, by Lemma 8.26 we have that for any integer k ,

$$\chi(1 + k g) = \left(\chi(1 + g)\right)^k. \quad (8.40)$$

Combining (8.37), (8.38), (8.39), and (8.40) yields (8.36) as desired. \blacksquare

Lemma 8.44. *Let $N \geq 3$ be an integer, then for any $\lambda \in L_N^\times$, we have $\lambda \neq -\lambda$.*

Proof. For a positive integer N , it is clear that there exists $\lambda \in L_N^\times$ such that $\lambda = -\lambda$, if and only if N is an even integer and $\gcd(N/2, N) = 1$. But this condition is satisfied only when $N = 2$. \blacksquare

Lemma 8.45. *Let $k \in \mathbb{Z}_{\geq 2}$ and $z \in \mathfrak{H}$, then we have*

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = C_k \sum_{n=1}^{\infty} n^{k-1} e(nz),$$

where $C_k := \frac{(-2\pi i)^k}{(k-1)!}$.

Remark 8.46. This is known as the Lipschitz's formula, and proofs inspired by Euler can be found for example on page 16 of [Bru+08] and page 49 of [KK07]. For a generalization of this classical summation formula, see [PP01]. Here we present a proof based on contour integral and the Poisson summation formula.

Proof. For fixed $k \in \mathbb{Z}_{\geq 2}$ and $z \in \mathfrak{H}$, let $f : \mathbb{R} \rightarrow \mathbb{C}$ be the function

$$f(t) := \frac{1}{(z+t)^k}.$$

Clearly f is a function of moderate decrease, i.e., there is a constant C such that $f(t) \leq \frac{C}{1+t^2}$. We then compute its Fourier transform

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(t) e(-\xi t) dt,$$

by applying the residue formula to the meromorphic function $g_\xi(\zeta) := f(t)e(-\xi\zeta) = \frac{1}{(z+\zeta)^k}e(-\xi\zeta)$ and a certain contour.

If $\xi \in \mathbb{R}_{<0}$, we take the contour C_R for $g_\xi(\zeta)$ to be the union of the line segment $[-R, R]$ and the semicircle centered at the origin of radius R in the upper half plane \mathfrak{H} , with the positive orientation, for a positive real number R . Note that the only pole of $g_\xi(\zeta)$ is $-z$, which lies in the lower half plane, $g_\xi(\zeta)$ is thus holomorphic in an open set containing both the contour C_R and its interior, hence

$$\int_{C_R} g_\xi(\zeta)d\zeta = 0.$$

Since the integral of $g_\xi(\zeta)$ on the circle of radius R is bounded above by $O(R^{-(k-1)})$, it tends to 0 when $R \rightarrow \infty$. Therefore, we obtain

$$\hat{f}(\xi) = 0, \text{ for all } \xi \in \mathbb{R}_{\leq 0}. \quad (8.41)$$

If $\xi \in \mathbb{R}_{>0}$, we take the contour C_R for $g_\xi(\zeta)$ to be the union of the line segment $[-R, R]$ and the semicircle centered at the origin of radius R in the lower half plane \mathfrak{H} , with the negative orientation (clockwise), for a positive real number R . Note that $g_\xi(\zeta)$ is holomorphic in an open set containing both the contour C_R and its interior, except at the pole $-z$ of order k . By the residue formula (negative orientation), we get

$$\int_{C_R} g_\xi(\zeta)d\zeta = -2\pi i \operatorname{res}_{-z} g_\xi,$$

where the residue at the pole of order k is given by the formula

$$\begin{aligned} \operatorname{res}_{-z} g_\xi &= \frac{1}{(k-1)!} \lim_{\zeta \rightarrow -z} \frac{d^{k-1}}{d\zeta^{k-1}} (\zeta + z)^k g_\xi(\zeta) \\ &= \frac{(-2\pi i \xi)^{k-1}}{(k-1)!} e(\xi z). \end{aligned}$$

Since the integral of $g_\xi(\zeta)$ on the circle of radius R is bounded above by $O(R^{-(k-1)})$, it tends to 0 when $R \rightarrow \infty$. Therefore we obtain

$$\hat{f}(\xi) = C_k \xi^{k-1} e(\xi z), \text{ for all } \xi \in \mathbb{R}_{>0}. \quad (8.42)$$

From Equation 8.41 and Equation 8.42, it is now clear that \hat{f} is also of moderate decrease. We apply the Poisson Summation formula to f and insert Equation 8.41 and Equation 8.42 to conclude as follows.

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) = C_k \sum_{n=1}^{\infty} n^{k-1} e(nz). \quad \blacksquare$$

Lemma 8.47. *Let $k \in \mathbb{Z}_{\geq 1}$, $n, N \in \mathbb{Z}$ such that $1 \leq n \leq N$. Then we have the following equation of the Hurwitz zeta function.*

$$\zeta(1 - k, n/N) = \frac{2(k-1)!}{(2\pi N)^k} \sum_{d=1}^N \cos\left(k\pi/2 - 2\pi dn/N\right) \zeta(k, d/N).$$

Proof. This is a special case of the functional equation of Hurwitz zeta function, when the usual complex variable s is taken as an integer. See Apostol theorem 12.8, page 261, with replacement of variables $r \rightsquigarrow d$, $k \rightsquigarrow N$, $s \rightsquigarrow k$, and $h \rightsquigarrow n$. Finally, since $k \in \mathbb{Z}_{\geq 1}$, we can insert $\Gamma(k) = (k-1)!$ into the formula. ■

Lemma 8.48. *For each $k \in \mathbb{Z}_{\geq 1}$, we have*

$$\zeta(1 - k, a) = -\frac{B_k(a)}{k}.$$

Proof. See Theorem 12.13 on page 264 in [Apo76], with $n \rightsquigarrow k-1 \in \mathbb{Z}_{\geq 0}$ here. ■

9 Examples

Example 9.1. Let $G = S_3$ and $X = \{1, 2, 3\}$. Let G act on X naturally, and ρ the corresponding permutation representation. For an element $g = (1, 3, 2) \in G$, we have $\rho(g)\mathbf{e}_i = \mathbf{e}_{g(i)}$. Therefore, the column representation of permutation matrices of $\rho(g)$ under the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ attached to X is

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 9.2. To better illustrate how to discover the size of the orbit, first we give a very generic case, that is, when $c(\Delta)$ is coprime to N . So we have actually $c(\Delta) = 1$. For convenience, we use the standard representatives to name the elements in \mathcal{R}_0 . Let $N = 9$ and Δ the orbit containing $(1, 0)$. Then $\Delta = \{(1, n) : 0 \leq n \leq 8\}$ has size $9 = \frac{9}{1}$. This example works for a general integer N , namely instead we will have in general $\Delta = \{(1, n) : 0 \leq n \leq N-1\}$.

Example 9.3. We will see more tricky cases following Ex. 9.2, when $c(\Delta)$ is bigger than 1. Let $N = 18 = 2 \cdot 3^2$ and Δ be the orbit containing $(6, 1)$, then we see that actually Δ contains only this element $(6, 1)$, in that $(6, 7)$ and $(6, 13)$ are not standard representatives, and they correspond to the same standard representative $(6, 1)$ by multiplying 7 (mod 18) consecutively. On the other hands, if we consider another close example where $N = 18$ and Δ is the orbit containing $(3, 1)$, then we see that actually Δ contains two elements $(3, 1)$ and $(3, 4)$, and we have $2 = \frac{18}{3^2}$. Yet in another example where $N = 18$ and Δ is the orbit containing $(2, 1)$, we see that Δ contains nine elements $(2, 2n+1)$, where $0 \leq n \leq 8$ and we have $9 = \frac{18}{2}$. This phenomenon suggests that the size of orbits is rather reduced by the square of $c(\Delta)$, where one needs to take gcd with N .

Example 9.4. We take the same material as in Example 9.3. This time, instead of focusing on different elements in the same orbits, we take a different point of view, by looking at the repetition of the same element.

If $c(\Delta) = 1$, there is actually no repetition of elements $(\bmod 18)$, i.e. $(1, n)$ are all different elements for $0 \leq n \leq 17$.

When $c(\Delta) = 2$, we also get no repetition $(\bmod 18)$, but this time things are a bit different: the orbit Δ contains only 9 elements, literally speaking there is a double repetition $(\bmod 18)$, for instance $(2, 1) = (2, 19)$. But this was somehow neglected due to the fact that $c(\Delta)|\Delta| \equiv 0 \pmod{18}$ has hidden the effect of repetition.

Now consider the case where $c(\Delta) = 3$, we can directly see the effect of repetition $(\bmod 18)$, as in $(3, 1) = (3, 7) = (3, 13)$. Furthermore, we see that the leap $3 = c(\Delta)$ in the second component between two consecutive elements, under the action by $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, with $2 = |\Delta|$ steps of the action by T , leads to the difference $6 = c(\Delta)|\Delta|$ between 1 and 7, namely the period of repetition. Moreover we also see that from 7 to 13, the factor $(\bmod 18)$ is still 7, and in general one has $7^n = (1 + 6)^n \equiv 1 + 6n \pmod{18}$.

Carrying on with this example, we see that when $c(\Delta) = 6$, we have that the orbit Δ has only one element $(6, 1)$, and the period of repetition should be also 6 since $(6, 1) = (6, 7)$. This comes from $6 = 6 \cdot 1 = c(\Delta)|\Delta|$. To conclude, it provides us with some clue that $c(\Delta)|\Delta|$ might be important to measure the period and $1 + c(\Delta)|\Delta|$ might be important as the multiplicative factor.

After doing more similar examples with a choice of \mathcal{R}_0 that is not the one fixed by the standard representatives, we can see that this notion turns out to be intrinsic.

Example 9.5. Let $N = 8$ and $\chi(n) := (-1)^{\lfloor \frac{n}{4} \rfloor}$ for all odd integers n and $\chi(n) := 0$ for all even integers n defines a primitive Dirichlet character $\chi \pmod{8}$. Consider an element $\gamma = \begin{pmatrix} -2^1 & -8 \\ 8 & 3 \end{pmatrix}$ which is not in the principal congruence subgroup $\Gamma(8)$. However, simple computation reveals that $\gamma \in \ker \rho$, where $\rho = \text{Ind}_{\Gamma_0(8)}^{\Gamma}(\chi)$. In general, a quadratic character, e.g. the quadratic residue symbol yields such examples. It is worthwhile to mention that quadratic characters in general yield such examples.

Example 9.6. Let $N = 36$ and $\gamma = \begin{pmatrix} * & * \\ 27 & * \end{pmatrix}$ an element in Γ . We know that the gcd of 36 and 27 is 9, so there should be an element $\gamma_0 = \begin{pmatrix} * & * \\ * & \lambda \end{pmatrix} \in \Gamma_0(36)$, such that $\gamma_0\gamma = \begin{pmatrix} * & * \\ 9 & * \end{pmatrix}$. To find out λ , we first note that $\frac{36}{9} = 4$ and $\frac{27}{9} = 3$ are coprime and $3 \cdot 3 \equiv 1 \pmod{4}$. But 3 is not coprime to 36, so we can add a multiple of 4 to 3, so that it becomes coprime to 36. For example, we can take λ to be $3 + 4 = 7$ and find out the rest part to construct such a matrix γ_0 .

Example 9.7. We give an explicit isomorphism to illustrate Proposition 5.5. Let N, l be positive integers, then we have

$$\begin{aligned} E_k(\rho_N^\times) &\cong \text{Hom}(\mathbf{1}, \mathcal{E}_k[\rho_{lN}^\times] \otimes \rho_N^\times) \\ f &\longmapsto (1 \mapsto f). \end{aligned}$$

Example 9.8. We look at the case where $N' = 6$. For $A := \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$ and $B := \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, they are both in $\Delta_{6,1}$. Let $\gamma_1 := \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ and $\gamma_2 := \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$, we have $\gamma_i \in \Gamma$ for $i = 1, 2$, and that

$$A = \gamma_1 B \gamma_2.$$

10 Appendices

We include here two notes that the author composed before carrying out the project on vector-valued Eisenstein series. These notes serve as archives about some of the difficulties the author once encountered, and a beginner's path to the vector-valued Eisenstein series. In particular, in writing these notes, two typos from each of the two classic books, [IK04] (resp. [Bru+08]), were found and corrected at Equation (10.29) (resp. below Equation (10.37)) independently by the author, which are important for other applications.

10.1 A: Fourier expansions of real analytic Eisenstein series This note is to investigate some classical aspects of real analytic Eisenstein series, which serves as part of the preparation for the research project on vector-valued real analytic Eisenstein series. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ be the full modular group and $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n \in \mathbb{Z}\}$ the stabilizer of the cusp point seen as a subgroup of Γ . In this manuscript, we denote the complex variable in the upper half plane by τ , $\mathrm{Re}(\tau)$ by x and $\mathrm{Im}(\tau)$ by y and hence $\tau = x + iy$ by convention. Note that there are at least two different definitions for the (real analytic) Eisenstein series $E_k(\tau, s)$.

From ‘‘average action’’ (or, Reynolds operator) point of view which takes a proper average through group action on the real analytic function $\mathrm{Im}(\tau)^s$, we can define

$$\begin{aligned} E_k(\tau, s) &:= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\tau)^s |k\gamma \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma\tau)^s (c\tau + d)^{-k} \\ &= \frac{1}{2} \sum_{\substack{c, d \in \mathbb{Z} \\ (c, d) = 1}} \frac{y^s}{(c\tau + d)^k |c\tau + d|^{2s}}. \end{aligned} \tag{10.1}$$

as our definition. It is clear that this sum is absolutely convergent if the condition

$$2\mathrm{Re}(s) + k > 2. \tag{10.2}$$

is satisfied. Since the number of pairs (c, d) with $N \leq |c\tau + d| < N + 1$ is $O(N)$ as $N \rightarrow \infty$, up to a constant multiple the series is bounded above by $\sum_{N=1}^{\infty} N^{-(2\mathrm{Re}(s)+k-1)}$, hence absolute convergent and it is the absolute convergence that guarantees the modularity of the sum. Since there are no modular forms of odd weights on the full modular group, we can thus assume that in this manuscript,

$$k \text{ is an even integer satisfying } k > 2 - 2\mathrm{Re}(s). \tag{10.3}$$

The Eisenstein series from the “function of lattice point of view” is defined as

$$G_k(\tau, s) := \frac{1}{2} \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{y^s}{(m\tau + n)^k |m\tau + n|^{2s}}. \quad (10.4)$$

which can be related to $E_k(\tau, s)$ via $G_k(\tau, s) = \zeta(2s + k)E_k(\tau, s)$. Separating the terms corresponding to $m = 0$ for the remaining ones, we find that $G_k(\tau, s)$ is equal to

$$\begin{aligned} & \zeta(2s + k)y^s \\ & + (-1)^{k/2}y^s \sum_{m=1}^{\infty} m^{-2s-k} \sum_{r \in \mathbb{Z}/m} \sum_{n \in \mathbb{Z}} \left(y - i(x + r/m) - in\right)^{-s-k} \left(y + i(x + r/m) + in\right)^{-s}. \end{aligned} \quad (10.5)$$

where x and y are the real part and imaginary part of τ respectively, as fixed notations from the beginning of this section.

Now we are in position to compute the Fourier expansion of real analytic Eisenstein series. First of all, we observe that $G_k(\tau, s)$ is 1-periodic with respect to τ . In particular, it has a Fourier expansion of the form

$$G_k(\tau, s) = \sum_{n \in \mathbb{Z}} c(n, y) e(nx)$$

for specific functions $c(n, y)$, where $e(x) := \exp(2\pi ix)$ for a real number x by convention. We compute it by applying the Poisson summation formula to the sum

$$\sum_{n \in \mathbb{Z}} \left(y - i(x + r/m) - in\right)^{-s-k} \left(y + i(x + r/m) + in\right)^{-s}.$$

Specifically, we set

$$f_r(t) := \left(y - i(x + r/m) - it\right)^{-s-k} \left(y + i(x + r/m) + it\right)^{-s}. \quad (10.6)$$

For each r , $f_r(t)$ is square integrable on \mathbb{R} , since $|f_r(t)|^2$ is bounded by $O(t^{-4})$. Its Fourier transform is given by

$$\hat{f}_r(\xi) := \int_{-\infty}^{\infty} f_r(t) e^{-2\pi i \xi t} dt.$$

Replacing $t + x + r/m$ in the integrand by a new dumb variable t , it is equal to

$$e\left(\xi(x + r/m)\right) \int_{-\infty}^{\infty} (b + it)^{-2\mu} (c - it)^{-2\nu} e^{-ipt} dt, \quad (10.7)$$

with

$$b = c \rightsquigarrow y, \quad \mu \rightsquigarrow \frac{s}{2}, \quad \nu \rightsquigarrow \frac{s+k}{2}, \quad p \rightsquigarrow 2\pi\xi$$

so that if $\xi \neq 0$, formula 3.384.9.6 of [GR07] allows us to evaluate this integral. The conditions to apply this formula are satisfied:

$$\operatorname{Re}(b) = \operatorname{Re}(c) = y > 0, \text{ since } \tau \in \mathbb{H}; \quad \operatorname{Re}(\mu + \nu) > 1, \text{ due to Condition (10.2).}$$

We thus obtain explicit formula for $\hat{f}_r(\xi)$ and in particular when $\xi = l$ is an integer, we have

$$\begin{aligned} \hat{f}_r(l) &= e(l(x + r/m)) \left(2\pi(2y)^{-s-k/2} \frac{(2\pi l)^{s+k/2-1}}{\Gamma(s+k)} W_{k/2, 1/2-s-k/2}(4\pi ly) \right) \\ &= \left(\frac{\pi^{s+k/2}}{\Gamma(s+k)} y^{-s-k/2} l^{s+k/2-1} W_{k/2, 1/2-s-k/2}(4\pi ly) e(lr/m) \right) e(lx) \end{aligned} \quad (10.8)$$

for positive integer l and

$$\begin{aligned} \hat{f}_r(l) &= e(l(x + r/m)) \left(2\pi(2y)^{-s-k/2} \frac{(-2\pi l)^{s+k/2-1}}{\Gamma(s)} W_{-k/2, 1/2-s-k/2}(-4\pi ly) \right) \\ &= \left(\frac{\pi^{s+k/2}}{\Gamma(s)} y^{-s-k/2} |l|^{s+k/2-1} W_{-k/2, 1/2-s-k/2}(4\pi |l|y) e(lr/m) \right) e(lx) \end{aligned} \quad (10.9)$$

for negative integer l .

Therefore the Poisson summation formula guarantees that

$$\begin{aligned} \sum_{n \in \mathbb{Z}} (y - i(x + r/m) - in)^{-s-k} (y + i(x + r/m) + in)^{-s} &= \sum_{n \in \mathbb{Z}} f_r(n) = \sum_{l \in \mathbb{Z}} \hat{f}_r(l) \\ &= \hat{f}_r(0) + \frac{\pi^{s+k/2}}{\Gamma(s+k)} y^{-s-k/2} \sum_{l \in \mathbb{Z}_{\geq 1}} \left(l^{s+k/2-1} W_{k/2, 1/2-s-k/2}(4\pi ly) e(lr/m) \right) e(lx) \\ &\quad + \frac{\pi^{s+k/2}}{\Gamma(s)} y^{-s-k/2} \sum_{l \in \mathbb{Z}_{\leq -1}} \left(|l|^{s+k/2-1} W_{-k/2, 1/2-s-k/2}(4\pi |l|y) e(lr/m) \right) e(lx). \end{aligned}$$

We insert this into (10.5), and isolate the Fourier coefficient $c(l, y)$ associated with $e(lx)$. For positive integer l , we obtain

$$\begin{aligned} c(l, y) &= (-1)^{k/2} y^s \sum_{m=1}^{\infty} m^{-2s-k} \sum_{r \in \mathbb{Z}/m} \frac{\pi^{s+k/2}}{\Gamma(s+k)} y^{-s-k/2} l^{s+k/2-1} W_{k/2, 1/2-s-k/2}(4\pi ly) e(lr/m) \\ &= (-1)^{k/2} \frac{\pi^{s+k/2} y^{-k/2}}{\Gamma(s+k)} l^{s+k/2-1} W_{k/2, 1/2-s-k/2}(4\pi ly) \sum_{m=1}^{\infty} m^{-2s-k} \sum_{r \in \mathbb{Z}/m} e(lr/m) \\ &= (-1)^{k/2} \frac{\pi^{s+k/2} y^{-k/2}}{\Gamma(s+k)} l^{s+k/2-1} W_{k/2, 1/2-s-k/2}(4\pi ly) \sum_{m=1}^{\infty} m^{-2s-k} (m \mathbf{1}_{m|l}) \\ &= (-1)^{k/2} \frac{\pi^{s+k/2} y^{-k/2}}{\Gamma(s+k)} W_{k/2, 1/2-s-k/2}(4\pi ly) l^{s+k/2-1} \sigma_{1-2s-k}(l) \\ &= (-1)^{k/2} \frac{\pi^{s+k/2} y^{-k/2}}{\Gamma(s+k)} W_{k/2, s+k/2-1/2}(4\pi ly) l^{-s-k/2} \sigma_{2s+k-1}(l), \end{aligned} \quad (10.10)$$

where the last step is because the Whittaker function $W_{\kappa,\mu}$ is an even function in μ . For negative integer l , similiary we obtain

$$c(l, y) = (-1)^{k/2} \frac{\pi^{s+k/2} y^{-k/2}}{\Gamma(s)} W_{-k/2, s+k/2-1/2}(4\pi|l|y)|l|^{-s-k/2} \sigma_{2s+k-1}(|l|). \quad (10.11)$$

Finally we are to compute the $l = 0$ -th Fourier coefficient $c(0, y)$. From (10.5) we find that it is equal to

$$\zeta(2s+k)y^s + (-1)^{k/2} y^s \sum_{m=1}^{\infty} m^{-2s-k} \sum_{r \in \mathbb{Z}/m} \hat{f}_r(0). \quad (10.12)$$

To this end, we have to evaluate $\hat{f}_r(0)$. From (10.7) we find that

$$\hat{f}_r(0) = \int_{-\infty}^{\infty} (y+it)^{-s} (y-it)^{-s-k} dt$$

is independent of r . We denote this constant by $I_k(s)$ and the $l = 0$ -th Fourier coefficient $c(0, y)$ is then equal to

$$\zeta(2s+k)y^s + (-1)^{k/2} \zeta(2s+k-1)y^s I_k(s). \quad (10.13)$$

When $k = 0$, we can simply interpret the integral $I_0(s)$ in terms of the Beta function. We have

$$I_0(s) = \int_{-\infty}^{\infty} (y^2 + t^2)^{-s} dt.$$

After changing the variable t to $u = (1 + \frac{t^2}{y^2})^{-1}$ we can transform the integral into

$$y^{1-2s} \int_0^1 u^{s-\frac{3}{2}} (1-u)^{-\frac{1}{2}} du,$$

which is by definition of the Beta function equal to $y^{1-2s} B(s - \frac{1}{2}, \frac{1}{2})$. Recall the relation between Beta and Gamma function, we have

$$\begin{aligned} I_0(s) &= y^{1-2s} \frac{\Gamma(s - \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(s)} \\ &= \sqrt{\pi} y^{1-2s} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)}. \end{aligned} \quad (10.14)$$

For the general case when $k \in \mathbb{Z}$, the strategy is to apply the partial derivative operator ∂_y to

$$I_k(s) = \int_{-\infty}^{\infty} (y+it)^{-s} (y-it)^{-s-k} dt.$$

Since both $\partial_y((y+it)^{-s}(y-it)^{-s-k})$ and $(y+it)^{-s}(y-it)^{-s-k}$ are continuous in t and y everywhere, we can interchange ∂_y and the integration. Therefore after a simple computation we get a recurrence relation between I_{k-1} , I_k and I_{k+1} for all the integers k :

$$\partial_y I_k(s) = -sI_{k-1}(s+1) - (s+k)I_{k+1}(s), \quad (10.15)$$

from which we can read two useful formulas for the case $k \leq -1$ and $k \geq 0$ respectively. If $k \leq -1$, from Condition (10.2) we know that $s \neq 1$, so that it is valid to write (10.15) with

$$I_{k-1}(s) = -\frac{1}{s-1} \left(\partial_y I_k(s-1) + (s+k-1)I_{k+1}(s-1) \right) \text{ for } k \in \mathbb{Z}_{\leq -1}. \quad (10.16)$$

If $k \geq 0$, from Condition (10.2) we know that $s \neq -k$, hence it is valid to write (10.15) with

$$I_{k+1}(s) = -\frac{1}{s+k} \left(\partial_y I_k(s) + sI_{k-1}(s+1) \right) \text{ for } k \in \mathbb{Z}_{\geq 0}. \quad (10.17)$$

We observe that if we could get $I_{-1}(s)$ explicitly, together with Formula (10.14) for $I_0(s)$ we can obtain $I_k(s)$ for all negative integers k , by induction through Formula (10.16); and similarly through Formula (10.17), we can obtain $I_k(s)$ for all positive integers k .

The problem is now reduced to $I_{-1}(s)$, but this can be evaluated directly. We have that

$$\begin{aligned} I_{-1}(s) &= \int_{-\infty}^{\infty} (y+it)^{-s}(y-it)^{-s+1} dt \\ &= yI_0(s) - i \int_{-\infty}^{\infty} (y^2+t^2)^{-s} t dt \\ &= yI_0(s) \\ &= \sqrt{\pi} y^{2-2s} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}, \end{aligned} \quad (10.18)$$

since the integrand $(y^2+t^2)^{-s}t$ is an odd function in t . Hence the remaining task is to clear up the results by induction.

From Formula (10.16), Expression (10.18) and Expression (10.14) for I_{-1} and I_0 , respectively, we find by induction that

$$I_k(s) = \sqrt{\pi} y^{-2s-k+1} \left(\prod_{j=k}^{k/2-1} (s+j) \right) \frac{\Gamma(s+k/2-1/2)}{\Gamma(s)} \text{ for } k = -2, -4, \dots \text{ and} \quad (10.19)$$

$$I_k(s) = \sqrt{\pi} y^{-2s-k+1} \left(\prod_{j=k}^{k/2-3/2} (s+j) \right) \frac{\Gamma(s+k/2)}{\Gamma(s)} \text{ for } k = -1, -3, \dots, \quad (10.20)$$

where we set the product equal to 1 if $k > k/2 - 1$ or $k > k/2 - 3/2$, respectively.

On the other hand, from Formula (10.17), Expression (10.14) and Expression (10.18) for I_0 and I_{-1} respectively, we find by induction that

$$I_k(s) = \sqrt{\pi}y^{-2s-k+1} \left(\prod_{j=0}^{k/2-1} (s+j) \right) \frac{\Gamma(s+k/2-1/2)}{\Gamma(s+k)} \text{ for } k = 2, 4, \dots \text{ and} \quad (10.21)$$

$$I_k(s) = \sqrt{\pi}y^{-2s-k+1} \left(\prod_{j=0}^{k/2-3/2} (s+j) \right) \frac{\Gamma(s+k/2)}{\Gamma(s+k)} \text{ for } k = 1, 3, \dots, \quad (10.22)$$

with the convention of defining the empty product to be 1 as mentioned above. In fact, with this convention we even have that (10.19) and (10.21) coincide at $k = 0$ with Expression (10.14). Now, we only need the results for an even integer k , due to Assumption (10.3). Inserting (10.19) and (10.21), respectively, into (10.13), we obtain the 0-th Fourier coefficient $c(0, y)$. Moreover, we can simplify and also unify the formulas for both positive and negative k by the following classic identities for Gamma functions:

$$\prod_{j=k}^{k/2-1} (s+j) = \frac{\Gamma(s+k/2)}{\Gamma(s+k)} \text{ if } k \leq 0, \quad \prod_{j=0}^{k/2-1} (s+j) = \frac{\Gamma(s+k/2)}{\Gamma(s)} \text{ if } k \geq 0, \quad (10.23)$$

and $\Gamma(s+k/2-1/2)\Gamma(s+k/2) = 2^{2-2s-k} \sqrt{\pi} \Gamma(2s+k-1)$.

where the last identity follows directly from the Legendre duplication formula for Gamma functions.

To summarize the computation in this section, we have the following proposition for the Fourier expansion of $G_k(\tau, s)$.

Proposition 10.1. *The Fourier expansion of $G_k(\tau, s)$ is given by*

$$\begin{aligned} & (-1)^{k/2} \frac{2^k \pi^{s+k}}{\Gamma(s)} \sum_{l \in \mathbb{Z}_{\leq -1}} \left(|l|^{-s} \sigma_{2s+k-1}(|l|) (4\pi|ly|)^{-k/2} W_{-k/2, s+k/2-1/2}(4\pi|ly|) \right) e(lx) \\ & + \left(\zeta(2s+k)y^s + (-1)^{k/2} 2^{2-2s-k} \pi \frac{\Gamma(2s+k-1)}{\Gamma(s)\Gamma(s+k)} \zeta(2s+k-1)y^{1-s-k} \right) \\ & + (-1)^{k/2} \frac{2^k \pi^{s+k}}{\Gamma(s+k)} \sum_{l \in \mathbb{Z}_{\geq 1}} \left(l^{-s} \sigma_{2s+k-1}(l) (4\pi ly)^{-k/2} W_{k/2, s+k/2-1/2}(4\pi ly) \right) e(lx) \end{aligned}$$

for all even integers k , where $x = \text{Re}(\tau)$ and $y = \text{Im}(\tau)$.

10.2 B: Classical Eisenstein series for $\Gamma_0(N)$ In this manuscript, we consider some classical aspects of Eisenstein series on the congruence subgroup $\Gamma_0(N)$, especially those twisted by a general Dirichlet character $\chi \bmod N$. We fix $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$ to be the stabilizer of the cusp point at infinity. In general we have the notion of modular forms on $\Gamma_0(N)$ for a Dirichlet character χ :

Definition 10.2 (Modular Forms of level N). Let χ be a Dirichlet character modulo N . A holomorphic function $f : \mathfrak{H} \rightarrow \mathbb{C}$ is said to be a modular form of weight k for the character χ on $\Gamma_0(N)$ if the following conditions (a) and (b) are satisfied:

$$\begin{aligned} \text{(a)} \quad & f|_{k,\chi}\gamma(z) = f(z) \text{ for all } \gamma \in \Gamma_0(N) \text{ and } z \in \mathfrak{H}, \\ \text{(b)} \quad & f|_k\gamma(z) = O(1) \text{ as } z \rightarrow i\infty \text{ for all } \gamma \in \text{SL}_2(\mathbb{Z}), \end{aligned} \tag{10.24}$$

where $f|_{k,\chi}\gamma : \mathfrak{H} \rightarrow \mathbb{C}$ is defined by

$$(f|_{k,\chi}\gamma)(z) = \bar{\chi}(d)(cz + d)^{-k} f(\gamma z) \tag{10.25}$$

and $f|_k\gamma : \mathfrak{H} \rightarrow \mathbb{C}$ defined by

$$(f|_k\gamma)(z) = (cz + d)^{-k} f(\gamma z)$$

Remark 10.3. Since $\gamma = -\text{id} \in \Gamma_0(N)$, if we plug it in the condition (a) of (10.24) we find a necessary condition for the existence of nonzero modular form of weight k and character χ on $\Gamma_0(N)$:

$$\chi(-1) = (-1)^k. \tag{10.26}$$

Under Assumption (10.26), we investigate a very important class of modular forms, namely Eisenstein series, of weight k and character χ on $\Gamma_0(N)$, represented by the series $E_{k,\chi} := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} 1|_{k,\chi}\gamma$. The idea of the construction of Eisenstein series and that of Poincaré series is very natural and is very well illustrated in [Bru+08] that we take an appropriate scale of averaging operation.

Definition 10.4 (Eisenstein series twisted by a Dirichlet character). Let $k > 2$ be a natural number. Let χ be a Dirichlet character modulo N satisfying (10.26). Then the function series

$$E_{k,\chi} := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} 1|_{k,\chi}\gamma,$$

where $f|_{k,\chi}\gamma$ is defined by the same formula as in (10.25), is absolutely convergent in the whole upper half plane \mathfrak{H} , which we call the Eisenstein series of weight k twisted by the character χ .

In the next section on the computation of Fourier expansion of Eisenstein series, we will see that actually all the Fourier coefficients of $E_{k,\chi}$ are nonzero, hence this construction of averaging operation is nonzero.

10.2.1 Fourier expansion In this section we state the Fourier expansion of Eisenstein series twisted by a Dirichlet character. We fix the notation $e(x) = \exp(2\pi ix)$ as common in analytic number theory and many other related mathematical topics. We adopt the basic theory for L -functions and carry out our computation on a zero-free region of L -functions.

Let $k > 2$ be an integer. Let $\chi \pmod{m}$ be a Dirichlet character satisfying (10.26). Recall the definition of Eisenstein series $E_{k,\chi}$ from Def. 10.4, whose Fourier coefficients are to be stated based on the following notations. We introduce the “analytic” factor $c_{k,\chi}$ involved in the Fourier coefficients by the formula

$$c_{k,\chi} = \frac{c_k}{m^k L(k, \chi)}, \text{ where } c_k = \frac{(-2\pi i)^k}{(k-1)!}, \quad (10.27)$$

and the “arithmetic” factor $a_{k,\chi}(n)$ for each n by

$$a_{k,\chi}(n) = \sum_{d|n} d^{k-1} \tau(\chi, \psi_d), \text{ where } \tau(\chi, \psi_a) = \sum_{b \in \mathbb{Z}/m} \chi(b) e(ab/m). \quad (10.28)$$

Here $\tau(\chi, \psi_a)$ is called the Gauss sum associated with a Dirichlet character χ and the additive character $\psi_a(b) := e(ab/m)$. We are now in position to state the Fourier coefficients.

Proposition 10.5. *Let k, χ be as from the beginning of this section. The 0-th holomorphic Fourier coefficient of $E_{k,\chi}(z)$ is 1, and for all positive integers n , the n -th coefficient is equal to $c_{k,\bar{\chi}} a_{k,\bar{\chi}}(n)$, where $c_{k,\bar{\chi}}$ is given explicitly in (10.27) and $a_{k,\bar{\chi}}(n)$ in (10.28).*

Given χ of modulus m from the beginning of this section, we denote by χ^* of modulus m^* the primitive character which induces χ . We find that $\tau(\chi, \psi_a)$ can be expressed in terms of the classical Gauss sum $\tau(\chi^*) = \sum_{b^* \in \mathbb{Z}/m^*} \chi^*(b^*) e(b^*/m^*)$ by

$$\tau(\chi, \psi_a) = \tau(\chi^*) \sum_{d|(a, m/m^*)} d \bar{\chi}^*(a/d) \mu(m/dm^*) \chi^*(m/dm^*) \quad (10.29)$$

Remark 10.6. Note that (10.29) differs from the one stated in the book [IK04] at Lemma 3.2 on page 48, which can be falsified with the example when χ is the non-primitive character mod 9 induced by the unique primitive character mod 3 and $a = 3$. In this case the LHS of the formula given in the book is nonzero while the RHS of it is zero.

We now employ Formula (10.29) to write down explicitly the Fourier expansion of $E_{k,\chi}$ as a corollary of Proposition 10.5:

Corollary 10.7. *Let k, χ be as from the beginning of this section. Let χ^* of modulus m^* be the primitive character that induces χ . Then, we have the Fourier expansion for Eisenstein series $E_{k,\chi}$ as*

$$E_{k,\chi}(z) = 1 + c_{k,\bar{\chi}} \tau(\bar{\chi}^*) \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \sum_{l|(d, \frac{m}{m^*})} l \chi^*(d/l) \mu(m/lm^*) \bar{\chi}^*(m/lm^*) \right) e(nz), \quad (10.30)$$

In particular, when χ is a primitive character, i.e. when $\chi = \chi^$, we have*

$$E_{k,\chi^*}(z) = 1 + c_{k,\bar{\chi}^*} \tau(\bar{\chi}^*) \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \chi^*(d) \right) e(nz), \quad (10.31)$$

where the constant $c_{k,\chi}$ is given in (10.27).

Finally we are led to the fact that Eisenstein series twisted by a Dirichlet character χ is an “old form”, i.e. $E_{k,\chi}(z)$ can be decomposed into a direct sum of one dimensional spaces spanned by $E_{k,\chi^*}(dz)$.

Proposition 10.8. *Let k, χ be as from the beginning of this section, then we have*

$$E_{k,\chi}(z) = \sum_{d|\frac{m}{m^*}} \frac{a_d}{\sum_{d|\frac{m}{m^*}} a_d} E_{k,\chi^*}(dz), \text{ where } a_d = d^k \mu(m/dm^*) \overline{\chi^*}(m/dm^*). \quad (10.32)$$

Moreover, these $E_{k,\chi^*}(dz)$ are linearly independent over \mathbb{C} .

We will prove it in the next section, but skip the part of linear independence, namely the fact that it is actually a direct sum.

10.2.2 Proof of the old-new form relation This section was planned to be divided into three parts, in order to prove the statements in Section 10.2.1, with each of them dedicated to deduce Fourier coefficients of $E_{k,\chi}$ from the definition and basic function theory, (10.29) from the inclusion-exclusion principle, and Proposition 10.8 from Corollary 10.7. For the main purpose of this manuscript, we now only provide the third part, namely deduction of Proposition 10.8 from Corollary 10.7.

To prove Proposition 10.8, it suffices to show that the coefficient of $e(nz)$ on both sides are equal for all non-negative integers n . For $n = 0$ this is clear from the expressions in (10.30) and (10.31) with the fact that

$$\sum_{d|\frac{m}{m^*}} \frac{a_d}{\sum_{d|\frac{m}{m^*}} a_d} = 1.$$

For the general terms, note that the contribution of the coefficients of $e(nz)$ on the RHS of (10.32) only comes from those of $E_{k,\chi^*}(dz)$ where d is a divisor of n , and in terms of expression (10.31) each contribution corresponds to the original coefficient of $e(\frac{n}{d}z)$ instead of $e(nz)$. Therefore summing them together we obtain that the coefficient of $e(nz)$ on the RHS of (10.32) is equal to

$$\frac{c_{k,\overline{\chi^*}\tau}(\overline{\chi^*})}{\sum_{d|\frac{m}{m^*}} a_d} \left(\sum_{d|(n,m/m^*)} d^k \mu(m/dm^*) \overline{\chi^*}(m/dm^*) \sum_{l|\frac{n}{d}} l^{k-1} \chi^*(l) \right). \quad (10.33)$$

As to the coefficient of $e(nz)$ on the LHS of (10.32), we simply copy that from (10.30), which is

$$c_{k,\overline{\chi}\tau}(\overline{\chi^*}) \left(\sum_{d|n} d^{k-1} \sum_{l|(d,\frac{m}{m^*})} l \chi^*(d/l) \mu(m/lm^*) \overline{\chi^*}(m/lm^*) \right). \quad (10.34)$$

Now the proof can be decomposed into two parts: first we show that

$$\frac{c_{k,\overline{\chi^*}\tau}(\overline{\chi^*})}{\sum_{d|\frac{m}{m^*}} a_d} = c_{k,\overline{\chi}\tau}(\overline{\chi^*});$$

and second we show that

$$\begin{aligned} \sum_{d|(n, \frac{m}{m^*})} d^k \mu(m/dm^*) \overline{\chi^*}(m/dm^*) \sum_{l|\frac{n}{d}} l^{k-1} \chi^*(l) \\ = \sum_{d|n} d^{k-1} \sum_{l|(d, \frac{m}{m^*})} l \chi^*(d/l) \mu(m/lm^*) \overline{\chi^*}(m/lm^*). \end{aligned} \quad (10.35)$$

For the first part, note that $L(k, \overline{\chi})$ and $L(k, \overline{\chi^*})$ have no pole in our setting $k > 2$, hence $c_{k, \overline{\chi}}$ and $c_{k, \overline{\chi^*}}$ are nonzero and that

$$\begin{aligned} \frac{c_{k, \overline{\chi^*}}}{c_{k, \overline{\chi}}} &= \frac{m^k}{m^{*k}} \frac{L(k, \overline{\chi})}{L(k, \overline{\chi^*})} = \frac{m^k}{m^{*k}} \prod_{\substack{p \text{ prime} \\ p|\frac{m}{m^*}}} \left(1 - \frac{\overline{\chi^*}(p)}{p^k}\right) = \frac{m^k}{m^{*k}} \sum_{d|\frac{m}{m^*}} \mu(d) \frac{\overline{\chi^*}(d)}{d^k} \\ &= \sum_{d|\frac{m}{m^*}} \left(\frac{m}{dm^*}\right)^k \mu(d) \overline{\chi^*}(d) = \sum_{d|\frac{m}{m^*}} d^k \mu\left(\frac{m}{dm^*}\right) \overline{\chi^*}\left(\frac{m}{dm^*}\right) = \sum_{d|\frac{m}{m^*}} a_d. \end{aligned}$$

For the second part, we note that on the RHS of (10.35) any pair of positive integers (d, l) appears in the double sum if and only if $d = lh$ for some positive integer h , such that $l|(n, \frac{m}{m^*})$ and that $h|\frac{n}{l}$, hence this double sum can also be written as

$$\sum_{l|(n, \frac{m}{m^*})} l \mu\left(\frac{m}{lm^*}\right) \overline{\chi^*}\left(\frac{m}{lm^*}\right) \sum_{h|\frac{n}{l}} (lh)^{k-1} \chi^*(h) = \sum_{l|(n, \frac{m}{m^*})} l^k \mu\left(\frac{m}{lm^*}\right) \overline{\chi^*}\left(\frac{m}{lm^*}\right) \sum_{h|\frac{n}{l}} h^{k-1} \chi^*(h),$$

which is the same as the LHS of (10.35), since the only difference between them are symbols of dummy variables in the same double sum. We are done with the second part of the proof, hence conclude Proposition 10.8.

10.2.3 Numerical falsification On page 17 of [Bru+08], it is mentioned that if χ is a non-trivial Dirichlet character and k a positive integer with $\chi(-1) = (-1)^k$, then there is an Eisenstein series having the Fourier expansion

$$\mathbb{G}_{k, \chi}(\tau) = c_k(\chi) + \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) d^{k-1} \right) e(n\tau), \text{ where } c_k(\chi) = \frac{1}{2} L(1-k, \chi). \quad (10.36)$$

satisfying the modularity condition

$$\mathbb{G}_{k, \chi}\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(a)(c\tau + d)^k \mathbb{G}_{k, \chi}(\tau) \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \quad (10.37)$$

In this section we provide a numerical evidence to falsify (10.37) and point out that if one changes $\chi(a)$ into $\overline{\chi}(a) = \chi(d)$ in (10.37), then it coincides with the results we predict, and is also verified by the computational evidence.

The example we take as a computational evidence here is the special case when $N = 5$, $k = 5$, the primitive character $\chi \bmod 5$ defined by $\chi(2) = i$ which satisfies Condition

(10.26). Moreover, we compute for a special point in the upper half plane, $\tau = i$ with $(a, b, c, d) = (2, 1, 5, 3)$. The output of the following Julia code has two terms: the first being relative error (i.e. $|\text{LHS}-\text{RHS}|$ divided by LHS) of (10.37), and the second that of its modified version replaced by $\bar{\chi}(a)$. It turns out from the computation that the relative error of (10.37) is close to 2 while that of the modified version is close to 0 up to an error of scale 10^{-15} . Note that the numerical issue for the variable tauone in the code is a bit tricky: it turns out that if we take the precision variable “prec” to be from 1000 to 20000 and the result is very close. However, if we use “//” instead of the usual “/” in Julia for umerical division, the result turns to be much more accurate: for prec 1000 we get error scale 10^{-73} and for prec 10000 we obtain 10^{-146} .

```
import Base.e
using Nemo

divisors(a::Int) = map(Int, divisors(ZZ(a)))

function divisors(a::fmpz)
    iszero(a) && return []

    divs = [one(ZZ)]
    isone(a) && return divs

    for (p,e) in factor(abs(a))
        p = fmpz(p)
        ndivs = deepcopy(divs)
        for i = 1:e
            map!((d) -> p*d, ndivs, ndivs)
            append!(divs, ndivs)
        end
    end

    return divs
end

CC = ComplexField(500)
ee(tau) = exppii(2*CC(tau))

function chi0(n)
    if n % 5 == 0
        return 0
    elseif n % 5 == 1
        return 1
    elseif n % 5 == 2
```

```

    return onei(CC)
elseif n % 5 == 3
    return (-1)*onei(CC)
elseif n % 5 == 4
    return -1
end
end
end

```

```
sigmachi(n) = sum(d^4*chi0(d) for d in divisors(n))
```

```

# the following variable ‘‘a0’’ is the constant Fourier coefficient
# which is
# equal to 1/2 L(1-k, chi) in the book’s formula. Note that such
# value of
# L-function can be computed through a finite sum of Hurwitz zeta
# function.

```

```

function eisenstein(tau,prec)
    a0 = (5^4)//2 * (zeta(CC(-4), CC(1//5)) + zeta(CC(-4), CC(2//5))
    * onei(CC)
    - zeta(CC(-4), CC(3//5)) * onei(CC) - zeta(CC(-4), CC(4//5)))
    return CC(a0) + sum(sigmachi(n)* ee(n*tau) for n in 1:prec)
end

```

```

# the following variables corresponding to the mathematical terms
# in (4.2)
# of the note Eis series of level N. ‘‘tautwo’’ is tau in the note
# which is
# taken as the imaginary unit i, and ‘‘tauone’’ is (a tau + b)/(c tau + d)
# which is exactly (2i + 1)/(5i + 3) = 13/34 + 1/34 i. ‘‘b0’’ is
# (c tau + d)^k
# which is (5i + 3)^5 in our setting.

```

```

tauone = 13//34 + 1//34*onei(CC)
tautwo = onei(CC)
b0 = (3 + 5*onei(CC))^5

```

```

# the first term in the output is the relative error of the book’s
# formula and
# the second one that of the modified version, namely by taking a
# bar in the
# original formula.

```


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