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The $\bar{\partial}$-equation on a non-reduced analytic space

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Abstract Let $X$ be a, possibly non-reduced, analytic space of pure dimension. We introduce a notion of $\bar{\partial}$-equation on $X$ and prove a Dolbeault–Grothendieck lemma. We obtain fine sheaves $A^q_X$ of $(0, q)$-currents, so that the associated Dolbeault complex yields a resolution of the structure sheaf $\mathcal{O}_X$. Our construction is based on intrinsic semi-global Koppelman formulas on $X$.

Mathematics Subject Classification 32A26 · 32A27 · 32B15 · 32C30

1 Introduction

Let $X$ be a smooth complex manifold of dimension $n$ and let $\mathcal{E}_X^{0,*}$ denote the sheaf of smooth $(0, *)$-forms. It is well-known that the Dolbeault complex

$$0 \to \mathcal{O}_X \overset{i}{\to} \mathcal{E}_X^{0,0} \overset{\bar{\partial}}{\to} \mathcal{E}_X^{0,1} \overset{\bar{\partial}}{\to} \cdots \overset{\bar{\partial}}{\to} \mathcal{E}_X^{0,n} \to 0 \quad (1.1)$$

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is exact, and hence provides a fine resolution of the structure sheaf $\mathcal{O}_X$. If $X$ is a reduced analytic space of pure dimension, then there is still a natural notion of “smooth forms”. In fact, assume that $X$ is locally embedded as $i: X \to \Omega$, where $\Omega$ is a pseudoconvex domain in $\mathbb{C}^N$. If $\text{Ker} \ i^*$ denotes the subsheaf of all smooth forms $\xi$ in ambient space such that $i^*\xi = 0$ on the regular part $X_{\text{reg}}$ of $X$, then one defines the sheaf $\mathcal{E}_X$ of smooth forms on $X$ simply as

$$\mathcal{E}_X := \mathcal{E}/\text{Ker} \ i^*.$$

It is well-known that this definition is independent of the choice of embedding of $X$. Currents on $X$ are defined as the duals of smooth forms with compact support. It is readily seen that the currents $\mu$ on $X$ so defined are in a one-to-one correspondence to the currents $\hat{\mu} = i_*\mu$ in ambient space such that $\hat{\mu}$ vanish on $\text{Ker} \ i_*$, see, e.g., [6]. There is an induced $\bar{\partial}$-operator on smooth forms and currents on $X$. In particular, (1.1) is a complex on $X$ but in general it is not exact. In [6], Samuelsson and the first author introduced, by means of intrinsic Koppelman formulas on $X$, fine sheaves $\mathcal{A}_X^q$ of $(0, *)$-currents that are smooth on $X_{\text{reg}}$ and with mild singularities at the singular part of $X$, such that

$$0 \to \mathcal{O}_X \xrightarrow{i} \mathcal{A}_X^0 \xrightarrow{\bar{\partial}} \mathcal{A}_X^1 \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}_X^n \to 0 \quad (1.2)$$

is exact, and thus a fine resolution of the structure sheaf $\mathcal{O}_X$. An immediate consequence is the representation

$$H^q(X, \mathcal{O}_X) = \frac{\text{Ker} \left( \mathcal{A}_X^0, q(X) \xrightarrow{\bar{\partial}} \mathcal{A}_X^{0, q+1}(X) \right)}{\text{Im} \left( \mathcal{A}_X^{0, q-1}(X) \xrightarrow{\bar{\partial}} \mathcal{A}_X^{0, q}(X) \right)}, \quad q \geq 1, \quad (1.3)$$

of sheaf cohomology, and so (1.3) is a generalization of the classical Dolbeault isomorphism. In special cases more qualitative information of the sheaves $\mathcal{A}_X^q$ are known, see, e.g., [5,23].

Starting with the influential works [28,29] by Pardon and Stern, there has been a lot of progress recently on the $L^2$-$\bar{\partial}$ theory on non-smooth (reduced) varieties; see, e.g., [15,27,31]. The point in these works, contrary to [6], is basically to determine the obstructions to solve $\bar{\partial}$ locally in $L^2$. For a more extensive list of references regarding the $\bar{\partial}$-equation on reduced singular varieties, see, e.g., [6].

In [17], a notion of the $\bar{\partial}$-equation on non-reduced local complete intersections was introduced, and which was further studied in [18]. We discuss below how their work relates to ours.

The aim of this paper is to extend the construction in [6] to a non-reduced pure-dimensional analytic space. The first basic problem is to find appropriate definitions of forms and currents on $X$. Let $X_{\text{reg}}$ be the part of $X$ where the underlying reduced space $Z$ is smooth, and in addition $\mathcal{O}_X$ is Cohen–Macaulay. On $X_{\text{reg}}$ the structure sheaf $\mathcal{O}_X$ has a structure as a free finitely generated $\mathcal{O}_Z$-module. More precisely, assume that we have a local embedding $i: X \to \Omega \subset \mathbb{C}^N$ and coordinates $(z, w)$ in $\Omega$ such that
$Z = \{ w = 0 \}$. Let $\mathcal{J}$ be the defining ideal sheaf for $X$ on $\Omega$. Then there are monomials $1, w^{a_1}, \ldots, w^{a_{v-1}}$ such that each $\phi$ in $\mathcal{O}_\Omega/\mathcal{J} \simeq \mathcal{O}_X$ has a unique representation

$$\phi = \hat{\phi}_0 \otimes 1 + \hat{\phi}_1 \otimes w^{a_1} + \cdots + \hat{\phi}_{v-1} \otimes w^{a_{v-1}}, \quad (1.4)$$

where $\hat{\phi}_j$ are in $\mathcal{O}_Z$. A reasonable notion of a smooth form on $X$ should admit a similar representation on $X_{\text{reg}}$ with smooth forms $\hat{\phi}_j$ on $Z$. We first introduce the sheaves $\mathcal{E}_X^{0,*}$ of smooth $(0, *)$-forms on $X$. By duality, we then obtain the sheaf $\mathcal{C}_X^{n,*}$ of $(n, *)$-currents. We are mainly interested in the subsheaf $\mathcal{P}_X^{n,*}$ of pseudomeromorphic currents, and especially, the even more restricted sheaf $\mathcal{W}_X^{n,*}$ of such currents with the so-called standard extension property, SEP, on $X$. A current with the SEP is, roughly speaking, determined by its restriction to any dense Zariski-open subset.

Of special interest is the sheaf $\omega_X^n \subset \mathcal{W}_X^{n,0}$ of $\overline{\partial}$-closed pseudomeromorphic $(n, 0)$-currents. In the reduced case this is precisely the sheaf of holomorphic $(n, 0)$-forms in the sense of Barlet–Henkin–Passare, see, e.g., [12, 16].

We have no definition of “smooth $(n, *)$-form” on $X$. In order to define $(0, *)$-currents, we use instead the sheaf $\omega_X^n$ in the following way. Any holomorphic function defines a morphism in $\mathcal{H}om(\omega_X^n, \omega_X^n)$, and it is a reformulation of a fundamental result of Roos [30], that this morphism is indeed injective, and generically surjective. In the reduced case, multiplication by a current in $\mathcal{W}_X^{0,*}$ induces a morphism in $\mathcal{H}om(\omega_X^n, \mathcal{V}_X^{n,*})$, and in fact $\mathcal{W}_X^{0,*} \rightarrow \mathcal{H}om(\omega_X^n, \mathcal{V}_X^{n,*})$ is an isomorphism. In the non-reduced case, we then take this as the definition of $\mathcal{W}_X^{0,*}$. It turns out that with this definition, on $X_{\text{reg}}$, any element of $\mathcal{W}_X^{0,*}$ admits a unique representation (1.4), where $\hat{\phi}_j$ are in $\mathcal{W}_Z^{0,*}$, see Sect. 6 below for details.

Given $\nu, \phi$ in $\mathcal{W}_X^{0,*}$ we say that $\overline{\partial}\nu = \phi$ if $\overline{\partial}(\nu \wedge h) = \phi \wedge h$ for all $h$ in $\omega_X^n$. Following [6] we introduce semi-global integral formulas and prove that if $\phi$ is a smooth $\overline{\partial}$-closed $(0, q + 1)$-form there is a locally current $\nu$ in $\mathcal{V}_X^{0,q}$ such that $\overline{\partial}\nu = \phi$. A crucial problem is to verify that the integral operators preserve smoothness on $X_{\text{reg}}$ so that the solution $\nu$ is indeed smooth on $X_{\text{reg}}$. By an iteration procedure as in [6] we can define sheaves $\mathcal{A}_X^{k} \subset \mathcal{V}_X^{0,k}$ and obtain our main result in this paper.

**Theorem 1.1** Let $X$ be an analytic space of pure dimension $n$. There are sheaves $\mathcal{A}_X^{k} \subset \mathcal{V}_X^{0,k}$ that are modules over $\mathcal{E}_X^{0,*}$, coinciding with $\mathcal{E}_X^{0,k}$ on $X_{\text{reg}}$, and such that (1.2) is a resolution of the structure sheaf $\mathcal{O}_X$.

The main contribution in this article compared to [6] is the development of a theory for smooth $(0, *)$-forms and various classes of $(n, *)$- and $(0, *)$-currents in the non-reduced case as is described above. This is done in Sects. 4–8. The construction of integral operators to provide solutions to $\overline{\partial}$ in Sect. 9 and the construction of the fine resolution of $\mathcal{O}_X$ in Sect. 11, which proves Theorem 1.1, are done pretty much in the same way as in [6]. The proof of the smoothness of the solutions of the regular part in Sect. 10 however becomes significantly more involved in the non-reduced case and requires completely new ideas. In Sect. 12 we discuss the relation to the results in [17,18] in case $X$ is a local complete intersection.
2 Pseudomeromophic currents

Let \( s_1, \ldots, s_m \) be coordinates in \( \mathbb{C}^m \), let \( \alpha \) be a smooth form with compact support, and let \( a_1, \ldots, a_r \) be positive integers, \( 0 \leq \ell \leq r \leq m \). Then

\[
\bar{\partial} \frac{1}{s_1^{a_1}} \wedge \cdots \wedge \bar{\partial} \frac{1}{s_\ell^{a_\ell}} \wedge \frac{\alpha}{s_{\ell+1} \cdots s_r}
\]

is a well-defined current that we call an elementary (pseudomeromorphic) current. Let \( Z \) be a reduced space of pure dimension. A current \( \tau \) is pseudomeromorphic on \( Z \) if, locally, it is the push-forward of a finite sum of elementary pseudomeromorphic currents under a sequence of modifications, simple projections, and open inclusions. The pseudomeromorphic currents define an analytic sheaf \( PM_Z \) on \( Z \). This sheaf was introduced in [8] and somewhat extended in [6]. If nothing else is explicitly stated, proofs of the properties listed below can be found in, e.g., [6].

If \( \tau \) is pseudomeromorphic and has support on an analytic subset \( V \), and \( h \) is a holomorphic function that vanishes on \( V \), then \( \bar{h} \tau = 0 \) and \( d \bar{h} \wedge \tau = 0 \).

Given a pseudomeromorphic current \( \tau \) and a subvariety \( V \) of some open subset \( U \subset Z \), the natural restriction to the open set \( U \setminus V \) of \( \tau \) has a natural extension to a pseudomeromorphic current on \( U \) that we denote by \( \mathbf{1}_{U \setminus V} \tau \). Throughout this paper we let \( \chi \) denote a smooth function on \( (0, \infty) \) that is 0 in a neighborhood of 0 and 1 in a neighborhood of \( \infty \). If \( h \) is a holomorphic tuple whose common zero set is \( V \), then

\[
\mathbf{1}_{U \setminus V} \tau = \lim_{\epsilon \to 0^+} \chi(|h|^2/\epsilon) \tau.
\]

This action of \( \mathbf{1}_V \) on the sheaf of pseudomeromorphic currents is a basic tool. In fact one can extend this calculus to all constructible sets so that (2.2) holds, see [8]. One readily checks that if \( \xi \) is a smooth form, then

\[
\mathbf{1}_V (\xi \wedge \tau) = \xi \wedge \mathbf{1}_V \tau.
\]

If \( f : Z' \to Z \) is a modification and \( \tau \) is in \( PM_Z \), then \( f_\# \tau \) is in \( PM_Z \). The same holds if \( f \) is a simple projection and \( \tau \) has compact support in the fiber direction. In any case we have

\[
\mathbf{1}_V f_\# \tau = f_\# (\mathbf{1}_{f^{-1} V} \tau).
\]

It is not hard to check that if \( \tau \) is in \( PM_Z \) and \( \tau' \) is in \( PM_{Z'} \), then \( \tau \otimes \tau' \) is in \( PM_{Z \times Z'} \), see, e.g., [4, Lemma 3.3]. If \( V \subset U \subset Z \) and \( V' \subset U' \subset Z' \), then

\[
(\mathbf{1}_V \tau) \otimes \mathbf{1}_{V'} \tau' = \mathbf{1}_{V \times V'} (\tau \otimes \tau').
\]
Another basic tool is the *dimension principle*, that states that if \( \tau \) is a pseudommeromorphic \((*,p)\)-current with support on an analytic set with codimension larger than \( p \), then \( \tau \) must vanish.

A pseudommeromorphic current \( \tau \) on \( Z \) has the *standard extension property*, SEP, if \( 1_V \tau = 0 \) for each germ \( V \) of an analytic set with positive codimension on \( Z \). The set \( \mathcal{W}_Z \) of all pseudommeromorphic currents on \( Z \) with the SEP is a subsheaf of \( \mathcal{PM}_Z \). By (2.3), \( \mathcal{W}_Z \) is closed under multiplication by smooth forms.

Let \( f \) be a holomorphic function (or a holomorphic section of a Hermitian line bundle), not vanishing identically on any irreducible component of \( Z \). Then \( 1/f \), a priori defined outside of \( \{f = 0\} \), has an extension as a pseudommeromorphic current, the principal value current, still denoted by \( 1/f \), such that \( 1_{\{f = 0\}}(1/f) = 0 \). The current \( 1/f \) has the SEP and

\[
\frac{1}{f} = \lim_{\epsilon \to 0^+} \chi(|f|^2/\epsilon) \frac{1}{f}.
\]

We say that a current \( a \) on \( Z \) is *almost semi-meromorphic* if there is a modification \( \pi: Z' \to Z \), a holomorphic section \( f \) of a line bundle \( L \to Z' \) and a smooth form \( \gamma \) with values in \( L \) such that \( a = \pi_*(\gamma/f) \), cf., [10, Section 4]. If \( a \) is almost semi-meromorphic, then it is clearly pseudommeromorphic. Moreover, it is smooth outside an analytic set \( V \subset Z \) of positive codimension, \( a \) is in \( \mathcal{W}_Z \), and in particular, \( a = \lim_{\epsilon \to 0^+} \chi(|h|/\epsilon) a \) if \( h \) is a holomorphic tuple that cuts out (an analytic set of positive codimension that contains) \( V \). The *Zariski singular support* of \( a \) is the Zariski closure of the set where \( a \) is not smooth.

One can multiply pseudommeromorphic currents by almost semi-meromorphic currents; and this fact will be crucial in defining \( \mathcal{W}_X^{0,*} \), when \( X \) is non-reduced. Notice that if \( a \) is almost semi-meromorphic in \( Z \) then it also is in any open \( U \subset Z \).

**Proposition 2.1** ([10, Theorem 4.8, Proposition 4.9]) Let \( Z \) be a reduced space, assume that \( a \) is an almost semi-meromorphic current in \( Z \), and let \( V \) be the Zariski singular support of \( a \).

(i) If \( \tau \) is a pseudommeromorphic current in \( U \subset Z \), then there is a unique pseudommeromorphic current \( a \wedge \tau \) in \( U \) that coincides with (the naturally defined current) \( a \wedge \tau \) in \( U \setminus V \) and such that \( 1_V(a \wedge \tau) = 0 \).

(ii) If \( W \subset U \) is any analytic subset, then

\[
1_W(a \wedge \tau) = a \wedge 1_W \tau. \tag{2.6}
\]

Notice that if \( h \) is a tuple that cuts out \( V \), then in view of (2.1),

\[
a \wedge \tau = \lim_{\epsilon \to 0^+} \chi(|h|^2/\epsilon) a \wedge \tau. \tag{2.7}
\]

It follows that if \( \xi \) is a smooth form, then

\[
\xi \wedge (a \wedge \tau) = (-1)^{\deg \xi \deg a} a \wedge (\xi \wedge \tau). \tag{2.8}
\]
For future reference we will need the following result.

**Proposition 2.2** Let $Z$ be a reduced space. Then $\mathcal{PM}_Z = \mathcal{W}_Z + \bar{\partial}\mathcal{W}_Z$.

**Proof** First assume that $Z$ is smooth. Since $\mathcal{W}_Z$ is closed under multiplication by smooth forms, so is $\mathcal{W}_Z + \bar{\partial}\mathcal{W}_Z$. The statement that $\mathcal{PM}_Z = \mathcal{W}_Z + \bar{\partial}\mathcal{W}_Z$ is local, and since both sides are closed under multiplication by cutoff functions, we may consider a pseudomeromorphic current $\mu$ with compact support in $\mathbb{C}^n$. If $\mu$ has bidegree $(\ast, 0)$, then it is in $\mathcal{W}_Z$ in view of the dimension principle. Thus we assume that $\mu$ has bidegree $(\ast, q)$ with $q \geq 1$. Let

$$K\mu(z) = \int k(\xi, z) \wedge \mu(\xi),$$

where $k$ is the Bochner–Martinelli kernel. Here (2.9) means that $K\mu = p_\ast(k \wedge \mu \otimes 1)$, where $p$ is the projection $\mathbb{C}^n_\xi \times \mathbb{C}^n_\zeta \to \mathbb{C}^n_\zeta, (\xi, z) \mapsto z$. Recall that we have the Koppelman formula $\mu = \delta K\mu + K(\delta \mu)$. It is thus enough to see that $K\mu$ in $\mathcal{W}_Z$ if $\mu$ is pseudomeromorphic. Let $\chi_\epsilon = \chi(|\zeta - z|^2/\epsilon)$. It is easy to see, by a blowup of $\mathbb{C}^n \times \mathbb{C}^n$ along the diagonal, that $k$ is almost semi-meromorphic on $\mathbb{C}^n \times \mathbb{C}^n$. Thus, by (2.7), if $k(\mu \otimes 1) \to k(\mu \otimes 1)$. In view of Proposition 2.1 it follows that $k(\mu \otimes 1)$ is pseudomeromorphic. Finally, if $W$ is a germ of a subvariety of $\mathbb{C}^n$ of positive codimension, then by (2.4) and (2.5),

$$1_W p_\ast(k \wedge \mu \otimes 1) = \lim_{\epsilon \to 0^+} p_\ast (1_{\mathbb{C}^n_\xi \times W} (\chi_\epsilon k \wedge (\mu \otimes 1)))$$

$$= \lim_{\epsilon \to 0^+} p_\ast (\chi_\epsilon k \wedge (1_{\mathbb{C}^n_\xi \times W} \mu \otimes 1))$$

$$= \lim_{\epsilon \to 0^+} p_\ast (\chi_\epsilon k \wedge (1_{\mathbb{C}^n_\xi \mu \otimes 1_W} 1)) = 0,$$

since $1_W 1 = 0$. Thus $K\mu$ is in $\mathcal{W}_Z$.

If $Z$ is not smooth, then we take a smooth modification $\pi: Z' \to Z$. For any $\mu$ in $\mathcal{PM}_Z$ there is some $\mu'$ in $\mathcal{PM}_{Z'}$ such that $\pi_\ast \mu' = \mu$, see [4, Proposition 1.2]. Since $\mu' = \tau + \bar{\partial}u$ with $\tau, u$ in $\mathcal{W}_{Z'}$, we have that $\mu = \pi_\ast \tau + \bar{\partial}\pi_\ast u$. \hfill $\square$

### 2.1 Pseudomeromorphic currents with support on a subvariety

Let $\Omega$ be an open set in $\mathbb{C}^N$ and let $Z$ be a (reduced) subvariety of pure dimension $n$. Let $\mathcal{PM}_Z^\Omega$ denote the sheaf of pseudomeromorphic currents $\tau$ on $\Omega$ with support on $Z$, and let $\mathcal{W}_Z^\Omega$ denote the subsheaf of $\mathcal{PM}_Z^\Omega$ of currents of bidegree $(N, \ast)$ with the SEP with respect to $Z$, i.e., such that $1_W \tau = 0$ for all germs $W$ of subvarieties of $Z$ of positive codimension. The sheaf $\mathcal{CH}^Z_\Omega$ of Coleff–Herrera currents on $Z$ is the subsheaf of $\mathcal{W}_Z^\Omega$ of $\bar{\partial}$-closed $(N, p)$-currents, where $p = N - n$.

**Remark 2.3** In [3,6] $\mathcal{CH}^Z_\Omega$ denotes the sheaf of pseudomeromorphic $(0, p)$-currents with support on $Z$ and the SEP with respect to $Z$. If this sheaf is tensored by the canonical bundle $K_\Omega$ we get the sheaf $\mathcal{CH}_\Omega^Z$ in this paper. Locally these sheaves are thus isomorphic via the mapping $\mu \mapsto \mu \wedge \alpha$, where $\alpha$ is a non-vanishing holomorphic $(N, 0)$-form. \hfill $\square$
We have the following direct consequence of Proposition 2.1.

**Proposition 2.4** Let $Z \subset \Omega$ be a subvariety of pure dimension, let $a$ be almost semi-meromorphic in $\Omega$, and assume that it is smooth generically on $Z$. If $\tau$ is in $\mathcal{W}_Z$, then $a \wedge \tau$ is in $\mathcal{W}_Z$ as well.

Assume that we have local coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^p$ in $\Omega$ such that $Z = \{w = 0\}$. We will use the short-hand notation

$$\bar{\partial} \frac{dw}{w^{\gamma+1}} := \bar{\partial} \frac{dw_1}{w_1^{\gamma_1+1}} \wedge \cdots \wedge \bar{\partial} \frac{dw_p}{w_p^{\gamma_p+1}}$$

for multiindices $\gamma = (\gamma_1, \ldots, \gamma_p)$ with $\gamma_j \geq 0$, and let $\gamma! := \gamma_1! \cdots \gamma_p!$. Notice that

$$\frac{1}{(2\pi i)^p} \bar{\partial} \frac{dw}{w^{\gamma+1}} \xi = \frac{1}{\gamma!} \int_z \frac{\partial^\gamma \xi}{\partial w^\gamma} (z, 0)$$

for test forms $\xi$. If $\tau$ is in $\mathcal{W}_Z$, then it follows by (2.5) and the fact that $\text{supp} \bar{\partial} (1/w^{\gamma+1}) = \{w = 0\}$ that $\tau \otimes \bar{\partial} (1/w^{\gamma+1})$ is in $\mathcal{W}_Z$. We have the following local structure result, see [11, Proposition 4.1 and (4.3)] and [10, Theorem 3.5].

**Proposition 2.5** Assume that we have local coordinates $(z, w)$ such that $Z = \{w = 0\}$. Then $\tau$ in $\mathcal{W}_Z$ has a unique representation as a finite sum

$$\tau = \sum_{\gamma} \tau_\gamma \wedge dz \otimes \bar{\partial} \frac{dw}{w^{\gamma+1}}, \quad \tau_\gamma \in \mathcal{W}_Z^{0,*},$$

(2.11)

where $dz := dz_1 \wedge \cdots \wedge dz_n$. If $\pi$ is the projection $(z, w) \mapsto z$, then

$$\tau_\gamma \wedge dz = (2\pi i)^{-p} \pi_*(w^\gamma \tau).$$

(2.12)

If in addition $\bar{\partial} \tau$ is in $\mathcal{W}_\Omega$ then its coefficients in the expansion (2.11) are $\bar{\partial} \tau_\gamma$, cf., (2.12). In particular, $\bar{\partial} \tau = 0$ if and only if $\bar{\partial} \tau_\gamma = 0$ for all $\gamma$.

Let us now consider the pairing between $\mathcal{W}_Z$ and germs $\phi$ at $Z$ of smooth (0, $\ast$)-forms. We assume that $Z$ is smooth and that we have coordinates $(z, w)$ as before, that $\tau$ is in $\mathcal{W}_\Omega$, and that (2.11) holds. Moreover, we assume that $\phi$ is a smooth (0, $\ast$)-form in a neighborhood of $Z$ in $\Omega$. For any positive integer $M$ we have the expansion

$$\phi = \sum_{|\alpha| < M} \phi_\alpha (z) \otimes w^\alpha + O\left(|w|^M\right) + O(\bar{w}, d\bar{w}),$$

(2.13)

where

$$\phi_\alpha (z) = \frac{1}{\alpha!} \frac{\partial^\alpha \phi}{\partial w^\alpha} (z, 0)$$
and $\mathcal{O}(\bar{w}, d\bar{w})$ denotes a sum of terms, each of which contains a factor $\bar{w}_j$ or $d\bar{w}_j$ for some $j$. If $M$ in (2.13) is chosen so that $\mathcal{O}(|w|^M)\tau = 0$, then

$$
\phi \wedge \tau = \sum_{\alpha \leq \gamma} \phi_{\alpha} \wedge \tau_{\gamma} \wedge dz \otimes \bar{\partial} \frac{d\bar{w}}{w^{\gamma - \alpha + 1}},
$$
i.e.,

$$
\phi \wedge \tau = \sum_{\ell \geq 0} \sum_{\gamma \geq 0} \phi_{\gamma} \wedge \tau_{\ell + \gamma} \wedge dz \otimes \bar{\partial} \frac{d\bar{w}}{w^{\ell + 1}}. \quad (2.14)
$$
Thus $\phi \wedge \tau = 0$ if and only if $\sum_{\gamma \geq 0} \phi_{\gamma} \wedge \tau_{\ell + \gamma} = 0$ for all $\ell$ (which is a finite number of conditions!).

### 2.2 Intrinsic pseudomeromorphic currents on a reduced subvariety

Currents on a reduced analytic space $Z$ are defined as the dual of the sheaf of test forms. If $i : Z \to Y$ is an embedding of a reduced space $Z$ into a smooth manifold $Y$, then the push-forward mapping $\tau \mapsto i_* \tau$ gives an isomorphism between currents $\tau$ on $Z$ and currents $\mu$ on $Y$ such that $\xi \wedge \mu = 0$ for all $\xi$ in $\mathcal{O}_Y$ such that $i^* \xi = 0$.

When defining pseudomeromorphic currents in the non-reduced case it is desirable that it coincides with the previous definition in case $Z$ is reduced. From [4, Theorem 1.1] we have the following description of pseudomeromorphicity from the point of view of an ambient smooth space.

**Proposition 2.6** Assume that we have an embedding $i : Z \to Y$ of a reduced space $Z$ into a smooth manifold $Y$.

(i) If $\tau$ is in $\mathcal{P}M_Z$, then $i_* \tau$ is in $\mathcal{P}M_Y$.

(ii) If $\tau$ is a current on $Z$ such that $i_* \tau$ is in $\mathcal{P}M_Y$ and $1_{Z^{\text{sing}}} i_* \tau = 0$, then $\tau$ is in $\mathcal{P}M_Z$.

Since $i_* (i^* \chi(|h|^2/\epsilon) \tau) = \chi(|h|^2/\epsilon) i_* \tau$ for any current $\tau$ on $Z$, we get by (2.1) that for a subvariety $V \subset U \subset Z$,

$$
1_V (i_* \tau) = i_* (1_V \tau), \quad (2.15)
$$
i.e., (2.4) holds also for an embedding $i : Z \to Y$. The condition $1_{Z^{\text{sing}}} i_* \tau = 0$ in (ii) is fulfilled if $i_* \tau$ has the SEP with respect to $Z$.

**Corollary 2.7** We have the isomorphism

$$
i_* : \mathcal{W}_Z^{n,*} \to \operatorname{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z),
$$
where $\mathcal{J}$ is the ideal defining $Z$ in $\Omega$.

Notice that $\operatorname{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$ is precisely the sheaf of $\mu$ in $\mathcal{W}_\Omega^Z$ such that $\mathcal{J} \mu = 0$. 

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In this way however, the first one is in general not minimal.

Let \( \phi \) and \( \phi_1, \ldots, \phi_m \) be currents in \( \mathcal{W}_Z \). If \( \phi = 0 \) on the set on \( Z_{\text{reg}} \) where \( \phi_1, \ldots, \phi_m \) are smooth, then \( \phi = 0. \)

3 Local embeddings of a non-reduced analytic space

Let \( X \) be an analytic space of pure dimension \( n \) with structure sheaf \( \mathcal{O}_X \) and let \( Z = X_{\text{red}} \) be the underlying reduced analytic space. For any point \( x \in X \) there is, by definition, an open set \( \Omega \subset \mathbb{C}^N \) and an ideal sheaf \( \mathcal{J} \subset \mathcal{O}_\Omega \) of pure dimension \( n \) with zero set \( Z \) such that \( \mathcal{O}_X \) is isomorphic to \( \mathcal{O}_\Omega/\mathcal{J} \), and all associated primes of \( \mathcal{J} \) at any point have dimension \( n \). We say that we have a local embedding \( i : X \to \Omega \subset \mathbb{C}^N \) at \( x \). There is a minimal such \( N \), called the Zariski embedding dimension \( \hat{N} \) of \( X \) at \( x \), and the associated embedding is said to be minimal. Any two minimal embeddings are identical up to a biholomorphism, and any embedding \( i : X \to \Omega \) has locally at \( x \) the form

\[
X \to \hat{\Omega} \to \Omega := \hat{\Omega} \times \mathcal{U}, \quad i = i \circ j, \tag{3.1}
\]

where \( j \) is minimal, \( \mathcal{U} \) is an open subset of \( \mathbb{C}^m_w, m = N - \hat{N} \), and the ideal in \( \Omega \) is \( \mathcal{J} = \mathcal{J} \otimes 1 + (w_1, \ldots, w_m) \). Notice that we then also have embeddings \( Z \to \hat{\Omega} \to \Omega \); however, the first one is in general not minimal.

Now consider a fixed local embedding \( i : X \to \Omega \subset \mathbb{C}^N \), assume that \( Z \) is smooth, and let \( (z, w) \) be coordinates in \( \Omega \) such that \( Z = \{ w = 0 \} \). We can identify \( \mathcal{O}_Z \) with holomorphic functions of \( z \), and we can define an injection

\[
\mathcal{O}_Z \to \mathcal{O}_X, \quad \phi(z) \mapsto \tilde{\phi}(z, w) = \phi(z).
\]

In this way \( \mathcal{O}_X \) becomes an \( \mathcal{O}_Z \)-module, which however depends on the choice of coordinates.

**Proposition 3.1** Assume that \( Z \) is smooth. Let \( \mathcal{O}_X \) have the \( \mathcal{O}_Z \)-module structure from a choice of local coordinates as above. Then \( \mathcal{O}_X \) is a coherent \( \mathcal{O}_Z \)-module, and \( \mathcal{O}_X \) is a free \( \mathcal{O}_Z \)-module at \( x \) if and only if \( \mathcal{O}_X \) is Cohen–Macaulay at \( x \).
Recall that \( f_1, \ldots, f_m \in R \) is a regular sequence on the \( R \)-module \( M \) if \( f_i \) is a non zero-divisor on \( M/(f_1, \ldots, f_{i-1}) \) for \( i = 1, \ldots, m \), and \( (f_1, \ldots, f_m)M \neq M \). If \( R \) is a local ring, then \( \text{depth}_R M \) is the maximal length \( d \) of a regular sequence \( f_1, \ldots, f_d \) such that \( f_1, \ldots, f_d \) are contained in the maximal ideal \( m \); furthermore, \( M \) is Cohen–Macaulay if \( \text{depth}_R M = \dim_R M \), where \( \dim_R M = \dim_R (R/\text{ann}_R M) \). If \( R \) is Cohen–Macaulay, and \( M \) has a finite free resolution over \( R \), then the Auslander–Buchsbaum formula, \([14, \text{Theorem 19.9}]\), gives that

\[
\text{depth}_R M + \text{pd}_R M = \dim_R R, \tag{3.2}
\]

where \( \text{pd}_R M \) is the length of a minimal free resolution of \( M \) over \( R \). In this case, \( M \) is Cohen–Macaulay as an \( R \)-module if and only if \( M \) has a free resolution over \( R \) of length codim \( M \).

**Remark 3.2** Notice that if we have a local embedding \( i : X \rightarrow \Omega \) as above, then the depth and dimension of \( \mathcal{O}_{X,x} = \mathcal{O}_{\Omega,x}/\mathcal{J} \) as an \( \mathcal{O}_{\Omega,x} \)-module coincide with the depth and dimension of \( \mathcal{O}_{X,x} \) as an \( \mathcal{O}_{X,x} \)-module. Thus \( \mathcal{O}_{X,x} \) is Cohen–Macaulay as an \( \mathcal{O}_{X,x} \)-module if and only if it is Cohen–Macaulay as an \( \mathcal{O}_{\Omega,x} \)-module, and this holds in turn if and only if \( \mathcal{O}_{\Omega,x}/\mathcal{J} \) has a free resolution of length \( n - n \). \[ \square \]

**Proof of Proposition 3.1** By the Nullstellensatz there is an \( M \) such that \( w^\alpha \) is in \( \mathcal{J} \) in some neighborhood of \( x \) if \( |\alpha| = M \). Let \( \mathcal{M} \subset \mathcal{O}_{\Omega} \) be the ideal generated by \( \{w^\alpha : |\alpha| = M\} \). Then \( \mathcal{M}' = \mathcal{O}_{\Omega}/\mathcal{M} \) is a free, finitely generated \( \mathcal{O}_{\Omega} \)-module. Thus, \( \mathcal{O}_{\Omega}/\mathcal{J} \simeq \mathcal{M}/\mathcal{J}\mathcal{M}' \) is a coherent \( \mathcal{O}_{Z} \)-module, which we note is generated by the finite set of monomials \( w^\alpha \) such that \( |\alpha| < M \).

We shall now show that

\[
\text{depth} \mathcal{O}_{X,x} \mathcal{O}_{X,x} = \text{depth} \mathcal{O}_{Z,x} \mathcal{O}_{X,x} \tag{3.3}
\]

and

\[
\dim \mathcal{O}_{X,x} \mathcal{O}_{X,x} = \dim \mathcal{O}_{Z,x} \mathcal{O}_{X,x}. \tag{3.4}
\]

We claim that a sequence \( f_1, \ldots, f_m \) in \( \mathcal{O}_{X,x} \) is regular (on \( \mathcal{O}_{X,x}\)) if and only if \( f_1, \ldots, f_m \in \mathcal{O}_{Z,x} \) is regular on \( \mathcal{O}_{X,x} \), where \( \tilde{f}_j(z) = f_j(z,0) \). In fact, since \( \mathcal{O}_{X,x} \) has pure dimension, a function \( g \in \mathcal{O}_{X,x} = \mathcal{O}_{\Omega,x}/\mathcal{J} \) is a non zero-divisor if and only if \( g \) is generically non-vanishing on each irreducible component of \( Z(\mathcal{J}) \). Thus \( f_1 \) is a non zero-divisor if and only if \( \tilde{f}_1 \) is. If it is, then \( \mathcal{O}_{X,x}/(f_1) = \mathcal{O}_{\Omega,x}/(\mathcal{J} + (f_1)) \) again has pure dimension. Thus the claim follows by induction, and the fact that \( Z(\mathcal{J} + (f_1, \ldots, f_k)) = Z(\mathcal{J} + (\tilde{f}_1, \ldots, \tilde{f}_k)) \). The claim immediately implies (3.3).

To see (3.4), we note first that \( \dim_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x} \) is just the usual (geometric) dimension of \( X \) or \( Z \), i.e., in this case, \( n \). Now, \( \text{ann}_{\mathcal{O}_{Z,x}} \mathcal{O}_{X,x} = \{0\} \), so \( \dim_{\mathcal{O}_{Z,x}} \mathcal{O}_{X,x} = \dim_{\mathcal{O}_{Z,x}} \mathcal{O}_{Z,x}/(\text{ann}_{\mathcal{O}_{Z,x}} \mathcal{O}_{X,x}) = \dim_{\mathcal{O}_{Z,x}} \mathcal{O}_{Z,x} = n \).

From (3.3) and (3.4) we conclude that \( \mathcal{O}_{X,x} \) is Cohen–Macaulay as an \( \mathcal{O}_{Z,x} \)-module if and only if it is Cohen–Macaulay (as an \( \mathcal{O}_{X,x} \)-module). Hence, by (3.2), with \( R = \mathcal{O}_{Z,x} \) and \( M = \mathcal{O}_{X,x} \),

\[
\text{depth}_{\mathcal{O}_{Z,x}} \mathcal{O}_{X,x} + \text{pd}_{\mathcal{O}_{Z,x}} \mathcal{O}_{X,x} = n,
\]
so $\mathcal{O}_{X,x}$ is Cohen–Macaulay as an $\mathcal{O}_Z$-module if and only if $\text{pd}\mathcal{O}_{Z,x} \mathcal{O}_{X,x} = 0$, that is, if and only if $\mathcal{O}_{X,x}$ is a free $\mathcal{O}_Z$-module.

In the proof above, we saw that $\mathcal{O}_X$ is generated (locally) as an $\mathcal{O}_Z$-module by all monomials $w^\alpha$ with $|\alpha| \leq M$ for some $M$.

**Corollary 3.3** Assume that $1, w^{\alpha_1}, \ldots, w^{\alpha_{v-1}}$ is a minimal set of generators at a given point $x$ (clearly 1 must be among the generators!). Then we have a unique representation (1.4) for each $\phi \in \mathcal{O}_{X,x}$ if and only if $\mathcal{O}_{X,x}$ is Cohen–Macaulay.

By coherence it follows that if $\mathcal{O}_{X,x}$ is free as an $\mathcal{O}_Z$-module, then $\mathcal{O}_{Z,x'}$ is free as an $\mathcal{O}_{Z,x'}$-module for all $x'$ in a neighborhood of $x$, and 1, $w^{\alpha_1}, \ldots, w^{\alpha_{v-1}}$ is a basis at each such $x'$.

**Example 3.4** Let $\mathcal{J}$ be the ideal in $\mathbb{C}^4$ generated by $(w_1^2, w_2^2, w_1 w_2, w_1 z_2 - w_2 z_1)$.

It is readily checked that $\mathcal{O}_X$ is a free $\mathcal{O}_Z$-module at a point on $Z = \{w_1 = w_2 = 0\}$ where $z_1$ or $z_2$ is $\neq 0$. If, say, $z_1 \neq 0$, then we can take 1, $w_1$ as generators. At the point $z = (0, 0)$, e.g., 1, $w_1$, $w_2$ form a minimal set of generators, and then $\mathcal{O}_X$ is not a free $\mathcal{O}_Z$-module, since there is a non-trivial relation between $w_1$ and $w_2$.

We claim that $\mathcal{O}_X$ has pure dimension. That is, we claim that there is no embedded associated prime ideal at $(0, 0)$; since $Z$ is irreducible, this is the same as saying that $\mathcal{J}$ is primary with respect to $Z$. To see the claim, let $\phi$ and $\psi$ be functions such that $\phi \psi$ is in $\mathcal{J}$ and $\psi$ is not in $\sqrt{\mathcal{J}}$. The latter assumption means, in view of the Nullstellensatz, that $\psi$ does not vanish identically on $Z$, i.e., $\psi = a(z) + \mathcal{O}(w)$, where $a$ does not vanish identically. Since in particular $\phi \psi$ must vanish on $Z$ it follows that $\phi = \mathcal{O}(w)$. It is now easy to see that $\phi$ is in $\mathcal{J}$. We conclude that $\mathcal{J}$ is primary.

The pure-dimensionality of $\mathcal{O}_X$ can also be rephrased in the following way: if $\phi$ is holomorphic and $0$ is generically, then $\phi = 0$. If we delete the generator $w_1 w_2$ from the definition of $\mathcal{J}$ in the example, then $\phi = w_1 w_2$ is 0 generically in $\mathcal{O}_\Omega/\mathcal{J}$ but is not identically zero. Thus $\mathcal{J}$ then has an embedded primary ideal at $(0, 0)$.

**Example 3.5** Let $\Omega = \mathbb{C}^2_{z,w}$ and $\mathcal{J} = (w^2)$ so that $Z = \{w = 0\}$. Then 1, $w$ is a basis for $\mathcal{O}_X = \mathcal{O}_{\mathbb{C}^2}/(w^2)$ so each function $\phi$ in $\mathcal{O}_X$ has a unique representation $a_0(z) \otimes 1 + a_1(z) \otimes w$. Let us consider the new coordinates $\zeta = z - w$, $\eta = w$. Then $\mathcal{J} = (\eta^2)$ and since

$$a_0(z) + a_1(z)w = a_0(\zeta + \eta) + a_1(\zeta + \eta)\eta = a_0(\zeta) + (\partial a_0/\partial \zeta)(\zeta)\eta + a_1(\zeta)\eta + \mathcal{J}$$

we have the representation $a_0(\zeta) \otimes 1 + (a_1(\zeta) + \partial a_0/\partial \zeta)(\zeta) \otimes \eta$ with respect to $(\zeta, \eta)$.

More generally, assume that, at a given point in $X_{reg} \subset \Omega$, we have two different choices $(z, w)$ and $(\zeta, \eta)$ of coordinates so that $Z = \{w = 0\} = \{\eta = 0\}$, and bases $1, \ldots, w^{\alpha_{v-1}}$ and $1, \ldots, \eta^{\beta_{v-1}}$ for $\mathcal{O}_X$ as a free module over $\mathcal{O}_Z$. Then there is a $v \times v$-matrix $L$ of holomorphic differential operators so that if $(a_j)$ is any tuple in $(\mathcal{O}_Z)^v$ and $(b_j) = L(a_j)$, then $a_0 \otimes 1 + \cdots + a_{v-1} \otimes w^{\alpha_{v-1}} = b_0 \otimes 1 + \cdots + b_{v-1} \otimes \eta^{\beta_{v-1}} + \mathcal{J}$.  

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4 Smooth \((0, *)\)-forms on a non-reduced space \(X\)

Let \(i : X \to \Omega\) be a local embedding of \(X\). In order to define the sheaf of smooth \((0, *)\)-forms on \(X\), in analogy with the reduced case, we have to state which smooth \((0, *)\)-forms \(\Phi\) in \(\Omega\) “vanish” on \(X\), or more formally, give a meaning to \(i^* \Phi = 0\). We will see, cf., Lemma 4.8 below, that the suitable requirement is that locally on \(X_{\text{reg}}\), \(\Phi\) belongs to \(\mathcal{E}_\Omega^{0, *} \mathcal{J} + \mathcal{E}_\Omega^{0, *} \mathcal{J}_Z + \mathcal{E}_\Omega^{0, *} \mathcal{J}_Z^Z\), where \(\mathcal{J}_Z\) is the ideal sheaf defining \(Z\). However, it turns out to be more convenient to represent the sheaf \(\ker i^*\) of such forms as the annihilator of certain residue currents, and this is the path we will follow. Moreover, these currents play a central role themselves later on.

The following classical duality result is fundamental for this paper; see, e.g., [3] for a discussion.

**Proposition 4.1** If \(\mathcal{J}\) has pure dimension, then

\[
\mathcal{J} = \text{ann} \mathcal{O}_\Omega \mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}H^Z_{\Omega}).
\]

That is, \(\phi\) is in \(\mathcal{J}\) if and only if \(\phi \mu = 0\) for all \(\mu\) in \(\mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}H^Z_{\Omega})\). It is also well-known, see, e.g., [3, Theorem 1.5], that

\[
\mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}H^Z_{\Omega}) \simeq \mathcal{E}xt^p(\mathcal{O}_\Omega/\mathcal{J}, K_{\Omega}),
\]

so \(\mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}H^Z_{\Omega})\) is a coherent analytic sheaf. Locally we thus have a finite number of generators \(\mu^1, \ldots, \mu^m\). In Example 6.9, we compute explicitly such generators for the ideal \(\mathcal{J}\) in Example 3.4.

Let \(\xi\) be a smooth \((0, *)\)-form in \(\Omega\). Without first giving meaning to \(i^*\), we define the sheaf \(\ker i^*\) by saying that \(\xi\) is in \(\ker i^*\) if

\[
\xi \wedge \mu = 0, \quad \mu \in \mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}H^Z_{\Omega}).
\]

Notice that if \(\xi\) is holomorphic, then, in view of the duality (4.1), \(\xi\) is in \(\ker i^*\) if and only if \(\xi\) is in \(\mathcal{J}\).

**Definition 4.2** We define the sheaf of smooth \((0, *)\)-forms on \(X\) as

\[
\mathcal{E}_X^{0, *} := \mathcal{E}_\Omega^{0, *}/\ker i^*.
\]

We will prove below that this sheaf is independent of the choice of embedding and thus intrinsic on \(X\).

Given \(\phi\) in \(\mathcal{E}_\Omega^{0, *}\), let \(i^* \phi\) be its image in \(\mathcal{E}_X^{0, *}\). In particular, \(i^* \xi = 0\) means that \(\xi\) belongs to \(\ker i^*\), which then motivates this notation. Notice that \(\ker i^*\) is a two-sided ideal in \(\mathcal{E}_\Omega^{0, *}\), i.e., if \(\phi\) is in \(\mathcal{E}_\Omega^{0, *}\) and \(\xi\) is in \(\ker i^*\), then \(\phi \wedge \xi\) and \(\xi \wedge \phi\) are in \(\ker i^*\). It follows that we have an induced wedge product on \(\mathcal{E}_X^{0, *}\) such that

\[
i^* (\phi \wedge \xi) = i^* \phi \wedge i^* \xi.
\]
Remark 4.3 It follows from Lemma 4.8 below that in case $X = Z$ is reduced, then $\xi$ is in $\text{Ker } i^*$ if and only its pullback to $X_{\text{reg}}$ vanishes. Thus our definition of $\mathcal{E}^{0,*}_X$ is consistent with the usual one in that case. \hfill $\Box$

Lemma 4.4 Using the notation of (3.1),

\[ \iota_*: \text{Hom}_{\hat{\Omega}}(\Omega/\hat{\Omega}, \mathcal{W}^Z_{\Omega}) \rightarrow \text{Hom}_{\Omega}(\Omega/\mathcal{J}, \mathcal{W}^Z_{\Omega}) \] (4.4)

is an isomorphism.

We can realize the mapping in (4.4) as the tensor product $\tau \mapsto \tau \wedge [w = 0]$, where $[w = 0]$ is the Lelong current in $\Omega$ associated with the submanifold $\{w = 0\}$.

Proof To begin with, $\iota_*$ maps pseudomeromorphic ($\hat{N}, \hat{p} + \ell$)-currents with support on $Z \subset \hat{\Omega}$ to pseudomeromorphic ($N, p + \ell$)-currents with support on $Z \subset \Omega$. If, in addition, $\tau$ has the SEP with respect to $Z$, then $\iota_\ast \tau$ has, as well by (2.15). Moreover, if $\tau$ is annihilated by $\hat{\mathcal{J}}$, then $\iota_\ast \tau$ is annihilated by $\mathcal{J} = \hat{\mathcal{J}} \otimes 1 + (w)$. Thus the mapping (4.4) is well-defined, and it is injective since $\iota$ is injective.

Now assume that $\mu$ is in $\text{Hom}(\Omega/\mathcal{J}, \mathcal{W}^Z_{\Omega})$. Arguing as in the proof of Corollary 2.7, we see that $\mu = \iota_\ast \hat{\mu}$ for a current $\hat{\mu}$ in $\mathcal{W}^Z_{\Omega}$. Since $\hat{\mathcal{J}} = \iota^* \mathcal{J}$ and $\mathcal{J} \mu = 0$, it follows that $\hat{\mathcal{J}} \mathcal{J} = 0$. Thus (4.4) is surjective. \hfill $\Box$

Since $\iota_*$ is injective, $\bar{\partial} \tau = 0$ if and only if $\bar{\partial} \iota_\ast \tau = 0$, and thus we get

Corollary 4.5 Using the notation of (3.1),

\[ \iota_*: \text{Hom}_{\hat{\Omega}}(\Omega/\hat{\mathcal{J}}, \mathcal{C}\mathcal{H}_{\hat{\Omega}}) \rightarrow \text{Hom}_{\Omega}(\Omega/\mathcal{J}, \mathcal{C}\mathcal{H}_{\Omega}) \] (4.5)

is an isomorphism.

Corollary 4.6 Using the notation in (3.1),

\[ \iota^*: \mathcal{E}^{0,*}_\Omega/\text{Ker } i^* \rightarrow \mathcal{E}^{0,*}_\hat{\Omega}/\text{Ker } j^*, \] (4.6)

is an isomorphism.

Proof It follows immediately from (4.5) that the mapping (4.6) is well-defined and injective. Given $\hat{\xi}$ in $\mathcal{E}^{0,*}_\Omega$, let $\xi = \hat{\xi} \otimes 1$. Then $\iota^* \xi = \hat{\xi}$ and so (4.6) is indeed surjective as well. \hfill $\Box$

It follows from (4.6) and (4.3) that the sheaf $\mathcal{E}^{0,*}_X$ is intrinsically defined on $X$. Since $\bar{\partial}$ maps $\text{Ker } i^*$ to $\text{Ker } i^*$, we have a well-defined operator $\bar{\partial}: \mathcal{E}^{0,*}_X \rightarrow \mathcal{E}^{0,*+1}_X$ such that $\bar{\partial}^2 = 0$. Unfortunately the sheaf complex so obtained is not exact in general, see, e.g., [6, Example 1.1] for a counterexample already in the reduced case.
4.1 Local representation on $X_{\text{reg}}$ of smooth forms

Recall that $X_{\text{reg}}$ is the open subset of $X$, where the underlying reduced space is smooth and $\mathcal{O}_X$ is Cohen–Macaulay. Let us fix some point in $X_{\text{reg}}$, and assume that we have local coordinates $(z, w)$ such that $Z = \{ w = 0 \}$. We also choose generators $1, w^{\alpha_1}, \ldots, w^{\alpha_\nu - 1}$ of $\mathcal{O}_X$ as a free $\mathcal{O}_Z$-module, which exist by Corollary 3.3, and generators $\mu^1, \ldots, \mu^m$ of $\mathcal{H}om(\mathcal{O}_Z/J, \mathcal{C}H^1_\mathcal{Z})$.

Notice that for each smooth $(0, *)$-form $\Phi$ in $\Omega$, $\Phi \mapsto \Phi \wedge \mu^\ell$ only depends on its class $\phi$ in $E_0^*, X$, and $\phi$ is in fact determined by these currents. By Proposition 2.5 each of these currents can (locally) be represented by a tuple of currents in $W_0^*, Z$. Putting all these tuples together, we get a tuple in $(W_0^*, Z)^M$, where $M = M_1 + \cdots + M_m$ and $M_j$ is the number of indices in (2.11) in the representation of $\mu^j$.

Recall from Corollary 3.3 that $\phi$ in $\mathcal{O}_X$ has a unique representative

\[ \hat{\phi} = \hat{\phi}_0 + \hat{\phi}_1 \otimes w^{\alpha_1} + \cdots + \hat{\phi}_{\nu-1} \otimes w^{\alpha_{\nu-1}}, \]  

where $\hat{\phi}_j$ are in $\mathcal{O}_Z$. We thus have an $\mathcal{O}_Z$-linear morphism

\[ T : (\mathcal{O}_Z)^{\nu} \rightarrow (\mathcal{O}_Z)^M. \]

The morphism is injective by Proposition 4.1, and the holomorphic matrix $T$ is therefore generically pointwise injective.

**Lemma 4.7** Each $\phi$ in $E^{0,*}_X$ has a unique representation (4.7) where $\hat{\phi}_j$ are in $E^{0,*}_Z$.

**Proof** To begin with notice that a given smooth $\phi$ must have at least one such representation. In fact, taking the finite Taylor expansion (2.13) we can forget about high order terms, since they must annihilate all the $\mu^j$, and the terms $\bar{w}$ and $d\bar{w}$ annihilate all the $\mu^j$ as well since they are pseudomeromorphic with support on $\{ w = 0 \}$. On the other hand, each $w^{\alpha}$ not in the set of generators must be of the form

\[ w^{\alpha} = a_0 + a_1 \otimes w^{\alpha_1} + \cdots + a_{\nu-1} \otimes w^{\alpha_{\nu-1}} + J, \]

and hence $\phi_\alpha \otimes w^{\alpha}$ is of the form (4.7). Thus the representation exists. To show uniqueness of the representation, we assume that $\hat{\phi}$ is in $\text{Ker } i^*$. Then the tuple $(\hat{\phi}_j)$ is mapped to 0 by the matrix $T$, and since $T$ is generically pointwise injective we conclude that each $\hat{\phi}_j$ vanishes. \qed

By the above proof we get

**Lemma 4.8** A smooth $(0, *)$-form $\xi$ in $\Omega$ is in $\text{Ker } i^*$ if and only if $\xi$ is in $E^{0,*}_\Omega J + E^{0,*}_\Omega \bar{J}_Z + E^{0,*}_\Omega d\bar{J}_Z$ on $X_{\text{reg}}$, where $J_Z$ is the radical sheaf of $Z$.

**Remark 4.9** This is not the same as saying that $\xi$ is in $E^{0,*}_\Omega J + E^{0,*}_\Omega \bar{J}_Z + E^{0,*}_\Omega d\bar{J}_Z$ at singular points. For a simple counterexample, consider $\phi = x\bar{y}$ on the reduced space $Z = \{ xy = 0 \} \subset \mathbb{C}^2$. © Springer
However, this can happen also when $Z$ is irreducible at a point. For example, the variety $Z = \{x^2y - z^2 = 0\} \subset \mathbb{C}^3$ is irreducible at 0, but there exist points arbitrarily close to 0 such that $(Z, z)$ is not irreducible. In this case, the ideal of smooth functions vanishing on $(Z, 0)$ is strictly larger than $\mathcal{E}^{0,0}_{\Omega}J_{Z,0} + \mathcal{E}^{0,0}_{\Omega}\tilde{J}_{Z,0}$ see [26, Proposition 9, Chapter IV], and [25, Theorem 3.10, Chapter VI].

Remark 4.10 It is easy to check that if we have the setting as in the discussion at the end of Sect. 3 but $(a_j)$ is instead a tuple in $\mathcal{E}^{0,*}_{Z}$, then we can still define $(b_j) = L(a_j)$ if we consider the derivatives in $L$ as Lie derivatives; in fact, since $a_j$ has no holomorphic differentials, $L$ only acts on the smooth coefficients, and it is easy to check that $a_0 \otimes 1 + \cdots + a_{\nu-1} \otimes w^{\alpha_{\nu-1}}$ and $b_0 \otimes 1 + \cdots + b_{\nu-1} \otimes \eta^{\beta_{\nu-1}}$ are equal modulo $\mathcal{E}^{0,*}_{\Omega}J + \mathcal{E}^{0,*}_{\Omega}\tilde{J}_{Z} + \mathcal{E}^{0,*}_{\Omega}d\tilde{J}_{Z}$, and thus define the same element in $\mathcal{E}^{0,*}_{X}$.

For future needs we prove in Sect. 6.1:

Lemma 4.11 The morphism $T$ is pointwise injective.

We can thus choose a holomorphic matrix $A$ such that

$$0 \to \mathcal{E}^{\nu}_{Z} \xrightarrow{T} \mathcal{E}^{M}_{Z} \xrightarrow{A} \mathcal{E}^{M'}_{Z} \tag{4.9}$$

is pointwise exact, and we can also find holomorphic matrices $S$ and $B$ such that

$$I = TS + BA. \tag{4.10}$$

5 Intrinsic $(n, *)$-currents on $X$

In analogy with the reduced case we have the following definition when $X$ is possibly non-reduced.

Definition 5.1 The sheaf $\mathcal{E}^{n,q}_{X}$ of $(n, q)$-currents on $X$ is the dual sheaf of $(0, n - q)$-test forms, i.e., forms in $\mathcal{E}^{0,n-q}_{X}$ with compact support.

Here, just as in the case of reduced spaces, cf., for example [19, Section 4.2], the space of smooth forms $\mathcal{E}^{n,q}_{X}$ is equipped with the quotient topology induced by a local embedding.

More concretely, this means that given an embedding $i : X \to \Omega$, currents $\psi$ in $\mathcal{E}^{n,q}_{X}$ precisely correspond to the $(N, N-n+q)$-currents $\tau$ on $\Omega$ that vanish on $\text{Ker} \ i^*$. Since $\text{Ker} \ i^*$ is a two-sided ideal in $\mathcal{E}^{0,*}_{\Omega}$ this holds if and only if $\xi \wedge \tau = 0$ for all $\xi$ in $\text{Ker} \ i^*$. It is natural to write $\tau = i_* \psi$ so that

$$i_* \psi \cdot \xi = \psi \cdot i^* \xi.$$

Clearly, we get a mapping $\tilde{\partial} : \mathcal{C}^{n,q}_{X} \to \mathcal{C}^{n,q+1}_{X}$ such that $\tilde{\partial}^2 = 0$.

Proposition 5.2 If $\tau$ is in $\mathcal{W}^{Z}_{\Omega}$ and $J\tau = 0$, then $\xi \wedge \tau = 0$ for all smooth $\xi$ such that $i^* \xi = 0$. 

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Proof. Because of the SEP it is enough to prove that $\xi \wedge \tau = 0$ on $X_{reg}$. By assumption, $J$ annihilates $\tau$, and by general properties of pseudomeromorphic currents, since $\tau$ has support on $Z$, $J_Z$ and $dJ_Z$ annihilate $\tau$. Thus the proposition follows by Lemma 4.8. \hfill $\Box$

Definition 5.3. An $(n, *)$-current $\psi$ on $X$ is in $W_{X}^{n,*}$ if $i_\ast \psi$ is in $\mathcal{H}om(\mathcal{O}_\Omega / J, W^Z_\Omega)$.

By definition we thus have the isomorphism

$$i_\ast : W_X^{n,*} \simeq \mathcal{H}om(\mathcal{O}_\Omega / J, W^Z_\Omega). \quad (5.1)$$

It follows from Lemma 4.4 that $W_X^{n,*}$ is intrinsically defined.

Remark 5.4. By Corollary 2.7, this definition is consistent with the previous definition of $W_X^{n,*}$ when $X$ is reduced. We cannot define $\mathcal{P}\mathcal{M}_X^{n,*}$ in the analogous simple way, cf., Remark 2.8. \hfill $\Box$

Definition 5.5. If $\psi$ is in $W_X^{n,*}$ and $a$ is an almost semi-meromorphic $(0, *)$-current on $\Omega$ that is generically smooth on $Z$, then the product $a \wedge \psi$ is a current in $W_X^{n,*}$ defined as follows: By definition, $i_\ast \psi$ is in $\mathcal{H}om(\mathcal{O}_\Omega / J, W^Z_\Omega)$ and by Proposition 2.4 and (2.8), one can define $a \wedge i_\ast \psi$ in $\mathcal{H}om(\mathcal{O}_\Omega / J, W^Z_\Omega)$; now $a \wedge \psi$ is the unique current in $W_X^{n,*}$ such that $i_\ast (a \wedge \psi) = a \wedge i_\ast \psi$.

By (2.7),

$$a \wedge \psi = \lim_{\epsilon \to 0^+} \chi(|h|^2/\epsilon) a \wedge \psi \quad (5.2)$$

if $h$ cuts out the Zariski singular support of $a$.

Definition 5.6. We let $\omega^0_X$ be the sheaf of $\bar{\partial}$-closed currents in $W_X^{n,0}$.

This sheaf corresponds via $i_\ast$ to $\bar{\partial}$-closed currents in $\mathcal{H}om(\mathcal{O}_\Omega / J, W^Z_\Omega)$ so we have the isomorphism

$$i_\ast : \omega^0_X \simeq \mathcal{H}om(\mathcal{O}_\Omega / J, \mathcal{C}H^Z_\Omega). \quad (5.3)$$

When $X$ is reduced $\omega^0_X$ is the sheaf of $(n, 0)$-forms that are $\bar{\partial}$-closed in the Barlet–Henkin–Passare sense. Let $\mu^1, \ldots, \mu^m$ be a set of generators for $\mathcal{H}om(\mathcal{O}_\Omega / J, \mathcal{C}H^Z_\Omega)$. They correspond via (5.3) to a set of generators $h^1, \ldots, h^m$ for the $\mathcal{O}_X$-module $\omega^n_X$.

We will also need a definition of $\mathcal{P}\mathcal{M}_X^{n,*}$. Let $\mathcal{F}_X$ be the subsheaf of $\mathcal{C}^\mathcal{n,*}_X$ of $\tau$ such that $i_\ast \tau$ is in $\mathcal{P}\mathcal{M}_\Omega^Z$. If $\tau$ is a section of $\mathcal{F}_X$ and $W$ is a subvariety of some open subset of $Z$, then $1_W i_\ast \tau$ is in $\mathcal{P}\mathcal{M}_\Omega^Z$, and by (2.3), $1_W i_\ast \tau$ is annihilated by $\mathcal{K}er i_\ast$. Hence we can define $1_W \tau$ as the unique current in $\mathcal{F}_X$ such that $i_\ast 1_W \tau = 1_W i_\ast \tau$. Clearly, $1_W \tau$ has support on $W$ and it is easily checked that the computational rule (2.3) holds also in $\mathcal{F}_X$. Moreover, $\mathcal{F}_X$ is closed under $\bar{\partial}$ since $\mathcal{P}\mathcal{M}_\Omega^Z$ is.

Definition 5.7. The sheaf $\mathcal{P}\mathcal{M}_X^{n,*}$ is the smallest subsheaf of $\mathcal{F}_X$ that contains $W_X^{n,*}$ and is closed under $\bar{\partial}$ and multiplication by $1_W$ for all germs $W$ of subvarieties of $Z$.

In view of Proposition 2.2 this definition coincides with the usual definition in case $X$ is reduced. It is readily checked that the dimension principle holds for $\mathcal{F}_X$, and hence it also holds for the (possibly smaller) sheaf $\mathcal{P}\mathcal{M}_X^{n,*}$, and in addition, (2.3) holds for forms $\xi$ in $\mathcal{E}^{0,*}_X$ and $\tau$ in $\mathcal{P}\mathcal{M}_X^{n,*}$. 

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6 Structure form on $X$

Let $i : X \to \Omega \subset \mathbb{C}^N$ be a local embedding as before, let $p = N - n$ be the codimension of $X$, and let $\mathcal{J}$ be the associated ideal sheaf on $\Omega$. In a slightly smaller set, still denoted $\Omega$, there is a free resolution

$$0 \to \mathcal{O}(E_{N_0}) \xrightarrow{f_{N_0}} \cdots \xrightarrow{f_3} \mathcal{O}(E_2) \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0)$$

(6.1)

of $\mathcal{O}_\Omega/\mathcal{J}$; here $E_k$ are trivial vector bundles over $\Omega$ and $E_0$ is the trivial line bundle. This resolution induces a complex of vector bundles

$$0 \to E_{N_0} \xrightarrow{f_{N_0}} \cdots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0$$

(6.2)

that is pointwise exact outside $Z$. Let $X_k$ be the set where $f_k$ does not have optimal rank. Then

$$\cdots \subset X_{k+1} \subset X_k \subset \cdots \subset X_{p+1} \subset X_p = \cdots = X_1 = Z;$$

these sets are independent of the choice of resolution and thus invariants of $\mathcal{O}_\Omega/\mathcal{J}$. Since $\mathcal{O}_\Omega/\mathcal{J}$ has pure codimension $p$,

$$\text{codim } X_k \geq k + 1, \quad \text{for } k \geq p + 1,$$

(6.3)

see [14, Corollary 20.14]. Thus there is a free resolution (6.1) if and only if $X_k = \emptyset$ for $k > N_0$. Unless $n = 0$ (which is not interesting in relation to the $\bar{\partial}$-equation), we can thus choose the resolution so that $N_0 \leq N - 1$. The variety $X$ is Cohen–Macaulay at a point $x$, i.e., the sheaf $\mathcal{O}_\Omega/\mathcal{J}$ is Cohen–Macaulay at $x$, if and only if $x \notin X_{p+1}$. Notice that $Z \setminus (X_{\text{reg}})_{\text{red}} = Z_{\text{sing}} \cup X_{p+1}$. The sets $X_k$ are independent of the choice of embedding, see [9, Lemma 4.2], and are thus intrinsic subvarieties of $Z = X_{\text{red}}$, and they reflect the complexity of the singularities of $X$.

Let us now choose Hermitian metrics on the bundles $E_k$. We then refer to (6.1) as a Hermitian resolution of $\mathcal{O}_\Omega/\mathcal{J}$ in $\Omega$. In $\Omega \setminus X_k$ we have a well-defined vector bundle morphism $\sigma_{k+1} : E_k \to E_{k+1}$, if we require that $\sigma_{k+1}$ vanishes on $(\text{Im } f_{k+1})^\perp$, takes values in $(\text{Ker } f_{k+1})^\perp$, and that $f_{k+1} \sigma_{k+1}$ is the identity on $\text{Im } f_{k+1}$. Following [7, Section 2] we define smooth $E_k$-valued forms

$$u_k = (\bar{\partial}\sigma_k) \cdots (\bar{\partial}\sigma_2) \sigma_1 = \sigma_k (\bar{\partial}\sigma_{k-1}) \cdots (\bar{\partial}\sigma_1)$$

(6.4)

in $\Omega \setminus X$; for the second equality, see [7, (2.3)]. We have that

$$f_1 u_1 = 1, \quad f_{k+1} u_{k+1} - \bar{\partial} u_k = 0, \quad k \geq 1,$$

in $\Omega \setminus X$. If $f := \oplus f_k$ and $u := \sum u_k$, then these relations can be written economically as $\nabla f u = 1$, where $\nabla f := f - \bar{\partial} f$. To make the algebraic machinery work properly one has to introduce a superstructure on the bundle $E =: \oplus E_k$ so that vectors in $E_{2k}$ are
even and vectors in \( E_{2k+1} \) are odd; hence \( f, \sigma := \oplus \sigma_k \), and \( u := \sum u_k \) are odd. For details, see [7]. It turns out that \( u \) has a (necessarily unique) almost semi-meromorphic extension \( U \) to \( \Omega \). The residue current \( R \) is defined by the relation

\[
\nabla_f U = 1 - R. \tag{6.5}
\]

It follows directly that \( R \) is \( \nabla_f \)-closed. In addition, \( R \) has support on \( Z \) and is a sum \( \sum R_k \), where \( R_k \) is a pseudomeromorphic \( E_k \)-valued current of bidegree \((0, k)\). It follows from the dimension principle that \( R = R_p + R_{p+1} + \cdots + R_N \). If we choose a free resolution that ends at level \( N - 1 \), then \( R_N = 0 \). If \( X \) is Cohen–Macaulay and \( N_0 = p \) in (6.1), then \( R = R_p \), and the \( \nabla_f \)-closedness implies that \( R \) is \( \bar{\partial} \)-closed.

If \( \phi \) is in \( J \) then \( \phi R = 0 \) and in fact, \( J = \text{ann} R \), see [7, Theorem 1.1].

**Remark 6.1** In case \( J \) is generated by the single non-trivial function \( f \), then we have the free resolution \( 0 \to \mathcal{O}_\Omega \xrightarrow{f} \mathcal{O}_\Omega \to \mathcal{O}_\Omega/(f) \to 0 \); thus \( U \) is just the principal value current \( 1/f \) and \( R = \bar{\partial}(1/f) \). More generally, if \( f = (f_1, \ldots, f_p) \) is a complete intersection, then

\[
R = \bar{\partial} \frac{1}{f_p} \wedge \cdots \wedge \bar{\partial} \frac{1}{f_1},
\]

where the right hand side is the so-called Coleff–Herrera product of \( f \), see for example [1, Corollary 3.5].

There are almost semi-meromorphic \( \alpha_k \) in \( \Omega \), cf., [7, Section 2] and the proof of [6, Proposition 3.3], that are smooth outside \( X_k \), such that

\[
R_{k+1} = \alpha_{k+1} R_k \tag{6.6}
\]

outside \( X_{k+1} \) for \( k \geq p \). In view of (6.3) and the dimension principle, \( 1_{X_{k+1}} R_{k+1} = 0 \) and hence (6.6) holds across \( X_{k+1} \), i.e., \( R_{k+1} \) is indeed equal to the product \( \alpha_{k+1} R_k \) in the sense of Proposition 2.1. In particular, it follows that \( R_k \) has the SEP with respect to \( Z \).

In this section, we let \((z_1, \ldots, z_N)\) denote coordinates on \( \mathbb{C}^N \), and let \( dz := dz_1 \wedge \cdots \wedge dz_N \).

**Lemma 6.2** There is a matrix of almost semi-meromorphic currents \( b \) such that

\[
R \wedge dz = b \mu, \tag{6.7}
\]

where \( \mu \) is a tuple of currents in \( \text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_Z^\infty) \).

**Proof** As in [6, Section 3], see also [32, Proposition 3.2], one can prove that \( R_p = \sigma_F \mu \), where \( \mu \) is a tuple of currents in \( \text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_Z^\infty) \) and \( \sigma_F \) is an almost semi-meromorphic current that is smooth outside \( X_{p+1} \).

Let \( b_p = \sigma_F \) and \( b_k = \alpha_k \cdots \alpha_{p+1} \sigma_F \) for \( k \geq p + 1 \). Then each \( b_k \) is almost semi-meromorphic, cf., [10, Section 4.1]. In view of (6.6) we have that \( R_k = b_k \mu \) outside \( X_{p+1} \) since \( b_k \) is smooth there. It follows by the SEP that it holds across \( X_{p+1} \) as well since \( R_k \) has the SEP with respect to \( Z \). We then take \( b = b_p + b_{p+1} + \cdots \). \( \square \)
By Proposition 2.4 we get

**Corollary 6.3** The current $R \wedge dz$ is in $\mathcal{H}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{W}_{\Omega}^2)$.

From Lemma 6.2, Corollary 6.3, (5.1), and (5.3) we get the following analogue to [6, Proposition 3.3]:

**Proposition 6.4** Let (6.1) be a Hermitian resolution of $\mathcal{O}_\Omega/\mathcal{J}$ in $\mathcal{W}_{\Omega}$, and let $R$ be the associated residue current. Then there exists a (unique) current $\omega$ in $W_{X}^n$ such that

$$i_*\omega = R \wedge dz.$$  \hfill (6.8)

There is a matrix $b$ of almost semi-meromorphic $(0,*)$-currents in $\Omega$, smooth outside of $X_{p+1}$, and a tuple $\vartheta$ of currents in $\Omega^n_X$ such that

$$\omega = b\vartheta.$$  \hfill (6.9)

More precisely, $\omega = \omega_0 + \omega_1 + \cdots + \omega_n$, where $\omega_k \in \mathcal{W}^{m,k}(X, E_{p+k})$, and if $f^j := f_{p+j}$, then

$$f^0\omega_0 = 0, \quad f^{j+1}\omega_{j+1} - \bar{\partial}\omega_j = 0, \text{ for } j \geq 0.$$  \hfill (6.10)

We will also use the short-hand notation $\nabla_f \omega = 0$. As in the reduced case, following [6], we say that $\omega$ is a *structure form* for $X$. The products in (6.9) are defined according to Definition 5.5.

**Remark 6.5** Recall that $X_{p+1} = \emptyset$ if $X$ is Cohen–Macaulay, so in that case $\omega = b\vartheta$, where $b$ is smooth. If we take a free resolution of length $p$, then $\omega = \omega_0$, and $\bar{\partial}\omega_0 = f^1\omega_1 = 0$, so $\omega$ is in $\Omega^n_X$. \hfill \qed

**Remark 6.6** If $X = \{ f = 0 \}$ is a reduced hypersurface in $\Omega$, then $R = \bar{\partial}(1/f)$ and $\omega$ is the classical Poincaré residue form on $X$ associated with $f$, which is a meromorphic form on $X$. More generally, if $X$ is reduced, since forms in $\Omega^n_X$ are then meromorphic, by (6.9), $\omega$ can be represented by almost semi-meromorphic forms on $X$.

We now consider the case when $X$ is non-reduced. We recall that a differential operator is a Noetherian operator for an ideal $J$ if $L\varphi \in \sqrt{J}$ for all $\varphi \in J$. It is proved by Björk, [13], see also [32, Theorem 2.2], that if $\mu \in \mathcal{H}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}\mathcal{H}_{\Omega})$, then there exists a Noetherian operator $L$ for $\mathcal{J}$ with meromorphic coefficients such that the action of $\mu$ on $\xi$ equals the integral of $L\xi$ over $Z$. By (5.3), the action of $h$ in $\omega^n_X$ on $\xi$ in $\phi^0_X$ can then be expressed as

$$h_\xi = \int_Z L\xi.$$  

1 In [6, Proposition 3.3], the sum ends with $\omega_{n-1}$ instead of $\omega_n$, which, as remarked above, one can indeed assume when $n \geq 1$ and the resolution is chosen to be of length $\leq N - 1$. 

\begin{footnotesize}
\textsuperscript{1} \end{footnotesize}
One can then verify using this formula and (6.9) that the action of the structure form \( \omega \) on a test form \( \xi \) in \( \phi_X^{0,*} \) equals

\[
\omega \cdot \xi = \int_Z \tilde{L} \xi,
\]
where \( \tilde{L} \) is now a tuple of Noetherian operators for \( J \) with almost semi-meromorphic coefficients, cf., [32, Section 4].

Notice that (6.1) gives rise to the dual Hermitian complex

\[
0 \to \mathcal{O}(E^*_0) \xrightarrow{f_1^*} \cdots \xrightarrow{f_p^*} \mathcal{O}(E^*_{p-1}) \xrightarrow{f_p^*} \mathcal{O}(E^*_p) \xrightarrow{f_{p+1}^*} \cdots .
\]

(6.11)

Let \( \xi = \xi_0 \wedge dz \) be a holomorphic section of the sheaf

\[
\mathcal{H}om(E_p, K_{\Omega}) \simeq \mathcal{O}(E^*_p) \otimes \mathcal{O}(-K_{\Omega})
\]
such that \( f_{p+1}^* \xi_0 = 0 \). Then \( \tilde{\delta}(\xi_0 \omega_0) = \pm \xi_0 \tilde{\delta} \omega_0 = \pm \xi_0 f_{p+1} \omega_1 = \pm (f_{p+1}^* \xi_0) \omega_1 = 0 \), so that \( \xi_0 \omega_0 \) is in \( \omega^n_X \). Moreover, if \( \xi_0 = f_p^* \eta \) for \( \eta \) in \( \mathcal{O}(E^*_{p-1}) \), then \( \xi_0 \omega_0 = f_p^* \eta \omega_0 = \pm \eta f_p \omega_0 = 0 \). We thus have a sheaf mapping

\[
\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_{\Omega})) \to \omega^n_X, \quad \xi_0 \wedge dz \mapsto \xi_0 \omega_0.
\]

(6.12)

**Proposition 6.7** The mapping (6.12) is an isomorphism, which establishes an intrinsic isomorphism

\[
\mathcal{E}xt^p(\mathcal{O}_{\Omega}/J, K_{\Omega}) \simeq \omega^n_X.
\]

(6.13)

**Proof** If \( h \) is in \( \omega^n_X \), then \( i_* h \) is in \( \mathcal{H}om(\mathcal{O}_{\Omega}/J, \mathcal{C}H^Z_{\Omega}) \). We have mappings

\[
\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_{\Omega})) \to \omega^n_X \xrightarrow{\sim} \mathcal{H}om(\mathcal{O}_{\Omega}/J, \mathcal{C}H^Z_{\Omega}),
\]

(6.14)

where the first mapping is (6.12), and the second is \( h \mapsto i_* h \). In view of (6.8), the composed mapping is \( \xi = \xi_0 \wedge dz \mapsto \xi R_p = \xi_0 R_p \wedge dz \). This mapping is an intrinsic isomorphism

\[
\mathcal{E}xt^p(\mathcal{O}_{\Omega}/J, K_{\Omega}) \simeq \mathcal{H}om(\mathcal{O}_{\Omega}/J, \mathcal{C}H^Z_{\Omega})
\]

according to [3, Theorem 1.5]. It follows that (6.12) also establishes an intrinsic isomorphism.

In particular it follows that \( \omega^n_X \) is coherent, and we have:

If \( \xi^1, \ldots, \xi^m \) are generators of \( \mathcal{H}^p(\mathcal{H}om(E_\bullet, K_{\Omega})) \), where \( \xi^\ell = \xi_0^\ell \wedge dz \), then \( h^\ell := \xi_0^\ell \omega_0, \ \ell = 1, \ldots, m \), generate the \( \mathcal{O}_X \)-module \( \omega^n_X \), and \( \mu^\ell = i_* h^\ell = \xi^\ell R_p \) generate the \( \mathcal{O}_{\Omega} \)-module \( \mathcal{H}om(\mathcal{O}_{\Omega}/J, \mathcal{C}H^Z_{\Omega}) \).

\[\text{2} \quad \text{There is a superstructure involved, with respect to which} \ R_p \ \text{has even degree, and therefore} \ dz \wedge R_p = R_p \wedge dz, \ \text{explaining the lack of a sign in the last equality, see [6] or [7].}\]
Remark 6.8 The isomorphism

\[ H^p(\text{Hom}(E_\bullet, K_\Omega)) \xrightarrow{\sim} \text{Hom}(\mathcal{O}_\Omega/\mathcal{I}, \mathcal{CH}_{\mathcal{O}_\Omega}^Z) \]  

(6.15)

was well-known since long ago, the contribution in [3] was the realization \( \xi \mapsto \xi R_p. \)

We give here an example where we can explicitly compute generators of \( \text{Hom}(\mathcal{O}_\Omega/\mathcal{I}, \mathcal{CH}_{\mathcal{O}_\Omega}^Z) \).

Example 6.9 Let \( \mathcal{J} \) be as in Example 3.4. We claim that \( \text{Hom}(\mathcal{O}_\Omega/\mathcal{I}, \mathcal{CH}_{\mathcal{O}_\Omega}^Z) \) is generated by

\[ \mu_1 := \bar{\partial} \frac{1}{w_1} \wedge \bar{\partial} \frac{1}{w_2} \wedge dz \wedge dw \quad \text{and} \quad \mu_2 := \left( z_1 \bar{\partial} \frac{1}{w_1} \wedge \bar{\partial} \frac{1}{w_2} + z_2 \bar{\partial} \frac{1}{w_1} \wedge \bar{\partial} \frac{1}{w_2} \right) \wedge dz \wedge dw. \]

In order to prove this claim, we use the comparison formula for residue currents from [21], which states that if \( \mathcal{O}(F_\bullet) \) and \( \mathcal{O}(E_\bullet) \) are free resolutions of \( \mathcal{O}_\Omega/I \) and \( \mathcal{O}_\Omega/J \), respectively, where \( I \) and \( J \) have codimension \( \geq p \), and \( a : F_\bullet \to E_\bullet \) is a morphism of complexes, then there exists a \( \text{Hom}(F_0, E_{p+1}) \)-valued current \( M_{p+1} \) such that \( R^E_p a_0 = a_p R^F_p + f_{p+1} M_{p+1} \). If \( \xi \) is in \( \text{Ker} f_{p+1}^* \), we thus get that

\[ \xi R^E_p a_0 = \xi a_p R^F_p. \]  

(6.16)

We will apply this with \( \mathcal{O}_\Omega(E_\bullet) \) as the free resolution

\[ 0 \to \mathcal{O}_\Omega \xrightarrow{f_3} \mathcal{O}_\Omega^4 \xrightarrow{f_2} \mathcal{O}_\Omega^4 \xrightarrow{f_1} \mathcal{O}_\Omega \to \mathcal{O}_\Omega/J \to 0, \]

where

\[ f_3 = \begin{bmatrix} w_2 \\ -w_1 \\ z_2 \\ -z_1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} z_2 & 0 & -w_2 & 0 \\ -z_1 & z_2 & w_1 & -w_2 \\ 0 & -z_1 & 0 & w_1 \\ -w_1 & -w_2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad f_1 = \begin{bmatrix} w_2^2 \\ w_1 w_2 \\ w_2^2 \\ z_2 w_1 - z_1 w_2 \end{bmatrix}. \]

and the Koszul complex \( (F, \delta_{w^2}) \) generated by \( w^2 := (w_1^2, w_2^2) \), which is a free resolution of \( \mathcal{O}/(w_1^2, w_2^2) \). We then take the morphism of complexes \( a : F_\bullet \to E_\bullet \) given by.
\[ a_2 = \begin{bmatrix} 0 \\ 0 \\ w_2 \\ w_1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad a_0 = [1]. \]

Since the current \( R_p^F \) is equal to the Coleff–Herrera product \( \tilde{\partial}(1/w_1^2) \wedge \tilde{\partial}(1/w_2^2) \), cf., Remark 6.1, we thus get by (6.16) and Remark 6.8 that \( \mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}H^Z_{\Omega}) \) is generated by

\[
(Ker f_3^*) a_2 \tilde{\partial} \frac{1}{w_1^2} \wedge \tilde{\partial} \frac{1}{w_2^2}.
\]

A straightforward calculation gives the generators \( \mu_1 \) and \( \mu_2 \) above. \( \square \)

### 6.1 Proof of Lemma 4.11

Since \( T \) is generically injective, it is clearly injective if \( n = 0 \). We are going to reduce to this case. Fix the point \( 0 \in Z \) and let \( \mathcal{I} \) be the ideal generated by \( z = (z_1, \ldots, z_n) \).

Let \( \mathcal{O}(E_\bullet) \) be a free Hermitian resolution of \( \mathcal{O}_\Omega/\mathcal{J} \) of minimal length \( p = N - n \) at 0 and let \( R^E \) be the associated residue current. Recall that the canonical isomorphism (6.15) is realized by \( \xi \mapsto \xi R_p^E \). Let \( F_\bullet \) be the Koszul complex generated by \( z \); then \( \mathcal{O}(F_\bullet) \) is a free resolution of \( \mathcal{O}_\Omega/\mathcal{I} \). Since \( \mathcal{J} \) and \( \mathcal{I} \) are Cohen–Macaulay and intersect properly in \( \Omega \), the complex \( \mathcal{O}_\Omega((E \otimes F)_\bullet) \) is a free resolution of \( \mathcal{O}_\Omega/(\mathcal{J} + \mathcal{I}) \), and the corresponding residue current is

\[
R_N^{E \otimes F} = R_p^E \wedge R_n^F
\]

according to [2, Theorem 4.2]. From [3, Theorem 1.5] again it follows that the canonical isomorphism

\[
\mathcal{H}^N(\mathcal{H}om((E \otimes F)_\bullet, K_\Omega)) \rightarrow \mathcal{H}om(\mathcal{O}_\Omega/(\mathcal{J} + \mathcal{I}), \mathcal{C}H^{[0]}_{\Omega})
\]

is given by \( \eta \mapsto \eta R_N^{E \otimes F} \).

Let \( \mu_1, \ldots, \mu_m \) be a minimal set of generators for \( \mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}H^Z_{\Omega}) \) at 0. Then \( \mu^i = \xi^i R_p^E \), where \( \xi^i \) is a minimal set of generators for \( \mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \). Notice that

\[
\mathcal{H}^N(\mathcal{H}om((E \otimes F)_\bullet, K_\Omega)) = \mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \otimes \mathcal{H}^0(\mathcal{H}om(F_\bullet, K_\Omega)).
\]

Since \( \mathcal{H}^n(\mathcal{H}om(F_\bullet, K_\Omega)) \) is generated by 1, it follows that \( \mathcal{H}^N(\mathcal{H}om((E \otimes F)_\bullet, K_\Omega)) \) is generated by \( \xi^j \otimes 1 \). We conclude that \( \mathcal{H}om(\mathcal{O}_\Omega/(\mathcal{J} + \mathcal{I}), \mathcal{C}H^{[0]}_{\Omega}) \) is generated by \( \xi^j \otimes 1 \cdot R_p^E \wedge R_n^F = \mu^j \wedge \mu^z \), \( j = 1, \ldots, m \), where \( R_n^F = \mu^z = \tilde{\partial}(1/z^1) \).

If \( 1, \ldots, w^{\nu-1} \) is a basis for \( \mathcal{O}_\Omega/\mathcal{J} \) as an \( \mathcal{O}_\Omega \)-module, then it is also a basis for \( \mathcal{O}_{X_0} := \mathcal{O}_\Omega/(\mathcal{J} + \mathcal{I}) \) as a module over \( \mathcal{O}_{[0]} \simeq \mathbb{C} \). Since \( \phi \tilde{\partial}(1/z^1) = \phi(0, \cdot) \tilde{\partial}(1/z^1) \),
we have that
\[
\phi(z, w) \mu^j \wedge \mu^z = \phi(0, w) \sum a_j^i (0) \bar{\partial} \frac{1}{w^{\ell + 1}} \wedge \bar{\partial} \frac{1}{z^1}.
\]

The morphism constructed in (4.8) for \(X_0\) instead of \(X\) is then \(T_0 = T(0)\), where \(T\) is the morphism (4.8) for \(X\). Thus \(T(0)\) is injective.

7 The intrinsic sheaf \(\mathcal{W}^{0, *}_X\) on \(X\)

Our aim is to find a fine resolution of \(\mathcal{O}_X\) and since the complex (1.1) is not exact in general when \(X\) is singular we have to consider larger fine sheaves; we first define sheaves \(\mathcal{W}^{0, *}_X \supset \mathcal{E}^{0, *}_X\) of \((0, *)\)-currents. Given a local embedding \(i: X \to \Omega\) at a point on \(X_{reg}\) and local coordinates \((z, w)\) as before, it is natural, in view of Lemma 4.7, to require that an element in \(\mathcal{W}^{0, *}_X\) shall have a unique representation
\[
\phi = \hat{\phi}_0 \otimes 1 + \hat{\phi}_1 \otimes w^{\alpha_1} + \cdots + \hat{\phi}_{v-1} \otimes w^{\alpha_{v-1}}, \tag{7.1}
\]
where \(\hat{\phi}_j\) are in \(\mathcal{W}^{0, *}_Z\). In view of Remark 4.10 we should expect that the same transformation rules hold as for smooth \((0, *)\)-forms. In particular it is then necessary that \(\mathcal{W}^{0, *}_Z\) is closed under the action of holomorphic differential operators, which in fact is true, see Proposition 7.11 below. We must also define a reasonable extension of these sheaves across \(X_{sing}\). Before we present our formal definition we make a preliminary observation.

**Lemma 7.1** If \(\phi\) has the form (7.1) and \(\tau\) is in \(\mathcal{H}(\mathcal{O}_\Omega / \mathcal{J}, \mathcal{C} \mathcal{H}^Z_\Omega)\), expressed in the form (2.11), then
\[
\phi \wedge \tau := \sum_i \sum_{\gamma \geq \alpha_i} \hat{\phi}_i \wedge \tau_\gamma \wedge d z \otimes \bar{\partial} \frac{d w}{w^{\gamma - \alpha_i + 1}} \tag{7.2}
\]
is in \(\mathcal{H}(\mathcal{O}_\Omega / \mathcal{J}, \mathcal{W}^Z_\Omega)\).

**Proof** The right hand side defines a current in \(\mathcal{W}^Z_\Omega\) since \(\hat{\phi}_i\) are in \(\mathcal{W}^{0, *}_Z\) and \(\tau_\gamma\) are in \(\mathcal{O}_Z\). We have to prove that it is annihilated by \(\mathcal{J}\). Take \(\xi\) in \(\mathcal{J}\). On the subset of \(Z\) where \(\hat{\phi}_0, \ldots, \hat{\phi}_{v-1}\) are all smooth, \(\phi \wedge \tau\), as defined above, is just multiplication of the smooth form \(\phi\) by \(\tau\), and thus \(\xi \phi \wedge \tau = 0\) there. We have a unique representation
\[
\xi \phi \wedge \tau = \sum_{\ell \geq 0} a_\ell (z) \wedge d z \otimes \bar{\partial} \frac{d w}{w^{\ell + 1}},
\]
with \(a_\ell\) in \(\mathcal{W}^{0, *}_Z\). Since \(a_\ell\) vanish on the set where all \(\hat{\phi}_j\) are smooth, we conclude from Proposition 2.9 that \(a_\ell\) vanish identically. It follows that \(\xi \phi \wedge \tau = 0\).
If $\phi$ has the form (7.1) in a neighborhood of some point $x \in X_{\text{reg}}$ and $h$ is in $\omega^n_X$, then we get an element $\phi \wedge h$ in $\mathcal{V}_X^{n*}$ defined by $i_*(\phi \wedge h) = \phi \wedge i_*h$. It follows that $\phi$ in this way defines an element in $\mathcal{H}om_{\mathcal{O}_X}(\omega^n_X, \mathcal{W}_X^{n*})$. This sheaf is global and invariantly defined and so we can make the following global definition.

**Definition 7.2** $\mathcal{W}_X^{0,*} = \mathcal{H}om_{\mathcal{O}_X}(\omega^n_X, \mathcal{W}_X^{n*})$.

If $\phi$ is in $\mathcal{W}_X^{0,*}$ and $h$ is in $\omega^n_X$, we consider $\phi(h)$ as the product of $\phi$ and $h$, and sometimes write it as $\phi \wedge h$.

Since $\mathcal{W}_X^{n,*}$ are $\mathcal{O}_X^{n,*}$-modules, $\mathcal{W}_X^{0,*}$ are as well. Before we investigate these sheaves further, we give some motivation for the definition. First notice that we have a natural injection, cf., Proposition 4.1,

\[ \mathcal{O}_X \rightarrow \mathcal{H}om(\omega^n_X, \omega^n_X), \quad \phi \mapsto (h \mapsto \phi h). \]  

(7.3)

**Theorem 7.3** The mapping (7.3) is an isomorphism in the Zariski-open subset of $X$ where it is $S_2$.

This is the subset of $X$ where $\text{codim} X_k \geq k + 2$, $k \geq p + 1$, cf., Sect. 6. Thus it contains all points $x$ such that $\mathcal{O}_{X,x}$ is Cohen–Macaulay. In particular, (7.3) is an isomorphism in $X_{\text{reg}}$.

Theorem 7.3 is a consequence of the results in [22]. If $X$ has pure dimension $p$, there is an injective mapping

\[ \mathcal{O}_X \rightarrow \mathcal{H}om\left(\mathcal{E}xt^p(\mathcal{O}_X, \mathcal{K}_\Omega), \mathcal{C}H^0_{\Omega}\right), \]  

(7.4)

which by [22, Theorem 1.2 and Remark 6.11] is an isomorphism if and only if $\mathcal{O}_X$ is $S_2$. Since the image of such a morphism must be annihilated by $\mathcal{J}$ by linearity, it is indeed a morphism

\[ \mathcal{O}_X \rightarrow \mathcal{H}om\left(\mathcal{E}xt^p(\mathcal{O}_X, \mathcal{K}_\Omega), \mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}H^0_{\Omega})\right). \]  

(7.5)

In view of (4.2) and (5.3), (7.5) corresponds to a morphism $\mathcal{O}_X \rightarrow \mathcal{H}om(\omega^n_X, \omega^n_X)$, and the fact that it is the morphism (7.3) is a rather simple consequence of the definition of the morphism (7.4) in [22, (6.9)].

As mentioned in the introduction, Theorem 7.3 can be seen as a reformulation of a classical result of Roos, [30], which is the same statement about the injection

\[ \mathcal{O}_\Omega/\mathcal{J} \rightarrow \mathcal{E}xt^p(\mathcal{E}xt^p(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{K}_\Omega), \mathcal{K}_\Omega); \]  

(7.6)

here we assume that the ideal has pure dimension. The equivalence of the morphisms (7.4) and (7.6) is discussed in [22, Corollary 1.4].

Let us now consider the case when $X$ is reduced. Since sections of $\omega^n_X$ are meromorphic, see [6, Example 2.8], and thus almost semi-meromorphic and generically smooth, by Proposition 2.4 (with $Z = X = \Omega$) we can extend (7.3) to a morphism

\[ \mathcal{W}_X^{0,*} \rightarrow \mathcal{H}om(\omega^n_X, \mathcal{W}_X^{n,*}). \]  

(7.7)
Lemma 7.4  When $X$ is reduced (7.7) is an isomorphism.

Thus Definition 7.2 is consistent with the previous definition of $\mathcal{W}_X^{0,*}$ when $X$ is reduced.

Proof  Clearly each $\phi$ in $\mathcal{W}_X^{0,*}$ defines an element $\alpha$ in $\text{Hom}(\Omega^n_X, \mathcal{W}_X^{0,*})$ by $h \mapsto \phi \wedge h$. If we apply this to a generically nonvanishing $h$ we see by the SEP that (7.7) is injective.

For the surjectivity, take $\alpha$ in $\text{Hom}(\Omega^n_X, \mathcal{W}_X^{0,*})$. If $h'$ is nonvanishing at a point on $X_{\text{reg}}$, then it generates $\Omega^n_X$ and thus $\alpha$ is determined by $\phi := \alpha h'$ there. By [10, Theorem 3.7], $\phi = \psi \wedge h'$ for a unique current $\psi$ in $\mathcal{W}_X^{0,*}$ so by $\mathcal{O}_X$-linearity $\alpha h = \psi \wedge h$ for any $h$. Hence, $\psi$ is well-defined as a current in $\mathcal{W}_X^{0,*}$ on $X_{\text{reg}}$.

We must verify that $\psi$ has an extension in $\mathcal{W}_X^{0,*}$ across $X_{\text{sing}}$. Since such an extension must be unique by the SEP, the statement is local on $X$. Thus we may assume that $\alpha$ is defined on the whole of $X$ and that there is a generically nonvanishing holomorphic $n$-form $\gamma$ on $X$. Then $\alpha \gamma$ is a section of $\mathcal{W}_X^{n,*}(X)$.

Let us choose a smooth modification $\pi: X' \to X$ that is biholomorphic outside $X_{\text{sing}}$. Then $\pi^* \gamma$ is a holomorphic $n$-form on $X'$ that is generically non-vanishing. We claim that there is a current $\tau$ in $\mathcal{W}_X^{n,0}(X')$ such that $\pi_* \tau = \alpha \gamma$. In fact, $\tau$ exists on $\pi^{-1}(X_{\text{reg}})$ since $\pi$ is a biholomorphism there. Moreover, by [4, Proposition 1.2], $\alpha h$ is the direct image of some pseudomeromorphic current $\tilde{\tau}$ on $X'$, and is therefore also the image of the (unique) current $\tau = 1_{\pi^{-1}(X_{\text{reg}})} \tilde{\tau}$ in $\mathcal{W}_X^{n,*}(X')$.

By [10, Theorem 3.7] again $\tau$ is locally of the form $\xi \wedge ds$, where $\xi$ is in $\mathcal{W}_X^{0,*}$ and $ds = ds_1 \wedge \cdots \wedge ds_n$ for some local coordinates $s$. Hence, $\tau$ is a $K^{1*}$-valued section of $\mathcal{W}_X^{0,*}(X')$, so $\tau/\pi^* \gamma$ is a section of $\mathcal{W}_X^{0,*}(X')$. Now $\Psi := \pi_*(\tau/\pi^* \gamma)$ is a section of $\mathcal{W}_X^{0,*}(X)$. On $X_{\text{reg}} \cap \{ \gamma \neq 0 \}$ we thus have that $\Psi \wedge \gamma = \pi_* \tau = \alpha \gamma = \psi \wedge \gamma$ and so $\Psi = \psi$ there. By the SEP it follows that $\Psi$ coincides with $\psi$ on $X_{\text{reg}}$ and is thus the desired pseudomeromorphic extension to $X$.

In view of (5.1) and (5.3) we have, given a local embedding $i: X \to \Omega$, the extrinsic representation

$$\mathcal{W}_X^{0,*} \simeq \text{Hom}\left( \text{Hom}\left( \mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_{\Omega}^{Z,12} \right), \text{Hom}\left( \mathcal{O}_\Omega/\mathcal{J}, \mathcal{W}_{\Omega}^{Z,12} \right) \right), \phi \mapsto (i_*h \mapsto i_*(\phi \wedge h)).$$

(7.8)

Lemma 7.5  Assume that $X_{\text{reg}} \to \Omega$ is a local embedding and $(z, w)$ coordinates as before. Each section $\phi$ in $\mathcal{W}_X^{0,*}$ has a unique representation (7.1) with $\phi_j$ in $\mathcal{W}_Z^{0,*}$.

A current with a representation (7.1) is considered as an element of $\mathcal{W}_X^{0,*} = \text{Hom}(\Omega^n_X, \mathcal{W}_X^{n,*})$ in view of the comment after Lemma 7.1.

Proof  From (4.9) we get an induced sequence

$$0 \to \mathcal{W}_Z^{0,*} \to T \to M \to M' \to 0,$$

(7.9)

which is also exact. In fact, $T$ in (7.9) is clearly injective, and by (4.10), if $\xi$ in $(\mathcal{W}_Z^{0,*})^M$ and $A\xi = 0$, then $T\eta = \xi$, if $\eta = S\xi$. 

\hfill \Box
Now take \( \phi \) in \( \mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*}) \). Let us choose a basis \( \mu^1, \ldots, \mu^m \) for \( \omega_X^n \) and let \( \tilde{\phi} \) be the element in \( (\mathcal{W}_Z^{0,*})^M \) obtained from the coefficients of \( \phi \mu^j \) when expressed as in (2.11), cf., Sect. 4.1. We claim that \( A\tilde{\phi} = 0 \). Taking this for granted, by the exactness of (7.9), \( \tilde{\phi} \) is the image of the tuple \( \hat{\phi} = S\tilde{\phi} \). Now \( \tilde{\phi} \wedge \mu^j = \phi \mu^j \) since they are represented by the same tuple in \( (\mathcal{W}_Z^{0,*})^M \). Thus \( \hat{\phi} \) gives the desired representation of \( \phi \).

In view of Proposition 2.9 it is enough to prove the claim where \( \tilde{\phi} \) is smooth. Let us therefore fix such a point, say 0, and show that \((A\tilde{\phi})(0) = 0\). From the proof of Lemma 4.11, if we let \( I \) be the ideal generated by \( z \), and let \( X_0 \) be defined by \( \mathcal{O}_{X_0} := \mathcal{O}_\Omega/(\mathcal{J} + I) \), then \( \mu^1 \wedge \mu^z, \ldots, \mu^m \wedge \mu^z \) generate \( \omega_{X_0}^0 \). If we let \( \phi_0 \) be the morphism in \( \mathcal{H}om(\omega_{X_0}^0, \omega_{X_0}^0) \) given by \( \phi_0(\mu^i \wedge \mu^z) := \phi \mu^i \wedge \mu^z \) (which indeed gives a well-defined such morphism), then, as in the proof of Lemma 4.11, \( \tilde{\phi}_0 = \tilde{\phi}(0) \).

In addition, the sequence (4.9) for \( X_0 \) is

\[
0 \to \mathbb{C}^v \to T(0)A(0)M' \to \mathbb{C}^M.
\]

Since \( X_0 \) is 0-dimensional, the morphism \( \mathcal{O}_{X_0} \to \mathcal{H}om(\omega_{X_0}, \omega_{X_0}) \) is an isomorphism by Theorem 7.3, and thus \( \phi_0 \) is given as multiplication by a function in \( \mathcal{O}_{X_0} \), which we also denote by \( \phi_0 \), i.e., \( \tilde{\phi}_0 = T(0)\phi_0 \). Hence, \( A(0)\tilde{\phi}_0 = A(0)T(0)\phi_0 = 0 \), and thus \((A\tilde{\phi})(0) = 0\). \( \square \)

**Example 7.6** (Meromorphic functions) Assume that we have a local embedding \( X \to \Omega \). Given meromorphic functions \( \Phi, \Phi' \) in \( \Omega \) that are holomorphic generically on \( Z \), we say that \( \Phi \sim \Phi' \) if and only if \( \Phi - \Phi' \) is in \( \mathcal{J} \) generically on \( Z \). If \( \Phi = A/B \) and \( \Phi' = A'/B' \), where \( B \) and \( B' \) are generically non-vanishing on \( Z \), the condition is precisely that \( AB' - A'B \) is in \( \mathcal{J} \). We say that such an equivalence class is a meromorphic function \( \phi \) on \( X \), i.e., \( \phi \) is in \( \mathcal{M}_X \). Clearly we have \( \mathcal{O}_X \subset \mathcal{M}_X \). We claim that

\[
\mathcal{M}_X \subset \mathcal{W}_X^{0,*}.
\]

To see this, first notice that if we take a representative \( \Phi \) in \( \mathcal{M}_\Omega \) of \( \phi \), then it can be considered as an almost semi-meromorphic current on \( \Omega \) with Zariski-singular support of positive codimension on \( Z \), since it is generically holomorphic on \( Z \). As in Definition 5.5 we therefore have a current \( \Phi \wedge h \) in \( \mathcal{W}_X^{n,0} \) for \( h \) in \( \omega_X^n \). Another representative \( \Phi' \) of \( \phi \) will give rise to the same current generically and hence everywhere by the SEP. Thus \( \phi \) defines a section of \( \mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*}) = \mathcal{W}_X^{0,*} \). \( \square \)

By definition, a current \( \phi \) in \( \mathcal{W}_X^{0,*} \) can be multiplied by a current \( h \) in \( \omega_X^n \), and the product \( \phi \wedge h \) lies in \( \mathcal{W}_X^{n,*} \). It will be crucial that we can extend to products by somewhat more general currents. Notice that \( \omega_X^n \) is a subsheaf of \( \mathcal{E}_X^{n,*} \), which is an \( \mathcal{E}_X^{0,*} \)-module. Thus, we can consider the subsheaf \( \mathcal{E}_X^{0,*}\omega_X^n \) of \( \mathcal{E}_X^{n,*} \) which consists of finite sums \( \sum \xi_i \wedge h_i \), where \( \xi_i \) are in \( \mathcal{E}_X^{0,*} \) and \( h_i \) are in \( \omega_X^n \).
Lemma 7.7 Each $\phi$ in $\mathcal{W}_X^{0,*} = \text{Hom}_{\mathcal{O}_X}(\omega^n_X, \mathcal{W}^{n,*}_X)$ has a unique extension to a morphism in $\text{Hom}_{\mathcal{O}_X}(\mathcal{E}^{0,*}_X, \mathcal{W}^{n,*}_X)$.

Proof The uniqueness follows by $\mathcal{E}^{0,*}_X$-linearity, i.e., if $b = \xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r$ is in $\mathcal{E}^{0,*}_X$, then one must have

$$\phi b = \sum_i (-1)^{\deg \xi_i \deg \phi} \xi_i \wedge \phi h_i.$$ \hfill (7.10)

We must check that this is well-defined, i.e., that the right hand side does not depend on the representation $\xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r$ of $b$. By the SEP, it is enough to prove this locally on $X_{\text{reg}}$, and we can then assume that $\phi$ has a representation (7.1). By Proposition 2.9, it is then enough to prove that it is well-defined assuming that $\hat{\phi}_0, \ldots, \hat{\phi}_{-1}$ in (7.1) are all smooth. In this case, the right hand side of (7.10) is simply the product of $\xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r = b$ by the smooth form $\phi$ in $\mathcal{E}^{0,*}_X$, and this product only depends on $b$. \hfill $\square$

Corollary 7.8 Let $\phi$ be a current in $\mathcal{W}_X^{0,*}$ and let $\alpha$ be a current in $\mathcal{W}^{n,*}_X$ of the form $\alpha = \sum a_i \wedge h_i$, where $a_i$ are almost semi-meromorphic $(0,*)$-currents on $\Omega$ which are generically smooth on $Z$, and $h_i$ are in $\omega^n_Z$. Then one has a well-defined product

$$\phi \wedge \alpha = \sum (-1)^{\deg a_i \deg \phi} a_i \wedge (\phi \wedge h_i).$$ \hfill (7.11)

Proof The right hand side of (7.11) exists as a current in $\mathcal{W}^{n,*}_X$, and we must prove is that it only depends on the current $\alpha$ and not on the representation $\sum a_i \wedge h_i$. Notice that all the $a_i$ are smooth outside some subvariety $V$ of $Z$ and there the right hand side of (7.11) is the product of $\xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r = b$ by the smooth form $\phi$ in $\mathcal{E}^{0,*}_X$, and this product only depends on $b$. \hfill $\square$

Remark 7.9 Recall from (6.9) that $\omega = b \vartheta$. If $\phi$ is in $\mathcal{W}_X^{0,*}$, then we can define the product $\phi \wedge \omega$ by Corollary 7.8.

Expressed extrinsically, if $\mu = i_* \vartheta$, and if we write $R \wedge dz = b \mu$ as in Lemma 6.2, then we can define the product $R \wedge dz \wedge \phi := b \mu \wedge \phi$ as a current in $\text{Hom}(\mathcal{O}_\Omega/\mathcal{F}, \mathcal{W}^Z_\Omega)$. \hfill $\square$

Lemma 7.10 Assume that $\phi$ is in $\mathcal{W}_X^{0,*}$, and that $\phi \wedge \omega = 0$ for some structure form $\omega$, where the product is defined by Remark 7.9. Then $\phi = 0$.

Proof Considering the component with values in $E_p$, we get that $\phi \wedge \omega_0 = 0$. By Proposition 6.7, any $h$ in $\omega^n_Z$ can be written as $h = \xi \omega_0$, where $\xi$ is a holomorphic section of $E^*_p$, so by $\partial$-linearity, $\phi \wedge h = 0$, i.e., $\phi = 0$. \hfill $\square$
Proposition 7.11 If $Z$ is smooth, then $W_Z$ is closed under holomorphic differential operators.

Proof Let $\tau$ be any current in $W_Z$. It suffices to prove that if $\zeta$ are local coordinates on $Z$, then $\partial \tau / \partial \zeta_1$ is in $W_Z$. Consider the current $\tau' = \tau \otimes \bar{\partial} \frac{dw}{2\pi i w^2}$ on the manifold $Y := Z \times \mathbb{C}_w$. Clearly $\tau'$ has support on $Z$, and it follows from (2.5) that $\tau'$ is in $W^Z_Y$. Let

$$p: (z, w) \mapsto \zeta = (z_1 + w, z_2, \ldots, z_n),$$

which is just a change of variables on $Y$ followed by a projection. It follows from (2.4) that $p^* \tau'$ is in $W_Z$. Since

$$\bar{\partial} \frac{dw}{2\pi i w^2} \cdot \xi(w) = \frac{\partial \xi}{\partial w}(0)$$

it is readily verified that $p^* \tau' = \partial \tau / \partial \zeta_1$, so we conclude that $\partial \tau / \partial \zeta_1$ is in $W_Z$. \qed

8 The $\bar{\partial}$-operator on $W^0_X$

We already know the meaning of $\bar{\partial}$ on $W^n_X$, and we now define $\bar{\partial}$ on $W^0_X$.

Definition 8.1 Assume that $\phi, v$ are in $W^0_X$, We say that $\bar{\partial} v = \phi$ if

$$\bar{\partial} (v \wedge h) = \phi \wedge h, \ h \in \omega^n_X.$$  (8.1)

If we have an embedding $X \to \Omega$, (8.1) means, cf., (7.8), that

$$\bar{\partial} (v \wedge \mu) = \phi \wedge \mu, \ \mu \in \operatorname{Hom} \left( \mathcal{O}_\Omega / \mathcal{J}, \mathcal{C}H^Z_0 \right).$$  (8.2)

In view of Remark 7.9 we can define the product $\phi \wedge \omega$ for $\phi$ in $W^0_X$.

Definition 8.2 We say that $v$ belongs to $\text{Dom} \bar{\partial}_X$ if $v$ is in $\text{Dom} \bar{\partial}$, i.e., $\bar{\partial} v = \phi$ for some $\phi$ and in addition $\bar{\partial} (v \wedge \omega)$, a priori only in $\mathcal{P}M^n_X$, is in $W^n_X$, for each structure form $\omega$ from any possible embedding.

If $X$ is Cohen–Macaulay, then any such $\omega$ is of the form $a_1 h^1 + \cdots + a_m h^m$, where $h^j$ are in $\omega^n_X$ and $a_j$ are smooth, see Remark 6.5, and hence $\text{Dom} \bar{\partial}_X$ coincides with $\text{Dom} \bar{\partial}$ in this case.
Example 8.3 Assume that \( v \) is in \( \mathcal{E}^{0,*}_X \) and \( \phi = \overline{\partial} v \) in the sense in Section 4. Then clearly
\[
\overline{\partial} (v \wedge \omega) = \phi \wedge \omega + (-1)^{\deg v} v \wedge \overline{\partial} \omega.
\]
Since \( \overline{\partial} \omega = f \omega \), and \( \mathcal{W}^{n,*}_X \) is closed under multiplication with forms in \( \mathcal{E}^{0,*}_X \), we get that \( \overline{\partial} (v \wedge \omega) \) is in \( \mathcal{W}^{n,*}_X \), so \( v \) is in Dom \( \overline{\partial} X \) and \( \overline{\partial} v = \phi \).

If \( w \) is in Dom \( \overline{\partial} X \) and \( v \) is in \( \mathcal{E}^{0,*}_X \), then
\[
\overline{\partial} (v \wedge w \wedge \omega) = \overline{\partial} v \wedge w \wedge \omega + (-1)^{\deg v} v \wedge \overline{\partial} (w \wedge \omega).
\]
Thus \( v \wedge w \) is in Dom \( \overline{\partial} X \), and the Leibniz rule \( \overline{\partial} (v \wedge w) = \overline{\partial} v \wedge w + (-1)^{\deg v} v \wedge \overline{\partial} w \) holds.

Let \( \chi_\delta = \chi (|h|^2/\delta) \) where \( h \) is a tuple of holomorphic functions that cuts out \( X_{\text{sing}} \).

Lemma 8.4 If \( v \) is in \( \mathcal{W}^{0,*}(X) \), and it is in Dom \( \overline{\partial} X \) on \( X_{\text{reg}} \), then \( v \) is in Dom \( \overline{\partial} X \) on all of \( X \) if and only if
\[
\overline{\partial} \chi_\delta \wedge v \wedge \omega \rightarrow 0, \quad \delta \rightarrow 0,
\]
for all structure forms \( \omega \). In this case,
\[
-\nabla_f (v \wedge \omega) = \overline{\partial} v \wedge \omega.
\]

Proof Since \( \mathcal{W}^{n,*}_X \) is closed under multiplication by \( f \), \( v \) is in Dom \( \overline{\partial} X \) if and only if \( \nabla_f (v \wedge \omega) \) is in \( \mathcal{W}^{n,*}_X \) for all structure forms \( \omega \). Since \( v \) is in Dom \( \overline{\partial} X \) on \( X_{\text{reg}} \), thus \( \nabla_f (v \wedge \omega) \) is in \( \mathcal{W}^{n,*}_X \) on \( X_{\text{reg}} \). By (2.2), \( \nabla_f (v \wedge \omega) \) is then in \( \mathcal{W}^{n,*}_X \) on all of \( X \) if and only if
\[
1_{X_{\text{reg}}} \nabla_f (v \wedge \omega) = \nabla_f (v \wedge \omega).
\]
By the Leibniz rule,
\[
\nabla_f (\chi_\delta v \wedge \omega) = -\overline{\partial} \chi_\delta \wedge v \wedge \omega + \chi_\delta \nabla_f (v \wedge \omega).
\]
Since \( v \) is in \( \mathcal{W}^{0,*}_X \), \( v \wedge \omega \) is in \( \mathcal{W}^{n,*}_X \), so the left hand side of (8.6) tends to \( \nabla_f (v \wedge \omega) \) when \( \delta \rightarrow 0 \), whereas the second term on the right hand side of (8.6) tends to \( 1_{X_{\text{reg}}} \nabla_f (v \wedge \omega) \). Thus (8.5) holds if and only if (8.3) does. Thus the first statement in the lemma is proved.

Recall, cf., (6.9), that \( \omega = b \partial \) where \( b \) is smooth on \( X_{\text{reg}} \) and \( \partial \) is in \( \omega^n_X \). By the Leibniz rule thus \( -\nabla_f (v \wedge \omega) = \overline{\partial} v \wedge \omega \) on \( X_{\text{reg}} \), since \( \nabla_f \omega = 0 \). Therefore, (8.6) is equivalent to \( -\nabla_f (\chi_\delta v \wedge \omega) = \overline{\partial} \chi_\delta \wedge v \wedge \omega + \chi_\delta \overline{\partial} v \wedge \omega \). If (8.3) holds, we therefore get (8.4) when \( \delta \rightarrow 0 \). □

Remark 8.5 In case \( X \) is reduced the definition of \( \overline{\partial} X \) is precisely the same as in [6]. However, the definition of \( \overline{\partial} v = \phi \) given here, for \( v, \phi \) in \( \mathcal{W}^{0,*}_X \), does not coincide with the definition in, e.g., [6]. In fact, that definition means that \( \overline{\partial} (v \wedge h) = \phi \wedge h \) for all smooth \( h \) in \( \omega^n_X \), which in general is a strictly weaker condition. For example, for
any weakly holomorphic function $v$, we have $\bar{\partial}(v \wedge h) = 0$ for all smooth $h$ in $\Omega^*_X$, while if $X$ is a reduced complete intersection, or more generally Cohen–Macaulay, then $\bar{\partial}(v \wedge h) = 0$ for all $h$ in $\Omega^*_X$ is equivalent to $v$ being strongly holomorphic, see [33, p. 124] and [2].

We conclude this section with a lemma that shows that $\bar{\partial}$ means what one should expect when $\phi, v$ are expressed with respect to a local basis $w^a j$ for $O_X$ over $O_Z$ as in Lemma 7.5.

**Lemma 8.6** Assume that we have a local embedding $X_{reg} \to \Omega$ and $\phi, v$ in $\mathcal{W}^0_\mathcal{X}$ represented as in (7.1). Then $\bar{\partial}v = \phi$ if and only if

$$\bar{\partial}v_j = \hat{\phi}_j, \quad j = 0, \ldots, v - 1. \quad (8.7)$$

**Proof** Let us use the notation from the proof of Lemma 7.5. Recall that $\hat{v} = S\hat{v}$. In view of (8.2) and (2.12), $\bar{\partial}\hat{v} = \bar{\partial}v$. Since $S$ is holomorphic therefore $\bar{\partial}v = S\bar{\partial}\hat{v} = \bar{\partial}(S\hat{v}) = \bar{\partial}\hat{v}$. \hfill \Box

**9 Solving $\bar{\partial}u = \phi$ on $X$**

We will find local solutions to the $\bar{\partial}$-equation on $X$ by means of integral formulas. We use the notation and machinery from [6, Section 5]. Let $i : X \to \Omega \subset \mathbb{C}^N$ be a local embedding such that $\Omega$ is pseudoconvex, let $\Omega' \subset \subset \Omega$ be a relatively compact subdomain of $\Omega$, and let $X' = X \cap \Omega'$.

**Theorem 9.1** There are integral operators

$$K : \mathcal{E}^{0,*+1}(X) \to \mathcal{W}^0_\mathcal{X} \cap \text{Dom } \bar{\partial}_X, \quad P : \mathcal{E}^{0,*}(X) \to \mathcal{E}^{0,*}(X')$$

such that, for $\phi \in \mathcal{E}^{0,k}(X)$,

$$\phi = \bar{\partial}K\phi + K(\bar{\partial}\phi) + P\phi. \quad (9.1)$$

The operators $K$ and $P$ are described below; they depend on a choice of weight $g$. Since $\Omega$ is Stein one can find such a weight $g$ that is holomorphic in $z$, by which we mean that it depends holomorphically on $z \in \Omega'$ and has no components containing any $d\bar{z}_i$, cf., Example 5.1 in [6]. In this case, $P\phi$ is holomorphic when $k = 0$, and vanishes when $k \geq 1$, i.e.,

$$\phi = \bar{\partial}K\phi + K(\bar{\partial}\phi), \quad \phi \in \mathcal{E}^{0,k}(X), \quad k \geq 1. \quad (9.2)$$

If $\bar{\partial}\phi = 0$ in $\Omega$, and $k \geq 1$, then $K\phi$ is a solution to $\bar{\partial}\phi = \phi$. If $k = 0$, then $\phi = P\phi$ is holomorphic. It follows that a smooth $\bar{\partial}$-closed function is holomorphic. In the reduced case this is a classical theorem of Malgrange [24]. In Sect. 10 we prove that $K\phi$ is smooth on $X_{reg}$.
We now turn to the definition of $K$ and $P$. For future need, in Sect. 11, we define them acting on currents in $\mathcal{W}^{0,\ast}(X)$ and not only on smooth forms. Let $\pi : \Omega_\xi \times \Omega'_\xi \rightarrow \Omega'_\xi$ be the natural projection. Let us choose a holomorphic Hefer form $3$. The tensor product $\bar{R}$ the tensor product of currents in the following way: first of all, by Remark 7.9, we can acting on currents in $(\text{Bochner–Martinelli form. Since we are only are concerned with $(0, \ast)$-forms, we will here assume that $H$ and $B$ only have holomorphic differentials in $\xi$, i.e., the factors $d\eta_i = d\zeta_i - dz_i$ in $H$ and $B$ in [6] should be replaced by just $d\xi_i$.

If $\gamma$ is a current in $\Omega_\xi \times \Omega'_\xi$ we let $(\gamma)_N$ be the component of bidegree $(N, \ast)$ in $\xi$ and $(0, \ast)$ in $\zeta$, and let $\vartheta(\gamma)$ be the current such that

$$\vartheta(\gamma) \wedge d\xi = (\gamma)_N. \quad (9.3)$$

Consider now $\mu$ in $\mathcal{H}om(\partial_{\Omega}/\mathcal{J}, \mathcal{W}_{\Omega}^{\ast})$ and $\phi$ in $\mathcal{W}_{X}^{0,\ast}$. We can give meaning to

$$(g \wedge H R(\xi))_N \wedge \phi(\xi) \wedge \mu(z) \quad (9.4)$$

as a tensor product of currents in the following way: first of all, by Remark 7.9, we can form the product $R(\xi) \wedge d\xi \wedge \phi(\xi)$ as a current in $\mathcal{W}_{\Omega}^{\ast}$. In view of [11, Corollary 4.7] the tensor product $R(\xi) \wedge d\xi \wedge \phi(\xi) \wedge \mu(z)$ is in $\mathcal{W}_{\Omega \times \Omega'}^{Z \times Z'}$, where $Z' = Z \cap \Omega'$. Finally, we multiply this with the smooth form $\vartheta(g \wedge H)$ to obtain (9.4). Similarly, outside of $\Delta$, the diagonal in $\Omega \times \Omega'$, where $B$ is smooth, we can define

$$(B \wedge g \wedge H R(\xi))_N \wedge \phi(\xi) \wedge \mu(z) \quad (9.5)$$

as a tensor product of currents.

**Lemma 9.2** For $\mu$ in $\mathcal{H}om(\partial_{\Omega}/\mathcal{J}, \mathcal{W}_{\Omega}^{\ast})$ and $\phi$ in $\mathcal{W}_{X}^{0,\ast}$, the current (9.5), a priori defined as a current in $\mathcal{W}_{\Omega}^{Z \times Z'}$, has an extension across $\Delta$. The current (9.4) and the extension of (9.5) depend $\partial_{\Omega}/\mathcal{J}$-bilinearly on $\mu$ and $\phi$, and are such that

$$K \phi \wedge \mu := \pi_\ast((B \wedge g \wedge H R(\xi))_N \wedge \phi(\xi) \wedge \mu(z)) \quad (9.6)$$

and

$$P \phi \wedge \mu := \pi_\ast((g \wedge H R(\xi))_N \wedge \phi(\xi) \wedge \mu(z)) \quad (9.7)$$

are in $\mathcal{H}om(\partial_{\Omega}/\mathcal{J}, \mathcal{W}_{\Omega}^{Z'})$.

It follows that $K \phi \wedge \mu$ and $P \phi \wedge \mu$ are $\mathbb{C}$-linear in $\phi$ and $\partial_{\Omega}/\mathcal{J}$-linear in $\mu$. In view of (7.8), by considering $\mu$ in $\mathcal{H}om(\partial_{\Omega}/\mathcal{J}, \mathcal{W}_{\Omega}^{Z'})$, we have defined linear operators

$$K : \mathcal{W}_{X}^{0,\ast + 1}(X) \rightarrow \mathcal{W}_{X}^{0,\ast}(X'), \quad P : \mathcal{W}_{X}^{0,\ast}(X) \rightarrow \mathcal{W}_{X}^{0,\ast}(X'). \quad (9.8)$$

**Proof of Lemma 9.2** In order to define the extension of (9.5) across $\Delta$, we note first that since $B$ is almost semi-meromorphic with Zariski singular support $\Delta$, $\vartheta(B \wedge g \wedge H)$

3 We are only concerned with the component $H^0$ of this form, so for simplicity we write just $H$. 

\[ \Theta \text{ Springer} \]
is an almost semi-meromorphic \((0, \ast)\)-current on \(\Omega_\zeta \times \Omega'_z\), which is smooth outside the diagonal. We can thus form the current \(\vartheta (B \wedge g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi(\zeta) \wedge \mu(z)\) in \(\mathcal{W}Z_{\Omega_\zeta} \times \mathcal{Z}'_{\Omega'_z}\), cf., Proposition 2.4, and this is the extension of (9.5) across \(\Delta\).

From the definitions above, it is clear that (9.4) and the extension of (9.5) are \(\mathcal{O}_{\Omega}\)-bilinear in \(\phi\) and \(\mu\). Both these currents are annihilated by \(J_\zeta\) and \(J_\zeta\), cf., (2.8), so they depend \(\mathcal{O}_{\Omega}/J\)-bilinearly. In view of (2.4) we conclude that (9.6) and (9.7) are in \(\mathcal{H}_{\Omega/Z, \mathcal{Z}'_{\Omega'_z}}\).

**Proposition 9.3** If \(\phi \in \mathcal{W}^{0,k}(X)\), then \(P\phi \in \mathcal{E}^{0,k}(X')\), and if in addition \(g\) is holomorphic in \(z\), then \(P\phi \in \mathcal{O}(X')\) if \(k = 0\) and vanishes if \(k \geq 1\).

**Proof** Since \(\vartheta (g \wedge H)\) is smooth, we get that

\[
\pi_*(\vartheta (g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi \wedge \mu(z)) = \pi_*(\vartheta (g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi) \wedge \mu(z),
\]

cf., for example [20, (5.1.2)]. Thus \(P\phi(z) = \pi_*( (g \wedge H R(\zeta))_N \wedge \phi)\) which is smooth on \(\Omega'\). If \(g\) depends holomorphically on \(z\), then \(P\phi\) is holomorphic in \(\Omega'\) if \(\phi\) is a \((0, 0)\)-current, and vanishes for degree reasons if \(\phi\) has positive degree. \(\square\)

We shall now approximate \(K\phi\) by smooth forms. Let \(B^\epsilon = \chi(|\zeta - z|^2/\epsilon)B\).

**Proposition 9.4** For any \(\phi \in \mathcal{W}^{0,k}(X)\), \(k \geq 1\),

\[
K^\epsilon \phi := \pi_*( (B^\epsilon \wedge g \wedge H R(\zeta))_N \wedge \phi) = \pi_*(\vartheta (B^\epsilon \wedge g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi)
\]

is in \(\mathcal{E}^{0,k-1}(X')\) and \(K^\epsilon \phi \to K\phi\) when \(\epsilon \to 0\).

The last statement means that

\[
K^\epsilon \phi \wedge \mu \to K\phi \wedge \mu, \quad \mu \in \mathcal{H}_{\Omega/Z, \mathcal{Z}'_{\Omega'_z}}.
\]

**Proof** Since \(B^\epsilon\) is smooth, the current we push forward is \(R(\zeta) \wedge \phi(\zeta)\) times a smooth form of \(\zeta\) and \(z\). Therefore \(K^\epsilon \phi\) is smooth. As in the proof of Proposition 9.3, we obtain since \(B^\epsilon\) is smooth that

\[
K^\epsilon \phi \wedge \mu = \pi_*( (B^\epsilon \wedge g \wedge H R(\zeta))_N \wedge \phi \wedge \mu(z)).
\]

By (5.2) applied to \(a = B\) we have that

\[
(B^\epsilon \wedge g \wedge H R(\zeta))_N \wedge \phi \wedge \mu(z) \to (B \wedge g \wedge H R(\zeta))_N \wedge \phi \wedge \mu(z)
\]

which implies (9.9). \(\square\)

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9.1 Proof of Theorem 9.1

By definition $K \phi$ and $P \phi$ are currents in $\mathcal{H}^0,*(X')$ such that (9.6) and (9.7) hold for $\mu$ in $\mathcal{H}\text{om}(\mathcal{O}_{\Omega}, J, \mathcal{C}\mathcal{H}^2_{\Omega})$. We claim that

$$K \phi \wedge R \wedge dz = \pi_s((B \wedge g \wedge HR(\xi))_N \wedge \phi \wedge R(z) \wedge dz) \quad (9.12)$$

and

$$P \phi \wedge R \wedge dz = \pi_s((g \wedge HR(\xi))_N \wedge \phi \wedge R(z) \wedge dz); \quad (9.13)$$

here the left hand sides are defined in view of Remark 7.9, whereas the right hand sides have meaning by Lemma 9.2 and the fact that $R(z) \wedge dz$ is in $\mathcal{H}\text{om}(\mathcal{O}_{\Omega}, J, \mathcal{C}\mathcal{H}^2_{\Omega})$ by Corollary 6.3.

Recall from Lemma 6.2 that $R \wedge dz = b \mu$, where $\mu$ is a tuple of currents in $\mathcal{H}\text{om}(\mathcal{O}_{\Omega}, J, \mathcal{C}\mathcal{H}^2_{\Omega})$ and $b$ is an almost semi-meromorphic matrix that is smooth generically on $Z'$. Therefore (9.12) and (9.13) hold where $b$ is smooth, in view of Lemma 7.7, and since both sides are in $\mathcal{H}\text{om}(\mathcal{O}_{\Omega}, J, \mathcal{C}\mathcal{H}^2_{\Omega})$, the equalities hold everywhere by the SEP.

As in [6] we let $R^\lambda = \tilde{\partial}|f|^{2\lambda} \wedge U$ for $\text{Re} \lambda \gg 0$. It has an analytic continuation to $\lambda = 0$ and $R = R^\lambda|_{\lambda=0}$. Notice that $R(z) \wedge B$ is well-defined since it is a tensor product with respect to the coordinates $z, \eta = \xi - z$. Also $R(z) \wedge R^\lambda(\xi) \wedge B$ admits such an analytic continuation and defines a pseudomeromorphic current\(^4\) when $\lambda = 0$. Let $B_{k,k-1}$ be the component of $B$ of bidegree $(k, k-1)$.

Lemma 9.5 For all $k$,

$$B_{k,k-1} \wedge H R^\lambda(\xi) \wedge R(z)|_{\lambda=0} = B_{k,k-1} \wedge H R(\xi) \wedge R(z). \quad (9.14)$$

Proof of Lemma 9.5 Notice that the equality holds outside $\Delta$. Let $T$ be the left hand side of (9.14). In view of Proposition 2.1 it is therefore enough to check that $1_\Delta T = 0$. Fix $j, k$ and let

$$T_\ell = B_{k,k-1} \wedge H R^\lambda(\xi) \wedge R(\xi)|_{\lambda=0}.$$ 

Clearly $T_\ell = 0$ if $\ell < p$ so first assume that $\ell = p$. Since $HR_j$ has bidegree $(j, j)$ in $\xi$, the current vanishes unless $j+k \leq N$. Thus the total antiholomorphic degree is $\leq N-n+N-1$. On the other hand, the current has support on $\Delta \cap Z \times Z \simeq Z \times \{pt\}$ which has codimension $N+N-n$. Thus it vanishes by the dimension principle.

We now prove by induction over $\ell \geq p$ that $1_\Delta T_\ell = 0$. Note that by (6.6), outside of $Z_\ell$, $R_\ell(z) = \alpha_\ell(z) R_{\ell-1}(z)$, where $\alpha_\ell(z)$ is smooth. Thus, outside of $Z_\ell \times \Omega, T_\ell$ is a smooth form times $T_{\ell-1}$, and thus, by induction and (2.3), $1_\Delta T_\ell$ has its support in $\Delta \cap (Z_\ell \times Z) \simeq Z_\ell \times \{pt\}$, which has codimension $\geq N+\ell+1$, see (6.3). On the other hand, the total antiholomorphic degree is $\leq \ell+j+k-1 \leq \ell+N-1$, so the current vanishes by the dimension principle. We conclude that (9.14) holds. \(\Box\)

\(^4\) One can consider this current as $R(z) \wedge B$ multiplied by the residue of the almost semi-meromorphic current $U$ in (6.5), cf., [10, Section 4.4].
By the same argument\textsuperscript{5} as for [6, (5.2)] we have the equality
\[
\nabla_{f(z)}((B \wedge g \wedge H R^\lambda(\zeta))_N \wedge R(z) \wedge dz) = [\Delta]' \wedge R(z) \wedge dz - (g \wedge H R^\lambda)_N \wedge R(z) \wedge dz,
\textup{(9.15)}
\]
also for our $R$, where $[\Delta]'$ denotes the part of $[\Delta]$ where $d\eta_i = d\xi_i - dz_i$ has been replaced\textsuperscript{6} by $d\xi_i$. In view of (9.14) we can put $\lambda = 0$ in (9.15), and then we get
\[
\nabla_{f(z)}((B \wedge g \wedge H R(\zeta))_N \wedge R(z) \wedge dz) = [\Delta]' \wedge R(z) \wedge dz - (H R(\zeta) \wedge g)_N \wedge R(z) \wedge dz.
\textup{(9.16)}
\]
Multiplying (9.16) by the smooth form $\phi$, and using (9.12) and (9.13), we get
\[
\phi \wedge R \wedge dz = -\nabla_f(K \phi \wedge R \wedge dz) + K(\bar{\partial} \phi) \wedge R \wedge dz + P \phi \wedge R \wedge dz,
\]
or equivalently,
\[
\phi \wedge \omega = -\nabla_f(K \phi \wedge \omega) + K(\bar{\partial} \phi) \wedge \omega + P \phi \wedge \omega.
\textup{(9.17)}
\]
Multiplying by suitable holomorphic $\xi_0$ in $E^*_p$ such that $f^*_p \xi_0 = 0$, cf., Proposition 6.7, we see that $\phi \wedge h = \bar{\partial}(K \phi \wedge h) + K(\bar{\partial} \phi) \wedge h + P \phi \wedge h$ for all $h$ in $\omega_X$. Thus by definition (9.1) holds.

Since $\mathcal{W}^{0,*}_X$ is closed under multiplication by $\partial_X$, we get that $\psi$ in $\mathcal{W}^{0,*}_X$ is in Dom $\partial_X$ if and only if $-\nabla_f(\psi \wedge \omega)$ is in $\mathcal{W}^{n,*}_X$. Thus, we conclude from (9.17) that $K \phi$ is in Dom $\partial_X$ since all the other terms but $-\nabla_f(K \phi \wedge \omega)$ are in $\mathcal{W}^{0,*}_X$.

\subsection{9.2 Intrinsic interpretation of $K$ and $P$}

So far we have defined $K$ and $P$ by means of currents in ambient space. We used this approach in order to avoid introducing push-forwards on a non-reduced space. However, we will sketch here how this can be done. We must first define the product space $X \times X'$. Given a local embedding $i : X \rightarrow \Omega$ as before, we have an embedding $(i \times i) : X \times X' \rightarrow \Omega \times \Omega'$ such that the structure sheaf is $\partial_{\Omega \times \Omega'}/(\mathcal{J}_X + \mathcal{J}_X')$. One can check that this sheaf is independent of the chosen embedding, i.e., $\partial_{X \times X'}$ is intrinsically defined. Thus we also have definitions of all the various sheaves on $X \times X'$ like $\partial_{X \times X'}^{0,*}$. The projection $p : X \times X' \rightarrow X'$ is determined by $p^* \phi : \mathcal{O}_{X'} \rightarrow \mathcal{O}_{X \times X'}$, which in turn is defined so that $p^* i^* \Phi = (i \times i)^* \pi^* \Phi$ for $\Phi$ in $\partial_{\Omega'}$, where $\pi : \Omega \times \Omega' \rightarrow \Omega'$ as before. Again one can check that this definition is independent of the embedding, and also extends to smooth $(0,*)$-forms $\phi$. Therefore, we have the well-defined mapping $p_* : \mathcal{C}^{2n,*+n}_{X \times X'} \rightarrow \mathcal{C}^{n,*}_{X'}$, and clearly
\[
i_* p_* = \pi_*(i \times i)_*.
\textup{(9.18)}
\]

\textsuperscript{5} There is a sign error in [6, (5.2)] due to $R(z) \wedge dz$ being odd with respect to the super structure. Since we here move $R(z) \wedge dz$ to the right, we get the correct sign.

\textsuperscript{6} This change is due to the fact that we do the same change of the differentials in the definition of $H$ and $B$ above.
As before we have the isomorphism

\[(i \times i)_* : \mathcal{W}_{X \times X'}^{2n,*} \cong \mathcal{H}om \left( \mathcal{O}_{\Omega X \times \Omega X'}/(\mathcal{J}_X + \mathcal{J}_X'), \mathcal{W}_{\Omega X \times \Omega X'}^{Z \times Z'} \right).\]

As in the proof of Lemma 9.2 we see that \( \pi_* \) maps a current in \( \mathcal{W}_{X \times X'}^{Z \times Z'} \) annihilated by \( J_X' \) to a current in \( \mathcal{H}om(\mathcal{O}_X/\mathcal{J}, \mathcal{W}_{\Omega X \times \Omega X'}^{Z \times Z'}) \). It follows by (9.18) that

\[p_* : \mathcal{W}_{X \times X'}^{2n,*+n} \rightarrow \mathcal{W}_{X'}^{n,*}.\]

Now, take \( h \) in \( \omega^n_{X'} \) and let \( \mu = i_* h \). Then, cf., the proof of Lemma 9.2,

\[(B \wedge g \wedge H R(\zeta))_N \wedge \phi(\zeta) \wedge \mu(z) = (i \times i)_* (\partial (B \wedge g \wedge H) \wedge \omega(\zeta) \wedge \phi(\zeta) \wedge h).\]

Thus we can define \( K \phi \) intrinsically by

\[K \phi \wedge h = p_* (\partial (B \wedge g \wedge H) \wedge \omega(\zeta) \wedge \phi(\zeta) \wedge h(z)).\]  
(9.19)

From above it follows that \( K \phi \wedge h \) is in \( \mathcal{W}_{X'}^{n,*} \). In the same way we can define \( P \phi \) by

\[P \phi \wedge h = p_* (\partial (g \wedge H) \wedge \omega(\zeta) \wedge \phi(\zeta) \wedge h(z)).\]  
(9.20)

It is natural to write

\[K \phi(z) = \int_{\zeta} \partial (B \wedge g \wedge H) \wedge \omega(\zeta) \wedge \phi(\zeta), \quad P \phi(z) = \int_{\zeta} \partial (g \wedge H) \wedge \omega(\zeta) \wedge \phi(\zeta),\]

although the formal meaning is given by (9.19) and (9.20).

### 10 Regularity of solutions on \( X_{\text{reg}}' \)

We have already seen, cf., Proposition 9.3, that \( P \phi \) is always a smooth form. We shall now prove that \( K \) preserves regularity on \( X_{\text{reg}}' \). More precisely,

**Theorem 10.1** If \( \phi \) in \( \mathcal{W}_X^{0,*} \) is smooth near a point \( x \in X_{\text{reg}}' \), then \( K \phi \) in Theorem 9.1 is smooth near \( x \).

Throughout this section, let us choose local coordinates \( (\zeta, \tau) \) and \( (z, w) \) at \( x \) corresponding to the variables \( \zeta \) and \( z \) in the integral formulas, so that \( Z = \{(\zeta, \tau) : \tau = 0\} \).

**Lemma 10.2** Let \( B^\epsilon := \chi(|\zeta - z|^2/\epsilon) B \), and assume that \( \phi \) has compact support in our coordinate neighborhood. Then \( K \phi \) can be approximated by the smooth forms

\[K^\epsilon \phi := \pi_* ((B^\epsilon \wedge g \wedge H R)_N \wedge \phi).\]
Notice that here we cut away the diagonal $\Delta'$ in $Z \times Z'$ times $\mathbb{C}_\tau \times \mathbb{C}_w$ in contrast to Proposition 9.4, where we only cut away the diagonal $\Delta$ in $\Omega \times \Omega'$.

**Proof** Clearly $B^\epsilon$ is smooth so that each $K^\epsilon \phi$ is smooth in a full neighborhood in $\Omega'$ of $x$. Let $T = \mu(z, w) \wedge (HR(\zeta, \tau) \wedge B \wedge g)_N \wedge \phi$, and let $W = \Delta' \times \mathbb{C}_\tau \times \mathbb{C}_w$. Since $\mu(z, w) \otimes R(\zeta, \tau)$ has support on $\{w = \tau = 0\}$, $T = \mathbf{1}_{\{w = \tau = 0\}}T$. Therefore, $W T = \mathbf{1}_W \mathbf{1}_{\{w = \tau = 0\}} T = 0$ since $W \cap \{w = \tau = 0\} \subset \Delta$ and $\Delta = 0$ by definition, cf., Proposition 2.1 (i). Now notice that $W T = 0$ implies (9.11) and in turn (9.9) with our present choice of $B^\epsilon$.

We first consider a simple but nontrivial example of Theorem 10.1.

**Example 10.3** Let $X = \mathbb{C}_\zeta \subset \mathbb{C}_2(\zeta, \tau)$ and $J = (\tau^m + 1)$. Then $R = \bar{\partial}(1/\tau^m + 1)$. For an arbitrary point $(z, w)$ we can choose the Hefer form

$$H = \frac{1}{2\pi i} \sum_{j=0}^m \tau^{m-k} w^k d\tau.$$ 

From the Bochner–Martinelli form $B$ we only get a contribution from the term

$$B_1 = \frac{1}{2\pi i} \frac{(\bar{z} - z) d\zeta + (\bar{\tau} - \bar{w}) d\tau}{|\zeta - z|^2 + |\tau - w|^2}.$$ 

Let $\Omega' \subset \subset \Omega$ be open balls with center at the origin, and let $\varphi = \varphi(|\zeta|^2 + |\tau|^2)$ be a smooth cutoff function with support in $\Omega$ that is $\equiv 1$ in a neighborhood of $\Omega'$. Then we can choose a holomorphic weight $g = \varphi + \cdots$, see, [6, Example 5.1] with respect to $\Omega'$, and with support in $\Omega$. Now $1, \tau, \ldots, \tau^m$ is a set of generators for $\mathcal{O}_X$ over $\mathcal{O}_Z$. Assume that

$$\phi = (\hat{\phi}_0(\zeta) \otimes 1 + \cdots + \hat{\phi}_m(\zeta) \otimes \tau^m)d\bar{\zeta}$$

is a smooth $(0, 1)$-form. We want to compute $K\phi$. We know that

$$K\phi = a_0(z) \otimes 1 + \cdots + a_m(z) \otimes w^m$$

with $a_k(z)$ in $V^{0,0}_Z$. By Lemma 10.2 and its proof, we have smooth $K^\epsilon \phi(z, w)$ in $\Omega'$ such that

$$K^\epsilon \phi \wedge dz \wedge dw \wedge \bar{\partial} \frac{1}{w^{m+1}} \to K\phi \wedge dz \wedge dw \wedge \bar{\partial} \frac{1}{w^{m+1}}.$$ 

It follows that

$$a_k(z) = \lim_{\epsilon \to 0} \frac{1}{k!} \frac{\partial^k}{\partial w^k} K^\epsilon \phi(z, w)|_{w=0}.$$
Notice that

\[( B \wedge g \wedge H R(\tau))_2 = B_1 \wedge g_{0,0} \wedge H \wedge \bar{\partial} \frac{1}{\tau^{m+1}} \]

\[= -\varphi \bar{\partial} \frac{1}{\tau^{m+1}} \wedge \frac{1}{(2\pi i)^2} \sum_{\ell = 0}^{m} \tau^{m-\ell} w^\ell \wedge \frac{(\bar{\zeta} - \bar{w})d\tau}{|\zeta - z|^2 + |\tau - w|^2} \]

\[= -\varphi \bar{\partial} \frac{d\tau}{\tau^{m+1}} \wedge \frac{1}{(2\pi i)^2} \sum_{\ell = 0}^{m} \tau^{m-\ell} w^\ell \wedge \frac{(\bar{\zeta} - \bar{\zeta})d\zeta}{|\zeta - z|^2 + |\tau - w|^2}. \]

For each fixed \( \epsilon > 0, |\zeta - z| > 0 \) on \( \text{supp} \chi_\epsilon \), cf., Lemma 10.2, we have

\[
K^\epsilon \phi(z, w) = \int_{\zeta, \tau} \varphi \left( \frac{1}{(2\pi i)^2} d\tau \right) \sum_{\ell = 0}^{m} \frac{d\tau}{\tau^{\ell+1}} \wedge w^\ell \chi_\epsilon \frac{(\bar{\zeta} - \bar{\zeta})d\zeta}{|\zeta - z|^2 + |\tau - w|^2} \wedge \sum_{k=0}^{m} \hat{\phi}_k(\zeta) \otimes \tau^k.
\]

(10.3)

A simple computation yields that

\[
K^\epsilon \phi(z, w) = \sum_{k=0}^{m} a_k^\epsilon(z) \otimes w^k + \mathcal{O}(\bar{w}),
\]

where

\[
a_k^\epsilon(z) = \frac{1}{2\pi i} \int_{\zeta} \varphi(|\zeta|^2) \chi_\epsilon \frac{\hat{\phi}_k(\zeta) d\zeta}{\zeta - z}.
\]

Letting \( \epsilon \) tend to 0 we get \( K \phi \) as in (10.1), where

\[
a_k(z) = \frac{1}{2\pi i} \int_{\zeta} \varphi(|\zeta|^2) \frac{\hat{\phi}_k(\zeta) d\zeta}{\zeta - z}.
\]

It is well-known that these Cauchy integrals \( a_k(z) \) are smooth solutions to \( \bar{\partial} v = \hat{\phi}_k d\bar{\zeta} \) in \( Z' = Z \cap \Omega' \). Thus \( K \phi \) is smooth.

\[\square\]

Remark 10.4 The terms \( \mathcal{O}(\bar{w}) \) in the expansion (10.4) of \( K^\epsilon \phi(z, w) \) do not converge to smooth functions in general when \( \epsilon \to 0 \). For a simple example, take \( \phi = \zeta d\bar{\zeta} \otimes \tau^m \). Then \( K^\epsilon \phi(0, w) \) tends to

\[
w^m \int \varphi(|\zeta|^2) \frac{1}{2\pi i} \frac{|\zeta|^2 d\zeta}{|\zeta|^2 + |w|^2}
\]

which is a smooth function of \( w \) plus (a constant times) \( w^m |w|^2 \log |w|^2 \), and thus not smooth. However, it is certainly in \( C^m \). One can check that \( K \phi(z, w) = \)

\[\square\] Springer
\[ K^\varepsilon \phi(z, w) = \sum_{k=0}^{m} \int_{\xi} \varphi(|\xi|^2) \chi_{\varepsilon} \frac{1}{2\pi i} \frac{(\bar{\xi} - \bar{z})\hat{\phi}_k(\xi) \, d\xi \wedge d\xi}{|\xi - z|^2 + |w|^2} \, w^k \sum_{j=0}^{m-k} \left( \frac{|w|^2}{|\xi - z|^2 + |w|^2} \right)^j. \]

It is now clear that we can let \( \varepsilon \to 0 \). By a simple computation we then get

\[ K\phi(z, w) = \sum_{k=0}^{m} C \hat{\phi}_k(z) \otimes w^k - \sum_{k=0}^{m} \int_{\xi} \varphi(|\xi|^2) \frac{1}{2\pi i} \frac{\hat{\phi}_k(\xi) \, d\xi \wedge d\xi}{|\xi - z|^2 + |w|^2} \, w^k \left( \frac{|w|^2}{|\xi - z|^2 + |w|^2} \right)^{m-k+1}. \]

Let \( \psi = \varphi \hat{\phi}_k \). Then the \( k \)th term in the second sum is equal to

\[ b(z, w) = \frac{1}{2\pi i} \int_{\xi} \psi(z + \xi) \frac{d\xi \wedge d\xi}{\xi} \, w^k \left( \frac{|w|^2}{|\xi|^2 + |w|^2} \right)^{m-k+1}. \]

If we integrate outside the unit disk, then we certainly get a smooth function. Thus it is enough to consider the integral over the disk. Moreover, if \( \psi(z + \xi) = O(|\xi|^M) \) for a large \( M \), then the integral is at least \( C^m \). By a Taylor expansion of \( \psi(z + \xi) \) at the point \( z \), we are thus reduced to consider

\[ \int_{|\xi| < 1} \frac{\xi^\alpha \bar{\xi}^\beta}{\xi} \left( \frac{|w|^2}{|\xi|^2 + |w|^2} \right)^{m-k+1}. \]

For symmetry reasons, they vanish, except when \( \alpha = \beta + 1 \). Thus we are left with

\[ \int_{|\xi| < 1} |\xi|^{2\beta} \left( \frac{|w|^2}{|\xi|^2 + |w|^2} \right)^{m-k+1} \, w^k = C w^k |w|^{2(m-k+1)} \int_0^1 \frac{s^\beta \, ds}{(s + |w|^2)^{m-k+1}} \]

for non-negative integers \( \beta \). The right hand side is a smooth function of \( w \) if \( \beta \leq m - k - 1 \) and a smooth function plus

\[ C w^k |w|^{2(\beta+1)} \log |w|^2 \]

if \( \beta \geq m - k \). The worst case therefore is when \( k = m \) and \( \beta = 0 \); then we have \( w^m |w|^2 \log |w|^2 \) that we encountered above.

**Proposition 10.5** Let \( z, w \) be coordinates at a point \( x \in X_{reg} \) such that \( Z = \{ w = 0 \} \) and \( x = (0, 0) \). If \( \phi \) is smooth, and has support where the local coordinates are defined, then

\[ \lim_{\varepsilon \to 0^+} K^\varepsilon \phi(z, w) \]
\[
v^\epsilon(z, w) = \int_\zeta \chi(|\zeta - z|^2/\epsilon)(HR \wedge B \wedge g)_N \wedge \phi,
\]
is smooth for \( \epsilon > 0 \), and for each multiindex \( \ell \) there is a smooth form \( v_\ell \) such that
\[
\partial^\ell_w v^\epsilon |_{w=0} \to v_\ell
\]
as currents on \( Z \).

Taking this proposition for granted we can conclude the proof of Theorem 10.1.

**Proof of Theorem 10.1** If \( \phi \equiv 0 \) in a neighborhood of \( x \in X'_{\text{reg}} \), then \( K\phi \) is smooth near \( x \), cf., the proof of Proposition 9.4. Thus, it is sufficient to prove Theorem 10.1 assuming that \( \phi \) is smooth and has support near \( x \).

Recall that given a minimal generating set \( 1, w^{\alpha_1}, \ldots, w^{\alpha_{r-1}} \), one gets the coefficients \( \hat{v}^\epsilon_j \) in the representation
\[
v^\epsilon = \hat{v}^\epsilon_0 \otimes 1 + \cdots + \hat{v}^\epsilon_{r-1} \otimes w^{\alpha_{r-1}}
\]
from \( \partial^\ell_w v^\epsilon |_{w=0}, |\ell| \leq M \) by a holomorphic matrix, cf., the proof of Lemma 4.7. It thus follows from Proposition 10.5 that there are smooth \( \hat{v}^\epsilon_j \) such that \( \hat{v}^\epsilon_j \to \hat{v}^\epsilon_j \) as currents on \( Z \). Let \( v = \hat{v}^\epsilon_0 \otimes 1 + \cdots + \hat{v}^\epsilon_{r-1} \otimes w^{\alpha_{r-1}} \). In view of (2.14), \( v^\epsilon \wedge \mu \to v \wedge \mu \) for all \( \mu \) in \( \text{Hom}(\Theta_{\Omega}/J, CH^Z_{\Omega}) \). From Lemma 10.2 we conclude that \( v \wedge \mu = K\phi \wedge \mu \) for all such \( \mu \). Thus \( K\phi = v \) in \( W^0_{X, \ast} \) and hence \( K\phi \) is smooth. \( \square \)

**Proof of Proposition 10.5** Assume that \( X \) is embedded in \( \Omega \subset \mathbb{C}^N_{\zeta, \tau} \). After a suitable rotation we can assume that \( Z \) is the graph \( \tau' = \psi(\zeta') \). The Bochner–Martinelli kernel in \( \Omega \) is rotation invariant, so it is
\[
B = \sigma + \sigma \wedge \overline{\partial} \sigma + \sigma \wedge (\overline{\partial} \sigma)^2 + \cdots,
\]
where
\[
\sigma = \frac{(\overline{\zeta}' - \overline{\zeta}) \cdot d\zeta' + (\overline{\tau}' - \overline{\tau}) \cdot d\tau'}{|\zeta' - \zeta|^2 + |\tau' - \tau|^2}.
\]
We now choose the new coordinates \( \zeta = \zeta', \tau = \tau' - \psi(\zeta') \) in \( \Omega \), so that \( Z = \{ (\zeta, \tau) ; \tau = 0 \} \).

Recall that on \( X_{\text{reg}} \) we have that \( R \wedge dz \) is a smooth form times \( \mu = (\mu_1, \ldots, \mu_m) \), where \( \mu_j \) is a generating set for \( \text{Hom}(\Theta_{\Omega}/J, CH^Z_{\Omega}) \). Thus we are to compute \( \partial^\ell_w |_{w=0} \) of integrals like
\[
\int_{\zeta, \tau} \overline{\partial} d\tau^{\alpha+1} \wedge B^\epsilon_k \wedge \phi(\zeta, z, w, \tau), \quad (10.5)
\]
where \( k \leq n \) and \( \phi \) is smooth with compact support near \( x \). It is clear that the symbols \( \overline{\tau}, \overline{w}, d\overline{\tau} \) can be omitted in the expression for
\[
B^\epsilon = \chi_\epsilon B = \chi(|\zeta - z|^2/\epsilon) B,
\]
since $\tilde{\tau}$ and $d\tilde{\tau}$ annihilate $\bar{\partial}(1/\tau^{\alpha+1})$, and since we only take holomorphic derivatives with respect to $w$ and set $w = 0$.

Let us write $\Psi(\xi) - \Psi(z) = A(\xi, z)\eta$, where $\eta := \xi - z$ is considered as a column matrix and $A$ is a holomorphic $(N - n) \times n$-matrix. Then

$$\sigma = \frac{\eta^* v}{|\xi - z|^2 + |\tau - w + \Psi(\xi) - \Psi(z)|^2},$$

where $v$ is the $(1, 0)$-form valued column matrix

$$v = d\xi + A^* d(\tau + \Psi(\xi)).$$

Since $\eta^* v$ is a $(1, 0)$-form we have that

$$B_k^\epsilon = \chi_\epsilon \frac{\eta^* v \wedge ((d\eta^*) v + \eta^* \bar{\partial} v)^{k-1}}{|\xi - z|^2 + |\tau - w + \Psi(\xi) - \Psi(z)|^2}.$$  

**Lemma 10.6** Let

$$\xi^i = \xi_1^i \frac{\partial}{\partial \xi_1} + \cdots + \xi_n^i \frac{\partial}{\partial \xi_n}$$

be smooth $(1, 0)$-vector fields, and let $L_i = L_{\xi_i}$ be the associated Lie derivatives for $i = 1, \ldots, \rho$. Let

$$\gamma_k := \eta^* v \wedge ((d\eta^*) v + \eta^* \bar{\partial} v)^{k-1}.$$  

If we have a modification $\pi : \tilde{W} \to \Omega \times \Omega$ such that locally $\pi^* \eta = \eta_0 \eta'$, where $\eta_0$ is a holomorphic function, then

$$\pi^* (L_1 \cdots L_\rho \gamma_k) = \eta_0^k \beta,$$

where $\beta$ is smooth.

Recall that if $a$ is a form, then $L_{\xi} a = d(\xi \lrcorner a) + \xi \lrcorner (da)$, and that $L_{\xi} (\beta \lrcorner a) = [\xi, \beta] \lrcorner a + \beta \lrcorner (L_{\xi} a)$ if $\beta$ is another vector field.

**Proof** Introduce a nonsense basis $e$ and its dual $e^*$ and consider the exterior algebra spanned by $e_j, e^*_\ell$, and the cotangent bundle. Let

$$c_\ell = \eta^* e \wedge ((d\eta^*) e)^{\ell-1}.$$  

Notice that $\gamma_k$ is a sum of terms like

$$(ve^* \lrcorner)^\ell c_\ell \wedge (\eta^* \bar{\partial} v)^{k-\ell}.$$
Since $L_i c_\ell = 0$ and $L_i(\eta^* b) = \eta^* L_i b$ it follows after a finite number of applications of $L_i$'s that we get

$$(v_1 e^*) \cdots (v_\ell e^*) c_\ell (\eta^* b_1) \cdots (\eta^* b_{k-\ell}),$$

where $v_j$ and $b_j$ are smooth. Since

$$\pi^* c_\ell = \eta_0^/(\eta')^* e \wedge (d(\eta')^* e)^{\ell-1},$$

the lemma now follows. \( \square \)

We note that $\eta^*(I + A^* A)\eta = |\zeta - z|^2 + |\psi(\zeta) - \psi(z)|^2$. Thus, differentiating (10.5) with respect to $w$, setting $w = 0$, and evaluating the residue with respect to $\tau$ using (2.10), we obtain a sum of integrals like

$$\int \chi_\epsilon \left( (\eta^* a_1) \cdots (\eta^* a_{r+1}) \wedge \phi \right) \frac{\Phi_{k+\ell}}{(\eta^*(I + A^* A)\eta)^{k+\ell+1}},$$

where $a_1, \ldots, a_{r+1}$ are column vectors of smooth functions. We must prove that the limit of such integrals when $\epsilon \to 0$ are smooth in $z$.

**Lemma 10.7** Let

$$I_{\ell+1}^{r,s} = \int \chi_\epsilon \left( (\eta^* a_1) \cdots (\eta^* a_r) g^2 \bar{\gamma}_k \wedge \phi \right) \frac{\Phi_{k+\ell}}{\Phi_{k+\ell+1}},$$

where $a_1, \ldots, a_r$ are tuples of smooth functions, $\bar{\gamma}_k = L_1 \cdots L_\rho \gamma_k$, where $L_i = L_{\xi_i}$ are Lie derivatives with respect to smooth $(1,0)$-vector fields $\xi^i$ as above for $i = 1, \ldots, \rho$, $\phi$ is a test form with support close to $z$, and $\Phi := \eta^*(I + A^* A)\eta$. If $r \geq 1$ and $r + s \geq \ell + 1$, then we have the relation

$$I_{\ell+1}^{r,s} = I_{\ell}^{r-1,s} + I_{\ell+1}^{r-1,s+1} + I_{\ell}^{r,s-1} + o(1) \quad (10.6)$$

when $\epsilon \to 0$.

**Proof** If

$$\bar{\xi} = a^i_r (I + A^* A)^{-r} \frac{\partial}{\partial \zeta},$$

and $L = L_{\bar{\xi}}$, then using that $\Phi = \eta^l (I + A^* A)^l \bar{\eta}$, one obtains that

$$L \Phi = \eta^* a_r + g^2 (|\eta|^2). \quad (10.7)$$

Thus

$$I_{\ell+1}^{r,s} = \int \chi_\epsilon \left( (\eta^* a_1) \cdots (\eta^* a_{r-1}) g^2 \bar{\gamma}_k \wedge \phi L \frac{1}{\Phi_{k+\ell+1}} + I_{\ell+1}^{r-1,s+1} \right.$$
in view of (10.7). We now integrate by parts $L$ in the integral. If a derivative with respect to $\zeta_j$ falls on some $\eta^*a_i$, we get a term $I^{r-1,s}_\ell$. If it falls on $O(|\eta|^{2s})$ we get either $I^{r,s-1}_\ell$ or $O(|\eta|^{2s})$, and this gives rise to another term $I^{r-1,s}_\ell$. If it falls on $\phi$ or $\tilde{\gamma}_k$ we get an additional term $I^{r-1,s}_\ell$. The only possibility left is when the derivative falls on $\chi_\epsilon = \chi(|\eta|^2/\epsilon)$. It remains to show that an integral of the form

$$\int_{\zeta,z} \chi'(|\eta|^2/\epsilon) \frac{(\eta^*a_1) \cdots (\eta^*a_{r-1})(\eta^*b) O(|\eta|^{2s}) \gamma_k \wedge \phi}{\Phi^{k+\ell}}$$

 tends to 0, where the factor $\eta^*b$ comes from the derivative of $|\eta|^2$. We now choose a resolution $\tilde{V} \to \Omega \times \Omega$ such that $\eta = \eta_0 \eta'$ where $\eta'$ is non-vanishing and $\eta_0$ is (locally) a monomial. Notice that $r^*\Phi = |\eta_0|^2\Phi'$ where $\Phi'$ is smooth and strictly positive. In view of Lemma 10.6 we thus obtain integrals of the form

$$\int_{\tilde{V}} \chi'(|\eta_0|^2 v/\epsilon) \frac{1}{\epsilon} \frac{\eta_0^{r+s-\ell}}{\eta_0^{k+\ell-s}} \alpha,$$

where $v$ is smooth and strictly positive and $\alpha$ is smooth.

In order to see that the limit of (10.8) tends to 0, we note first that if we let

$$\tilde{\chi}(s) = s \chi'(s) + \chi(s),$$

then just as $\chi$, $\tilde{\chi}$ is also a smooth function on $[0, \infty)$ that is 0 in a neighborhood of 0 and 1 in a neighborhood of $\infty$. By assumption, $r + s - \ell - 1 \geq 0$. Since the principal value current $1/f^m$ acting on a test form $\beta$ can be defined as

$$\lim_{\epsilon \to 0^+} \int \chi(|f|^2 v/\epsilon) \frac{\beta}{f^m}$$

for any cut-off function as above, the principal value current $1/\eta_0^{k+\ell-s}$ acting on $\eta_0^{r+s-\ell-1} \alpha$ equals

$$\lim_{\epsilon \to 0^+} \int_{\tilde{V}} \chi(|\eta_0|^2 v/\epsilon) \frac{\eta_0^{r+s-\ell-1}}{\eta_0^{k+\ell-s}} \alpha = \lim_{\epsilon \to 0^+} \int_{\tilde{V}} \tilde{\chi}(|\eta_0|^2 v/\epsilon) \frac{\eta_0^{r+s-\ell-1}}{\eta_0^{k+\ell-s}} \alpha.$$

Taking the difference between the left and right hand side, we conclude that (10.8) tends to 0 when $\epsilon \to 0$. \hfill $\square$

Now we can conclude the proof of Proposition 10.5. From the beginning we have $I^{0,0}_\ell$. After repeated applications of (10.6) we end up with

$$I^{0,\ell}_\ell + I^{0,\ell-1}_{\ell-1} + \cdots + I^{0,0}_0 + o(1).$$
However, any of these integrals has an integrable kernel even when $\epsilon = 0$. This means that we are back to the case in [6, Lemma 6.2], and so the limit integral is smooth in $z$. \hfill \Box

11 A fine resolution of $O_X$

We first consider a generalization of Theorem 9.1.

**Lemma 11.1** Assume that $\phi \in \mathcal{W}^{0,k}(X) \cap \mathcal{O}_X^{0,k}(X_{\text{reg}}) \cap \text{Dom } \bar{\partial}X$ and that $K \phi$ is in $\text{Dom } \bar{\partial}X$ (or just in $\text{Dom } \partial$). Then (9.1) holds on $X'$.

**Proof** Let $\chi_\delta$ be functions as before that cut away $X_{\text{sing}}$. From Koppelman’s formula (9.1) for smooth forms we have

$$
\chi_\delta \phi \wedge h = \bar{\partial}(K(\chi_\delta \phi)) \wedge h + K(\chi_\delta \bar{\partial} \phi) \wedge h + P(\chi_\delta \phi) \wedge h + K(\bar{\partial} \chi_\delta \wedge \phi) \wedge h, \quad h \in \omega^n_X,
$$

for $z \in X_{\text{reg}}'$. Clearly the left hand side tends to $\phi \wedge h$ when $\delta \to 0$. From Lemma 9.2 it follows that $K(\chi_\delta \phi) \wedge h \to K \phi \wedge h$. Thus the first term on the right hand side of (11.1) tends to $\bar{\partial}(K \phi) \wedge h$. In the same way the second and third terms on the right hand side tend to $K(\bar{\partial} \phi) \wedge h$ and $P \phi \wedge h$, respectively. It remains to show that the last term tends to 0. If $z$ belongs to a fixed compact subset of $X_{\text{reg}}'$, then $B$ is smooth in (9.5) when $\zeta$ is in $\text{supp } \bar{\partial} \chi_\delta$ for small $\delta$. Hence it suffices to see that

$$
R(\zeta) \wedge d \zeta \wedge \bar{\partial} \chi_\delta \wedge \phi(\zeta) \wedge i_\ast h \to 0,
$$

and since this is a tensor product of currents, it suffices to see that

$$
R(\zeta) \wedge d \zeta \wedge \bar{\partial} \chi_\delta \wedge \phi(\zeta) \to 0,
$$

or equivalently, $\omega(\zeta) \wedge \bar{\partial} \chi_\delta \wedge \phi(\zeta) \to 0$, which follows by Lemma 8.4 since $\phi$ is in $\text{Dom } \bar{\partial}X$. We have thus proved that

$$
\chi_\delta \phi \wedge h = \chi_\delta \bar{\partial}(K \phi) \wedge h + \chi_\delta K(\bar{\partial} \phi) \wedge h + \chi_\delta P \phi \wedge h.
$$

The first term on the right hand side is equal to $\bar{\partial}(\chi_\delta K \phi \wedge h) - \bar{\partial} \chi_\delta \wedge K \phi \wedge h$, where the latter term tends to 0 if $K \phi$ is in $\text{Dom } \bar{\partial}X$ or just in $\text{Dom } \partial$, cf., Lemma 8.4. Thus we get

$$
\phi \wedge h = \bar{\partial}(K \phi) \wedge h + K(\bar{\partial} \phi) \wedge h + P \phi \wedge h, \quad h \in \omega^n_X,
$$

which precisely means that (9.1) holds. \hfill \Box

**Definition 11.2** We say that a $(0, q)$-current $\phi$ on an open set $U \subset X$ is a section of $\mathcal{O}^q_X$ over $U$, $\phi \in \mathcal{O}^q(U)$, if, for every $x \in U$, the germ $\phi_x$ can be written as a finite sum of terms

$$
\xi_{\nu} \wedge K_{\nu}(\cdots \xi_2 \wedge K_2(\xi_1 \wedge K_1(\xi_0))),
$$

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where \( \xi_j \) are smooth \((0, *)\)-forms and \( K_j \) are integral operators with kernels \( k_j(\zeta, z) \) at \( x \), defined as above, and such that \( \xi_j \) has compact support in the set where \( z \mapsto k_j(\zeta, z) \) is defined.

Clearly \( \mathcal{A}_X^* \) is closed under multiplication by \( \xi \) in \( \mathcal{E}_X^{0,*} \). It follows from (9.8) that \( \mathcal{A}_X^* \) is a subsheaf of \( \mathcal{W}_X^* \) and from Theorem 10.1 that \( \mathcal{A}_X^k = \mathcal{E}_X^{0,*} \) on \( X_{reg} \). Clearly any operator \( K \) as above maps \( \mathcal{A}^{k+1}_X \to \mathcal{A}^k_X \).

**Lemma 11.3** If \( \phi \) is in \( \mathcal{A}_X \), then \( \phi \) and \( K\phi \) are in Dom \( \bar{\partial}_X \).

**Proof** Notice that [6, Lemma 6.4] holds in our case by verbatim the same proof, since we have access to the dimension principle for (tensor products of) pseudomeromorphic \((n, *)\)-currents, and the computation rule (2.3), cf., the comment after Definition 5.7. Since \( \mathcal{A}_X^* = \mathcal{E}_X^{0,*} \) on \( X_{reg} \) it is enough by Lemma 8.4 to check that \( \bar{\partial}\chi^\delta \wedge \omega \wedge \phi \to 0 \), and this precisely follows from [6, Lemma 6.4].

In view of Lemmas 11.1 and 11.3 we have

**Proposition 11.4** Let \( K, P \) be integral operators as in Theorem 9.1. Then

\[
K : \mathcal{A}^{k+1}_X(\mathcal{E}) \to \mathcal{A}^k(X'), \quad P : \mathcal{A}^k_X(\mathcal{E}) \to \mathcal{E}^{0,k}(X'),
\]

and the Koppelman formula (9.1) holds.

**Proof of Theorem 1.1** By definition, it is clear that \( \mathcal{A}^k_X \) are modules over \( \mathcal{E}^{0,k}_X \), and by Theorem 10.1, \( \mathcal{A}^k_X \) coincides with \( \mathcal{E}^{0,k}_X \) on \( X_{reg} \). Since we have access to Koppelman formulas, precisely as in the proof of [6, Theorem 1.2] we can verify that \( \bar{\partial} : \mathcal{A}^k_X \to \mathcal{A}^k_{X+1} \).

It remains to prove that (1.2) is exact. We choose locally a weight \( g \) that is holomorphic in \( z \), so the term \( P\phi \) vanishes if \( \phi \) is in \( \mathcal{A}^k_X, k \geq 1 \), and is holomorphic in \( z \) when \( k = 0 \). Assume that \( \phi \) is in \( \mathcal{A}^k_X \) and \( \bar{\partial}\phi = 0 \). If \( k \geq 1 \), then \( \bar{\partial}K\phi = \phi \), and if \( k = 0 \), then \( \phi = P\phi \).

\( \square \)

### 11.1 Global solvability

Assume that \( E \to X \) is a holomorphic vector bundle; this means that the transition matrices have entries in \( \mathcal{O}_X \). For instance if we have a global embedding \( i : X \to \Omega \) and a holomorphic vector bundle \( F \to \Omega \), then \( F \) defines a vector bundle \( i^*F \to X \). The sheaves \( \mathcal{A}_X^*(E) \) give rise to a fine resolution of the sheaf \( \mathcal{O}_X(E) \), and by standard homological algebra we have the isomorphisms

\[
H^q(X, \mathcal{O}(E)) = \frac{\text{Ker} (\mathcal{A}^q(X, E) \to \mathcal{A}^{q+1}(X, E))}{\text{Im} (\mathcal{A}^{q-1}(X, E) \to \mathcal{A}^q(X, E))}, \quad q \geq 1.
\]

Thus, if \( \phi \in \mathcal{A}^{q+1}(X, E), \bar{\partial}\phi = 0 \), and its canonical cohomology class vanishes, then the equation \( \bar{\partial}\psi = \phi \) has a global solution in \( \mathcal{A}^q(X, E) \). In particular, the equation

\( \square \) Springer
is always solvable if $X$ is Stein. If for instance $X$ is a pure-dimensional projective variety $i: X \to \mathbb{P}^N$, then the $\bar{\partial}$-equation is solvable, e.g., if $E$ is a sufficiently ample line bundle.

12 Locally complete intersections

Let us consider the special case when $X$ locally is a complete intersection, i.e., given a local embedding $i: X \to \Omega \subset \mathbb{C}^N$ there are global sections $f_j$ of $\mathcal{O}(d_j) \to \mathbb{P}^N$ such that $\mathcal{J} = (f_1, \ldots, f_p)$, where $p = N - n$. In particular, $Z = \{ f_1 = \cdots = f_p = 0 \}$. In this case $\text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{C}\mathcal{H}_\Omega)$ is generated by the single current

$$
\mu = \bar{\partial} \frac{1}{f_p} \wedge \cdots \wedge \bar{\partial} \frac{1}{f_1} \wedge dz_1 \wedge \cdots \wedge dz_N,
$$

see, e.g., [3]. Each smooth $(0, q)$-form $\phi$ in $\mathcal{E}^{0,q}_X$ is thus represented by a current $\Phi \wedge \mu$, where $\Phi$ is smooth in a neighborhood of $Z$ and $i^* \Phi = \phi$. Moreover, $X$ is Cohen–Macaulay so $X_{\text{reg}}$ coincides with the part of $X$ where $Z$ is regular, and $\bar{\partial} \phi = \psi$ if and only if $\bar{\partial} (\phi \wedge \mu) = \psi \wedge \mu$.

Henkin and Polyakov introduced, see [17, Definition 1.3], the notion of residual currents $\phi$ of bidegree $(0, q)$ on a locally complete intersection $X \subset \mathbb{P}^N$, and the $\bar{\partial}$-equation $\bar{\partial} \psi = \phi$. Their currents $\phi$ correspond to our $\phi$ in $\mathcal{E}^{0,q}_X$ and their $\bar{\partial}$-operator on such currents coincides with ours.

**Remark 12.1** In [18] Henkin and Polyakov consider a global reduced complete intersection $X \subset \mathbb{P}^N$. They prove, by a global explicit formula, that if $\phi$ is a global $\bar{\partial}$-closed smooth $(0, q)$-form with values in $\mathcal{O}(\ell)$, $\ell = d_1 + \cdots + d_p - N - 1$, then there is a smooth solution to $\bar{\partial} \psi = \phi$ at least on $X_{\text{reg}}$, if $1 \leq q \leq n - 1$. When $q = n$ a necessary obstruction term occurs. However, their meaning of “$\bar{\partial}$-closed” is that locally there is a representative $\Phi$ of $\phi$ and smooth $g_j$ such that $\bar{\partial} \Phi = g_1 f_1 + \cdots + g_p f_p$. If this holds, then clearly $\bar{\partial} \phi = 0$. The converse implication is not true, see Example 12.2 below. It is not clear to us whether their formula gives a solution under the weaker assumption that $\bar{\partial} \phi = 0$, neither do we know whether their solution admits some intrinsic extension across $X_{\text{sing}}$ as a current on $X$.

**Example 12.2** Let $X = \{ f = 0 \} \subset \Omega \subset \mathbb{C}^{n+1}$ be a reduced hypersurface, and assume that $df \neq 0$ on $X_{\text{reg}}$, so that $\mathcal{J} = (f)$. Let $\phi$ be a smooth $(0, q)$-form in a neighborhood of some point $x$ on $X$ such that $\bar{\partial} \phi = 0$. We claim that $\bar{\partial} u = \phi$ has a smooth solution $u$ if and only if $\phi$ has a smooth representative $\Phi$ in ambient space such that $\Phi = fg$ for some smooth form $g$. In fact, if such a $\Phi$ exists then $0 = f \bar{\partial} g$ and thus $\bar{\partial} g = 0$. Therefore, $g = \bar{\partial} \gamma$ for some smooth $\gamma$ (in a Stein neighborhood of $x$ in ambient space) and hence $\bar{\partial} (\Phi - f \gamma) = 0$. Thus there is a smooth $U$ such that $\bar{\partial} U = \Phi - f \gamma$; this means that $u = i^* U$ is a smooth solution to $\bar{\partial} u = \phi$. Conversely, if $u$ is a smooth solution, then $u = i^* U$ for some smooth $U$ in ambient space, and thus $\Phi = \bar{\partial} U$ is a representative of $\phi$ in ambient space. Thus $\bar{\partial} \Phi = fg$ (with $g = 0$).
There are examples of hypersurfaces $X$ where there exist smooth $\phi$ with $\bar{\partial}\phi = 0$ that do not admit smooth solutions to $\bar{\partial}u = \phi$, see, e.g., [6, Example 1.1]. It follows that such a $\phi$ cannot have a representative $\Phi$ in ambient space as above.  

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