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Constructing KMS states from infinite-dimensional spectral triples

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ABSTRACT

We construct KMS-states from Li_1-summable semifinite spectral triples and show that in several important examples the construction coincides with well-known direct constructions of KMS-states for naturally defined flows. Under further summability assumptions the constructed KMS-state can be computed in terms of Dixmier traces. For closed manifolds, we recover the ordinary Lebesgue integral. For Cuntz–Pimsner algebras with their gauge flow, the construction produces KMS-states from traces on the coefficient algebra and recovers the Laca–Neshveyev correspondence. For a discrete group acting on its Stone–Čech boundary, we recover the Patterson–Sullivan measures on the Stone-Čech boundary for a flow defined from the Radon–Nikodym cocycle.

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1. Introduction

The construction of the JLO cocycle [25,35,36] from θ-summable spectral triples [12] has from the start been closely linked with the idea of KMS states. A θ-summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on a C^*-algebra A gives rise to a state $\phi(a) := \mathrm{Tr}(ae^{-\mathcal{D}^2})$ on A and under suitable conditions this is a KMS-state on the saturation of A by the $\mathbb{R}$-action defined from the wave operators $e^{it\mathcal{D}^2}$. By [36] the JLO-cocycle can be defined starting from this KMS-state. On the other hand, [12] shows that a finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on a C^*-algebra A defines a tracial state on A. Similar constructions were studied in [61].

The idea since then has been to understand the measure theory associated to θ-summable spectral triples in terms of 'twisted traces', and more specifically KMS states. Indeed this idea was present early in the development, [36]. Two viewpoints make it interesting to study states associated with spectral triples having specified summability degrees: the associated states obstruct summability degrees, and the states provide a notion of measure theory.

In this paper we present a construction of KMS states from Li_1-summable spectral triples. By definition, a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is Li_1-summable if and only if $e^{-t|\mathcal{D}|}$ is trace class for t large enough — a slight strengthening of being θ-summable.

It is an important observation that large classes of examples of θ-summable spectral triples are also Li_1-summable.

For the spectral triple defined from a Dirac operator on a closed manifold, our construction recovers the Lebesgue integral. For Cuntz–Pimsner algebras we also relate our construction to previous work of Laca and Neshveyev [45], and the authors [30,54]. We also examine spectral triples arising from certain Hilbert space valued cocycles on discrete groups.

In the examples we consider, the KMS-states are associated to flows that are well-suited to the geometries. This is usually not the case for the KMS-state $\phi(a) = \mathrm{Tr}(ae^{-\mathcal{D}^2})$ associated with a θ-summable spectral triple. It is our hope that

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our construction provides a more natural approach to the KMS-states appearing in the JLO-cocycle and that in the future it will have a bearing on the index theory of Li_1-summable spectral triples.

1.1. Main results

We now state our main results. All our results make sense for general semifinite spectral triples, and so we fix a semifinite trace $\mathcal{T}$ for this discussion.

First, we state the main technical construction of KMS-states from Li_1-summable spectral triples. After that, we state the implications of this construction to more specific examples. We use the notation $P_{\mathcal{D}}$ for the non-negative spectral projection of $\mathcal{D}$, i.e. $P_{\mathcal{D}} := \chi_{[0,\infty)}(\mathcal{D})$. If for some $\beta_{\mathcal{D}} \geq 0$, $\mathcal{T}(P_{\mathcal{D}} \mathrm{e}^{-t\mathcal{D}})$ **is finite for** $t > \beta_{\mathcal{D}}$ **and diverges as** $t \searrow \beta_{\mathcal{D}}$, we say that $\mathcal{D}$ has positive $\mathcal{T}$-essential spectrum. We define the C^*-algebra $A_{\mathcal{D}}$ as the saturation of A under the action of the wave group $\mathrm{e}^{it\mathcal{D}}$, that is

$$A_{\mathcal{D}} := C^* \left(\cup_{t \in \mathbb{R}} \sigma_t(A) \right), \quad \text{where} \quad \sigma_t(a) := \mathrm{e}^{it\mathcal{D}} a \mathrm{e}^{-it\mathcal{D}}.$$

At this stage, we formulate our results in terms of $A_{\mathcal{D}}$. In Section 3.2 we refine the construction to a smaller C^*-algebra. In examples, the construction often applies to A directly. Recall from [3, Definition 5.3.1] that a state ϕ on an $\mathbb{R}$-C^*-algebra $\sigma : \mathbb{R} \curvearrowright A$ is said to be KMS at inverse temperature β if $\phi(ab) = \phi(\sigma_{-i\beta}(b)a)$ for a, b from an $\mathbb{R}$-invariant norm dense *-subalgebra of A. If ϕ is a state on an $\mathbb{R}$-von Neumann algebra $\sigma : \mathbb{R} \curvearrowright A$ we say that it is KMS if the same condition holds on an $\mathbb{R}$-invariant σ-weakly dense *-subalgebra of A.

The following theorem is the main result of the paper.

Theorem 1. *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ *be a unital* Li_1-summable semifinite spectral triple such that $\mathcal{D}$ has positive $\mathcal{T}$-essential spectrum (see Definition 3.1) and is β-analytic (see Definition 3.18). Define

$$\beta_{\mathcal{D}} := \inf\{t > 0 : \mathcal{T}(P_{\mathcal{D}} \mathrm{e}^{-t\mathcal{D}}) < \infty\}.$$

For any extended limit $\omega \in L^\infty(\beta_{\mathcal{D}}, \infty)^*$ *as* $t \to \beta_{\mathcal{D}}$ *(see Definition 3.14), we define the state* ϕ_ω *on* $A_{\mathcal{D}}$ *as*

$$\phi_\omega(a) := \omega\text{-}\lim_{t \to \beta} \frac{\mathcal{T}(P_{\mathcal{D}} a \mathrm{e}^{-t\mathcal{D}})}{\mathcal{T}(P_{\mathcal{D}} \mathrm{e}^{-t\mathcal{D}})}.$$

Then ϕ_ω *is a KMS-state at inverse temperature* $\beta_{\mathcal{D}}$ *for the* $\mathbb{R}$-action defined from σ_t. In particular, if $\beta_{\mathcal{D}} = 0$ then ϕ_ω is a tracial state on A.

If $\beta_{\mathcal{D}} = 0$, *and there is a decreasing function* $\psi : [0, \infty) \to (0, \infty)$ *with regular variation of index* -1, *satisfying the conditions* (4.2) *and* (4.3), *and for some* $d > 0$ *we have that* $\mu_{\mathcal{T}}(t, P_{\mathcal{D}} \mathcal{D}) \sim \psi(t)^{-1/d}$ *as* $t \to \infty$, *then for any exponentiation invariant extended limit* ω *as* $t \to \infty$,

$$\phi_{\tilde{\omega}}(a) = \mathcal{T}_{\omega, \psi}(P_{\mathcal{D}} a(1 + \mathcal{D}^2)^{-d/2}),$$

where $\tilde{\omega}$ *is an extended limit as* $t \to 0$ *defined in Theorem 4.9, and* $\mathcal{T}_{\omega, \psi}$ *is the Dixmier trace defined from* $\mathcal{T}$ *and* ω *on the weak ideal* $\mathcal{L}_\psi(\mathcal{N}) := \{T \in \mathbb{K}_{\mathcal{N}} : \mu_{\mathcal{T}}(t, T) = O(\psi(t))\}$.

The first part of this result can be found as Corollary 3.21 in the body of the text and the second part as Corollary 4.10.

Remark 1.1. If $\beta_{\mathcal{D}} = 0$, any unital Li_1-summable semifinite spectral triple with $\mathcal{T}(P_{\mathcal{D}}) = \infty$ has positive $\mathcal{T}$-essential spectrum and is β-analytic. Therefore, Theorem 1 shows that any unital Li_1-summable semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ with $\mathcal{T}(P_{\mathcal{D}} \mathrm{e}^{-t\mathcal{D}}) < \infty$ for $t > 0$ and $\mathcal{T}(P_{\mathcal{D}}) = \infty$ gives rise to a tracial state on A. This extends a result of Voiculescu [61, Proposition 4.6]. For details on this case, see Theorem 3.22.

The following three results compute the KMS-state in specific examples.

Theorem 2. *Let* M *be a closed Riemannian manifold,* $\mathcal{A} := C^\infty(M)$, $\mathcal{D}$ *be a Dirac operator on a Clifford bundle* $S \to M$ *and* $\mathcal{H} := L^2(M, S)$. *Then the KMS-state* ϕ_ω *constructed in Theorem 1 is independent of* ω *and is a tracial state on* $C(M)$ *that takes the form*

$$\phi_\omega(a) = \fint_M a \, \mathrm{d}V,$$

where $\mathrm{d}V$ *denotes the volume measure defined from the Riemannian metric on* M *and* $\fint$ *the normalized integral.*

This result appears as Theorem 5.1 in the body of the text.

Theorem 3. *Let* A *be a unital* C^*-algebra, E *be a strictly W-regular fgp bi-Hilbertian bimodule (see Definitions 2.21 and 2.25) and* $(\mathcal{O}_E, \Xi_A, \mathcal{D})$ *the associated unbounded* (O_E, A)-cycle as in [30]. If τ is a positive trace on A, then the semifinite spectral triple $(\mathcal{O}_E, \Xi_A \otimes_A L^2(A, \tau), \mathcal{D} \otimes 1_A, (\mathrm{End}_A^*(\Xi_A) \otimes 1)'', \mathrm{Tr}_\tau)$ is Li_1-summable.

Moreover, if τ is critical for E (see Definition 6.2), the assumptions in Theorem 1 are satisfied and the state ϕ_ω is KMS for the gauge action on O_E. If τ satisfies the Laca–Neshveyev condition for $\alpha \geq 0$ (see Definition 6.7), then ϕ_ω is independent of ω and takes the form $\phi_\omega = \phi_{LN,\tau}$ where $\phi_{LN,\tau}$ is the KMS-state defined from τ via the Laca–Neshveyev correspondence.

This result is found in Section 6. We also discuss extensions of these results to more general A–A-correspondences in Section 6.3 dispensing the assumption of strict W-regularity.

Theorem 4. *Let Γ be a discrete group and $c : \Gamma \to \mathcal{H}_0$ a Hilbert space valued proper 1-cocycle defining a length function of at most exponential growth. The semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ constructed from c in Section 2.2.4 is an Li_1-summable semifinite spectral triple on $C_b(\Gamma) \rtimes \Gamma$. Moreover, if c is critical (see Definition 3.10) the assumptions of Theorem 1 are satisfied and the associated KMS-state ϕ_ω on $C(\partial_{SC}\Gamma) \rtimes \Gamma$ is given by*

$$\phi_\omega\left(\sum_{g \in \Gamma} a_g \lambda_g\right) = \int_{\partial\Gamma} a_e \, \mathrm{d}\mu_\omega,$$

where μ_ω is a quasi-invariant Patterson–Sullivan measure on the Stone–Cech boundary $\partial_{SC}\Gamma$. The state ϕ_ω extends to a KMS-state on the von Neumann algebra $L^\infty(\partial_{SC}\Gamma, \mu_\omega)\overline{\rtimes}\Gamma$ where it is KMS with inverse temperature 1 for the $\mathbb{R}$-action defined from the Radon–Nikodym cocycle

$$\sigma_t\left(\sum_{g \in \Gamma} a_g \lambda_g\right) := \sum_{g \in \Gamma} \left(\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}\right)^{it} a_g \lambda_g.$$

This result appears as Theorem 5.10. Our method extends to proper quasi-cocycles, and as such would allow for the construction of KMS-states from semifinite spectral triples with possible K-homological content on a-TT-menable groups.

Remark 1.2. We will prove that the spectral triple of a length function (which is K-homologically trivial) gives rise to the same KMS state as that appearing in Theorem 4.

1.2. Connection to some earlier work

Here we show how our approach relates to some results obtained by Connes in [13, Section IV.8.α, Theorem 4]. Connes proves that θ-summable Fredholm modules can be lifted to θ-summable spectral triples. We show that Connes' result can be extended to Li_s-summability for $0 < s \leq 1$, and discuss obstructions to summability properties of K-homology classes. For terminology and notations concerning summability and operator ideals, the reader is referred forward to Section 2.1.

Recall [5,15] that a semifinite Fredholm module is a collection $(\mathcal{A}, \mathcal{H}, F, \mathcal{N}, \mathcal{T})$ where $\mathcal{A}$ acts on the Hilbert space $\mathcal{H}$ by operators from $\mathcal{N}$ and $F \in \mathcal{N}$ is an operator with $a(F - F^*), a(F^2 - 1), [F, a] \in \mathbb{K}_\mathcal{T}$ for all $a \in \mathcal{A}$. We say that $(\mathcal{A}, \mathcal{H}, F, \mathcal{N}, \mathcal{T})$ is unital if $\mathcal{A}$ acts unitally. A unital semifinite Fredholm is said to be Li_s-summable if $[F, a] \in \mathrm{Li}_s(\mathcal{T})$ for all $a \in \mathcal{A}$ and $F^2 - 1, F - F^* \in \mathrm{Li}_{2s}(\mathcal{T})$. If the same conditions holds with $\mathrm{Li}_s(\mathcal{T})$ replaced by $\mathcal{L}^p(\mathcal{T})$, and $\mathrm{Li}_{2s}(\mathcal{T})$ by $\mathcal{L}^{p/2}(\mathcal{T})$, we say that $(\mathcal{A}, \mathcal{H}, F, \mathcal{N}, \mathcal{T})$ is p-summable. If $(\mathcal{A}, \mathcal{H}, F, \mathcal{N}, \mathcal{T})$ is a semifinite Fredholm module we say that a semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is a lift if $F - \mathrm{sign}(\mathcal{D}) \in \mathbb{K}_\mathcal{T}$.

Theorem 5. *Let $s \in (0, 1]$ and $(\mathcal{A}, \mathcal{H}, F, \mathcal{N}, \mathcal{T})$ be a unital semifinite Li_s-summable Fredholm module with $F^2 = 1$ and $F = F^*$. Assume that $\mathcal{A}$ is countably generated. Then there is a self-adjoint operator $\mathcal{D}$ affiliated with $\mathcal{N}$ making $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ into a unital semifinite Li_s-summable spectral triple with*

$$F = F_\mathcal{D} := \mathcal{D}|\mathcal{D}|^{-1}.$$

Moreover, $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ satisfies that $a \, \mathrm{Dom}(|\mathcal{D}|^{1/s}) \subseteq \mathrm{Dom}(|\mathcal{D}|^{1/s})$ and $[|\mathcal{D}|^{1/s}, a]$ has a bounded extension for all $a \in \mathcal{A}$.

This theorem is found in [13, Section IV.8.α, Theorem 4] in the special case $s = 1/2$ and $\mathcal{N} = \mathbb{B}(\mathcal{H})$. We will not give the full details of the proof in the general case, but merely indicate how Connes' proof extends. The starting point of Connes' proof is a reduction to the case that $\mathcal{A}$ contains F and is generated by a countable group of unitaries Γ generated by a countable set of unitaries $(u^\mu)_{\mu \in \mathbb{N}}$. This argument extends to a general von Neumann algebra $\mathcal{N}$. Connes introduces the operator

$$G := \sum_{\mu \in \mathbb{N}} \frac{[F, u^\mu]^*[F, u^\mu]}{2^\mu \|[F, u^\mu]^*[F, u^\mu]\|_{\mathrm{Li}_1}}.$$

Since $[F, u^\mu] \in \mathrm{Li}_{1/2}$ for all μ, the series converges in Li_1. The proof proceeds by using an average procedure Θ over the group Γ applied to G and Connes proves that $\mathcal{D} := F\Theta(G)^{-1/2}$ fulfils the statement of the theorem. For general $s \in (0, 1]$, the proof goes mutatis mutandis using the operator

$$G_s := \sum_{\mu \in \mathbb{N}} \frac{([F, u^\mu]^*[F, u^\mu])^{2s}}{2^\mu \|([F, u^\mu]^*[F, u^\mu])^{2s}\|_{\mathrm{Li}_1}} \in \mathrm{Li}_1(\mathcal{T}),$$

and setting $\mathcal{D} := F\Theta(G)^{-s}$.

In the special case $s = 1$, we obtain that $(\mathcal{A}, \mathcal{H}, F, \mathcal{N}, \mathcal{T})$ lifts to a unital semifinite Li_s-summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ which is Lipschitz regular, i.e. for all $a \in \mathcal{A}$ the commutator $[|\mathcal{D}|, a]$ is bounded.

The lifting theorem for Li_s-summable spectral triples (Theorem 5) stands in sharp contrast to the finitely summable setup, or even the $\mathrm{Li}_{(0),s}$-summable setup. The two upcoming theorems show that a statement as in Theorem 5 could not extend to the ideal $\mathrm{Li}_{(0),1}$.

Theorem 6. *Let A be a unital C*-algebra with no tracial states and $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a unital semifinite spectral triple on A defining a non-trivial class in $KK_1(A, \mathbb{K}_{\mathcal{T}})$. Then $P_{\mathcal{D}}(i \pm \mathcal{D})^{-1} \notin \mathrm{Li}_{(0),1}(\mathcal{T})$.*

Proof. Consider a unital C*-algebra A and an $\mathrm{Li}_{(0),1}$-summable unital semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ on A defining a non-trivial class in $KK_1(A, \mathbb{K}_{\mathcal{T}})$. In particular, $\mathcal{T}(P_{\mathcal{D}}) = \infty$; otherwise $P_{\mathcal{D}} \in \mathbb{K}_{\mathcal{T}}$ which contradicts the non-triviality of the $KK_1(A, \mathbb{K}_{\mathcal{T}})$-class defined by $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$. By Remark 1.1 all assumptions of Theorem 1 reduces to $\mathcal{T}(P_{\mathcal{D}}) = \infty$ in the $\mathrm{Li}_{(0),1}$-summable case. Therefore, the existence of $\mathrm{Li}_{(0),1}$-summable unital semifinite spectral triples on A being non-trivial in KK implies that A admits a tracial state. This argument shows that if A admits no tracial states, it admits no $\mathrm{Li}_{(0),1}$-summable unital semifinite spectral triple. In fact, a careful inspection of the results used show that as soon as there is a unital semifinite spectral triple on A with $P_{\mathcal{D}}(i \pm \mathcal{D})^{-1} \in \mathrm{Li}_{(0),1}(\mathcal{T})$, there is an associated tracial state on A. The theorem follows. $\square$

There are several C*-algebras carrying no traces, for instance any purely infinite C*-algebra. Using Theorems 1 and 6, we will give an example of a finitely summable Fredholm module that can not lift to an $\mathrm{Li}_{(0),s}$-summable spectral triple. In particular, lifting of finite summability and $\mathrm{Li}_{(0),s}$-summability fails in general.

Theorem 7. *There is a C*-algebra A with $K^1(A) \neq 0$, such that for a dense *-subalgebra $\mathcal{A} \subseteq A$ we can represent any $x \in K^1(A)$ by a Fredholm module $(\mathcal{A}, \mathcal{H}_x, F_x)$ with $F_x^2 = 1$, $F_x^* = F_x$ and for any $a \in \mathcal{A}$, $[F_x, a]$ is of finite rank. Moreover, any lift $(\mathcal{A}, \mathcal{H}_x, \mathcal{D}_x)$ of $(\mathcal{A}, \mathcal{H}_x, F_x)$ will satisfy that $(1 + \mathcal{D}_x^2)^{-1/2} \notin \mathrm{Li}_{(0),1}(\mathcal{H})$.*

Proof. Consider the Cuntz algebra $A = O_N$ and $\mathcal{A}$ the *-algebra generated by isometries $S_1, S_2, \ldots, S_N \in O_N$ with orthogonal ranges. It is a well known fact that $K^1(O_N) \cong \mathbb{Z}/(N-1)\mathbb{Z} \neq 0$. By [29], we can represent the generator of $K^1(O_N) \cong \mathbb{Z}/(N-1)\mathbb{Z}$ by the Fredholm module $(\mathcal{A}, L^2(O_N, \phi), 2P - 1)$ where ϕ is the KMS-state on O_N and P is the orthogonal projection onto the closed linear span of S_μ, where μ ranges over all finite words on the alphabet $\{1, \ldots, N\}$. By the results of [29, Section 2.2], $[2P - 1, a] = 2[P, a]$ is finite rank for all $a \in \mathcal{A}$. The first statement of the theorem follows.

There are no tracial states on O_N since

$$1_{O_N} = \frac{1}{N-1}(N-1)1_{O_N} = \frac{1}{N-1}\left(\sum_{j=1}^{N} S_j^* S_j - \sum_{j=1}^{N} S_j S_j^*\right) = \frac{1}{N-1}\sum_{j=1}^{N}[S_j^*, S_j].$$

We can now deduce the second statement of the theorem from Theorem 6. $\square$

Remark 1.3. It is not of importance that $K^1(O_N)$ is torsion for the argument in Theorem 7 to work. In [29], non-torsion examples satisfying the conclusions of Theorem 7 can be found. The reader should also note that the proof of Theorem 7 obstructs all lifts $(\mathcal{A}, \mathcal{H}_x, \mathcal{D}_x)$ of $(\mathcal{A}, \mathcal{H}_x, F_x)$ with $P_{\mathcal{D}_x}(1 + \mathcal{D}_x^2)^{-1/2} \in \mathrm{Li}_{(0),1}(\mathcal{H})$.

In the nonunital case, the techniques of [4] will likely be required. The substantial technical considerations in the nonunital case goes beyond this paper, and is left to future work.

1.3. Structure of the paper

Section 2 recalls the basics of (unital) semifinite spectral triples and their summability. We also recall our main examples from the literature in this section for later use.

Section 3 presents our construction of KMS states from Li_1-summable spectral triples. We close the section by discussing connections to modular spectral triples. We consider the case $\beta_{\mathcal{D}} = 0$ in Section 4 and compute the tracial states constructed in Section 3 by means of Dixmier traces. In Section 5 we apply the techniques of Section 3 to the examples.

The final Section 6 examines the construction of KMS states for Cuntz–Pimsner algebras. In this case we apply our ideas to derive obstructions to the existence of fgp bi-Hilbertian bimodule structures compatible with the underlying correspondence of the Cuntz–Pimsner algebra.

1.4. Notations

$\mathcal{N}$	semifinite von Neumann algebra
$\mathcal{T}$	positive, faithful, normal, semifinite trace on $\mathcal{N}$
$\mathbb{K}_{\mathcal{N}}$	ideal of $\mathcal{T}$-compact operators
$\mathrm{End}_A^*(X)$	C^*-algebra of adjointable endomorphisms of an A-Hilbert C^*-module X
$\mathbb{K}_A(X)$	C^*-algebra of compact endomorphisms of an A-Hilbert C^*-module X
$\mathcal{A}$	$*$-algebra
$\mathcal{A}'$	the commutant of an algebra $\mathcal{A}$
A	C^*-closure of an algebra $\mathcal{A}$
$P_{\mathcal{D}} := \chi_{[0,\infty)}(\mathcal{D})$	non-negative spectral projection of an operator $\mathcal{D}$
$A_{\mathcal{D}}$	saturation of $A/\{a \in A : P_{\mathcal{D}} a P_{\mathcal{D}} \in \mathbb{K}_{\mathcal{N}}\}$ under the action of the wave group $\mathrm{e}^{it\mathcal{D}}$
$F_{\mathcal{D}} := 2P_{\mathcal{D}} - 1$	
$\mu_{\mathcal{T}}(\cdot, T)$	singular values function of an operator T affiliated with $\mathcal{N}$
$n_{\mathcal{T}}(\cdot, T)$	distribution function of an operator T affiliated with $\mathcal{N}$
$\mathrm{Li}_s(\mathcal{T}), \mathrm{Li}_{(0),s}(\mathcal{T})$	ideals of compact operators in Definition 2.6
$\mathcal{L}_\psi(\mathcal{T}), \mathcal{L}_{(0),\psi}(\mathcal{T})$	ideals of compact operators in Definition 2.6
$\mathcal{T}_{\omega,\psi}(T)$	Dixmier trace on $\mathcal{L}_\psi(\mathcal{T})$
$\mathfrak{T}(A)$	set of positive traces on a unital C^*-algebra A
$L^\infty(a, \infty),\ a \geq 0$	space of essentially bounded functions on (a, ∞) equipped with the essential supremum norm
$C_0(a, \infty),\ a \geq 0$	subspace of $L^\infty(a, \infty)$ of all continuous functions vanishing at infinity
$\omega\text{-}\lim_{t\to\infty} f(t)$	value of an extended limit ω on a function f
$\ell^\infty(\mathbb{N})$	space of bounded sequences equipped with the supremum norm
$\omega\text{-}\lim_{k\to\infty} x_k$	value of an extended limit ω on a sequence x
$\mathfrak{L}_g$	transfer operator defined by formula (2.13)
$f \sim g$	for two functions or sequences f and g if $f = g + o(f)$ and $g = f + o(g)$

2. Preliminaries

Before entering into the body of the paper, we recall some basic definitions that we will require and provide some examples that motivated this work. These examples will be studied further in the later sections of the paper. The results will be formulated for semifinite spectral triples. We do however remark that there are several examples of 'vanilla' spectral triples that will be used throughout the paper.

2.1. Semifinite spectral triples and summability

To set the stage for the paper, we summarize the basic definitions and properties of semifinite spectral triples. The reader familiar with semifinite spectral triples and symmetrically normed operator ideals can skip this subsection.

We let $\mathcal{N}$ denote a semifinite von Neumann algebra and we fix a positive, faithful, normal, semifinite trace $\mathcal{T}$ on $\mathcal{N}$. The $\mathcal{T}$-compact operators are denoted by $\mathbb{K}_{\mathcal{N}}$. The C^*-algebra $\mathbb{K}_{\mathcal{N}}$ can be defined as the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\mathcal{T}(E) < \infty$. Equivalently, one can define $\mathbb{K}_{\mathcal{N}} := \{T \in \mathcal{N} : \mu_{\mathcal{T}}(t, T) = o(1) \text{ as } t \to \infty\}$ where the singular value function $\mu_{\mathcal{T}}(t, T)$ is defined as

$$\mu_{\mathcal{T}}(t, T) := \inf\{\|T(1 - E)\|_{\mathcal{N}} : \text{ where } E \in \mathcal{N} \text{ is a projection with } \mathcal{T}(E) \leq t\}. \tag{2.1}$$

Definition 2.1. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ consists of

- A $*$-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$ as operators in $\mathcal{N} \subseteq \mathbb{B}(\mathcal{H})$, that is, we have a specified $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{N}$. $\mathcal{D} : \mathrm{dom}\,\mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ which is affiliated with $\mathcal{N}$ such that for all $a \in \mathcal{A}$ we have $a \cdot \mathrm{dom}\mathcal{D} \subset \mathrm{dom}\mathcal{D}$ and

    1. $[\mathcal{D}, \pi(a)] := \mathcal{D}\pi(a) - \pi(a)\mathcal{D}$ initially defined on $\mathrm{dom}(\mathcal{D})$ is bounded in operator norm.
    2. $\pi(a)(1 + \mathcal{D}^2)^{-1/2} \in \mathbb{K}_{\mathcal{N}}$.

Remark 2.2. Sometimes we write a spectral triple as a collection of three objects $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. In this case, it is implicitly assumed that $\mathcal{N} = \mathbb{B}(L^2(M, S))$ and $\mathcal{T}$ is the standard trace.

Remark 2.3. If in addition to the data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ we have specified an operator $\gamma \in \mathbb{B}(\mathcal{H})$ with $\gamma = \gamma^*$, $\gamma^2 = 1$, $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$ on $\mathrm{Dom}(\mathcal{D})$, and for all $a \in \mathcal{A}$ we have $\gamma\pi(a) = \pi(a)\gamma$, we call the semifinite spectral triple even, or sometimes graded. If γ has not been specified, we say that the semifinite spectral triple is odd, or ungraded. This distinction plays an important role in the topological properties of the spectral triple, but since this paper deals with measure theory it will not play a role in this paper.

Remark 2.4. We will nearly always dispense with the representation π, treating $\mathcal{A}$ as a subalgebra of $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$.

Remark 2.5. In the sequel we assume that the algebra $\mathcal{A}$ is unital and that $1 \in \mathcal{A}$ acts as the identity of the Hilbert space. In particular, the operator $(1 + \mathcal{D}^2)^{-1/2}$ is a $\mathcal{T}$-compact operator. To emphasize this assumption, we refer to the data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ as a *unital* semifinite spectral triple.

Examples of semifinite spectral triples often satisfy a finer summability structure, i.e. a refinement of the condition $(1 + \mathcal{D}^2)^{-1/2} \in \mathbb{K}_\mathcal{N}$. We formulate such conditions in terms of symmetrically quasi-normed operator ideals. We will use the Schatten ideals, the Li-ideals and more generally weak ideals.

Definition 2.6. Let $\mathcal{N}$ denote a semifinite von Neumann algebra and $\mathcal{T}$ a positive, faithful, normal, semifinite trace on $\mathcal{N}$. For parameters $p, d \in [1, \infty)$ and $s > 0$ we define the following operator ideals.

- $\mathcal{L}^p(\mathcal{T}) := \{T \in \mathbb{K}_\mathcal{N} : \mu_\mathcal{T}(\cdot, T) \in L^p(0, \infty)\}$.
- $\mathcal{L}^{(d, \infty)}(\mathcal{T}) := \{T \in \mathbb{K}_\mathcal{N} : \mu_\mathcal{T}(\cdot, T) = O(t^{-1/d}) \text{ as } t \to \infty\}$.
- $\mathrm{Li}_s(\mathcal{T}) := \{T \in \mathbb{K}_\mathcal{N} : \mu_\mathcal{T}(t, T) = O((\log(t))^{-s}) \text{ as } t \to \infty\}$.
- $\mathrm{Li}_{(0),s}(\mathcal{T}) := \{T \in \mathbb{K}_\mathcal{N} : \mu_\mathcal{T}(t, T) = o((\log(t))^{-s}) \text{ as } t \to \infty\}$.
- If $\psi : [0, \infty) \to (0, \infty)$ is a decreasing function satisfying that $\sup_{t>0} \frac{\psi(t)}{\psi(2t)} < \infty$, we define the associated weak ideal

$$\mathcal{L}_\psi(\mathcal{T}) := \{T \in \mathbb{K}_\mathcal{N} : \mu_\mathcal{T}(t, T) = O(\psi(t))\},$$

and its separable subspace

$$\mathcal{L}_{(0),\psi}(\mathcal{T}) := \{T \in \mathbb{K}_\mathcal{N} : \mu_\mathcal{T}(t, T) = o(\psi(t))\},$$

The condition $\sup_{t>0} \frac{\psi(t)}{\psi(2t)} < \infty$ guarantees that $\mathcal{L}_\psi(\mathcal{T})$ is a vector space, and in fact even a quasi-Banach space in the quasi-norm $\| \cdot \|_\mathcal{N} + \| \cdot \|_{\mathcal{L}_\psi}$, where

$$\|T\|_{\mathcal{L}_\psi} := \sup_{t>0} \frac{\mu_\mathcal{T}(t, T)}{\psi(t)}.$$

Note that $\mathrm{Li}_s(\mathcal{T}) = \mathcal{L}_\psi(\mathcal{T})$ and $\mathrm{Li}_{(0),s}(\mathcal{T}) = \mathcal{L}_{(0),\psi}(\mathcal{T})$ for $\psi(t) := (\log(2 + t))^{-s}$. It is immediate from the definition that $\mathcal{L}^p(\mathcal{T}) \subseteq \mathrm{Li}_{(0),s}(\mathcal{T})$ for any p and s. More generally, if $\psi_1, \psi_2 : [0, \infty) \to (0, \infty)$ are two decreasing functions satisfying that $\sup_{t>0} \frac{\psi_j(t)}{\psi_j(2t)} < \infty$, then $\mathcal{L}_{\psi_1}(\mathcal{T}) \subseteq \mathcal{L}_{\psi_2}(\mathcal{T})$ as soon as $\psi_1 = O(\psi_2)$.

Remark 2.7. Our definition of symmetrically normed operator ideals in the semifinite setting differs slightly from the standard definition unless $\mathcal{N}$ is atomic. In the usual definition, the symmetrically normed operator ideals are defined from operators affiliated with $\mathcal{N}$ that potentially are unbounded. Since we only use bounded operators from these ideals, we have incorporated this fact in our definition.

Definition 2.8. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a unital semifinite spectral triple.

- $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is said to be p-summable if $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^p(\mathcal{T})$.
- $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is said to be (d, ∞)-summable if $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(d, \infty)}(\mathcal{T})$.
- $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is said to be Li_s-summable if $(1 + \mathcal{D}^2)^{-1/2} \in \mathrm{Li}_s(\mathcal{T})$.
- $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is said to be $\mathrm{Li}_{(0),s}$-summable if $(1 + \mathcal{D}^2)^{-1/2} \in \mathrm{Li}_{(0),s}(\mathcal{T})$.
- $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is said to be ψ-summable if $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}_\psi(\mathcal{T})$.

The standard terminology in the literature for the special case $s = 1/2$ is to refer to $\mathrm{Li}_{1/2}$-summability as weak θ-summability and to $\mathrm{Li}_{(0),1/2}$-summability as θ-summability. Since $\mathcal{L}^{(d, \infty)}(\mathcal{T}) \subseteq \mathcal{L}^p(\mathcal{T})$ for all $p > d$, (d, ∞)-summability refines p-summability.

The notion of (d, ∞)-summability is a noncommutative generalization of being d-dimensional as the spectral triple defined from a Dirac operator on a closed d-dimensional manifold (as in Section 2.2.1) is (d, ∞)-summable. We shall see an abundance of Li_1-summable, truly noncommutative, examples where p-summability and (d, ∞)-summability fails for all p and d.

The notion of ψ-summability generalizes both (d, ∞)-summability and Li_1-summability, and appears naturally in examples of (semi-) group actions on manifolds (see Section 5.1 and [17,31]). We will make use of this notion in Section 4 where certain conditions on ψ allow one to compute the tracial state defined from a ψ-summable unital semifinite spectral triples in terms of Dixmier traces on $\mathcal{L}_\psi(\mathcal{T})$.

Remark 2.9. It is readily verified that Li_s-summability is equivalent to

$$\mathcal{T}(\mathrm{e}^{-t|\mathcal{D}|^{1/s}}) < \infty, \quad \text{for } t > t_0 \text{ for some critical value } t_0,$$

and that $\mathrm{Li}_{(0),s}$-summability is equivalent to

$$\mathcal{T}(\mathrm{e}^{-t|\mathcal{D}|^{1/s}}) < \infty, \quad \text{for } t > 0.$$

In particular, $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is θ-summable if and only if

$$\mathcal{T}(\mathrm{e}^{-t\mathcal{D}^2}) < \infty, \quad \text{for all } t > 0.$$

Historically, θ-summability has been studied more in depth than Li_1-summability. This can in part be explained from the two facts that the JLO-cocycle only requires θ-summability and classically, the heat operator $\mathrm{e}^{-t\mathcal{D}^2}$ is geometrically more interesting than $\mathrm{e}^{-t|\mathcal{D}|}$ to study on a manifold. The two operators $\mathrm{e}^{-t\mathcal{D}^2}$ and $\mathrm{e}^{-t|\mathcal{D}|}$ can be compared by explicit integral formulas, see [27, Chapter 4]. We will exploit the observation that large classes of examples of θ-summable spectral triples are also Li_1-summable.

In the bulk of the paper, we are interested in computing asymptotics of heat traces of the form $\mathcal{T}(B\mathrm{e}^{-t|\mathcal{D}|^{1/s}})$ for $B \in \mathcal{N}$ as t approaches a critical value. When $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is $\mathrm{Li}_{(0),s}$-summable, the critical value of t is 0, and in several classical examples (e.g. on closed manifolds) the heat trace $\mathcal{T}(B\mathrm{e}^{-t|\mathcal{D}|^{1/s}})$ admits an asymptotic expansion. The following result is useful for relating heat trace asymptotics to zeta function asymptotics in the case of the nice behaviour appearing when $t_0 = 0$.

Theorem 2.10. Let $s \in (0, 1]$. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be an $\mathrm{Li}_{(0),s}$-summable semifinite spectral triple and $B \in \mathcal{N}$. The following are equivalent.

1. There are constants $p^{\mathrm{heat}} > 0$, $\epsilon > 0$ and $c_B^{\mathrm{heat}} \in \mathbb{C}$ such that

$$\mathcal{T}(B\mathrm{e}^{-t|\mathcal{D}|^{1/s}}) = c_B^{\mathrm{heat}} t^{-sp^{\mathrm{heat}}} + O(t^{-sp^{\mathrm{heat}}+\epsilon}), \quad \text{as } t \to 0.$$

2. The ζ-function $\zeta(z; B, |\mathcal{D}|^{1/s}) := \mathcal{T}(B|\mathcal{D}|^{-z/s})$ is well-defined for large $\mathrm{Re}(z)$ and there are constants $p^\zeta > 0$, $\epsilon' > 0$, $c_B^\zeta \in \mathbb{C}$ and a function $f = f(z)$ holomorphic in the region $\mathrm{Re}(z) > sp^\zeta - \epsilon'$ such that

$$\zeta(z; B, |\mathcal{D}|^{1/s}) = \frac{1}{\Gamma(sp^\zeta)}\frac{c_B^\zeta}{z - sp^\zeta} + f(z).$$

In this case, $p^{\mathrm{heat}} = p^\zeta$ and $c_B^{\mathrm{heat}} = c_B^\zeta$. Moreover, if the conditions above hold for one $s \in (0, 1]$, it holds for all $s \in (0, 1]$.
　　If $B = 1$ and either of the conditions above hold, then $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is p^ζ-summable.

The proof of Theorem 2.10 follows by noting that $\Gamma(z)\zeta(z; B, |\mathcal{D}|^{1/s})$ is the Mellin transform of $\mathcal{T}(B\mathrm{e}^{-t|\mathcal{D}|^{1/s}})$ and using [32, Proposition 5.1].
　　Recall that we use the notation $P_\mathcal{D} := \chi_{[0,\infty)}(\mathcal{D})$ for the non-negative spectral projection of $\mathcal{D}$. Let us state a fundamental lemma on the commutators of A with the function of $\mathcal{D}$ defined by

$$F_\mathcal{D} := 2P_\mathcal{D} - 1.$$

Note that $F_\mathcal{D}$ differs from the phase $\mathcal{D}|\mathcal{D}|^{-1}$ by the $\mathcal{T}$-finite kernel projection of $\mathcal{D}$.

Lemma 2.11. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a semifinite spectral triple. Then for any $a \in A$

$$[F_\mathcal{D}, a] \in \mathbb{K}_\mathcal{N}.$$

Moreover if $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is unital, then if $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is p-summable, then $[F_\mathcal{D}, a] \in \mathcal{L}^p(\mathcal{T})$ for all $a \in \mathcal{A}$, and if $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is Li_s-summable, then $[F_\mathcal{D}, a] \in \mathrm{Li}_s(\mathcal{T})$ for all $a \in \mathcal{A}$. More generally, if $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is ψ-summable then $[F_\mathcal{D}, a] \in \mathcal{L}_\psi(\mathcal{T})$ for all $a \in \mathcal{A}$.

Proof. The proof of the operator inequality $-\|[\mathcal{D}, a]\|\,|\mathcal{D}|^{-1} \leq [F_\mathcal{D}, a] \leq \|[\mathcal{D}, a]\|\,|\mathcal{D}|^{-1}$ for invertible $\mathcal{D}$ and $a = -a^*$ is found in the proof of [57, Proposition 1]. The assertion follows from the definition of p-, Li_s- and ψ-summability, resp.
　　In the non-invertible case we replace $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ by

$$\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{H} \oplus \mathcal{H}, \mathcal{D}_\mu = \begin{pmatrix} \mathcal{D} & \mu \\ \mu & -\mathcal{D} \end{pmatrix}, M_2(\mathcal{N}), \mathcal{T} \otimes \mathrm{Tr}_{M_2}\right), \tag{2.2}$$

for $\mu \in [0, 1]$. When $\mu > 0$ we are back in the invertible case. Let Q be the strong limit $\lim_{\mu \to 0} P_{\mathcal{D}_\mu}$. The proof of [5, Proposition 2.25] shows that for any $a \in \mathcal{A}$ and all $\mu \in [0, 1]$ we have

$$[P_{\mathcal{D}_\mu} - Q, a] \in \mathcal{L}_\psi(\mathcal{T})$$

whenever $(\mu^2 + \mathcal{D}^2)^{-1/2} a \in \mathcal{L}_\psi(\mathcal{T})$. Hence $[Q, a] \in \mathcal{L}_\psi(\mathcal{T})$ as well. Writing P_0 for the projection onto $\ker(\mathcal{D})$, whenever $(\mu^2 + \mathcal{D}^2)^{-1/2} a \in \mathcal{L}_\psi(\mathcal{T})$ we necessarily have $P_0 a \in \mathcal{L}_\psi(\mathcal{T})$. Finally, the equality

$$2Q - 1 = \begin{pmatrix} F_\mathcal{D} & 0 \\ 0 & -F_\mathcal{D} \end{pmatrix} + \begin{pmatrix} -P_0 & P_0 \\ P_0 & P_0 \end{pmatrix},$$

shows that $[F_\mathcal{D}, a] \in \mathcal{L}_\psi(\mathcal{T})$. $\quad\square$

2.1.1. Semifinite spectral triples from unbounded Kasparov modules

For several kinds of C^*-algebras one can capture the noncommutative geometry through an unbounded Kasparov module. This is a bivariant generalization of spectral triples. Localizing an unbounded Kasparov module in a positive trace gives rise to a semifinite spectral triple as in Theorem 2.12. Several of the examples in this paper arises in this way. We briefly recall this construction, which has been informally used for some years.

Let A and B be unital C^*-algebras. A unital unbounded (B, A)-Kasparov module is a collection $(\mathcal{B}, X, \mathcal{D})$ where

- $\mathcal{B} \subseteq B$ is a dense $*$-subalgebra,
- X is an A-Hilbert C^*-module carrying a left action of B as adjointable operators,
- $\mathcal{D}$ is an A-linear, densely defined, self-adjoint, regular operator on X with A-compact resolvent $(i \pm \mathcal{D})^{-1} \in \mathbb{K}_A(X)$ and
- for $a \in \mathcal{B}$ the operator $[\mathcal{D}, a]$ is defined on $\mathrm{dom}(\mathcal{D})$ and is bounded in the norm on X.

If τ is a positive trace on A, we write $L^2(X, \tau) := X \otimes_A L^2(A, \tau)$ where $L^2(A, \tau)$ is the GNS-representation associated with τ.

For $\xi, \eta \in X$, we write $\Theta_{\xi,\eta}$ for the rank one operator $\Theta_{\xi,\eta}(\nu) = \xi(\eta|\nu)_A$. The von Neumann algebra $\mathcal{N}_\tau(X) := (\mathrm{End}_A^*(X) \otimes 1_A)'' \subseteq \mathbb{B}(L^2(X, \tau))$ coincides with the weak closure of the set of operator spanned by $\{\Theta_{\xi,\eta} \otimes 1_A : \xi, \eta \in X\}$ and carries a positive, normal, semifinite trace Tr_τ characterized by $\mathrm{Tr}_\tau(\Theta_{\xi,\eta} \otimes 1_A) := \tau((\eta|\xi)_A)$, see [45, Section 3]. The following theorem also appears in [48].

Theorem 2.12. *Let* $(\mathcal{B}, X_A, \mathcal{D})$ *be a unital unbounded* (B, A)-*Kasparov module and* $\tau : A \to \mathbb{C}$ *a faithful norm densely defined norm lower semicontinuous tracial weight. Then the data* $(\mathcal{B}, L^2(X, \tau), \mathcal{D} \otimes 1, \mathcal{N}_\tau(X), \mathrm{Tr}_\tau)$ *defines a semifinite spectral triple. The von Neumann algebra is* $\mathcal{N}_\tau(X) = (\mathrm{End}_A^*(X) \otimes 1)''$ *and* $\mathrm{Tr}_\tau : \mathcal{N}_\tau(X) \to \mathbb{C}$ *is the (positive faithful semifinite normal) trace dual to the normal extension of* τ *to* $A'' \subseteq \mathbb{B}(L^2(A, \tau))$.

Proof. The operator $\mathcal{D} \otimes 1$ is self-adjoint by [46, Proposition 9.10]. The commutant of $\mathcal{N}_\tau(X)$ in $X \otimes_A L^2(A, \tau)$ is the algebra A'' (acting by right multiplication). Every unitary in A'' thus preserves the domain of $\mathcal{D} \otimes 1$ and so $\mathcal{D} \otimes 1$ is affiliated to $\mathcal{N}_\tau(X)$. Plainly commutators of $\mathcal{D} \otimes 1$ with $\mathcal{B}$ remain bounded. Since we start with an unbounded Kasparov module, the operator $(1 + \mathcal{D}^2)^{-1/2} \in \mathbb{K}_A(X)$. So we can approximate $(1 + \mathcal{D}^2)^{-1/2}$ in norm by finite rank operators $\sum_j \Theta_{x_j, y_j}$ and we can take the $x_j, y_j \in X$. Hence $(1 + (\mathcal{D} \otimes 1)^2)^{-1/2} = (1 + \mathcal{D}^2)^{-1/2} \otimes 1$ is in the norm closure of the finite trace operators in $\mathcal{N}_\tau(X)$, and so $\mathcal{T}$-compact. $\quad\square$

A conceptual viewpoint is that $(\mathcal{B}, L^2(X, \tau), \mathcal{D} \otimes 1_A, (\mathrm{End}_A^*(X) \otimes 1_A)'', \mathrm{Tr}_\tau)$ is a semi-finite refinement of the unbounded Kasparov product of $(\mathcal{B}, X, \mathcal{D})$ with the Morita morphism $A \to \mathbb{K}_A(X)$ and the $*$-homomorphism $\mathbb{K}_A(X) \to \mathbb{K}_{(\mathrm{End}_A^*(X) \otimes 1_A)''}$. There is a close relationship between the semifinite index and the Kasparov product, described in [5,38].

We remark at this stage that there is to date no general theory of symmetrically quasi-normed operator ideals in Hilbert C^*-modules, and, as such, no satisfactory way of describing summability. In concrete applications, it is possible to circumvent this problem by choosing a frame on X, implicitly using the machinery of [51]. In the examples of most relevance to this paper the spectrum of $\mathcal{D}$ is discrete, and we can use the following proposition to study summability.

Proposition 2.13. *Let* $(\mathcal{B}, X, \mathcal{D})$ *be a unital unbounded* (B, A)-*Kasparov module, where* $\mathcal{D}$ *has discrete spectrum* $\sigma(\mathcal{D}) \subseteq \mathbb{R}$, *and* τ *be a positive trace on* A. *Set* $P_\lambda := \chi_{\{\lambda\}}(\mathcal{D}) \in \mathbb{K}_A(X)$ *for* $\lambda \in \sigma(\mathcal{D})$. *Then it holds that*

1. $(\mathcal{B}, L^2(X, \tau), \mathcal{D} \otimes 1_A, \mathcal{N}_\tau(X), \mathrm{Tr}_\tau)$ *is* Li_s-*summable if and only if*

$$\sum_{\lambda \in \sigma(\mathcal{D})} e^{-t|\lambda|^{1/s}} \mathrm{Tr}_\tau(P_\lambda) < \infty,$$

for t *large enough.*

2. $(\mathcal{B}, L^2(X, \tau), \mathcal{D} \otimes 1_A, \mathcal{N}_\tau(X), \mathrm{Tr}_\tau)$ *is* p-*summable if and only if*

$$\sum_{\lambda \in \sigma(\mathcal{D})} (1 + \lambda^2)^{-p/2} \mathrm{Tr}_\tau(P_\lambda) < \infty.$$

The proof follows from the following formula:

$$\mathrm{Tr}_\tau(f(\mathcal{D})) = \sum_{\lambda \in \sigma(\mathcal{D})} f(\lambda) \mathrm{Tr}_\tau(P_\lambda),$$

which holds for every positive Borel function f.

2.2. Examples

To give some further context before entering into the main construction of this paper, let us recall some well known examples that we will further explore later on in the paper. The focus in our presentation is on Li_1-summability and heat traces. We remark that the constructions in this subsection are rather lengthy, and the reader familiar with the literature can at a first read restrict themself to glancing through this subsection.

2.2.1. Dirac Operators on closed manifolds

The prototypical example of a spectral triple arises from Dirac operators on a closed Riemannian manifold M. We can work with a rather general type of Dirac operators: if $S \to M$ is a Clifford module on M and $\slashed{D}$ is a first order elliptic operator acting on $C^\infty(M, S)$ being symmetric in the L^2-inner product and $\slashed{D}^2$ is a Laplacian type operator,[1] we say that $\slashed{D}$ is a Dirac operator. In this case, the closure of $\slashed{D}$ in its graph norm defines a self-adjoint operator on $L^2(M, S)$ that we by an abuse of notation also denote by $\slashed{D}$. It is well-known that $(C^\infty(M), L^2(M, S), \slashed{D})$ is a spectral triple on $C^\infty(M)$. We summarize the main properties of its heat traces in the following proposition.

Proposition 2.14. *Let M be an n-dimensional Riemannian closed manifold, $\slashed{D}$ a Dirac operator on M, and $(C^\infty(M), L^2(M, S), \slashed{D})$ the associated spectral triple. This spectral triple is (n, ∞)-summable and for any classical zero-th order pseudo-differential operator A on S with principal symbol $a \in C^\infty(S^*M, \text{End}(S))$ and every $s \in (0, 1]$ we have*

$$\text{Tr}_{L^2(M,S)}(Ae^{-t|\slashed{D}|^{1/s}}) = \Gamma(sn+1)c_n^s t^{-sn} \int_{S^*M} \text{Tr}_S(a)\, dV + O(t^{-sn+\epsilon}), \quad \text{as } t \to 0,$$

for some dimensional constant $c_n > 0$ and $\epsilon > 0$. Here $\text{Tr}_S(a) \in C^\infty(S^*M)$ denotes the fibrewise trace of a.

The dimensional constant c_n is determined by the Weyl law for $|\slashed{D}|$ describing the ordered sequence $(\lambda_k(|\slashed{D}|))_{k \in \mathbb{N}}$ of eigenvalues as

$$\lambda_k(|\slashed{D}|) = \frac{1}{c_n \text{vol}(M)^{1/n} \text{rank}(S)^{1/n}} k^{1/n} + O(k^{1/n-\epsilon_0}),$$

for some $\epsilon_0 > 0$. The Weyl law is proven in many places, for instance [26]. The general heat trace asymptotics follows from Theorem 2.10 and [26]. We return to this example below in Example 3.5 and Section 5.1 .

Later on in the paper, we will make use of a modification of the spectral triple coming from a Dirac operator that is also compatible with non-isometric semigroup actions. Similar constructions were previously considered in [17,31].

Definition 2.15. Let $\psi : [0, \infty) \to (0, \infty)$ be a positive measurable function.

- We say that ψ is regularly varying of index ρ if for all $\lambda > 0$

$$\lim_{t \to \infty} \frac{\psi(\lambda t)}{\psi(t)} = \lambda^\rho. \tag{2.3}$$

- We say that ψ is smoothly regularly varying of index ρ if $\psi \in C^\infty$ and for any $k \in \mathbb{N}$,

$$\lim_{t \to \infty} \frac{t^k \psi^{(k)}(t)}{\psi(t)} = \rho(\rho - 1) \cdots (\rho - k + 1). \tag{2.4}$$

A regularly varying function satisfies $\sup_{t>0} \frac{\psi(t)}{\psi(2t)} < \infty$ and there is an associated weak ideal $\mathcal{L}_\psi$ as in Definition 2.6. By [31, Lemma 7.1], any smoothly regularly varying function ψ satisfies that $\partial_t^k \psi(t) = O(\psi(t)(1+t^2)^{-k/2})$ so if ψ additionally is bounded, ψ belongs to the Hörmander class S^0.

Smooth regular variation is a strengthening of having regular variation − a condition used below in Section 4 in the context of defining and computing Dixmier traces. See more also in [31]. By [2, Theorem 1.8.2], any regularly varying function asymptotically behaves like a smoothly regularly varying function. Smooth regular variation allows for defining associated classes of pseudo-differential operators and computing Dixmier traces of geometric operators by means of a Connes trace theorem, see [31, Section 7 and 9].

For a decreasing smoothly varying function $\psi : [0, \infty) \to (0, \infty)$ with $\lim_{t \to 0} \psi(t) = 0$, we define the self-adjoint operator

$$\slashed{D}_\psi := F_{\slashed{D}} \psi(|\slashed{D}|^n)^{-1}.$$

It follows from [31, Proposition 10.1] that $\slashed{D}_\psi \in L^0_{\psi^{-1}}(M, S)$ (see [31] for the meaning of this symbol) and its ψ-principal symbol is $c_S(\xi)|\xi|^{-1}\psi(|\xi|)^{-1}$, where $c_S : T^*M \to \text{End}(S)$ denotes Clifford multiplication. The next result follows from [31, Proposition 10.3].

---

[1] I.e. the symbol of $\slashed{D}^2$ coincides with the Riemannian metric as a function on T^*M.

Proposition 2.16. *Let M be an n-dimensional Riemannian closed manifold, $\not{D}$ a Dirac operator on M, and ψ as above with*

$$\psi(t)^{-1} = O(t^{1/n}), \quad \text{as } t \to \infty.$$

Then $(C^\infty(M), L^2(M, S), \not{D}_\psi)$ is a ψ-summable spectral triple whose associated K-homology class coincides with that of $(C^\infty(M), L^2(M, S), \not{D})$. If $\psi(t)^{-1} = o(t^{1/n})$, then for any $a \in C^\infty(M)$, the operator $[\not{D}_\psi, a]$ is compact with

$$\mu(t, [\not{D}_\psi, a]) = O(t^{-1/n}\psi(t)^{-1}), \quad \text{as } t \to \infty.$$

We return to the problem of computing heat traces involving $\not{D}_\psi$ below in Section 4 and Section 5.1.

The spectral triple $(C^\infty(M), L^2(M, S), \not{D}_\psi)$ does in some cases extend to a crossed product by a semigroup action. For simplicity, we consider an action of $\mathbb{N}$ by a local diffeomorphism $g : M \to M$. Following [31], we say that g acts conformally if there is a function $c_g \in C^\infty(M, \mathbb{R}_{>0})$ such that $g^*g_M = c_g g_M$ where g_M denotes the Riemannian metric on M. We say that g lifts to the Clifford bundle $S \to M$ if there is a unitary Clifford linear morphism $u_g : g^*S \to S$. For simplicity, we assume M to be connected and define $N := \#g^{-1}(\{x\})$ for some $x \in M$. Since g is a local diffeomorphism and M is connected, N is independent of the choice of x.

If $g : M \to M$ is a surjective local diffeomorphism, acting conformally and lifting to S, we can define the isometry

$$V_g : L^2(M, S) \to L^2(M, S), \quad V_g \xi := c_g^{n/4} N^{-1/2} u_g(\xi \circ g). \tag{2.5}$$

The isometry V_g satisfies the following for $a \in C(M)$:

$$V_g a V_g^* = (a \circ g) V_g V_g^*, \quad \text{and} \quad V_g^* a V_g = \mathfrak{L}_g(a),$$

where $\mathfrak{L}_g(a)(x) := \sum_{g(y)=x} a(y)$. See more in [17, Proposition 8.3]. Define $\mathcal{A}$ as the $*$-algebra generated by $C^\infty(M)$ and V_g. By [17, Proposition 8.6], the C^*-closure of $\mathcal{A}$ is the image in a representation on $L^2(M, S)$ of the Cuntz–Pimsner algebra O_{E_g} defined from $E_g := C(M)V_g$ (see more in Examples 2.30 and 6.6). The space $E_g = C(M)V_g \subseteq \mathbb{B}(L^2(M, S))$ is considered as a bimodule over $C(M)$ and is an fgp bi-Hilbertian $C(M)$-bimodule because E_g can be identified with $C(M)$ with the bimodule structure $(afb)(x) = a(x)f(x)b(g(x))$ for $a, b, f \in C(M)$, for details, see Example 2.30 or [17, Section 8].

Definition 2.17. Let $\psi : [0, \infty) \to (0, \infty)$ be a decreasing function and $g : M \to M$ a local diffeomorphism of a Riemannian manifold.

- We say that ψ and g are compatible if there is a constant $C \geq 0$ such that for all $x \in M$ and $\xi \in T_x^*M$, the differential Dg satisfies

$$\left|\psi(|(Dg)_x^T\xi|) - \psi(|\xi|)\right| \leq C\psi(|\xi|)^2.$$

- We say that g acts isometrically if Dg acts isometrically on each fibre.

The following results pose restrictions on a function ψ compatible with a local diffeomorphism which acts non-isometrically.

Proposition 2.18. *Let $\psi : [0, \infty) \to (0, \infty)$ be a decreasing function with $\lim_{t\to\infty} \psi(t) = 0$ and $g : M \to M$ a compatible local diffeomorphism of a Riemannian manifold. If ψ has regular variation, then either g acts isometrically or ψ has regular variation of index 0.*

Proof. Assume that g acts non-isometrically, we shall prove that in this case ψ has regular variation of index 0. Since ψ is decreasing, it suffices to show that $\lim_{t\to\infty} \frac{\psi(rt)}{\psi(t)} = 1$ for some $r \neq 1$ by [31, Proposition 2.15]. If g acts non-isometrically, there is a point $x \in M$ and a unit vector $\xi_0 \in T_x^*M$ such that $|(Dg)_x^T\xi_0|_{g(x)} \neq 1$. Set

$$r := |(Dg)_x^T\xi_0|_{g(x)} \neq 1.$$

Since ψ is compatible with g, there is a constant $C \geq 0$ such that

$$\left|\psi(|(Dg)_x^T\xi|) - \psi(|\xi|)\right| \leq C\psi(|\xi|)^2$$

and by setting $\xi = t\xi_0$, we arrive at the inequality

$$|\psi(rt) - \psi(t)| \leq C\psi(t)^2.$$

After dividing by $\psi(t)$, taking the limit $t \to \infty$ and using that $\lim_{t\to\infty} \psi(t) = 0$ we arrive at the desired equality $\lim_{t\to\infty} \frac{\psi(rt)}{\psi(t)} = 1$. $\square$

A prototypical example of a function with regular variation of index 0 is $\psi(t) := \log^k(2 + t)$, for $k \in \mathbb{Z}$. This function satisfies the condition $\lim_{t\to\infty} \psi(t) = 0$ appearing in Proposition 2.18 if $k < 0$. This function is compatible with any conformal local diffeomorphism and has been considered in the context of constructing spectral triples in [17,28,29,31,49].

Proposition 2.19. *Let M be an n-dimensional Riemannian closed manifold, $\slashed{D}$ a Dirac operator on $S \to M$, $g : M \to M$ a surjective local diffeomorphism acting conformally and lifting to S and $\psi : [0, \infty) \to (0, \infty)$ a decreasing function compatible with g, having smooth regular variation and satisfying $\lim_{t \to \infty} \psi(t) = 0$ and $\psi(t)^{-1} = O(t^{1/n})$ as $t \to \infty$. Let $\mathcal{A}$ denote the $*$-algebra generated by $C^\infty(M)$ and the isometry V_g from Eq. (2.5) . Then $(\mathcal{A}, L^2(M, S), \slashed{D}_\psi)$ is a unital ψ-summable spectral triple.*

This result follows in the same manner as in [17, Section 8] and [31, Theorem 10.6]. Clearly, if $\psi(t)^{-1} = O(\log(t))$ as $t \to \infty$, then $(\mathcal{A}, L^2(M, S), \slashed{D}_\psi)$ is Li_1-summable. The reader should note that the K-homology class $[(\mathcal{A}, L^2(M, S), \slashed{D}_\psi)] \in K^*(O_{E_g})$, where O_{E_g} is the Cuntz–Pimsner algebra of the module $E_g = C(M)V_g$, discussed later. The class $[(\mathcal{A}, L^2(M, S), \slashed{D}_\psi)]$ restricts to $[(C^\infty(M), L^2(M, S), \slashed{D})] \in K^*(C(M))$. Thus the class $[(C^\infty(M), L^2(M, S), \slashed{D})] \in K^*(C(M))$ obstructs $[(\mathcal{A}, L^2(M, S), \slashed{D}_\psi)] \in K^*(O_{E_g})$ being a Kasparov product with the Cuntz–Pimsner boundary extension in $KK^1(O_{E_g}, C(M))$ — we will return to study this boundary extension and its associated semifinite spectral triples below in Section 2.2.3.

2.2.2. Graph C*-algebras

A class of examples carrying interesting noncommutative geometries with a well-studied set of KMS-states is that of graph C^*-algebras. With a finite directed graph $G = (V, E)$, with edge set E and vertex set V, one associates a C^*-algebra $C^*(G)$ [53]. For simplicity we suppose that we have no sources nor sinks. The C^*-algebra $C^*(G)$ is generated by partial isometries $(S_e)_{e \in E}$ and projections $(p_v)_{v \in V}$ satisfying the relations

$$S_e^* S_e = p_{r(e)}, \quad \text{and} \quad p_v = \sum_{s(e)=v} S_e S_e^*,$$

where $r(e)$ denotes the range of the vertex e and $s(e)$ its source. The C^*-algebra $C^*(G)$ can be described as a Cuntz–Pimsner algebra in several ways, a class of C^*-algebras carrying noncommutative geometries that we will study in the next subsection. In this subsection, we focus on a construction of noncommutative geometries along an orbit in the infinite path space of G — a construction based in the model of $C^*(G)$ as a groupoid C^*-algebra. The associated noncommutative geometries come from [29]. As explained in [30] the groupoid model can be seen as a Cuntz–Pimsner model of $C^*(G)$ using the one-sided infinite path space

$$\Omega_G := \{x = e_1 e_2 \cdots \in E^{\mathbb{N}} : s(e_j) = r(e_{j+1}) \, \forall j\}.$$

The path space Ω_G is a compact Hausdorff space in the subspace topology $\Omega_G \subseteq E^{\mathbb{N}}$. It carries a shift mapping $\sigma_G : \Omega_G \to \Omega_G$, $\sigma_G(e_1 e_2 e_3 \cdots) := e_2 e_3 \cdots$. The shift mapping is a surjective local homeomorphism. We can define an étale groupoid $\mathcal{G}_G$ over Ω_G by

$$\mathcal{G}_G := \left\{ (x, n, y) \in \Omega_G \times \mathbb{Z} \times \Omega_G : \exists k \geq \max(0, -n) \text{ such that } \sigma_G^{n+k}(x) = \sigma_G^k(y) \right\}.$$

The range mapping is defined by $r(x, n, y) = x$, the source mapping as $s(x, n, y) := y$ and the product by

$$(x, n, y)(y, m, z) := (x, n + m, z).$$

The topology of $\mathcal{G}_G$ is uniquely determined by declaring the groupoid to be étale and the mappings $(x, n, y) \mapsto n$ and

$$\kappa_G(x, n, y) := \min\{k \geq \max(0, -n) : \sigma_G^{n+k}(x) = \sigma_G^k(y)\},$$

to be continuous. There is an isomorphism $\pi_G : C^*(G) \to C^*(\mathcal{G}_G)$ determined by defining $\pi_G(S_e)$ to be the characteristic function of the set $\{(x, 1, \sigma_G(x)) : x \in C_e\}$ where $C_e := \{e_1 e_2 \cdots \in \Omega_G : e_1 = e\}$. See [16].

Define the function $\alpha_0 : \{(n, k) \in \mathbb{Z} \times \mathbb{N} : n + k \geq 0\} \to \mathbb{Z}$ by

$$\alpha_0(n, k) := \begin{cases} n, & k = 0 \\ -|n| - k, & k > 0. \end{cases}$$

For a point $y \in \Omega_G$, we define the discrete set

$$\mathcal{V}_y := d^{-1}(\{y\}) = \{(x, n) : (x, n, y) \in \mathcal{G}_G\}.$$

The set $\mathcal{V}_y$ is the union of all forward orbits of all backward orbits of y under σ_G where we keep track of the lag n. Since $\mathcal{V}_y$ is a fibre of the domain mapping, point evaluation in y induces a representation $\pi_y : C^*(G) \to \mathbb{B}(\ell^2(\mathcal{V}_y))$. If G is primitive $C^*(G)$ is simple and π_y is faithful. At the level of the generators,

$$\pi_y(S_e)\delta_{(x,n)} = \begin{cases} \delta_{(ex, n+1)}, & r(x) = s(e), \\ 0, & r(x) \neq s(e). \end{cases}$$

Proposition 2.20. *Define the operator $\mathcal{D}_y$ densely on $\ell^2(\mathcal{V}_y)$ as the self-adjoint operator with*

$$\mathcal{D}_y \delta_{(x,n)} := \alpha_0(n, \kappa(x, n, y))\delta_{(x,n)}.$$

The triple $(C_c(\mathcal{G}_G), \ell^2(\mathcal{V}_y), \mathcal{D}_y)$ is an Li_1-summable spectral triple. Moreover, under the isomorphism $K^1(C^*(G)) \cong \mathbb{Z}^E/(1 - A_{\mathrm{edge}})\mathbb{Z}^E$, of odd K-homology with the cokernel of the edge adjacency matrix, the class $[(C_c(\mathcal{G}_G), \ell^2(\mathcal{V}_y), \mathcal{D}_y)]$ is mapped to the element $\delta_e \mod (1 - A_{\mathrm{edge}})\mathbb{Z}^E$ where $\delta_e \in \mathbb{Z}^E$ denotes the basis element corresponding to $e \in E$.

Proof. It is proven in [29, Theorem 5.2.3] that $(C_c(\mathcal{G}_G), \ell^2(\mathcal{V}_y), \mathcal{D}_y)$ is a spectral triple whose K-homology class corresponds to the element $\delta_e \mod (1 - A_{\text{edge}})\mathbb{Z}^E$ under $K^1(C^*(G)) \cong \mathbb{Z}^E/(1 - A_{\text{edge}})\mathbb{Z}^E$. It remains to prove that $(C_c(\mathcal{G}_G), \ell^2(\mathcal{V}_y), \mathcal{D}_y)$ is Li_1-summable. We compute that

$$\text{Tr}(e^{-t|\mathcal{D}_y|}) = \sum_{(x,n)\in\mathcal{V}_y} e^{-t(|n|+\kappa_G(x,n,y))} = \sum_{n\in\mathbb{Z}} \sum_{k=\max(0,-n)}^{\infty} \#\{(x,n) : \kappa_G(x,n,y) = k\} e^{-t(|n|+k)}.$$

However, if (x, n) is such that $\kappa_G(x, n, y) = k$ the path x is determined by y except for its first $n + k$ steps so $\#\{(x, n) : \kappa_G(x, n, y) = k\} \leq |E|^{n+k} \leq e^{\log(|E|)(|n|+k)}$. We conclude that

$$\text{Tr}(e^{-t|\mathcal{D}_y|}) < \infty \quad \text{if} \quad t > \log(|E|). \quad \square$$

We return to this example below in Example 3.6 and Section 5.2 .

2.2.3. Cuntz–Pimsner algebras

In this subsection we consider the construction of semi-finite spectral triples on Cuntz–Pimsner algebras – a broad class of examples which include both Cuntz–Krieger algebras and crossed products by $\mathbb{Z}$. Quite general techniques for constructing spectral triples for these algebras were developed in a series of papers (in rough chronological order) [28–30,54,55]. We consider the set up of [54] and [30], which provide a means of lifting data from the (unital) coefficient algebra of a bi-Hilbertian bimodule to its Cuntz–Pimsner algebra.

We start with a unital, separable C^*-algebra A, and a finitely generated projective (fgp) bi-Hilbertian bimodule E over A, i.e. a module fulfilling the conditions of the following definition.

Definition 2.21. An fgp bi-Hilbertian bimodule E over A is an A-bimodule equipped with the following structures:

- E has both left and right A-valued inner products which induce equivalent norms on E.
- The left and right actions are both injective and adjointable.
- E is finitely generated and projective as both a left and a right module.

To separate the left and the right structures, we write E_A when we want to emphasize the right module structure and $(\cdot|\cdot)_A$ for the right inner product. Similarly, $_AE$ denotes the left module defined from E and $_A(\cdot|\cdot)$ the left inner product.

The algebraic Fock space $\mathcal{F}_E^{\text{alg}}$ is the algebraic direct sum of the A-modules $E^{\otimes_A k}$. The Fock space $\mathcal{F}_E$ is defined as the right A-Hilbert C^*-module completion of $\mathcal{F}_E^{\text{alg}}$. The Cuntz–Toeplitz algebra $\mathcal{T}_E \subseteq \text{End}_A^*(\mathcal{F}_E)$ is the C^*-algebra generated by the creation operators $T_\mu \xi := \mu \otimes \xi$ for $\mu \in \mathcal{F}_E^{\text{alg}}$. The Cuntz–Pimsner algebra O_E is defined from the short exact sequence

$$0 \to \mathbb{K}_A(\mathcal{F}_E) \to \mathcal{T}_E \to O_E \to 0.$$

We call this short exact sequence the defining extension of O_E.

A set $(e_j)_{j=1}^N \subset E$ of vectors is a frame for the right module E_A if for all $e \in E$ we have $e = \sum_j e_j(e_j|e)_A$, and similarly for a left frame $(f_k)_{k=1}^N$. Since E is finitely generated and projective, there exist left and right frames and we can for simplicity assume that they have the same cardinality. For e and f in the right Hilbert module E_A, we denote the associated rank-one operator by $\Theta_{e,f} := e(f|\cdot)$. Then the frame condition can be expressed as

$$\sum_{j=1}^N \Theta_{e_j,e_j} = \text{Id}_E$$

and similarly for f_σ. The frame $(e_j)_{j=1}^N$ induces a frame for $E_A^{\otimes k}$, namely $(e_\rho)_{|\rho|=k}$ where ρ is a multi-index and $e_\rho = e_{\rho_1} \otimes \cdots \otimes e_{\rho_k}$.

We define the right Watatani index of $E^{\otimes k}$ as the element of A given by

$$e^{\beta_k} = \sum_{|\rho|=k} {}_A(e_\rho|e_\rho) = \sum_{|\rho'|=k-1} {}_A(e_{\rho'}e^\beta|e_{\rho'}). \tag{2.6}$$

The right Watatani index is positive, central and since the left action is injective, also invertible. Therefore β_k is a well defined self-adjoint central element in A. The key assumptions we make concern the asymptotic behaviour of the right Watatani indices. In [54, Section 3.2] we define an A-bilinear functional $\Phi_\infty : O_E \to A$. This functional gives us an A-valued inner product on O_E. The construction of Φ_∞ begins by defining

$$\Phi_k : \text{End}_A^*(E^{\otimes k}) \to A, \qquad \Phi_k(T) := \sum_{|\rho|=k} {}_A(Te_\rho|e_\rho). \tag{2.7}$$

Here we use the notation $\text{End}_A^*(E^{\otimes k})$ for the C^*-algebra of A-linear adjointable operators on $E^{\otimes k}$. It follows from [40, Lemma 2.16] that Φ_k does not depend on the choice of frame. We note that $e^{\beta_k} = \Phi_k(\text{Id}_{E^{\otimes k}})$. Since Φ_k is independent of the choice of frame, so is e^{β_k}. We extend the functional Φ_k to a mapping $\text{End}_A^*(\mathcal{F}_E) \to A$ by compressing along the orthogonally complemented submodule $E^{\otimes k} \subseteq \mathcal{F}_E$. To obtain a good "limiting functional" $\Phi_\infty(T) := \lim_{k\to\infty} \Phi_k(T)e^{-\beta_k}$ on the Cuntz–Toeplitz algebra, we impose the following condition on the Watatani indices.

Definition 2.22. Let E be an fgp bi-Hilbertian bimodule over the unital C^*-algebra A. We say that E is W-regular if for every $k \in \mathbb{N}$ and $\nu \in E^{\otimes k}$ there exists a $\tilde{\nu} \in E^{\otimes k}$ satisfying

$$\|e^{-\beta_n} \nu e^{\beta_{n-k}} - \tilde{\nu}\|_{E^{\otimes k}} \to 0 \quad \text{as } n \to \infty.$$

In [54] the reader can find several examples of Cuntz–Pimsner algebras for which a stronger version of W-regularity as defined in Definition 2.22 holds. There are no known examples of modules that are not W-regular. When E is W-regular, [54, Proposition 3.5] guarantees that $\lim_{k \to \infty} \Phi_k(T)e^{-\tilde{\beta}_k}$ is well defined for T from the $*$-algebra generated by the set of creation operators $\{T_\nu : \nu \in \mathcal{F}_E^{\text{alg}}\}$ and is continuous in the C^*-norm. In Section 6.3 we shall see that there are ways around W-regularity, and even the existence of an A-valued left inner product, when constructing semifinite spectral triples giving rise to KMS-states.

We thus obtain a unital positive A-bilinear functional $\Phi_\infty : \mathcal{T}_E \to A$. The functional Φ_∞ annihilates the compact endomorphisms, and descends to a well-defined functional on the Cuntz–Pimsner algebra $\mathcal{O}_E$. By an abuse of notation, we also denote this functional by $\Phi_\infty : \mathcal{O}_E \to A$. Since Φ_k and e^{β_k} do not depend on the choice of frame, neither does Φ_∞. We define the inner product

$$(S_1|S_2)_A := \Phi_\infty(S_1^* S_2), \qquad S_1, S_2 \in \mathcal{O}_E.$$

When computing these inner products, the following fact is useful.

Lemma 2.23. Let E be a W-regular fgp bi-Hilbertian bimodule. For homogeneous elements $\mu, \nu \in \mathcal{F}_E^{\text{alg}}$ we have

$$\Phi_\infty(S_\mu S_\nu^*) = \lim_{k \to \infty} {}_A(\mu | e^{-\beta_k} \nu e^{\beta_{k-|\nu|}}) = {}_A(\mu | \tilde{\nu}). \tag{2.8}$$

In particular, if $S \in \mathcal{O}_E$ is homogeneous of degree $n \neq 0$, then $\Phi_\infty(S) = 0$.

Completing $\mathcal{O}_E$ modulo the vectors of zero length (with respect to Φ_∞) yields a right A-Hilbert C^*-module that we denote by Ξ_A. The module Ξ_A carries a left action of $\mathcal{O}_E$ given by extending the multiplication action of $\mathcal{O}_E$ on itself. By considering the linear span of the image of the generators S_ν, $\nu \in \mathcal{F}_E^{\text{alg}}$, inside the module Ξ_A, we obtain an isometrically embedded and complemented copy of the Fock space. We let Q be the projection on this copy of the Fock space.

Theorem 2.24 (*Proposition 3.14 of [54]*). *Let E be a W-regular fgp bi-Hilbertian bimodule over a unital C^*-algebra A. The tuple $(\mathcal{O}_E, \Xi_A, 2Q - 1)$ is an odd Kasparov module representing the class of the defining extension*

$$0 \to \mathbb{K}_A(F_E) \to \mathcal{T}_E \to \mathcal{O}_E \to 0. \tag{2.9}$$

To construct an unbounded representative of $(\mathcal{O}_E, \Xi_A, 2Q - 1)$, we will add an additional assumption regarding the fine structure of the operation $\nu \mapsto \tilde{\nu}$ in the definition of W-regularity (see Definition 2.22). Assuming W-regularity, we can define the operator $\mathfrak{q}_k : E^{\otimes k} \to E^{\otimes k}$ by

$$\mathfrak{q}_k \nu := \tilde{\nu} = \lim_{n \to \infty} e^{-\beta_n} \nu e^{\beta_{n-k}}.$$

Definition 2.25. Let E be an fpg bi-Hilbertian bimodule over the unital C^*-algebra A. We say that E is strictly W-regular if it is W-regular and for any k, we can write $\mathfrak{q}_k = c_k P_k = P_k c_k$ where $P_k \in \text{End}_A^*(E^{\otimes k})$ is a (necessarily A-bilinear) projection and c_k is given by left-multiplication by an element in A.

Remark 2.26. As with W-regularity, the reader can in [54] find several examples of Cuntz–Pimsner algebras for which strict W-regularity holds. There are no known examples of modules that are not strictly W-regular.

Remark 2.27. If there is a decomposition $\mathfrak{q}_k = c_k P_k$ as in the definition of strict W-regularity, [30, Lemma 3.8] shows that it is unique and of a very specific form. Indeed, each c_k is central, invertible and $c_k = \Phi_k(P_k)^{-1}$. Strict W-regularity is readily verified in practice using [30, Lemma 3.8]. For instance, if β_1 is central for the module action on E, $c_k = e^{-\beta_k} = e^{-k\beta_1}$ is central for the module action on E and $P_k = 1_{E^{\otimes k}}$.

When E is strictly W-regular, an unbounded self-adjoint regular operator $\mathcal{D}_\alpha$ on Ξ_A is constructed in [30] making $(\mathcal{O}_E, \Xi_A, \mathcal{D}_\alpha)$ into an unbounded Kasparov module representing the KK-class of the defining extension (2.9). The operator $\mathcal{D}_\alpha$ is of the form

$$\mathcal{D}_\alpha = \sum_{n \in \mathbb{Z}} \sum_{r \geq \max\{0, n\}} \alpha(n, r) P_{n, r}$$

where $\alpha : \mathbb{Z} \times \mathbb{N} \to [0, \infty)$ is a function with certain Lipschitz properties (see [30, Remark 3.20]), and the $P_{n,r}$ are projections on finitely generated projective subspaces, [30]. While the particular choice of function α does not matter much, we will take the function

$$\alpha(n, r) = \begin{cases} n & n = r \\ -(2r - n) & \text{otherwise} \end{cases}.$$

The projections $P_{n,r}$ form a sequence of mutually orthogonal projections satisfying that the direct sum $\oplus_{r=\max(0,n)}^{r_0} P_{n,r}$ is the projection onto the A-linear span of $\{S_\mu S_\nu^* : |\mu|-|\nu|=n, \max(0,n) \le r \le r_0\}$. In particular, $P_{n,n}$ is the projection onto $E^{\otimes n}$ for $n \in \mathbb{N}$. More precisely, the projections are defined by

$$P_{n,r} := \begin{cases} Q_{n,r} - Q_{n,r-1}, & r > \max\{0,n\} \\ Q_{n,r}, & r = \max\{0,n\} \end{cases} \tag{2.10}$$

where the projections $Q_{n,r}$ are defined in terms of the right frame $(e_j)_{j=1}^N$ and the left frame $(f_j)_{j=1}^N$ as

$$Q_{n,r} := \sum_{|\rho|-|\sigma|=n,\, |\rho|=r} \Theta_{W_{e_\rho, c_{|\sigma|}^{-1/2} P_F f_\sigma},\, W_{e_\rho, c_{|\sigma|}^{-1/2} P_F f_\sigma}} \tag{2.11}$$

where $P_F = \oplus P_k$ is the projection on the Fock module coming from Definition 2.25. Here we have written $W_{\xi,\eta} \in \Xi_A$ for the element defined from $S_\xi S_\eta^* \in O_E$ where $\xi \in E^{\otimes r}$ and $\eta \in E^{\otimes k}$. For details, see [30, Lemma 3.10 and Proposition 3.11].

To obtain a semifinite spectral triple, we localize (O_E, Ξ_A, D_α) in a positive trace on A. Following Theorem 2.12, we consider the semifinite spectral triple

$$(O_E, L^2(\Xi_A, \tau), D_\alpha, N_\tau(\Xi_A), \mathrm{Tr}_\tau).$$

Here $L^2(\Xi_A, \tau) := \Xi_A \otimes_A L^2(A, \tau)$ and Tr_τ is the dual trace on $N_\tau(\Xi_A) := (\mathrm{End}_A^*(\Xi_A) \otimes 1)''$ which satisfies $\mathrm{Tr}_\tau(\Theta_{e,f}) = \tau((f|e)_A)$, [45,60].

Lemma 2.28. *Assume that the fgp bi-Hilbertian bimodule E is strictly W-regular and that τ is a positive trace on A. Then the semifinite spectral triple*

$$(O_E, L^2(\Xi_A, \tau), D_\alpha, N_\tau(\Xi_A), \mathrm{Tr}_\tau)$$

is Li_1-summable.

Remark 2.29. The assumptions on the existence of limiting behaviour for the Watatani indices are really just for convenience here. These assumptions relate to existence and behaviour of norm limits, but we have passed to the 'measurable setting' and so really only need weak limits. We will explore this point of view in Section 6.3.

Proof. We need to prove that the following expression is finite for t large enough:

$$\mathrm{Tr}_\tau(e^{-t|D_\alpha|}) = \sum_{n \in \mathbb{Z}} \sum_{r \ge \max\{0,n\}} e^{-t|\alpha(n,r)|} \mathrm{Tr}_\tau(P_{n,r}).$$

By definition (see (2.10)), $P_{n,r} = Q_{n,r} - Q_{n,r-1}$ when $r > \max(0,n)$ and $P_{n,r} = Q_{n,r}$ when $r = \max(0,n)$ and $Q_{n,r}$ is defined as in (2.11). Using the computations of [30, Lemma 2.8], we see that

$$\mathrm{Tr}_\tau(Q_{n,r}) = \sum_{|\rho|-|\sigma|=n,\, |\rho|=r} \tau\left((W_{e_\rho, c_{|\sigma|}^{-1/2} P_F f_\sigma} | W_{e_\rho, c_{|\sigma|}^{-1/2} P_F f_\sigma})_A\right) \tag{2.12}$$

$$= \sum_{|\rho|-|\sigma|=n,\, |\rho|=r} \tau \circ \Phi_\infty \left(S_{c_{|\sigma|}^{-1/2} P_F f_\sigma} S_{e_\rho}^* S_{e_\rho} S_{c_{|\sigma|}^{-1/2} P_F f_\sigma}^*\right)$$

$$= \sum_{|\rho|-|\sigma|=n,\, |\rho|=r} \tau\left({}_A(P_F f_\sigma | P_F f_\sigma (e_\rho|e_\rho)_A)\right).$$

Using the fact that the elements of the frame have norm bounded by 1, we see that

$$|\mathrm{Tr}_\tau(Q_{n,r})| \le \sum_{|\rho|-|\sigma|=n,\, |\rho|=r} |\tau\left({}_A(P_F f_\sigma | P_F f_\sigma (e_\rho|e_\rho)_A)\right)|$$

$$\le \sum_{|\rho|-|\sigma|=n,\, |\rho|=r} \|{}_A(P_F f_\sigma | P_F f_\sigma (e_\rho|e_\rho)_A)\| \le N^{2r-n} \le e^{\log(N)|\alpha(n,r)|},$$

where N is the number of elements in the left frame and the right frame. We can now estimate

$$\left|\mathrm{Tr}_\tau(e^{-t|D_\alpha|})\right| = \left|\sum_{n \in \mathbb{Z}} \sum_{r > \max\{0,n\}} e^{-t|\alpha(n,r)|} \mathrm{Tr}_\tau(Q_{n,r} - Q_{n,r-1})\right|$$

$$+ \left|\sum_{n \in \mathbb{Z}} \sum_{r = \max\{0,n\}} e^{-t|\alpha(n,r)|} \mathrm{Tr}_\tau(Q_{n,r})\right|$$

$$\leq 2 \sum_{n \in \mathbb{Z}} \sum_{r \geq \max\{0,n\}} e^{-t|\alpha(n,r)|} |\mathrm{Tr}_\tau(Q_{n,r})|$$

$$\leq 2 \sum_{n \in \mathbb{Z}} \sum_{r \geq \max\{0,n\}} e^{-(t-\log(N))|\alpha(n,r)|} < \infty,$$

if $t > \log(N)$. $\quad \square$

We return to Cuntz–Pimsner algebras below in Example 3.7 and Section 6 . Let us discuss a special case of Cuntz–Pimsner algebras arising on a commutative coefficient algebra.

Example 2.30. Let Y be a compact Hausdorff space and $g : Y \to Y$ a surjective local homeomorphism. We consider the module $E_g = C(Y)$ with the bimodule action

$$(afb)(x) = a(x)f(x)b(g(x)), \quad a, b \in C(Y), \ f \in E_g.$$

This is an fgp bi-Hilbertian bimodule in the inner products

$$_{C(Y)}(f_1, f_2) = f_1 \overline{f_2} \quad \text{and} \quad (f_1|f_2)_{C(Y)} := \mathfrak{L}_g(\overline{f_1} f_2),$$

where $\mathfrak{L}_g : C(Y) \to C(Y)$ is the transfer operator

$$\mathfrak{L}_g(f)(x) := \sum_{y \in g^{-1}(x)} f(y). \tag{2.13}$$

The Cuntz–Pimsner algebra O_{E_g} can be realized as a groupoid C^*-algebra as in [16] (see also [17, Theorem 3.2]) over Y. The solenoid

$$X = \{x = y_1 y_2 \cdots \in Y^{\mathbb{N}} : g(y_{k+1}) = y_k \ \forall k\},$$

equipped with the shift mapping is a Smale space under suitable assumptions on g. For its relation to O_{E_g} see [17].

The module E_g has a right frame $(e_j)_{j=1}^N$ where $e_j = \sqrt{\chi_j}$ for a partition of unity $(\chi_j)_{j=1}^N$ subordinate to an open covering $(U_j)_{j=1}^N$ of Y such that $g|_{U_j}$ is injective for all j. Using this partition of unity, one sees that $\beta_k = 0$ for all k. It follows that E_g is a strictly W-regular module.

The case that $g : M \to M$ was a surjective local diffeomorphism acting conformally was considered in Section 2.2.1. However, the spectral triple considered Proposition 2.19 on O_{E_g} differs greatly from the semifinite spectral triples considered in Lemma 2.28 — the latter are in the image of the boundary mapping in $KK_1(O_{E_g}, C(M))$ defined from Eq. (2.9) while the former is not if $[\not{D}] \neq 0 \in K^*(C(M))$.

Another class of examples already considered arises from a finite graph G as in Section 2.2.2 where the shift mapping $\sigma_G : \Omega_G \to \Omega_G$ is a surjective local homeomorphism and $C^*(G) \cong O_{E_{\sigma_G}}$. The spectral triples in Proposition 2.20 arise from the construction of Lemma 2.28 by taking the trace $\tau : C(\Omega_G) \to \mathbb{C}$ to be defined from point evaluation in y.

2.2.4. Group C*-algebras

We now turn our attention to examples coming from the reduced group C^*-algebra of a discrete group. A well known construction associates a spectral triple with a length function on the group, we consider this example and a semifinite modification thereof which is possible for a-T-menable groups, i.e. groups with the Haagerup property. The methods extend to a-TT-menable groups, a class of groups containing all hyperbolic groups, see more in [50, Chapter 7.2].

Let Γ denote a countable discrete group. Recall that a length function $\ell : \Gamma \to \mathbb{R}_{\geq 0}$ is a function satisfying $\ell(e) = 0$ for $e \in \Gamma$ the identity, $\ell(\gamma) = \ell(\gamma^{-1})$ and $\ell(\gamma \gamma') \leq \ell(\gamma) + \ell(\gamma')$ for all group elements $\gamma, \gamma' \in \Gamma$. We say that ℓ is a proper length function if ℓ is a proper function, i.e. the set $\{\gamma \in \Gamma : \ell(\gamma) \leq R\}$ is finite for any $R \geq 0$. If there exists a constant $\beta \geq 0$ such that $\#\{\gamma \in \Gamma : \ell(\gamma) \leq R\} = O(e^{\beta R})$ as $R \to \infty$ we say that (Γ, ℓ) has at most exponential growth.

Define the operator $\mathcal{D}_\ell$ densely on $\ell^2(\Gamma)$ as the self-adjoint operator with

$$\mathcal{D}_\ell \delta_\gamma := \ell(\gamma) \delta_\gamma.$$

The space of compactly supported functions $c_c(\Gamma)$ is a core for $\mathcal{D}_\ell$. We define the $*$-algebra $c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma$ as the $*$-algebra generated by multiplication operators (by bounded functions on Γ) $c_b(\Gamma) \subseteq \mathbb{B}(\ell^2(\Gamma))$ and all left translation operators.

Proposition 2.31. *Let ℓ be a proper length function on Γ. The triple $(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma, \ell^2(\Gamma), \mathcal{D}_\ell)$ is a spectral triple defining the trivial class in the K-homology of the C^*-algebra $c_b(\Gamma) \rtimes_r \Gamma$. Moreover, if (Γ, ℓ) has at most exponential growth the spectral triple $(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma, \ell^2(\Gamma), \mathcal{D}_\ell)$ is Li_1-summable.*

Proof. Since ℓ is proper, it is clear that $\mathcal{D}_\ell$ has compact resolvent and if (Γ, ℓ) has at most exponential growth, then there is $C > 0$ such that

$$\mathrm{Tr}(e^{-t|\mathcal{D}_\ell|}) = \sum_{\gamma \in \Gamma} e^{-t\ell(\gamma)} = \sum_{n=0}^\infty \#\{\gamma \in \Gamma : \ell(\gamma) = n\} e^{-tn} \leq C \sum_{n=0}^\infty e^{-(t-\beta)n} = \frac{C}{1 - e^{\beta - t}} < \infty.$$

To show that $(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma, \ell^2(\Gamma), \mathcal{D}_\ell)$ is a spectral triple, it remains to show that $c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma$ preserves the domain of $\mathcal{D}_\ell$ and has bounded commutators with $\mathcal{D}_\ell$. Domain preservation is clear. For an element $a\lambda_\gamma \in c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma$ and a function $f \in c_c(\Gamma)$ we compute that

$$[\mathcal{D}_\ell, a\lambda_\gamma]f(g) = a(g)(\ell(g) - \ell(\gamma^{-1}g))f(\gamma^{-1}g).$$

It follows that

$$\|[\mathcal{D}_\ell, a\lambda_\gamma]\|_{\mathbb{B}(\ell^2(\Gamma))} \leq \|a\|_{c_b(\Gamma)} \sup_{g \in \Gamma} |\ell(g) - \ell(\gamma^{-1}g)| \leq \|a\|_{c_b(\Gamma)} \ell(\gamma). \quad \square$$

The K-homology class of $(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma, \ell^2(\Gamma), \mathcal{D}_\ell)$ is trivial. We shall now consider a topologically more interesting semifinite spectral triple that can be constructed on groups with the Haagerup property. We are grateful to Branimir Ćaćić for sharing this construction with us. Similar ideas appeared in [37, Appendix B].

Let Γ be a discrete group with the Haagerup property. Then there is a proper isometric action of Γ on a real Hilbert space $\mathcal{H}_\Gamma$. By the Mazur–Ulam theorem there exists an orthogonal representation

$$\pi_\Gamma : \Gamma \to O(\mathcal{H}_\Gamma)$$

on the Hilbert space $\mathcal{H}_\Gamma$ and a proper cocycle c_Γ for π_Γ, meaning that $c_\Gamma : \Gamma \to \mathcal{H}$ is a proper function satisfying the cocycle identity

$$c_\Gamma(\gamma_1\gamma_2) = c_\Gamma(\gamma_1) - \pi_\Gamma(\gamma_1)c_\Gamma(\gamma_2). \tag{2.14}$$

The cocycle identity allows us to define a length function on Γ by

$$\ell(\gamma) := \|c_\Gamma(\gamma)\|_{\mathcal{H}_\Gamma}.$$

Since c_Γ is proper, so is ℓ.

Remark 2.32. The existence of a proper isometric action of a group Γ on a Hilbert space is equivalent to Γ having the Haagerup property, also known as a-T-menability. Our construction extends to the case when there exists an orthogonal representation $\pi_\Gamma : \Gamma \to O(\mathcal{H}_\Gamma)$ and a proper quasi-cocycle $c_\Gamma : \Gamma \to \mathcal{H}_\Gamma$. That is, when

$$Q(c_\Gamma) := \sup_{\gamma_1, \gamma_2} \|c_\Gamma(\gamma_1\gamma_2) - c_\Gamma(\gamma_1) + \pi_\Gamma(\gamma_1)c_\Gamma(\gamma_2)\|_{\mathcal{H}_\Gamma} < \infty.$$

In this case, $\ell(\gamma) := \|c_\Gamma(\gamma)\|_{\mathcal{H}_\Gamma}$ could fail to be a length function but still satisfies $\ell(\gamma\gamma') \leq \ell(\gamma) + \ell(\gamma') + Q(c_\Gamma)$ which suffices for our purposes. The existence of a proper quasi-cocycle on a Hilbert space is equivalent to Γ being a-TT-menable. Hyperbolic groups are a-TT-menable. For notational simplicity, we restrict our attention to cocycles.

Definition 2.33. Let $\mathbb{C}\ell(\mathcal{H}_\Gamma)$ denote the complex Clifford algebra of $\mathcal{H}_\Gamma$ and assume that $\mathfrak{c}_S : \mathbb{C}\ell(\mathcal{H}) \to \mathbb{B}(S_\mathcal{H})$ is a representation of $\mathbb{C}\ell(\mathcal{H})$. We say that a unitary representation $\pi_S : \Gamma \to U(S_\mathcal{H})$ is a lift of π_Γ to $S_\mathcal{H}$ if for all $v \in \mathcal{H}$ and $g \in \Gamma$ we have

$$\pi_S(g)\mathfrak{c}_S(v)\pi_S(g^{-1}) = \mathfrak{c}_S(\pi_\Gamma(g)v). \tag{2.15}$$

When $\mathcal{H}$ is finite dimensional this is just the well-known Clifford algebra, but when $\mathcal{H}$ is infinite dimensional we refer to [7,63] for a description of this algebra.

For a representation $S_\mathcal{H}$ of $\mathbb{C}\ell(\mathcal{H})$ we consider the new Hilbert space $\ell^2(\Gamma, S_\mathcal{H})$. Assuming that π_S lifts π_Γ to $S_\mathcal{H}$ the Hilbert space $\ell^2(\Gamma, S_\mathcal{H})$ carries a representation of Γ defined by

$$\tilde{\pi} : \Gamma \to U(\ell^2(\Gamma, S_\mathcal{H})), \qquad (\tilde{\pi}(g)f)(\gamma) = \pi_S(g)f(g^{-1}\gamma). \tag{2.16}$$

On the Hilbert space $\ell^2(\Gamma, S_\mathcal{H})$ define a self-adjoint operator $\mathcal{D}_c$ by declaring

$$(\mathcal{D}_cf)(\gamma) := \mathfrak{c}_S(c_\Gamma(\gamma))f(\gamma), \quad f \in c_c(\Gamma, S_\mathcal{H}). \tag{2.17}$$

Since $\mathfrak{c}_S(v)^2 = \|v\|_\mathcal{H}^2$ for all $v \in \mathcal{H}$, the domain for $\mathcal{D}_c$ can be deduced from

$$(\mathcal{D}_c^2f)(\gamma) = \ell(\gamma)^2 f(\gamma).$$

The compatibility requirement Equation (2.15) and cocycle property Equation (2.14) imply that for $g, \gamma \in \Gamma$ we have

$$\pi_S(g)\mathfrak{c}_S(c_\Gamma(\gamma)) = \mathfrak{c}_S(\pi(g)c_\Gamma(\gamma))\pi_S(g) = \mathfrak{c}_S(c_\Gamma(g))\pi_S(g) + \mathfrak{c}_S(c_\Gamma(g\gamma))\pi_S(g).$$

Then the commutator of $\mathcal{D}_c$ and a group element is

$$([\mathcal{D}_c, \tilde{\pi}(g)]f)(\gamma) = \mathfrak{c}_S(c_\Gamma(\gamma))\pi_S(g)f(g^{-1}\gamma) - \pi_S(g)\mathfrak{c}_S(c_\Gamma(g^{-1}\gamma))f(g^{-1}\gamma)$$
$$= \mathfrak{c}_S(c_\Gamma(g))\pi_S(g)f(g^{-1}\gamma) = (\mathfrak{c}_S(c_\Gamma(g))\tilde{\pi}(g))(f)(\gamma).$$

Hence the commutators between $\mathcal{D}_c$ and group elements are bounded. It is moreover clear that these commutators lie in $\mathcal{N}_0 \rtimes \Gamma$ where $\mathcal{N}_0 = \mathfrak{c}_S(\mathbb{C}\ell(\mathcal{H}))''$. We define $\mathcal{N}$ as the von Neumann algebraic tensor product $\mathbb{B}(\ell^2(\Gamma))\bar{\otimes}\mathcal{N}_0$.

Finally,

$$(1 + \mathcal{D}_c^2)^{-1} \in \mathbb{K}(\ell^2(\Gamma)) \otimes 1 \subset \mathbb{K}(\ell^2(\Gamma)) \otimes \mathbb{K}_\tau$$

where $\mathbb{K}_\tau$ is the compacts in $\mathcal{N}_0$ for a choice of normalized positive trace τ. Let Tr_τ be the trace on $\mathcal{N}$ defined from the trace τ on $\mathcal{N}_0$. We conclude that $(1 + \mathcal{D}_c^2)^{-1} \in \mathbb{K}_{\mathrm{Tr}_\tau}$. Finally, if ℓ has at most exponential growth then $\mathrm{Tr}_\tau(e^{-t|\mathcal{D}_c|}) = \sum_{\gamma \in \Gamma} e^{-t\ell(\gamma)} < \infty$ for t large enough. As such, $(i \pm \mathcal{D}_c)^{-1} \in \mathrm{Li}_1$ if ℓ has at most exponential growth. We conclude the following result.

Proposition 2.34. *Assume that* $\mathfrak{c}_S : \mathbb{C}\ell(\mathcal{H}) \to \mathbb{B}(S_\mathcal{H})$ *is a representation of* $\mathbb{C}\ell(\mathcal{H})$ *and that the unitary representation* $\pi_S : \Gamma \to U(S_\mathcal{H})$ *lifts* π_Γ *to* $S_\mathcal{H}$. *Let* $c_b(\Gamma)$ *be the (continuous) bounded functions on* Γ, *and define a representation of* $c_b(\Gamma) \rtimes \Gamma$ *on* $\ell^2(\Gamma, S_\mathcal{H})$ *by*

$$\hat{\pi}_S(a\lambda_g)f(\gamma) = a(\gamma)[\tilde{\pi}(g)f](\gamma),$$

where $\tilde{\pi}$ *is as in* (2.16). *Then the triple* $(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma, \ell^2(\Gamma, S_\mathcal{H}), \mathcal{D}_c, \mathcal{N}, \mathrm{Tr}_\tau)$ *is a semifinite spectral triple which is* Li_1-*summable if* ℓ *has at most exponential growth.*

We return to the example of this subsubsection in Example 3.9 and Section 5.3 .

Example 2.35. The following example of a proper group cocycle shows the construction's geometric advantage compared to only using a length function. Consider the trivial action of the discrete group $\Gamma = \mathbb{Z}^n$ on $\mathcal{H}_\Gamma = \mathbb{R}^n$. The inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is additive and proper, and therefore a proper group cocycle for the trivial action. The semifinite spectral triple associated with a finite dimensional Clifford representation $\mathfrak{c}_S : \mathbb{R}^n \to \mathrm{End}_\mathbb{C}(S)$ can when restricted to $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$ be identified with the semifinite spectral triple $(C^\infty(\mathbb{T}^n), L^2(\mathbb{T}^n, S), \slashed{D}_{\mathbb{T}^n} \otimes 1_S, \mathbb{B}(L^2(\mathbb{T}^n)) \otimes \mathbb{C}\ell_n, \mathrm{Tr}_\tau)$ using Fourier theory on the dual torus $\mathbb{T}^n = \widehat{\mathbb{Z}^n}$. We observe that to extract K-homological content from this construction we need to specify a grading if n is even. In particular, it is unclear how to interpret the construction above in K-homology when $\mathcal{H}_\Gamma$ is infinite-dimensional.

3. KMS States constructed from Li₁-summable spectral triples

This section contains the fundamental technical construction of the paper. Starting from a semifinite Li_1-summable spectral triple, we use the associated algebra of Toeplitz operators to construct an action from the operator $\mathcal{D}$ and a KMS-state from the operator $|\mathcal{D}|$.

3.1. The positive part of the spectrum and heat traces

Throughout this section we suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is a unital semifinite spectral triple. The spectral triple can be semifinite, in which case we let $\mathcal{T}$ denote the given positive faithful normal semifinite trace. In general $\mathcal{T}$ is not unique, and coincides with a non-zero multiple of the operator trace in the "usual" non-semifinite case $\mathcal{N} = \mathbb{B}(\mathcal{H})$. We write $\mathbb{K}_\mathcal{N}$ for the compacts for $\mathcal{T}$. Again, $\mathbb{K}_\mathcal{N}$ coincides with the usual compacts in the case of the type I factor $\mathcal{N} = \mathbb{B}(\mathcal{H})$.

We write $\mathcal{N}^+$ for the von Neumann algebra $P_\mathcal{D}\mathcal{N}P_\mathcal{D}$. By an abuse of notation, we write $\mathcal{T}$ also for the induced faithful normal semifinite trace on $\mathcal{N}^+$. The $\mathcal{T}$-compacts on $\mathcal{N}^+$ will be denoted by $\mathbb{K}_\mathcal{N}^+$.

Definition 3.1. The operator $\mathcal{D}$ is said to have positive $\mathcal{T}$-essential spectrum if for some $\beta \in [0, \infty)$, we have $\mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) < \infty$ for $t > \beta$ and $\mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) \nearrow \infty$ as $t \searrow \beta$.

Proposition 3.2. *Let* $\mathcal{D}$ *be a self-adjoint densely defined operator affiliated with* $\mathcal{N}$ *satisfying that* $(1 + \mathcal{D}^2)^{-1/2} \in \mathrm{Li}_1(\mathcal{T})$. *We define the number*

$$\beta_\mathcal{D} := \inf\{t > 0 : \mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) < \infty\}. \tag{3.1}$$

Then $\beta_\mathcal{D} \in [0, \infty)$ *and* $\mathcal{D}$ *has positive* $\mathcal{T}$-*essential spectrum if and only if*

$$\lim_{t \searrow \beta_\mathcal{D}} \mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) = \infty.$$

In particular, if $\beta_\mathcal{D} = 0$ *then* $\mathcal{D}$ *has positive* $\mathcal{T}$-*essential spectrum if and only if* $\mathcal{T}(P_\mathcal{D}) = \infty$.

Proof. By definition, if $(1 + \mathcal{D}^2)^{-1/2} \in \mathrm{Li}_1$ then $\mathcal{T}(e^{-t|\mathcal{D}|}) < \infty$ for t large enough. Therefore, $\mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) = \mathcal{T}(P_\mathcal{D}e^{-t|\mathcal{D}|}) < \infty$ for t large enough and $\beta_\mathcal{D} := \inf\{t > 0 : \mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) < \infty\}$ will be a number in $[0, \infty)$. By definition, $\mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) < \infty$ for $t > \beta_\mathcal{D}$ and if $\lim_{t \searrow \beta_\mathcal{D}} \mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) = \infty$ then $\mathcal{D}$ has positive $\mathcal{T}$-essential spectrum with $\beta = \beta_\mathcal{D}$. Conversely, if $\mathcal{D}$ has positive $\mathcal{T}$-essential spectrum there is a $\beta \in [0, \infty)$ with $\mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) < \infty$ for $t > \beta$ and $\mathcal{T}(P_\mathcal{D}e^{-t\mathcal{D}}) \nearrow \infty$ as $t \searrow \beta$, and in this case it is clear that $\beta = \beta_\mathcal{D}$. $\square$

Remark 3.3. We will often impose the assumption of positive $\mathcal{T}$-essential spectrum. If for some $\beta \in [0, \infty)$, $\mathcal{T}((1 - P_{\mathcal{D}})e^{t\mathcal{D}}) < \infty$ for $t > \beta$ and $\mathcal{T}((1 - P_{\mathcal{D}})e^{t\mathcal{D}}) \nearrow \infty$ as $t \searrow \beta$, we can equally well use $-\mathcal{D}$ in our construction.

Proposition 3.4. *Let $\mathcal{D}$ be a self-adjoint densely defined operator affiliated with $\mathcal{N}$ satisfying that $(1 + \mathcal{D}^2)^{-1/2} \in \mathrm{Li}_1(\mathcal{T})$. Then $\mathcal{D}$ has positive $\mathcal{T}$-essential spectrum if and only if both of the following conditions fail:*

1. *$P_{\mathcal{D}}$ has finite $\mathcal{T}$-trace.*
2. *There exists a $p > 0$ such that $P_{\mathcal{D}}e^{-|\mathcal{D}|} \in \mathcal{L}^p(\mathcal{T}) \setminus \cap_{q>p}\mathcal{L}^q(\mathcal{T})$.*

The reader should note that conditions 1. and 2. are mutually exclusive.

Proof. If $\mathcal{D}$ has positive $\mathcal{T}$-essential spectrum, then clearly 1. fails. Also 2. fails if $\mathcal{D}$ has positive $\mathcal{T}$-essential spectrum because condition 2. is equivalent to $\beta_{\mathcal{D}} = p$ and $\lim_{t \searrow p} \mathcal{T}(P_{\mathcal{D}}e^{-t\mathcal{D}})$ being finite.

Conversely, if Conditions 1. and 2. fail, then either $\beta_{\mathcal{D}} = 0$ and $\lim_{t \searrow 0} \mathcal{T}(P_{\mathcal{D}}e^{-t\mathcal{D}})$ must be infinite not to violate $\mathcal{T}(P_{\mathcal{D}})$ being infinite or $\beta_{\mathcal{D}} > 0$ and the set $\{p > 0 : P_{\mathcal{D}}e^{-|\mathcal{D}|} \in \mathcal{L}^p(\mathcal{T})\}$ is open (due to condition 2. failing) showing that $\mathcal{T}(P_{\mathcal{D}}e^{-t\mathcal{D}}) \nearrow \infty$ as $t \searrow \beta_{\mathcal{D}}$. $\square$

Example 3.5. Dirac operators on closed manifolds, as considered in Section 2.2.1 , have positive essential spectrum. In this case, we can compute $\beta_{\mathcal{D}} = 0$ and the leading term in the heat trace asymptotics using Proposition 2.14. If $(C^\infty(M), L^2(M, S), \slashed{D})$ is the spectral triple associated with a Dirac operator, Proposition 2.14 implies that

$$\mathrm{Tr}_{L^2(M,S)}(P_{\mathcal{D}}e^{-t|\slashed{D}|}) = n!c_n t^{-n} \int_{S^*M} \mathrm{Tr}_S(p_{\slashed{D}})\mathrm{d}V + O(t^{-n+\epsilon}), \quad \text{as } t \to 0,$$

where $p_{\slashed{D}}$ is the principal symbol of the zeroth order pseudo-differential operator $P_{\slashed{D}}$. A direct computation shows that $p_{\slashed{D}}(x, \xi) = \frac{1}{2}(c_S(\xi) + 1)$ for $x \in M$ and $\xi \in S_x^*M$. Here $c_S : T^*M \to \mathrm{End}(S)$ denotes Clifford multiplication. More generally, Proposition 2.14 allows us to conclude that for $a \in C^\infty(M)$,

$$\mathrm{Tr}_{L^2(M,S)}(P_{\mathcal{D}}ae^{-t|\slashed{D}|}) = n!c_n t^{-n} \int_{S^*M} \mathrm{Tr}_S(p_{\slashed{D}})a\,\mathrm{d}V + O(t^{-n+\epsilon}) =$$

$$= n!c_n t^{-n} \int_M \int_{S_x^*M} a(x)\mathrm{Tr}_S(p_{\slashed{D}}(x, \xi))\,\mathrm{d}V_{S_x^*M}(\xi)\mathrm{d}V(x) + O(t^{-n+\epsilon}) =$$

$$= \tilde{c}_n t^{-n} \int_M a\,\mathrm{d}V + O(t^{-n+\epsilon}), \quad \text{as } t \to 0,$$

for a new constant $\tilde{c}_n > 0$, depending only on the dimension of M and the rank of S. In the last equality we used that $p_{\slashed{D}} - 1/2$ is an antisymmetric function under the involution $(x, \xi) \mapsto (x, -\xi)$ of S^*M and therefore

$$\int_{S_x^*M} \mathrm{Tr}_S(p_{\slashed{D}}(x, \xi))\,\mathrm{d}V_{S_x^*M} = \int_{S_x^*M} \mathrm{Tr}_S(p_{\slashed{D}}(x, -\xi))\,\mathrm{d}V_{S_x^*M} =$$

$$= \frac{\mathrm{rank}(S)}{2} \int_{S_x^*M} \mathrm{d}V_{S_x^*M} = \frac{\mathrm{rank}(S)\pi^{\dim(M)/2}}{\Gamma(\dim(M)/2)}.$$

Example 3.6. The spectral triples for graph C^*-algebras from Proposition 2.20 also have positive essential spectrum. We compute $\beta_{\mathcal{D}}$ and the heat trace asymptotics assuming that G is primitive. In this example, $P_{\mathcal{D}_y}$ is the projection onto the subspace $\ell^2(\mathcal{V}_y \cap \kappa_G^{-1}(0))$. We use the notation

$$\mathcal{V}_y^+ := \mathcal{V}_y \cap \kappa_G^{-1}(0) = \{(x, n) \in \mathcal{V}_y : n \geq 0, \ \sigma_G^n(x) = y\}. \tag{3.2}$$

The space $P_{\mathcal{D}_y}\ell^2(\mathcal{V}_y)$ is therefore spanned by the orthonormal basis $(\delta_{(x,n)})_{(x,n) \in \mathcal{V}_y^+}$. Note that if $\sigma_G^n(x) = y$ then x is uniquely determined by y and a finite path $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ with $x = \sigma y$. Note that paths of the form $x = \sigma y$, with $s(\sigma_n) = r(y)$, exhaust all possible $x \in \sigma_G^{-n}(\{y\})$. Using that $P_{\mathcal{D}_y}\mathcal{D}_y\delta_{(x,n)} = n\delta_{(x,n)}$ for $(x, n) \in \mathcal{V}_y^+$, we compute that

$$\mathrm{Tr}(P_{\mathcal{D}_y}e^{-t|\mathcal{D}_y|}) = \sum_{n=0}^\infty \sum_{x \in \sigma_G^{-n}(\{y\})} e^{-tn} = \sum_{n=0}^\infty \#\{\sigma \in E_n : s(\sigma_n) = r(y)\}e^{-tn}.$$

Let A denote the edge adjacency matrix of G and $r_\sigma(A)$ its spectral radius. If G is primitive, (i.e. all entries of A^k are positive for some integer $k > 0$) we let $w \in \mathbb{C}^E$ its ℓ^2-normalized Perron–Frobenius vector. It follows from [54, Lemma 3.7] that there is an $\alpha_0 \in [0, 1)$ such that

$$\#\{\sigma \in E_n : s(\sigma_n) = r(y)\} = \|w\|_{\ell^1}w_{r(y)}r_\sigma(A)^n + O((\alpha_0 r_\sigma(A))^n),$$

as $n \to \infty$. We can conclude that there is a function f holomorphic in $\mathrm{Re}(t) > \log r_\sigma(A) + \log \alpha_0$ such that

$$\mathrm{Tr}(P_{\mathcal{D}_y} \mathrm{e}^{-t|\mathcal{D}_y|}) = \frac{\|w\|_{\ell^1} w_{r(y)}}{1 - r_\sigma(A)\mathrm{e}^{-t}} + f(t).$$

Therefore, $\mathrm{Tr}(P_{\mathcal{D}_y}\mathrm{e}^{-t|\mathcal{D}_y|}) - \frac{\|w\|_{\ell^1} w_{r(y)}}{t - \log(r_\sigma(A))}$ has a holomorphic extension to $\mathrm{Re}(t) > \log r_\sigma(A) + \log \alpha_0$ whenever G is primitive.

More generally, if G is primitive, the method above shows that for two finite paths μ and ν we can compute that

$$\mathrm{Tr}(P_{\mathcal{D}_y} S_\mu S_\nu^* \mathrm{e}^{-t|\mathcal{D}_y|}) = \delta_{\mu,\nu} \sum_{n=0}^{\infty} \sum_{x \in \sigma_G^{-n}(\{y\})} \|S_\mu^* \delta_{(x,n)}\|_{\ell^2}^2 \mathrm{e}^{-tn}$$

$$= \sum_{n=|\mu|}^{\infty} \#\{\sigma \in E_{n-|\mu|} : r(\sigma_1) = s(\mu),\ s(\sigma_{n-|\mu|}) = r(y)\} \mathrm{e}^{-tn}$$

$$+ \delta_{\mu,\nu} \sum_{n=0}^{|\mu|-1} \sum_{x \in \sigma_G^{-n}(\{y\})} \|S_\mu^* \delta_{(x,n)}\|_{\ell^2}^2 \mathrm{e}^{-tn}$$

$$= \delta_{\mu,\nu} w_{s(\mu)} w_{r(y)} \frac{\mathrm{e}^{-t|\mu|}}{1 - r_\sigma(A)\mathrm{e}^{-t}} + \delta_{\mu,\nu} f_{\mu,y}(t),$$

for a function $f_{\mu,y}$ holomorphic in $\mathrm{Re}(t) > \log r_\sigma(A) + \log \alpha_0$. We conclude that

$$\mathrm{Tr}(P_{\mathcal{D}_y} S_\mu S_\nu^* \mathrm{e}^{-t|\mathcal{D}_y|}) - \delta_{\mu,\nu} w_{d(\mu)} w_{r(y)} \frac{r_\sigma(A)^{-|\mu|}}{t - \log(r_\sigma(A))},$$

has a holomorphic extension to $\mathrm{Re}(t) > \log r_\sigma(A) + \log \alpha_0$ whenever G is primitive. As such, $\beta_{\mathcal{D}_y} = \log(r_\sigma(A))$ and $\mathcal{D}_y$ has positive essential spectrum.

Example 3.7. Let O_E be a Cuntz–Pimsner algebra defined from a strictly W-regular (recall Definition 2.25) finitely generated and projective bi-Hilbertian bimodule E_A and a positive trace τ on the coefficient algebra A. The semifinite spectral triple considered in Lemma 2.28 also has positive Tr_τ-essential spectrum assuming a criticality condition on τ that we formulate below (see Definition 6.2). The heat trace asymptotics are slightly more involved, and we compute these explicitly in Section 6.2 under a condition on τ previously studied by Laca–Neshveyev [45] in the context of KMS-states. However, for a general τ we can proceed as in the proof of Lemma 2.28 to deduce the following.

Proposition 3.8. For any strictly W-regular fgp bi-Hilbertian bimodule E over the unital C^*-algebra and a positive trace τ on A,

$$\mathrm{Tr}_\tau(P_{\mathcal{D}_\alpha} \mathrm{e}^{-t\mathcal{D}_\alpha}) = \sum_{n=0}^{\infty} \mathrm{e}^{-tn} \tau_*(E^{\otimes_A n}),$$

where $\tau_* : K_0(A) \to \mathbb{R}$ denotes the map induced by τ on K-theory. In particular, $\mathrm{Tr}_\tau(P_{\mathcal{D}_\alpha} \mathrm{e}^{-t\mathcal{D}_\alpha})$ does not depend on the choice of inner products on E but only on τ and the bimodule structure on E.

Proof. We compute that

$$\mathrm{Tr}_\tau(P_{\mathcal{D}} \mathrm{e}^{-t|\mathcal{D}_\alpha|}) = \sum_{n=0}^{\infty} \mathrm{e}^{-t|\psi(n,n)|} \mathrm{Tr}_\tau(P_{n,n}) = \sum_{n=0}^{\infty} \mathrm{e}^{-tn} \mathrm{Tr}_\tau(Q_{n,n}) = \sum_{n=0}^{\infty} \sum_{|\rho|=n} \mathrm{e}^{-tn} \tau((e_\rho|e_\rho)_A).$$

On the other hand, $\tau_*(E^{\otimes_A n}) = (\tau \otimes \mathrm{Tr}_{M_{N(n)}})(p_{E^{\otimes_A n}})$ where $p_{E^{\otimes_A n}} \in M_{N(n)}(A)$ is a projection representing $E^{\otimes_A n}$. Using the choice of frame $(e_j)_{j=1}^N$, we can take $N(n) := N^n$ and $p_{E^{\otimes_A n}} := ((e_\mu|e_\nu)_A)_{|\mu|=|\nu|=n}$. In this choice of representing projection,

$$\tau_*(E^{\otimes_A n}) = (\tau \otimes \mathrm{Tr}_{M_{N(n)}})(p_{E^{\otimes_A n}}) = \mathrm{Tr}_{M_{N(n)}}((\tau((e_\mu|e_\nu)_A)_{|\mu|=|\nu|=n})) = \sum_{|\rho|=n} \tau((e_\rho|e_\rho)_A). \quad \square$$

This computation shows that it is in general difficult to compute $\beta_{\mathcal{D}}$. In this case $\beta_{\mathcal{D}}$ depends on the asymptotic properties of the sequence $(\tau_*(E^{\otimes_A n}))_{n \in \mathbb{N}}$ as $n \to \infty$.

For a simple tensor $\sigma \in E^{\otimes m}$ write $\sigma = \underline{\sigma}\overline{\sigma}$, where the initial segment $\underline{\sigma}$ will be of a length understood from context ($|\underline{\sigma}| = |\mu|$ in the next computation). With this notation, we can compute our functional on a typical $S_\mu S_\nu^* \in O_E$, where

$\mu \in E^{\otimes k}$, $\nu \in E^{\otimes l}$ are simple tensors. We find

$$\mathrm{Tr}_\tau(P_{\mathcal{D}} S_\mu S_\nu^* \mathrm{e}^{-t|\mathcal{D}_\alpha|}) = \delta_{|\mu|,|\nu|} \sum_{n=0}^\infty \mathrm{e}^{-tn} \mathrm{Tr}_\tau(S_\mu S_\nu^* Q_{n,n}) \tag{3.3}$$

$$= \delta_{|\mu|,|\nu|} \sum_{n=|\mu|}^\infty \sum_{|\sigma|=n} \mathrm{e}^{-tn} \tau((S_\mu^* e_\sigma | S_\nu^* e_\sigma)_{E^{\otimes(n-|\mu|)}})$$

$$= \delta_{|\mu|,|\nu|} \sum_{n=|\mu|}^\infty \sum_{|\sigma|=n} \mathrm{e}^{-tn} \tau(((\mu|e_{\underline{\sigma}})_{E^{|\mu|}} e_{\overline{\sigma}} \mid (\nu|e_{\underline{\sigma}})_{E^{|\mu|}} e_{\overline{\sigma}})_{E^{\otimes(n-|\mu|)}}).$$

Example 3.9. Consider a length function ℓ on a countable group Γ as in Section 2.2.4.

Definition 3.10. We define the critical value of (Γ, ℓ) as

$$\beta(\Gamma, \ell) := \inf\{t \geq 0 : \sum_{\gamma \in \Gamma} \mathrm{e}^{-t\ell(\gamma)} < \infty\}.$$

If $\sum_{\gamma \in \Gamma} \mathrm{e}^{-t\ell(\gamma)} \nearrow \infty$ as $t \searrow \beta(\Gamma, \ell)$, we say that ℓ is critical.

It follows directly from Definition 3.1 that the operator $\mathcal{D}_\ell$ appearing in Proposition 2.31 has positive essential spectrum as long as ℓ is critical. The heat trace of an element $a\lambda_g \in c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma$ is given by

$$\mathrm{Tr}(P_{\mathcal{D}_\ell} a\lambda_g \mathrm{e}^{-t|\mathcal{D}_\ell|}) = \mathrm{Tr}(a\lambda_g \mathrm{e}^{-t|\mathcal{D}_\ell|}) = \delta_{e,g} \sum_{\gamma \in \Gamma} a(\gamma) \mathrm{e}^{-t\ell(\gamma)}.$$

Similar computations can be carried out for the semifinite spectral triple constructed in Proposition 2.34 using a Hilbert space valued cocycle c_Γ. Note that

$$P_{\mathcal{D}_c} f(g) = \frac{1}{2}\left(\frac{\mathfrak{c}_S(c_\Gamma(g))}{\|c_\Gamma(g)\|_{\mathcal{H}_\Gamma}} + 1 \right).$$

Therefore, the heat trace of an element $a\lambda_g \in c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma$ is given by

$$\mathrm{Tr}_\tau(P_{\mathcal{D}_c} a\lambda_g \mathrm{e}^{-t|\mathcal{D}_c|}) = \frac{1}{2} \sum_{\gamma \in \Gamma} \left\langle \delta_\gamma, \tau\left(\left(\frac{\mathfrak{c}_S(c_\Gamma(g))}{\|c_\Gamma(g)\|_{\mathcal{H}_\Gamma}} + 1 \right) a(g^{-1}\gamma) \delta_{g\gamma} \right) \right\rangle \mathrm{e}^{-t\ell(\gamma)} =$$

$$= \frac{1}{2} \delta_{e,g} \sum_{\gamma \in \Gamma} a(\gamma) \mathrm{e}^{-t\ell(\gamma)} = \frac{1}{2} \mathrm{Tr}(P_{\mathcal{D}_\ell} a\lambda_g \mathrm{e}^{-t|\mathcal{D}_\ell|}).$$

Here we use that $\tau(\mathfrak{c}_S(v)) = 0$ for any $v \in \mathcal{H}_\Gamma$ which holds due to the fact that we can pick a $w \in \mathcal{H}_\Gamma$ orthogonal to v and compute that

$$\tau(\mathfrak{c}_S(v)) = \tau(\mathfrak{c}_S(w)\mathfrak{c}_S(v)\mathfrak{c}_S(w)) = -\tau(\mathfrak{c}_S(w)^2 \mathfrak{c}_S(v)) = -\tau(\mathfrak{c}_S(v)).$$

If the length function $\ell(\gamma) := \|c_\Gamma(\gamma)\|_{\mathcal{H}_\Gamma}$ associated with c_Γ is critical, we say that c_Γ is critical. We conclude that the semifinite spectral triple from Proposition 2.34 has positive essential spectrum if c_Γ is critical.

3.2. To-plitz or not To-plitz

We proceed under the same working conditions as in the previous section to construct states from spectral triples. Recall that $\mathcal{N}^+ = P_{\mathcal{D}} \mathcal{N} P_{\mathcal{D}}$ and that $\mathbb{K}_{\mathcal{N}}^+ = P_{\mathcal{D}} \mathbb{K}_{\mathcal{N}} P_{\mathcal{D}}$.

Definition 3.11. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a semifinite spectral triple. We define the Toeplitz algebra of $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ as

$$T_A := P_{\mathcal{D}} A P_{\mathcal{D}} + \mathbb{K}_{\mathcal{N}}^+ \subseteq \mathcal{N}^+.$$

The saturated Toeplitz algebra of $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is defined by

$$T_{A,\mathcal{D}} := C^*\left(\bigcup_{s \in \mathbb{R}} \mathrm{e}^{is\mathcal{D}} T_A \mathrm{e}^{-is\mathcal{D}} \right) = C^*\left(\bigcup_{s \in \mathbb{R}} \mathrm{e}^{is\mathcal{D}} P_{\mathcal{D}} A P_{\mathcal{D}} \mathrm{e}^{-is\mathcal{D}} + \mathbb{K}_{\mathcal{N}}^+ \right) \subseteq \mathcal{N}^+.$$

Proposition 3.12. *The Toeplitz algebra T_A of a unital semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is a C^*-algebra.*

Proof. It follows from Lemma 2.11 that $[P_\mathcal{D}, a] \in \mathbb{K}_\mathcal{N}$ for all $a \in A$. Therefore, the mapping

$$\beta_\mathcal{D} : A \to \mathcal{N}^+/\mathbb{K}_\mathcal{N}^+, \quad a \mapsto P_\mathcal{D} a P_\mathcal{D} \mod \mathbb{K}_\mathcal{N}^+,$$

is a $*$-homomorphism and $\beta_\mathcal{D}(A) \subseteq \mathcal{N}^+/\mathbb{K}_\mathcal{N}^+$ is a closed C^*-subalgebra. By definition, T_A is the preimage of $\beta_\mathcal{D}(A)$ under the quotient mapping $\mathcal{N}^+ \to \mathcal{N}^+/\mathbb{K}_\mathcal{N}^+$ and is therefore a C^*-algebra. $\square$

The reader should note that Proposition 3.12 also holds in the non-unital setting because it only relies on Lemma 2.11 which holds non-unitally.

Proposition 3.13. *The saturated Toeplitz algebra of* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ *carries an* $\mathbb{R}$-*action* $\sigma^+ : \mathbb{R} \to \mathrm{Aut}(T_{A,\mathcal{D}})$ *defined by*

$$\sigma_s^+(T) := P_\mathcal{D} e^{is\mathcal{D}} T e^{-is\mathcal{D}} P_\mathcal{D} = P_\mathcal{D} e^{is|\mathcal{D}|} T e^{-is|\mathcal{D}|} P_\mathcal{D}, \qquad s \in \mathbb{R}, \ T \in T_{A,\mathcal{D}}.$$

This proposition is a consequence of that $T_{A,\mathcal{D}}$ is constructed as the saturation of T_A under the action σ^+ extended to $\mathcal{N}^+$. We note that if we identify $T_{A,\mathcal{D}}$ with a subalgebra of $\mathcal{N}$, we can also write $\sigma_s^+(T) := e^{is\mathcal{D}} T e^{-is\mathcal{D}} = e^{is|\mathcal{D}|} T e^{-is|\mathcal{D}|}$.

Define a one-parameter family of states $(\phi_{t,0})_{t > \beta}$ on $T_{A,\mathcal{D}}$ by

$$\phi_{t,0}(T) := \frac{\mathcal{T}(P_\mathcal{D} T e^{-t\mathcal{D}})}{\mathcal{T}(P_\mathcal{D} e^{-t\mathcal{D}})}.$$

We shall compose the family $(\phi_{t,0})_{t > \beta}$ with an "extended limit" as $t \to \beta$:

Definition 3.14. An extended limit as $t \to \beta$ is a state $\omega \in L^\infty(\beta, \infty)^*$ such that $\omega(f) = 0$ whenever $\lim_{t \to \beta} f(t) = 0$. For an extended limit ω and $f \in L^\infty(\beta, \infty)$, we write

$$\omega\text{-}\lim_{t \to \beta} f(t) := \omega(f).$$

Let ω be any extended limit and define

$$\phi_{\omega,0} : T_{A,\mathcal{D}} \to \mathbb{C}, \qquad \phi_{\omega,0}(T) := \omega \circ \phi_{t,0}(T) = \omega\text{-}\lim_{t \to \beta} \phi_{t,0}(T).$$

Lemma 3.15. *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ *be a semifinite* Li_1-*summable spectral triple with positive* $\mathcal{T}$-*essential spectrum (see Definition* 3.1*). For any extended limit* ω, *the functional* $\phi_{\omega,0}$ *is a state on* $T_{A,\mathcal{D}}$. *Moreover* $\phi_{\omega,0}(T) = 0$ *for all* $T \in \mathbb{K}_\mathcal{N}^+$.

Proof. It is immediate that $\phi_{\omega,0}$ is a state. For the statement that $\phi_{\omega,0}(T) = 0$ for all $T \in \mathbb{K}_\mathcal{N}^+$, we observe that since $\phi_{\omega,0}$ is a state, it is also norm-continuous. It therefore suffices to prove that $\phi_{\omega,0}(T) = 0$ for all projections $T \in \mathcal{N}^+$ with $\mathcal{T}(T) < \infty$. For such T, we can estimate that

$$\phi_{t,0}(T) = \frac{\mathcal{T}(P_\mathcal{D} T e^{-t\mathcal{D}})}{\mathcal{T}(P_\mathcal{D} e^{-t\mathcal{D}})} \leq \frac{\mathcal{T}(T)}{\mathcal{T}(P_\mathcal{D} e^{-t\mathcal{D}})}.$$

Since, $\mathcal{T}(P_\mathcal{D} e^{-t\mathcal{D}}) \nearrow \infty$ as $t \searrow \beta$, it follows that $\lim_{t \to \beta} \phi_{t,0}(T) = 0$. We conclude that for all projections $T \in \mathcal{N}^+$, with $\mathcal{T}(T) < \infty$, and any extended limit ω as $t \to \beta$, $\omega \circ \phi_{t,0}(T) = 0$. $\square$

Due to Lemma 3.15, we can make the following definition.

Definition 3.16. Define the C^*-algebra $A_\mathcal{D} := T_{A,\mathcal{D}}/\mathbb{K}_\mathcal{N}^+$ and the state ϕ_ω on $A_\mathcal{D}$ as

$$\phi_\omega(T \mod \mathbb{K}_\mathcal{N}^+) := \phi_{\omega,0}(T).$$

The state $\phi_{\omega,0}$ also restricts to a state on T_A, and Lemma 3.15 implies that $\phi_{\omega,0}|_{T_A}$ descends to a state on A via the $*$-epimorphism $\beta_\mathcal{D} : A \to T_A/\mathbb{K}_\mathcal{N}^+$ (see the proof of Proposition 3.12). To analyse the situation of the state on $A_\mathcal{D}$ versus that on A, we consider the following ideal

$$I := \{a \in A : P_\mathcal{D} a P_\mathcal{D} \in \mathbb{K}_\mathcal{N}^+\}$$

so that $P_\mathcal{D} I P_\mathcal{D} = P_\mathcal{D} A P_\mathcal{D} \cap \mathbb{K}_\mathcal{N}^+$. Since $T_A \subseteq T_{A,\mathcal{D}}$, we obtain a commuting diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K}_\mathcal{N}^+ & \longrightarrow & T_A & \longrightarrow & A/I & \longrightarrow & 0 \\
& & \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & \mathbb{K}_\mathcal{N}^+ & \longrightarrow & T_{A,\mathcal{D}} & \longrightarrow & A_\mathcal{D} & \longrightarrow & 0,
\end{array}
$$

with exact rows. The mapping $A/I \to A_\mathcal{D}$ is indeed injective by the four lemma. We identify A/I with a subalgebra of $A_\mathcal{D}$. The induced mapping $\gamma : A \to A_\mathcal{D}$ is compatible with the states ϕ_ω and $\phi_{\omega,0}$ in the sense that $\phi_{\omega,0}(P_\mathcal{D} a P_\mathcal{D}) = \phi_\omega(\gamma(a))$ for $a \in A$.

The $\mathbb{R}$-action $\sigma_s^+(T) := e^{is\mathcal{D}} T e^{-is\mathcal{D}}$ on $T_{A,\mathcal{D}}$ induces an $\mathbb{R}$-action on A/I. The following proposition follows from the construction of $T_{A,\mathcal{D}}$ as the saturation of T_A under σ^+.

Proposition 3.17. *Let $\beta \in \mathbb{R}$. The algebra $A_{\mathcal{D}}$ carries an $\mathbb{R}$-action $\sigma : \mathbb{R} \to \mathrm{Aut}(A_{\mathcal{D}})$ defined by declaring the quotient mapping $T_{A,\mathcal{D}} \to A_{\mathcal{D}}$ to be equivariant. The C^*-algebra $A_{\mathcal{D}}$ is the saturation of A/I under the action σ, i.e. $A_{\mathcal{D}}$ is generated in $\mathcal{N}^+/\mathbb{K}_{\mathcal{N}}^+$ by $\cup_{s\in\mathbb{R}}\sigma_s(A/I)$.*

The aim of our construction is to obtain a KMS state on A, or, failing that, on A/I. As a first step we introduce conditions ensuring that we at least get a KMS state on $T_{A,\mathcal{D}}$, and so on $A_{\mathcal{D}}$. To this end we make the following assumption

Definition 3.18. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a semifinite spectral triple. We say that a subset $S \subset \mathcal{A}$ is analytically generating at β if the set $P_{\mathcal{D}}SP_{\mathcal{D}} + \mathbb{K}^+$ generates the Toeplitz algebra T_A as a C^*-algebra and if $\beta \neq 0$, then for each $a \in S$ there is an $\epsilon \neq 0$ with $\epsilon\beta > 0$ satisfying that $\mathrm{e}^{(\beta+\epsilon)\mathcal{D}}P_{\mathcal{D}}aP_{\mathcal{D}}\mathrm{e}^{-(\beta+\epsilon)\mathcal{D}} \in \mathcal{N}^+$.*
We say that the semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is β-analytic if it admits an analytically generating set at β.

We note that this condition is empty if $\beta = 0$. The condition of being β-analytic is just requiring that we have enough analytic elements in $T_{A,\mathcal{D}}$ to verify the KMS condition. Indeed we have the following.

Proposition 3.19. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a semifinite spectral triple and $\beta \in \mathbb{R} \setminus \{0\}$. Consider the following statements:*

  (i) The semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is β-analytic.
  (ii) There is a dense σ^+-invariant subspace $T_{A,\mathcal{D}}^0 \subseteq T_{A,\mathcal{D}}$ of elements satisfying that for any $T \in T_{A,\mathcal{D}}^0$ there is an $\epsilon \neq 0$ with $\epsilon\beta > 0$ such that the function $f_T : \mathbb{R} \to \mathcal{N}, f_T(t) := \sigma_t^+(T)$ has a bounded holomorphic extension to the strip $\{z \in \mathbb{C} : \mathrm{Im}(z) \in (-\beta - \epsilon, 0)\}$.
  (iii) There is a dense σ-invariant subspace $A_{\mathcal{D}}^0 \subseteq A_{\mathcal{D}}$ of elements satisfying that for any $a \in A_{\mathcal{D}}$ there is an $\epsilon \neq 0$ with $\epsilon\beta > 0$ such that the function $f_a : \mathbb{R} \to \mathcal{N}, f_a(t) := \sigma_t(a)$ has a bounded holomorphic extension to the strip $\{z \in \mathbb{C} : \mathrm{Im}(z) \in (-\beta - \epsilon, 0)\}$.

It holds that (i) implies (ii) which implies (iii). If $I := \{a \in A : P_{\mathcal{D}}aP_{\mathcal{D}} \in \mathbb{K}_{\mathcal{N}}^+\} = 0$, (iii) implies (i).

Proof. It is clear that (i) implies (ii) since for an analytically generating set S at β we can take $T_{A,\mathcal{D}}^0$ to be the σ^+-invariant $*$-algebra generated by $P_{\mathcal{D}}SP_{\mathcal{D}} \cup \mathcal{F}_{\mathcal{D}}$, where $\mathcal{F}_{\mathcal{D}} \subseteq \mathbb{K}_{\mathcal{N}}^+$ is the dense two sided ideal in $\mathcal{N}$ generated by the spectral projections of $\mathcal{D}$ over compact intervals in $\mathbb{R}$. The implication (ii)$\Rightarrow$(iii) is seen from taking $A_{\mathcal{D}}^0 := T_{A,\mathcal{D}}^0/(T_{A,\mathcal{D}}^0 \cap \mathbb{K}_{\mathcal{N}}^+)$.

If $I = 0$, the set $S = A_{\mathcal{D}}^0 \cap A$ is dense in A, and for every $s \in S$ we find that there is an $\epsilon \neq 0$ with $\epsilon\beta > 0$ such that $\mathrm{e}^{(\beta+\epsilon)|\mathcal{D}|}s\mathrm{e}^{-(\beta+\epsilon)|\mathcal{D}|}$ is a bounded operator in $\mathcal{N}$. Thus $P_{\mathcal{D}}SP_{\mathcal{D}} + \mathbb{K}_{\mathcal{N}}^+$ plainly generates T_A and for each $a \in S$ there is an $\epsilon \neq 0$ with $\epsilon\beta > 0$ satisfying that $\mathrm{e}^{(\beta+\epsilon)\mathcal{D}}P_{\mathcal{D}}aP_{\mathcal{D}}\mathrm{e}^{-(\beta+\epsilon)\mathcal{D}} \in \mathcal{N}^+$. We conclude that S is analytically generating at β and that (iii) implies (i) if $I = 0$. $\square$

Proposition 3.20. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a $\beta_{\mathcal{D}}$-analytic Li_1-summable spectral triple with positive $\mathcal{T}$-essential spectrum. For any extended limit ω as $t \to \beta_{\mathcal{D}}$, the state $\phi_{\omega,0}$ is a $\beta_{\mathcal{D}}$-KMS state on $T_{A,\mathcal{D}}$ for the one-parameter group σ^+.*

Proof. It follows from Proposition 3.19 that the dense subalgebra $T_{A,\mathcal{D}}^0 \subseteq T_{A,\mathcal{D}}$ consists of $\beta_{\mathcal{D}}$-analytic elements of $T_{A,\mathcal{D}}$. The twisted trace property relative to the one-parameter group σ^+ holds on $T_{A,\mathcal{D}}^0$ by direct computation: for all $T_1, T_2 \in T_{A,\mathcal{D}}^0$

$$\phi_\omega(T_1 T_2) = \omega\text{-}\lim_{t\to\beta_{\mathcal{D}}} \frac{\mathcal{T}(T_1 T_2 \mathrm{e}^{-t|\mathcal{D}|})}{\mathcal{T}(P_{\mathcal{D}}\mathrm{e}^{-t|\mathcal{D}|})} = \omega\text{-}\lim_{t\to\beta_{\mathcal{D}}} \frac{\mathcal{T}(\mathrm{e}^{t|\mathcal{D}|}T_2\mathrm{e}^{-t|\mathcal{D}|}T_1\mathrm{e}^{-t|\mathcal{D}|})}{\mathcal{T}(P_{\mathcal{D}}\mathrm{e}^{-t|\mathcal{D}|})} = \phi_\omega(\sigma_{-i\beta}(T_2)T_1).$$

By definition, $\phi_{\omega,0}$ is a $\beta_{\mathcal{D}}$-KMS-state for σ^+. $\square$

Corollary 3.21. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a $\beta_{\mathcal{D}}$-analytic Li_1-summable spectral triple with positive $\mathcal{T}$-essential spectrum. For any extended limit ω as $t \to \beta_{\mathcal{D}}$, the state ϕ_ω is a $\beta_{\mathcal{D}}$-KMS state on $A_{\mathcal{D}}$ for the one-parameter group σ, defined in Proposition 3.17.*

Proof. This follows from Proposition 3.19 and Proposition 3.20 because $\phi_{\omega,0}$ vanishes on the compacts and the fact that ϕ_ω is induced from $\phi_{\omega,0}$. $\square$

In practice, for a $\beta_{\mathcal{D}}$-analytic Li_1-summable spectral triple with positive $\mathcal{T}$-essential spectrum, we will want to check that in fact $T_{A,\mathcal{D}} = T_A$. In this case $A_{\mathcal{D}} = A/I$ and ϕ_ω induces a KMS-state on A/I. In the often occurring special case $P_{\mathcal{D}}AP_{\mathcal{D}} \cap \mathbb{K}_{\mathcal{N}}^+ = 0$, we obtain a KMS state on the algebra A. In practice, these things are all checkable and we will do so in several examples in the subsequent sections.

The special case $\beta = 0$ has been addressed by Voiculescu, [61, Proposition 4.6] under the assumption that $\mathcal{D}$ is positive. When $\beta_{\mathcal{D}} = 0$, β-analyticity is superfluous. By Proposition 3.2, when $\beta_{\mathcal{D}} = 0$, positive $\mathcal{T}$-essential spectrum is equivalent to $\mathcal{T}(P_{\mathcal{D}}) = \infty$.

Theorem 3.22. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a unital Li_1-summable semifinite spectral triple with $\beta_{\mathcal{D}} = 0$ and $\mathcal{T}(P_{\mathcal{D}}) = \infty$. Then A has a tracial state. Indeed, for any extended limit ω as $t \to 0$,*

$$\phi_\omega(a) := \omega\text{-}\lim_{t \to 0} \frac{\mathcal{T}(P_{\mathcal{D}} a e^{-t\mathcal{D}})}{\mathcal{T}(P_{\mathcal{D}} e^{-t\mathcal{D}})}$$

is a tracial state.

Theorem 3.22 applies to unital $\mathrm{Li}_{(0),1}$-summable semifinite spectral triples with $\mathcal{T}(P_{\mathcal{D}}) = \infty$ since $\mathrm{Li}_{(0),1}$-summability implies $\beta_{\mathcal{D}} = 0$.

Remark 3.23. If the heat trace has an asymptotic expansion as in Theorem 2.10, then Theorem 3.22 can be further simplified. Assume that there is a $p > 0$ and that for any $a \in \mathcal{A}$, there is a $\phi_0(a) \in \mathbb{C}$ such that $\phi_0(1) \neq 0$ and

$$\mathcal{T}(P_{\mathcal{D}} a e^{-t|\mathcal{D}|}) = \phi_0(a) t^{-p} + O(t^{-p+\epsilon}), \quad \text{as } t \to 0,$$

for some $\epsilon > 0$ (which can depend on a). Since $\phi_0(1) \neq 0$, it follows that ϕ_0 is continuous in the C^*-norm on $\mathcal{A}$ and $\phi_\omega(a) = \frac{\phi_0(a)}{\phi_0(1)}$ for all $a \in \mathcal{A}$. In fact, $\phi_0(a) = \Gamma(p)\mathrm{Res}_{z=p}\zeta(z; P_{\mathcal{D}} a, |\mathcal{D}|)$.

The construction of the state ϕ_ω in Corollary 3.21 involves the operator $P_{\mathcal{D}}$. We shall now provide a result allowing us to remove $P_{\mathcal{D}}$ from the definition of ϕ_ω in the presence of certain symmetries on the spectral triple. The result provides a checkable set of conditions to compute ϕ_ω by means of asymptotics of $e^{-t|\mathcal{D}|}$.

Lemma 3.24. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a unital Li_1-summable semifinite spectral triple and $\beta \geq 0$ a number such that $\mathcal{T}(e^{-t|D|}) < \infty$ for $t > \beta$ and $\mathcal{T}(e^{-t|\mathcal{D}|}) \nearrow \infty$ as $t \searrow \beta$. Assume that there exist self-adjoint operators $\gamma_1, \ldots, \gamma_N \in \mathcal{N} \cap \mathcal{A}'$ such that*

1. *$\sum_{j=1}^N \gamma_j^2 = 1$;*
2. *$\gamma_j \mathrm{Dom}(\mathcal{D}) \subseteq \mathrm{Dom}(\mathcal{D})$ and $[\mathcal{D}, \gamma_j]_+ := \mathcal{D}\gamma_j + \gamma_j\mathcal{D}$ has a bounded extension to $\mathcal{H}$;*
3. *For $j = 1, \ldots, N$ and some $\epsilon > 0$, the function $t \mapsto e^{-t|\mathcal{D}|}\gamma_j e^{t|\mathcal{D}|}$ extends to a norm continuous function from the interval $[\beta, \beta + \epsilon)$ to $\mathcal{N}$ with*

$$\lim_{t \to \beta} e^{-t|\mathcal{D}|}\gamma_j e^{t|\mathcal{D}|} = \gamma_j.$$

Then $\beta_{\mathcal{D}} = \beta$ and $\mathcal{D}$ has positive $\mathcal{T}$-essential spectrum. Moreover, for any extended limit ω as $t \to \beta_{\mathcal{D}}$,

$$\phi_\omega(a) = \omega\text{-}\lim_{t \to \beta_{\mathcal{D}}} \frac{\mathcal{T}(a e^{-t|\mathcal{D}|})}{\mathcal{T}(e^{-t|\mathcal{D}|})}.$$

Example 3.25. Before proceeding with the proof of the lemma, we give some examples of how the operators $\gamma_1, \ldots, \gamma_N \in \mathcal{N} \cap \mathcal{A}'$ can arise. The most trivial instance is when $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ is even, in which case the grading γ will satisfy the conditions of Lemma 3.24.

Here is a more geometric example. Let $(C^\infty(M), L^2(M, S), \slashed{D})$ denote the spectral triple defined from a Dirac operator on a closed Riemannian manifold M as in Proposition 2.14. If we take a collection $X_1, \ldots, X_N \in C^\infty(M, TM)$ of vector fields spanning the tangent bundle TM in all points, the collection of Clifford multiplication operators

$$\gamma_j := c_S(X_j)\left(\sum_{k=1}^N c_S(X_k)^2\right)^{-1/2}, \quad j = 1, \ldots, N,$$

is readily verified to satisfy the conditions of Lemma 3.24. This construction extends to semi-finite spectral triples defined from the fibrewise Dirac operator of a Riemannian spin^c submersion $\pi : M \to B$ (see more in [39]) and a measure on B by taking $X_1, \ldots, X_N$ to be vertical vector fields spanning the vertical tangent bundle $\ker d\pi$ in all points of M.

Proof. Recall that $F_{\mathcal{D}} := \mathcal{D}|\mathcal{D}|^{-1}$ modulo a finite trace projection. For any self-adjoint $a \in \mathcal{A}$, we write

$$\mathcal{T}(F_{\mathcal{D}} a e^{-t|\mathcal{D}|}) = \frac{1}{2}\mathcal{T}\left((F_{\mathcal{D}} a + a F_{\mathcal{D}})e^{-t|\mathcal{D}|}\right) = \frac{1}{2}\mathcal{T}\left(\sum_{j=1}^N \gamma_j(F_{\mathcal{D}} a + a F_{\mathcal{D}})e^{-t|\mathcal{D}|}\gamma_j\right) =$$

$$= \frac{1}{2}\mathcal{T}\left(\sum_{j=1}^N \gamma_j(F_{\mathcal{D}} a + a F_{\mathcal{D}})\gamma_j e^{-t|\mathcal{D}|}\right) +$$

$$+ \frac{1}{2}\mathcal{T}\left(\sum_{j=1}^N \gamma_j(F_{\mathcal{D}} a + a F_{\mathcal{D}})(e^{-t|\mathcal{D}|}\gamma_j e^{t|\mathcal{D}|} - \gamma_j)e^{-t|\mathcal{D}|}\right) =$$

$$= -\frac{1}{2}\mathcal{T}\left(\sum_{j=1}^{N}\gamma_j^2(F_{\mathcal{D}}a + aF_{\mathcal{D}})e^{-t|\mathcal{D}|}\right) +$$

$$-\frac{1}{2}\mathcal{T}\left(\sum_{j=1}^{N}\gamma_j([F_{\mathcal{D}},\gamma_j]_+a + a[F_{\mathcal{D}},\gamma_j]_+)\gamma_j e^{-t|\mathcal{D}|}\right)$$

$$+\frac{1}{2}\mathcal{T}\left(\sum_{j=1}^{N}\gamma_j(F_{\mathcal{D}}a + aF_{\mathcal{D}})(e^{-t|\mathcal{D}|}\gamma_j e^{t|\mathcal{D}|} - \gamma_j)e^{-t|\mathcal{D}|}\right) =$$

$$= -\mathcal{T}\left(F_{\mathcal{D}}ae^{-t|\mathcal{D}|}\right) - \frac{1}{2}\mathcal{T}\left(\sum_{j=1}^{N}\gamma_j([F_{\mathcal{D}},\gamma_j]_+a + a[F_{\mathcal{D}},\gamma_j]_+)\gamma_j e^{-t|\mathcal{D}|}\right)$$

$$+\frac{1}{2}\mathcal{T}\left(\sum_{j=1}^{N}\gamma_j(F_{\mathcal{D}}a + aF_{\mathcal{D}})(e^{-t|\mathcal{D}|}\gamma_j e^{t|\mathcal{D}|} - \gamma_j)e^{-t|\mathcal{D}|}\right).$$

Since $[\mathcal{D},\gamma_j]_+$ is bounded, $[F_{\mathcal{D}},\gamma_j]_+$ is compact and an approximation argument by finite $\mathcal{T}$-rank operators shows that

$$\tau\left(\sum_{j=1}^{N}\gamma_j([F_{\mathcal{D}},\gamma_j]_+a + a[F_{\mathcal{D}},\gamma_j]_+)\gamma_j e^{-t|\mathcal{D}|}\right) = o(\mathcal{T}(e^{-t|\mathcal{D}|})),$$

as $t \searrow \beta$. By norm continuity of $t \mapsto e^{-t|\mathcal{D}|}\gamma_j e^{t|\mathcal{D}|}$ we can also deduce that

$$\tau\left(\sum_{j=1}^{N}\gamma_j(F_{\mathcal{D}}a + aF_{\mathcal{D}})(e^{-t|\mathcal{D}|}\gamma_j e^{t|\mathcal{D}|} - \gamma_j)e^{-t|\mathcal{D}|}\right) = o(\mathcal{T}(e^{-t|\mathcal{D}|})),$$

as $t \searrow \beta$. In conclusion, for a self-adjoint a,

$$\mathcal{T}(F_{\mathcal{D}}ae^{-t|\mathcal{D}|}) = -\mathcal{T}(F_{\mathcal{D}}ae^{-t|\mathcal{D}|}) + o(\mathcal{T}(e^{-t|\mathcal{D}|})).$$

We can conclude that $\tau(F_{\mathcal{D}}ae^{-t|\mathcal{D}|}) = o(\mathcal{T}(e^{-t|\mathcal{D}|}))$ as $t \searrow \beta$. Since $2P_{\mathcal{D}} = F_{\mathcal{D}} + 1$, we have for any $a \in \mathcal{A}$ that

$$\mathcal{T}(ae^{-t|\mathcal{D}|}) = 2\mathcal{T}(P_{\mathcal{D}}ae^{-t|\mathcal{D}|}) + o(\mathcal{T}(e^{-t|\mathcal{D}|})), \quad \text{as } t \searrow \beta.$$

In particular $\beta = \beta_{\mathcal{D}}$ and $\mathcal{T}(P_{\mathcal{D}}e^{-t|\mathcal{D}|}) \nearrow \infty$ as $t \searrow \beta$. We compute that

$$\frac{\mathcal{T}(ae^{-t|\mathcal{D}|})}{\mathcal{T}(e^{-t|\mathcal{D}|})} = \frac{\mathcal{T}(P_{\mathcal{D}}ae^{-t|\mathcal{D}|})}{\mathcal{T}(P_{\mathcal{D}}e^{-t|\mathcal{D}|})} + o(1), \quad \text{as } t \searrow \beta.$$

In particular, for any extended limit ω as $t \to \beta$,

$$\omega\text{-}\lim_{t\to\beta}\frac{\mathcal{T}(ae^{-t|\mathcal{D}|})}{\mathcal{T}(e^{-t|\mathcal{D}|})} = \omega\text{-}\lim_{t\to\beta}\frac{\mathcal{T}(P_{\mathcal{D}}ae^{-t|\mathcal{D}|})}{\mathcal{T}(P_{\mathcal{D}}e^{-t|\mathcal{D}|})}.$$

This concludes the proof of the lemma. $\square$

3.3. Modular spectral triples and modular index theory

Modular spectral triples and their (equivariant) index theory were considered in [6,8,10], with the definition laid out most clearly in [56, Definition 2.1]. These were defined in order to study the (equivariant) index theory of KMS weights associated to periodic flows, so one might wonder how modular spectral triples fit into our scheme.

Given a KMS state $\psi : B \to \mathbb{C}$ with inverse temperature β for a one-parameter group $\sigma : \mathbb{R} \to \text{Aut}(B)$ on a unital C^*-algebra and a faithful expectation onto the fixed point algebra $\Phi : B \to B^\sigma$, we can emulate the constructions that inspired the definition of modular spectral triples.

First we construct the right C^*-module X over B^σ by completing B in the norm coming from the inner product

$$(x|y)_{B^\sigma} = \Phi(x^*y).$$

Then we can use [45] to construct $\text{Tr}_\psi : \mathbb{K}_{B^\sigma}(X) \to \mathbb{C}$ the trace dual to $\psi|_{B^\sigma}$. The action σ induces a one parameter unitary group on $L^2(X, \psi)$, and we let $\mathcal{D}$ be the generator of this one parameter group. By [6], when the action σ is periodic, the data

$$(B, L^2(X, \psi), \mathcal{D}, \mathbb{K}_{B^\sigma}(X)'', \phi_{\mathcal{D}}),$$

where $\phi_{\mathcal{D}}(a) := \mathrm{Tr}_{\psi}(e^{-\beta \mathcal{D}/2} a e^{-\beta \mathcal{D}/2})$, gives us a modular spectral triple. The operator $\mathcal{D}$ is affiliated to the semifinite algebra $(\mathbb{K}(X)^{\sigma^{\phi_{\mathcal{D}}}})''$.

Proposition 3.26. *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ *be a semifinite spectral triple such that for all* $t > \beta$

$$\mathcal{T}(P_{\mathcal{D}} e^{-t|\mathcal{D}|}) < \infty$$

and $\lim_{t \searrow \beta} \mathcal{T}(P_{\mathcal{D}} e^{-t\mathcal{D}}) = \infty$. *Assume that* $P_{\mathcal{D}} A P_{\mathcal{D}} \cap \mathbb{K}_{\mathcal{N}} = \{0\}$ *and the spectral triple is* β*-analytic. Then we obtain the KMS state* $\phi_{\omega} : A_{\mathcal{D}} \to \mathbb{C}$, *and provided that* $\mathcal{D}$ *has discrete spectrum we obtain an expectation* $\Phi : A_{\mathcal{D}} \to A_{\mathcal{D}}^{\sigma}$ *onto the fixed point subalgebra.*

Provided that the group σ *is periodic we then obtain a finitely summable modular spectral triple*

$$(\mathcal{A}_{\mathcal{D}}, L^2(A_{\mathcal{D}}, \phi_{\omega}), \mathcal{D}, (\mathbb{K}_{A_{\mathcal{D}}^{\sigma}}(A_{\mathcal{D}}))'', \phi_{\mathcal{D}}),$$

where the operator $\mathcal{D}$ *generates the one-parameter group induced by* σ *on* $L^2(A_{\mathcal{D}}, \phi_{\omega})$ *and* $\phi_{\mathcal{D}} := \mathrm{Tr}_{\phi_{\omega}}(e^{-\beta \mathcal{D}/2} \cdot e^{-\beta \mathcal{D}/2})$. *The spectral dimension is 1.*

Proof. The existence of the KMS state ϕ_{ω} comes from Corollary 3.21.

In general the action σ is real, but assuming that the operator $\mathcal{D}$ has discrete spectrum, the action will factor through a (compact) torus. To see this, one takes a rational basis of the eigenvalues (possibly an infinite basis), and takes a product over the circles corresponding to these individual actions.

Consequently, by averaging over this torus, there is an expectation $\Phi : \mathcal{N} \to \mathcal{N}^{\sigma}$ onto the fixed point algebra for $t \mapsto (T \mapsto e^{it\mathcal{D}} T e^{-it\mathcal{D}})$. Of course $\mathcal{D}$ is affiliated to the fixed-point algebra.

Finally if the action σ is periodic then [6] proves that we have a modular spectral triple, and that $\phi_{\mathcal{D}}((1+\mathcal{D}^2)^{-s/2}) < \infty$ for $s > 1$. $\square$

For circle actions there is a local index formula in twisted cyclic theory, but for real actions factoring through a torus there is not.

One serious issue that comes up in this more general setting is the compactness of the resolvent of $\mathcal{D}$, and determining summability. We leave this issue to another place.

4. The KMS-state ϕ_{ω} and Dixmier traces

In the present section we discuss a relation between the trace ϕ_{ω} from Theorem 3.22 and Dixmier traces. For a decreasing function $\psi : [0, \infty) \to (0, \infty)$ we denote $\Psi(t) := \int_0^t \psi(s) ds$. Let $\mathcal{L}_{\psi}(\mathcal{T})$ be the principal ideal defined as in Definition 2.6 . Let E_T be the spectral projection of an operator T affiliated with $\mathcal{N}$ and let $n_{\mathcal{T}}(s, T) := \mathcal{T}(E_{|T|}(s, \infty))$ be its distribution function.

The following result extends [47, Lemma 12.6.3].

Lemma 4.1. *Let* ψ *be a regularly varying function of index* -1. *Let* $T \in \mathcal{L}_{\psi}(\mathcal{T})$ *be strictly positive and* $\mu_{\mathcal{T}}(t, T) \sim \psi(t)$, $t \to \infty$. *For every* $q > 0$ *we have*

$$\mathcal{T}(e^{-T^{-q}/t}) \sim \Gamma(1 + \frac{1}{q}) \psi^{-1}(t^{-\frac{1}{q}}), \ t \to \infty.$$

Proof. The assumption $\mu_{\mathcal{T}}(t, T) \sim \psi(t)$ implies $\mu_{\mathcal{T}}(t, T^q) \sim [\psi(t)]^q$, $t \to \infty$. Since the distribution function is an inverse of the singular values function, it follows that $n_{\mathcal{T}}(s, T^q) \sim \psi^{-1}(s^{\frac{1}{q}})$, $s \to 0+$. Next we have $\mathcal{T}(E_{T^{-q}}(t)) \sim n_{\mathcal{T}}(1/t, T^q)$, $t \to \infty$. Thus,

$$\mathcal{T}(E_{T^{-q}}(t)) \sim \psi^{-1}(t^{-\frac{1}{q}}), \ t \to \infty.$$

Since ψ is varying regularly with index -1, [2, Theorem 1.5.12] implies that ψ^{-1} varies regularly with index -1, too. Thus, $\mathcal{T}(E_{T^{-q}})$ varies regularly with index $\frac{1}{q}$. Writing the heat trace as a Laplace transform

$$\mathcal{T}(e^{-T^{-q}/t}) = \int_0^{\infty} e^{-z/t} \, d\mathcal{T}(E_{T^{-q}}(z))$$

and using the Karamata theorem [44, Chapter IV, Theorem 8.1] we obtain

$$\mathcal{T}(e^{-T^{-q}/t}) \sim \Gamma(1 + \frac{1}{q}) \psi^{-1}(t^{-\frac{1}{q}}), \ t \to \infty. \ \square$$

Let $P_a : L^{\infty}(0, \infty) \to L^{\infty}(0, \infty)$ be the exponentiation operator defined by $(P_a f)(t) := f(t^a), \ t > 0$.

Definition 4.2 ([9]). An extended limit ω as $t \to \infty$ on $L^\infty(0, \infty)$ is said to be exponentiation invariant if

$$\omega\text{-}\lim_{t\to\infty}(P_a f)(t) = \omega\text{-}\lim_{t\to\infty} f(t)$$

for every $f \in L^\infty(0, \infty)$ and every $a > 0$.

Definition 4.3. Let $\psi : [0, \infty) \to (0, \infty)$ be a regularly varying function of index -1. For any extended limit ω as $t \to \infty$ on $L^\infty(0, \infty)$ a linear extension of the weight

$$\mathfrak{T}_{\omega,\psi}(T) := \omega\text{-}\lim_{t\to\infty} \frac{1}{\Psi(t)} \int_0^t \mu_{\mathfrak{T}}(s, T)\,ds, \quad 0 \leq T \in \mathcal{L}_\psi(\mathfrak{T})$$

is said to be a Dixmier trace on $\mathcal{L}_\psi(\mathfrak{T})$.

Remark 4.4. Usually Dixmier traces are defined on Lorentz ideals corresponding to the function Ψ (which are strictly larger than $\mathcal{L}_\psi(\mathfrak{T})$) by exactly the same formula as in Definition 4.3 (see e.g. [13,18,47]). Then, Dixmier traces on $\mathcal{L}_\psi(\mathfrak{T})$ are restrictions of those on Lorentz ideal to $\mathcal{L}_\psi(\mathfrak{T})$. Since we do not deal with Lorentz ideals here, it is convenient to define Dixmier traces directly on $\mathcal{L}_\psi(\mathfrak{T})$.

It should be pointed out that on Lorentz ideals to define Dixmier traces one needs an additional assumption on ω: either dilation invariance [18,47] or exponentiation invariance [24,59]. As it was shown in [58, Lemma 16 and Theorem 17] these requirements are redundant on $\mathcal{L}_\psi(\mathfrak{T})$.

Also note that the condition of the regular variation of ψ is not optimal. The existence of Dixmier traces was proved in [19] under the milder assumption that

$$\liminf_{t\to\infty} \frac{\Psi(2t)}{\Psi(t)} = 1.$$

However, in this setting one needs an additional assumption on ω that $\omega\text{-}\lim_{t\to\infty} \frac{\Psi(2t)}{\Psi(t)} = 1$, termed Ψ-compatibility in [43]. Since we assume ψ to be a regularly varying function of index -1, it follows that

$$\lim_{t\to\infty} \frac{\Psi(2t)}{\Psi(t)} = 1.$$

Thus, every extended limit is Ψ-compatible.

The proof of the following theorem is the same as that of [47, Theorem 8.5.1] and thus omitted. Note however, that in [47] the result was proved for Lorentz ideals and required dilation invariance of the extended limit ω. For the case of $\mathcal{L}_\psi(\mathfrak{T})$ one can refer to [58, Lemma 15] to remove this assumption.

Theorem 4.5. Let $f \in C^2[0, \infty)$ be a bounded function such that $f(0) = f'(0) = 0$. Let $T \in \mathcal{L}_\psi(\mathfrak{T})$ be positive and let $B \in \mathcal{N}$. For every extended limit ω as $t \to \infty$ on $L^\infty(0, \infty)$ we have

$$\omega\text{-}\lim_{t\to\infty}\left(\frac{1}{\Psi(t)}\int_1^t \mathfrak{T}(f(sT)B)\frac{ds}{s^2}\right) = \int_0^\infty f(s)\frac{ds}{s^2} \cdot \omega\text{-}\lim_{t\to\infty}\left(\frac{1}{\Psi(t)}\int_1^t \mathfrak{T}(e^{-(sTB)^{-1}})\frac{ds}{s^2}\right).$$

Below we will need the relation between generalized heat kernels and Dixmier traces on $\mathcal{L}_\psi(\mathfrak{T})$, which was proved in [24] under the additional assumption that

$$A_\Psi(\alpha) := \lim_{t\to\infty} \frac{\Psi(t^\alpha)}{\Psi(t)} \quad \text{exists for every } \alpha > 0. \tag{4.1}$$

Recall the notation $\Psi(t) = \int_0^t \psi(s)ds$. The corresponding (natural) assumption on ψ is that

$$\alpha \cdot \lim_{t\to\infty} \frac{\psi(t^\alpha)t^{\alpha-1}}{\psi(t)} \quad \text{exists for every } \alpha > 0. \tag{4.2}$$

Note that condition (4.2) implies the condition (4.1) [24, Proposition 1.7]. By the L'Hôpital rule, the number appearing in Eq. (4.2) equals $A_\Psi(\alpha)$.

Let $H : L^\infty(0, \infty) \to L^\infty(0, \infty)$ be the Cesaro mean defined as follows:

$$(Hf)(t) := \frac{1}{t}\int_0^t f(s)\,ds, \ t > 0.$$

Let $M_\Psi : L^\infty(0, \infty) \to L^\infty(0, \infty)$ be the Cesaro mean twisted by Ψ, that is

$$(M_\Psi f)(t) := [(H(f \circ \Psi^{-1})) \circ \Psi](t) = \frac{1}{\Psi(t)}\int_0^t f(s)\psi(s)\,ds, \ t > 0.$$

Lemma 4.6. *If ψ satisfies condition* (4.2), *then*

$$M_\psi \circ P_a - P_a \circ M_\psi : L^\infty(0, \infty) \to C_0(0, \infty)$$

for every $a > 0$.

Proof. For $f \in L^\infty(0, \infty)$ we have

$$(M_\psi \circ P_a f)(t) = \frac{1}{\Psi(t)} \int_0^t f(s^a) \psi(s) \, ds = \frac{1}{\Psi(t)} \int_0^{t^a} f(s) \psi(s^{1/a}) \frac{s^{1/a-1}}{a} \, ds.$$

Using conditions (4.1) and (4.2) we obtain

$$(M_\psi \circ P_a f)(t) \in A_\psi(a) A_\psi(\tfrac{1}{a}) \frac{1}{\Psi(t^a)} \int_0^{t^a} f(s) \psi(s) \, ds + C_0.$$

Direct calculations show that $A_\psi(a) A_\psi(\tfrac{1}{a}) = 1$. Thus,

$$(M_\psi \circ P_a - P_a \circ M_\psi) f \in C_0. \quad \square$$

Lemma 4.7. *If $\psi : [0, \infty) \to (0, \infty)$ is a decreasing function with regular variation of index -1, then for any $d > 0$, as $t \to \infty$,*

$$\psi(t) = o((\log(t))^{-d}).$$

Proof. Since ψ is decreasing and $\lim_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 2^{-1}$, we can for any $\epsilon \in (0, 1)$ find a constant $C_\epsilon > 0$ such that for $t \in (2^k, 2^{k+1}]$,

$$\psi(t) \leq C_\epsilon (2 - \epsilon)^{-k} = C_\epsilon 2^{-\log_2(2-\epsilon)k} \leq C_\epsilon t^{-\log_2(2-\epsilon)}.$$

For details, see [31, Proposition 2.21]. Therefore, by taking $\epsilon < 2 - \sqrt{2}$ we see that $\psi(t) = o(t^{-1/2})$ which implies $\psi(t) = o((\log(t))^{-d})$. $\quad \square$

For the next result we need to assume the (stronger) condition, that the function ψ satisfies

$$\lim_{t\to\infty} \frac{t^2 \psi(t)}{\psi^{-1}(1/t)} = c, \tag{4.3}$$

for some constant $c > 0$.

Remark 4.8. For every $k \in \mathbb{Z}$, the functions $\psi(t) = \frac{\log^k t}{t}$ and $\psi(t) = \frac{\log^k(\log t)}{t \cdot \log t}$ satisfy condition (4.3). The functions $\psi = \Psi'$ with $\Psi(t) = e^{\log^\beta t}$ do not satisfy (4.3) for any $\beta > 0$.

In the following result we use a singular values function of an unbounded operator affiliated with $\mathbb{N}$. For such operators the formula (2.1) cannot be used. The singular values function of an operator T affiliated with $\mathbb{N}$ is defined [23] as follows:

$$\mu_{\mathbb{T}}(t, T) := \inf\{s \geq 0 : n_{\mathbb{T}}(s, T) \leq t\}.$$

Theorem 4.9. *Let $d > 0$ and $\psi : [0, \infty) \to (0, \infty)$ be a decreasing function with regular variation of index -1, satisfying conditions* (4.2) *and* (4.3). *Assume that $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbb{N}, \mathbb{T})$ is a unital semifinite spectral triple such that*

1. *$\mathbb{T}$ is infinite;*
2. *$\mathcal{D}$ is positive;*
3. *$\mu_{\mathbb{T}}(t, \mathcal{D}) \sim \psi(t)^{-1/d}$.*

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbb{N}, \mathbb{T})$ is $\mathrm{Li}_{(0),1}$-summable with positive $\mathbb{T}$-essential spectrum. Moreover, for every $a \in \mathcal{A}_\mathcal{D}$ and every exponentiation invariant extended limit ω as $t \to \infty$ we have

$$\phi_{\tilde\omega}(a) = \mathbb{T}_{\omega,\psi}(a(1 + \mathcal{D}^2)^{-d/2}),$$

where $\tilde\omega := \omega \circ (J M_\psi J)$ and $J : L^\infty(0, \infty) \to L^\infty(0, \infty)$ is defined as the pullback along $t \mapsto t^{-1}$.

Proof. By Lemma 4.7 $\psi(t) = o((\log(t))^{-d})$. Therefore, $\mu_{\mathbb{T}}(t, \mathcal{D}) = o(\log(t))$ and the semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathbb{N}, \mathbb{T})$ is $\mathrm{Li}_{(0),1}$-summable. Since $\mathbb{T}$ is infinite, $\mathbb{T}(e^{-t\mathcal{D}}) \nearrow \infty$ as $t \searrow 0$. The operator $\mathcal{D}$ therefore has positive $\mathbb{T}$-essential spectrum. Thus,

$$\phi_{\tilde\omega}(a) = \tilde\omega\text{-}\lim_{t\to 0} \frac{\mathbb{T}(a e^{-t\mathcal{D}})}{\mathbb{T}(e^{-t\mathcal{D}})}$$

is a trace on $\mathcal{A}$ by Theorem 3.22. The assumption on $\mathcal{D}$ implies that $\mu_{\mathcal{T}}(t, (1 + \mathcal{D}^2)^{-d/2}) \sim \psi(t)$, $t \to \infty$. Applying Lemma 4.1 with $T = (1 + \mathcal{D}^2)^{-d/2}$ and $q = 1/d$ yields

$$\mathcal{T}(e^{-\mathcal{D}/t}) \sim \Gamma(1 + d)\psi^{-1}(t^{-d}), \ t \to \infty.$$

Using the properties of extended limits, the definitions of P_a and J and Lemma 4.6, we obtain

$$
\begin{aligned}
\phi_{\tilde{\omega}}(a) &= \frac{1}{\Gamma(1 + d)} \cdot \omega\text{-}\lim_{t \to \infty} (J \circ M_\Psi) \left(\frac{\mathcal{T}(ae^{-\mathcal{D}/t})}{\psi^{-1}(t^{-d})} \right) \\
&= \frac{1}{\Gamma(1 + d)} \cdot \omega\text{-}\lim_{t \to \infty} (J \circ M_\Psi \circ P_d) \left(\frac{\mathcal{T}(ae^{-(t\mathcal{D}^{-d})^{-1/d}})}{\psi^{-1}(1/t)} \right) \\
&= \frac{1}{\Gamma(1 + d)} \cdot \omega\text{-}\lim_{t \to \infty} (J \circ P_d \circ M_\Psi) \left(\frac{\mathcal{T}(ae^{-(t\mathcal{D}^{-d})^{-1/d}})}{\psi^{-1}(1/t)} \right).
\end{aligned}
\tag{4.4}
$$

Using the definition of M_Ψ and assumption (4.3) we obtain

$$
\begin{aligned}
M_\Psi \left(\frac{\mathcal{T}(ae^{-(t\mathcal{D}^{-d})^{-1/d}})}{\psi^{-1}(1/t)} \right) &= \frac{1}{\Psi(t)} \int_0^t \frac{\mathcal{T}(ae^{-(s\mathcal{D}^{-d})^{-1/d}})}{\psi^{-1}(1/s)} \psi(s)\, ds \\
&\in \frac{1}{\Psi(t)} \int_0^t \mathcal{T}(ae^{-(s\mathcal{D}^{-d})^{-1/d}}) \frac{ds}{s^2} + C_0(0, \infty).
\end{aligned}
\tag{4.5}
$$

Since ω is exponentiation invariant extended limit, combining (4.4) and (4.5) we obtain

$$\phi_{\tilde{\omega}}(a) = \frac{1}{\Gamma(1 + d)} \cdot \omega\text{-}\lim_{t \to \infty} \left(\frac{1}{\Psi(1/t)} \int_1^{1/t} \mathcal{T}(ae^{-(s\mathcal{D}^{-d})^{-1/d}}) \frac{ds}{s^2} \right).$$

Now we apply Theorem 4.5 twice with $T = \mathcal{D}^{-d}$, $f(x) = e^{-x^{-1/d}}$ and then with $f(x) = e^{-x^{-1}}$. Since

$$\int_0^\infty e^{-x^{-q}} \frac{dx}{x} = \Gamma(1 + \frac{1}{q}),$$

we obtain

$$\phi_{\tilde{\omega}}(a) = \omega\text{-}\lim_{t \to \infty} J \left(\frac{1}{\Psi(t)} \int_1^t \mathcal{T}(ae^{-(s\mathcal{D}^{-d})^{-1}}) \frac{ds}{s^2} \right) = \mathcal{T}_{\omega, \psi} \left(a(1 + \mathcal{D}^2)^{-d/2} \right),$$

where the last equality follows from [24, Theorem 4.7]. □

We can now provide sufficient conditions on a general $\mathrm{Li}_{(0),1}$-summable unital semifinite spectral triples ensuring a relation between the trace ϕ_ω of Theorem 3.22 and Dixmier traces.

Corollary 4.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \mathcal{T})$ be a unital semifinite spectral triple with $\mathcal{T}(P_\mathcal{D}) = \infty$. Assume that there is a number $d > 0$ and a decreasing function $\psi : [0, \infty) \to (0, \infty)$ with regular variation of index -1 satisfying conditions (4.2) and (4.3) and*

$$\mu(t, P_\mathcal{D} \mathcal{D}) \sim \psi(t)^{-1/d}.$$

Then, $\beta_\mathcal{D} = 0$ and for any exponentiation invariant extended limit ω as $t \to \infty$ and $a \in A_\mathcal{D}$,

$$\phi_{\tilde{\omega}}(a) = \mathcal{T}_{\omega, \psi}(P_\mathcal{D} a(1 + \mathcal{D}^2)^{-d/2}),$$

where $\tilde{\omega}$ is as in Theorem 4.9.

The corollary follows immediately from Theorem 4.9 applied to the unital semifinite spectral triple $(\mathfrak{T}, P_\mathcal{D}\mathcal{H}, P_\mathcal{D}\mathcal{D}, \mathcal{N}^+, \mathcal{T})$ where $\mathfrak{T}$ is the $*$-algebra generated by $P_\mathcal{D}AP_\mathcal{D}$.

Example 4.11. Let us revisit the spectral triple constructed in Proposition 2.16 . If $\psi : [0, \infty) \to (0, \infty)$ is a smoothly varying function with $\lim_{t \to 0} \psi(t) = 0$ and $\psi(t)^{-1} = O(t^{1/n})$ as $t \to \infty$, and $\slashed{D}$ a Dirac operator on a Riemannian closed n-dimensional manifold M, a ψ-summable spectral triple $(C^\infty(M), L^2(M, S), \slashed{D}_\psi)$ was constructed in Proposition 2.16, where $\slashed{D}_\psi := F_{\slashed{D}}\psi(|\slashed{D}|^n)^{-1}$. The Weyl law for $|\slashed{D}|$ and Theorem 2.10 applied to $B = P_{\slashed{D}}$ shows that the order of the spectral asymptotics of $|\slashed{D}|$ coincides with the order of the spectral asymptotics of $P_{\slashed{D}}\slashed{D}$ so $\mu(t, P_{\slashed{D}}\slashed{D}_\psi) \sim c\psi(t)^{-1}$ for some constant $c > 0$.

If ψ has regular variation of index $-d$ for a $d > 0$, Corollary 4.10 shows that the tracial state on $C(M)$ defined from the spectral triple $(C^\infty(M), L^2(M, S), \slashed{D}_\psi)$ takes the form

$$\phi_\omega(a) = c' \operatorname{Tr}_{\omega, \psi}(P_{\slashed{D}} a\psi(|\slashed{D}|^n)^{1/d}),$$

for some proportionality constant c'. Applying Connes' trace theorem as in [31, Theorem 9.1], it follows that

$$\phi_\omega(a) = \fint_M a \, dV,$$

where dV denotes the Riemannian volume measure on M and $\fint_M$ the normalized integral.

The computation above requires ψ to have strictly negative index of regular variation. We note that by Proposition 2.18, the computation above can only extend to the spectral triple from Proposition 2.19 on a crossed product by a local diffeomorphism if the local diffeomorphism acts isometrically.

5. The KMS-state ϕ_ω in examples

We are now in a state where we can compute the KMS-states associated with the spectral triples considered in Section 2.2 . The computations for the KMS-states associated with the unbounded Kasparov cycle on Cuntz–Pimsner algebras considered in Section 2.2.3 are more involved and dedicated a separate section, Section 6 .

5.1. Dirac operators on closed manifolds

For a closed manifold M with a Dirac operator $\slashed{D}$ acting on a Clifford bundle $S \to M$, we consider the spectral triple $(C^\infty(M), L^2(M, S), \slashed{D})$ as in Proposition 2.14 . The following theorem can be deduced immediately from Example 3.5 or from Corollary 4.10 and Connes' trace theorem for pseudo-differential operators.

Theorem 5.1. Let $(C^\infty(M), L^2(M, S), \slashed{D})$ be the spectral triple associated with a Dirac operator on a closed manifold, ω an extended limit as $t \to 0$ and ϕ_ω the associated tracial state from Theorem 3.22 . The trace ϕ_ω is the normalized volume integral on M, i.e. for $a \in C(M)$,

$$\phi_\omega(a) = \fint_M a \, dV.$$

For completeness, let us describe the Toeplitz algebras $T_{C(M)}$, $T_{C(M),\slashed{D}}$ and the flow σ on $C(M)_{\slashed{D}}$ in this example. We remark that since ϕ_ω is a trace in this case, the flow induced from our construction is irrelevant for the study of ϕ_ω but it could nevertheless serve to clarify the constructions of Section 3.2. The relevant algebras are all contained in the C^*-algebra $\Psi^0_{C^*}(M, S)$ defined as the C^*-closure of the $*$-algebra $\Psi^0_{cl}(M, S)$ of zeroth order classical pseudo-differential operators acting on $L^2(M, S)$. It is well known that $\Psi^0_{C^*}(M, S)$ fits into a short exact sequence

$$0 \to \mathbb{K}(L^2(M, S)) \to \Psi^0_{C^*}(M, S) \xrightarrow{\text{symb}} C(S^*M, \text{End}(S)) \to 0,$$

where symb denotes the continuous extension of the principal symbol mapping $\Psi^0_{cl}(M, S) \to C^\infty(S^*M, \text{End}(S))$. Since $P_{\slashed{D}}$ is a projection in $\Psi^0_{cl}(M, S)$, we can consider the C^*-algebra $\Psi^0_{C^*,+}(M, S) := P_{\slashed{D}} \Psi^0_{C^*}(M, S) P_{\slashed{D}}$ and we obtain a short exact sequence

$$0 \to \mathbb{K}(P_{\slashed{D}} L^2(M, S)) \to \Psi^0_{C^*,+}(M, S) \xrightarrow{\text{symb}} C(S^*M, \text{End}(p_{\slashed{D}}S)) \to 0,$$

where $p_{\slashed{D}} \in C^\infty(S^*M, \text{End}(S))$ denotes the principal symbol of $P_{\slashed{D}}$. The algebras $T_{C(M)}$ and $T_{C(M),\slashed{D}}$ are characterized by the following commuting diagram with exact rows

The composition of the mappings in the right most column coincides with the pull back homomorphism $C(M) \to C(S^*M)$ composed with the inclusion $C(S^*M) \subseteq C(S^*M, \text{End}(p_{\slashed{D}}S))$.

To describe the flow σ on $C(M)_{\slashed{D}}$, we describe it on $C^\infty(S^*M, \text{End}(p_{\slashed{D}}S))$. Surjectivity of the principal symbol mapping implies that any $a \in C^\infty(S^*M, \text{End}(p_{\slashed{D}}S))$ is the symbol of an operator $A \in P_{\slashed{D}} \Psi^0_{cl}(M, S) P_{\slashed{D}}$. By Egorov's theorem [21] (see also [33, Section IV] and [20]), $e^{is\slashed{D}} A e^{-is\slashed{D}}$ is again an element of $P_{\slashed{D}} \Psi^0_{cl}(M, S) P_{\slashed{D}}$ and the expression $\sigma_s(a) := \text{symb}(e^{is\slashed{D}} A e^{-is\slashed{D}})$ gives a well defined flow on $C^\infty(S^*M, \text{End}(p_{\slashed{D}}S))$. Again by Egorov's theorem, using that $\sigma_s(a) = \text{symb}(e^{is|\slashed{D}|} A e^{-is|\slashed{D}|})$ for $A \in P_{\slashed{D}} \Psi^0_{cl}(M, S) P_{\slashed{D}}$, we have that $\sigma_s(a) = g_s^*(a)$ where $g_s : S^*M \to S^*M$ is the Hamiltonian flow associated with the

symbol $|\xi|$ of $|\slashed{D}|$. On the cosphere bundle, this Hamiltonian flow coincides with the geodesic flow. We conclude that the flow σ is induced by geodesic flow and $C(M)_{\slashed{D}} \subseteq C(S^*M)$ is a closed subalgebra invariant under geodesic flow. This discussion should be compared with that in [14].

As in Section 2.2.1, we consider a local diffeomorphism $g : M \to M$. We assume that g acts conformally and lifts to $S \to M$. Since g acts conformally, $|(Dg)_x^T \xi| = c_g^{1/2}|\xi|$ and it is readily verified that g is compatible with the decreasing function $\psi(t) := \frac{1}{\log(1+t^{2/n})}$ from the mean value theorem and $\psi'(t) = O\left(\psi(t)^2\right)$. We use the notation $\slashed{D}_{\log} := \slashed{D}_\psi$ for this particular choice of ψ. Note that

$$\slashed{D}_{\log} = F_{\slashed{D}} \log(1 + \slashed{D}^2) \quad \text{and} \quad \mathrm{e}^{-t|\slashed{D}_{\log}|} = (1 + \slashed{D}^2)^{-t}.$$

By Proposition 2.19, we arrive at a spectral triple $(\mathcal{A}, L^2(M, S), \slashed{D}_{\log})$ where $\mathcal{A}$ is the $*$-algebra generated by $C^\infty(M)$ and an isometry V_g. Let us compute KMS-state constructed from $(\mathcal{A}, L^2(M, S), \slashed{D}_{\log})$. Before diverting into this computation, we recall that the C^*-closure of $\mathcal{A}$ coincides with the image of a representation of O_{E_g}. As such, we can write elements of $\mathcal{A}$ as linear span of elements of the form $S_\mu S_\nu^*$ where $\mu = a_1 V_g \cdots a_k V_g$ and $\nu = b_1 V_g \cdots b_l V_g$, where $a_1, \ldots, a_k, b_1, \ldots, b_l \in C^\infty(M)$.

Proposition 5.2. Set $S := C^\infty(M) \cup C^\infty(M)V_g \subseteq \mathcal{A}$. For any $\beta \in \mathbb{R}$, the set S is an analytically generating set at β for $(\mathcal{A}, L^2(M, S), \slashed{D}_{\log})$.

Proof. For notational convenience, write $\mathcal{D} = \slashed{D}_{\log}$. The set S generates $\mathcal{A}$, so $P_{\mathcal{D}} S P_{\mathcal{D}} + \mathbb{K}$ generates $T_{\mathcal{A}}$. For any $\beta \in \mathbb{R}$, and $a \in S$

$$\mathrm{e}^{-\beta \mathcal{D}} P_{\mathcal{D}} a P_{\mathcal{D}} \mathrm{e}^{\beta \mathcal{D}} = (1 + \slashed{D}^2)^{-\beta} P_{\slashed{D}} a P_{\slashed{D}} (1 + \slashed{D}^2)^{\beta}.$$

The proposition follows from that $\mathrm{Dom}((1 + \slashed{D}^2)^\beta) = H^{2\beta}(M, S)$ as Banach spaces and any $a \in S$ extends to a continuous operator on the Sobolev spaces $H^{2\beta}(M, S)$ for all β. $\square$

Theorem 5.3. Let M be a connected n-dimensional Riemannian manifold, $\slashed{D}$ a Dirac operator on $S \to M$, and $g : M \to M$ a local diffeomorphism acting conformally and lifting to S. Then the spectral triple $(\mathcal{A}, L^2(M, S), \slashed{D}_{\log})$ is Li_1-summable, has positive essential spectrum with $\beta_{\mathcal{D}} = n/2$ and is $n/2$-analytic.

Assume for all $m \in \mathbb{N}_+$, that the set of fixed points

$$\{x \in M : g^m(x) = x\},$$

has measure zero with respect to the volume measure. Then the state ϕ_ω on A constructed from Corollary 3.21 is independent of ω and takes the form

$$\phi_\omega(S_\mu S_\nu^*) = \delta_{|\mu|,|\nu|} \int_M \mathfrak{L}_g(c_g^{n/2} b_k^* \mathfrak{L}_g(c_g^{n/2} b_{k-1}^* \mathfrak{L}_g(\cdots \mathfrak{L}_g(c_g^{n/2} b_1^* a_1) a_2) \cdots a_{k-1}) a_k) \, \mathrm{d}V, \tag{5.1}$$

For $\mu = a_1 V_g \cdots a_k V_g$ and $\nu = b_1 V_g \cdots b_l V_g$. Here $\mathrm{d}V$ denotes the Riemannian volume form.

The state ϕ_ω viewed as a state on O_{E_g} via its representation on $L^2(M, S)$ is KMS for the action defined by $\gamma_t(aV_g) := c_g^{itn/2} aV_g$ with inverse temperature 1.

We note that the state Theorem 5.3 is a KMS-state on a Cuntz–Pimsner algebra, but not for its gauge action. If $c_g(x) < 1$ for some $x \in M$, the generator of γ is not positive on E_g and the Laca–Neshveyev correspondence [45] does not apply in the context of O_{E_g} with the action γ.

In the case $k = l = 0$, the formula (5.1) should be interpreted as $\phi_\omega(a) = \int_M a \, \mathrm{d}V$ for $a \in C^\infty(M)$. This special case follows from Connes' trace theorem.

Proof. For $t > n/2$, standard techniques of pseudo-differential operators show that the integral kernel K_t of the trace class operator $P_{\slashed{D}}(1 + \slashed{D}^2)^{-t}$ belongs to $C(M \times M, \mathrm{END}(S)) \cap C^\infty(M \times M \setminus \Delta_M, \mathrm{END}(S))$. Here $\Delta_M \subseteq M \times M$ denotes the diagonal and $\mathrm{END}(S)$ denotes the big endomorphism bundle defined by $\mathrm{END}(S)_{(x,y)} := \mathrm{Hom}(S_x, S_y)$ for $(x, y) \in M \times M$. By [17, Proposition 8.3], V_g^* takes the form

$$V_g^* = N^{1/2} \mathfrak{L}_{S,g} c_g^{-n/4},$$

where $\mathfrak{L}_{S,g}$ is the L^2-extension of the operator

$$\mathfrak{L}_{S,g} : C(M, S) \to C(M, S), \quad \mathfrak{L}_{S,g}\xi(x) := \sum_{g(y)=x} u_g(y)^{-1} \xi(y).$$

Take $a_1, \ldots, a_k, b_1, \ldots, b_l \in C^\infty(M)$ and write $\mu = a_1 V_g \cdots a_k V_g$ and $\nu = b_1 V_g \cdots b_l V_g$. We introduce the notation $\tilde{a}_j := a_j c_g^{n/4}$ and $\tilde{b}_j := b_j c_g^{-n/4}$. We can compute for $t > n/2$ that

$$
\mathrm{Tr}_{L^2(M,S)}(P_{\not{D}} S_\mu S_\nu^* e^{-t|\not{D}_{\log}|}) = \mathrm{Tr}_{L^2(M,S)}(a_1 V_g \cdots a_k V_g V_g^* b_l^* V_g^* \cdots V_g^* b_1^* P_{\not{D}} (1 + \not{D}^2)^{-t})
$$

$$
= N^{-(k-l)/2} \, \mathrm{Tr}_{L^2(M,S)}(a_1 c_g^{n/4} u_g g^* \cdots a_k c_g^{n/4} u_g g^* \mathfrak{L}_{S,g} c_g^{-n/4} b_l^* \cdots c_g^{-n/4} \mathfrak{L}_{S,g} b_1^* P_{\not{D}} (1 + \not{D}^2)^{-t})
$$

$$
= N^{-(k-l)/2} \int_M \sum_{(x_1, \ldots, x_{k+l}) \in M(x,k,l)} \left(\prod_{j=1}^k \tilde{a}_j(x_j) \right) \left(\prod_{j=k+1}^{k+l} \tilde{b}_{j-k}(x_j)^* \right) K_t(x_{k+1}, x_1) \mathrm{d}V(x),
$$

where $M(x, k, l) \subseteq M^{k+l}$ is defined as the $k + l$-tuples $(x_1, \ldots, x_{k+l})$ such that for $j = 1, \ldots, k$, $x_j = g^{j-1}(x)$, $x_k = g(x_{k+l})$ and for $j = k+1, \ldots, k+l-1$, $g(x_j) = x_{j+1}$. Note that $M(x, k, l)$ is finite for all x, because g is a local homeomorphism, and that $x_1 = x$ for $(x_1, \ldots, x_{k+l}) \in M(x, k, l)$.

Define $M^0(x, k, l) \subseteq M(x, k, l)$ as the $k + l$-tuples $(x_1, \ldots, x_{k+l})$ where $x_{k+1} = x_1$. The set $M^0(x, k, l)$ can be characterized as the $k + l$-tuples $(x_1, \ldots, x_{k+l})$ with $x = x_1 = x_{k+1}$ and $x_k = g^l(x_{k+l})$ such that for $j = 1, \ldots, k$, $x_j = g^{j-1}(x)$, and for $j = k+1, \ldots, k+l-1$, $x_{j+1} = g(x_j)$. In particular, if $M^0(x, k, l)$ is non-empty then $g^k(x) = g^l(x)$. In other words, $(x_1, \ldots, x_{k+l}) \in M^0(x, k, l)$ if and only if $g^k(x) = g^l(x)$, and $x_j = g^{j-1}(x)$ for $j = 1, \ldots, k$ and $x_{k+j} = g^{j-1}(x)$ for $j = 1, \ldots, l$. Therefore, $M^0(x, k, l)$ contains at most one element. In particular, since $M^0(x, k, l)$ is non-empty then $g^k(x) = g^l(x)$ and the set of fixed points $\{x \in M : g^m(x) = x\}$ has measure zero for all $m \in \mathbb{N}_+$ by assumptions, then

$$
M^0(x, k, l) = \emptyset \quad \text{if } k \neq l \text{ a.e. in } x.
$$

As t approaches $n/2$, the integral kernel K_t localizes (up to lower order term) to the diagonal and the leading order terms come from the sum over $M^0(x, k, l)$. The Weyl law for $\not{D}^2$ and an explicit pseudo-differential computation of the principal symbol of $P_{\not{D}} (1 + \not{D}^2)^{-t}$ imply that there are a constant c and an $\epsilon > 0$ only depending on $\not{D}$ such that

$$
\mathrm{Tr}_{L^2(M,S)}(P_{\not{D}} S_\mu S_\nu^* e^{-t|\not{D}_{\log}|}) =
$$

$$
= c \delta_{k,l} (t - n/2)^{-1} \int_M \sum_{(x_1, \ldots, x_{k+l}) \in M^0(x,k,l)} \left(\prod_{j=1}^k \tilde{a}_j(x_j) \right) \left(\prod_{j=k+1}^{k+l} \tilde{b}_{j-k}(x_j)^* \right) \mathrm{d}V(x) + f_{\mu,\nu}(t),
$$

where $f_{\mu,\nu}$ is holomorphic on a neighbourhood of the interval $[n/2 - \epsilon, n/2]$.

Recall the notation $\tilde{a}_j := a_j c_g^{n/4}$ and $\tilde{b}_j := b_j c_g^{-n/4}$. For $k = l$ and $(x_1, \ldots, x_{k+l}) \in M^0(x, k, k)$ we write

$$
\left(\prod_{j=1}^k \tilde{a}_j(x_j) \right) \left(\prod_{j=k+1}^{2k} \tilde{b}_{j-k}(x_j)^* \right) = \prod_{j=1}^k \tilde{a}_j(g^{j-1}(x)) \tilde{b}_j^*(g^{j-1}(x)) =
$$

$$
= \prod_{j=1}^k a_j(g^{j-1}(x)) b_j^*(g^{j-1}(x)) = \left([a_1 b_1^*][g^*(a_2 b_2^*)][(g^2)^*(a_2 b_2^*)] \cdots [(g^k)^*(a_k b_k^*)] \right)(x).
$$

By the same argument that $V_g^* = N^{-1/2} \mathfrak{L}_{S,g} c_g^{-n/4}$, we can partially integrate $\int_M a g^*(b) \mathrm{d}V = \int_M \mathfrak{L}_g(c_g^{n/2} a) b \, \mathrm{d}V$ for $a, b \in C(M)$. By partially integrating $k - 1$ times we deduce that for some function $f_{\mu,\nu}$ holomorphic on a neighbourhood of the interval $[n/2 - \epsilon, n/2]$.

$$
\mathrm{Tr}_{L^2(M,S)}(P_{\not{D}} S_\mu S_\nu^* e^{-t|\not{D}_{\log}|}) =
$$

$$
= c \delta_{k,l} (t - n/2)^{-1} \int_M [a_1 b_1^*][g^*(a_2 b_2^*)][(g^2)^*(a_2 b_2^*)] \cdots [(g^k)^*(a_k b_k^*)] \mathrm{d}V + f_{\mu,\nu}(t) =
$$

$$
= c \delta_{k,l} (t - n/2)^{-1} \int_M \mathfrak{L}_g(c_g^{n/2} b_k^* \mathfrak{L}_g(c_g^{n/2} b_{k-1}^* \mathfrak{L}_g(\cdots \mathfrak{L}_g(c_g^{n/2} b_1^* a_1) a_2) \cdots a_{k-1}) a_k) \mathrm{d}V + f_{\mu,\nu}(t)
$$

We conclude that formula (5.1) holds.

It remains to show that ϕ_ω defines a KMS-state on $\mathcal{O}_{E_g}$ for the action defined by $\gamma_t(a V_g) := c_g^{itn/2} a V_g$. Let τ denote the tracial state on $C(M)$ defined from integrating against the volume form and $L \in \mathrm{End}^*_{C(M)}(E_g)$ the generator of γ_t, i.e. $L = \frac{n}{2} \log(c_g)$. Some yoga with inner products shows that for $\mu = a_1 V_g \otimes \cdots \otimes a_k V_g$, $\nu = b_1 V_g \otimes \cdots \otimes b_k V_g \in E_g^{\otimes C(M)^k}$, we have the computation

$$
\phi_\omega(S_\mu S_\nu^*) = \int_M \mathfrak{L}_g(c_g^{n/2} b_k^* \mathfrak{L}_g(c_g^{n/2} b_{k-1}^* \mathfrak{L}_g(\cdots \mathfrak{L}_g(c_g^{n/2} b_1^* a_1) a_2) \cdots a_{k-1}) a_k) \mathrm{d}V =
$$

$$
= \tau \left(b_1 V_g \otimes \cdots \otimes b_k V_g | e^{-L}(a_1 V_g) \otimes \cdots \otimes e^{-L}(a_k V_g) \right)_{E_g^{\otimes C(M)^k}} = \phi_\omega(S_\nu^* \gamma_i(S_\mu)).
$$

We conclude that $\phi_\omega(ab) = \phi_\omega(b \gamma_i(a))$ for $a, b \in \mathcal{A}$ and ϕ_ω is a KMS-state on $\mathcal{O}_{E_g}$ in the action γ. $\quad \square$

5.2. Graph C*-algebras

For a finite graph G we consider the spectral triple on $C^*(G)$ constructed in Proposition 2.20 from the choice of a point $y \in \Omega_G$ in the infinite path space. We will assume that G is primitive, in which case $C^*(G)$ is simple. For an element $(x, n) \in \mathcal{V}_y$ and a finite path μ, we compute that

$$e^{is\mathcal{D}_y}P_{\mathcal{D}_y}S_\mu P_{\mathcal{D}_y}e^{-is\mathcal{D}_y}\delta_{(x,n)} = e^{is|\mu|}P_{\mathcal{D}_y}S_\mu P_{\mathcal{D}_y}\delta_{(x,n)}.$$

It follows that $\sigma_t(S_\mu S_\nu^*) = e^{is(|\mu|-|\nu|)}S_\mu S_\nu^*$ and σ coincides with the gauge action on the graph C^*-algebra $C^*(G)$. We can conclude that $C^*(G) = C^*(G)_{\mathcal{D}_y}$ is closed under the flow σ. The reader can recall from Example 3.6 that $P_{\mathcal{D}_y}\ell^2(\mathcal{V}_y) = \ell^2(\mathcal{V}_y^+)$ where $\mathcal{V}_y^+$ is defined in Eq. (3.2). Moreover, $T_{C^*(G)} \subseteq \mathbb{B}(\ell^2(\mathcal{V}_y^+))$ is the C^*-algebra generated by the creation operators

$$T_e\delta_{(x,n)} := \begin{cases} \delta_{(ex,n)}, & r(e) = s(x), \\ 0, & r(e) \neq s(x). \end{cases}$$

Proposition 5.4. *Let $\beta \in \mathbb{R}$. The set $S = \{S_e : e \in E\} \subseteq C_c(\mathcal{G}_G)$ is an analytically generating set at β for $(C_c(\mathcal{G}_G), \ell^2(\mathcal{V}_y), \mathcal{D}_y)$.*

Proof. Since $P_{\mathcal{D}_y}S_eP_{\mathcal{D}_y} = T_e$, it is clear that S satisfies that $P_{\mathcal{D}_y}SP_{\mathcal{D}_y} + \mathbb{K}(\ell^2(\mathcal{V}_y^+))$ generates the Toeplitz algebra $T_{C^*(G)}$ as a C^*-algebra. Moreover, $e^{\beta\mathcal{D}_y}P_{\mathcal{D}}S_eP_{\mathcal{D}}e^{-\beta\mathcal{D}_y} = e^\beta P_{\mathcal{D}}S_eP_{\mathcal{D}}$ is bounded and the proposition follows. □

We conclude the following theorem from Example 3.6.

Theorem 5.5. *Let G be a finite primitive graph with edge adjacency matrix A and $y \in \Omega_G$. For any extended limit ω as $t \to \log r_\sigma(A)$, the KMS-state ϕ_ω on $C^*(G)$ associated with the spectral triple $(C_c(\mathcal{G}_G), \ell^2(\mathcal{V}_y), \mathcal{D}_y)$ (see Proposition 2.20) as in Corollary 3.21 is given by*

$$\phi_\omega(S_\mu S_\nu^*) = \delta_{\mu,\nu}\frac{w_{s(\mu)}}{\|w\|_{\ell^1}}r_\sigma(A)^{-|\mu|},$$

where ν and μ are finite paths and $w \in \mathbb{C}^E$ is the ℓ^2-normalized Perron–Frobenius vector. The state ϕ_ω is KMS for the gauge action and its inverse temperature is $\log(r_\sigma(A))$.

The KMS-state ϕ_ω on $C^*(G)$ in Theorem 5.5 is the unique KMS-state by [22]. Numerous authors present constructions of this state and for more general graphs, eg. [1,11,34,42].

5.3. Group C*-algebras

For the reduced group C^*-algebra of a countable discrete group we considered two types of (semifinite) spectral triples in Section 2.2.4 . We now compute the associated KMS-states.

We fix a length function ℓ on the discrete countable group Γ. For technical simplicity, we assume that Γ is an exact group ensuring that Γ acts amenably on its Stone–Cech boundary $\partial_{SC}\Gamma$ (see [52]). We assume that (Γ, ℓ) is of at most exponential growth and that ℓ is critical (see Definition 3.10). Let $(c_b(\Gamma) \rtimes^{alg} \Gamma, \ell^2(\Gamma), \mathcal{D}_\ell)$ denote the associated Li$_1$-summable spectral triple as in Proposition 2.31. Since $\mathcal{D}_\ell \geq 0$, we have that

$$T_{c_b(\Gamma)\rtimes_r\Gamma} = c_b(\Gamma) \rtimes_r \Gamma + \mathbb{K}(\ell^2(\Gamma)) = c_b(\Gamma) \rtimes_r \Gamma.$$

The last equality follows from that $c_0(\Gamma) \rtimes_r \Gamma = \mathbb{K}(\ell^2(\Gamma))$. We conclude that we have a short exact sequence

$$0 \to \mathbb{K}(\ell^2(\Gamma)) \to T_{c_b(\Gamma)\rtimes_r\Gamma} \to C(\partial_{SC}\Gamma) \rtimes_r \Gamma \to 0.$$

The flow σ^+ on $T_{c_b(\Gamma)\rtimes_r\Gamma} = c_b(\Gamma) \rtimes_r \Gamma$ is given on an element $a\lambda_g \in c_b(\Gamma) \rtimes \Gamma$ by

$$\sigma_s^+(a\lambda_g) = e^{is(\ell(\cdot)-\ell(g^{-1}\cdot))}a\lambda_g, \tag{5.2}$$

and we conclude that $T_{c_b(\Gamma)\rtimes_r\Gamma}$ is invariant under σ^+. Therefore $T_{c_b(\Gamma)\rtimes_r\Gamma} = T_{c_b(\Gamma)\rtimes_r\Gamma,\mathcal{D}_\ell}$.

Proposition 5.6. *Let $\beta \in \mathbb{R}$. The $*$-algebra $c_b(\Gamma) \rtimes^{alg} \Gamma$ is an analytically generating set at β for $(c_b(\Gamma) \rtimes^{alg} \Gamma, \ell^2(\Gamma), \mathcal{D}_\ell)$.*

Proof. Note that $P_{\mathcal{D}_\ell} = 1$ because $\mathcal{D}_\ell$ is positive. For $a\lambda_g \in c_b(\Gamma) \rtimes^{alg} \Gamma$, we compute that

$$e^{\beta\mathcal{D}_\ell}a\lambda_g e^{-\beta\mathcal{D}_\ell} = e^{-\beta(\ell(\cdot)-\ell(g^{-1}\cdot))}a\lambda_g.$$

Since $\ell(\gamma^{-1}) = \ell(\gamma)$ we have that $\|e^{-\beta(\ell(\cdot)-\ell(g^{-1}\cdot))}a\|_{c_b(\Gamma)} \leq e^{|\beta|\ell(g^{-1})}\|a\|_{c_b(\Gamma)}$, and it holds that $e^{\beta\mathcal{D}_\ell}a\lambda_g e^{-\beta\mathcal{D}_\ell} \in c_b(\Gamma)\rtimes^{alg}\Gamma$. The proposition follows. □

Proposition 5.7. *Let $\beta \in \mathbb{R}$. The $*$-algebra $c_b(\Gamma) \rtimes^{alg} \Gamma$ is an analytically generating set at β for $(c_b(\Gamma) \rtimes^{alg} \Gamma, \ell^2(\Gamma, S_\mathcal{H}), \mathcal{D}_c, \mathcal{N}, \mathrm{Tr}_\tau)$.*

Proof. For $a\lambda_g \in c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma$, we compute for $f \in \ell^2(\Gamma, S_{\mathcal{H}})$ that

$$\mathrm{e}^{\beta|\mathcal{D}_c|}\hat{\pi}_S(a\lambda_g)\mathrm{e}^{-\beta|\mathcal{D}_c|}f(\gamma) = \mathrm{e}^{-\beta(\ell(\gamma)-\ell(g^{-1}\gamma))}a(\gamma)[\tilde{\pi}(g)f](\gamma).$$

The estimate $\|\mathrm{e}^{-\beta(\ell(\cdot)-\ell(g^{-1}\cdot))}a\|_{c_b(\Gamma)} \leq \mathrm{e}^{|\beta|\ell(g^{-1})}\|a\|_{c_b(\Gamma)}$ shows that $\mathrm{e}^{\beta|\mathcal{D}_c|}\hat{\pi}_S(a\lambda_g)\mathrm{e}^{-\beta|\mathcal{D}_c|} \in \hat{\pi}\left(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma\right)$. Therefore,

$$\mathrm{e}^{\beta\mathcal{D}_c}P_{\mathcal{D}_c}(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma)P_{\mathcal{D}_c}\mathrm{e}^{-\beta\mathcal{D}_c}$$
$$= P_{\mathcal{D}_c}\mathrm{e}^{\beta|\mathcal{D}_c|}(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma)\mathrm{e}^{-\beta|\mathcal{D}_c|}P_{\mathcal{D}_c} \subseteq P_{\mathcal{D}_c}(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma)P_{\mathcal{D}_c} \subseteq \mathcal{N}^+,$$

and the proposition follows. $\quad\square$

Definition 5.8. If ω is an extended limit as $t \to \beta(\Gamma, \ell)$, and ℓ is critical, we define the Patterson–Sullivan measure μ_ω on the Stone-Čech boundary $\partial_{SC}\Gamma$ as

$$\int_{\partial_{SC}\Gamma} a \, \mathrm{d}\mu_\omega := \omega\text{-}\lim_{t \to \beta(\Gamma,\ell)} \frac{\sum_{\gamma \in \Gamma}\tilde{a}(\gamma)\mathrm{e}^{-t\ell(\gamma)}}{\sum_{\gamma \in \Gamma}\mathrm{e}^{-t\ell(\gamma)}},$$

for a function $a \in C(\partial_{SC}\Gamma)$ and where $\tilde{a} \in c_b(\Gamma)$ is any function with $a = \tilde{a} \mod c_0(\Gamma)$.

Remark 5.9. It is possible to define the Patterson–Sullivan measure μ_ω as an extended weak*-limit of the family of probability measures on Γ

$$\mu_t = \frac{\sum_{\gamma \in \Gamma}\delta_\gamma\mathrm{e}^{-t\ell(\gamma)}}{\sum_{\gamma \in \Gamma}\mathrm{e}^{-t\ell(\gamma)}}.$$

In the literature, Patterson–Sullivan measures are usually defined as weak* accumulation points of $(\mu_t)_{t>\beta(\Gamma,\ell)}$ but we allow for a slightly more general construction with extended limits. A priori, μ_ω is a probability measure on the Stone-Čech compactification of Γ. Since the support of μ_ω is contained in the closed subspace $\partial_{SC}\Gamma$ we consider μ_ω as a measure on $\partial_{SC}\Gamma$.

Theorem 5.10. *Let Γ be a discrete group and ϕ_ω the KMS-state on $C(\partial_{SC}\Gamma) \rtimes_r \Gamma$ constructed as in Corollary 3.21 using an extended limit ω as $t \to \beta(\Gamma, \ell)$ and either of the following two semifinite spectral triples:*

- *The spectral triple*

  $$(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma, \ell^2(\Gamma), \mathcal{D}_\ell)$$

  associated with a critical length function of at most exponential growth as in Proposition 2.31 .
- *The semifinite spectral triple $(c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma, \ell^2(\Gamma, S_{\mathcal{H}}), \mathcal{D}_c, \mathcal{N}, \mathrm{Tr}_\tau)$ associated with a critical proper Hilbert space valued cocycle of at most exponential growth as in Proposition 2.34 .*

Then ϕ_ω is given in terms of the Patterson–Sullivan measure μ_ω by

$$\phi_\omega(a\lambda_g) = \delta_{e,g}\int_{\partial_{SC}\Gamma} a \, \mathrm{d}\mu_\omega.$$

The state ϕ_ω is KMS at inverse temperature $\beta(\Gamma, \ell)$ for the flow on $C(\partial_{SC}\Gamma) \rtimes_r \Gamma$ induced by the action σ^+ on $c_b(\Gamma) \rtimes_r \Gamma$ given in Eq. (5.2). Moreover, ϕ_ω extends to a KMS-state at inverse temperature 1 on the von Neumann algebra $L^\infty(\partial_{SC}\Gamma, \mu_\omega)\overline{\rtimes}\Gamma$ with its Radon–Nikodym flow

$$\sigma_s^{RN}(a\lambda_g) = \left(\frac{\mathrm{d}g_*\mu_\omega}{\mathrm{d}\mu_\omega}\right)^{is}a\lambda_g.$$

Proof. By the computations of Example 3.9 , the spectral triple associated with a length function as in Proposition 2.31 have the same heat traces as the semifinite spectral triple associated with a proper Hilbert space valued cocycle as in Proposition 2.34. In both cases, Example 3.9 shows that for $\tilde{a}\lambda_g \in c_b(\Gamma) \rtimes^{\mathrm{alg}} \Gamma$ we have

$$\phi_{t,0}(\tilde{a}\lambda_g) = \delta_{e,g}\frac{\sum_{\gamma \in \Gamma}\tilde{a}(\gamma)\mathrm{e}^{-t\ell(\gamma)}}{\sum_{\gamma \in \Gamma}\mathrm{e}^{-t\ell(\gamma)}}.$$

It follows that $\phi_\omega(a\lambda_g) = \delta_{e,g}\int_{\partial_{SC}\Gamma} a \, \mathrm{d}\mu_\omega$ in both cases.

To relate ϕ_ω to the Radon–Nikodym flow, we first show that μ_ω is strictly positive, i.e. that $\mu_\omega(U) > 0$ for any open set $U \subset \partial_{SC}\Gamma$. For any open set $U \subset \partial_{SC}\Gamma$, the translates $(\gamma U)_{\gamma \in \Gamma}$ cover $\partial_{SC}\Gamma$. If $\mu_\omega(U) = 0$, then by quasi-invariance $\mu_\omega(\gamma U) = 0$ which contradicts μ_ω being a probability measure.

The fact that μ_ω is strictly positive ensures that the Radon–Nikodym derivatives $\frac{dg_*\mu_\omega}{d\mu_\omega}$ are well defined and strictly positive. The mapping $g \mapsto \frac{dg_*\mu_\omega}{d\mu_\omega}$ is a cocycle, i.e. for $h, g \in \Gamma$,

$$\frac{d(gh)_*\mu_\omega}{d\mu_\omega} = \frac{dg_*\mu_\omega}{d\mu_\omega}(g^{-1})^* \left[\frac{dh_*\mu_\omega}{d\mu_\omega}\right].$$

The proof is completed by computing that for $a\lambda_g, b\lambda_h \in C(\partial_{SC}\Gamma) \rtimes^{\mathrm{alg}} \Gamma$, we have the identity

$$\phi_\omega(a\lambda_g b\lambda_h) = \delta_{h,g^{-1}}\phi_\omega(ah^*(b)) = \delta_{h,g^{-1}}\int_{\partial_{SC}\Gamma} ah^*(b)d\mu_\omega =$$

$$= \delta_{h,g^{-1}}\int_{\partial_{SC}\Gamma} h^*\left(g^*(a)b\right)d\mu_\omega = \delta_{h,g^{-1}}\int_{\partial_{SC}\Gamma} g^*(a)b\,d(h_*\mu_\omega) =$$

$$= \delta_{h,g^{-1}}\int_{\partial_{SC}\Gamma} bg^*(a)\frac{dh_*\mu_\omega}{d\mu_\omega}d\mu_\omega =$$

$$= \delta_{h,g^{-1}}\int_{\partial_{SC}\Gamma} b(h^{-1})^*\left(a\left(\frac{dg_*\mu_\omega}{d\mu_\omega}\right)^{-1}\right)d\mu_\omega =$$

$$= \phi_\omega\left(b\lambda_h\left(\frac{dg_*\mu_\omega}{d\mu_\omega}\right)^{-1}a\lambda_g\right) = \phi_\omega\left(b\lambda_h\sigma_{s=i}^{RN}(a\lambda_g)\right)$$

In the third last identity we used the cocycle identity implying that if $hg = e$, then

$$(h^{-1})^*\left(\frac{dg_*\mu_\omega}{d\mu_\omega}\right)\frac{dh_*\mu_\omega}{d\mu_\omega} = 1. \quad \square$$

Remark 5.11. The reader should note that $\phi_\omega|_{C_r^*(\Gamma)}$ coincides with the ℓ^2-trace.

6. KMS-states on Cuntz–Pimsner algebras with their gauge action

In this section we consider the constructions of Corollary 3.21 in a broad class of examples which include both Cuntz–Krieger algebras and crossed products by $\mathbb{Z}$. Here we use the techniques from Section 3 in conjunction with those from Section 2.2.3 to analyse the KMS states on Cuntz–Pimsner algebras, and compare them to the Laca–Neshveyev correspondence establishing a bijection between KMS-states on Cuntz–Pimsner algebras and tracial states on its coefficient algebra.

6.1. KMS-States on Cuntz–Pimsner algebras from traces on the coefficient algebra

Firstly, we shall show that a critical positive trace on A (see Definition 6.2) gives rise to a KMS-state on the Cuntz–Pimsner algebra O_E assuming that E is strictly W-regular. Recall Lemma 2.28 giving the construction of the Li$_1$-summable semifinite spectral triple $(O_E, L^2(\Xi_A, \tau), \mathcal{D}_\alpha, \mathcal{N}_\tau(\Xi_A), \mathrm{Tr}_\tau)$, where $\mathcal{N}_\tau(\Xi_A) := (\mathrm{End}_A^*(\Xi_A) \otimes_A 1)''$.

Lemma 6.1. Let E be a strictly W-regular fgp bi-Hilbertian bimodule over A, $\beta \in \mathbb{R}$ and τ a positive trace on A. The set $S = \{S_e : e \in E\} \subset O_E$ is an analytically generating set at β for $(O_E, L^2(\Xi_A, \tau), \mathcal{D}_\alpha, \mathcal{N}_\tau(\Xi_A), \mathrm{Tr}_\tau)$.

Proof. Since $T_{O_E,\mathcal{D}}$ is precisely the Toeplitz algebra T_E, it is immediate that the set of operators $\{PS_eP : e \in E\}$ generates $T_{O_E,\mathcal{D}}$. The analyticity condition on the Fock space is likewise obvious from the computation

$$e^{\beta\mathcal{D}_\alpha}P_{\mathcal{D}_\alpha}S_eP_{\mathcal{D}_\alpha}e^{-\beta\mathcal{D}_\alpha} = e^\beta P_{\mathcal{D}_\alpha}S_eP_{\mathcal{D}_\alpha}. \quad \square$$

Definition 6.2. Let E be an A-bimodule which is fgp from the right and τ a positive trace on A. We define the critical value of (E, τ) as

$$\beta(E, \tau) := \inf\{t \geq 0 : \sum_{n=0}^{\infty} \tau_*(E^{\otimes_A n})e^{-tn} < \infty\}.$$

We say that τ is critical for E if

$$\lim_{t \searrow \beta(E,\tau)} \sum_{n=0}^{\infty} \tau_*(E^{\otimes_A n})e^{-tn} = \infty.$$

Remark 6.3. Note that the critical value of a trace and the notion of it being critical only depends on $[E] \in KK_0(A, A)$, in fact only on the sequence $(\mathrm{ch}_0(E^{\otimes_A n}))_{n \in \mathbb{N}} \subseteq HC_0(A)$ in cyclic homology. Just as in the proof of Lemma 2.28 we obtain the estimate $0 \leq \beta(E, \tau) \leq \log(N)$, where N is the number of elements in the left frame and the right frame.

It follows from Definition 3.1 and Proposition 3.8 that the Li_1-summable semifinite spectral triple $(\mathcal{O}_E, L^2(\Xi_A, \tau), \mathcal{D}_\alpha, \mathcal{N}_\tau(\Xi_A), \mathrm{Tr}_\tau)$ of Lemma 2.28 has positive Tr_τ-essential spectrum if and only if τ is critical.

By construction, the projection $P_{\mathcal{D}_\alpha}$ is the projection onto the Fock module $\mathcal{F}_E$ and therefore $\mathcal{N}_\tau(\Xi_A)^+ = (\mathrm{End}_A^*(\mathcal{F}_E) \otimes_A 1)''$ is a subalgebra of $\mathbb{B}(L^2(\mathcal{F}_E, \tau))$. We let N denote the number operator on $L^2(\mathcal{F}_E, \tau)$ — the self-adjoint operator defined from $N|_{E^{\otimes n} \otimes_A L^2(A, \tau)} = n\mathrm{Id}_{E^{\otimes n} \otimes_A L^2(A, \tau)}$. The next proposition follows from the definition of the Toeplitz algebra of a semifinite spectral triple.

Proposition 6.4. *The Toeplitz algebra $T_{\mathcal{O}_E}$ of the semifinite spectral triple*

$$(\mathcal{O}_E, L^2(\Xi_A, \tau), \mathcal{D}_\alpha, \mathcal{N}_\tau(\Xi_A), \mathrm{Tr}_\tau)$$

is given by

$$T_{\mathcal{O}_E} = \mathcal{T}_E \otimes_A 1_A + \mathbb{K}_{(\mathrm{End}_A^*(\mathcal{F}_E) \otimes_A 1)''},$$

where $\mathcal{T}_E \subseteq \mathrm{End}_A^*(\mathcal{F}_E)$ is the Cuntz–Toeplitz algebra of E. Moreover, the action σ^+ preserves $T_{\mathcal{O}_E}$ and is generated by the number operator in the sense that for $\mu \in E^k$, $\nu \in E^{\otimes l}$ and $K \in \mathbb{K}_{(\mathrm{End}_A^*(\mathcal{F}_E) \otimes_A 1)''}$, we have

$$\sigma_s^+(T_\mu T_\nu^* + K) = e^{is(|\mu| - |\nu|)} T_\mu T_\nu^* + e^{isN} K e^{-isN}.$$

An immediate consequence of Proposition 6.4 is that $\mathcal{O}_E = T_{\mathcal{O}_E}/\mathbb{K}_{(\mathrm{End}_A^*(\mathcal{F}_E))''}$ and that the action σ on $\mathcal{O}_E$ coincides with the gauge action $\sigma_s(S_\mu S_\nu^*) = e^{is(|\mu| - |\nu|)} S_\mu S_\nu^*$. The following theorem is readily deduced from the computations of Example 3.7 .

Theorem 6.5. *Let E be a strictly W-regular fgp bi-Hilbertian bimodule over A. For any positive trace τ on A which is critical for E and any extended limit ω as $t \to \beta(E, \tau)$, the KMS-state ϕ_ω on $\mathcal{O}_E$ associated with the semifinite spectral triple $(\mathcal{O}_E, L^2(\Xi_A, \tau), \mathcal{D}_\alpha, \mathcal{N}_\tau(\Xi_A), \mathrm{Tr}_\tau)$ as in Corollary 3.21 is given by*

$$\phi_{\tau,\omega}(S_\mu S_\nu^*) = \delta_{|\mu|,|\nu|} e^{-\beta(E,\tau)|\mu|} \cdot \omega\text{-}\lim_{t \to \beta(E,\tau)} \frac{\sum_{n=0}^\infty \mathrm{Tr}_\tau^{E^{\otimes n}} ((\nu|\mu)_{E^{|\mu|}}) e^{-tn}}{\sum_{n=0}^\infty \mathrm{Tr}_\tau^{E^{\otimes n}}(1) e^{-tn}},$$

where $\nu \in E^{\otimes k}$ and $\mu \in E^{\otimes l}$. The state $\phi_{\tau,\omega}$ is KMS for the gauge action on $\mathcal{O}_E$ and its inverse temperature is $\beta(E, \tau)$.

Proof. By Lemma 6.1 the semifinite spectral triple $(\mathcal{O}_E, L^2(\Xi_A, \tau), \mathcal{D}_\alpha, \mathcal{N}_\tau(\Xi_A), \mathrm{Tr}_\tau)$ is β-analytic for any $\beta \in \mathbb{R}$. By Definition 3.1 and Proposition 3.8 it has positive Tr_τ-essential spectrum. Thus the state $\phi_{\tau,\omega}$ as in Corollary 3.21 is KMS for the gauge action on $\mathcal{O}_E$ with inverse temperature $\beta(E, \tau)$. Using the computation in Eq. (3.3), we see that

$$\phi_{\tau,\omega}(S_\mu S_\nu^*) = \delta_{|\mu|,|\nu|} \cdot \omega\text{-}\lim_{t \to \beta(E,\tau)} \frac{\sum_{n=|\mu|}^\infty \sum_{|\sigma|=n} e^{-tn} \tau\left(((\mu|e_{\underline{\sigma}})_{E^{|\mu|}} e_{\overline{\sigma}} \mid (\nu|e_{\underline{\sigma}})_{E^{|\mu|}} e_{\overline{\sigma}})_{E^{\otimes(n-|\mu|)}}\right)}{\sum_{n=0}^\infty \sum_{|\rho|=n} \tau((e_\rho|e_\rho)_A) e^{-tn}},$$

for $\nu \in E^{\otimes k}$ and $\mu \in E^{\otimes l}$. However, this expression can be vastly simplified using that $\phi_{\tau,\omega}$ is KMS. Using

$$\phi_{\tau,\omega}(S_\mu S_\nu^*) = \delta_{|\mu|,|\nu|} e^{-\beta(E,\tau)|\mu|} \phi_{\tau,\omega}(S_\nu^* S_\mu) = \delta_{|\mu|,|\nu|} e^{-\beta(E,\tau)|\mu|} \phi_{\tau,\omega}((\nu|\mu)_{E^{\otimes|\mu|}}),$$

we obtain

$$\phi_{\tau,\omega}(S_\mu S_\nu^*) = \delta_{|\mu|,|\nu|} e^{-\beta(E,\tau)|\mu|} \phi_\omega((\nu|\mu)_{E^{\otimes|\mu|}})$$

$$= \delta_{|\mu|,|\nu|} e^{-\beta(E,\tau)|\mu|} \cdot \omega\text{-}\lim_{t \to \beta(E,\tau)} \frac{\sum_{n=0}^\infty \sum_{|\sigma|=n} e^{-tn} \tau\left((e_\sigma \mid (\nu|\mu)_{E^{|\mu|}} e_\sigma)_{E^{\otimes n}}\right)}{\sum_{n=0}^\infty \sum_{|\rho|=n} \tau((e_\rho|e_\rho)_A) e^{-tn}}$$

$$= \delta_{|\mu|,|\nu|} e^{-\beta(E,\tau)|\mu|} \cdot \omega\text{-}\lim_{t \to \beta(E,\tau)} \frac{\sum_{n=0}^\infty \mathrm{Tr}_\tau^{E^{\otimes n}} ((\nu|\mu)_{E^{|\mu|}}) e^{-tn}}{\sum_{n=0}^\infty \mathrm{Tr}_\tau^{E^{\otimes n}}(1) e^{-tn}}. \quad \square$$

The reader should note that in the formula computing $\phi_{\tau,\omega}$, it is only the right inner product on E that appears.

Example 6.6. Let us consider the construction from Theorem 6.5 in a specific example. As in Example 2.30 , we consider a compact Hausdorff space Y, a surjective local homeomorphism $g : Y \to Y$ and the associated bimodule E_g. Let us compute $\phi_{\tau,\omega}$ starting from a positive trace τ on $C(Y)$, i.e. a positive measure λ on Y. By the argument of Theorem 6.5, the KMS-condition on $\phi_{\tau,\omega}$ guarantees that it suffices to describe $\phi_{\tau,\omega}(a)$ for $a \in C(Y)$. By the Riesz representation theorem,

$$\phi_{\tau,\omega}(a) = \int_Y a \, d\lambda_\omega,$$

for a probability measure λ_ω. We compute that for $a \in C(Y)$,

$$\mathrm{Tr}_\tau^{E^{\otimes n}}(a) = \sum_{|\sigma|=n} \tau((e_\sigma | ae_\sigma)_{C(Y)}) = \int_Y \mathfrak{L}_g^n(a) \, d\lambda = \int_Y a \, d[(\mathfrak{L}_g^n)_* \lambda].$$

From Theorem 6.5, we conclude that λ_ω is given by an extended weak* limit of measures

$$\lambda_\omega = \omega\text{-}\lim_{t \to \beta(E_g, \tau)} \frac{\sum_{n=0}^\infty e^{-tn} (\mathfrak{L}_g^n)_* \lambda}{\sum_{n=0}^\infty e^{-tn} [(\mathfrak{L}_g^n)_* \lambda](Y)}.$$

The KMS-condition on $\phi_{\tau,\omega}$ translates into $(\mathfrak{L}_g)_* \lambda_\omega = e^{\beta(E,\tau)} \lambda_\omega$ which is readily verified for the measure λ_ω.

We remark that the construction above is reminiscent of the method in [62] to construct equilibrium measures. In the case that g is mixing, i.e. for all open subsets $U, V \subseteq Y$ there is an $N \geq 0$ such that $g^n(U) \cap V \neq \emptyset$ for all $n \geq N$, then there exists a unique KMS-state on O_{E_g} (see [17, Theorem 6.1]). In particular, for mixing g, the KMS-state $\phi_{\tau,\omega}$ on O_{E_g} does not depend on the choice of trace τ.

6.2. The Toeplitz construction vs the Laca–Neshveyev correspondence

In the previous subsection we saw that there is a mapping from the set of positive critical traces on a C^*-algebra A to the set of KMS-states on the Cuntz–Pimsner algebra O_E when E is strictly W-regular. As Example 6.6 shows, this mapping is not injective in general, but it is surjective in some cases (e.g. when g is mixing). We now compare our construction to the bijective correspondence between a certain set of tracial states on A with KMS-states on the Cuntz–Pimsner algebra O_E first discovered by Laca–Neshveyev [45].

Definition 6.7. The positive trace $\tau : A \to \mathbb{C}$ satisfies the Laca–Neshveyev condition for $\alpha \geq 0$ if

$$\mathrm{Tr}_\tau^E(L_a) = e^\alpha \tau(a),$$

where L_a denotes the left action of a on E.

For notational simplicity we often write $\mathrm{Tr}_\tau^E(a)$ instead of $\mathrm{Tr}_\tau^E(L_a)$. Given a positive trace $\tau : A \to \mathbb{C}$ satisfying the Laca–Neshveyev condition, it was proven by Laca–Neshveyev [45, Theorem 2.1 and 2.5] that the expression

$$\phi_{LN,\tau}(S_\mu S_\nu^*) := \delta_{|\mu|,|\nu|} e^{-\alpha|\mu|} \tau((\nu|\mu)_A),$$

defines an α-KMS state on O_E. Moreover, Laca–Neshveyev proved that the construction $\tau \mapsto \phi_{LN,\tau}$ is a bijection between tracial states on A satisfying the Laca–Neshveyev condition for $\alpha \geq 0$ and α-KMS states on O_E.

The work of Laca–Neshveyev [45] gives more context to the construction in Theorem 6.5 . For a unital C^*-algebra A, we let $\mathfrak{T}(A)$ denote the set of positive traces on A. If E is an A–A-correspondence which is finitely generated and projective as a right module, we can also define $\mathfrak{CT}_{E,\alpha}(A)$ as the set of positive critical traces τ with $\beta(E, \tau) = \alpha$. We also define $\mathfrak{LN}_{E,\alpha}(A)$ as the set of positive traces satisfying the Laca–Neshveyev condition.

Following [45, Discussion proceeding Definition 2.3], we define

$$F_{E,\alpha} : \mathfrak{T}(A) \to \mathfrak{T}(A), \quad F_{E,\alpha}\tau(a) = \mathrm{Tr}_\tau^E(a) e^{-\alpha}. \tag{6.1}$$

Proposition 6.8. *For any positive trace τ on a unital C^*-algebra and an A–A-correspondence E which is fgp from the right, it holds that*

$$F_{E,\alpha}^n \tau(a) = e^{-\alpha n} \mathrm{Tr}_\tau^{E^{\otimes n}}(a), \quad n \in \mathbb{N}_+.$$

Proof. By definition, $F_{E,\alpha}\tau(a) = e^{-\alpha} \sum_{j=1}^N \tau(e_j | ae_j)_A$ for a right frame $(e_j)_{j=1}^N$ of E. A direct computation shows that

$$F_{E,\alpha}^n \tau(a) = e^{-\alpha n} \sum_{|\sigma|=n} \tau(e_\sigma | ae_\sigma)_A = e^{-\alpha n} \mathrm{Tr}_\tau^{E^{\otimes n}}(a). \quad \square$$

Proposition 6.9. *Let E be an A–A-correspondence which is fgp from the right. Then $\tau \in \mathfrak{T}(A)$ is a fixed point of $F_{E,\alpha}$ if and only if τ satisfies the Laca–Neshveyev condition from Definition 6.7 .*

Proposition 6.9 is a direct consequence of Definition 6.7 and the formula (6.1).

Proposition 6.10. *Let E be an A–A-correspondence which is fgp from the right. If $\tau \in \mathfrak{T}(A)$ is a tracial state satisfying the Laca–Neshveyev condition for $\alpha \geq 0$, then*

$$\tau_*(E^{\otimes_A n}) = e^{\alpha n} F_{E,\alpha}^n \tau(1) = e^{\alpha n}.$$

Proof. The proposition follows from the computation

$$\tau_*(E^{\otimes_A n}) = \mathrm{Tr}^{E^{\otimes n}}_\tau(1) = e^{\alpha n}F^n_{E,\alpha}(\tau)(1),$$

and Proposition 6.9. □

Proposition 6.11. *Let E be an A–A-correspondence which is fgp from the right and $\alpha \geq 0$. Then the following holds:*

1. *A positive trace $\tau \in \mathfrak{T}(A)$ is critical for α if and only if the positive trace*

$$S^t_{E,\alpha}\tau := \sum_{n=0}^\infty e^{-tn}F^n_{E,\alpha}\tau,$$

   is finite for $t > 0$ and satisfies $S^t_{E,\alpha}\tau(1) \to \infty$ as $t \to 0$. In particular, we have an inclusion of sets

$$\mathfrak{LN}_{E,\alpha}(A) \subseteq \mathfrak{CT}_{E,\alpha}(A).$$

2. *For any extended limit ω as $t \to 0$, the mapping*

$$S^\omega_{E,\alpha} : \mathfrak{CT}_{E,\alpha}(A) \to \mathfrak{LN}_{E,\alpha}(A), \quad S^\omega_{E,\alpha}\tau := \omega\text{-}\lim_{t\to 0} \frac{S^t_{E,\alpha}\tau}{S^t_{E,\alpha}\tau(1)},$$

   is well defined. Moreover, $(S^\omega_{E,\alpha})^2\tau = \frac{S^\omega_{E,\alpha}\tau}{S^\omega_{E,\alpha}\tau(1)}$ and $S^\omega_{E,\alpha}$ surjects onto the set of tracial states in $\mathfrak{LN}_{E,\alpha}(A)$.

Proof. Statement 1 is an immediate consequence of Proposition 6.10. The first part of statement 2 follows from the computation

$$S^t_{E,\alpha}\tau(1) - F_{E,\alpha}S^t_{E,\alpha}\tau(1) = \tau(1) = o(S^t_{E,\alpha}\tau(1)), \quad \text{as } t \to 0 \text{ for } \tau \in \mathfrak{CT}_{E,\alpha}(A).$$

The second part of statement 2 follows from the fact that Proposition 6.9 implies that for $\tau \in \mathfrak{LN}_{E,\alpha}(A)$,

$$S^t_{E,\alpha}\tau := (1 - e^{-t})^{-1}\tau. \quad \square$$

Remark 6.12. For Example 6.6, the mapping $F_{E,\alpha}$ takes the form $F_{E,\alpha}\tau = e^{-\alpha}\mathfrak{L}^*_g\tau$. In particular, the computation of Example 6.6 is a special case of the constructions in Proposition 6.11. We shall see that this holds in general below in Proposition 6.14.

Using our previous results, Proposition 3.8 and Proposition 6.10 , we can deduce a computation of heat traces.

Proposition 6.13. *Let E be a strictly W-regular fpg bi-Hilbertian bimodule over a unital C^*-algebra A. If τ is a tracial state on A satisfying the Laca–Neshveyev condition for $\alpha \geq 0$, then*

$$\mathrm{Tr}_\tau(P_\mathcal{D}e^{-t|\mathcal{D}_\alpha|}) = \frac{1}{1 - e^{\alpha-t}}.$$

In particular, for any tracial state τ satisfying the Laca–Neshveyev condition for $\alpha \geq 0$ the semifinite spectral triple $(\mathcal{O}_E, L^2(\Xi_A, \tau), \mathcal{D}_\alpha, \mathcal{N}_\tau(\Xi_A), \mathrm{Tr}_\tau)$ has positive Tr_τ-essential spectrum with $\beta_\mathcal{D} = \beta(E, \tau) = \alpha$.

Proof. We compute that

$$\mathrm{Tr}_\tau(P_\mathcal{D}e^{-t|\mathcal{D}_\alpha|}) = \sum_{n=0}^\infty \tau_*(E^{\otimes_A n})e^{-tn} = \sum_{n=0}^\infty e^{-(t-\alpha)n} = \frac{1}{1 - e^{\alpha-t}}.$$

In the first step we used Proposition 3.8 and in the second step we used Proposition 6.10. □

We can now reformulate Theorem 6.5 in terms of the map $F_{E,\alpha}$ and the constructions of Proposition 6.11.

Proposition 6.14. *Assume that E is a strictly W-regular fgp bi-Hilbertian bimodule over A. Let $\alpha \geq 0$, ω be an extended limit as $t \to \alpha$ and $\tau \in \mathfrak{CT}_{E,\alpha}(A)$ a critical trace. The KMS-state $\phi_{\tau,\omega}$ defined from Theorem 6.5 takes the following form:*

$$\phi_{\tau,\omega}(S_\mu S^*_\nu) = \delta_{|\mu|,|\nu|}e^{-\alpha|\mu|}S^{\omega_\alpha}_{E,\alpha}\tau((\nu|\mu)_A), \quad \mu, \nu \in \mathcal{F}^{\text{alg}}_E,$$

where ω_α is the extended limit at 0 obtained from translating ω by α.

Proof. The KMS-condition on O_E reduces the proof to showing that $\phi_{\tau,\omega}(a) = S^{\omega_\alpha}_{E,\alpha}\tau(a)$ for $a \in A$, just as in the proof of Theorem 6.5. Using Proposition 6.8 we can compute for $a \in A$ that

$$\phi_{\tau,\omega}(a) = \omega_0\text{-}\lim_{t\to\alpha}\frac{\sum_{n=0}^\infty \mathrm{Tr}^{E^{\otimes n}}_\tau(a)e^{-tn}}{\sum_{n=0}^\infty \mathrm{Tr}^{E^{\otimes n}}_\tau(1)e^{-tn}} = \omega\text{-}\lim_{t\to\alpha}\frac{\sum_{n=0}^\infty F^n_{E,\alpha}\tau(a)e^{-(t-\alpha)n}}{\sum_{n=0}^\infty F^n_{E,\alpha}\tau(1)e^{-(t-\alpha)n}} = S^{\omega_\alpha}_{E,\alpha}\tau(a). \quad \square$$

Let $KMS_\alpha(O_E)$ denote the set of α-KMS states on O_E for the gauge action and $\mathfrak{LNS}_{E,\alpha}(A)$ for the set of tracial states satisfying the Laca–Neshveyev condition.

Theorem 6.15. *Let $\alpha \geq 0$ and let ω be an extended limit as $t \to \alpha$. Assume that E is a strictly W-regular fgp bi-Hilbertian bimodule over A. The mapping*

$$\mathfrak{LNS}_{E,\alpha}(A) \to KMS_\alpha(O_E), \quad \tau \mapsto \phi_{\tau,\omega},$$

defined from Theorem 6.5, and revisited in Proposition 6.14, is a well defined bijection of sets. More precisely, $\phi_{\tau,\omega} : O_E \to \mathbb{C}$ is a KMS_α state for the gauge action, which is independent of ω and coincides with $\phi_{LN,\tau}$.

Proof. By Theorem 6.5 and Proposition 6.13, the mapping $\tau \mapsto \phi_{\tau,\omega}$ is a well defined mapping from the set of positive traces on A satisfying the Laca–Neshveyev condition for α and α-KMS-states on O_E for the gauge action. By the Laca–Neshveyev correspondence, the KMS-state $\phi_{\tau,\omega}$ is uniquely determined by the trace $\phi_{\tau,\omega}|_A$. We can therefore deduce that $\phi_{\tau,\omega} = \phi_{LN,\tau}$ and the Theorem upon proving the identity $\phi_{\tau,\omega}|_A = \tau$. This statement follows immediately from Proposition 6.14 and the second part of Proposition 6.11. $\square$

There are some quasi-invariance assumptions on traces that allows us to compare the KMS-states constructed in Theorem 6.5 and the Laca–Neshveyev correspondence to a simpler construction involving Φ_∞. While the construction of Φ_∞ depends on the left inner product on E, the quasi-invariance condition we impose also depends on the left inner product. The following quasi-invariance condition is a refinement of the notion of E-invariant functionals from [54].

Definition 6.16. Let $\alpha \geq 0$ and E a finitely generated projective bi-Hilbertian bimodule. We say that a positive trace τ on A is α-quasi-invariant with respect to E and the extended limit $\omega_0 \in \ell^\infty(\mathbb{N})^*$ if for all $n \in \mathbb{N}$ and $\mu, \nu \in E^{\otimes n}$ we have

$$\mathrm{e}^{-\alpha|\mu|}\tau((\nu|\mu)_A) = \omega_0\text{-}\lim_{k\to\infty}\tau(\Phi_k(T_\mu T_\nu^*)\mathrm{e}^{-\beta_k}) = \omega_0\text{-}\lim_{k\to\infty}\tau(\,_A(\mu|\nu\mathrm{e}^{\beta_{k-|\nu|}})\mathrm{e}^{-\beta_k}). \tag{6.2}$$

If τ is α-quasi-invariant with respect to E and some extended limit, we simply say that τ is α-quasi-invariant with respect to E.

Note, that Φ_k are defined on formula (2.7) and the paragraph after.

Observe that if E is W-regular then the limit in the definition of quasi-invariance exists, and so is independent of the extended limit ω_0.

Remark 6.17. Just like in [54, Lemma 4.2], if E is full as a right module, then any positive functional $\tau : A \to \mathbb{C}$ which is quasi-invariant in the sense of Definition 6.16 is a positive trace. To see this, observe that for all $\mu, \nu \in \mathcal{F}_E^{\mathrm{alg}}$ and $a \in A$, the centrality of the Watatani indices $\mathrm{e}^{\beta_k} \in A$ (see formula (2.6) for the definition) gives

$$\mathrm{e}^{-\alpha|\mu|}\tau((\nu|\mu)_A a) = \mathrm{e}^{-\alpha|\mu|}\tau((\nu|\mu a)_A) = \omega_0\text{-}\lim_{k\to\infty}\tau(\,_A(\mu a|\nu\mathrm{e}^{\beta_{k-|\nu|}})\mathrm{e}^{-\beta_k})$$
$$= \omega_0\text{-}\lim_{k\to\infty}\tau(\,_A(\mu|\nu a^*\mathrm{e}^{\beta_{k-|\nu|}})\mathrm{e}^{-\beta_k}) = \mathrm{e}^{-\alpha|\mu|}\tau((\nu a^*|\mu)_A)$$
$$= \mathrm{e}^{-\alpha|\mu|}\tau(a(\nu|\mu)_A).$$

Example 6.18. We consider the module E_g over $C(Y)$ defined from a surjective local homeomorphism $g : Y \to Y$ as in Example 6.6 . In this case, $\beta_n = 0$ for all n and quasi-invariance of a positive trace τ given by a positive measure λ on Y is equivalent to the condition

$$(\mathfrak{L}_g)_*\lambda = \mathrm{e}^\alpha\lambda.$$

Another computation shows that this condition is equivalent to the Laca–Neshveyev condition. In particular, for the module E_g, quasi-invariance is equivalent to satisfying the Laca–Neshveyev condition.

Example 6.19. Let us consider the Cuntz algebra O_N defined as the Cuntz–Pimsner algebra of the $\mathbb{C}$-bimodule $\mathbb{C}^N$. In this case, $(\nu|\mu)_A = \,_A(\mu|\nu)$ for all $\mu, \nu \in (\mathbb{C}^N)^{\otimes n}$ and $\beta_n = n\log(N)$. Therefore, quasi-invariance of a trace τ on $\mathbb{C}$ is equivalent to

$$\mathrm{e}^\alpha\tau = N\tau.$$

That is, any non-zero trace on $\mathbb{C}$ is $\log(N)$-quasi invariant.

Our immediate aim is to connect the quasi-invariance of Definition 6.16 with the condition imposed by Laca–Neshveyev, [45].

Lemma 6.20. *Let E be an fgp bi-Hilbertian bimodule. Suppose that τ satisfies the α-quasi-invariance condition of Definition 6.16 with respect to E. Then τ satisfies the Laca–Neshveyev condition for α.*

Proof. This is a computation using Definition 6.16 of quasi-invariance and a frame (e_j) for E_A. So

$$\mathrm{Tr}_\tau^E(L_a) = \mathrm{Tr}_\tau\left(\sum_j a\Theta_{e_j,e_j}\right) = \sum_j \tau((e_j|ae_j)_A) = \mathrm{e}^\alpha \omega_0\text{-}\lim_{k\to\infty}\tau(\Phi_k(\sum_j T_{ae_j}T_{e_j}^*)\mathrm{e}^{-\beta_k})$$

$$= \mathrm{e}^\alpha \omega_0\text{-}\lim_{k\to\infty}\tau(\sum_j a_A(e_j|e_j\mathrm{e}^{\beta_{k-1}})\mathrm{e}^{-\beta_k}) = \mathrm{e}^\alpha \omega_0\text{-}\lim_{k\to\infty}\tau(a\mathrm{e}^{\beta_k}\mathrm{e}^{-\beta_k}) = \mathrm{e}^\alpha \tau(a). \quad \square$$

If $\tau : A \to \mathbb{C}$ satisfies the α-quasi-invariance condition of Definition 6.16, then we can rewrite the Laca–Neshveyev KMS state $\phi_{LN,\tau} : O_E \to \mathbb{C}$ as

$$\phi_{LN,\tau}(T_\mu T_\nu^*) = \delta_{|\mu|,|\nu|}\mathrm{e}^{-\alpha|\mu|}\tau((\mu|\nu)_A) = \omega_0\text{-}\lim_{k\to\infty}\tau(\Phi_k(T_\mu T_\nu^*)\mathrm{e}^{-\beta_k}).$$

This computation proves the following. Note that the next result does not require any W-regularity from the module E.

Proposition 6.21. *Let E be an fgp bi-Hilbertian bimodule and consider an α-quasi-invariant positive trace $\tau : A \to \mathbb{C}$ with respect to E. The state on O_E defined by*

$$S_\mu S_\nu^* \mapsto \omega_0\text{-}\lim_{k\to\infty}\tau(\Phi_k(T_\mu T_\nu^*)\mathrm{e}^{-\beta_k}), \tag{6.3}$$

is α-KMS for the gauge action on O_E and coincides with $\phi_{LN,\tau}$.

If E is W-regular, the definition $\Phi_\infty(S_\mu S_\nu^*) := \lim_{k\to\infty}\Phi_k(T_\mu T_\nu^*)\mathrm{e}^{-\beta_k}$ shows that the state in Eq. (6.3) coincides with $\tau \circ \Phi_\infty$. We conclude the following.

Corollary 6.22. *Let $\tau : A \to \mathbb{C}$ be an α-quasi-invariant positive trace with respect to an fgp bi-Hilbertian bimodule E. Assume that E is a W-regular. Then, the state $\tau \circ \Phi_\infty$ on O_E is α-KMS for the gauge action on O_E and coincides with $\phi_{LN,\tau}$.*

By Theorem 6.15 and Corollary 6.22, we have that $\phi_{\omega,\tau} = \tau \circ \Phi_\infty$ for any α-quasi-invariant positive trace τ and extended limit ω at α assuming that E is strictly regular.

6.3. Obstructions to bi-Hilbertian bimodule structures

In the two previous subsections, we assumed our A-bimodule E to be an fgp bi-Hilbertian bimodule, and imposed the additional assumption of strict W-regularity (see Definitions 2.22 and 2.25). The assumptions allowed us to construct a semifinite spectral triple from a Kasparov module relying on both the left and the right inner product on E. Instead, we can just use [45] to proceed from a KMS-state directly to a semifinite spectral triple whose associated KMS-state as in Corollary 3.21 coincides with the original KMS-state. In order to compare the indirect approach for strictly W-regular modules to the direct approach from the KMS-state we will need to extend our module to von Neumann algebra coefficients, and along the way we derive obstructions to having the structure of a strictly W-regular bi-Hilbertian bimodule structure on an A–A-correspondence.

We suppose that we have a finitely generated projective right A-module E_A, with A unital, and carrying a unital left action of A. Let $\tau : A \to \mathbb{C}$ be a faithful positive trace satisfying the Laca–Neshveyev condition for $\alpha \geq 0$ (see Definition 6.7), and define the associated KMS-state on O_E by

$$\phi_{LN,\tau}(S_\mu S_\nu^*) := \delta_{|\mu|,|\nu|}\tau((\nu|\mu)_A)\mathrm{e}^{-\alpha|\mu|}, \quad \mu, \nu \in \mathcal{F}_E^{\mathrm{alg}}.$$

By construction, $\phi_{LN,\tau}|_A = \tau$.

The Cuntz–Pimsner algebra O_E acts on the GNS-space $L^2(O_E, \phi_{LN,\tau})$ by left multiplication and, since $\phi_{LN,\tau}$ restricts to a trace on A, A acts by both left and right multiplication on $L^2(O_E, \phi_{LN,\tau})$. For notational simplicity, we identify τ with its normal extension to A''. Note that A'' is independent of whether we take the bicommutant in $L^2(O_E, \phi_{LN,\tau})$ or $L^2(A, \tau)$ and by faithfulness of τ we can identify A with its image under the GNS-representation and obtain an inclusion $A \subseteq A''$.

Proposition 6.23. *Let E be an fgp right A-Hilbert C^*-module with a left unital action of A, τ a faithful positive trace on A satisfying the Laca–Neshveyev condition and let $P_0 : L^2(O_E, \phi_{LN,\tau}) \to L^2(A, \tau) \subset L^2(O_E, \phi_{LN,\tau})$ denote the orthogonal projection. It then holds that*

$$A'' = P_0 O_E'' P_0.$$

Proof. It is clear that $A'' \subseteq P_0 O_E'' P_0$. To prove the converse inclusion, take $T \in P_0 O_E'' P_0$ and write $T = P_0 T_0 P_0$ where T_0 is the WOT-limit of a net $T_\lambda = \sum_j S_{\mu_{\lambda,j}} S_{\nu_{\lambda,j}}^* \in O_E$. We have that

$$P_0 T_\lambda P_0 = \sum_{j:|\mu_{\lambda,j}|=|\nu_{\lambda,j}|=0} S_{\mu_{\lambda,j}} S_{\nu_{\lambda,j}}^*,$$

so $P_0 T_\lambda P_0 \in A$ and $T \in A''$. $\quad \square$

We can define a conditional expectation

$$\tilde{\Phi}_\infty : O_E'' \to A'', \quad \tilde{\Phi}_\infty(S) := P_0 S P_0,$$

which is well defined by Proposition 6.23. Using the expectation $\tilde{\Phi}_\infty$ we can define a right module $\Xi_{A''}$ by completing O_E in the norm defined by the inner product

$$(S_1|S_2)_{A''} := \tilde{\Phi}_\infty(S_1^* S_2), \quad S_1, S_2 \in O_E.$$

It is clear that $L^2(O_E, \phi_{LN,\tau}) = L^2(\Xi_{A''}, \tau)$. The construction of $\Xi_{A''}$ does not require E_A to be biHilbertian, just an A–A-correspondence. The following result follows from the relations defining the Cuntz–Pimsner algebra and the fact that $\tilde{\Phi}_\infty$ is a conditional expectation.

Proposition 6.24. *For $\mu \in E^{\otimes k}$ and $\nu \in E^{\otimes l}$,*

$$\tilde{\Phi}_\infty(S_\mu^* S_\nu) = \delta_{|\mu|,|\nu|}(\mu|\nu)_A.$$

In particular, the map $\mu \mapsto S_\mu$ extends to an A''-linear isometric embedding $\mathcal{F}_E \otimes_A A'' \to \Xi_{A''}$ of the Fock module.

We now turn to describing the WOT-closure of E inside O_E''. We will identify E with an A-sub-bimodule of O_E via $\mu \mapsto S_\mu$.

Lemma 6.25. *Let A be a unital C^*-algebra. Let E_A be a finitely generated projective right A-module with a unital left action, and suppose that $\tau : A \to \mathbb{C}$ satisfies the Laca–Neshveyev condition on E for $\alpha \geq 0$. Then*

$$E'' := \overline{E}^{\text{WOT}} \subset O_E''$$

is an A''-bimodule. Moreover, the following hold:

1. *As right A''-modules, $E'' \cong E \otimes_A A''$ and the isomorphism is an isomorphism of A''-Hilbert C^*-modules when equipping E'' with the right inner product*

   $$(\mu|\nu)_{A''} := \tilde{\Phi}_\infty(S_\mu^* S_\nu), \quad \mu, \nu \in E''.$$

2. *The right A''-Hilbert C^*-module E'' is finitely generated and projective.*
3. *If E is finitely generated and projective as a left A-module, then E'' is finitely generated and projective as a left A''-module.*
4. *If the implication*

   $$P_0 e e^* P_0 = 0 \Rightarrow e = 0 \quad \forall e \in E'', \tag{6.4}$$

   holds, the expression

   $$_A(e|f) := P_0 S_e S_f^* P_0$$

   gives a left inner product on E'' making it into a bi-Hilbertian bimodule. The right Watatani index of E'' is 1 and $\mathfrak{q}_k = \text{Id}_{(E'')^{\otimes k}}$ for all k (and so is invertible).
5. *If E'' is a finitely generated projective module from the left and the implication (6.4) holds, then E'' is a strictly W-regular fgp bi-Hilbertian bimodule over A''.*

Proof. The initial statement of the lemma is clear since E is an A-sub-bimodule of O_E, so its WOT closure is a bimodule over A''. To prove statement 1. we note that $E \otimes_A A'' \cong EA'' \subseteq O_E''$. Moreover, using that E is finitely generated and projective, it follows that $E'' = EA''$ and therefore $E'' \cong E \otimes_A A''$ follows. Statement 1. now follows from Proposition 6.24. Statement 2. follows from statement 1. because E is finitely generated and projective over A. Statement 3. is proven in a similar way as Statement 1., indeed if E is finitely generated and projective as a left A-module then $E'' \cong A'' \otimes_A E$ as a left A-module.

Statement 4. is less trivial. Assuming that the implication (6.4) holds, it is straight-forward to verify that the left and right actions are compatible, i.e. that the A-action from the left/right is adjointable for the right/left inner product. For E'' to be a bi-Hilbertian bimodule it remains to show that the norm arising from the left inner product is equivalent to the norm arising from the right inner product. For any $e \in E$, we compute that

$$\|_A(e|e)\|_A = \|P_0 S_e S_e^* P_0\|_{L^2(A,\tau)} = \|S_e^* P_0 S_e\|_{L^2(E^*,\phi_\tau)} = \|S_e^* S_e\|_{L^2(E^*,\phi_\tau)} =$$
$$= \|(e|e)_A\|_{L^2(E^*,\phi_\tau)} = \|(e|e)_A\|_A \tag{6.5}$$

where the norm of $S_e^* S_e$ is attained by $S_e^* P_0 S_e$ on $L^2(E^*, \phi_\tau)$ by the assumption that $_A(\cdot|\cdot)$ is positive definite. Eq. (6.5) shows that the two norms $\sqrt{\|(\cdot|\cdot)_A\|_A}$ and $\sqrt{\|_A(\cdot|\cdot)\|_A}$ on E are equivalent.

To finalize the proof of statement 4., we compute the right Watatani index of E, which exists because E'' is finitely generated projective by statement 2. We compute on $L^2(A, \tau) \subset L^2(O_E, \phi_\tau)$ that

$$\langle a, \sum_j {}_A(e_j|e_j)a \rangle = \langle a, \sum_j P_0 \pi^{(1)}(\Theta_{e_j,e_j}) P_0 a \rangle = \phi_\tau(a^* P_0 \pi^{(1)}(\text{Id}_E) P_0 a) = \phi_\tau(a^* 1_A a) = \tau(a^* a).$$

By the faithfulness of τ and Cuntz–Pimsner covariance we can now deduce that the right Watatani index is equal to 1_A. This immediately implies that $q_k = \mathrm{Id}_{(E'')^{\otimes k}}$ for all k.

Finally, statement 5. follows from that under the stated assumptions, E'' is an fgp bi-Hilbertian bimodule (using statements 1., 2. and 4.) and by statement 4. q_k satisfies the condition in Definition 2.25, so E'' is strictly W-regular. $\square$

Remark 6.26. There are examples of A–A-correspondences E that are fgp from the right but not fgp from the left such that E'' is fgp from the left. These examples come from (certain) self-similar dynamical systems, see [41].

Here is a simple example. Let $A = C([0, 1])$,

$$\gamma_1 : [0, 1] \to [0, 1] \quad \gamma_1(x) = x/2, \qquad \gamma_2 : [0, 1] \to [0, 1] \quad \gamma_2(x) = 1/2 + x/2,$$

and $E = C(\{(\gamma_1(x), x) : x \in [0, 1]\} \cup \{(\gamma_2(x), x) : x \in [0, 1]\})$. The correspondence structure is defined for $a, b \in A$ and $e \in E$ by

$$(a \cdot e \cdot b)(\gamma_j(x), x) = a(\gamma_j(x))e(\gamma_j(x), x)b(x) \quad \text{and} \quad (e_1|e_2)_A(x) = \sum_{j=1,2} \overline{e_1(\gamma_j(x), x)}e(\gamma_j(x), x).$$

The graphs of γ_1 and γ_2 in $[0, 1] \times [0, 1]$ are disjoint and their respective characteristic functions χ_1 and χ_2 are elements of E. One checks directly that $\{\chi_1, \chi_2\}$ is a right frame for E_A, and since $(\chi_1|\chi_2) = 0$, there is an isomorphism of right Hilbert C^*-modules $E_A \cong C[0, 1] \oplus C[0, 1]$. We conclude that E_A is fgp from the right. Using the frame $\{\chi_1, \chi_2\}$, we can identify $_AE \cong C[0, 1/2] \oplus C[1/2, 1]$ as a left $C[0, 1]$-module. As a left module, $_AE$ is therefore finitely generated but clearly not projective as the rank of $E_x := E/C_0([0, 1] \setminus \{x\})E$ is discontinuous at $x = 1/2$.

We shall now see that it is even impossible for a left inner product compatible with the right inner product to exist. Let $0 \leq \phi \in A$ be 1 on $[0, 1/2]$. Then if we have a compatible left A-valued inner product

$$_A(\chi_1|\chi_2)(x) = {}_A(\phi \cdot \chi_1|\chi_2)(x) = \phi(x){}_A(\chi_1|\chi_2)(x) = {}_A(\chi_1|\chi_2)(x)\phi(x) = {}_A(\chi_1|\phi \cdot \chi_2)(x).$$

Taking the infimum over such ϕ we see that the support of $_A(\chi_1|\chi_2)$ is contained in $\{1/2\}$. Then for arbitrary $a, b \in A$

$$_A(a\chi_1 + b\chi_2|a\chi_1 + b\chi_2)$$
$$= a_A(\chi_1|\chi_1)a^* + b_A(\chi_2|\chi_2)b^* + a(1/2)b^*(1/2)_A(\chi_1|\chi_2) + b(1/2)a^*(1/2)_A(\chi_2|\chi_1).$$

From here one can show that any inner product taking values in the continuous functions takes values in the functions vanishing at $1/2$. Then one shows that the associated norm cannot be equivalent to the right inner product.

The situation is better for E''. We consider the trace $\tau(a) := \int_0^1 a(x)\mathrm{d}x$ on $C[0, 1]$. A short computation shows that $\mathrm{Tr}_\tau^E = 2\tau$ so τ satisfies the Laca–Neshveyev condition for $\alpha = \log(2)$ and extends to a KMS-state on O_E at inverse temperature $\log(2)$. It is readily verified that $C[0, 1]'' = L^\infty[0, 1]$, $E'' \cong L^\infty[0, 1] \oplus L^\infty[0, 1]$ as a right module and $E'' = L^\infty[0, 1/2] \oplus L^\infty[1/2, 1]$ as a left module. In particular, E'' is an fgp bi-Hilbertian bimodule over $L^\infty[0, 1]$.

The same discussion applies to any 'graph separated' iterated function system satisfying the open set condition. See [41] for more details.

Theorem 6.27. *Let E_A be a right A-Hilbert C^*-module with a unital left action and assume that E is finitely generated both as a left and a right A-module and E'' is finitely generated and projective both as a left and a right A''-module. Then E_A has a left inner product such that E is an fgp bi-Hilbertian bimodule, has finite right Watatani index and is W-regular with q_k invertible for all k if and only if $\tilde{\Phi}_\infty$ is faithful and $\tilde{\Phi}_\infty(O_E) \subseteq A$.*

We remark that if E is W-regular and q_k is invertible for all k, then E is strictly W-regular by [30, Lemma 3.8].

Proof. First suppose that $\tilde{\Phi}_\infty$ is faithful and $\tilde{\Phi}_\infty(O_E) \subseteq A$. Lemma 6.25 shows that E'' has a left inner product making E (not just E'') bi-Hilbertian with right Watatani index 1_A and (invertible) $q_k = \mathrm{Id}_{(E)^{\otimes k}}$. Since E, both as a left and a right module, is finitely generated and admits a Hilbert C^*-module structure it is also projective.

Conversely, if E is an fgp bi-Hilbertian bimodule with finite right Watatani index and is W-regular with invertible q_k, then [54] proves that the map $\Phi_\infty(S_\mu S_\nu^*) = {}_A(\mu|q_{|\nu|}(\nu))$ is a (faithful) conditional expectation. That it agrees with $\tilde{\Phi}_\infty$ is a computation. $\square$

Remark 6.28. The issue with q_k being non-invertible is as follows. Since $q_k = q_1 \otimes q_1 \otimes \cdots \otimes q_1$, the non-invertibility occurs with q_1. Supposing O_E to be strictly W-regular, we can define the right module Ξ_A as the completion of O_E for the norm coming from Φ_∞. Then $\tilde{\Phi}_\infty(S_e S_f^*) = {}_A(e|q_1(f))$, and so if q_1 is not invertible, we do not get a left inner product in this way. See [54, Example 3.10] for an example where we have strict W-regularity with q_1 not being invertible.

Heuristically, one should view the passage to Ξ_A as erasing the information about the left inner product on E, corresponding to the kernel of q_1. On the other hand, if q_k is invertible for all k then we can replace our left inner products $_A(\cdot|\cdot)^{E^{\otimes k}}$ by $_A(\cdot|q_k(\cdot))^{E^{\otimes k}}$ and obtain an equivalent inner product structure with right Watatani index 1.

We can now relate our constructions above back to KMS-states.

Proposition 6.29. *Let A be a unital C*-algebra, E a finitely generated projective right A-Hilbert C*-module with a unital adjointable left A-action, $\alpha \geq 0$ and τ a positive trace on A. We assume that this data satisfies the following conditions:*

- *τ satisfies the Laca–Neshveyev condition.*
- *The A''-bimodule E'' is finitely generated and projective from the left.*
- *The implication* (6.4) *holds.*

We define a semifinite spectral triple $(O_E, L^2(O_E, \phi_{LN,\tau}), \mathcal{D}_\alpha, \mathcal{N}, \mathcal{T})$ using that $L^2(O_E, \phi_{LN,\tau}) = L^2(O_{E''}, \phi_{LN,\tau}) = L^2(\Xi_{A''}, \tau)$ and pulling back the semi-finite spectral triple defined from the fgp bi-Hilbertian A''-bimodule E'' as in Lemma 2.28 along the inclusion $O_E \to O_{E''}$. This semi-finite spectral triple is Li_1-summable, α-analytic, has positive essential $\mathcal{T}$-spectrum with $\beta_{\mathcal{D}_\alpha} = \alpha$ and its associated KMS-state for the gauge action (as in Corollary 3.21) coincides with $\phi_{LN,\tau}$.

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