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The $\bar{\partial}$-equation on a non-reduced analytic space

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Abstract Let X be a, possibly non-reduced, analytic space of pure dimension. We introduce a notion of $\bar{\partial}$-equation on X and prove a Dolbeault–Grothendieck lemma. We obtain fine sheaves $\mathcal{A}_X^q$ of $(0, q)$-currents, so that the associated Dolbeault complex yields a resolution of the structure sheaf $\mathscr{O}_X$. Our construction is based on intrinsic semi-global Koppelman formulas on X.

Mathematics Subject Classification 32A26 · 32A27 · 32B15 · 32C30

1 Introduction

Let X be a smooth complex manifold of dimension n and let $\mathscr{E}_X^{0,*}$ denote the sheaf of smooth $(0, *)$-forms. It is well-known that the Dolbeault complex

$$0 \to \mathscr{O}_X \xrightarrow{i} \mathscr{E}_X^{0,0} \xrightarrow{\bar{\partial}} \mathscr{E}_X^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{E}_X^{0,n} \to 0 \tag{1.1}$$

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is exact, and hence provides a fine resolution of the structure sheaf $\mathscr{O}_X$. If X is a reduced analytic space of pure dimension, then there is still a natural notion of "smooth forms". In fact, assume that X is locally embedded as $i\colon X \to \Omega$, where Ω is a pseudoconvex domain in $\mathbb{C}^N$. If $\mathcal{K}er\, i^*$ denotes the subsheaf of all smooth forms ξ in ambient space such that $i^*\xi = 0$ on the regular part X_{reg} of X, then one defines the sheaf $\mathscr{E}_X$ of smooth forms on X simply as

$$\mathscr{E}_X := \mathscr{E}_\Omega / \mathcal{K}er\, i^*.$$

It is well-known that this definition is independent of the choice of embedding of X. Currents on X are defined as the duals of smooth forms with compact support. It is readily seen that the currents μ on X so defined are in a one-to-one correspondence to the currents $\hat{\mu} = i_*\mu$ in ambient space such that $\hat{\mu}$ vanish on $\mathcal{K}er\, i_*$, see, e.g., [6]. There is an induced $\bar\partial$-operator on smooth forms and currents on X. In particular, (1.1) is a complex on X but in general it is not exact. In [6], Samuelsson and the first author introduced, by means of intrinsic Koppelman formulas on X, fine sheaves $\mathscr{A}_X^*$ of $(0,*)$-currents that are smooth on X_{reg} and with mild singularities at the singular part of X, such that

$$0 \to \mathscr{O}_X \xrightarrow{i} \mathscr{A}_X^0 \xrightarrow{\bar\partial} \mathscr{A}_X^1 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{A}_X^n \to 0 \tag{1.2}$$

is exact, and thus a fine resolution of the structure sheaf $\mathscr{O}_X$. An immediate consequence is the representation

$$H^q(X, \mathscr{O}_X) = \frac{\mathrm{Ker}\left(\mathscr{A}^{0,q}(X) \xrightarrow{\bar\partial} \mathscr{A}^{0,q+1}(X)\right)}{\mathrm{Im}\left(\mathscr{A}^{0,q-1}(X) \xrightarrow{\bar\partial} \mathscr{A}^{0,q}(X)\right)}, \quad q \geq 1, \tag{1.3}$$

of sheaf cohomology, and so (1.3) is a generalization of the classical Dolbeault isomorphism. In special cases more qualitative information of the sheaves $\mathscr{A}_X^q$ are known, see, e.g., [5,23].

Starting with the influential works [28,29] by Pardon and Stern, there has been a lot of progress recently on the L^2-$\bar\partial$ theory on non-smooth (reduced) varieties; see, e.g., [15,27,31]. The point in these works, contrary to [6], is basically to determine the obstructions to solve $\bar\partial$ locally in L^2. For a more extensive list of references regarding the $\bar\partial$-equation on reduced singular varieties, see, e.g., [6].

In [17], a notion of the $\bar\partial$-equation on non-reduced local complete intersections was introduced, and which was further studied in [18]. We discuss below how their work relates to ours.

The aim of this paper is to extend the construction in [6] to a non-reduced pure-dimensional analytic space. The first basic problem is to find appropriate definitions of forms and currents on X. Let X_{reg} be the part of X where the underlying reduced space Z is smooth, and in addition $\mathscr{O}_X$ is Cohen–Macaulay. On X_{reg} the structure sheaf $\mathscr{O}_X$ has a structure as a free finitely generated $\mathscr{O}_Z$-module. More precisely, assume that we have a local embedding $i\colon X \to \Omega \subset \mathbb{C}^N$ and coordinates (z, w) in Ω such that

$Z = \{w = 0\}$. Let $\mathcal{J}$ be the defining ideal sheaf for X on Ω. Then there are monomials $1, w^{\alpha_1}, \ldots, w^{\alpha_{\nu-1}}$ such that each ϕ in $\mathscr{O}_\Omega/\mathcal{J} \simeq \mathscr{O}_X$ has a unique representation

$$\phi = \hat\phi_0 \otimes 1 + \hat\phi_1 \otimes w^{\alpha_1} + \cdots + \hat\phi_{\nu-1} \otimes w^{\alpha_{\nu-1}}, \tag{1.4}$$

where $\hat\phi_j$ are in $\mathscr{O}_Z$. A reasonable notion of a smooth form on X should admit a similar representation on X_{reg} with smooth forms $\hat\phi_j$ on Z. We first introduce the sheaves $\mathscr{E}_X^{0,*}$ of smooth $(0,*)$-forms on X. By duality, we then obtain the sheaf $\mathcal{C}_X^{n,*}$ of $(n,*)$-currents. We are mainly interested in the subsheaf $\mathcal{PM}_X^{n,*}$ of pseudomeromorphic currents, and especially, the even more restricted sheaf $\mathcal{W}_X^{n,*}$ of such currents with the so-called standard extension property, SEP, on X. A current with the SEP is, roughly speaking, determined by its restriction to any dense Zariski-open subset.

Of special interest is the sheaf $\omega_X^n \subset \mathcal{W}_X^{n,0}$ of $\bar\partial$-closed pseudomeromorphic $(n,0)$-currents. In the reduced case this is precisely the sheaf of holomorphic $(n,0)$-forms in the sense of Barlet–Henkin–Passare, see, e.g., [12,16].

We have no definition of "smooth $(n,*)$-form" on X. In order to define $(0,*)$-currents, we use instead the sheaf ω_X^n in the following way. Any holomorphic function defines a morphism in $\mathcal{H}om(\omega_X^n, \omega_X^n)$, and it is a reformulation of a fundamental result of Roos [30], that this morphism is indeed injective, and generically surjective. In the reduced case, multiplication by a current in $\mathcal{W}_X^{0,*}$ induces a morphism in $\mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$, and in fact $\mathcal{W}_X^{0,*} \to \mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$ is an isomorphism. In the non-reduced case, we then take this as the definition of $\mathcal{W}_X^{0,*}$. It turns out that with this definition, on X_{reg}, any element of $\mathcal{W}_X^{0,*}$ admits a unique representation (1.4), where $\hat\phi_j$ are in $\mathcal{W}_Z^{0,*}$, see Sect. 6 below for details.

Given v, ϕ in $\mathcal{W}_X^{0,*}$ we say that $\bar\partial v = \phi$ if $\bar\partial(v \wedge h) = \phi \wedge h$ for all h in ω_X^n. Following [6] we introduce semi-global integral formulas and prove that if ϕ is a smooth $\bar\partial$-closed $(0, q+1)$-form there is locally a current v in $\mathcal{W}_X^{0,q}$ such that $\bar\partial v = \phi$. A crucial problem is to verify that the integral operators preserve smoothness on X_{reg} so that the solution v is indeed smooth on X_{reg}. By an iteration procedure as in [6] we can define sheaves $\mathscr{A}_X^k \subset \mathcal{W}_X^{0,k}$ and obtain our main result in this paper.

Theorem 1.1 *Let X be an analytic space of pure dimension n. There are sheaves $\mathscr{A}_X^k \subset \mathcal{W}_X^{0,k}$ that are modules over $\mathscr{E}_X^{0,*}$, coinciding with $\mathscr{E}_X^{0,k}$ on X_{reg}, and such that* (1.2) *is a resolution of the structure sheaf $\mathscr{O}_X$.*

The main contribution in this article compared to [6] is the development of a theory for smooth $(0,*)$-forms and various classes of $(n,*)$- and $(0,*)$-currents in the non-reduced case as is described above. This is done in Sects. 4–8. The construction of integral operators to provide solutions to $\bar\partial$ in Sect. 9 and the construction of the fine resolution of $\mathscr{O}_X$ in Sect. 11, which proves Theorem 1.1, are done pretty much in the same way as in [6]. The proof of the smoothness of the solutions of the regular part in Sect. 10 however becomes significantly more involved in the non-reduced case and requires completely new ideas. In Sect. 12 we discuss the relation to the results in [17,18] in case X is a local complete intersection.

2 Pseudomeromorphic currents

Let $s_1, \ldots, s_m$ be coordinates in $\mathbb{C}^m$, let α be a smooth form with compact support, and let $a_1, \ldots, a_r$ be positive integers, $0 \le \ell \le r \le m$. Then

$$\bar\partial \frac{1}{s_1^{a_1}} \wedge \cdots \wedge \bar\partial \frac{1}{s_\ell^{a_\ell}} \wedge \frac{\alpha}{s_{\ell+1}^{a_{\ell+1}} \cdots s_r^{a_r}}$$

is a well-defined current that we call an *elementary (pseudomeromorphic) current*. Let Z be a reduced space of pure dimension. A current τ is *pseudomeromorphic* on Z if, locally, it is the push-forward of a finite sum of elementary pseudomeromorphic currents under a sequence of modifications, simple projections, and open inclusions. The pseudomeromorphic currents define an analytic sheaf $\mathcal{PM}_Z$ on Z. This sheaf was introduced in [8] and somewhat extended in [6]. If nothing else is explicitly stated, proofs of the properties listed below can be found in, e.g., [6].

If τ is pseudomeromorphic and has support on an analytic subset V, and h is a holomorphic function that vanishes on V, then $\bar h\tau = 0$ and $d\bar h \wedge \tau = 0$.

Given a pseudomeromorphic current τ and a subvariety V of some open subset $\mathcal{U} \subset Z$, the natural restriction to the open set $\mathcal{U}\setminus V$ of τ has a natural extension to a pseudomeromorphic current on $\mathcal{U}$ that we denote by $\mathbf{1}_{\mathcal{U}\setminus V}\tau$. Throughout this paper we let χ denote a smooth function on $[0, \infty)$ that is 0 in a neighborhood of 0 and 1 in a neighborhood of ∞. If h is a holomorphic tuple whose common zero set is V, then

$$\mathbf{1}_{\mathcal{U}\setminus V}\tau = \lim_{\epsilon\to 0^+} \chi(|h|^2/\epsilon)\tau. \tag{2.1}$$

Notice that $\mathbf{1}_V\tau := (1 - \mathbf{1}_{\mathcal{U}\setminus V})\tau$ is also pseudomeromorphic and has support on V. If W is another analytic set, then

$$\mathbf{1}_V \mathbf{1}_W \tau = \mathbf{1}_{V\cap W}\tau. \tag{2.2}$$

This action of $\mathbf{1}_V$ on the sheaf of pseudomeromorphic currents is a basic tool. In fact one can extend this calculus to all constructible sets so that (2.2) holds, see [8]. One readily checks that if ξ is a smooth form, then

$$\mathbf{1}_V(\xi \wedge \tau) = \xi \wedge \mathbf{1}_V\tau. \tag{2.3}$$

If $f\colon Z' \to Z$ is a modification and τ is in $\mathcal{PM}_{Z'}$ then $f_*\tau$ is in $\mathcal{PM}_Z$. The same holds if f is a simple projection and τ has compact support in the fiber direction. In any case we have

$$\mathbf{1}_V f_*\tau = f_*(\mathbf{1}_{f^{-1}V}\tau). \tag{2.4}$$

It is not hard to check that if τ is in $\mathcal{PM}_Z$ and τ' is in $\mathcal{PM}_{Z'}$, then $\tau\otimes\tau'$ is in $\mathcal{PM}_{Z\times Z'}$, see, e.g., [4, Lemma 3.3]. If $V \subset \mathcal{U} \subset Z$ and $V' \subset \mathcal{U}' \subset Z'$, then

$$(\mathbf{1}_V\tau)\otimes \mathbf{1}_{V'}\tau' = \mathbf{1}_{V\times V'}(\tau\otimes\tau'). \tag{2.5}$$

Another basic tool is the *dimension principle*, that states that if τ is a pseudomeromorphic $(*, p)$-current with support on an analytic set with codimension larger than p, then τ must vanish.

A pseudomeromorphic current τ on Z has the *standard extension property*, SEP, if $\mathbf{1}_V\tau = 0$ for each germ V of an analytic set with positive codimension on Z. The set $\mathcal{W}_Z$ of all pseudomeromorphic currents on Z with the SEP is a subsheaf of $\mathcal{PM}_Z$. By (2.3), $\mathcal{W}_Z$ is closed under multiplication by smooth forms.

Let f be a holomorphic function (or a holomorphic section of a Hermitian line bundle), not vanishing identically on any irreducible component of Z. Then $1/f$, a priori defined outside of $\{f = 0\}$, has an extension as a pseudomeromorphic current, the principal value current, still denoted by $1/f$, such that $\mathbf{1}_{\{f=0\}}(1/f) = 0$. The current $1/f$ has the SEP and

$$\frac{1}{f} = \lim_{\epsilon\to 0^+} \chi(|f|^2/\epsilon)\frac{1}{f}.$$

We say that a current a on Z is *almost semi-meromorphic* if there is a modification $\pi\colon Z' \to Z$, a holomorphic section f of a line bundle $L \to Z'$ and a smooth form γ with values in L such that $a = \pi_*(\gamma/f)$, cf., [10, Section 4]. If a is almost semi-meromorphic, then it is clearly pseudomeromorphic. Moreover, it is smooth outside an analytic set $V \subset Z$ of positive codimension, a is in $\mathcal{W}_Z$, and in particular, $a = \lim_{\epsilon\to 0^+} \chi(|h|/\epsilon)a$ if h is a holomorphic tuple that cuts out (an analytic set of positive codimension that contains) V. The *Zariski singular support* of a is the Zariski closure of the set where a is not smooth.

One can multiply pseudomeromorphic currents by almost semi-meromorphic currents; and this fact will be crucial in defining $\mathcal{W}_X^{0,*}$, when X is non-reduced. Notice that if a is almost semi-meromorphic in Z then it also is in any open $\mathcal{U} \subset Z$.

Proposition 2.1 ([10, Theorem 4.8, Proposition 4.9]) *Let Z be a reduced space, assume that a is an almost semi-meromorphic current in Z, and let V be the Zariski singular support of a.*

(i) *If τ is a pseudomeromorphic current in $\mathcal{U} \subset Z$, then there is a unique pseudomeromorphic current $a \wedge \tau$ in $\mathcal{U}$ that coincides with (the naturally defined current) $a \wedge \tau$ in $\mathcal{U}\setminus V$ and such that $\mathbf{1}_V(a \wedge \tau) = 0$.*
(ii) *If $W \subset \mathcal{U}$ is any analytic subset, then*

$$\mathbf{1}_W(a \wedge \tau) = a \wedge \mathbf{1}_W\tau. \tag{2.6}$$

Notice that if h is a tuple that cuts out V, then in view of (2.1),

$$a \wedge \tau = \lim_{\epsilon\to 0^+} \chi(|h|^2/\epsilon)a \wedge \tau. \tag{2.7}$$

It follows that if ξ is a smooth form, then

$$\xi \wedge (a \wedge \tau) = (-1)^{\deg\xi \deg a} a \wedge (\xi \wedge \tau). \tag{2.8}$$

For future reference we will need the following result.

Proposition 2.2 *Let Z be a reduced space. Then $\mathcal{PM}_Z = \mathcal{W}_Z + \bar\partial \mathcal{W}_Z$.*

Proof First assume that Z is smooth. Since $\mathcal{W}_Z$ is closed under multiplication by smooth forms, so is $\mathcal{W}_Z + \bar\partial\mathcal{W}_Z$. The statement that $\mathcal{PM}_Z = \mathcal{W}_Z + \bar\partial\mathcal{W}_Z$ is local, and since both sides are closed under multiplication by cutoff functions, we may consider a pseudomeromorphic current μ with compact support in $\mathbb{C}^n$. If μ has bidegree $(*, 0)$, then it is in $\mathcal{W}_Z$ in view of the dimension principle. Thus we assume that μ has bidegree $(*, q)$ with $q \geq 1$. Let

$$K\mu(z) = \int_\zeta k(\zeta, z) \wedge \mu(\zeta), \tag{2.9}$$

where k is the Bochner–Martinelli kernel. Here (2.9) means that $K\mu = p_*(k \wedge \mu \otimes 1)$, where p is the projection $\mathbb{C}^n_\zeta \times \mathbb{C}^n_z \to \mathbb{C}^n_z, \quad (\zeta, z) \mapsto z$. Recall that we have the Koppelman formula $\mu = \bar\partial K\mu + K(\bar\partial\mu)$. It is thus enough to see that $K\mu$ is in $\mathcal{W}_Z$ if μ is pseudomeromorphic. Let $\chi_\epsilon = \chi(|\zeta - z|^2/\epsilon)$. It is easy to see, by a blowup of $\mathbb{C}^n \times \mathbb{C}^n$ along the diagonal, that k is almost semi-meromorphic on $\mathbb{C}^n \times \mathbb{C}^n$. Thus, by (2.7), $\chi_\epsilon k \wedge (\mu \otimes 1) \to k \wedge (\mu \otimes 1)$. In view of Proposition 2.1 it follows that $k \wedge (\mu \otimes 1)$ is pseudomeromorphic. Finally, if W is a germ of a subvariety of $\mathbb{C}^n$ of positive codimension, then by (2.4) and (2.5),

$$\begin{aligned}
\mathbf{1}_W p_*(k \wedge \mu \otimes 1) &= \lim_{\epsilon\to 0^+} p_*\left(\mathbf{1}_{\mathbb{C}^n\times W}(\chi_\epsilon k \wedge (\mu \otimes 1))\right) \\
&= \lim_{\epsilon\to 0^+} p_*\left(\chi_\epsilon k \wedge (\mathbf{1}_{\mathbb{C}^n\times W}\mu \otimes 1)\right) \\
&= \lim_{\epsilon\to 0^+} p_*\left(\chi_\epsilon k \wedge (\mathbf{1}_{\mathbb{C}^n}\mu \otimes \mathbf{1}_W 1)\right) = 0,
\end{aligned}$$

since $\mathbf{1}_W 1 = 0$. Thus $K\mu$ is in $\mathcal{W}_Z$.

If Z is not smooth, then we take a smooth modification $\pi : Z' \to Z$. For any μ in $\mathcal{PM}_Z$ there is some μ' in $\mathcal{PM}_{Z'}$ such that $\pi_*\mu' = \mu$, see [4, Proposition 1.2]. Since $\mu' = \tau + \bar\partial u$ with τ, u in $\mathcal{W}_{Z'}$, we have that $\mu = \pi_*\tau + \bar\partial\pi_* u$. □

2.1 Pseudomeromorphic currents with support on a subvariety

Let Ω be an open set in $\mathbb{C}^N$ and let Z be a (reduced) subvariety of pure dimension n. Let $\mathcal{PM}^Z_\Omega$ denote the sheaf of pseudomeromorphic currents τ on Ω with support on Z, and let $\mathcal{W}^Z_\Omega$ denote the subsheaf of $\mathcal{PM}^Z_\Omega$ of currents of bidegree $(N, *)$ with the SEP with respect to Z, i.e., such that $\mathbf{1}_W\tau = 0$ for all germs W of subvarieties of Z of positive codimension. The sheaf $\mathcal{CH}^Z_\Omega$ of Coleff–Herrera currents on Z is the subsheaf of $\mathcal{W}^Z_\Omega$ of $\bar\partial$-closed (N, p)-currents, where $p = N - n$.

Remark 2.3 In [3,6] $\mathcal{CH}^\Omega_Z$ denotes the sheaf of pseudomeromorphic $(0, p)$-currents with support on Z and the SEP with respect to Z. If this sheaf is tensored by the canonical bundle K_Ω we get the sheaf $\mathcal{CH}^Z_\Omega$ in this paper. Locally these sheaves are thus isomorphic via the mapping $\mu \mapsto \mu \wedge \alpha$, where α is a non-vanishing holomorphic $(N, 0)$-form. □

We have the following direct consequence of Proposition 2.1.

Proposition 2.4 *Let $Z \subset \Omega$ be a subvariety of pure dimension, let a be almost semi-meromorphic in Ω, and assume that it is smooth generically on Z. If τ is in $\mathcal{W}_\Omega^Z$, then $a \wedge \tau$ is in $\mathcal{W}_\Omega^Z$ as well.*

Assume that we have local coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^p$ in Ω such that $Z = \{w = 0\}$. We will use the short-hand notation

$$\bar{\partial} \frac{dw}{w^{\gamma+\mathbf{1}}} := \bar{\partial} \frac{dw_1}{w_1^{\gamma_1+1}} \wedge \cdots \wedge \bar{\partial} \frac{dw_p}{w_p^{\gamma_p+1}}$$

for multiindices $\gamma = (\gamma_1, \ldots, \gamma_p)$ with $\gamma_j \geq 0$, and let $\gamma! := \gamma_1! \cdots \gamma_p!$. Notice that

$$\frac{1}{(2\pi i)^p} \bar{\partial} \frac{dw}{w^{\gamma+\mathbf{1}}} . \xi = \frac{1}{\gamma!} \int_z \frac{\partial^\gamma \xi}{\partial w^\gamma}(z, 0) \tag{2.10}$$

for test forms ξ. If τ is in $\mathcal{W}_Z$, then it follows by (2.5) and the fact that $\operatorname{supp} \bar{\partial}(1/w^{\gamma+\mathbf{1}}) = \{w = 0\}$ that $\tau \otimes \bar{\partial}(1/w^{\gamma+\mathbf{1}})$ is in $\mathcal{W}_\Omega^Z$. We have the following local structure result, see [11, Proposition 4.1 and (4.3)] and [10, Theorem 3.5].

Proposition 2.5 *Assume that we have local coordinates (z, w) such that $Z = \{w = 0\}$. Then τ in $\mathcal{W}_\Omega^Z$ has a unique representation as a finite sum*

$$\tau = \sum_\gamma \tau_\gamma \wedge dz \otimes \bar{\partial} \frac{dw}{w^{\gamma+\mathbf{1}}}, \quad \tau_\gamma \in \mathcal{W}_Z^{0,*}, \tag{2.11}$$

where $dz := dz_1 \wedge \cdots \wedge dz_n$. If π is the projection $(z, w) \mapsto z$, then

$$\tau_\gamma \wedge dz = (2\pi i)^{-p} \pi_*(w^\gamma \tau). \tag{2.12}$$

If in addition $\bar{\partial}\tau$ is in $\mathcal{W}_\Omega^Z$ then its coefficients in the expansion (2.11) are $\bar{\partial}\tau_\gamma$, cf., (2.12). In particular, $\bar{\partial}\tau = 0$ if and only if $\bar{\partial}\tau_\gamma = 0$ for all γ.

Let us now consider the pairing between $\mathcal{W}_\Omega^Z$ and germs ϕ at Z of smooth $(0, *)$-forms. We assume that Z is smooth and that we have coordinates (z, w) as before, that τ is in $\mathcal{W}_\Omega^Z$, and that (2.11) holds. Moreover, we assume that ϕ is a smooth $(0, *)$-form in a neighborhood of Z in Ω. For any positive integer M we have the expansion

$$\phi = \sum_{|\alpha|<M} \phi_\alpha(z) \otimes w^\alpha + \mathscr{O}\left(|w|^M\right) + \mathscr{O}(\bar{w}, d\bar{w}), \tag{2.13}$$

where

$$\phi_\alpha(z) = \frac{1}{\alpha!} \frac{\partial \phi}{\partial w^\alpha}(z, 0)$$

and $\mathscr{O}(\bar{w}, d\bar{w})$ denotes a sum of terms, each of which contains a factor $\bar{w}_j$ or $d\bar{w}_j$ for some j. If M in (2.13) is chosen so that $\mathscr{O}(|w|^M)\tau = 0$, then

$$\phi \wedge \tau = \sum_{\alpha \leq \gamma} \phi_\alpha \wedge \tau_\gamma \wedge dz \otimes \bar{\partial} \frac{dw}{w^{\gamma - \alpha + \mathbf{1}}},$$

i.e.,

$$\phi \wedge \tau = \sum_{\ell \geq 0} \sum_{\gamma \geq 0} \phi_\gamma \wedge \tau_{\ell + \gamma} \wedge dz \otimes \bar{\partial} \frac{dw}{w^{\ell + \mathbf{1}}}. \tag{2.14}$$

Thus $\phi \wedge \tau = 0$ if and only if $\sum_{\gamma \geq 0} \phi_\gamma \wedge \tau_{\ell+\gamma} = 0$ for all ℓ (which is a finite number of conditions!).

2.2 Intrinsic pseudomeromorphic currents on a reduced subvariety

Currents on a reduced analytic space Z are defined as the dual of the sheaf of test forms. If $i : Z \to Y$ is an embedding of a reduced space Z into a smooth manifold Y, then the push-forward mapping $\tau \mapsto i_*\tau$ gives an isomorphism between currents τ on Z and currents μ on Y such that $\xi \wedge \mu = 0$ for all ξ in $\mathscr{E}_Y$ such that $i^*\xi = 0$.

When defining pseudomeromorphic currents in the non-reduced case it is desirable that it coincides with the previous definition in case Z is reduced. From [4, Theorem 1.1] we have the following description of pseudomeromophicity from the point of view of an ambient smooth space.

Proposition 2.6 *Assume that we have an embedding $i : Z \to Y$ of a reduced space Z into a smooth manifold Y.*

(i) *If τ is in $\mathcal{PM}_Z$, then $i_*\tau$ is in $\mathcal{PM}_Y$.*
(ii) *If τ is a current on Z such that $i_*\tau$ is in $\mathcal{PM}_Y$ and $\mathbf{1}_{Z_{sing}}(i_*\tau) = 0$, then τ is in $\mathcal{PM}_Z$.*

Since $i_*(i^*\chi(|h|^2/\epsilon)\tau) = \chi(|h|^2/\epsilon)i_*\tau$ for any current τ on Z, we get by (2.1) that for a subvariety $V \subset \mathcal{U} \subset Z$,

$$\mathbf{1}_V(i_*\tau) = i_*(\mathbf{1}_V\tau), \tag{2.15}$$

i.e., (2.4) holds also for an embedding $i : Z \to Y$. The condition $\mathbf{1}_{Z_{sing}}(i_*\tau) = 0$ in (ii) is fulfilled if $i_*\tau$ has the SEP with respect to Z.

Corollary 2.7 *We have the isomorphism*

$$i_* : \mathcal{W}_Z^{n,*} \to \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z),$$

where $\mathcal{J}$ is the ideal defining Z in Ω.

Notice that $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$ is precisely the sheaf of μ in $\mathcal{W}_\Omega^Z$ such that $\mathcal{J}\mu = 0$.

Proof The map i_* is injective, since it is injective on any currents, and it maps into $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$ by (2.15).

To see that i_* is surjective, we take a μ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$. We assume first that we are on Z_{reg}, with local coordinates such that $Z_{\mathrm{reg}} = \{w = 0\}$. If ξ is in $\mathscr{E}_\Omega^{0,*}$ and $i^*\xi = 0$, then ξ is a sum of forms with a factor $d\bar{w}_j$, w_j or $\bar{w}_j$. Since $w_j \in \mathcal{J}$, w_j annihilates μ by assumption, and since w_j vanishes on the support of μ, $\bar{w}_j$ and $d\bar{w}_j$ annihilate μ since μ is pseudomeromorphic. Thus, $\mu.\xi = 0$, so $\mu = i_*\tau$ for some current τ on Z. By Proposition 2.6 (ii), τ is pseudomeromorphic, and by (2.15), has the SEP, i.e., τ is in $\mathcal{W}_Z^{n,*}$. □

Remark 2.8 We do not know whether $i_*\tau \in \mathcal{PM}_\Omega^Z$ implies that $\tau \in \mathcal{PM}_Z$. □

By [11, Proposition 3.12 and Theorem 3.14], we get

Proposition 2.9 *Let φ and $\phi_1, \ldots, \phi_m$ be currents in $\mathcal{W}_Z$. If $\varphi = 0$ on the set on Z_{reg} where $\phi_1, \ldots, \phi_m$ are smooth, then $\varphi = 0$.*

3 Local embeddings of a non-reduced analytic space

Let X be an analytic space of pure dimension n with structure sheaf $\mathscr{O}_X$ and let $Z = X_{red}$ be the underlying reduced analytic space. For any point $x \in X$ there is, by definition, an open set $\Omega \subset \mathbb{C}^N$ and an ideal sheaf $\mathcal{J} \subset \mathscr{O}_\Omega$ of pure dimension n with zero set Z such that $\mathscr{O}_X$ is isomorphic to $\mathscr{O}_\Omega/\mathcal{J}$, and all associated primes of $\mathcal{J}$ at any point have dimension n. We say that we have a local embedding $i\colon X \to \Omega \subset \mathbb{C}^N$ at x. There is a minimal such N, called the Zariski embedding dimension $\hat{N}$ of X at x, and the associated embedding is said to be minimal. Any two minimal embeddings are identical up to a biholomorphism, and any embedding $i\colon X \to \Omega$ has locally at x the form

$$X \xrightarrow{j} \widehat{\Omega} \xrightarrow{\iota} \Omega := \widehat{\Omega} \times \mathcal{U}, \quad i = \iota \circ j, \tag{3.1}$$

where j is minimal, $\mathcal{U}$ is an open subset of $\mathbb{C}_w^m$, $m = N - \hat{N}$, and the ideal in Ω is $\mathcal{J} = \widehat{\mathcal{J}} \otimes 1 + (w_1, \ldots, w_m)$. Notice that we then also have embeddings $Z \to \widehat{\Omega} \to \Omega$; however, the first one is in general not minimal.

Now consider a fixed local embedding $i\colon X \to \Omega \subset \mathbb{C}^N$, assume that Z is smooth, and let (z, w) be coordinates in Ω such that $Z = \{w = 0\}$. We can identify $\mathscr{O}_Z$ with holomorphic functions of z, and we can define an injection

$$\mathscr{O}_Z \to \mathscr{O}_X, \quad \phi(z) \mapsto \tilde{\phi}(z, w) = \phi(z).$$

In this way $\mathscr{O}_X$ becomes an $\mathscr{O}_Z$-module, which however depends on the choice of coordinates.

Proposition 3.1 *Assume that Z is smooth. Let $\mathscr{O}_X$ have the $\mathscr{O}_Z$-module structure from a choice of local coordinates as above. Then $\mathscr{O}_X$ is a coherent $\mathscr{O}_Z$-module, and $\mathscr{O}_X$ is a free $\mathscr{O}_Z$-module at x if and only if $\mathscr{O}_X$ is Cohen–Macaulay at x.*

Recall that $f_1, \ldots, f_m \in R$ is a *regular sequence* on the R-module M if f_i is a non zero-divisor on $M/(f_1, \ldots, f_{i-1})$ for $i = 1, \ldots, m$, and $(f_1, \ldots, f_m)M \neq M$. If R is a local ring, then $\operatorname{depth}_R M$ is the maximal length d of a regular sequence $f_1, \ldots, f_d$ such that $f_1, \ldots, f_d$ are contained in the maximal ideal $\mathfrak{m}$; furthermore, M is *Cohen–Macaulay* if $\operatorname{depth}_R M = \dim_R M$, where $\dim_R M = \dim_R(R/\operatorname{ann}_R M)$. If R is Cohen–Macaulay, and M has a finite free resolution over R, then the *Auslander–Buchsbaum* formula, [14, Theorem 19.9], gives that

$$\operatorname{depth}_R M + \operatorname{pd}_R M = \dim_R R, \tag{3.2}$$

where $\operatorname{pd}_R M$ is the length of a minimal free resolution of M over R. In this case, M is Cohen–Macaulay as an R-module if and only if M has a free resolution over R of length $\operatorname{codim} M$.

Remark 3.2 Notice that if we have a local embedding $i\colon X \to \Omega$ as above, then the depth and dimension of $\mathscr{O}_{X,x} = \mathscr{O}_{\Omega,x}/\mathcal{J}$ as an $\mathscr{O}_{\Omega,x}$-module coincide with the depth and dimension of $\mathscr{O}_{X,x}$ as an $\mathscr{O}_{X,x}$-module. Thus $\mathscr{O}_{X,x}$ is Cohen–Macaulay as an $\mathscr{O}_{X,x}$-module if and only if it is Cohen–Macaulay as an $\mathscr{O}_{\Omega,x}$-module, and this holds in turn if and only if $\mathscr{O}_{\Omega,x}/\mathcal{J}$ has a free resolution of length $N - n$. □

Proof of Proposition 3.1 By the Nullstellensatz there is an M such that w^α is in $\mathcal{J}$ in some neighborhood of x if $|\alpha| = M$. Let $\mathcal{M} \subset \mathscr{O}_\Omega$ be the ideal generated by $\{w^\alpha;\ |\alpha| = M\}$. Then $\mathcal{M}' = \mathscr{O}_\Omega/\mathcal{M}$ is a free, finitely generated $\mathscr{O}_Z$-module. Thus, $\mathscr{O}_\Omega/\mathcal{J} \simeq \mathcal{M}'/\mathcal{J}\mathcal{M}'$ is a coherent $\mathscr{O}_Z$-module, which we note is generated by the finite set of monomials w^α such that $|\alpha| < M$.

We shall now show that

$$\operatorname{depth}_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} = \operatorname{depth}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} \tag{3.3}$$

and

$$\dim_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} = \dim_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x}. \tag{3.4}$$

We claim that a sequence $f_1, \ldots, f_m$ in $\mathscr{O}_{X,x}$ is regular (on $\mathscr{O}_{X,x}$) if and only if $\tilde{f}_1, \ldots, \tilde{f}_m \in \mathscr{O}_{Z,x}$ is regular on $\mathscr{O}_{X,x}$, where $\tilde{f}_j(z) = f_j(z, 0)$. In fact, since $\mathscr{O}_{X,x}$ has pure dimension, a function $g \in \mathscr{O}_{X,x} = \mathscr{O}_{\Omega,x}/\mathcal{J}$ is a non zero-divisor if and only if g is generically non-vanishing on each irreducible component of $Z(\mathcal{J})$. Thus f_1 is a non zero-divisor if and only if $\tilde{f}_1$ is. If it is, then $\mathscr{O}_{X,x}/(f_1) = \mathscr{O}_{\Omega,x}/(\mathcal{J} + (f_1))$ again has pure dimension. Thus the claim follows by induction, and the fact that $Z(\mathcal{J} + (f_1, \ldots, f_k)) = Z(\mathcal{J} + (\tilde{f}_1, \ldots, \tilde{f}_k))$. The claim immediately implies (3.3).

To see (3.4), we note first that $\dim_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x}$ is just the usual (geometric) dimension of X or Z, i.e., in this case, n. Now, $\operatorname{ann}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} = \{0\}$, so $\dim_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} = \dim_{\mathscr{O}_{Z,x}} \mathscr{O}_{Z,x}/(\operatorname{ann}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x}) = \dim_{\mathscr{O}_{Z,x}} \mathscr{O}_{Z,x} = n$.

From (3.3) and (3.4) we conclude that $\mathscr{O}_{X,x}$ is Cohen–Macaulay as an $\mathscr{O}_{Z,x}$-module if and only if it is Cohen–Macaulay (as an $\mathscr{O}_{X,x}$-module). Hence, by (3.2), with $R = \mathscr{O}_{Z,x}$ and $M = \mathscr{O}_{X,x}$,

$$\operatorname{depth}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} + \operatorname{pd}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} = n,$$

so $\mathscr{O}_{X,x}$ is Cohen–Macaulay as an $\mathscr{O}_{Z,x}$-module if and only if $\mathrm{pd}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} = 0$, that is, if and only if $\mathscr{O}_{X,x}$ is a free $\mathscr{O}_{Z,x}$-module. □

In the proof above, we saw that $\mathscr{O}_X$ is generated (locally) as an $\mathscr{O}_Z$-module by all monomials w^α with $|\alpha| \le M$ for some M.

Corollary 3.3 *Assume that* $1, w^{\alpha_1}, \dots, w^{\alpha_{\nu-1}}$ *is a minimal set of generators at a given point x (clearly* 1 *must be among the generators!). Then we have a unique representation* (1.4) *for each $\phi \in \mathscr{O}_{X,x}$ if and only if $\mathscr{O}_{X,x}$ is Cohen–Macaulay.*

By coherence it follows that if $\mathscr{O}_{X,x}$ is free as an $\mathscr{O}_{Z,x}$-module, then $\mathscr{O}_{Z,x'}$ is free as an $\mathscr{O}_{Z,x'}$-module for all x' in a neighborhood of x, and $1, w^{\alpha_1}, \dots, w^{\alpha_{\nu-1}}$ is a basis at each such x'.

Example 3.4 Let $\mathscr{J}$ be the ideal in $\mathbb{C}^4$ generated by $(w_1^2, w_2^2, w_1 w_2, w_1 z_2 - w_2 z_1)$. It is readily checked that $\mathscr{O}_X$ is a free $\mathscr{O}_Z$-module at a point on $Z = \{w_1 = w_2 = 0\}$ where z_1 or z_2 is $\neq 0$. If, say, $z_1 \neq 0$, then we can take $1, w_1$ as generators. At the point $z = (0, 0)$, e.g., $1, w_1, w_2$ form a minimal set of generators, and then $\mathscr{O}_X$ is not a free $\mathscr{O}_Z$-module, since there is a non-trivial relation between w_1 and w_2.

We claim that $\mathscr{O}_X$ has pure dimension. That is, we claim that there is no embedded associated prime ideal at $(0, 0)$; since Z is irreducible, this is the same as saying that $\mathscr{J}$ is primary with respect to Z. To see the claim, let ϕ and ψ be functions such that $\phi\psi$ is in $\mathscr{J}$ and ψ is not in $\sqrt{\mathscr{J}}$. The latter assumption means, in view of the Nullstellensatz, that ψ does not vanish identically on Z, i.e., $\psi = a(z) + \mathscr{O}(w)$, where a does not vanish identically. Since in particular $\phi\psi$ must vanish on Z it follows that $\phi = \mathscr{O}(w)$. It is now easy to see that ϕ is in $\mathscr{J}$. We conclude that $\mathscr{J}$ is primary. □

The pure-dimensionality of $\mathscr{O}_X$ can also be rephrased in the following way: *If ϕ is holomorphic and is* 0 *generically, then* $\phi = 0$. If we delete the generator $w_1 w_2$ from the definition of $\mathscr{J}$ in the example, then $\phi = w_1 w_2$ is 0 generically in $\mathscr{O}_\Omega/\mathscr{J}$ but is not identically zero. Thus $\mathscr{J}$ then has an embedded primary ideal at $(0, 0)$.

Example 3.5 Let $\Omega = \mathbb{C}^2_{z,w}$ and $\mathscr{J} = (w^2)$ so that $Z = \{w = 0\}$. Then $1, w$ is a basis for $\mathscr{O}_X = \mathscr{O}_{\mathbb{C}^2}/(w^2)$ so each function ϕ in $\mathscr{O}_X$ has a unique representation $a_0(z) \otimes 1 + a_1(z) \otimes w$. Let us consider the new coordinates $\zeta = z - w$, $\eta = w$. Then $\mathscr{J} = (\eta^2)$ and since

$$a_0(z) + a_1(z)w = a_0(\zeta + \eta) + a_1(\zeta + \eta)\eta = a_0(\zeta) + (\partial a_0/\partial\zeta)(\zeta)\eta + a_1(\zeta)\eta + \mathscr{J}$$

we have the representation $a_0(\zeta) \otimes 1 + (a_1(\zeta) + \partial a_0/\partial\zeta)(\zeta) \otimes \eta$ with respect to (ζ, η). □

More generally, assume that, at a given point in $X_{reg} \subset \Omega$, we have two different choices (z, w) and (ζ, η) of coordinates so that $Z = \{w = 0\} = \{\eta = 0\}$, and bases $1, \dots, w^{\alpha_{\nu-1}}$ and $1, \dots, \eta^{\beta_{\nu-1}}$ for $\mathscr{O}_X$ as a free module over $\mathscr{O}_Z$. Then there is a $\nu \times \nu$-matrix L of holomorphic differential operators so that if (a_j) is any tuple in $(\mathscr{O}_Z)^\nu$ and $(b_j) = L(a_j)$, then $a_0 \otimes 1 + \cdots + a_{\nu-1} \otimes w^{\alpha_{\nu-1}} = b_0 \otimes 1 + \cdots + b_{\nu-1} \otimes \eta^{\beta_{\nu-1}} + \mathscr{J}$.

4 Smooth $(0, *)$-forms on a non-reduced space X

Let $i: X \to \Omega$ be a local embedding of X. In order to define the sheaf of smooth $(0,*)$-forms on X, in analogy with the reduced case, we have to state which smooth $(0,*)$-forms Φ in Ω "vanish" on X, or more formally, give a meaning to $i^*\Phi = 0$. We will see, cf., Lemma 4.8 below, that the suitable requirement is that locally on X_{reg}, Φ belongs to $\mathscr{E}_\Omega^{0,*}\mathcal{J} + \mathscr{E}_\Omega^{0,*}\bar{\mathcal{J}}_Z + \mathscr{E}_\Omega^{0,*} d\bar{\mathcal{J}}_Z$, where $\mathcal{J}_z$ is the ideal sheaf defining Z. However, it turns out to be more convenient to represent the sheaf $\mathcal{K}er\, i^*$ of such forms as the annihilator of certain residue currents, and this is the path we will follow. Moreover, these currents play a central role themselves later on.

The following classical duality result is fundamental for this paper; see, e.g., [3] for a discussion.

Proposition 4.1 *If $\mathcal{J}$ has pure dimension, then*

$$\mathcal{J} = \operatorname{ann}_{\mathscr{O}_\Omega} \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z). \tag{4.1}$$

That is, ϕ is in $\mathcal{J}$ if and only if $\phi\mu = 0$ for all μ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$. It is also well-known, see, e.g., [3, Theorem 1.5], that

$$\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z) \simeq \mathcal{E}xt^{\,p}(\mathscr{O}_\Omega/\mathcal{J}, K_\Omega), \tag{4.2}$$

so $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ is a coherent analytic sheaf. Locally we thus have a finite number of generators $\mu^1, \dots, \mu^m$. In Example 6.9, we compute explicitly such generators for the ideal $\mathcal{J}$ in Example 3.4.

Let ξ be a smooth $(0,*)$-form in Ω. Without first giving meaning to i^*, we define the sheaf $\mathcal{K}er\, i^*$ by saying that ξ is in $\mathcal{K}er\, i^*$ if

$$\xi \wedge \mu = 0, \quad \mu \in \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z).$$

Notice that if ξ is holomorphic, then, in view of the duality (4.1), ξ is in $\mathcal{K}er\, i^*$ if and only if ξ is in $\mathcal{J}$.

Definition 4.2 We define the sheaf of smooth $(0,*)$-forms on X as

$$\mathscr{E}_X^{0,*} := \mathscr{E}_\Omega^{0,*}/\mathcal{K}er\, i^*. \tag{4.3}$$

We will prove below that this sheaf is independent of the choice of embedding and thus intrinsic on X.

Given ϕ in $\mathscr{E}_\Omega^{0,*}$, let $i^*\phi$ be its image in $\mathscr{E}_X^{0,*}$. In particular, $i^*\xi = 0$ means that ξ belongs to $\mathcal{K}er\, i^*$, which then motivates this notation. Notice that $\mathcal{K}er\, i^*$ is a two-sided ideal in $\mathscr{E}_\Omega^{0,*}$, i.e., if ϕ is in $\mathscr{E}_\Omega^{0,*}$ and ξ is in $\mathcal{K}er\, i^*$, then $\phi \wedge \xi$ and $\xi \wedge \phi$ are in $\mathcal{K}er\, i^*$. It follows that we have an induced wedge product on $\mathscr{E}_X^{0,*}$ such that

$$i^*(\phi \wedge \xi) = i^*\phi \wedge i^*\xi.$$

Remark 4.3 It follows from Lemma 4.8 below that in case $X = Z$ is reduced, then ξ is in $\mathcal{K}er\, i^*$ if and only its pullback to X_{reg} vanishes. Thus our definition of $\mathscr{E}_X^{0,*}$ is consistent with the usual one in that case. □

Lemma 4.4 *Using the notation of* (3.1),

$$\iota_*\colon \mathcal{H}om_{\mathscr{O}_{\widehat{\Omega}}}(\mathscr{O}_{\widehat{\Omega}}/\widehat{\mathcal{J}}, \mathcal{W}^Z_{\widehat{\Omega}}) \to \mathcal{H}om_{\mathscr{O}_\Omega}(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}^Z_\Omega) \tag{4.4}$$

is an isomorphism.

We can realize the mapping in (4.4) as the tensor product $\tau \mapsto \tau \wedge [w = 0]$, where $[w = 0]$ is the Lelong current in Ω associated with the submanifold $\{w = 0\}$.

Proof To begin with, ι_* maps pseudomeromorphic $(\hat{N}, \hat{p} + \ell)$-currents with support on $Z \subset \widehat{\Omega}$ to pseudomeromorphic $(N, p + \ell)$-currents with support on $Z \subset \Omega$. If, in addition, τ has the SEP with respect to Z, then $\iota_*\tau$ has, as well by (2.15). Moreover, if τ is annihilated by $\widehat{\mathcal{J}}$, then $\iota_*\tau$ is annihilated by $\mathcal{J} = \widehat{\mathcal{J}} \otimes 1 + (w)$. Thus the mapping (4.4) is well-defined, and it is injective since ι is injective.

Now assume that μ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}^Z_\Omega)$. Arguing as in the proof of Corollary 2.7, we see that $\mu = \iota_*\hat{\mu}$ for a current $\hat{\mu}$ in $\mathcal{W}^Z_{\widehat{\Omega}}$. Since $\widehat{\mathcal{J}} = \iota^*\mathcal{J}$ and $\mathcal{J}\mu = 0$, it follows that $\widehat{\mathcal{J}}\hat{\mu} = 0$. Thus (4.4) is surjective. □

Since ι_* is injective, $\bar{\partial}\tau = 0$ if and only if $\bar{\partial}\iota_*\tau = 0$, and thus we get

Corollary 4.5 *Using the notation of* (3.1),

$$\iota_*\colon \mathcal{H}om_{\mathscr{O}_{\widehat{\Omega}}}(\mathscr{O}_{\widehat{\Omega}}/\widehat{\mathcal{J}}, \mathcal{CH}^Z_{\widehat{\Omega}}) \to \mathcal{H}om_{\mathscr{O}_\Omega}(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}^Z_\Omega) \tag{4.5}$$

is an isomorphism.

Corollary 4.6 *Using the notation in* (3.1),

$$\iota^*\colon \mathscr{E}^{0,*}_\Omega/\mathcal{K}er\, i^* \to \mathscr{E}^{0,*}_{\widehat{\Omega}}/\mathcal{K}er\, j^*, \tag{4.6}$$

is an isomorphism.

Proof It follows immediately from (4.5) that the mapping (4.6) is well-defined and injective. Given $\widehat{\xi}$ in $\mathscr{E}^{0,*}_{\widehat{\Omega}}$, let $\xi = \widehat{\xi} \otimes 1$. Then $\iota^*\xi = \widehat{\xi}$ and so (4.6) is indeed surjective as well. □

It follows from (4.6) and (4.3) that the sheaf $\mathscr{E}_X^{0,*}$ is intrinsically defined on X. Since $\bar{\partial}$ maps $\mathcal{K}er\, i^*$ to $\mathcal{K}er\, i^*$, we have a well-defined operator $\bar{\partial}\colon \mathscr{E}_X^{0,*} \to \mathscr{E}_X^{0,*+1}$ such that $\bar{\partial}^2 = 0$. Unfortunately the sheaf complex so obtained is not exact in general, see, e.g., [6, Example 1.1] for a counterexample already in the reduced case.

4.1 Local representation on X_{reg} of smooth forms

Recall that X_{reg} is the open subset of X, where the underlying reduced space is smooth and $\mathscr{O}_X$ is Cohen–Macaulay. Let us fix some point in X_{reg}, and assume that we have local coordinates (z, w) such that $Z = \{w = 0\}$. We also choose generators $1, w^{\alpha_1}, \dots, w^{\alpha_{\nu-1}}$ of $\mathscr{O}_X$ as a free $\mathscr{O}_Z$-module, which exist by Corollary 3.3, and generators $\mu^1, \dots, \mu^m$ of $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}^Z_\Omega)$.

Notice that for each smooth $(0,*)$-form Φ in Ω, $\Phi \mapsto \Phi \wedge \mu^\ell$ only depends on its class ϕ in $\mathscr{E}_X^{0,*}$, and ϕ is in fact determined by these currents. By Proposition 2.5 each of these currents can (locally) be represented by a tuple of currents in $\mathcal{W}_Z^{0,*}$. Putting all these tuples together, we get a tuple in $(\mathcal{W}_Z^{0,*})^M$, where $M = M_1 + \cdots + M_m$ and M_j is the number of indices in (2.11) in the representation of μ^j.

Recall from Corollary 3.3 that ϕ in $\mathscr{O}_X$ has a unique representative

$$\hat{\phi} = \hat{\phi}_0 + \hat{\phi}_1 \otimes w^{\alpha_1} + \cdots + \hat{\phi}_{\nu-1} \otimes w^{\alpha_{\nu-1}}, \tag{4.7}$$

where $\hat{\phi}_j$ are in $\mathscr{O}_Z$. We thus have an $\mathscr{O}_Z$-linear morphism

$$T\colon (\mathscr{O}_Z)^\nu \to (\mathscr{O}_Z)^M. \tag{4.8}$$

The morphism is injective by Proposition 4.1, and the holomorphic matrix T is therefore generically pointwise injective.

Lemma 4.7 *Each ϕ in $\mathscr{E}_X^{0,*}$ has a unique representation* (4.7) *where $\hat{\phi}_j$ are in $\mathscr{E}_Z^{0,*}$.*

Proof To begin with notice that a given smooth ϕ must have at least one such representation. In fact, taking the finite Taylor expansion (2.13) we can forget about high order terms, since they must annihilate all the μ^j, and the terms $\bar{w}$ and $d\bar{w}$ annihilate all the μ^j as well since they are pseudomeromorphic with support on $\{w = 0\}$. On the other hand, each w^α not in the set of generators must be of the form

$$w^\alpha = a_0 + a_1 \otimes w^{\alpha_1} + \cdots + a_{\nu-1} \otimes w^{\alpha_{\nu-1}} + \mathcal{J},$$

and hence $\phi_\alpha \otimes w^\alpha$ is of the form (4.7). Thus the representation exists. To show uniqueness of the representation, we assume that $\hat{\phi}$ is in $\mathcal{K}er\, i^*$. Then the tuple $(\hat{\phi}_j)$ is mapped to 0 by the matrix T, and since T is generically pointwise injective we conclude that each $\hat{\phi}_j$ vanishes. □

By the above proof we get

Lemma 4.8 *A smooth $(0,*)$-form ξ in Ω is in $\mathcal{K}er\, i^*$ if and only if ξ is in $\mathscr{E}_\Omega^{0,*}\mathcal{J} + \mathscr{E}_\Omega^{0,*}\bar{\mathcal{J}}_Z + \mathscr{E}_\Omega^{0,*} d\bar{\mathcal{J}}_Z$ on X_{reg}, where $\mathcal{J}_Z$ is the radical sheaf of Z.*

Remark 4.9 This is *not* the same as saying that ξ is in $\mathscr{E}_\Omega^{0,*}\mathcal{J} + \mathscr{E}_\Omega^{0,*}\bar{\mathcal{J}}_Z + \mathscr{E}_\Omega^{0,*} d\bar{\mathcal{J}}_Z$ at singular points. For a simple counterexample, consider $\phi = x\bar{y}$ on the reduced space $Z = \{xy = 0\} \subset \mathbb{C}^2$.

However, this can happen also when Z is irreducible at a point. For example, the variety $Z = \{x^2 y - z^2 = 0\} \subset \mathbb{C}^3$ is irreducible at 0, but there exist points arbitrarily close to 0 such that (Z, z) is not irreducible. In this case, the ideal of smooth functions vanishing on $(Z, 0)$ is strictly larger than $\mathscr{E}_\Omega^{0,0} \mathcal{J}_{Z,0} + \mathscr{E}_\Omega^{0,0} \bar{\mathcal{J}}_{Z,0}$ see [26, Proposition 9, Chapter IV], and [25, Theorem 3.10, Chapter VI]. $\square$

Remark 4.10 It is easy to check that if we have the setting as in the discussion at the end of Sect. 3 but (a_j) is instead a tuple in $\mathscr{E}_Z^{0,*}$, then we can still define $(b_j) = L(a_j)$ if we consider the derivatives in L as Lie derivatives; in fact, since a_j has no holomorphic differentials, L only acts on the smooth coefficients, and it is easy to check that $a_0 \otimes 1 + \cdots + a_{\nu-1} \otimes w^{\alpha_{\nu-1}}$ and $b_0 \otimes 1 + \cdots + b_{\nu-1} \otimes \eta^{\beta_{\nu-1}}$ are equal modulo $\mathscr{E}_\Omega^{0,*} \mathcal{J} + \mathscr{E}_\Omega^{0,*} \bar{\mathcal{J}}_Z + \mathscr{E}_\Omega^{0,*} d\bar{\mathcal{J}}_Z$, and thus define the same element in $\mathscr{E}_X^{0,*}$. $\square$

For future needs we prove in Sect. 6.1:

Lemma 4.11 *The morphism T is pointwise injective.*

We can thus choose a holomorphic matrix A such that

$$0 \to \mathscr{O}_Z^\nu \xrightarrow{T} \mathscr{O}_Z^M \xrightarrow{A} \mathscr{O}_Z^{M'} \tag{4.9}$$

is pointwise exact, and we can also find holomorphic matrices S and B such that

$$I = TS + BA. \tag{4.10}$$

5 Intrinsic $(n, *)$-currents on X

In analogy with the reduced case we have the following definition when X is possibly non-reduced.

Definition 5.1 The sheaf $\mathcal{C}_X^{n,q}$ of (n, q)-currents on X is the dual sheaf of $(0, n-q)$-test forms, i.e., forms in $\mathscr{E}_X^{0,n-q}$ with compact support.

Here, just as in the case of reduced spaces, cf., for example [19, Section 4.2], the space of smooth forms $\mathscr{E}_X^{0,n-q}$ is equipped with the quotient topology induced by a local embedding.

More concretely, this means that given an embedding $i\colon X \to \Omega$, currents ψ in $\mathcal{C}_X^{n,q}$ precisely correspond to the $(N, N-n+q)$-currents τ on Ω that vanish on $\mathcal{K}er\, i^*$. Since $\mathcal{K}er\, i^*$ is a two-sided ideal in $\mathscr{E}_\Omega^{0,*}$ this holds if and only if $\xi \wedge \tau = 0$ for all ξ in $\mathcal{K}er\, i^*$. It is natural to write $\tau = i_* \psi$ so that

$$i_* \psi . \xi = \psi . i^* \xi.$$

Clearly, we get a mapping $\bar{\partial}\colon \mathcal{C}_X^{n,q} \to \mathcal{C}_X^{n,q+1}$ such that $\bar{\partial}^2 = 0$.

Proposition 5.2 *If τ is in $\mathcal{W}_\Omega^Z$ and $\mathcal{J}\tau = 0$, then $\xi \wedge \tau = 0$ for all smooth ξ such that $i^* \xi = 0$.*

Proof Because of the SEP it is enough to prove that $\xi\wedge\tau=0$ on X_{reg}. By assumption, $\mathcal{J}$ annihilates τ, and by general properties of pseudomeromorphic currents, since τ has support on Z, $\bar{\mathcal{J}}_Z$ and $d\bar{\mathcal{J}}_Z$ annihilate τ. Thus the proposition follows by Lemma 4.8. □

Definition 5.3 An $(n,*)$-current ψ on X is in $\mathcal{W}_X^{n,*}$ if $i_*\psi$ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z)$.

By definition we thus have the isomorphism

$$i_*\colon \mathcal{W}_X^{n,*}\simeq \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z). \tag{5.1}$$

It follows from Lemma 4.4 that $\mathcal{W}_X^{n,*}$ is intrinsically defined.

Remark 5.4 By Corollary 2.7, this definition is consistent with the previous definition of $\mathcal{W}_X^{n,*}$ when X is reduced. We cannot define $\mathcal{PM}_X^{n,*}$ in the analogous simple way, cf., Remark 2.8. □

Definition 5.5 If ψ is in $\mathcal{W}_X^{n,*}$ and a is an almost semi-meromorphic $(0,*)$-current on Ω that is generically smooth on Z, then the product $a\wedge\psi$ is a current in $\mathcal{W}_X^{n,*}$ defined as follows: By definition, $i_*\psi$ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z)$ and by Proposition 2.4 and (2.8), one can define $a\wedge i_*\psi$ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z)$; now $a\wedge\psi$ is the unique current in $\mathcal{W}_X^{n,*}$ such that $i_*(a\wedge\psi)=a\wedge i_*\psi$.

By (2.7),

$$a\wedge\psi=\lim_{\epsilon\to 0^+}\chi(|h|^2/\epsilon)a\wedge\psi \tag{5.2}$$

if h cuts out the Zariski singular support of a.

Definition 5.6 We let ω_X^n be the sheaf of $\bar\partial$-closed currents in $\mathcal{W}_X^{n,0}$.

This sheaf corresponds via i_* to $\bar\partial$-closed currents in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z)$ so we have the isomorphism

$$i_*\colon \omega_X^n\simeq \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{CH}_\Omega^Z). \tag{5.3}$$

When X is reduced ω_X^n is the sheaf of $(n,0)$-forms that are $\bar\partial$-closed in the Barlet–Henkin–Passare sense. Let $\mu^1,\dots,\mu^m$ be a set of generators for $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{CH}_\Omega^Z)$. They correspond via (5.3) to a set of generators $h^1,\dots,h^m$ for the $\mathscr{O}_X$-module ω_X^n.

We will also need a definition of $\mathcal{PM}_X^{n,*}$. Let $\mathcal{F}_X$ be the subsheaf of $\mathcal{C}_X^{n,*}$ of τ such that $i_*\tau$ is in $\mathcal{PM}_\Omega^Z$. If τ is a section of $\mathcal{F}_X$ and W is a subvariety of some open subset of Z, then $\mathbf{1}_W i_*\tau$ is in $\mathcal{PM}_\Omega^Z$, and by (2.3), $\mathbf{1}_W i_*\tau$ is annihilated by $\mathcal{K}er\, i^*$. Hence we can define $\mathbf{1}_W\tau$ as the unique current in $\mathcal{F}_X$ such that $i_*\mathbf{1}_W\tau=\mathbf{1}_W i_*\tau$. Clearly, $\mathbf{1}_W\tau$ has support on W and it is easily checked that the computational rule (2.3) holds also in $\mathcal{F}_X$. Moreover, $\mathcal{F}_X$ is closed under $\bar\partial$ since $\mathcal{PM}_\Omega^Z$ is.

Definition 5.7 The sheaf $\mathcal{PM}_X^{n,*}$ is the smallest subsheaf of $\mathcal{F}_X$ that contains $\mathcal{W}_X^{n,*}$ and is closed under $\bar\partial$ and multiplication by $\mathbf{1}_W$ for all germs W of subvarieties of Z.

In view of Proposition 2.2 this definition coincides with the usual definition in case X is reduced. It is readily checked that the dimension principle holds for $\mathcal{F}_X$, and hence it also holds for the (possibly smaller) sheaf $\mathcal{PM}_X^{n,*}$, and in addition, (2.3) holds for forms ξ in $\mathscr{E}_X^{0,*}$ and τ in $\mathcal{PM}_X^{n,*}$.

6 Structure form on X

Let $i\colon X \to \Omega \subset \mathbb{C}^N$ be a local embedding as before, let $p = N - n$ be the codimension of X, and let $\mathcal{J}$ be the associated ideal sheaf on Ω. In a slightly smaller set, still denoted Ω, there is a free resolution

$$0 \to \mathscr{O}(E_{N_0}) \xrightarrow{f_{N_0}} \cdots \xrightarrow{f_3} \mathscr{O}(E_2) \xrightarrow{f_2} \mathscr{O}(E_1) \xrightarrow{f_1} \mathscr{O}(E_0) \tag{6.1}$$

of $\mathscr{O}_\Omega/\mathcal{J}$; here E_k are trivial vector bundles over Ω and E_0 is the trivial line bundle. This resolution induces a complex of vector bundles

$$0 \to E_{N_0} \xrightarrow{f_{N_0}} \cdots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \tag{6.2}$$

that is pointwise exact outside Z. Let X_k be the set where f_k does not have optimal rank. Then

$$\cdots \subset X_{k+1} \subset X_k \subset \cdots \subset X_{p+1} \subset X_p = \cdots = X_1 = Z;$$

these sets are independent of the choice of resolution and thus invariants of $\mathscr{O}_\Omega/\mathcal{J}$. Since $\mathscr{O}_\Omega/\mathcal{J}$ has *pure* codimension p,

$$\operatorname{codim} X_k \geq k+1, \quad \text{for}\, k \geq p+1, \tag{6.3}$$

see [14, Corollary 20.14]. Thus there is a free resolution (6.1) if and only if $X_k = \emptyset$ for $k > N_0$. Unless $n = 0$ (which is not interesting in relation to the $\bar\partial$-equation), we can thus choose the resolution so that $N_0 \leq N - 1$. The variety X is Cohen–Macaulay at a point x, i.e., the sheaf $\mathscr{O}_\Omega/\mathcal{J}$ is Cohen–Macaulay at x, if and only if $x \notin X_{p+1}$. Notice that $Z\setminus(X_{reg})_{red} = Z_{sing} \cup X_{p+1}$. The sets X_k are independent of the choice of embedding, see [9, Lemma 4.2], and are thus intrinsic subvarieties of $Z = X_{red}$, and they reflect the complexity of the singularities of X.

Let us now choose Hermitian metrics on the bundles E_k. We then refer to (6.1) as a *Hermitian resolution* of $\mathscr{O}_\Omega/\mathcal{J}$ in Ω. In $\Omega\setminus X_k$ we have a well-defined vector bundle morphism $\sigma_{k+1}\colon E_k \to E_{k+1}$, if we require that σ_{k+1} vanishes on $(\operatorname{Im} f_{k+1})^\perp$, takes values in $(\mathcal{K}er\ f_{k+1})^\perp$, and that $f_{k+1}\sigma_{k+1}$ is the identity on $\operatorname{Im} f_{k+1}$. Following [7, Section 2] we define smooth E_k-valued forms

$$u_k = (\bar\partial\sigma_k)\cdots(\bar\partial\sigma_2)\sigma_1 = \sigma_k(\bar\partial\sigma_{k-1})\cdots(\bar\partial\sigma_1) \tag{6.4}$$

in $\Omega\setminus X$; for the second equality, see [7, (2.3)]. We have that

$$f_1 u_1 = 1, \quad f_{k+1}u_{k+1} - \bar\partial u_k = 0, \quad k \geq 1,$$

in $\Omega\setminus X$. If $f := \oplus f_k$ and $u := \sum u_k$, then these relations can be written economically as $\nabla_f u = 1$, where $\nabla_f := f - \bar\partial$. To make the algebraic machinery work properly one has to introduce a superstructure on the bundle $E =: \oplus E_k$ so that vectors in E_{2k} are even and vectors in E_{2k+1} are odd; hence f, $\sigma := \oplus\sigma_k$, and $u := \sum u_k$ are odd. For details, see [7]. It turns out that u has a (necessarily unique) almost semi-meromorphic extension U to Ω. The residue current R is defined by the relation

$$\nabla_f U = 1 - R. \tag{6.5}$$

It follows directly that R is ∇_f-closed. In addition, R has support on Z and is a sum $\sum R_k$, where R_k is a pseudomeromorphic E_k-valued current of bidegree $(0, k)$. It follows from the dimension principle that $R = R_p + R_{p+1} + \cdots + R_N$. If we choose a free resolution that ends at level $N-1$, then $R_N = 0$. If X is Cohen–Macaulay and $N_0 = p$ in (6.1), then $R = R_p$, and the ∇_f-closedness implies that R is $\bar\partial$-closed.

If ϕ is in $\mathcal{J}$ then $\phi R = 0$ and in fact, $\mathcal{J} = \text{ann}\, R$, see [7, Theorem 1.1].

Remark 6.1 In case $\mathcal{J}$ is generated by the single non-trivial function f, then we have the free resolution $0 \to \mathscr{O}_\Omega \xrightarrow{f} \mathscr{O}_\Omega \to \mathscr{O}_\Omega/(f) \to 0$; thus U is just the principal value current $1/f$ and $R = \bar\partial(1/f)$. More generally, if $f = (f_1, \ldots, f_p)$ is a complete intersection, then

$$R = \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1},$$

where the right hand side is the so-called Coleff–Herrera product of f, see for example [1, Corollary 3.5]. □

There are almost semi-meromorphic α_k in Ω, cf., [7, Section 2] and the proof of [6, Proposition 3.3], that are smooth outside X_k, such that

$$R_{k+1} = \alpha_{k+1} R_k \tag{6.6}$$

outside X_{k+1} for $k \geq p$. In view of (6.3) and the dimension principle, $\mathbf{1}_{X_{k+1}} R_{k+1} = 0$ and hence (6.6) holds across X_{k+1}, i.e., R_{k+1} is indeed equal to the product $\alpha_{k+1} R_k$ in the sense of Proposition 2.1. In particular, it follows that R_k has the SEP with respect to Z.

In this section, we let $(z_1, \ldots, z_N)$ denote coordinates on $\mathbb{C}^N$, and let $dz := dz_1 \wedge \cdots \wedge dz_N$.

Lemma 6.2 *There is a matrix of almost semi-meromorphic currents b such that*

$$R \wedge dz = b\mu, \tag{6.7}$$

where μ is a tuple of currents in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$.

Proof As in [6, Section 3], see also [32, Proposition 3.2], one can prove that $R_p = \sigma_F \mu$, where μ is a tuple of currents in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ and σ_F is an almost semi-meromorphic current that is smooth outside X_{p+1}.

Let $b_p = \sigma_F$ and $b_k = \alpha_k \cdots \alpha_{p+1}\sigma_F$ for $k \geq p+1$. Then each b_k is almost semi-meromorphic, cf., [10, Section 4.1]. In view of (6.6) we have that $R_k = b_k\mu$ outside X_{p+1} since b_k is smooth there. It follows by the SEP that it holds across X_{p+1} as well since R_k has the SEP with respect to Z. We then take $b = b_p + b_{p+1} + \cdots$. □

By Proposition 2.4 we get

Corollary 6.3 *The current $R \wedge dz$ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$.*

From Lemma 6.2, Corollary 6.3, (5.1), and (5.3) we get the following analogue to [6, Proposition 3.3]:

Proposition 6.4 *Let (6.1) be a Hermitian resolution of $\mathscr{O}_\Omega/\mathcal{J}$ in Ω, and let R be the associated residue current. Then there exists a (unique) current ω in $\mathcal{W}_X^{n,*}$ such that*

$$i_*\omega = R \wedge dz. \tag{6.8}$$

There is a matrix b of almost semi-meromorphic $(0,*)$-currents in Ω, smooth outside of X_{p+1}, and a tuple ϑ of currents in ω_X^n such that

$$\omega = b\vartheta. \tag{6.9}$$

More precisely, $\omega = \omega_0 + \omega_1 + \cdots + \omega_n$,[1] where $\omega_k \in \mathcal{W}^{n,k}(X, E_{p+k})$, and if $f^j := f_{p+j}$, then

$$f^0\omega_0 = 0, \quad f^{j+1}\omega_{j+1} - \bar\partial\omega_j = 0, \textit{ for } j \geq 0. \tag{6.10}$$

We will also use the short-hand notation $\nabla_f\omega = 0$. As in the reduced case, following [6], we say that ω is a *structure form* for X. The products in (6.9) are defined according to Definition 5.5.

Remark 6.5 Recall that $X_{p+1} = \emptyset$ if X is Cohen–Macaulay, so in that case $\omega = b\vartheta$, where b is smooth. If we take a free resolution of length p, then $\omega = \omega_0$, and $\bar\partial\omega_0 = f^1\omega_1 = 0$, so ω is in ω_X^n. □

Remark 6.6 If $X = \{f = 0\}$ is a reduced hypersurface in Ω, then $R = \bar\partial(1/f)$ and ω is the classical Poincaré residue form on X associated with f, which is a meromorphic form on X. More generally, if X is reduced, since forms in ω_X^n are then meromorphic, by (6.9), ω can be represented by almost semi-meromorphic forms on X.

We now consider the case when X is non-reduced. We recall that a differential operator is a Noetherian operator for an ideal $\mathcal{J}$ if $\mathcal{L}\varphi \in \sqrt{\mathcal{J}}$ for all $\varphi \in \mathcal{J}$. It is proved by Björk, [13], see also [32, Theorem 2.2], that if $\mu \in \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$, then there exists a Noetherian operator $\mathcal{L}$ for $\mathcal{J}$ with meromorphic coefficients such that the action of μ on ξ equals the integral of $\mathcal{L}\xi$ over Z. By (5.3), the action of h in ω_X^n on ξ in $\mathscr{E}_X^{0,*}$ can then be expressed as

$$h.\xi = \int_Z \mathcal{L}\xi.$$

[1] In [6, Proposition 3.3], the sum ends with ω_{n-1} instead of ω_n, which, as remarked above, one can indeed assume when $n \geq 1$ and the resolution is chosen to be of length $\leq N-1$.

One can then verify using this formula and (6.9) that the action of the structure form ω on a test form ξ in $\mathscr{E}_X^{0,*}$ equals

$$\omega.\xi = \int_Z \tilde{\mathcal{L}}\xi,$$

where $\tilde{\mathcal{L}}$ is now a tuple of Noetherian operators for $\mathcal{J}$ with almost semi-meromorphic coefficients, cf., [32, Section 4]. □

Notice that (6.1) gives rise to the dual Hermitian complex

$$0 \to \mathscr{O}(E_0^*) \xrightarrow{f_1^*} \cdots \to \mathscr{O}(E_{p-1}^*) \xrightarrow{f_p^*} \mathscr{O}(E_p^*) \xrightarrow{f_{p+1}^*} \cdots . \tag{6.11}$$

Let $\xi = \xi_0 \wedge dz$ be a holomorphic section of the sheaf

$$\mathcal{H}om(E_p, K_\Omega) \simeq \mathscr{O}(E_p^*) \otimes \mathscr{O}(K_\Omega)$$

such that $f_{p+1}^*\xi_0 = 0$. Then $\bar\partial(\xi_0\omega_0) = \pm\xi_0\bar\partial\omega_0 = \pm\xi_0 f_{p+1}\omega_1 = \pm(f_{p+1}^*\xi_0)\omega_1 = 0$, so that $\xi_0\omega_0$ is in ω_X^n. Moreover, if $\xi_0 = f_p^*\eta$ for η in $\mathscr{O}(E_{p-1}^*)$, then $\xi_0\omega_0 = f_p^*\eta\omega_0 = \pm\eta f_p\omega_0 = 0$. We thus have a sheaf mapping

$$\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \to \omega_X^n, \quad \xi_0 \wedge dz \mapsto \xi_0\omega_0. \tag{6.12}$$

Proposition 6.7 *The mapping* (6.12) *is an isomorphism, which establishes an intrinsic isomorphism*

$$\mathcal{E}xt^{\,p}(\mathscr{O}_\Omega/\mathcal{J}, K_\Omega) \simeq \omega_X^n. \tag{6.13}$$

Proof If h is in ω_X^n, then i_*h is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$. We have mappings

$$\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \to \omega_X^n \xrightarrow{\simeq} \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z), \tag{6.14}$$

where the first mapping is (6.12), and the second is $h \mapsto i_*h$. In view of (6.8), the composed mapping is $\xi = \xi_0 \wedge dz \mapsto \xi R_p = \xi_0 R_p \wedge dz$.[2] This mapping is an intrinsic isomorphism

$$\mathcal{E}xt^{\,p}(\mathscr{O}_\Omega/\mathcal{J}, K_\Omega) \simeq \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$$

according to [3, Theorem 1.5]. It follows that (6.12) also establishes an intrinsic isomorphism. □

In particular it follows that ω_X^n is coherent, and we have:
If $\xi^1, \ldots, \xi^m$ are generators of $\mathcal{H}^p(\mathcal{H}om(E_\bullet^*, K_\Omega)))$, where $\xi^\ell = \xi_0^\ell \wedge dz$, then $h^\ell := \xi_0^\ell\omega_0$, $\ell = 1, \ldots, m$, generate the $\mathscr{O}_X$-module ω_X^n, and $\mu^\ell = i_*h^\ell = \xi^\ell R_p$ generate the $\mathscr{O}_\Omega$-module $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$.

[2] There is a superstructure involved, with respect to which R_p has even degree, and therefore $dz \wedge R_p = R_p \wedge dz$, explaining the lack of a sign in the last equality, see [6] or [7].

Remark 6.8 The isomorphism

$$\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \xrightarrow{\simeq} \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z) \tag{6.15}$$

was well-known since long ago, the contribution in [3] was the realization $\xi \mapsto \xi R_p$.□

We give here an example where we can explicitly compute generators of $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$.

Example 6.9 Let $\mathcal{J}$ be as in Example 3.4. We claim that $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ is generated by

$$\mu_1 := \bar\partial \frac{1}{w_1} \wedge \bar\partial \frac{1}{w_2} \wedge dz \wedge dw \text{ and } \mu_2 := \left(z_1 \bar\partial \frac{1}{w_1^2} \wedge \bar\partial \frac{1}{w_2} + z_2 \bar\partial \frac{1}{w_1} \wedge \bar\partial \frac{1}{w_2^2} \right) \wedge dz \wedge dw.$$

In order to prove this claim, we use the comparison formula for residue currents from [21], which states that if $\mathscr{O}(F_\bullet)$ and $\mathscr{O}(E_\bullet)$ are free resolutions of $\mathscr{O}_\Omega/\mathcal{I}$ and $\mathscr{O}_\Omega/\mathcal{J}$, respectively, where $\mathcal{I}$ and $\mathcal{J}$ have codimension $\geq p$, and $a : F_\bullet \to E_\bullet$ is a morphism of complexes, then there exists a $\mathcal{H}om(F_0, E_{p+1})$-valued current M_{p+1} such that $R_p^E a_0 = a_p R_p^F + f_{p+1} M_{p+1}$. If ξ is in $\mathcal{K}er\, f_{p+1}^*$, we thus get that

$$\xi R_p^E a_0 = \xi a_p R_p^F. \tag{6.16}$$

We will apply this with $\mathscr{O}_\Omega(E_\bullet)$ as the free resolution

$$0 \to \mathscr{O}_\Omega \xrightarrow{f_3} \mathscr{O}_\Omega^4 \xrightarrow{f_2} \mathscr{O}_\Omega^4 \xrightarrow{f_1} \mathscr{O}_\Omega \to \mathscr{O}_\Omega/\mathcal{J} \to 0,$$

where

$$f_3 = \begin{bmatrix} w_2 \\ -w_1 \\ z_2 \\ -z_1 \end{bmatrix}, f_2 = \begin{bmatrix} z_2 & 0 & -w_2 & 0 \\ -z_1 & z_2 & w_1 & -w_2 \\ 0 & -z_1 & 0 & w_1 \\ -w_1 & -w_2 & 0 & 0 \end{bmatrix} \quad \text{and}$$
$$f_1 = \begin{bmatrix} w_1^2 & w_1 w_2 & w_2^2 & z_2 w_1 - z_1 w_2 \end{bmatrix},$$

and the Koszul complex $(F, \delta_{\mathbf{w}^2})$ generated by $\mathbf{w}^2 := (w_1^2, w_2^2)$, which is a free resolution of $\mathscr{O}/(w_1^2, w_2^2)$. We then take the morphism of complexes $a : F_\bullet \to E_\bullet$ given by

$$a_2 = \begin{bmatrix} 0 \\ 0 \\ w_2 \\ w_1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } a_0 = \begin{bmatrix} 1 \end{bmatrix}.$$

Since the current R_2^F is equal to the Coleff–Herrera product $\bar{\partial}(1/w_1^2) \wedge \bar{\partial}(1/w_2^2)$, cf., Remark 6.1, we thus get by (6.16) and Remark 6.8 that $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ is generated by

$$(\mathcal{K}er\ f_3^*)a_2\bar{\partial}\frac{1}{w_1^2} \wedge \bar{\partial}\frac{1}{w_2^2}.$$

A straightforward calculation gives the generators μ_1 and μ_2 above. $\square$

6.1 Proof of Lemma 4.11

Since T is generically injective, it is clearly injective if $n = 0$. We are going to reduce to this case. Fix the point $0 \in Z$ and let $\mathcal{I}$ be the ideal generated by $z = (z_1, \dots, z_n)$.

Let $\mathscr{O}(E_\bullet)$ be a free Hermitian resolution of $\mathscr{O}_\Omega/\mathcal{J}$ of minimal length $p = N - n$ at 0 and let R^E be the associated residue current. Recall that the canonical isomorphism (6.15) is realized by $\xi \mapsto \xi R_p^E$. Let $F_\bullet$ be the Koszul complex generated by z; then $\mathscr{O}(F_\bullet)$ is a free resolution of $\mathscr{O}_\Omega/\mathcal{I}$. Since $\mathcal{J}$ and $\mathcal{I}$ are Cohen–Macaulay and intersect properly in Ω, the complex $\mathscr{O}_\Omega((E \otimes F)_\bullet)$ is a free resolution of $\mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I})$, and the corresponding residue current is

$$R_N^{E\otimes F} = R_p^E \wedge R_n^F$$

according to [2, Theorem 4.2]. From [3, Theorem 1.5] again it follows that the canonical isomorphism

$$\mathcal{H}^N(\mathcal{H}om((E \otimes F)_\bullet, K_\Omega)) \to \mathcal{H}om(\mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I}), \mathcal{CH}_\Omega^{\{0\}})$$

is given by $\eta \mapsto \eta R_N^{E\otimes F}$.

Let $\mu^1, \dots, \mu^m$ be a minimal set of generators for $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ at 0. Then $\mu^j = \xi^j R_p^E$, where ξ^j is a minimal set of generators for $\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega))$. Notice that

$$\mathcal{H}^N(\mathcal{H}om((E \otimes F)_\bullet, K_\Omega)) = \mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \otimes_{\mathscr{O}} \mathcal{H}^n(\mathcal{H}om(F_\bullet, \mathscr{O}_\Omega)).$$

Since $\mathcal{H}^n(\mathcal{H}om(F_\bullet, \mathscr{O}_\Omega))$ is generated by 1, it follows that $\mathcal{H}^N(\mathcal{H}om((E \otimes F)_\bullet, K_\Omega))$ is generated by $\xi^j \otimes 1$. We conclude that $\mathcal{H}om(\mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I}), \mathcal{CH}_\Omega^{\{0\}})$ is generated by $\xi^j \otimes 1 \cdot R_p^E \wedge R_n^F = \mu^j \wedge \mu^z$, $j = 1, \dots, m$, where $R_n^F = \mu^z = \bar{\partial}(1/z^{\mathbf{1}})$.

If $1, \dots, w^{\alpha_{\nu-1}}$ is a basis for $\mathscr{O}_\Omega/\mathcal{J}$ as an $\mathscr{O}_Z$-module, then it is also a basis for $\mathscr{O}_{X_0} := \mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I})$ as a module over $\mathscr{O}_{\{0\}} \simeq \mathbb{C}$. Since $\phi\bar{\partial}(1/z^{\mathbf{1}}) = \phi(0, \cdot)\bar{\partial}(1/z^{\mathbf{1}})$

we have that

$$\phi(z,w)\mu^j \wedge \mu^z = \phi(z,w)\sum a_\ell^j(z)\bar\partial\frac{1}{w^{\ell+\mathbf{1}}} \wedge \bar\partial\frac{1}{z^{\mathbf{1}}}$$
$$= \phi(0,w)\sum a_\ell^j(0)\bar\partial\frac{1}{w^{\ell+\mathbf{1}}} \wedge \bar\partial\frac{1}{z^{\mathbf{1}}}.$$

The morphism constructed in (4.8) for X_0 instead of X is then $T_0 = T(0)$, where T is the morphism (4.8) for X. Thus $T(0)$ is injective.

7 The intrinsic sheaf $\mathcal{W}_X^{0,*}$ on X

Our aim is to find a fine resolution of $\mathscr{O}_X$ and since the complex (1.1) is not exact in general when X is singular we have to consider larger fine sheaves; we first define sheaves $\mathcal{W}_X^{0,*} \supset \mathscr{E}_X^{0,*}$ of $(0,*)$-currents. Given a local embedding $i\colon X \to \Omega$ at a point on X_{reg} and local coordinates (z,w) as before, it is natural, in view of Lemma 4.7, to require that an element in $\mathcal{W}_X^{0,*}$ shall have a unique representation

$$\phi = \widehat{\phi}_0 \otimes 1 + \widehat{\phi}_1 \otimes w^{\alpha_1} + \cdots + \widehat{\phi}_{\nu-1} \otimes w^{\alpha_{\nu-1}}, \tag{7.1}$$

where $\widehat{\phi}_j$ are in $\mathcal{W}_Z^{0,*}$. In view of Remark 4.10 we should expect that the same transformation rules hold as for smooth $(0,*)$-forms. In particular it is then necessary that $\mathcal{W}_Z^{0,*}$ is closed under the action of holomorphic differential operators, which in fact is true, see Proposition 7.11 below. We must also define a reasonable extension of these sheaves across X_{sing}. Before we present our formal definition we make a preliminary observation.

Lemma 7.1 *If ϕ has the form (7.1) and τ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$, expressed in the form (2.11), then*

$$\phi \wedge \tau := \sum_i \sum_{\gamma \ge \alpha_i} \widehat{\phi}_i \wedge \tau_\gamma \wedge dz \otimes \bar\partial\frac{dw}{w^{\gamma-\alpha_i+\mathbf{1}}} \tag{7.2}$$

is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$.

Proof The right hand side defines a current in $\mathcal{W}_\Omega^Z$ since $\widehat{\phi}_i$ are in $\mathcal{W}_Z^{0,*}$ and τ_γ are in $\mathscr{O}_Z$. We have to prove that it is annihilated by $\mathcal{J}$. Take ξ in $\mathcal{J}$. On the subset of Z where $\widehat{\phi}_0, \dots, \widehat{\phi}_{\nu-1}$ are all smooth, $\phi\wedge\tau$, as defined above, is just multiplication of the smooth form ϕ by τ, and thus $\xi\phi\wedge\tau = 0$ there. We have a unique representation

$$\xi\phi\wedge\tau = \sum_{\ell\ge 0} a_\ell(z) \wedge dz \otimes \bar\partial\frac{dw}{w^{\ell+\mathbf{1}}},$$

with a_ℓ in $\mathcal{W}_Z^{0,*}$. Since a_ℓ vanish on the set where all $\widehat{\phi}_j$ are smooth, we conclude from Proposition 2.9 that a_ℓ vanish identically. It follows that $\xi\phi\wedge\tau = 0$. $\square$

If ϕ has the form (7.1) in a neighborhood of some point $x \in X_{reg}$ and h is in ω_X^n, then we get an element $\phi \wedge h$ in $\mathcal{W}_X^{n,*}$ defined by $i_*(\phi \wedge h) = \phi \wedge i_* h$. It follows that ϕ in this way defines an element in $\mathcal{H}om_{\mathscr{O}_X}(\omega_X^n, \mathcal{W}_X^{n,*})$. This sheaf is global and invariantly defined and so we can make the following global definition.

Definition 7.2 $\mathcal{W}_X^{0,*} = \mathcal{H}om_{\mathscr{O}_X}(\omega_X^n, \mathcal{W}_X^{n,*})$.

If ϕ is in $\mathcal{W}_X^{0,*}$ and h is in ω_X^n, we consider $\phi(h)$ as the product of ϕ and h, and sometimes write it as $\phi \wedge h$.

Since $\mathcal{W}_X^{n,*}$ are $\mathscr{E}_X^{0,*}$-modules, $\mathcal{W}_X^{0,*}$ are as well. Before we investigate these sheaves further, we give some motivation for the definition. First notice that we have a natural injection, cf., Proposition 4.1,

$$\mathscr{O}_X \to \mathcal{H}om\left(\omega_X^n, \omega_X^n\right), \quad \phi \mapsto (h \mapsto \phi h). \tag{7.3}$$

Theorem 7.3 *The mapping* (7.3) *is an isomorphism in the Zariski-open subset of* X *where it is* S_2.

This is the subset of X where $\operatorname{codim} X_k \geq k+2$, $k \geq p+1$, cf., Sect. 6. Thus it contains all points x such that $\mathscr{O}_{X,x}$ is Cohen–Macaulay. In particular, (7.3) is an isomorphism in X_{reg}.

Theorem 7.3 is a consequence of the results in [22]. If X has pure dimension p, there is an injective mapping

$$\mathscr{O}_X \to \mathcal{H}om\left(\mathcal{E}xt^{\,p}(\mathscr{O}_X, K_\Omega), \mathcal{CH}_\Omega^Z\right), \tag{7.4}$$

which by [22, Theorem 1.2 and Remark 6.11] is an isomorphism if and only if $\mathscr{O}_X$ is S_2. Since the image of such a morphism must be annihilated by $\mathcal{J}$ by linearity, it is indeed a morphism

$$\mathscr{O}_X \to \mathcal{H}om\left(\mathcal{E}xt^{\,p}(\mathscr{O}_X, K_\Omega), \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)\right). \tag{7.5}$$

In view of (4.2) and (5.3), (7.5) corresponds to a morphism $\mathscr{O}_X \to \mathcal{H}om(\omega_X^n, \omega_X^n)$, and the fact that it is the morphism (7.3) is a rather simple consequence of the definition of the morphism (7.4) in [22, (6.9)].

As mentioned in the introduction, Theorem 7.3 can be seen as a reformulation of a classical result of Roos, [30], which is the same statement about the injection

$$\mathscr{O}_\Omega/\mathcal{J} \to \mathcal{E}xt^{\,p}\left(\mathcal{E}xt^{\,p}(\mathscr{O}_\Omega/\mathcal{J}, K_\Omega), K_\Omega\right); \tag{7.6}$$

here we assume that the ideal has pure dimension. The equivalence of the morphisms (7.4) and (7.6) is discussed in [22, Corollary 1.4].

Let us now consider the case when X is reduced. Since sections of ω_X^n are meromorphic, see [6, Example 2.8], and thus almost semi-meromorphic and generically smooth, by Proposition 2.4 (with $Z = X = \Omega$) we can extend (7.3) to a morphism

$$\mathcal{W}_X^{0,*} \to \mathcal{H}om\left(\omega_X^n, \mathcal{W}_X^{n,*}\right). \tag{7.7}$$

Lemma 7.4 *When X is reduced (7.7) is an isomorphism.*

Thus Definition 7.2 is consistent with the previous definition of $\mathcal{W}_X^{0,*}$ when X is reduced.

Proof Clearly each ϕ in $\mathcal{W}_X^{0,*}$ defines an element α in $\mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$ by $h \mapsto \phi\wedge h$. If we apply this to a generically nonvanishing h we see by the SEP that (7.7) is injective.

For the surjectivity, take α in $\mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$. If h' is nonvanishing at a point on X_{reg}, then it generates ω_X^n and thus α is determined by $\phi := \alpha h'$ there. By [10, Theorem 3.7], $\phi = \psi \wedge h'$ for a unique current ψ in $\mathcal{W}_X^{0,*}$ so by $\mathscr{O}_X$-linearity $\alpha h = \psi \wedge h$ for any h. Hence, ψ is well-defined as a current in $\mathcal{W}_X^{0,*}$ on X_{reg}.

We must verify that ψ has an extension in $\mathcal{W}_X^{0,*}$ across X_{sing}. Since such an extension must be unique by the SEP, the statement is local on X. Thus we may assume that α is defined on the whole of X and that there is a generically nonvanishing holomorphic n-form γ on X. Then $\alpha\gamma$ is a section of $\mathcal{W}^{n,*}(X)$.

Let us choose a smooth modification $\pi : X' \to X$ that is biholomorphic outside X_{sing}. Then $\pi^*\gamma$ is a holomorphic n-form on X' that is generically non-vanishing. We claim that there is a current τ in $\mathcal{W}^{n,0}(X')$ such that $\pi_*\tau = \alpha\gamma$. In fact, τ exists on $\pi^{-1}(X_{reg})$ since π is a biholomorphism there. Moreover, by [4, Proposition 1.2], αh is the direct image of some pseudomeromorphic current $\tilde\tau$ on X', and is therefore also the image of the (unique) current $\tau = \mathbf{1}_{\pi^{-1}(X_{reg})}\tilde\tau$ in $\mathcal{W}^{n,*}(X')$.

By [10, Theorem 3.7] again τ is locally of the form $\xi \wedge ds$, where ξ is in $\mathcal{W}_{X'}^{0,*}$ and $ds = ds_1 \wedge \cdots \wedge ds_n$ for some local coordinates s. Hence, τ is a $K_{X'}$-valued section of $\mathcal{W}^{0,*}(X')$, so $\tau/\pi^*\gamma$ is a section of $\mathcal{W}^{0,*}(X')$. Now $\Psi := \pi_*(\tau/\pi^*\gamma)$ is a section of $\mathcal{W}^{0,*}(X)$. On $X_{reg} \cap \{\gamma \neq 0\}$ we thus have that $\Psi \wedge \gamma = \pi_*\tau = \alpha\gamma = \psi \wedge \gamma$ and so $\Psi = \psi$ there. By the SEP it follows that Ψ coincides with ψ on X_{reg} and is thus the desired pseudomeromorphic extension to X. □

In view of (5.1) and (5.3) we have, given a local embedding $i : X \to \Omega$, the extrinsic representation

$$\mathcal{W}_X^{0,*} \simeq \mathcal{H}om\left(\mathcal{H}om\left(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z\right), \mathcal{H}om\left(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z\right)\right), \phi \mapsto (i_*h \mapsto i_*(\phi\wedge h)). \tag{7.8}$$

Lemma 7.5 *Assume that $X_{reg} \to \Omega$ is a local embedding and (z, w) coordinates as before. Each section ϕ in $\mathcal{W}_X^{0,*}$ has a unique representation (7.1) with $\widehat{\phi}_j$ in $\mathcal{W}_Z^{0,*}$.*

A current with a representation (7.1) is considered as an element of $\mathcal{W}_X^{0,*} = \mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$ in view of the comment after Lemma 7.1.

Proof From (4.9) we get an induced sequence

$$0 \to \left(\mathcal{W}_Z^{0,*}\right)^\nu \xrightarrow{T} \left(\mathcal{W}_Z^{0,*}\right)^M \xrightarrow{A} \left(\mathcal{W}_Z^{0,*}\right)^{M'}, \tag{7.9}$$

which is also exact. In fact, T in (7.9) is clearly injective, and by (4.10), if ξ in $(\mathcal{W}_Z^{0,*})^M$ and $A\xi = 0$, then $T\eta = \xi$, if $\eta = S\xi$.

Now take ϕ in $\mathcal{H}om(\omega^n_X, \mathcal{W}^{n,*}_X)$. Let us choose a basis $\mu^1, \ldots, \mu^m$ for ω^n_X and let $\tilde{\phi}$ be the element in $(\mathcal{W}^{0,*}_Z)^M$ obtained from the coefficients of $\phi\mu^j$ when expressed as in (2.11), cf., Sect. 4.1. We claim that $A\tilde{\phi} = 0$. Taking this for granted, by the exactness of (7.9), $\tilde{\phi}$ is the image of the tuple $\hat{\phi} = S\tilde{\phi}$. Now $\hat{\phi} \wedge \mu^j = \phi\mu^j$ since they are represented by the same tuple in $(\mathcal{W}^{0,*}_Z)^M$. Thus $\hat{\phi}$ gives the desired representation of ϕ.

In view of Proposition 2.9 it is enough to prove the claim where $\tilde{\phi}$ is smooth. Let us therefore fix such a point, say 0, and show that $(A\tilde{\phi})(0) = 0$. From the proof of Lemma 4.11, if we let $\mathcal{I}$ be the ideal generated by z, and let X_0 be defined by $\mathscr{O}_{X_0} := \mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I})$, then $\mu^1 \wedge \mu^z, \ldots, \mu^m \wedge \mu^z$ generate $\omega^0_{X_0}$. If we let ϕ_0 be the morphism in $\mathcal{H}om(\omega^0_{X_0}, \omega^0_{X_0})$ given by $\phi_0(\mu^i \wedge \mu^z) := \phi\mu^i \wedge \mu^z$ (which indeed gives a well-defined such morphism), then, as in the proof of Lemma 4.11, $\tilde{\phi}_0 = \tilde{\phi}(0)$. In addition, the sequence (4.9) for X_0 is

$$0 \to \mathbb{C}^\nu \xrightarrow{T(0)} \mathbb{C}^M \xrightarrow{A(0)} \mathbb{C}^{M'}.$$

Since X_0 is 0-dimensional, the morphism $\mathscr{O}_{X_0} \to \mathcal{H}om(\omega_{X_0}, \omega_{X_0})$ is an isomorphism by Theorem 7.3, and thus ϕ_0 is given as multiplication by a function in $\mathscr{O}_{X_0}$, which we also denote by ϕ_0, i.e., $\tilde{\phi}_0 = T(0)\hat{\phi}_0$. Hence, $A(0)\tilde{\phi}_0 = A(0)T(0)\hat{\phi}_0 = 0$, and thus $(A\tilde{\phi})(0) = 0$. □

Example 7.6 (Meromorphic functions) Assume that we have a local embedding $X \to \Omega$. Given meromorphic functions Φ, Φ' in Ω that are holomorphic generically on Z, we say that $\Phi \sim \Phi'$ if and only if $\Phi - \Phi'$ is in $\mathcal{J}$ generically on Z. If $\Phi = A/B$ and $\Phi' = A'/B'$, where B and B' are generically non-vanishing on Z, the condition is precisely that $AB' - A'B$ is in $\mathcal{J}$. We say that such an equivalence class is a meromorphic function ϕ on X, i.e., ϕ is in $\mathcal{M}_X$. Clearly we have $\mathscr{O}_X \subset \mathcal{M}_X$. We claim that

$$\mathcal{M}_X \subset \mathcal{W}^{0,*}_X.$$

To see this, first notice that if we take a representative Φ in $\mathcal{M}_\Omega$ of ϕ, then it can be considered as an almost semi-meromorphic current on Ω with Zariski-singular support of positive codimension on Z, since it is generically holomorphic on Z. As in Definition 5.5 we therefore have a current $\Phi \wedge h$ in $\mathcal{W}^{n,0}_X$ for h in ω^n_X. Another representative Φ' of ϕ will give rise to the same current generically and hence everywhere by the SEP. Thus ϕ defines a section of $\mathcal{H}om(\omega^n_X, \mathcal{W}^{n,*}_X) = \mathcal{W}^{0,*}_X$. □

By definition, a current ϕ in $\mathcal{W}^{0,*}_X$ can be multiplied by a current h in ω^n_X, and the product $\phi \wedge h$ lies in $\mathcal{W}^{n,*}_X$. It will be crucial that we can extend to products by somewhat more general currents. Notice that ω^n_X is a subsheaf of $\mathcal{C}^{n,*}_X$, which is an $\mathscr{E}^{0,*}_X$-module. Thus, we can consider the subsheaf $\mathscr{E}^{0,*}_X \omega^n_X$ of $\mathcal{C}^{n,*}_X$ which consists of finite sums $\sum \xi_i \wedge h_i$, where ξ_i are in $\mathscr{E}^{0,*}_X$ and h_i are in ω^n_X.

Lemma 7.7 *Each ϕ in $\mathcal{W}_X^{0,*} = \mathcal{H}om_{\mathscr{O}_X}(\omega_X^n, \mathcal{W}_X^{n,*})$ has a unique extension to a morphism in $\mathcal{H}om_{\mathscr{E}_X^{0,*}}(\mathscr{E}_X^{0,*}\omega_X^n, \mathcal{W}_X^{n,*})$.*

Proof The uniqueness follows by $\mathscr{E}_X^{0,*}$-linearity, i.e., if $b = \xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r$ is in $\mathscr{E}_X^{0,*}\omega_X^n$, then one must have

$$\phi b = \sum_i (-1)^{(\deg \xi_i)(\deg \phi)} \xi_i \wedge \phi h_i. \tag{7.10}$$

We must check that this is well-defined, i.e., that the right hand side does not depend on the representation $\xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r$ of b. By the SEP, it is enough to prove this locally on X_{reg}, and we can then assume that ϕ has a representation (7.1). By Proposition 2.9, it is then enough to prove that it is well-defined assuming that $\widehat{\phi}_0, \ldots, \widehat{\phi}_{\nu-1}$ in (7.1) are all smooth. In this case, the right hand side of (7.10) is simply the product of $\xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r = b$ by the smooth form ϕ in $\mathscr{E}_X^{0,*}$, and this product only depends on b. $\square$

Corollary 7.8 *Let ϕ be a current in $\mathcal{W}_X^{0,*}$ and let α be a current in $\mathcal{W}_X^{n,*}$ of the form $\alpha = \sum a_i \wedge h_i$, where a_i are almost semi-meromorphic $(0, *)$-currents on Ω which are generically smooth on Z, and h_i are in ω_X^n. Then one has a well-defined product*

$$\phi \wedge \alpha = \sum (-1)^{(\deg a_i)(\deg \phi)} a_i \wedge (\phi \wedge h_i). \tag{7.11}$$

Proof The right hand side of (7.11) exists as a current in $\mathcal{W}_X^{n,*}$, and we must prove is that it only depends on the current α and not on the representation $\sum a_i \wedge h_i$. Notice that all the a_i are smooth outside some subvariety V of Z and there the right hand side of (7.11) is the product of ϕ and α in $\mathscr{E}_X^{0,*}\omega_X^n$, cf., Lemma 7.7. It follows by the SEP that the right hand side only depends on α. $\square$

Remark 7.9 Recall from (6.9) that $\omega = b\vartheta$. If ϕ is in $\mathcal{W}_X^{0,*}$, then we can define the product $\phi \wedge \omega$ by Corollary 7.8.

Expressed extrinsically, if $\mu = i_*\vartheta$, and if we write $R \wedge dz = b\mu$ as in Lemma 6.2, then we can define the product $R \wedge dz \wedge \phi := b\mu \wedge \phi$ as a current in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$. $\square$

Lemma 7.10 *Assume that ϕ is in $\mathcal{W}_X^{0,*}$, and that $\phi \wedge \omega = 0$ for some structure form ω, where the product is defined by Remark 7.9. Then $\phi = 0$.*

Proof Considering the component with values in E_p, we get that $\phi \wedge \omega_0 = 0$. By Proposition 6.7, any h in ω_X^n can be written as $h = \xi\omega_0$, where ξ is a holomorphic section of E_p^*, so by $\mathscr{O}$-linearity, $\phi \wedge h = 0$, i.e., $\phi = 0$. $\square$

We end this section with the following result, first part of [10, Theorem 3.7]. We include here a different proof than the one in [10], since we believe the proof here is instructive.

Proposition 7.11 *If Z is smooth, then $\mathcal{W}_Z$ is closed under holomorphic differential operators.*

Proof Let τ be any current in $\mathcal{W}_Z$. It suffices to prove that if ζ are local coordinates on Z, then $\partial\tau/\partial\zeta_1$ is in $\mathcal{W}_Z$. Consider the current

$$\tau' = \tau \otimes \bar\partial \frac{dw}{2\pi i w^2}$$

on the manifold $Y := Z \times \mathbb{C}_w$. Clearly τ' has support on Z, and it follows from (2.5) that τ' is in $\mathcal{W}_Y^Z$. Let

$$p : (z, w) \mapsto \zeta = (z_1 + w, z_2, \dots, z_n),$$

which is just a change of variables on Y followed by a projection. It follows from (2.4) that $p_*\tau'$ is in $\mathcal{W}_Z$. Since

$$\bar\partial \frac{dw}{2\pi i w^2} \cdot \xi(w) = \frac{\partial \xi}{\partial w}(0)$$

it is readily verified that $p_*\tau' = \partial\tau/\partial\zeta_1$, so we conclude that $\partial\tau/\partial\zeta_1$ is in $\mathcal{W}_Z$. □

8 The $\bar\partial$-operator on $\mathcal{W}_X^{0,*}$

We already know the meaning of $\bar\partial$ on $\mathcal{W}_X^{n,*}$, and we now define $\bar\partial$ on $\mathcal{W}_X^{0,*}$.

Definition 8.1 Assume that ϕ, v are in $\mathcal{W}_X^{0,*}$, We say that $\bar\partial v = \phi$ if

$$\bar\partial(v \wedge h) = \phi \wedge h, \quad h \in \omega_X^n. \tag{8.1}$$

If we have an embedding $X \to \Omega$, (8.1) means, cf., (7.8), that

$$\bar\partial(v \wedge \mu) = \phi \wedge \mu, \quad \mu \in \mathcal{H}om\left(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z\right). \tag{8.2}$$

In view of Remark 7.9 we can define the product $\phi \wedge \omega$ for ϕ in $\mathcal{W}_X^{0,*}$.

Definition 8.2 We say that v belongs to $\operatorname{Dom} \bar\partial_X$ if v is in $\operatorname{Dom} \bar\partial$, i.e., $\bar\partial v = \phi$ for some ϕ and in addition $\bar\partial(v \wedge \omega)$, a priori only in $\mathcal{PM}_X^{n,*}$, is in $\mathcal{W}_X^{n,*}$, for each structure form ω from any possible embedding.

If X is Cohen–Macaulay, then any such ω is of the form $a_1 h^1 + \cdots + a_m h^m$, where h^j are in ω_X^n and a_j are smooth, see Remark 6.5, and hence $\operatorname{Dom} \bar\partial_X$ coincides with $\operatorname{Dom} \bar\partial$ in this case.

Example 8.3 Assume that v is in $\mathscr{E}_X^{0,*}$ and $\phi = \bar\partial v$ in the sense in Section 4. Then clearly

$$\bar\partial(v \wedge \omega) = \phi \wedge \omega + (-1)^{\deg v} v \wedge \bar\partial\omega.$$

Since $\bar\partial\omega = f\omega$, and $\mathcal{W}_X^{n,*}$ is closed under multiplication with forms in $\mathscr{E}_X^{0,*}$, we get that $\bar\partial(v \wedge \omega)$ is in $\mathcal{W}_X^{n,*}$, so v is in Dom $\bar\partial_X$ and $\bar\partial_X v = \phi$.

If w is in Dom $\bar\partial_X$ and v is in $\mathscr{E}_X^{0,*}$, then

$$\bar\partial(v \wedge w \wedge \omega) = \bar\partial v \wedge w \wedge \omega + (-1)^{\deg v} v \wedge \bar\partial(w \wedge \omega).$$

Thus $v \wedge w$ is in Dom $\bar\partial_X$, and the Leibniz rule $\bar\partial(v \wedge w) = \bar\partial v \wedge w + (-1)^{\deg v} v \wedge \bar\partial w$ holds. □

Let $\chi_\delta = \chi(|h|^2/\delta)$ where h is a tuple of holomorphic functions that cuts out X_{sing}.

Lemma 8.4 *If v is in $\mathcal{W}^{0,*}(X)$, and it is in Dom $\bar\partial_X$ on X_{reg}, then v is in Dom $\bar\partial_X$ on all of X if and only if*

$$\bar\partial\chi_\delta \wedge v \wedge \omega \to 0, \quad \delta \to 0, \tag{8.3}$$

for all structure forms ω. In this case,

$$-\nabla_f(v \wedge \omega) = \bar\partial v \wedge \omega. \tag{8.4}$$

Proof Since $\mathcal{W}_X^{n,*}$ is closed under multiplication by f, v is in Dom $\bar\partial_X$ if and only if $\nabla_f(v \wedge \omega)$ is in $\mathcal{W}_X^{n,*}$ for all structure forms ω. Since v is in Dom $\bar\partial_X$ on X_{reg}, thus $\nabla_f(v \wedge \omega)$ is in $\mathcal{W}_X^{n,*}$ on X_{reg}. By (2.2), $\nabla_f(v \wedge \omega)$ is then in $\mathcal{W}_X^{n,*}$ on all of X if and only if

$$\mathbf{1}_{X_{\mathrm{reg}}}\nabla_f(v \wedge \omega) = \nabla_f(v \wedge \omega). \tag{8.5}$$

By the Leibniz rule,

$$\nabla_f(\chi_\delta v \wedge \omega) = -\bar\partial\chi_\delta \wedge v \wedge \omega + \chi_\delta\nabla_f(v \wedge \omega). \tag{8.6}$$

Since v is in $\mathcal{W}_X^{0,*}$, $v \wedge \omega$ is in $\mathcal{W}_X^{n,*}$, so the left hand side of (8.6) tends to $\nabla_f(v \wedge \omega)$ when $\delta \to 0$, whereas the second term on the right hand side of (8.6) tends to $\mathbf{1}_{X_{\mathrm{reg}}}\nabla_f(v \wedge \omega)$. Thus (8.5) holds if and only if (8.3) does. Thus the first statement in the lemma is proved.

Recall, cf., (6.9), that $\omega = b\vartheta$ where b is smooth on X_{reg} and ϑ is in ω_X^n. By the Leibniz rule thus $-\nabla_f(v \wedge \omega) = \bar\partial v \wedge \omega$ on X_{reg}, since $\nabla_f\omega = 0$. Therefore, (8.6) is equivalent to $-\nabla_f(\chi_\delta v \wedge \omega) = \bar\partial\chi_\delta \wedge v \wedge \omega + \chi_\delta\bar\partial v \wedge \omega$. If (8.3) holds, we therefore get (8.4) when $\delta \to 0$. □

Remark 8.5 In case X is reduced the definition of $\bar\partial_X$ is precisely the same as in [6]. However, the definition of $\bar\partial v = \phi$ given here, for v, ϕ in $\mathcal{W}_X^{0,*}$, does *not* coincide with the definition in, e.g., [6]. In fact, that definition means that $\bar\partial(v \wedge h) = \phi \wedge h$ for all *smooth* h in ω_X^n, which in general is a strictly weaker condition. For example, for any weakly holomorphic function v, we have $\bar\partial(v\wedge h)=0$ for all smooth h in ω_X^n, while if X is a reduced complete intersection, or more generally Cohen–Macaulay, then $\bar\partial(v\wedge h)=0$ for all h in ω_X^n is equivalent to v being strongly holomorphic, see [33, p. 124] and [2]. $\square$

We conclude this section with a lemma that shows that $\bar\partial$ means what one should expect when ϕ, v are expressed with respect to a local basis w^{α_j} for $\mathscr{O}_X$ over $\mathscr{O}_Z$ as in Lemma 7.5.

Lemma 8.6 *Assume that we have a local embedding $X_{reg}\to\Omega$ and ϕ, v in $\mathcal{W}_X^{0,*}$ represented as in (7.1). Then $\bar\partial v=\phi$ if and only if*

$$\bar\partial\hat v_j=\hat\phi_j,\quad j=0,\dots,\nu-1. \tag{8.7}$$

Proof Let us use the notation from the proof of Lemma 7.5. Recall that $\hat v=S\tilde v$. In view of (8.2) and (2.12), $\widetilde{\bar\partial v}=\bar\partial\tilde v$. Since S is holomorphic therefore $\widehat{\bar\partial v}=S\widetilde{\bar\partial v}=S\bar\partial\tilde v=\bar\partial(S\tilde v)=\bar\partial\hat v$. $\square$

9 Solving $\bar\partial u=\phi$ on X

We will find local solutions to the $\bar\partial$-equation on X by means of integral formulas. We use the notation and machinery from [6, Section 5]. Let $i\colon X\to\Omega\subset\mathbb{C}^N$ be a local embedding such that Ω is pseudoconvex, let $\Omega'\subset\subset\Omega$ be a relatively compact subdomain of Ω, and let $X'=X\cap\Omega'$.

Theorem 9.1 *There are integral operators*

$$K\colon\mathscr{E}^{0,*+1}(X)\to\mathcal{W}^{0,*}(X')\cap Dom\,\bar\partial_X,\quad P\colon\mathscr{E}^{0,*}(X)\to\mathscr{E}^{0,*}(X')$$

such that, for $\phi\in\mathscr{E}^{0,k}(X)$,

$$\phi=\bar\partial K\phi+K(\bar\partial\phi)+P\phi. \tag{9.1}$$

The operators K and P are described below; they depend on a choice of weight g. Since Ω is Stein one can find such a weight g that is holomorphic in z, by which we mean that it depends holomorphically on $z\in\Omega'$ and has no components containing any $d\bar z_i$, cf., Example 5.1 in [6]. In this case, $P\phi$ is holomorphic when $k=0$, and vanishes when $k\geq 1$, i.e.,

$$\phi=\bar\partial K\phi+K(\bar\partial\phi),\quad\phi\in\mathscr{E}^{0,k}(X),\quad k\geq 1. \tag{9.2}$$

If $\bar\partial\phi=0$ in Ω, and $k\geq 1$, then $K\phi$ is a solution to $\bar\partial v=\phi$. If $k=0$, then $\phi=P\phi$ is holomorphic. It follows that a smooth $\bar\partial$-closed function is holomorphic. In the reduced case this is a classical theorem of Malgrange [24]. In Sect. 10 we prove that $K\phi$ is smooth on X_{reg}.

We now turn to the definition of K and P. For future need, in Sect. 11, we define them acting on currents in $\mathcal{W}^{0,*}(X)$ and not only on smooth forms. Let $\pi : \Omega_\zeta \times \Omega'_z \to \Omega'_z$ be the natural projection. Let us choose a holomorphic Hefer form[3] H, a smooth weight g with compact support in Ω with respect to $z \in \Omega' \subset\subset \Omega$, and let B be the Bochner–Martinelli form. Since we are only are concerned with $(0, *)$-forms, we will here assume that H and B only have holomorphic differentials in ζ, i.e., the factors $d\eta_i = d\zeta_i - dz_i$ in H and B in [6] should be replaced by just $d\zeta_i$.

If γ is a current in $\Omega_\zeta \times \Omega'_z$ we let $(\gamma)_N$ be the component of bidegree $(N, *)$ in ζ and $(0, *)$ in z, and let $\vartheta(\gamma)$ be the current such that

$$\vartheta(\gamma) \wedge d\zeta = (\gamma)_N. \tag{9.3}$$

Consider now μ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}^Z_\Omega)$ and ϕ in $\mathcal{W}^{0,*}_X$. We can give meaning to

$$(g \wedge HR(\zeta))_N \wedge \phi(\zeta) \wedge \mu(z) \tag{9.4}$$

as a tensor product of currents in the following way: first of all, by Remark 7.9, we can form the product $R(\zeta) \wedge d\zeta \wedge \phi(\zeta)$ as a current in $\mathcal{W}^Z_\Omega$. In view of [11, Corollary 4.7] the tensor product $R(\zeta) \wedge d\zeta \wedge \phi(\zeta) \wedge \mu(z)$ is in $\mathcal{W}^{Z\times Z'}_{\Omega_\zeta \times \Omega'_z}$, where $Z' = Z \cap \Omega'$. Finally, we multiply this with the smooth form $\vartheta(g \wedge H)$ to obtain (9.4). Similarly, outside of Δ, the diagonal in $\Omega \times \Omega'$, where B is smooth, we can define

$$(B \wedge g \wedge HR(\zeta))_N \wedge \phi(\zeta) \wedge \mu(z) \tag{9.5}$$

as a tensor product of currents.

Lemma 9.2 *For μ in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}^{Z'}_{\Omega'})$ and $\phi \in \mathcal{W}^{0,*}(X)$, the current* (9.5)*, a priori defined as a current in $\mathcal{W}^{Z\times Z'\setminus\Delta}_{\Omega_\zeta\times\Omega'_z\setminus\Delta}$ has an extension across Δ. The current* (9.4) *and the extension of* (9.5) *depend $\mathscr{O}_\Omega/\mathcal{J}$-bilinearly on μ and ϕ, and are such that*

$$K\phi \wedge \mu := \pi_*\big((B \wedge g \wedge HR(\zeta))_N \wedge \phi(\zeta) \wedge \mu(z)\big) \tag{9.6}$$

and

$$P\phi \wedge \mu := \pi_*\big((g \wedge HR(\zeta))_N \wedge \phi(\zeta) \wedge \mu(z)\big) \tag{9.7}$$

are in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}^{Z'}_{\Omega'})$.

It follows that $K\phi\wedge\mu$ and $P\phi\wedge\mu$ are $\mathbb{C}$-linear in ϕ and $\mathscr{O}_{\Omega'}/\mathcal{J}$-linear in μ. In view of (7.8), by considering μ in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{CH}^{Z'}_{\Omega'})$, we have defined linear operators

$$K : \mathcal{W}^{0,*+1}(X) \to \mathcal{W}^{0,*}(X'), \quad P : \mathcal{W}^{0,*}(X) \to \mathcal{W}^{0,*}(X'). \tag{9.8}$$

Proof of Lemma 9.2 In order to define the extension of (9.5) across Δ, we note first that since B is almost semi-meromorphic with Zariski singular support Δ, $\vartheta(B\wedge g\wedge H)$

[3] We are only concerned with the component H^0 of this form, so for simplicity we write just H.

is an almost semi-meromorphic $(0,*)$-current on $\Omega_\zeta \times \Omega'_z$, which is smooth outside the diagonal. We can thus form the current $\vartheta(B \wedge g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi(\zeta) \wedge \mu(z)$ in $\mathcal{W}^{Z \times Z'}_{\Omega_\zeta \times \Omega'_z}$, cf., Proposition 2.4, and this is the extension of (9.5) across Δ.

From the definitions above, it is clear that (9.4) and the extension of (9.5) are $\mathscr{O}_\Omega$-bilinear in ϕ and μ. Both these currents are annihilated by $\mathcal{J}_z$ and $\mathcal{J}_\zeta$, cf., (2.8), so they depend $\mathscr{O}_\Omega/\mathcal{J}$-bilinearly. In view of (2.4) we conclude that (9.6) and (9.7) are in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}^{Z'}_{\Omega'})$. $\square$

Proposition 9.3 *If $\phi \in \mathcal{W}^{0,k}(X)$, then $P\phi \in \mathscr{E}^{0,k}(X')$, and if in addition g is holomorphic in z, then $P\phi \in \mathscr{O}(X')$ if $k = 0$ and vanishes if $k \geq 1$.*

Proof Since $\vartheta(g \wedge H)$ is smooth, we get that

$$\begin{aligned} &\pi_*\big(\vartheta(g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi \wedge \mu(z)\big) \\ &\quad = \pi_*\big(\vartheta(g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi\big) \wedge \mu(z) = \pi_*\big((g \wedge HR)_N \wedge \phi\big) \wedge \mu(z), \end{aligned}$$

cf., for example [20, (5.1.2)]. Thus $P\phi(z) = \pi_*\big((g \wedge HR(\zeta))_N \wedge \phi\big)$ which is smooth on Ω'. If g depends holomorphically on z, then $P\phi$ is holomorphic in Ω' if ϕ is a $(0,0)$-current, and vanishes for degree reasons if ϕ has positive degree. $\square$

We shall now approximate $K\phi$ by smooth forms. Let $B^\epsilon = \chi(|\zeta - z|^2/\epsilon)B$.

Proposition 9.4 *For any $\phi \in \mathcal{W}^{0,k}(X)$, $k \geq 1$,*

$$K^\epsilon\phi := \pi_*\big((B^\epsilon \wedge g \wedge HR(\zeta))_N \wedge \phi\big) = \pi_*\big(\vartheta(B^\epsilon \wedge g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi\big)$$

is in $\mathscr{E}^{0,k-1}(X')$ and $K^\epsilon\phi \to K\phi$ when $\epsilon \to 0$.

The last statement means that

$$K^\epsilon\phi \wedge \mu \to K\phi \wedge \mu, \quad \mu \in \mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{CH}^{Z'}_{\Omega'}). \tag{9.9}$$

Proof Since B^ϵ is smooth, the current we push forward is $R(\zeta) \wedge \phi(\zeta)$ times a smooth form of ζ and z. Therefore $K^\epsilon\phi$ is smooth. As in the proof of Proposition 9.3, we obtain since B^ϵ is smooth that

$$K^\epsilon\phi \wedge \mu = \pi_*\big((B^\epsilon \wedge g \wedge HR(\zeta))_N \wedge \phi \wedge \mu(z)\big). \tag{9.10}$$

By (5.2) applied to $a = B$ we have that

$$(B^\epsilon \wedge g \wedge HR(\zeta))_N \wedge \phi \wedge \mu(z) \to (B \wedge g \wedge HR(\zeta))_N \wedge \phi \wedge \mu(z) \tag{9.11}$$

which implies (9.9). $\square$

9.1 Proof of Theorem 9.1

By definition $K\phi$ and $P\phi$ are currents in $\mathcal{W}^{0,*}(X')$ such that (9.6) and (9.7) hold for μ in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{CH}_{\Omega'}^{Z'})$. We claim that

$$K\phi \wedge R \wedge dz = \pi_*\big((B \wedge g \wedge HR(\zeta))_N \wedge \phi \wedge R(z) \wedge dz\big) \tag{9.12}$$

and

$$P\phi \wedge R \wedge dz = \pi_*\big((g \wedge HR(\zeta))_N \wedge \phi \wedge R(z) \wedge dz\big); \tag{9.13}$$

here the left hand sides are defined in view of Remark 7.9, whereas the right hand sides have meaning by Lemma 9.2 and the fact that $R(z) \wedge dz$ is in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}_{\Omega'}^{Z'})$ by Corollary 6.3.

Recall from Lemma 6.2 that $R \wedge dz = b\mu$, where μ is a tuple of currents in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{CH}_{\Omega'}^{Z'})$ and b is an almost semi-meromorphic matrix that is smooth generically on Z'. Therefore (9.12) and (9.13) hold where b is smooth, in view of Lemma 7.7, and since both sides are in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}_{\Omega'}^{Z'})$, the equalities hold everywhere by the SEP.

As in [6] we let $R^\lambda = \bar\partial |f|^{2\lambda} \wedge U$ for $\mathrm{Re}\,\lambda \gg 0$. It has an analytic continuation to $\lambda = 0$ and $R = R^\lambda|_{\lambda=0}$. Notice that $R(z) \wedge B$ is well-defined since it is a tensor product with respect to the coordinates $z, \eta = \zeta - z$. Also $R(z) \wedge R^\lambda(\zeta) \wedge B$ admits such an analytic continuation and defines a pseudomeromorphic current[4] when $\lambda = 0$. Let $B_{k,k-1}$ be the component of B of bidegree $(k, k-1)$.

Lemma 9.5 *For all k,*

$$B_{k,k-1} \wedge HR^\lambda(\zeta) \wedge R(z)|_{\lambda=0} = B_{k,k-1} \wedge HR(\zeta) \wedge R(z). \tag{9.14}$$

Proof of Lemma 9.5 Notice that the equality holds outside Δ. Let T be the left hand side of (9.14). In view of Proposition 2.1 it is therefore enough to check that $\mathbf{1}_\Delta T = 0$. Fix j, k and let

$$T_\ell = B_{k,k-1} \wedge HR_j^\lambda(\zeta) \wedge R_\ell(z)|_{\lambda=0}.$$

Clearly $T_\ell = 0$ if $\ell < p$ so first assume that $\ell = p$. Since HR_j has bidegree (j, j) in ζ, the current vanishes unless $j + k \le N$. Thus the total antiholomorphic degree is $\le N - n + N - 1$. On the other hand, the current has support on $\Delta \cap Z \times Z \simeq Z \times \{pt\}$ which has codimension $N + N - n$. Thus it vanishes by the dimension principle.

We now prove by induction over $\ell \ge p$ that $\mathbf{1}_\Delta T_\ell = 0$. Note that by (6.6), outside of Z_ℓ, $R_\ell(z) = \alpha_\ell(z) R_{\ell-1}(z)$, where $\alpha_\ell(z)$ is smooth. Thus, outside of $Z_\ell \times \Omega$, T_ℓ is a smooth form times $T_{\ell-1}$, and thus, by induction and (2.3), $\mathbf{1}_\Delta T_\ell$ has its support in $\Delta \cap (Z_\ell \times Z) \simeq Z_\ell \times \{pt\}$, which has codimension $\ge N + \ell + 1$, see (6.3). On the other hand, the total antiholomorphic degree is $\le \ell + j + k - 1 \le \ell + N - 1$, so the current vanishes by the dimension principle. We conclude that (9.14) holds. □

4 One can consider this current as $R(z) \wedge B$ multiplied by the residue of the almost semi-meromorphic current U in (6.5), cf., [10, Section 4.4].

By the same argument[5] as for [6, (5.2)] we have the equality

$$\nabla_{f(z)}\big((B\wedge g\wedge HR^\lambda(\zeta))_N\wedge R(z)\wedge dz\big)=[\Delta]'\wedge R(z)\wedge dz-(g\wedge HR^\lambda)_N\wedge R(z)\wedge dz, \tag{9.15}$$

also for our R, where $[\Delta]'$ denotes the part of $[\Delta]$ where $d\eta_i=d\zeta_i-dz_i$ has been replaced[6] by $d\zeta_i$. In view of (9.14) we can put $\lambda=0$ in (9.15), and then we get

$$\nabla_{f(z)}\big((B\wedge g\wedge HR(\zeta))_N\wedge R(z)\wedge dz\big)=[\Delta]'\wedge R(z)\wedge dz-(HR(\zeta)\wedge g)_N\wedge R(z)\wedge dz. \tag{9.16}$$

Multiplying (9.16) by the smooth form ϕ, and using (9.12) and (9.13), we get

$$\phi\wedge R\wedge dz=-\nabla_f(K\phi\wedge R\wedge dz)+K(\bar\partial\phi)\wedge R\wedge dz+P\phi\wedge R\wedge dz,$$

or equivalently,

$$\phi\wedge\omega=-\nabla_f(K\phi\wedge\omega)+K(\bar\partial\phi)\wedge\omega+P\phi\wedge\omega. \tag{9.17}$$

Multiplying by suitable holomorphic ξ_0 in E_p^* such that $f_{p+1}^*\xi_0=0$, cf., Proposition 6.7, we see that $\phi\wedge h=\bar\partial(K\phi\wedge h)+K(\bar\partial\phi)\wedge h+P\phi\wedge h$ for all h in ω_X. Thus by definition (9.1) holds.

Since $\mathcal{W}_X^{0,*}$ is closed under multiplication by $\mathscr{O}_X$, we get that ψ in $\mathcal{W}_X^{0,*}$ is in $\operatorname{Dom}\bar\partial_X$ if and only if $-\nabla_f(\psi\wedge\omega)$ is in $\mathcal{W}_X^{n,*}$. Thus, we conclude from (9.17) that $K\phi$ is in $\operatorname{Dom}\bar\partial_X$ since all the other terms but $-\nabla_f(K\phi\wedge\omega)$ are in $\mathcal{W}_X^{0,*}$.

9.2 Intrinsic interpretation of K and P

So far we have defined K and P by means of currents in ambient space. We used this approach in order to avoid introducing push-forwards on a non-reduced space. However, we will sketch here how this can be done. We must first define the product space $X\times X'$. Given a local embedding $i: X\to\Omega$ as before, we have an embedding $(i\times i)\colon X\times X'\to\Omega\times\Omega'$ such that the structure sheaf is $\mathscr{O}_{\Omega\times\Omega'}/(\mathcal{J}_X+\mathcal{J}_{X'})$. One can check that this sheaf is independent of the chosen embedding, i.e., $\mathscr{O}_{X\times X'}$ is intrinsically defined. Thus we also have definitions of all the various sheaves on $X\times X'$ like $\mathscr{E}^{0,*}_{X\times X'}$. The projection $p\colon X\times X'\to X'$ is determined by $p^*\phi\colon\mathscr{O}_{X'}\to\mathscr{O}_{X\times X'}$, which in turn is defined so that $p^*i^*\Phi=(i\times i)^*\pi^*\Phi$ for Φ in $\mathscr{O}_{\Omega'}$, where $\pi:\Omega\times\Omega'\to\Omega'$ as before. Again one can check that this definition is independent of the embedding, and also extends to smooth $(0,*)$-forms ϕ. Therefore, we have the well-defined mapping $p_*\colon\mathcal{C}^{2n,*+n}_{X\times X'}\to\mathcal{C}^{n,*}_{X'}$, and clearly

$$i_*p_*=\pi_*(i\times i)_*. \tag{9.18}$$

[5] There is a sign error in [6, (5.2)] due to $R(z)\wedge dz$ being odd with respect to the super structure. Since we here move $R(z)\wedge dz$ to the right, we get the correct sign.

[6] This change is due to the fact that we do the same change of the differentials in the definition of H and B above.

As before we have the isomorphism

$$(i\times i)_*\colon \mathcal{W}^{2n,*}_{X\times X'} \simeq \mathcal{H}om\left(\mathscr{O}_{\Omega\times\Omega'}/(\mathcal{J}_X+\mathcal{J}_{X'}), \mathcal{W}^{Z\times Z'}_{\Omega\times\Omega'}\right).$$

As in the proof of Lemma 9.2 we see that π_* maps a current in $\mathcal{W}^{Z\times Z'}_{\Omega\times\Omega'}$ annihilated by $\mathcal{J}_{X'}$ to a current in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}^{Z'}_{\Omega'})$. It follows by (9.18) that

$$p_*\colon \mathcal{W}^{2n,*+n}_{X\times X'} \to \mathcal{W}^{n,*}_{X'}.$$

Now, take h in $\omega^n_{X'}$ and let $\mu = i_* h$. Then, cf., the proof of Lemma 9.2,

$$(B\wedge g\wedge HR(\zeta))_N\wedge\phi(\zeta)\wedge\mu(z) = (i\times i)_*\big(\vartheta(B\wedge g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta)\wedge h\big).$$

Thus we can define $K\phi$ intrinsically by

$$K\phi\wedge h = p_*\left(\vartheta(B\wedge g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta)\wedge h(z)\right). \tag{9.19}$$

From above it follows that $K\phi\wedge h$ is in $\mathcal{W}^{n,*}_{X'}$. In the same way we can define $P\phi$ by

$$P\phi\wedge h = p_*\left(\vartheta(g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta)\wedge h(z)\right). \tag{9.20}$$

It is natural to write

$$K\phi(z) = \int_\zeta \vartheta(B\wedge g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta), \quad P\phi(z) = \int_\zeta \vartheta(g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta),$$

although the formal meaning is given by (9.19) and (9.20).

10 Regularity of solutions on X_{reg}

We have already seen, cf., Proposition 9.3, that $P\phi$ is always a smooth form. We shall now prove that K preserves regularity on X_{reg}. More precisely,

Theorem 10.1 *If ϕ in $\mathcal{W}^{0,*}_X$ is smooth near a point $x\in X'_{\mathrm{reg}}$, then $K\phi$ in Theorem 9.1 is smooth near x.*

Throughout this section, let us choose local coordinates (ζ,τ) and (z,w) at x corresponding to the variables ζ and z in the integral formulas, so that $Z=\{(\zeta,\tau);\ \tau=0\}$.

Lemma 10.2 *Let $B^\epsilon := \chi(|\zeta-z|^2/\epsilon)B$, and assume that ϕ has compact support in our coordinate neighborhood. Then $K\phi$ can be approximated by the smooth forms*

$$K^\epsilon\phi := \pi_*\big((B^\epsilon\wedge g\wedge HR)_N\wedge\phi\big).$$

Notice that here we cut away the diagonal Δ' in $Z \times Z'$ times $\mathbb{C}_\tau \times \mathbb{C}_w$ in contrast to Proposition 9.4, where we only cut away the diagonal Δ in $\Omega \times \Omega'$.

Proof Clearly B^ϵ is smooth so that each $K^\epsilon \phi$ is smooth in a full neighborhood in Ω' of x. Let $T = \mu(z, w) \wedge (HR(\zeta, \tau) \wedge B \wedge g)_N \wedge \phi$, and let $W = \Delta' \times \mathbb{C}_\tau \times \mathbb{C}_w$. Since $\mu(z, w) \otimes R(\zeta, \tau)$ has support on $\{w = \tau = 0\}$, $T = \mathbf{1}_{\{w=\tau=0\}} T$. Therefore, $\mathbf{1}_W T = \mathbf{1}_W \mathbf{1}_{\{w=\tau=0\}} T = 0$ since $W \cap \{w = \tau = 0\} \subset \Delta$ and $\mathbf{1}_\Delta T = 0$ by definition, cf., Proposition 2.1 (i). Now notice that $\mathbf{1}_W T = 0$ implies (9.11) and in turn (9.9) with our present choice of B^ϵ. □

We first consider a simple but nontrivial example of Theorem 10.1.

Example 10.3 Let $X = \mathbb{C}_\zeta \subset \mathbb{C}^2_{\zeta,\tau}$ and $\mathcal{J} = (\tau^{m+1})$. Then $R = \bar\partial(1/\tau^{m+1})$. For an arbitrary point (z, w) we can choose the Hefer form

$$H = \frac{1}{2\pi i} \sum_{j=0}^{m} \tau^{m-k} w^k d\tau.$$

From the Bochner–Martinelli form B we only get a contribution from the term

$$B_1 = \frac{1}{2\pi i} \frac{(\bar\zeta - \bar z)d\zeta + (\bar\tau - \bar w)d\tau}{|\zeta - z|^2 + |\tau - w|^2}.$$

Let $\Omega' \subset\subset \Omega$ be open balls with center at the origin, and let $\varphi = \varphi(|\zeta|^2 + |\tau|^2)$ be a smooth cutoff function with support in Ω that is $\equiv 1$ in a neighborhood of $\overline{\Omega'}$. Then we can choose a holomorphic weight $g = \varphi + \cdots$, see, [6, Example 5.1] with respect to Ω', and with support in Ω. Now $1, \tau, \ldots, \tau^m$ is a set of generators for $\mathscr{O}_X$ over $\mathscr{O}_Z$. Assume that

$$\phi = (\hat\phi_0(\zeta) \otimes 1 + \cdots + \hat\phi_m(\zeta) \otimes \tau^m) d\bar\zeta$$

is a smooth (0, 1)-form. We want to compute $K\phi$. We know that

$$K\phi = a_0(z) \otimes 1 + \cdots + a_m(z) \otimes w^m \tag{10.1}$$

with $a_k(z)$ in $\mathcal{W}_Z^{0,0}$. By Lemma 10.2 and its proof, we have smooth $K^\epsilon \phi(z, w)$ in Ω' such that

$$K^\epsilon \phi \wedge dz \wedge dw \wedge \bar\partial \frac{1}{w^{m+1}} \to K\phi \wedge dz \wedge dw \wedge \bar\partial \frac{1}{w^{m+1}}. \tag{10.2}$$

It follows that

$$a_k(z) = \lim_{\epsilon \to 0} \frac{1}{k!} \frac{\partial^k}{\partial w^k} K^\epsilon \phi(z, w)\Big|_{w=0}.$$

Notice that

$$
\begin{aligned}
(B \wedge g \wedge HR(\tau))_2 &= B_1 \wedge g_{0,0} \wedge H \wedge \bar\partial \frac{1}{\tau^{m+1}} \\
&= -\varphi \bar\partial \frac{1}{\tau^{m+1}} \wedge \frac{1}{(2\pi i)^2} \sum_{\ell=0}^{m} \tau^{m-\ell} w^{\ell} d\tau \wedge \frac{(\bar\zeta - \bar z) d\zeta + (\bar\tau - \bar w) d\tau}{|\zeta - z|^2 + |\tau - w|^2} \\
&= -\varphi \bar\partial \frac{d\tau}{\tau^{m+1}} \wedge \frac{1}{(2\pi i)^2} \sum_{\ell=0}^{m} \tau^{m-\ell} w^{\ell} \wedge \frac{(\bar\zeta - \bar z) d\zeta}{|\zeta - z|^2 + |\tau - w|^2}.
\end{aligned}
$$

For each fixed $\epsilon > 0$, $|\zeta - z| > 0$ on $\operatorname{supp} \chi_\epsilon$, cf., Lemma 10.2, so we have

$$
\begin{aligned}
&K^\epsilon \phi(z, w) \\
&= \int_{\zeta,\tau} \varphi \frac{1}{(2\pi i)^2} \sum_{\ell=0}^{m} \bar\partial \frac{d\tau}{\tau^{\ell+1}} \wedge w^{\ell} \chi_\epsilon \frac{(\bar\zeta - \bar z) d\bar\zeta \wedge d\zeta}{|\zeta - z|^2 + |\tau - w|^2} \wedge \sum_{k=0}^{m} \hat\phi_k(\zeta) \otimes \tau^k.
\end{aligned}
\tag{10.3}
$$

A simple computation yields that

$$
K^\epsilon \phi(z, w) = \sum_{k=0}^{m} a_k^\epsilon(z) \otimes w^k + \mathscr{O}(\bar w), \tag{10.4}
$$

where

$$
a_k^\epsilon(z) = \frac{1}{2\pi i} \int_\zeta \varphi(|\zeta|^2) \chi_\epsilon \frac{\hat\phi_k(\zeta) d\bar\zeta \wedge d\zeta}{\zeta - z}.
$$

Letting ϵ tend to 0 we get $K\phi$ as in (10.1), where

$$
a_k(z) = \frac{1}{2\pi i} \int_\zeta \varphi(|\zeta|^2) \frac{\hat\phi_k(\zeta) d\bar\zeta \wedge d\zeta}{\zeta - z}.
$$

It is well-known that these Cauchy integrals $a_k(z)$ are smooth solutions to $\bar\partial v = \hat\phi_k d\bar z$ in $Z' = Z \cap \Omega'$. Thus $K\phi$ is smooth. $\square$

Remark 10.4 The terms $\mathscr{O}(\bar w)$ in the expansion (10.4) of $K^\epsilon \phi(z, w)$ do *not* converge to smooth functions in general when $\epsilon \to 0$. For a simple example, take $\phi = \zeta d\bar\zeta \otimes \tau^m$. Then $K^\epsilon \phi(0, w)$ tends to

$$
w^m \int \varphi(|\zeta|^2) \frac{1}{2\pi i} \frac{|\zeta|^2 d\bar\zeta \wedge d\zeta}{|\zeta|^2 + |w|^2}
$$

which is a smooth function of w plus (a constant times) $w^m |w|^2 \log |w|^2$, and thus not smooth. However, it is certainly in C^m. One can check that $K\phi(z, w) =$

$\lim_{\epsilon\to 0^+} K^\epsilon\phi(z,w)$ exists pointwise and defines a function in at least C^m and that our solution can be computed from this limit. In fact, by a more precise computation we get from (10.3) that

$$K^\epsilon\phi(z,w)=\sum_{k=0}^m\int_\zeta \varphi(|\zeta|^2)\chi_\epsilon\frac{1}{2\pi i}\frac{(\bar\zeta-\bar z)\hat\phi_k(\zeta)d\bar\zeta\wedge d\zeta}{|\zeta-z|^2+|w|^2}w^k\sum_{j=0}^{m-k}\left(\frac{|w|^2}{|\zeta-z|^2+|w|^2}\right)^j.$$

It is now clear that we can let $\epsilon\to 0$. By a simple computation we then get

$$\begin{aligned}K\phi(z,w)&=\sum_{k=0}^m C\hat\phi_k(z)\otimes w^k\\&\quad-\sum_{k=0}^m\int_\zeta\varphi(|\zeta|^2)\frac{1}{2\pi i}\frac{\hat\phi_k(\zeta)d\bar\zeta\wedge d\zeta}{\zeta-z}w^k\left(\frac{|w|^2}{|\zeta-z|^2+|w|^2}\right)^{m-k+1}.\end{aligned}$$

Let $\psi=\varphi\hat\phi_k$. Then the kth term in the second sum is equal to

$$b(z,w)=\frac{1}{2\pi i}\int_\zeta\frac{\psi(z+\zeta)d\bar\zeta\wedge d\zeta}{\zeta}w^k\left(\frac{|w|^2}{|\zeta|^2+|w|^2}\right)^{m-k+1}.$$

If we integrate outside the unit disk, then we certainly get a smooth function. Thus it is enough to consider the integral over the disk. Moreover, if $\psi(z+\zeta)=\mathscr{O}(|\zeta|^M)$ for a large M, then the integral is at least C^m. By a Taylor expansion of $\psi(z+\zeta)$ at the point z, we are thus reduced to consider

$$\int_{|\zeta|<1}\frac{\zeta^\alpha\bar\zeta^\beta}{\zeta}\left(\frac{|w|^2}{|\zeta|^2+|w|^2}\right)^{m-k+1}.$$

For symmetry reasons, they vanish, except when $\alpha=\beta+1$. Thus we are left with

$$\int_{|\zeta|<1}|\zeta|^{2\beta}\left(\frac{|w|^2}{|\zeta|^2+|w|^2}\right)^{m-k+1}w^k=Cw^k|w|^{2(m-k+1)}\int_0^1\frac{s^\beta ds}{(s+|w|^2)^{m-k+1}}$$

for non-negative integers β. The right hand side is a smooth function of w if $\beta\le m-k-1$ and a smooth function plus

$$Cw^k|w|^{2(\beta+1)}\log|w|^2$$

if $\beta\ge m-k$. The worst case therefore is when $k=m$ and $\beta=0$; then we have $w^m|w|^2\log|w|^2$ that we encountered above. □

Proposition 10.5 *Let z,w be coordinates at a point $x\in X_{reg}$ such that $Z=\{w=0\}$ and $x=(0,0)$. If ϕ is smooth, and has support where the local coordinates are defined, then*

$$v^\epsilon(z,w)=\int_\zeta \chi(|\zeta-z|^2/\epsilon)(HR\wedge B\wedge g)_N\wedge\phi,$$

is smooth for $\epsilon>0$, and for each multiindex ℓ there is a smooth form v_ℓ such that

$$\partial_w^\ell v^\epsilon|_{w=0}\to v_\ell$$

as currents on Z.

Taking this proposition for granted we can conclude the proof of Theorem 10.1.

Proof of Theorem 10.1 If $\phi\equiv 0$ in a neighborhood of $x\in X'_{\mathrm{reg}}$, then $K\phi$ is smooth near x, cf., the proof of Proposition 9.4. Thus, it is sufficient to prove Theorem 10.1 assuming that ϕ is smooth and has support near x.

Recall that given a minimal generating set $1, w^{\alpha_1},\dots,w^{\alpha_{\nu-1}}$, one gets the coefficients $\hat v_j^\epsilon$ in the representation

$$v^\epsilon=\hat v_0^\epsilon\otimes 1+\cdots+\hat v_{\nu-1}^\epsilon\otimes w^{\alpha_{\nu-1}}$$

from $\partial_w^\ell v^\epsilon|_{w=0}$, $|\ell|\le M$ by a holomorphic matrix, cf., the proof of Lemma 4.7. It thus follows from Proposition 10.5 that there are smooth $\hat v_j$ such that $\hat v_j^\epsilon\to\hat v_j$ as currents on Z. Let $v=\hat v_0\otimes 1+\cdots+\hat v_{\nu-1}\otimes w^{\alpha_{\nu-1}}$. In view of (2.14), $v^\epsilon\wedge\mu\to v\wedge\mu$ for all μ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{CH}_\Omega^Z)$. From Lemma 10.2 we conclude that $v\wedge\mu=K\phi\wedge\mu$ for all such μ. Thus $K\phi=v$ in $\mathcal{W}_X^{0,*}$ and hence $K\phi$ is smooth. □

Proof of Proposition 10.5 Assume that X is embedded in $\Omega\subset\mathbb{C}^N_{\zeta',\tau'}$. After a suitable rotation we can assume that Z is the graph $\tau'=\psi(\zeta')$. The Bochner–Martinelli kernel in Ω is rotation invariant, so it is

$$B=\sigma+\sigma\wedge\bar\partial\sigma+\sigma\wedge(\bar\partial\sigma)^2+\cdots,$$

where

$$\sigma=\frac{(\bar\zeta'-\bar z')\cdot d\zeta'+(\bar\tau'-\bar w')\cdot d\tau'}{|\zeta'-z'|^2+|\tau'-w'|^2}.$$

We now choose the new coordinates $\zeta=\zeta'$, $\tau=\tau'-\psi(\zeta')$ in Ω, so that $Z=\{(\zeta,\tau);\ \tau=0\}$.

Recall that on X_{reg} we have that $R\wedge dz$ is a smooth form times $\mu=(\mu_1,\dots,\mu_m)$, where μ_j is a generating set for $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{CH}_\Omega^Z)$. Thus we are to compute $\partial_w^\ell|_{w=0}$ of integrals like

$$\int_{\zeta,\tau}\bar\partial\frac{d\tau}{\tau^{\alpha+\mathbf{1}}}\wedge B_k^\epsilon\wedge\phi(\zeta,z,w,\tau), \tag{10.5}$$

where $k\le n$ and ϕ is smooth with compact support near x. It is clear that the symbols $\bar\tau,\bar w,d\bar\tau$ can be omitted in the expression for

$$B^\epsilon=\chi_\epsilon B=\chi(|\zeta-z|^2/\epsilon)B,$$

since $\bar\tau$ and $d\bar\tau$ annihilate $\bar\partial(1/\tau^{\alpha+1})$, and since we only take holomorphic derivatives with respect to w and set $w=0$.

Let us write $\psi(\zeta)-\psi(z)=A(\zeta,z)\eta$, where $\eta:=\zeta-z$ is considered as a column matrix and A is a holomorphic $(N-n)\times n$-matrix. Then

$$\sigma=\frac{\eta^*\nu}{|\zeta-z|^2+|\tau-w+\psi(\zeta)-\psi(z)|^2},$$

where ν is the $(1,0)$-form valued column matrix

$$\nu=d\zeta+A^*d(\tau+\psi(\zeta)).$$

Since $\eta^*\nu$ is a $(1,0)$-form we have that

$$B_k^\epsilon=\chi_\epsilon\frac{\eta^*\nu\wedge((d\eta^*)\nu+\eta^*\bar\partial\nu)^{k-1}}{(|\zeta-z|^2+|\tau-w+\psi(\zeta)-\psi(z)|^2)^k}.$$

Lemma 10.6 *Let*

$$\xi^i=\xi_1^i\frac{\partial}{\partial\zeta_1}+\cdots+\xi_n^i\frac{\partial}{\partial\zeta_n}$$

be smooth $(1,0)$*-vector fields, and let* $L_i=L_{\xi^i}$ *be the associated Lie derivatives for* $i=1,\ldots,\rho$. *Let*

$$\gamma_k:=\eta^*\nu\wedge((d\eta^*)\nu+\eta^*\bar\partial\nu)^{k-1}.$$

If we have a modification $\pi:\tilde W\to\Omega\times\Omega$ *such that locally* $\pi^*\eta=\eta_0\eta'$, *where* η_0 *is a holomorphic function, then*

$$\pi^*(L_1\cdots L_\rho\gamma_k)=\bar\eta_0^k\beta,$$

where β *is smooth.*

Recall that if a is a form, then $L_\xi a=d(\xi\neg a)+\xi\neg(da)$, and that $L_\xi(\beta\neg a)=[\xi,\beta]\neg a+\beta\neg(L_\xi a)$ if β is another vector field.

Proof Introduce a nonsense basis e and its dual e^* and consider the exterior algebra spanned by e_j, e_ℓ^*, and the cotangent bundle. Let

$$c_\ell=\eta^*e\wedge((d\eta^*)e)^{\ell-1}.$$

Notice that γ_k is a sum of terms like

$$(\nu e^*\neg)^\ell c_\ell\wedge(\eta^*\bar\partial\nu)^{k-\ell}.$$

Since $L_i c_\ell = 0$ and $L_i(\eta^* b) = \eta^* L_i b$ it follows after a finite number of applications of L_i's that we get

$$(\nu_1 e^*)\neg \cdots (\nu_\ell e^*)\neg c_\ell (\eta^* b_1) \cdots (\eta^* b_{k-\ell}),$$

where ν_j and b_j are smooth. Since

$$\pi^* c_\ell = \bar{\eta}_0^\ell (\eta')^* e \wedge (d(\eta')^* e)^{\ell-1},$$

the lemma now follows. □

We note that $\eta^*(I + A^* A)\eta = |\zeta - z|^2 + |\psi(\zeta) - \psi(z)|^2$. Thus, differentiating (10.5) with respect to w, setting $w = 0$, and evaluating the residue with respect to τ using (2.10), we obtain a sum of integrals like

$$\int_\zeta \chi_\epsilon \frac{(\eta^* a_1) \cdots (\eta^* a_{t+1}) \wedge \gamma_k \wedge \phi}{(\eta^*(I + A^* A)\eta)^{k+t+1}},$$

where $a_1, \ldots, a_{t+1}$ are column vectors of smooth functions. We must prove that the limit of such integrals when $\epsilon \to 0$ are smooth in z.

Lemma 10.7 *Let*

$$I_\ell^{r,s} = \int \chi_\epsilon \frac{(\eta^* a_1) \cdots (\eta^* a_r) \mathscr{O}(|\eta|^{2s}) \tilde{\gamma}_k \wedge \phi}{\Phi^{k+\ell}},$$

where $a_1, \ldots, a_r$ are tuples of smooth functions, $\tilde{\gamma}_k = L_1 \cdots L_\rho \gamma_k$, where $L_i = L_{\xi_i}$ are Lie derivatives with respect to smooth $(1,0)$-vector fields ξ^i as above for $i = 1, \ldots, \rho$, ϕ is a test form with support close to z, and $\Phi := \eta^*(I + A^* A)\eta$. If $r \geq 1$ and $r + s \geq \ell + 1$, then we have the relation

$$I_{\ell+1}^{r,s} = I_\ell^{r-1,s} + I_{\ell+1}^{r-1,s+1} + I_\ell^{r,s-1} + o(1) \tag{10.6}$$

when $\epsilon \to 0$.

Proof If

$$\xi = a_r^t (I + A^* A)^{-t} \frac{\partial}{\partial \zeta},$$

and $L = L_\xi$, then using that $\Phi = \eta^t (I + A^* A)^t \bar{\eta}$, one obtains that

$$L\Phi = \eta^* a_r + \mathscr{O}(|\eta|^2). \tag{10.7}$$

Thus

$$I_{\ell+1}^{r,s} = \int \chi_\epsilon (\eta^* a_1) \cdots (\eta^* a_{r-1}) \mathscr{O}(|\eta|^{2s}) \tilde{\gamma}_k \wedge \phi L \frac{1}{\Phi^{k+\ell}} + I_{\ell+1}^{r-1,s+1}$$

in view of (10.7). We now integrate by parts by L in the integral. If a derivative with respect to ζ_j falls on some $\eta^* a_i$, we get a term $I_\ell^{r-1,s}$. If it falls on $\mathscr{O}(|\eta|^{2s})$ we get either $\mathscr{O}(|\eta|^{2(s-1)})$ times $\eta^* b$, for some tuple b of smooth functions, and this gives rise to the term $I_\ell^{r,s-1}$ or $\mathscr{O}(|\eta|^{2s})$, and this gives rise to another term $I_\ell^{r-1,s}$. If it falls on ϕ or $\tilde{\gamma}_k$ we get an additional term $I_\ell^{r-1,s}$. The only possibility left is when the derivative falls on $\chi_\epsilon = \chi(|\eta|^2/\epsilon)$. It remains to show that an integral of the form

$$\int_{\zeta,z} \chi'(|\eta|^2/\epsilon) \frac{(\eta^* a_1)\cdots(\eta^* a_{r-1})(\eta^* b)}{\epsilon} \frac{\mathscr{O}(|\eta|^{2s})\gamma_k \wedge \phi}{\Phi^{k+\ell}}$$

tends to 0, where the factor $\eta^* b$ comes from the derivative of $|\eta|^2$. We now choose a resolution $\widetilde{V} \to \Omega \times \Omega$ such that $\eta = \eta_0 \eta'$ where η' is non-vanishing and η_0 is (locally) a monomial. Notice that $\pi^* \Phi = |\eta_0|^2 \Phi'$ where Φ' is smooth and strictly positive. In view of Lemma 10.6 we thus obtain integrals of the form

$$\int_{\widetilde{V}} \chi'(|\eta_0|^2 v/\epsilon) \frac{1}{\epsilon} \frac{\bar{\eta}_0^{r+s-\ell}}{\eta_0^{k+\ell-s}} \alpha, \tag{10.8}$$

where v is smooth and strictly positive and α is smooth.

In order to see that the limit of (10.8) tends to 0, we note first that if we let

$$\tilde{\chi}(s) = s\chi'(s) + \chi(s),$$

then just as χ, $\tilde{\chi}$ is also a smooth function on $[0,\infty)$ that is 0 in a neighborhood of 0 and 1 in a neighborhood of ∞. By assumption, $r+s-\ell-1 \geq 0$. Since the principal value current $1/f^m$ acting on a test form β can be defined as

$$\lim_{\epsilon \to 0^+} \int \chi(|f|^2 v/\epsilon) \frac{\beta}{f^m}$$

for any cut-off function as above, the principal value current $1/\eta_0^{k+\ell-s}$ acting on $\bar{\eta}_0^{r+s-\ell-1}\alpha$ equals

$$\lim_{\epsilon \to 0^+} \int_{\widetilde{V}} \chi\left(|\eta_0|^2 v/\epsilon\right) \frac{\bar{\eta}_0^{r+s-\ell-1}}{\eta_0^{k+\ell-s}} \alpha = \lim_{\epsilon \to 0^+} \int_{\widetilde{V}} \tilde{\chi}\left(|\eta_0|^2 v/\epsilon\right) \frac{\bar{\eta}_0^{r+s-\ell-1}}{\eta_0^{k+\ell-s}} \alpha.$$

Taking the difference between the left and right hand side, we conclude that (10.8) tends to 0 when $\epsilon \to 0$. □

Now we can conclude the proof of Proposition 10.5. From the beginning we have $I_\ell^{\ell,0}$. After repeated applications of (10.6) we end up with

$$I_\ell^{0,\ell} + I_{\ell-1}^{0,\ell-1} + \cdots + I_0^{0,0} + o(1).$$

However, any of these integrals has an integrable kernel even when $\epsilon = 0$. This means that we are back to the case in [6, Lemma 6.2], and so the limit integral is smooth in z. $\square$

11 A fine resolution of $\mathscr{O}_X$

We first consider a generalization of Theorem 9.1.

Lemma 11.1 *Assume that $\phi \in \mathcal{W}^{0,k}(X) \cap \mathscr{E}_X^{0,k}(X_{reg}) \cap \mathrm{Dom}\,\bar\partial_X$ and that $K\phi$ is in $\mathrm{Dom}\,\bar\partial_X$ (or just in $\mathrm{Dom}\,\bar\partial$). Then* (9.1) *holds on X'.*

Proof Let χ_δ be functions as before that cut away X_{sing}. From Koppelman's formula (9.1) for smooth forms we have

$$\chi_\delta\phi\wedge h = \bar\partial(K(\chi_\delta\phi))\wedge h + K(\chi_\delta\bar\partial\phi)\wedge h + P(\chi_\delta\phi)\wedge h + K(\bar\partial\chi_\delta\wedge\phi)\wedge h,\ \ h\in\omega_X^n, \tag{11.1}$$

for $z\in X'_{reg}$. Clearly the left hand side tends to $\phi\wedge h$ when $\delta\to 0$. From Lemma 9.2 it follows that $K(\chi_\delta\phi)\wedge h\to K\phi\wedge h$. Thus the first term on the right hand side of (11.1) tends to $\bar\partial(K\phi)\wedge h$. In the same way the second and third terms on the right hand side tend to $K(\bar\partial\phi)\wedge h$ and $P\phi\wedge h$, respectively. It remains to show that the last term tends to 0. If z belongs to a fixed compact subset of X'_{reg}, then B is smooth in (9.5) when ζ is in $\mathrm{supp}\,\bar\partial\chi_\delta$ for small δ. Hence it suffices to see that

$$R(\zeta)\wedge d\zeta\wedge\bar\partial\chi_\delta\wedge\phi(\zeta)\wedge i_*h\to 0,$$

and since this is a tensor product of currents, it suffices to see that

$$R(\zeta)\wedge d\zeta\wedge\bar\partial\chi_\delta\wedge\phi(\zeta)\to 0,$$

or equivalently, $\omega(\zeta)\wedge\bar\partial\chi_\delta\wedge\phi(\zeta)\to 0$, which follows by Lemma 8.4 since ϕ is in $\mathrm{Dom}\,\bar\partial_X$. We have thus proved that

$$\chi_\delta\phi\wedge h = \chi_\delta\bar\partial(K\phi)\wedge h + \chi_\delta K(\bar\partial\phi)\wedge h + \chi_\delta P\phi\wedge h.$$

The first term on the right hand side is equal to $\bar\partial(\chi_\delta K\phi\wedge h) - \bar\partial\chi_\delta\wedge K\phi\wedge h$, where the latter term tends to 0 if $K\phi$ is in $\mathrm{Dom}\,\bar\partial_X$ or just in $\mathrm{Dom}\,\bar\partial$, cf., Lemma 8.4. Thus we get

$$\phi\wedge h = \bar\partial(K\phi)\wedge h + K(\bar\partial\phi)\wedge h + P\phi\wedge h,\ \ h\in\omega_X^n,$$

which precisely means that (9.1) holds. $\square$

Definition 11.2 We say that a $(0,q)$-current ϕ on an open set $\mathcal{U}\subset X$ is a section of $\mathscr{A}_X^q$ over $\mathcal{U}$, $\phi\in\mathscr{A}^q(\mathcal{U})$, if, for every $x\in\mathcal{U}$, the germ ϕ_x can be written as a finite sum of terms

$$\xi_\nu\wedge K_\nu(\cdots\xi_2\wedge K_2(\xi_1\wedge K_1(\xi_0))),$$

where ξ_j are smooth $(0,*)$-forms and K_j are integral operators with kernels $k_j(\zeta, z)$ at x, defined as above, and such that ξ_j has compact support in the set where $z \mapsto k_j(\zeta, z)$ is defined.

Clearly $\mathscr{A}_X^*$ is closed under multiplication by ξ in $\mathscr{E}_X^{0,*}$. It follows from (9.8) that $\mathscr{A}_X^*$ is a subsheaf of $\mathcal{W}_X^{0,*}$ and from Theorem 10.1 that $\mathscr{A}_X^k = \mathscr{E}_X^{0,*}$ on X_{reg}. Clearly any operator K as above maps $\mathscr{A}_X^{*+1} \to \mathscr{A}_X^*$.

Lemma 11.3 *If ϕ is in $\mathscr{A}_X$, then ϕ and $K\phi$ are in Dom $\bar\partial_X$.*

Proof Notice that [6, Lemma 6.4] holds in our case by verbatim the same proof, since we have access to the dimension principle for (tensor products of) pseudomeromorphic $(n,*)$-currents, and the computation rule (2.3), cf., the comment after Definition 5.7. Since $\mathscr{A}_X^* = \mathscr{E}_X^{0,*}$ on $X_{\rm reg}$ it is enough by Lemma 8.4 to check that $\bar\partial\chi_\delta \wedge \omega \wedge \phi \to 0$, and this precisely follows from [6, Lemma 6.4]. □

In view of Lemmas 11.1 and 11.3 we have

Proposition 11.4 *Let K, P be integral operators as in Theorem 9.1. Then*

$$K\colon \mathscr{A}^{k+1}(X) \to \mathscr{A}^k(X'), \quad P\colon \mathscr{A}^k(X) \to \mathscr{E}^{0,k}(X'),$$

and the Koppelman formula (9.1) holds.

Proof of Theorem 1.1 By definition, it is clear that $\mathscr{A}_X^k$ are modules over $\mathscr{E}_X^{0,k}$, and by Theorem 10.1, $\mathscr{A}_X^k$ coincides with $\mathscr{E}_X^{0,k}$ on $X_{\rm reg}$. Since we have access to Koppelman formulas, precisely as in the proof of [6, Theorem 1.2] we can verify that $\bar\partial\colon \mathscr{A}_X^k \to \mathscr{A}_X^{k+1}$.

It remains to prove that (1.2) is exact. We choose locally a weight g that is holomorphic in z, so the term $P\phi$ vanishes if ϕ is in $\mathscr{A}_X^k$, $k \geq 1$, and is holomorphic in z when $k = 0$. Assume that ϕ is in $\mathscr{A}_X^k$ and $\bar\partial\phi = 0$. If $k \geq 1$, then $\bar\partial K\phi = \phi$, and if $k = 0$, then $\phi = P\phi$. □

11.1 Global solvability

Assume that $E \to X$ is a holomorphic vector bundle; this means that the transition matrices have entries in $\mathscr{O}_X$. For instance if we have a global embedding $i\colon X \to \Omega$ and a holomorphic vector bundle $F \to \Omega$, then F defines a vector bundle $i^*F \to X$. The sheaves $\mathscr{A}_X^*(E)$ give rise to a fine resolution of the sheaf $\mathscr{O}_X(E)$, and by standard homological algebra we have the isomorphisms

$$H^q(X, \mathscr{O}(E)) = \frac{\text{Ker}\,(\mathscr{A}^q(X,E) \xrightarrow{\bar\partial} \mathscr{A}^{q+1}(X,E))}{\text{Im}\,(\mathscr{A}^{q-1}(X,E) \xrightarrow{\bar\partial} \mathscr{A}^q(X,E))}, \quad q \geq 1.$$

Thus, if $\phi \in \mathscr{A}^{q+1}(X,E)$, $\bar\partial\phi = 0$, and its canonical cohomology class vanishes, then the equation $\bar\partial\psi = \phi$ has a global solution in $\mathscr{A}^q(X,E)$. In particular, the equation is always solvable if X is Stein. If for instance X is a pure-dimensional projective variety $i\colon X \to \mathbb{P}^N$, then the $\bar{\partial}$-equation is solvable, e.g., if E is a sufficiently ample line bundle.

12 Locally complete intersections

Let us consider the special case when X locally is a complete intersection, i.e., given a local embedding $i\colon X \to \Omega \subset \mathbb{C}^N$ there are global sections f_j of $\mathscr{O}(d_j) \to \mathbb{P}^N$ such that $\mathcal{J} = (f_1, \dots, f_p)$, where $p = N - n$. In particular, $Z = \{f_1 = \cdots = f_p = 0\}$. In this case $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega)$ is generated by the single current

$$\mu = \bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1} \wedge dz_1 \wedge \cdots \wedge dz_N,$$

see, e.g., [3]. Each smooth $(0, q)$-form ϕ in $\mathscr{E}_X^{0,q}$ is thus represented by a current $\Phi \wedge \mu$, where Φ is smooth in a neighborhood of Z and $i^*\Phi = \phi$. Moreover, X is Cohen–Macaulay so X_{reg} coincides with the part of X where Z is regular, and $\bar{\partial}\phi = \psi$ if and only if $\bar{\partial}(\phi \wedge \mu) = \psi \wedge \mu$.

Henkin and Polyakov introduced, see [17, Definition 1.3], the notion of *residual currents ϕ of bidegree* $(0, q)$ on a locally complete intersection $X \subset \mathbb{P}^N$, and the $\bar{\partial}$-equation $\bar{\partial}\psi = \phi$. Their currents ϕ correspond to our ϕ in $\mathscr{E}_X^{0,q}$ and their $\bar{\partial}$-operator on such currents coincides with ours.

Remark 12.1 In [18] Henkin and Polyakov consider a global reduced complete intersection $X \subset \mathbb{P}^N$. They prove, by a global explicit formula, that if ϕ is a global $\bar{\partial}$-closed smooth $(0, q)$-form with values in $\mathscr{O}(\ell)$, $\ell = d_1 + \cdots d_p - N - 1$, then there is a smooth solution to $\bar{\partial}\psi = \phi$ at least on X_{reg}, if $1 \le q \le n-1$. When $q = n$ a necessary obstruction term occurs. However, their meaning of "$\bar{\partial}$-closed" is that locally there is a representative Φ of ϕ and smooth g_j such that $\bar{\partial}\Phi = g_1 f_1 + \cdots + g_p f_p$. If this holds, then clearly $\bar{\partial}\phi = 0$. The converse implication is *not* true, see Example 12.2 below. It is not clear to us whether their formula gives a solution under the weaker assumption that $\bar{\partial}\phi = 0$, neither do we know whether their solution admits some intrinsic extension across X_{sing} as a current on X. □

Example 12.2 Let $X = \{f = 0\} \subset \Omega \subset \mathbb{C}^{n+1}$ be a reduced hypersurface, and assume that $df \neq 0$ on X_{reg}, so that $\mathcal{J} = (f)$. Let ϕ be a smooth $(0, q)$-form in a neighborhood of some point x on X such that $\bar{\partial}\phi = 0$. We claim that $\bar{\partial}u = \phi$ has a smooth solution u if and only if ϕ has a smooth representative Φ in ambient space such that $\bar{\partial}\Phi = fg$ for some smooth form g. In fact, if such a Φ exists then $0 = f\bar{\partial}g$ and thus $\bar{\partial}g = 0$. Therefore, $g = \bar{\partial}\gamma$ for some smooth γ (in a Stein neighborhood of x in ambient space) and hence $\bar{\partial}(\Phi - f\gamma) = 0$. Thus there is a smooth U such that $\bar{\partial}U = \Phi - f\gamma$; this means that $u = i^*U$ is a smooth solution to $\bar{\partial}u = \phi$. Conversely, if u is a smooth solution, then $u = i^*U$ for some smooth U in ambient space, and thus $\Phi = \bar{\partial}U$ is a representative of ϕ in ambient space. Thus $\bar{\partial}\Phi = fg$ (with $g = 0$).

There are examples of hypersurfaces X where there exist smooth ϕ with $\bar\partial\phi = 0$ that do not admit smooth solutions to $\bar\partial u = \phi$, see, e.g., [6, Example 1.1]. It follows that such a ϕ cannot have a representative Φ in ambient space as above. □

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