Robust inversion based fault estimation for discrete-time LPV systems

Downloaded from: https://research.chalmers.se, 2019-11-16 18:35 UTC

Citation for the original published paper (version of record):
Robust inversion based fault estimation for discrete-time LPV systems
IEEE Transactions on Automatic Control, 57(6): 1581-1586
http://dx.doi.org/10.1109/TAC.2011.2176160

N.B. When citing this work, cite the original published paper.

©2012 IEEE. Personal use of this material is permitted.
However, permission to reprint/republish this material for advertising or promotional purposes
or for creating new collective works for resale or redistribution to servers or lists, or to
reuse any copyrighted component of this work in other works must be obtained from
the IEEE.

This document was downloaded from http://research.chalmers.se, where it is available in accordance with the IEEE PSPB

(article starts on next page)
Robust Inversion Based Fault Estimation for Discrete-Time LPV Systems

Balázs Kulcsár, Member, IEEE, and Michel Verhaegen

Abstract—The article presents a state-space based Fault Diagnosis (FD) method for discrete-time, affine Linear Parameter Varying (LPV) systems. The goal of the technical note is to develop a robust and dynamic inversion based technique for systems with parameter varying representations when an additive, exogenous disturbance signal perturbs the system. After applying geometric concepts for explicit fault inversion, a robust strategy is proposed to attenuate the effect of the unknown disturbance input signal on the fault estimation error. The proposed robust observer is derived as a solution of off-line Linear Matrix Inequality (LMI) conditions. The technical note demonstrates the viability of the novel methodology through a numerical example.

Index Terms—Discrete-time linear parameter varying (LPV) systems, dynamic inversion, fault detection and isolation, fault diagnosis (FD), robust observers.

I. INTRODUCTION

Fault Diagnosis (FD) is of capital importance in the design of complex and safety related systems. Since in FD recognition the occurring malfunction always conveys substantial information, the idea of reconstructing faults attracted significant research attention, e.g., [1]. Hence, instead of designing a controller robust enough to cover all possible effects of failures, FD techniques suggest the direct use of the detected fault signals, e.g. [2]. Even though, alternative nominal techniques have been developed, in most cases disturbance or modeling error do influence the quality of FD. Hence, various robust methods have been deployed, such as disturbance decoupling or other robust approaches, e.g. [3], [4]. FD for nonlinear systems has recently become an active research field. [5] gives an overview on nonlinear FD techniques considering both existing algebraic (e.g. [6]) and geometric approaches [7]. Besides, robust nonlinear FD has been identified as a relevant future research direction [5].

A rich set of linear methodologies has been developed for control, estimation and diagnosis of Linear Parameter Varying (LPV) model representations, such as [8]. LPV models with appropriate parametrization can be alternative to nonlinear system description. Methods for FD in LPV systems became attractive since they can significantly reduce computational complexity respectively result in explicit diagnosis solutions. In LPV FD literature, two main concepts have been proposed. First, indirect methods provide a residual signal (fault indicator). A nominal technique for continuous-time and affine LPV systems is described in [9]. Robust observer based indirect solutions are presented in [10]–[12]. Second, conceptually different (direct) approaches have been introduced in LPV fault estimation e.g. by means of nominal system inversion [13].

This technical note contributes to the field of fault estimation for discrete-time, affine LPV systems. Motivated by and connected to the concise techniques in [9], [13], [14], this work presents a discrete-time and dynamic inversion based FD solution. The present methodology robustly reconstructs faults under the effect of unknown exogenous but norm bounded disturbance signals. This technical note, for a wide class of discrete-time LPV systems, allows one to consider a certain type of parameter variation at the plant output, as well as at the fault and disturbance input directions. Quadratically stabilizing and disturbance attenuating fault estimation is achieved by assigning a dynamic FD observer structure to the solution, namely a Luenberger type of observer. Even though, the original discrete-time LPV dynamics is affine in the scheduling parameter, the robust inversion procedure results in generic LPV model structures. Given the observer structure and the inverse system dynamics, the technical note shows how to attenuate disturbance signals on the performance output of the fault estimation problem, i.e. on the fault estimation error.

The technical note is divided into three main parts. After the introductory part, Section II formulates the robust and dynamic inversion problem to be solved. The main result of the technical note is discussed in Section III. The elaborated methodology is demonstrated by a numerical example in Section IV. Remarks and further concerns conclude the technical note. Technical derivations are tabulated to the Appendix.

II. PROBLEM FORMULATION AND NOTATIONS

Consider the following system by

\[ x(k+1) = \sum_{j=1}^{n_p} p^{(j)}_k \left( A^{(j)} x(k) + B^{(j)}_2 u(k) \right) + E^{(j)} f(k) + B^{(j)}_1 d(k) \]

\[ y(k) = \sum_{j=1}^{n_p} p^{(j)}_k \left( C^{(j)}_2 x(k) + D^{(j)}_2 u(k) + D^{(j)}_1 d(k) \right) \]

where \( f, u, x, y, g, d, p \in \mathbb{R}^{n_f, n_m, n_f, n_m, n_p} \) are the fault, control input, state, output, disturbance, and scheduling parameter signals with appropriate dimensional, known, and real valued state-space matrices \( A^{(j)}, E^{(j)}, B^{(j)}_1, B^{(j)}_2, C^{(j)}_2, D^{(j)}_1, D^{(j)}_2, J = 1 \ldots n_p \) with \( p^{(j)} = 1 \) and \( t \geq n_f \). The scheduling parameter \( p \) belongs to a compact polytope \( \mathcal{P} \subseteq \mathbb{R}^{n_f} \) with vertices \( \mathcal{P} \) defined by \( p \in [\mu, \bar{p}] \). In case of the sensor fault, we suggest to use pseudo actuation [15] in order to yield the above form. Up to the time sample \( k \), the fault-free \( p^{(j)} \), \( u(k) \), and \( y(k) \) are measurable but the fault input \( f(k) \) and the 2-norm bounded disturbance signal \( d(k) \) are unknown.

For the sake of simplicity, we assume in the remaining part of Section II that \( u(k) = d(k) = 0 \). The goal of the problem statement section is to show how to express the fault signal \( f(k) \) purely by measurable quantities and model parameter information.

Assumption 1: We assume, the kernel space of the affine parameter dependent output map, is parameter independent (constant) and will be denoted by \( \mathcal{C} = ker \left( \sum_{j=1}^{n_p} p^{(j)}_k C^{(j)}_2 \right) \). The \( \mathcal{C} \) is full column rank \( \forall p \in \mathcal{P}, \forall \bar{k} \in \bar{L} \). In other words, the scheduling trajectories are such that they never cause any changes in the kernel space directions of the measurement map, though, the (2) is parameter dependent.

Assumption 2: \( \sum_{j=1}^{n_p} p^{(j)}_k E^{(j)} \) is full column rank \( \forall p \in \mathcal{P} \). Alternatively, the kernel space of the fault insertion map is the origin for all possible parameter trajectories.

In the sequel, an algorithm is provided to show how to over-parametrize the parameter varying fault direction.
Algorithm 1:
Step 1 Consider \( \sum_{j=1}^{n_p} p_{k}^{(j)} E^{(j)} f(k) = E \tilde{f}(k), \tilde{f}(k) \in \mathbb{R}^{n_\tau} \) with \( E = \text{Im} \{ \bigoplus_{j=1}^{n_p} E^{(j)} \} \) where \( \bigoplus \) and \( \text{Im} \{ \cdot \} \) denote the subspace dilation and the computation of the range space, respectively. \( E \) is an orthonormal basis representation of \( E \).

Step 2 In virtue of Assumption 2, \( \tilde{f}(k) \) can be unambiguously projected back to \( f(k) \) by \( P(k) = \left( \sum_{j=1}^{n_p} p_{k}^{(j)} E^{(j)} \right)^{-1} E \), where \( (-)^{-1} \) stands for the pseudo inverse.

Assumption 3: System (1), (2) is reversible [16], [17], quadratically detectable [18] and (strong) invertible, i.e. \( \mathcal{R}^c_k = \{ 0 \} \) [19], with parameter invariant relative degree vector. \( \mathcal{R}^c_k \) is the largest fault controllability subspace in the kernel space of the output map \( C \).

Lemma 1: Given the system in (1)–(2) with \( u(k) = d(k) = 0 \). If Assumptions 1–3 are fulfilled, based on Algorithm 1, the inverse system dynamics can be written as

\[
x_2(k + 1) = A(k) x_2(k) + B(k + \kappa) y(k, k + \kappa) + \mathbf{C}(k) x_2(k) + D(k + \kappa) y(k, k + \kappa)
\]

where the coefficient matrices \( A(k), B(k + \kappa), C(k), D(k + \kappa) \) (see (16), (17)) are generic parameter varying coefficient matrices. Furthermore, \( x_2(k) \) characterizes the dynamics within the largest fault invariant subspace contained in \( C \), and \( y(k, k + \kappa) \) is the measured output vector along the future time horizon up to \( \kappa \) denoting the maximal element of the relative degree vector.

Proof: \( \mathcal{R}^c_k \) can be constructed by [14], based on the fault controllability subspace \( A(k), B(k + \kappa), C(k), D(k + \kappa) \) invariant subspace denoted by \( \mathcal{V} \subseteq C \). The invertibility condition in Assumption 3 can be reformulated as a left inverter condition by \( \mathcal{V} \cap E = \{ 0 \} \). The left invertibility condition then gives rise to a nonsingular, parameter independent coordinate transformation [13], \( T : \mathbb{R}^n \rightarrow \mathbb{R}^{n_\tau} \) splits the original LPV representation into two parts. \( T = [V^{\perp} \ V^*] \), where \( V^{\perp} \) and \( V^* \) are bases for the orthogonal complement of \( V \) denoted by \( V^{\perp} \), and \( V^* \), respectively. First, we apply Step 1 of Algorithm 1, then \( T \) to the original system in (1)–(2).

Then, the following form can be obtained:

\[
x_1(k + 1) = \sum_{j=1}^{n_p} p_{k}^{(j)} (A_1^{(j)} x_1(k) + A_2^{(j)} x_2(k)) = E_1 \tilde{f}(k)
\]

\[
x_2(k + 1) = \sum_{j=1}^{n_p} p_{k}^{(j)} (A_2^{(j)} x_1(k) + A_2^{(j)} x_2(k))
\]

\[
y(k) = \sum_{j=1}^{n_p} p_{k}^{(j)} \tilde{C}_2^{(j)} x_1(k)
\]

where the LPV system is decomposed in two interconnected dynamics, \( x_1 \) characterizes the dynamics in \( V^{\perp} \) with \( \dim (V^{\perp}) = n_\tau \) and \( x_2 \) refers to the dynamics in \( V^* \) with \( \dim (V^*) = n - n_\tau \).

With an appropriately chosen feedback term \( F_2^{(j)} = -A_1^{(j)} \) is an obvious choice and \( f_s(k) \) represents a synthetic fault signal. Thus, substituting (8) into (5) reads as

\[
x_1(k + 1) = \sum_{j=1}^{n_p} p_{k}^{(j)} A_1^{(j)} x_1(k) + E_1 f_s(k)
\]

\[
x_2(k + 1) = \sum_{j=1}^{n_p} p_{k}^{(j)} A_2^{(j)} x_2(k)
\]

\[
y(k) = \sum_{j=1}^{n_p} p_{k}^{(j)} \tilde{C}_2^{(j)} x_1(k)
\]

Unlike the subsystem in (10), the dynamics in (9) can be completely characterized by the output measurement (11) and the scheduling parameter \( \kappa \), \( f_s(k) \) can therefore be reconstructed simply based on measurable information. Therefore, denote the rows of the output map \( \tilde{C}_2^{(j)} \) as \( \tilde{c}_i^{(j)} \) and \( y(k, k + \kappa) = \sum_{i=1}^{n_\tau} \tilde{c}_i^{(j)} y_i(k, k + \kappa - 1) = \sum_{i=1}^{n_\tau} \sum_{j=1}^{n_p} p_{k}^{(j)} \tilde{c}_i^{(j)} A_1^{(j)} x_1(k) \), where we collect output information as long as we do not exceed sample \( \kappa_i = 1 \).

Condensing the measurements of the output channel \( i \) into a vector \( y \) results in

\[
y(k, k + \kappa - 1) = \sum_{i=1}^{n_\tau} \tilde{c}_i^{(j)} y_i(k, k + \kappa - 1)
\]

Repeating the above procedure for all output channels, we can write

\[
y(k, k + \kappa - 1) = \left[ y_1(k, k + \kappa - 1) \right]
\]

At time sample \( k + \kappa \), the effect of the fault term \( f_s(k) \) appears in the output equation under the form of

\[
y(k, k + \kappa - 1) = \sum_{i=1}^{n_\tau} \tilde{c}_i^{(j)} y_i(k, k + \kappa - 1) = \sum_{i=1}^{n_\tau} \tilde{c}_i^{(j)} y_i(k, k + \kappa - 1)
\]

Consequently, by substituting the observations from (13) into (14) if \( \sum_{i=1}^{n_\tau} \tilde{c}_i^{(j)} y_i(k, k + \kappa - 1) - \sum_{i=1}^{n_\tau} \tilde{c}_i^{(j)} y_i(k, k + \kappa - 1)
\]

Now, the dynamics of the inverse system w.r.t. the fault input can be read as a non-affine LPV system defined between the new system inputs \( y(k, k + \kappa) \) and the inverse system’s output \( f \).

The dynamics in equations (9)–(11) and replace the state vector \( x(k) \) by the measurements based on (13).

The dynamics of the inverse system is characterized by (10) and can be rewritten into the form of (16), respectively. Furthermore, the combination of (15) and the feedback term in eq. (8) together with Step 2 from Algorithm 1 results in (17). Accordingly

\[
x_2(k + 1) = \sum_{j=1}^{n_p} p_{k}^{(j)} A_2^{(j)} x_2(k) + \sum_{j=1}^{n_p} p_{k}^{(j)} A_2^{(j)} \tilde{c}_i^{(j)} y_i(k, k + \kappa - 1) S_S y(k, k + \kappa - 1)
\]

Footnote 1: The \( \kappa_i = 1 \ldots \ell \) is assumed to be constant for all possible scheduling trajectory in any time interval.
\[ f(k) = -P(k) \sum_{j=1}^{n_p} p_{k}^{(j)} (E_1)^{i} A_{12}^{(j)} x_2(k) + P(k) \hat{c}^{(j)}(k) \]
\[ + P(k) \hat{\xi}_{[k,k+n-1]}^{(j)} \left( I_{n_x} - \hat{C}_{[k,k+n-1]} A_{[k,k+n-1]} \right) Y_{[k,k+n]} \]

with the selector \( \hat{S}_a \) defined as \( Y_{[k,k+n-1]} = \hat{S}_a Y_{[k,k+n]} \). \( I_{n_x} \) stands for the identity matrix with dimension \( n_x = \sum_{i=1}^{k} \). This completes the proof.

The above-mentioned nominal procedure for discrete-time LPV fault inversion might be sensitive to exogenous disturbance signals. If complete disturbance decoupling is not possible, this technical note shows an efficient way for inversion based and disturbance rejecting observer design via a Luenberger observer. We shall refer to matrix inequalities by the symbol \( \succ 0 \), denoting that matrix \( A \) is positive definite. The disturbance signal \( d(k) \) with bounded 2-norm is denoted by \( ||d||_2 = \left( \sum_{l=0}^{n} d(l)^T d(l) \right)^{1/2} \) and \( d_{[k,k+n]} = [d(k)^T \ldots d(k + n - 1)^T]^T \). The superscript \((\cdot)^T\) denotes the transpose of the real matrix and \( \ast \) represents the symmetric term of the matrix inequality.

### III. ROBUST FAULT INVERSION

The following Lemma extends the results of Lemma 1., first, by considering non-zero input and disturbance signals. Second, it explains the way of obtaining a system description for robust and model-based observer design.

**Lemma 2:** Given the system in equations (1)–(2). If Assumptions 1–3, are fulfilled, based on Algorithm 1, the inverse system dynamics can be written as

\[
\begin{bmatrix}
  x(k + \kappa + 1) \\
  f(k + \kappa) \\
  y(k + \kappa)
\end{bmatrix} = \begin{bmatrix}
  A(k) & B_1(k) & B_2(k) \\
  C_1(k) & D_{11}(k) & D_{12}(k) \\
  C_2(k) & D_{21}(k + \kappa) & D_{22}(k + \kappa)
\end{bmatrix} \begin{bmatrix}
  x(k - \kappa) \\
  d_{[k-\kappa,n-\kappa]} \\
  u_{[k-n,k-\kappa]}
\end{bmatrix}
\]

(18)

where the coefficient matrices \( A(k), B(k), C_1(k), D_{11}(k), D_{12}(k), C_2(k - \kappa), D_{21}(k - \kappa), D_{22}(k - \kappa) \), [see (28)–(31)] are generic parameter varying coefficients. Furthermore, \( d_{[k-\kappa,n-\kappa]} \) and \( u_{[k-n,k-\kappa]} \) are the vectorized values of

\[ \text{disturbance and of the augmented input signals for the time interval } k - \kappa \text{ to } k, \text{ respectively.} \]

**Proof:** Recall the system representation in (1)–(2) with Step 1 of Algorithm 1 and apply the coordinate transformation \( T \) with the feedback design in (8), then

\[
x_{1}(k + 1) = \sum_{j=1}^{n_p} p_{k}^{(j)} \left( A_{11}^{(j)} x_1(k) + B_{11}^{(j)} u(k) + B_{11}^{(j)} d(k) \right) + E_1 f_s(k)
\]
\[
x_{2}(k + 1) = \sum_{j=1}^{n_p} p_{k}^{(j)} \left( A_{21}^{(j)} x_1(k) + A_{22}^{(j)} x_2(k) + B_{22}^{(j)} u(k) \right) + B_{22}^{(j)} d(k)
\]
\[
y(k) = \sum_{j=1}^{n_p} p_{k}^{(j)} \left( \hat{c}_{2}^{(j)} x_1(k) + D_{22}^{(j)} u(k) \right) + D_{22}^{(j)} d(k)
\]

(19)

(20)

(21)

Denote the rows of \( \hat{c}_{2}^{(j)} \) by \( c_{i}^{(j)} \) and of \( D_{22}^{(j)} \) by \( d_{22,i}^{(j)} \) where \( i = 1 \ldots l, j = 1 \ldots n_p \). Define

\[
O_{[k,k+n]} = \begin{bmatrix}
  \sum_{j=1}^{n_p} p_{k}^{(j)} c_{1}^{(j)} \ldots \sum_{j=1}^{n_p} p_{k}^{(j)} c_{l}^{(j)} A_{11}^{(j)} \\
  \vdots \\
  \sum_{j=1}^{n_p} p_{k}^{(j)} c_{1}^{(j)} \ldots \sum_{j=1}^{n_p} p_{k}^{(j)} c_{l}^{(j)} E_1
\end{bmatrix}
\]

(22)

\[
\hat{\xi}_{[k,k+n]} = \begin{bmatrix}
  \sum_{j=1}^{n_p} p_{k}^{(j)} c_{1}^{(j)} \ldots \sum_{j=1}^{n_p} p_{k}^{(j)} c_{l}^{(j)} A_{11}^{(j)} \\
  \vdots \\
  \sum_{j=1}^{n_p} p_{k}^{(j)} c_{1}^{(j)} \ldots \sum_{j=1}^{n_p} p_{k}^{(j)} c_{l}^{(j)} E_1
\end{bmatrix}
\]

(23)

In (24) and (25), shown at the bottom of the page, \( O_{s-\kappa} \) is a zero block and \( \kappa \) is the largest relative degree number \( \forall i = 1 \ldots l. \) Therefore

\[
y(k + \kappa) = O_{[k,k+n]} x_{1}(k) + T_{u,[k,k+n]} u_{[k,k+n]}
\]
\[
+ T_{d,[k,k+n]} d_{[k,k+n]} + \hat{\xi}_{[k,k+n]} f_s(k)
\]
\[
f_s(k) = \hat{c}_{2}^{(j)} x_1(k) - O_{[k,k+n]} x_{1}(k)
\]
\[
+ T_{u,[k,k+n]} u_{[k,k+n]}
\]
\[
- T_{d,[k,k+n]} d_{[k,k+n]}
\]

(26)

(27)

Together with the dynamic evolution of \( x_2 \) in (20), (27) defines the inverse system w.r.t. the synthetic fault input \( f_s(k). \) Instead of directly using the inverse system for robust detection filter design, we revert to

\[ \text{In (24)–(25) the amount of information (i.e. the number of } u \text{ and } d \text{ samples) influencing the output at time sample } y(k + \kappa) \text{ is changing due to fact that } \kappa \text{ is different } \forall i = 1 \ldots l. \text{ In fact, the Toeplitz structures in (24)–(25), shown at the bottom of the page, might have zero padded columns depending on } \kappa, \text{ as well as on the relative degree vectors associated to the input and disturbance terms.} \]
the original system description, (19)–(21). By feeding back the information \( f_r(k) \) (and using the Step 2 in Algorithm 1) as well as considering \( \hat{u}_{k,k+n} \), we can write

\[
x_1(k+1) = \left( \sum_{j=1}^{N_p} p^{(j)}_k A_1^{(j)} + E_1 \hat{e}_{k,k+n}^T \hat{e}_{k,k+n} \right) x_1(k)
+ \left[ \sum_{j=1}^{N_p} p^{(j)}_k B_1^{(j)} \right] y(k) - E_1 \hat{e}_{k,k+n} \hat{e}_{k,k+n} T_{d,k,k+n} \]
\[
\times \left[ d(k) \right] + E_1 \hat{e}_{k,k+n} y(k) + \hat{u}(k)
= A_1(k + \kappa) x_1(k)
+ B_1(k + \kappa) d(k)
+ B_2(k + \kappa) \hat{u}(k)
\tag{28}
\]

\[
x_2(k+1) = \sum_{j=1}^{N_p} p^{(j)}_k (A_2^{(j)} x_1(k) + A_{22}^{(j)} x_2(k))
+ B_{22}^{(j)} d(k) + B_{22}^{(k)} (k)
= A_2(k + \kappa) x_1(k) + B_{22}^{(j)} x_2(k)
+ B_2(k + \kappa) d(k)
+ B_2(k + \kappa) \hat{u}(k)
\tag{29}
\]

\[
y(k) = \sum_{j=1}^{N_p} p^{(j)}_k \left( c_1^{(j)} x_1(k) + D_{21}^{(j)} u(k) + D_{22}^{(j)} d(k) \right)
= \hat{C}_0 x_1(k) + D_{22}(k) \hat{u}_{k,k+n}
\tag{30}
\]

where \( \hat{u}_{k,k+n} \in \mathbb{R}^{\tilde{n}_m} = \sum_{i=1}^{\ell} (1 : \tilde{n}_d), \hat{d} \in \mathbb{R}^{\tilde{n}_d}, \hat{u}_{k,k+n} \in \mathbb{R}^{\tilde{n}_d}, \hat{d} \in \mathbb{R}^{\tilde{n}_d}, \hat{u}_{k,k+n} \in \mathbb{R}^{\tilde{n}_d} \).

Apart the fulfillment of the predefined objective in inequality (33), the closed-loop observer has to preserve asymptotic stability. Note that the above criteria fit to the generic problem formulation of the proposed observer in (33). As a result of Lemma 2, the inverse system naturally structures delays when replacing the fault input with measurement. Moreover, throughout \( \hat{K}(k) \), we encounter past scheduling measurements in the robust observer synthesis.

**Theorem 1:** Given the system in (18), the quadratic objective function in (33) together with the observer structure in (34)–(36), all being the proposed observer results in a asymptotically unbiased estimate of \( \hat{f} \) when \( \hat{f} = 0 \). Furthermore, in case of \( \hat{d} \neq 0 \), it attenuates in induced \( L_2 \) norm sense the disturbance on the fault estimation error \( \Delta f \) up to a given \( \gamma > 0 \) with \( G \neq 0 \) if the following matrix inequalities are feasible:

\[
\begin{bmatrix}
- P & 0 \\
0 & - \mathbb{I}_{n_d} \gamma^2
\end{bmatrix}
\begin{bmatrix}
C_0^T & G^T \hat{A}_1(k) \\
C_1(k) & D_{11}(k)
\end{bmatrix}
\begin{bmatrix}
G & 0 \\
0 & - \mathbb{I}_{n_f}
\end{bmatrix}
\geq 0
\tag{37}
\]

with \( P \succ 0 \) and \( \hat{A}_1(k) = A(k) - K \hat{C}_2(k - \kappa) + B_1(k) + \hat{K}(k) D_{21}(k) \). 

**Proof:** Consider the dynamics of the state reconstruction error by

\[
e_{k}(k - \kappa) = \left( \hat{A}(k) - K \hat{C}_2(k - \kappa) \right) e_r(k - \kappa)
\tag{38}
\]

\[
\Delta f(k - \kappa) \triangleq f(k - \kappa) - \hat{f}(k - \kappa)
\tag{39}
\]
Define a Lyapunov function with constant $P^T = P \succ 0$ for the state reconstruction error by

$$V(k - \kappa) = e_x(k - \kappa)^T P e_x(k - \kappa) \succ 0, \quad e_x(k - \kappa) \neq 0. \quad (40)$$

Consequently, to achieve asymptotic stability, we intend to find $P$ and $\mathcal{K}(\kappa)$ based on the dissipativity condition and the performance objective [defined in (33)]

$$V(k - \kappa + 1) - V(k - \kappa) + \Delta f(k - \kappa)^T \Delta f(k - \kappa) - \gamma^2 d(k_{\kappa+n_{d}}) \leq 0. \quad (41)$$

By using the dynamics of the error equation and rearranging the terms, we can write

$$\mathcal{L}_1 = A(k)^T P A(k) - P + \mathcal{C}_1(k)^T \mathcal{C}_1(k)$$

$$\mathcal{L}_2 = \tilde{B}(k)^T P \tilde{B}(k) + \mathcal{D}_1(k)^T \mathcal{D}_1(k)$$

$$\mathcal{L}_3 = \mathcal{D}_1(k)^T P \mathcal{C}_1(k) + \mathcal{D}_1(k)^T \mathcal{D}_1(k) - \gamma^2 \mathcal{I}_{n_{\text{d}}}$$

$$[e_x(k - \kappa - 1)^T \mathcal{L}_1 e_x(k - \kappa - 1) \leq 0. \quad (45)$$

Equivalently, we can write

$$[\tilde{A}(k)^T \tilde{B}(k)^T \mathcal{C}_1(k)^T \mathcal{D}_1(k)^T \mathcal{I}_{n_{\text{f}}} \mathcal{C}_1(k) \mathcal{D}_1(k)]$$

$$[\mathcal{I}_{n_{\text{f}}} \mathcal{I}_{n_{\text{d}}} \mathcal{I}_{n_{\text{d}}} + \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} 0 & \gamma^2 \mathcal{I}_{n_{\text{d}}} \end{bmatrix} \leq 0. \quad (46)$$

Applying the Schur decomposition and a weighted over-bound on $P_{\text{d}}$ by $P - \hat{G}^T \hat{G} - \gamma^2 \mathcal{I}_{\text{d}}$, we arrive to the matrix inequality in (37).

Remark 1: The disturbance rejection problem can also be formulated as a minimization problem considering $\gamma$ as a decision parameter. Moreover, the inverse system in (18) cannot preserve affinity in the scheduling parameters anymore. Consequently, solution of the feasibility problem in (37) has to be approximated. Instead of solving the disturbance rejection problem for each $k \in \mathbb{P}$, we propose to use a grid-based method [18]. Finally, this approximate solution can be performed off-line.

IV. NUMERICAL EXAMPLE

The above inversion based and disturbance attenuating fault estimator method is illustrated on a simulation example. A third-order discrete-time LPV model of an unbalanced, nonlinear dc motor [23] is used in the sequel.

Given the discrete-time state-space LPV representation by

$$
\begin{bmatrix}
\theta(k + 1) \\
\omega(k + 1) \\
I(k + 1)
\end{bmatrix} =
\begin{bmatrix}
1 & T_{e} & 0 \\
-p_k \cdot T_{e} M g t / J & 1 - T_{e} b / J & T_{e} K / J \\
0 & -T_{e} K / L & 1 - T_{e} B / L
\end{bmatrix}
\begin{bmatrix}
\theta(k) \\
\omega(k) \\
I(k)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0.1 \\
T_{e} / L
\end{bmatrix} u(k) + \begin{bmatrix}
0 \\
0.1 \\
0
\end{bmatrix} d(k) + \begin{bmatrix}
p_k \\
0 \\
0
\end{bmatrix} f(k) \quad (47)
$$

Methods have been tested and compared when a series of step disturbance act upon the dc motor system. Results of a robust model-based technique derived form [21] is compared to a nominal and robust dynamic inversion estimation techniques. The first row of the plots in Fig. 1 shows the values of the signals $u(k), g(k)$ and $d(k)$. Simulation results in the second row of Fig. 1 depict the achieved results with the nominal and robust inversion based methods, and with the existing robust model-based fault estimator, respectively. Comparing the nominal and robust inversion based results, one can conclude that the ro-

![Simulation Results](image)

Fig. 1. Simulation results. The first row depicts the $u(k)$ (dotted line), $g(k)$ (dashed and dashdot lines) and $1000 \cdot d(k)$ (solid line). The second row is a comparative plot of nominal inversion based (grey dashed line), robust inversion based with parameter independent (dotted line) and dependent (solid line) gains, and existing robust fault estimation (grey solid line) methods.
burst solution attenuates the effect of disturbance more than the nominal one. Robust inversion based techniques with parameter varying gain, \( \lambda(k) \) slightly over-performs the observer with constant gain matrix \( \lambda \) in terms of disturbance rejection, which can also be seen at time sample 96 (differences between dotted and solid lines). Contrasting the robust inversion based alternatives (with \( \lambda \) and \( \lambda(k) \)) for FD to the existing and model-based robust observer design (grey solid line), one can see that the robust inversion based solution could react faster to show the presence of the fault, although, in this specific case the existing technique was more robust against disturbances.

V. CONCLUSION

The technical note shows a novel and robust algorithm for discrete-time LPV fault estimation. This method is proposed to use inversion based concepts and to describe how to attenuate an additive disturbance signal on the fault reconstruction error in an induced \( \mathcal{L}_2 \) norm sense. Dynamic inversion w.r.t additive fault input does not preserve the originally affine parameter varying structure, and introduces (relative degree dependent) time delay in the inverted system description. In case of an additive and exogenous disturbance signal, a robust objective can be formulated. With the help of a Luenberger fault observer, the technical note presented a robust and inversion based LPV FD method. Since the availability of fault-free scheduling signal is of paramount importance for the underlying design process, further research can be carried out to investigate the effect of impair scheduling measurement.

APPENDIX

Algorithm 2 (Relative Degree Vector Computation): Denote by \( c(i) \) the \( i^{th} \) row of \( C^{(j)} \), \( i = 1 \ldots l, j = 1 \ldots n_p \). Given the system in (9)-(10), the relative degree \( \kappa_i \) associated to the output channel \( i \) w.r.t the fault input \( E_1 \) can be algorithmized as, \( S(i) = \sum_{j=0}^{\kappa_i} p_j^{(i)} k_j^{(i)} \), \( S(i) = \sum_{j=0}^{\kappa_i} p_j^{(i)} k_j^{(i)} \sum_{j=0}^{\kappa_i} p_j^{(i)} A_{11}^{(i)} \), \( S(i) = \sum_{j=0}^{\kappa_i} p_j^{(i)} A_{11}^{(i)} \), where \( (\cdot) \) denotes the forward shift operator, until we reach \( \kappa_i \) where \( \sum_{j=0}^{\kappa_i} p_j^{(i)} E_1 \neq 0 \) but \( \forall l < \kappa_i, \sum_{j=0}^{\kappa_i} p_j^{(i)} A_{11}^{(i)} E_1 = 0 \forall p_{\kappa_i} \in \mathcal{P} \). Performing this computation to all output channel, we can define the relative degree vector \( \hat{\kappa} = [\kappa_1 \ldots \kappa_l] \). The maximal element of the vector will be denoted by \( \kappa = \max (\hat{\kappa}) \).

ACKNOWLEDGMENT

The authors thank the anonymous reviewers for helpful comments, and Aje.

REFERENCES