# The $\bar{\partial}$-equation on a non-reduced analytic space

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# The $\bar{\partial}$-equation on a non-reduced analytic space

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**Abstract** Let $X$ be a, possibly non-reduced, analytic space of pure dimension. We introduce a notion of $\bar{\partial}$-equation on $X$ and prove a Dolbeault–Grothendieck lemma. We obtain fine sheaves $\mathcal{A}_X^q$ of $(0, q)$-currents, so that the associated Dolbeault complex yields a resolution of the structure sheaf $\mathscr{O}_X$. Our construction is based on intrinsic semi-global Koppelman formulas on $X$.

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## 1 Introduction

Let $X$ be a smooth complex manifold of dimension $n$ and let $\mathscr{E}_X^{0,*}$ denote the sheaf of smooth $(0, *)$-forms. It is well-known that the Dolbeault complex

$$0 \to \mathscr{O}_X \xrightarrow{i} \mathscr{E}_X^{0,0} \xrightarrow{\bar{\partial}} \mathscr{E}_X^{0,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{E}_X^{0,n} \to 0 \tag{1.1}$$

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is exact, and hence provides a fine resolution of the structure sheaf $\mathscr{O}_X$. If $X$ is a reduced analytic space of pure dimension, then there is still a natural notion of "smooth forms". In fact, assume that $X$ is locally embedded as $i\colon X \to \Omega$, where $\Omega$ is a pseudoconvex domain in $\mathbb{C}^N$. If $\mathcal{K}er\, i^*$ denotes the subsheaf of all smooth forms $\xi$ in ambient space such that $i^*\xi = 0$ on the regular part $X_{reg}$ of $X$, then one defines the sheaf $\mathscr{E}_X$ of smooth forms on $X$ simply as

$$\mathscr{E}_X := \mathscr{E}_\Omega / \mathcal{K}er\, i^*.$$

It is well-known that this definition is independent of the choice of embedding of $X$. Currents on $X$ are defined as the duals of smooth forms with compact support. It is readily seen that the currents $\mu$ on $X$ so defined are in a one-to-one correspondence to the currents $\hat{\mu} = i_*\mu$ in ambient space such that $\hat{\mu}$ vanish on $\mathcal{K}er\, i_*$, see, e.g., [6]. There is an induced $\bar\partial$-operator on smooth forms and currents on $X$. In particular, (1.1) is a complex on $X$ but in general it is not exact. In [6], Samuelsson and the first author introduced, by means of intrinsic Koppelman formulas on $X$, fine sheaves $\mathscr{A}_X^*$ of $(0,*)$-currents that are smooth on $X_{reg}$ and with mild singularities at the singular part of $X$, such that

$$0 \to \mathscr{O}_X \xrightarrow{i} \mathscr{A}_X^0 \xrightarrow{\bar\partial} \mathscr{A}_X^1 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{A}_X^n \to 0 \tag{1.2}$$

is exact, and thus a fine resolution of the structure sheaf $\mathscr{O}_X$. An immediate consequence is the representation

$$H^q(X, \mathscr{O}_X) = \frac{\operatorname{Ker}\big(\mathscr{A}^{0,q}(X) \xrightarrow{\bar\partial} \mathscr{A}^{0,q+1}(X)\big)}{\operatorname{Im}\big(\mathscr{A}^{0,q-1}(X) \xrightarrow{\bar\partial} \mathscr{A}^{0,q}(X)\big)}, \quad q \geq 1, \tag{1.3}$$

of sheaf cohomology, and so (1.3) is a generalization of the classical Dolbeault isomorphism. In special cases more qualitative information of the sheaves $\mathscr{A}_X^q$ are known, see, e.g., [5,23].

Starting with the influential works [28,29] by Pardon and Stern, there has been a lot of progress recently on the $L^2$-$\bar\partial$ theory on non-smooth (reduced) varieties; see, e.g., [15,27,31]. The point in these works, contrary to [6], is basically to determine the obstructions to solve $\bar\partial$ locally in $L^2$. For a more extensive list of references regarding the $\bar\partial$-equation on reduced singular varieties, see, e.g., [6].

In [17], a notion of the $\bar\partial$-equation on non-reduced local complete intersections was introduced, and which was further studied in [18]. We discuss below how their work relates to ours.

The aim of this paper is to extend the construction in [6] to a non-reduced pure-dimensional analytic space. The first basic problem is to find appropriate definitions of forms and currents on $X$. Let $X_{reg}$ be the part of $X$ where the underlying reduced space $Z$ is smooth, and in addition $\mathscr{O}_X$ is Cohen–Macaulay. On $X_{reg}$ the structure sheaf $\mathscr{O}_X$ has a structure as a free finitely generated $\mathscr{O}_Z$-module. More precisely, assume that we have a local embedding $i\colon X \to \Omega \subset \mathbb{C}^N$ and coordinates $(z, w)$ in $\Omega$ such that

$Z=\{w=0\}$. Let $\mathcal{J}$ be the defining ideal sheaf for $X$ on $\Omega$. Then there are monomials $1, w^{\alpha_1}, \dots, w^{\alpha_{\nu-1}}$ such that each $\phi$ in $\mathscr{O}_\Omega/\mathcal{J} \simeq \mathscr{O}_X$ has a unique representation

$$\phi = \hat\phi_0 \otimes 1 + \hat\phi_1 \otimes w^{\alpha_1} + \cdots + \hat\phi_{\nu-1} \otimes w^{\alpha_{\nu-1}}, \tag{1.4}$$

where $\hat\phi_j$ are in $\mathscr{O}_Z$. A reasonable notion of a smooth form on $X$ should admit a similar representation on $X_{reg}$ with smooth forms $\hat\phi_j$ on $Z$. We first introduce the sheaves $\mathscr{E}_X^{0,*}$ of smooth $(0,*)$-forms on $X$. By duality, we then obtain the sheaf $\mathcal{C}_X^{n,*}$ of $(n,*)$-currents. We are mainly interested in the subsheaf $\mathcal{PM}_X^{n,*}$ of pseudomeromorphic currents, and especially, the even more restricted sheaf $\mathcal{W}_X^{n,*}$ of such currents with the so-called standard extension property, SEP, on $X$. A current with the SEP is, roughly speaking, determined by its restriction to any dense Zariski-open subset.

Of special interest is the sheaf $\omega_X^n \subset \mathcal{W}_X^{n,0}$ of $\bar\partial$-closed pseudomeromorphic $(n,0)$-currents. In the reduced case this is precisely the sheaf of holomorphic $(n,0)$-forms in the sense of Barlet–Henkin–Passare, see, e.g., [12,16].

We have no definition of "smooth $(n,*)$-form" on $X$. In order to define $(0,*)$-currents, we use instead the sheaf $\omega_X^n$ in the following way. Any holomorphic function defines a morphism in $\mathcal{H}om(\omega_X^n, \omega_X^n)$, and it is a reformulation of a fundamental result of Roos [30], that this morphism is indeed injective, and generically surjective. In the reduced case, multiplication by a current in $\mathcal{W}_X^{0,*}$ induces a morphism in $\mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$, and in fact $\mathcal{W}_X^{0,*} \to \mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$ is an isomorphism. In the non-reduced case, we then take this as the definition of $\mathcal{W}_X^{0,*}$. It turns out that with this definition, on $X_{reg}$, any element of $\mathcal{W}_X^{0,*}$ admits a unique representation (1.4), where $\hat\phi_j$ are in $\mathcal{W}_Z^{0,*}$, see Sect. 6 below for details.

Given $v, \phi$ in $\mathcal{W}_X^{0,*}$ we say that $\bar\partial v = \phi$ if $\bar\partial(v \wedge h) = \phi \wedge h$ for all $h$ in $\omega_X^n$. Following [6] we introduce semi-global integral formulas and prove that if $\phi$ is a smooth $\bar\partial$-closed $(0,q+1)$-form there is locally a current $v$ in $\mathcal{W}_X^{0,q}$ such that $\bar\partial v = \phi$. A crucial problem is to verify that the integral operators preserve smoothness on $X_{reg}$ so that the solution $v$ is indeed smooth on $X_{reg}$. By an iteration procedure as in [6] we can define sheaves $\mathscr{A}_X^k \subset \mathcal{W}_X^{0,k}$ and obtain our main result in this paper.

**Theorem 1.1** *Let $X$ be an analytic space of pure dimension $n$. There are sheaves $\mathscr{A}_X^k \subset \mathcal{W}_X^{0,k}$ that are modules over $\mathscr{E}_X^{0,*}$, coinciding with $\mathscr{E}_X^{0,k}$ on $X_{reg}$, and such that* (1.2) *is a resolution of the structure sheaf $\mathscr{O}_X$.*

The main contribution in this article compared to [6] is the development of a theory for smooth $(0,*)$-forms and various classes of $(n,*)$- and $(0,*)$-currents in the non-reduced case as is described above. This is done in Sects. 4–8. The construction of integral operators to provide solutions to $\bar\partial$ in Sect. 9 and the construction of the fine resolution of $\mathscr{O}_X$ in Sect. 11, which proves Theorem 1.1, are done pretty much in the same way as in [6]. The proof of the smoothness of the solutions of the regular part in Sect. 10 however becomes significantly more involved in the non-reduced case and requires completely new ideas. In Sect. 12 we discuss the relation to the results in [17,18] in case $X$ is a local complete intersection.

## 2 Pseudomeromorphic currents

Let $s_1, \dots, s_m$ be coordinates in $\mathbb{C}^m$, let $\alpha$ be a smooth form with compact support, and let $a_1, \dots, a_r$ be positive integers, $0 \le \ell \le r \le m$. Then

$$\bar\partial \frac{1}{s_1^{a_1}} \wedge \cdots \wedge \bar\partial \frac{1}{s_\ell^{a_\ell}} \wedge \frac{\alpha}{s_{\ell+1}^{a_{\ell+1}} \cdots s_r^{a_r}}$$

is a well-defined current that we call an *elementary (pseudomeromorphic) current*. Let $Z$ be a reduced space of pure dimension. A current $\tau$ is *pseudomeromorphic* on $Z$ if, locally, it is the push-forward of a finite sum of elementary pseudomeromorphic currents under a sequence of modifications, simple projections, and open inclusions. The pseudomeromorphic currents define an analytic sheaf $\mathcal{PM}_Z$ on $Z$. This sheaf was introduced in [8] and somewhat extended in [6]. If nothing else is explicitly stated, proofs of the properties listed below can be found in, e.g., [6].

If $\tau$ is pseudomeromorphic and has support on an analytic subset $V$, and $h$ is a holomorphic function that vanishes on $V$, then $\bar h \tau = 0$ and $d\bar h \wedge \tau = 0$.

Given a pseudomeromorphic current $\tau$ and a subvariety $V$ of some open subset $\mathcal{U} \subset Z$, the natural restriction to the open set $\mathcal{U} \setminus V$ of $\tau$ has a natural extension to a pseudomeromorphic current on $\mathcal{U}$ that we denote by $\mathbf{1}_{\mathcal{U}\setminus V}\tau$. Throughout this paper we let $\chi$ denote a smooth function on $[0, \infty)$ that is 0 in a neighborhood of 0 and 1 in a neighborhood of $\infty$. If $h$ is a holomorphic tuple whose common zero set is $V$, then

$$\mathbf{1}_{\mathcal{U}\setminus V}\tau = \lim_{\epsilon \to 0^+} \chi(|h|^2/\epsilon)\tau. \tag{2.1}$$

Notice that $\mathbf{1}_V \tau := (1 - \mathbf{1}_{\mathcal{U}\setminus V})\tau$ is also pseudomeromorphic and has support on $V$. If $W$ is another analytic set, then

$$\mathbf{1}_V \mathbf{1}_W \tau = \mathbf{1}_{V\cap W}\tau. \tag{2.2}$$

This action of $\mathbf{1}_V$ on the sheaf of pseudomeromorphic currents is a basic tool. In fact one can extend this calculus to all constructible sets so that (2.2) holds, see [8]. One readily checks that if $\xi$ is a smooth form, then

$$\mathbf{1}_V(\xi \wedge \tau) = \xi \wedge \mathbf{1}_V \tau. \tag{2.3}$$

If $f : Z' \to Z$ is a modification and $\tau$ is in $\mathcal{PM}_{Z'}$ then $f_*\tau$ is in $\mathcal{PM}_Z$. The same holds if $f$ is a simple projection and $\tau$ has compact support in the fiber direction. In any case we have

$$\mathbf{1}_V f_*\tau = f_*(\mathbf{1}_{f^{-1}V}\tau). \tag{2.4}$$

It is not hard to check that if $\tau$ is in $\mathcal{PM}_Z$ and $\tau'$ is in $\mathcal{PM}_{Z'}$, then $\tau \otimes \tau'$ is in $\mathcal{PM}_{Z\times Z'}$, see, e.g., [4, Lemma 3.3]. If $V \subset \mathcal{U} \subset Z$ and $V' \subset \mathcal{U}' \subset Z'$, then

$$(\mathbf{1}_V \tau) \otimes \mathbf{1}_{V'}\tau' = \mathbf{1}_{V\times V'}(\tau \otimes \tau'). \tag{2.5}$$

Another basic tool is the *dimension principle*, that states that if $\tau$ is a pseudomeromorphic $(*, p)$-current with support on an analytic set with codimension larger than $p$, then $\tau$ must vanish.

A pseudomeromorphic current $\tau$ on $Z$ has the *standard extension property*, SEP, if $\mathbf{1}_V \tau = 0$ for each germ $V$ of an analytic set with positive codimension on $Z$. The set $\mathcal{W}_Z$ of all pseudomeromorphic currents on $Z$ with the SEP is a subsheaf of $\mathcal{PM}_Z$. By (2.3), $\mathcal{W}_Z$ is closed under multiplication by smooth forms.

Let $f$ be a holomorphic function (or a holomorphic section of a Hermitian line bundle), not vanishing identically on any irreducible component of $Z$. Then $1/f$, a priori defined outside of $\{f = 0\}$, has an extension as a pseudomeromorphic current, the principal value current, still denoted by $1/f$, such that $\mathbf{1}_{\{f=0\}}(1/f) = 0$. The current $1/f$ has the SEP and

$$\frac{1}{f} = \lim_{\epsilon \to 0^+} \chi(|f|^2/\epsilon) \frac{1}{f}.$$

We say that a current $a$ on $Z$ is *almost semi-meromorphic* if there is a modification $\pi : Z' \to Z$, a holomorphic section $f$ of a line bundle $L \to Z'$ and a smooth form $\gamma$ with values in $L$ such that $a = \pi_*(\gamma/f)$, cf., [10, Section 4]. If $a$ is almost semi-meromorphic, then it is clearly pseudomeromorphic. Moreover, it is smooth outside an analytic set $V \subset Z$ of positive codimension, $a$ is in $\mathcal{W}_Z$, and in particular, $a = \lim_{\epsilon \to 0^+} \chi(|h|/\epsilon) a$ if $h$ is a holomorphic tuple that cuts out (an analytic set of positive codimension that contains) $V$. The *Zariski singular support* of $a$ is the Zariski closure of the set where $a$ is not smooth.

One can multiply pseudomeromorphic currents by almost semi-meromorphic currents; and this fact will be crucial in defining $\mathcal{W}_X^{0,*}$, when $X$ is non-reduced. Notice that if $a$ is almost semi-meromorphic in $Z$ then it also is in any open $\mathcal{U} \subset Z$.

**Proposition 2.1** ([10, Theorem 4.8, Proposition 4.9]) *Let $Z$ be a reduced space, assume that $a$ is an almost semi-meromorphic current in $Z$, and let $V$ be the Zariski singular support of $a$.*

(i) *If $\tau$ is a pseudomeromorphic current in $\mathcal{U} \subset Z$, then there is a unique pseudomeromorphic current $a \wedge \tau$ in $\mathcal{U}$ that coincides with (the naturally defined current) $a \wedge \tau$ in $\mathcal{U} \setminus V$ and such that $\mathbf{1}_V(a \wedge \tau) = 0$.*

(ii) *If $W \subset \mathcal{U}$ is any analytic subset, then*

$$\mathbf{1}_W(a \wedge \tau) = a \wedge \mathbf{1}_W \tau. \tag{2.6}$$

Notice that if $h$ is a tuple that cuts out $V$, then in view of (2.1),

$$a \wedge \tau = \lim_{\epsilon \to 0^+} \chi(|h|^2/\epsilon) a \wedge \tau. \tag{2.7}$$

It follows that if $\xi$ is a smooth form, then

$$\xi \wedge (a \wedge \tau) = (-1)^{\deg \xi \deg a} a \wedge (\xi \wedge \tau). \tag{2.8}$$

For future reference we will need the following result.

**Proposition 2.2** *Let $Z$ be a reduced space. Then $\mathcal{PM}_Z = \mathcal{W}_Z + \bar\partial \mathcal{W}_Z$.*

*Proof* First assume that $Z$ is smooth. Since $\mathcal{W}_Z$ is closed under multiplication by smooth forms, so is $\mathcal{W}_Z + \bar\partial\mathcal{W}_Z$. The statement that $\mathcal{PM}_Z = \mathcal{W}_Z + \bar\partial\mathcal{W}_Z$ is local, and since both sides are closed under multiplication by cutoff functions, we may consider a pseudomeromorphic current $\mu$ with compact support in $\mathbb{C}^n$. If $\mu$ has bidegree $(*, 0)$, then it is in $\mathcal{W}_Z$ in view of the dimension principle. Thus we assume that $\mu$ has bidegree $(*, q)$ with $q \geq 1$. Let

$$K\mu(z) = \int_\zeta k(\zeta, z) \wedge \mu(\zeta), \tag{2.9}$$

where $k$ is the Bochner–Martinelli kernel. Here (2.9) means that $K\mu = p_*(k \wedge \mu \otimes 1)$, where $p$ is the projection $\mathbb{C}^n_\zeta \times \mathbb{C}^n_z \to \mathbb{C}^n_z, \quad (\zeta, z) \mapsto z$. Recall that we have the Koppelman formula $\mu = \bar\partial K\mu + K(\bar\partial\mu)$. It is thus enough to see that $K\mu$ is in $\mathcal{W}_Z$ if $\mu$ is pseudomeromorphic. Let $\chi_\epsilon = \chi(|\zeta - z|^2/\epsilon)$. It is easy to see, by a blowup of $\mathbb{C}^n \times \mathbb{C}^n$ along the diagonal, that $k$ is almost semi-meromorphic on $\mathbb{C}^n \times \mathbb{C}^n$. Thus, by (2.7), $\chi_\epsilon k \wedge (\mu \otimes 1) \to k \wedge (\mu \otimes 1)$. In view of Proposition 2.1 it follows that $k \wedge (\mu \otimes 1)$ is pseudomeromorphic. Finally, if $W$ is a germ of a subvariety of $\mathbb{C}^n$ of positive codimension, then by (2.4) and (2.5),

$$\begin{aligned}
\mathbf{1}_W p_*(k \wedge \mu \otimes 1) &= \lim_{\epsilon\to 0^+} p_*\left(\mathbf{1}_{\mathbb{C}^n \times W}(\chi_\epsilon k \wedge (\mu \otimes 1))\right) \\
&= \lim_{\epsilon\to 0^+} p_*\left(\chi_\epsilon k \wedge (\mathbf{1}_{\mathbb{C}^n \times W}\mu \otimes 1)\right) \\
&= \lim_{\epsilon\to 0^+} p_*\left(\chi_\epsilon k \wedge (\mathbf{1}_{\mathbb{C}^n}\mu \otimes \mathbf{1}_W 1)\right) = 0,
\end{aligned}$$

since $\mathbf{1}_W 1 = 0$. Thus $K\mu$ is in $\mathcal{W}_Z$.

If $Z$ is not smooth, then we take a smooth modification $\pi : Z' \to Z$. For any $\mu$ in $\mathcal{PM}_Z$ there is some $\mu'$ in $\mathcal{PM}_{Z'}$ such that $\pi_*\mu' = \mu$, see [4, Proposition 1.2]. Since $\mu' = \tau + \bar\partial u$ with $\tau, u$ in $\mathcal{W}_{Z'}$, we have that $\mu = \pi_*\tau + \bar\partial\pi_* u$. □

### 2.1 Pseudomeromorphic currents with support on a subvariety

Let $\Omega$ be an open set in $\mathbb{C}^N$ and let $Z$ be a (reduced) subvariety of pure dimension $n$. Let $\mathcal{PM}^Z_\Omega$ denote the sheaf of pseudomeromorphic currents $\tau$ on $\Omega$ with support on $Z$, and let $\mathcal{W}^Z_\Omega$ denote the subsheaf of $\mathcal{PM}^Z_\Omega$ of currents of bidegree $(N, *)$ with the SEP with respect to $Z$, i.e., such that $\mathbf{1}_W \tau = 0$ for all germs $W$ of subvarieties of $Z$ of positive codimension. The sheaf $\mathcal{CH}^Z_\Omega$ of Coleff–Herrera currents on $Z$ is the subsheaf of $\mathcal{W}^Z_\Omega$ of $\bar\partial$-closed $(N, p)$-currents, where $p = N - n$.

*Remark 2.3* In [3,6] $\mathcal{CH}^\Omega_Z$ denotes the sheaf of pseudomeromorphic $(0, p)$-currents with support on $Z$ and the SEP with respect to $Z$. If this sheaf is tensored by the canonical bundle $K_\Omega$ we get the sheaf $\mathcal{CH}^Z_\Omega$ in this paper. Locally these sheaves are thus isomorphic via the mapping $\mu \mapsto \mu \wedge \alpha$, where $\alpha$ is a non-vanishing holomorphic $(N, 0)$-form. □

We have the following direct consequence of Proposition 2.1.

**Proposition 2.4** *Let $Z \subset \Omega$ be a subvariety of pure dimension, let $a$ be almost semi-meromorphic in $\Omega$, and assume that it is smooth generically on $Z$. If $\tau$ is in $\mathcal{W}_\Omega^Z$, then $a \wedge \tau$ is in $\mathcal{W}_\Omega^Z$ as well.*

Assume that we have local coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^p$ in $\Omega$ such that $Z = \{w = 0\}$. We will use the short-hand notation

$$\bar\partial \frac{dw}{w^{\gamma+\mathbf{1}}} := \bar\partial \frac{dw_1}{w_1^{\gamma_1+1}} \wedge \cdots \wedge \bar\partial \frac{dw_p}{w_p^{\gamma_p+1}}$$

for multiindices $\gamma = (\gamma_1, \ldots, \gamma_p)$ with $\gamma_j \geq 0$, and let $\gamma! := \gamma_1! \cdots \gamma_p!$. Notice that

$$\frac{1}{(2\pi i)^p} \bar\partial \frac{dw}{w^{\gamma+\mathbf{1}}}.\xi = \frac{1}{\gamma!} \int_z \frac{\partial^\gamma \xi}{\partial w^\gamma}(z, 0) \tag{2.10}$$

for test forms $\xi$. If $\tau$ is in $\mathcal{W}_Z$, then it follows by (2.5) and the fact that $\operatorname{supp} \bar\partial(1/w^{\gamma+\mathbf{1}}) = \{w = 0\}$ that $\tau \otimes \bar\partial(1/w^{\gamma+\mathbf{1}})$ is in $\mathcal{W}_\Omega^Z$. We have the following local structure result, see [11, Proposition 4.1 and (4.3)] and [10, Theorem 3.5].

**Proposition 2.5** *Assume that we have local coordinates $(z, w)$ such that $Z = \{w = 0\}$. Then $\tau$ in $\mathcal{W}_\Omega^Z$ has a unique representation as a finite sum*

$$\tau = \sum_\gamma \tau_\gamma \wedge dz \otimes \bar\partial \frac{dw}{w^{\gamma+\mathbf{1}}}, \quad \tau_\gamma \in \mathcal{W}_Z^{0,*}, \tag{2.11}$$

*where $dz := dz_1 \wedge \cdots \wedge dz_n$. If $\pi$ is the projection $(z, w) \mapsto z$, then*

$$\tau_\gamma \wedge dz = (2\pi i)^{-p} \pi_*(w^\gamma \tau). \tag{2.12}$$

If in addition $\bar\partial\tau$ is in $\mathcal{W}_\Omega^Z$ then its coefficients in the expansion (2.11) are $\bar\partial\tau_\gamma$, cf., (2.12). In particular, $\bar\partial\tau = 0$ if and only if $\bar\partial\tau_\gamma = 0$ for all $\gamma$.

Let us now consider the pairing between $\mathcal{W}_\Omega^Z$ and germs $\phi$ at $Z$ of smooth $(0, *)$-forms. We assume that $Z$ is smooth and that we have coordinates $(z, w)$ as before, that $\tau$ is in $\mathcal{W}_\Omega^Z$, and that (2.11) holds. Moreover, we assume that $\phi$ is a smooth $(0, *)$-form in a neighborhood of $Z$ in $\Omega$. For any positive integer $M$ we have the expansion

$$\phi = \sum_{|\alpha|<M} \phi_\alpha(z) \otimes w^\alpha + \mathscr{O}\left(|w|^M\right) + \mathscr{O}(\bar w, d\bar w), \tag{2.13}$$

where

$$\phi_\alpha(z) = \frac{1}{\alpha!} \frac{\partial\phi}{\partial w^\alpha}(z, 0)$$

and $\mathscr{O}(\bar{w}, d\bar{w})$ denotes a sum of terms, each of which contains a factor $\bar{w}_j$ or $d\bar{w}_j$ for some $j$. If $M$ in (2.13) is chosen so that $\mathscr{O}(|w|^M)\tau = 0$, then

$$\phi \wedge \tau = \sum_{\alpha \leq \gamma} \phi_\alpha \wedge \tau_\gamma \wedge dz \otimes \bar{\partial} \frac{dw}{w^{\gamma-\alpha+\mathbf{1}}},$$

i.e.,

$$\phi \wedge \tau = \sum_{\ell \geq 0} \sum_{\gamma \geq 0} \phi_\gamma \wedge \tau_{\ell+\gamma} \wedge dz \otimes \bar{\partial} \frac{dw}{w^{\ell+\mathbf{1}}}. \tag{2.14}$$

Thus $\phi \wedge \tau = 0$ if and only if $\sum_{\gamma \geq 0} \phi_\gamma \wedge \tau_{\ell+\gamma} = 0$ for all $\ell$ (which is a finite number of conditions!).

### 2.2 Intrinsic pseudomeromorphic currents on a reduced subvariety

Currents on a reduced analytic space $Z$ are defined as the dual of the sheaf of test forms. If $i : Z \to Y$ is an embedding of a reduced space $Z$ into a smooth manifold $Y$, then the push-forward mapping $\tau \mapsto i_*\tau$ gives an isomorphism between currents $\tau$ on $Z$ and currents $\mu$ on $Y$ such that $\xi \wedge \mu = 0$ for all $\xi$ in $\mathscr{E}_Y$ such that $i^*\xi = 0$.

When defining pseudomeromorphic currents in the non-reduced case it is desirable that it coincides with the previous definition in case $Z$ is reduced. From [4, Theorem 1.1] we have the following description of pseudomeromophicity from the point of view of an ambient smooth space.

**Proposition 2.6** *Assume that we have an embedding $i : Z \to Y$ of a reduced space $Z$ into a smooth manifold $Y$.*

(i) *If $\tau$ is in $\mathcal{PM}_Z$, then $i_*\tau$ is in $\mathcal{PM}_Y$.*
(ii) *If $\tau$ is a current on $Z$ such that $i_*\tau$ is in $\mathcal{PM}_Y$ and $\mathbf{1}_{Z_{sing}}(i_*\tau) = 0$, then $\tau$ is in $\mathcal{PM}_Z$.*

Since $i_*(i^*\chi(|h|^2/\epsilon)\tau) = \chi(|h|^2/\epsilon) i_*\tau$ for any current $\tau$ on $Z$, we get by (2.1) that for a subvariety $V \subset \mathcal{U} \subset Z$,

$$\mathbf{1}_V(i_*\tau) = i_*(\mathbf{1}_V\tau), \tag{2.15}$$

i.e., (2.4) holds also for an embedding $i : Z \to Y$. The condition $\mathbf{1}_{Z_{sing}}(i_*\tau) = 0$ in (ii) is fulfilled if $i_*\tau$ has the SEP with respect to $Z$.

**Corollary 2.7** *We have the isomorphism*

$$i_* : \mathcal{W}_Z^{n,*} \to \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z),$$

*where $\mathcal{J}$ is the ideal defining $Z$ in $\Omega$.*

Notice that $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$ is precisely the sheaf of $\mu$ in $\mathcal{W}_\Omega^Z$ such that $\mathcal{J}\mu = 0$.

*Proof* The map $i_*$ is injective, since it is injective on any currents, and it maps into $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$ by (2.15).

To see that $i_*$ is surjective, we take a $\mu$ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$. We assume first that we are on $Z_{\text{reg}}$, with local coordinates such that $Z_{\text{reg}} = \{w = 0\}$. If $\xi$ is in $\mathscr{E}_\Omega^{0,*}$ and $i^*\xi = 0$, then $\xi$ is a sum of forms with a factor $d\bar w_j$, $w_j$ or $\bar w_j$. Since $w_j \in \mathcal{J}$, $w_j$ annihilates $\mu$ by assumption, and since $w_j$ vanishes on the support of $\mu$, $\bar w_j$ and $d\bar w_j$ annihilate $\mu$ since $\mu$ is pseudomeromorphic. Thus, $\mu.\xi = 0$, so $\mu = i_*\tau$ for some current $\tau$ on $Z$. By Proposition 2.6 (ii), $\tau$ is pseudomeromorphic, and by (2.15), has the SEP, i.e., $\tau$ is in $\mathcal{W}_Z^{n,*}$. □

*Remark 2.8* We do not know whether $i_*\tau \in \mathcal{PM}_\Omega^Z$ implies that $\tau \in \mathcal{PM}_Z$. □

By [11, Proposition 3.12 and Theorem 3.14], we get

**Proposition 2.9** *Let $\varphi$ and $\phi_1, \ldots, \phi_m$ be currents in $\mathcal{W}_Z$. If $\varphi = 0$ on the set on $Z_{reg}$ where $\phi_1, \ldots, \phi_m$ are smooth, then $\varphi = 0$.*

## 3 Local embeddings of a non-reduced analytic space

Let $X$ be an analytic space of pure dimension $n$ with structure sheaf $\mathscr{O}_X$ and let $Z = X_{red}$ be the underlying reduced analytic space. For any point $x \in X$ there is, by definition, an open set $\Omega \subset \mathbb{C}^N$ and an ideal sheaf $\mathcal{J} \subset \mathscr{O}_\Omega$ of pure dimension $n$ with zero set $Z$ such that $\mathscr{O}_X$ is isomorphic to $\mathscr{O}_\Omega/\mathcal{J}$, and all associated primes of $\mathcal{J}$ at any point have dimension $n$. We say that we have a local embedding $i\colon X \to \Omega \subset \mathbb{C}^N$ at $x$. There is a minimal such $N$, called the Zariski embedding dimension $\hat N$ of $X$ at $x$, and the associated embedding is said to be minimal. Any two minimal embeddings are identical up to a biholomorphism, and any embedding $i\colon X \to \Omega$ has locally at $x$ the form

$$X \xrightarrow{j} \widehat\Omega \xrightarrow{\iota} \Omega := \widehat\Omega \times \mathcal{U}, \quad i = \iota \circ j, \tag{3.1}$$

where $j$ is minimal, $\mathcal{U}$ is an open subset of $\mathbb{C}_w^m$, $m = N - \hat N$, and the ideal in $\Omega$ is $\mathcal{J} = \widehat{\mathcal{J}} \otimes 1 + (w_1, \ldots, w_m)$. Notice that we then also have embeddings $Z \to \widehat\Omega \to \Omega$; however, the first one is in general not minimal.

Now consider a fixed local embedding $i\colon X \to \Omega \subset \mathbb{C}^N$, assume that $Z$ is smooth, and let $(z, w)$ be coordinates in $\Omega$ such that $Z = \{w = 0\}$. We can identify $\mathscr{O}_Z$ with holomorphic functions of $z$, and we can define an injection

$$\mathscr{O}_Z \to \mathscr{O}_X, \quad \phi(z) \mapsto \tilde\phi(z, w) = \phi(z).$$

In this way $\mathscr{O}_X$ becomes an $\mathscr{O}_Z$-module, which however depends on the choice of coordinates.

**Proposition 3.1** *Assume that $Z$ is smooth. Let $\mathscr{O}_X$ have the $\mathscr{O}_Z$-module structure from a choice of local coordinates as above. Then $\mathscr{O}_X$ is a coherent $\mathscr{O}_Z$-module, and $\mathscr{O}_X$ is a free $\mathscr{O}_Z$-module at $x$ if and only if $\mathscr{O}_X$ is Cohen–Macaulay at $x$.*

Recall that $f_1, \dots, f_m \in R$ is a *regular sequence* on the $R$-module $M$ if $f_i$ is a non zero-divisor on $M/(f_1, \dots, f_{i-1})$ for $i = 1, \dots, m$, and $(f_1, \dots, f_m)M \neq M$. If $R$ is a local ring, then $\operatorname{depth}_R M$ is the maximal length $d$ of a regular sequence $f_1, \dots, f_d$ such that $f_1, \dots, f_d$ are contained in the maximal ideal $\mathfrak{m}$; furthermore, $M$ is *Cohen–Macaulay* if $\operatorname{depth}_R M = \dim_R M$, where $\dim_R M = \dim_R(R/\operatorname{ann}_R M)$. If $R$ is Cohen–Macaulay, and $M$ has a finite free resolution over $R$, then the *Auslander–Buchsbaum* formula, [14, Theorem 19.9], gives that

$$\operatorname{depth}_R M + \operatorname{pd}_R M = \dim_R R, \tag{3.2}$$

where $\operatorname{pd}_R M$ is the length of a minimal free resolution of $M$ over $R$. In this case, $M$ is Cohen–Macaulay as an $R$-module if and only if $M$ has a free resolution over $R$ of length $\operatorname{codim} M$.

*Remark 3.2* Notice that if we have a local embedding $i : X \to \Omega$ as above, then the depth and dimension of $\mathscr{O}_{X,x} = \mathscr{O}_{\Omega,x}/\mathcal{J}$ as an $\mathscr{O}_{\Omega,x}$-module coincide with the depth and dimension of $\mathscr{O}_{X,x}$ as an $\mathscr{O}_{X,x}$-module. Thus $\mathscr{O}_{X,x}$ is Cohen–Macaulay as an $\mathscr{O}_{X,x}$-module if and only if it is Cohen–Macaulay as an $\mathscr{O}_{\Omega,x}$-module, and this holds in turn if and only if $\mathscr{O}_{\Omega,x}/\mathcal{J}$ has a free resolution of length $N - n$. □

*Proof of Proposition 3.1* By the Nullstellensatz there is an $M$ such that $w^\alpha$ is in $\mathcal{J}$ in some neighborhood of $x$ if $|\alpha| = M$. Let $\mathcal{M} \subset \mathscr{O}_\Omega$ be the ideal generated by $\{w^\alpha;\ |\alpha| = M\}$. Then $\mathcal{M}' = \mathscr{O}_\Omega/\mathcal{M}$ is a free, finitely generated $\mathscr{O}_Z$-module. Thus, $\mathscr{O}_\Omega/\mathcal{J} \simeq \mathcal{M}'/\mathcal{J}\mathcal{M}'$ is a coherent $\mathscr{O}_Z$-module, which we note is generated by the finite set of monomials $w^\alpha$ such that $|\alpha| < M$.

We shall now show that

$$\operatorname{depth}_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} = \operatorname{depth}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} \tag{3.3}$$

and

$$\dim_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} = \dim_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x}. \tag{3.4}$$

We claim that a sequence $f_1, \dots, f_m$ in $\mathscr{O}_{X,x}$ is regular (on $\mathscr{O}_{X,x}$) if and only if $\tilde{f}_1, \dots, \tilde{f}_m \in \mathscr{O}_{Z,x}$ is regular on $\mathscr{O}_{X,x}$, where $\tilde{f}_j(z) = f_j(z, 0)$. In fact, since $\mathscr{O}_{X,x}$ has pure dimension, a function $g \in \mathscr{O}_{X,x} = \mathscr{O}_{\Omega,x}/\mathcal{J}$ is a non zero-divisor if and only if $g$ is generically non-vanishing on each irreducible component of $Z(\mathcal{J})$. Thus $f_1$ is a non zero-divisor if and only if $\tilde{f}_1$ is. If it is, then $\mathscr{O}_{X,x}/(f_1) = \mathscr{O}_{\Omega,x}/(\mathcal{J} + (f_1))$ again has pure dimension. Thus the claim follows by induction, and the fact that $Z(\mathcal{J} + (f_1, \dots, f_k)) = Z(\mathcal{J} + (\tilde{f}_1, \dots, \tilde{f}_k))$. The claim immediately implies (3.3).

To see (3.4), we note first that $\dim_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x}$ is just the usual (geometric) dimension of $X$ or $Z$, i.e., in this case, $n$. Now, $\operatorname{ann}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} = \{0\}$, so $\dim_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} = \dim_{\mathscr{O}_{Z,x}} \mathscr{O}_{Z,x}/(\operatorname{ann}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x}) = \dim_{\mathscr{O}_{Z,x}} \mathscr{O}_{Z,x} = n$.

From (3.3) and (3.4) we conclude that $\mathscr{O}_{X,x}$ is Cohen–Macaulay as an $\mathscr{O}_{Z,x}$-module if and only if it is Cohen–Macaulay (as an $\mathscr{O}_{X,x}$-module). Hence, by (3.2), with $R = \mathscr{O}_{Z,x}$ and $M = \mathscr{O}_{X,x}$,

$$\operatorname{depth}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} + \operatorname{pd}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} = n,$$

so $\mathscr{O}_{X,x}$ is Cohen–Macaulay as an $\mathscr{O}_{Z,x}$-module if and only if $\mathrm{pd}_{\mathscr{O}_{Z,x}} \mathscr{O}_{X,x} = 0$, that is, if and only if $\mathscr{O}_{X,x}$ is a free $\mathscr{O}_{Z,x}$-module. □

In the proof above, we saw that $\mathscr{O}_X$ is generated (locally) as an $\mathscr{O}_Z$-module by all monomials $w^\alpha$ with $|\alpha| \leq M$ for some $M$.

**Corollary 3.3** *Assume that* $1, w^{\alpha_1}, \ldots, w^{\alpha_{\nu-1}}$ *is a minimal set of generators at a given point $x$ (clearly* 1 *must be among the generators!). Then we have a unique representation* (1.4) *for each $\phi \in \mathscr{O}_{X,x}$ if and only if $\mathscr{O}_{X,x}$ is Cohen–Macaulay.*

By coherence it follows that if $\mathscr{O}_{X,x}$ is free as an $\mathscr{O}_{Z,x}$-module, then $\mathscr{O}_{Z,x'}$ is free as an $\mathscr{O}_{Z,x'}$-module for all $x'$ in a neighborhood of $x$, and $1, w^{\alpha_1}, \ldots, w^{\alpha_{\nu-1}}$ is a basis at each such $x'$.

*Example 3.4* Let $\mathcal{J}$ be the ideal in $\mathbb{C}^4$ generated by $(w_1^2, w_2^2, w_1 w_2, w_1 z_2 - w_2 z_1)$. It is readily checked that $\mathscr{O}_X$ is a free $\mathscr{O}_Z$-module at a point on $Z = \{w_1 = w_2 = 0\}$ where $z_1$ or $z_2$ is $\neq 0$. If, say, $z_1 \neq 0$, then we can take $1, w_1$ as generators. At the point $z = (0, 0)$, e.g., $1, w_1, w_2$ form a minimal set of generators, and then $\mathscr{O}_X$ is not a free $\mathscr{O}_Z$-module, since there is a non-trivial relation between $w_1$ and $w_2$.

We claim that $\mathscr{O}_X$ has pure dimension. That is, we claim that there is no embedded associated prime ideal at $(0, 0)$; since $Z$ is irreducible, this is the same as saying that $\mathcal{J}$ is primary with respect to $Z$. To see the claim, let $\phi$ and $\psi$ be functions such that $\phi\psi$ is in $\mathcal{J}$ and $\psi$ is not in $\sqrt{\mathcal{J}}$. The latter assumption means, in view of the Nullstellensatz, that $\psi$ does not vanish identically on $Z$, i.e., $\psi = a(z) + \mathscr{O}(w)$, where $a$ does not vanish identically. Since in particular $\phi\psi$ must vanish on $Z$ it follows that $\phi = \mathscr{O}(w)$. It is now easy to see that $\phi$ is in $\mathcal{J}$. We conclude that $\mathcal{J}$ is primary. □

The pure-dimensionality of $\mathscr{O}_X$ can also be rephrased in the following way: *If $\phi$ is holomorphic and is* 0 *generically, then* $\phi = 0$. If we delete the generator $w_1 w_2$ from the definition of $\mathcal{J}$ in the example, then $\phi = w_1 w_2$ is 0 generically in $\mathscr{O}_\Omega/\mathcal{J}$ but is not identically zero. Thus $\mathcal{J}$ then has an embedded primary ideal at $(0, 0)$.

*Example 3.5* Let $\Omega = \mathbb{C}^2_{z,w}$ and $\mathcal{J} = (w^2)$ so that $Z = \{w = 0\}$. Then $1, w$ is a basis for $\mathscr{O}_X = \mathscr{O}_{\mathbb{C}^2}/(w^2)$ so each function $\phi$ in $\mathscr{O}_X$ has a unique representation $a_0(z) \otimes 1 + a_1(z) \otimes w$. Let us consider the new coordinates $\zeta = z - w$, $\eta = w$. Then $\mathcal{J} = (\eta^2)$ and since

$$a_0(z) + a_1(z)w = a_0(\zeta + \eta) + a_1(\zeta + \eta)\eta = a_0(\zeta) + (\partial a_0/\partial\zeta)(\zeta)\eta + a_1(\zeta)\eta + \mathcal{J}$$

we have the representation $a_0(\zeta) \otimes 1 + (a_1(\zeta) + \partial a_0/\partial\zeta)(\zeta) \otimes \eta$ with respect to $(\zeta, \eta)$. □

More generally, assume that, at a given point in $X_{reg} \subset \Omega$, we have two different choices $(z, w)$ and $(\zeta, \eta)$ of coordinates so that $Z = \{w = 0\} = \{\eta = 0\}$, and bases $1, \ldots, w^{\alpha_{\nu-1}}$ and $1, \ldots, \eta^{\beta_{\nu-1}}$ for $\mathscr{O}_X$ as a free module over $\mathscr{O}_Z$. Then there is a $\nu \times \nu$-matrix $L$ of holomorphic differential operators so that if $(a_j)$ is any tuple in $(\mathscr{O}_Z)^\nu$ and $(b_j) = L(a_j)$, then $a_0 \otimes 1 + \cdots + a_{\nu-1} \otimes w^{\alpha_{\nu-1}} = b_0 \otimes 1 + \cdots + b_{\nu-1} \otimes \eta^{\beta_{\nu-1}} + \mathcal{J}$.

## 4 Smooth $(0,*)$-forms on a non-reduced space $X$

Let $i\colon X \to \Omega$ be a local embedding of $X$. In order to define the sheaf of smooth $(0,*)$-forms on $X$, in analogy with the reduced case, we have to state which smooth $(0,*)$-forms $\Phi$ in $\Omega$ "vanish" on $X$, or more formally, give a meaning to $i^*\Phi = 0$. We will see, cf., Lemma 4.8 below, that the suitable requirement is that locally on $X_{reg}$, $\Phi$ belongs to $\mathscr{E}_\Omega^{0,*}\mathcal{J} + \mathscr{E}_\Omega^{0,*}\bar{\mathcal{J}}_Z + \mathscr{E}_\Omega^{0,*}d\bar{\mathcal{J}}_Z$, where $\mathcal{J}_z$ is the ideal sheaf defining $Z$. However, it turns out to be more convenient to represent the sheaf $\mathcal{K}er\, i^*$ of such forms as the annihilator of certain residue currents, and this is the path we will follow. Moreover, these currents play a central role themselves later on.

The following classical duality result is fundamental for this paper; see, e.g., [3] for a discussion.

**Proposition 4.1** *If $\mathcal{J}$ has pure dimension, then*

$$\mathcal{J} = \operatorname{ann}_{\mathscr{O}_\Omega}\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z). \tag{4.1}$$

That is, $\phi$ is in $\mathcal{J}$ if and only if $\phi\mu = 0$ for all $\mu$ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$. It is also well-known, see, e.g., [3, Theorem 1.5], that

$$\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z) \simeq \mathcal{E}xt^{\,p}(\mathscr{O}_\Omega/\mathcal{J}, K_\Omega), \tag{4.2}$$

so $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ is a coherent analytic sheaf. Locally we thus have a finite number of generators $\mu^1, \ldots, \mu^m$. In Example 6.9, we compute explicitly such generators for the ideal $\mathcal{J}$ in Example 3.4.

Let $\xi$ be a smooth $(0,*)$-form in $\Omega$. Without first giving meaning to $i^*$, we define the sheaf $\mathcal{K}er\, i^*$ by saying that $\xi$ is in $\mathcal{K}er\, i^*$ if

$$\xi \wedge \mu = 0, \quad \mu \in \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z).$$

Notice that if $\xi$ is holomorphic, then, in view of the duality (4.1), $\xi$ is in $\mathcal{K}er\, i^*$ if and only if $\xi$ is in $\mathcal{J}$.

**Definition 4.2** We define the sheaf of smooth $(0,*)$-forms on $X$ as

$$\mathscr{E}_X^{0,*} := \mathscr{E}_\Omega^{0,*}/\mathcal{K}er\, i^*. \tag{4.3}$$

We will prove below that this sheaf is independent of the choice of embedding and thus intrinsic on $X$.

Given $\phi$ in $\mathscr{E}_\Omega^{0,*}$, let $i^*\phi$ be its image in $\mathscr{E}_X^{0,*}$. In particular, $i^*\xi = 0$ means that $\xi$ belongs to $\mathcal{K}er\, i^*$, which then motivates this notation. Notice that $\mathcal{K}er\, i^*$ is a two-sided ideal in $\mathscr{E}_\Omega^{0,*}$, i.e., if $\phi$ is in $\mathscr{E}_\Omega^{0,*}$ and $\xi$ is in $\mathcal{K}er\, i^*$, then $\phi \wedge \xi$ and $\xi \wedge \phi$ are in $\mathcal{K}er\, i^*$. It follows that we have an induced wedge product on $\mathscr{E}_X^{0,*}$ such that

$$i^*(\phi \wedge \xi) = i^*\phi \wedge i^*\xi.$$

*Remark 4.3* It follows from Lemma 4.8 below that in case $X = Z$ is reduced, then $\xi$ is in $\mathcal{K}er\, i^*$ if and only its pullback to $X_{reg}$ vanishes. Thus our definition of $\mathscr{E}_X^{0,*}$ is consistent with the usual one in that case. □

**Lemma 4.4** *Using the notation of* (3.1),

$$\iota_*\colon \mathcal{H}om_{\mathscr{O}_{\widehat\Omega}}(\mathscr{O}_{\widehat\Omega}/\widehat{\mathcal{J}}, \mathcal{W}_{\widehat\Omega}^Z) \to \mathcal{H}om_{\mathscr{O}_\Omega}(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z) \tag{4.4}$$

*is an isomorphism.*

We can realize the mapping in (4.4) as the tensor product $\tau \mapsto \tau \wedge [w = 0]$, where $[w = 0]$ is the Lelong current in $\Omega$ associated with the submanifold $\{w = 0\}$.

*Proof* To begin with, $\iota_*$ maps pseudomeromorphic $(\hat N, \hat p + \ell)$-currents with support on $Z \subset \widehat\Omega$ to pseudomeromorphic $(N, p + \ell)$-currents with support on $Z \subset \Omega$. If, in addition, $\tau$ has the SEP with respect to $Z$, then $\iota_*\tau$ has, as well by (2.15). Moreover, if $\tau$ is annihilated by $\widehat{\mathcal{J}}$, then $\iota_*\tau$ is annihilated by $\mathcal{J} = \widehat{\mathcal{J}} \otimes 1 + (w)$. Thus the mapping (4.4) is well-defined, and it is injective since $\iota$ is injective.

Now assume that $\mu$ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$. Arguing as in the proof of Corollary 2.7, we see that $\mu = \iota_*\hat\mu$ for a current $\hat\mu$ in $\mathcal{W}_{\widehat\Omega}^Z$. Since $\widehat{\mathcal{J}} = \iota^*\mathcal{J}$ and $\mathcal{J}\mu = 0$, it follows that $\widehat{\mathcal{J}}\hat\mu = 0$. Thus (4.4) is surjective. □

Since $\iota_*$ is injective, $\bar\partial\tau = 0$ if and only if $\bar\partial\iota_*\tau = 0$, and thus we get

**Corollary 4.5** *Using the notation of* (3.1),

$$\iota_*\colon \mathcal{H}om_{\mathscr{O}_{\widehat\Omega}}(\mathscr{O}_{\widehat\Omega}/\widehat{\mathcal{J}}, \mathcal{CH}_{\widehat\Omega}^Z) \to \mathcal{H}om_{\mathscr{O}_\Omega}(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z) \tag{4.5}$$

*is an isomorphism.*

**Corollary 4.6** *Using the notation in* (3.1),

$$\iota^*\colon \mathscr{E}_\Omega^{0,*}/\mathcal{K}er\, i^* \to \mathscr{E}_{\widehat\Omega}^{0,*}/\mathcal{K}er\, j^*, \tag{4.6}$$

*is an isomorphism.*

*Proof* It follows immediately from (4.5) that the mapping (4.6) is well-defined and injective. Given $\widehat\xi$ in $\mathscr{E}_{\widehat\Omega}^{0,*}$, let $\xi = \widehat\xi \otimes 1$. Then $\iota^*\xi = \widehat\xi$ and so (4.6) is indeed surjective as well. □

It follows from (4.6) and (4.3) that the sheaf $\mathscr{E}_X^{0,*}$ is intrinsically defined on $X$. Since $\bar\partial$ maps $\mathcal{K}er\, i^*$ to $\mathcal{K}er\, i^*$, we have a well-defined operator $\bar\partial\colon \mathscr{E}_X^{0,*} \to \mathscr{E}_X^{0,*+1}$ such that $\bar\partial^2 = 0$. Unfortunately the sheaf complex so obtained is not exact in general, see, e.g., [6, Example 1.1] for a counterexample already in the reduced case.

### 4.1 Local representation on $X_{reg}$ of smooth forms

Recall that $X_{reg}$ is the open subset of $X$, where the underlying reduced space is smooth and $\mathscr{O}_X$ is Cohen–Macaulay. Let us fix some point in $X_{reg}$, and assume that we have local coordinates $(z, w)$ such that $Z = \{w = 0\}$. We also choose generators $1, w^{\alpha_1}, \dots, w^{\alpha_{\nu-1}}$ of $\mathscr{O}_X$ as a free $\mathscr{O}_Z$-module, which exist by Corollary 3.3, and generators $\mu^1, \dots, \mu^m$ of $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}^Z_\Omega)$.

Notice that for each smooth $(0,*)$-form $\Phi$ in $\Omega$, $\Phi \mapsto \Phi \wedge \mu^\ell$ only depends on its class $\phi$ in $\mathscr{E}_X^{0,*}$, and $\phi$ is in fact determined by these currents. By Proposition 2.5 each of these currents can (locally) be represented by a tuple of currents in $\mathcal{W}_Z^{0,*}$. Putting all these tuples together, we get a tuple in $(\mathcal{W}_Z^{0,*})^M$, where $M = M_1 + \dots + M_m$ and $M_j$ is the number of indices in (2.11) in the representation of $\mu^j$.

Recall from Corollary 3.3 that $\phi$ in $\mathscr{O}_X$ has a unique representative

$$\hat{\phi} = \hat{\phi}_0 + \hat{\phi}_1 \otimes w^{\alpha_1} + \dots + \hat{\phi}_{\nu-1} \otimes w^{\alpha_{\nu-1}}, \tag{4.7}$$

where $\hat{\phi}_j$ are in $\mathscr{O}_Z$. We thus have an $\mathscr{O}_Z$-linear morphism

$$T\colon (\mathscr{O}_Z)^\nu \to (\mathscr{O}_Z)^M. \tag{4.8}$$

The morphism is injective by Proposition 4.1, and the holomorphic matrix $T$ is therefore generically pointwise injective.

**Lemma 4.7** *Each $\phi$ in $\mathscr{E}_X^{0,*}$ has a unique representation (4.7) where $\hat{\phi}_j$ are in $\mathscr{E}_Z^{0,*}$.*

*Proof* To begin with notice that a given smooth $\phi$ must have at least one such representation. In fact, taking the finite Taylor expansion (2.13) we can forget about high order terms, since they must annihilate all the $\mu^j$, and the terms $\bar{w}$ and $d\bar{w}$ annihilate all the $\mu^j$ as well since they are pseudomeromorphic with support on $\{w = 0\}$. On the other hand, each $w^\alpha$ not in the set of generators must be of the form

$$w^\alpha = a_0 + a_1 \otimes w^{\alpha_1} + \dots + a_{\nu-1} \otimes w^{\alpha_{\nu-1}} + \mathcal{J},$$

and hence $\phi_\alpha \otimes w^\alpha$ is of the form (4.7). Thus the representation exists. To show uniqueness of the representation, we assume that $\hat{\phi}$ is in $\mathcal{K}er\, i^*$. Then the tuple $(\hat{\phi}_j)$ is mapped to 0 by the matrix $T$, and since $T$ is generically pointwise injective we conclude that each $\hat{\phi}_j$ vanishes. □

By the above proof we get

**Lemma 4.8** *A smooth $(0,*)$-form $\xi$ in $\Omega$ is in $\mathcal{K}er\, i^*$ if and only if $\xi$ is in $\mathscr{E}_\Omega^{0,*}\mathcal{J} + \mathscr{E}_\Omega^{0,*}\bar{\mathcal{J}}_Z + \mathscr{E}_\Omega^{0,*} d\bar{\mathcal{J}}_Z$ on $X_{reg}$, where $\mathcal{J}_Z$ is the radical sheaf of $Z$.*

*Remark 4.9* This is *not* the same as saying that $\xi$ is in $\mathscr{E}_\Omega^{0,*}\mathcal{J} + \mathscr{E}_\Omega^{0,*}\bar{\mathcal{J}}_Z + \mathscr{E}_\Omega^{0,*} d\bar{\mathcal{J}}_Z$ at singular points. For a simple counterexample, consider $\phi = x\bar{y}$ on the reduced space $Z = \{xy = 0\} \subset \mathbb{C}^2$.

However, this can happen also when $Z$ is irreducible at a point. For example, the variety $Z = \{x^2y - z^2 = 0\} \subset \mathbb{C}^3$ is irreducible at 0, but there exist points arbitrarily close to 0 such that $(Z, z)$ is not irreducible. In this case, the ideal of smooth functions vanishing on $(Z, 0)$ is strictly larger than $\mathscr{E}_\Omega^{0,0}\mathcal{J}_{Z,0} + \mathscr{E}_\Omega^{0,0}\bar{\mathcal{J}}_{Z,0}$ see [26, Proposition 9, Chapter IV], and [25, Theorem 3.10, Chapter VI]. □

*Remark 4.10* It is easy to check that if we have the setting as in the discussion at the end of Sect. 3 but $(a_j)$ is instead a tuple in $\mathscr{E}_Z^{0,*}$, then we can still define $(b_j) = L(a_j)$ if we consider the derivatives in $L$ as Lie derivatives; in fact, since $a_j$ has no holomorphic differentials, $L$ only acts on the smooth coefficients, and it is easy to check that $a_0 \otimes 1 + \cdots + a_{\nu-1} \otimes w^{\alpha_{\nu-1}}$ and $b_0 \otimes 1 + \cdots + b_{\nu-1} \otimes \eta^{\beta_{\nu-1}}$ are equal modulo $\mathscr{E}_\Omega^{0,*}\mathcal{J} + \mathscr{E}_\Omega^{0,*}\bar{\mathcal{J}}_Z + \mathscr{E}_\Omega^{0,*}d\bar{\mathcal{J}}_Z$, and thus define the same element in $\mathscr{E}_X^{0,*}$. □

For future needs we prove in Sect. 6.1:

**Lemma 4.11** *The morphism $T$ is pointwise injective.*

We can thus choose a holomorphic matrix $A$ such that

$$0 \to \mathscr{O}_Z^\nu \xrightarrow{T} \mathscr{O}_Z^M \xrightarrow{A} \mathscr{O}_Z^{M'} \tag{4.9}$$

is pointwise exact, and we can also find holomorphic matrices $S$ and $B$ such that

$$I = TS + BA. \tag{4.10}$$

## 5 Intrinsic $(n, *)$-currents on $X$

In analogy with the reduced case we have the following definition when $X$ is possibly non-reduced.

**Definition 5.1** The sheaf $\mathcal{C}_X^{n,q}$ of $(n, q)$-currents on $X$ is the dual sheaf of $(0, n-q)$-test forms, i.e., forms in $\mathscr{E}_X^{0,n-q}$ with compact support.

Here, just as in the case of reduced spaces, cf., for example [19, Section 4.2], the space of smooth forms $\mathscr{E}_X^{0,n-q}$ is equipped with the quotient topology induced by a local embedding.

More concretely, this means that given an embedding $i : X \to \Omega$, currents $\psi$ in $\mathcal{C}_X^{n,q}$ precisely correspond to the $(N, N-n+q)$-currents $\tau$ on $\Omega$ that vanish on $\mathcal{K}er\, i^*$. Since $\mathcal{K}er\, i^*$ is a two-sided ideal in $\mathscr{E}_\Omega^{0,*}$ this holds if and only if $\xi \wedge \tau = 0$ for all $\xi$ in $\mathcal{K}er\, i^*$. It is natural to write $\tau = i_*\psi$ so that

$$i_*\psi.\xi = \psi.i^*\xi.$$

Clearly, we get a mapping $\bar\partial : \mathcal{C}_X^{n,q} \to \mathcal{C}_X^{n,q+1}$ such that $\bar\partial^2 = 0$.

**Proposition 5.2** *If $\tau$ is in $\mathcal{W}_\Omega^Z$ and $\mathcal{J}\tau = 0$, then $\xi \wedge \tau = 0$ for all smooth $\xi$ such that $i^*\xi = 0$.*

*Proof* Because of the SEP it is enough to prove that $\xi\wedge\tau=0$ on $X_{reg}$. By assumption, $\mathcal{J}$ annihilates $\tau$, and by general properties of pseudomeromorphic currents, since $\tau$ has support on $Z$, $\bar{\mathcal{J}}_Z$ and $d\bar{\mathcal{J}}_Z$ annihilate $\tau$. Thus the proposition follows by Lemma 4.8. □

**Definition 5.3** An $(n,*)$-current $\psi$ on $X$ is in $\mathcal{W}_X^{n,*}$ if $i_*\psi$ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z)$.

By definition we thus have the isomorphism

$$i_*\colon \mathcal{W}_X^{n,*}\simeq \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z). \tag{5.1}$$

It follows from Lemma 4.4 that $\mathcal{W}_X^{n,*}$ is intrinsically defined.

*Remark 5.4* By Corollary 2.7, this definition is consistent with the previous definition of $\mathcal{W}_X^{n,*}$ when $X$ is reduced. We cannot define $\mathcal{PM}_X^{n,*}$ in the analogous simple way, cf., Remark 2.8. □

**Definition 5.5** If $\psi$ is in $\mathcal{W}_X^{n,*}$ and $a$ is an almost semi-meromorphic $(0,*)$-current on $\Omega$ that is generically smooth on $Z$, then the product $a\wedge\psi$ is a current in $\mathcal{W}_X^{n,*}$ defined as follows: By definition, $i_*\psi$ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z)$ and by Proposition 2.4 and (2.8), one can define $a\wedge i_*\psi$ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z)$; now $a\wedge\psi$ is the unique current in $\mathcal{W}_X^{n,*}$ such that $i_*(a\wedge\psi)=a\wedge i_*\psi$.

By (2.7),

$$a\wedge\psi=\lim_{\epsilon\to 0^+}\chi(|h|^2/\epsilon)a\wedge\psi \tag{5.2}$$

if $h$ cuts out the Zariski singular support of $a$.

**Definition 5.6** We let $\omega_X^n$ be the sheaf of $\bar\partial$-closed currents in $\mathcal{W}_X^{n,0}$.

This sheaf corresponds via $i_*$ to $\bar\partial$-closed currents in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{W}_\Omega^Z)$ so we have the isomorphism

$$i_*\colon \omega_X^n\simeq \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{CH}_\Omega^Z). \tag{5.3}$$

When $X$ is reduced $\omega_X^n$ is the sheaf of $(n,0)$-forms that are $\bar\partial$-closed in the Barlet–Henkin–Passare sense. Let $\mu^1,\dots,\mu^m$ be a set of generators for $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{CH}_\Omega^Z)$. They correspond via (5.3) to a set of generators $h^1,\dots,h^m$ for the $\mathscr{O}_X$-module $\omega_X^n$.

We will also need a definition of $\mathcal{PM}_X^{n,*}$. Let $\mathcal{F}_X$ be the subsheaf of $\mathcal{C}_X^{n,*}$ of $\tau$ such that $i_*\tau$ is in $\mathcal{PM}_\Omega^Z$. If $\tau$ is a section of $\mathcal{F}_X$ and $W$ is a subvariety of some open subset of $Z$, then $\mathbf{1}_W i_*\tau$ is in $\mathcal{PM}_\Omega^Z$, and by (2.3), $\mathbf{1}_W i_*\tau$ is annihilated by $\mathcal{K}er\, i^*$. Hence we can define $\mathbf{1}_W\tau$ as the unique current in $\mathcal{F}_X$ such that $i_*\mathbf{1}_W\tau=\mathbf{1}_W i_*\tau$. Clearly, $\mathbf{1}_W\tau$ has support on $W$ and it is easily checked that the computational rule (2.3) holds also in $\mathcal{F}_X$. Moreover, $\mathcal{F}_X$ is closed under $\bar\partial$ since $\mathcal{PM}_\Omega^Z$ is.

**Definition 5.7** The sheaf $\mathcal{PM}_X^{n,*}$ is the smallest subsheaf of $\mathcal{F}_X$ that contains $\mathcal{W}_X^{n,*}$ and is closed under $\bar\partial$ and multiplication by $\mathbf{1}_W$ for all germs $W$ of subvarieties of $Z$.

In view of Proposition 2.2 this definition coincides with the usual definition in case $X$ is reduced. It is readily checked that the dimension principle holds for $\mathcal{F}_X$, and hence it also holds for the (possibly smaller) sheaf $\mathcal{PM}_X^{n,*}$, and in addition, (2.3) holds for forms $\xi$ in $\mathscr{E}_X^{0,*}$ and $\tau$ in $\mathcal{PM}_X^{n,*}$.

## 6 Structure form on $X$

Let $i: X \to \Omega \subset \mathbb{C}^N$ be a local embedding as before, let $p = N - n$ be the codimension of $X$, and let $\mathcal{J}$ be the associated ideal sheaf on $\Omega$. In a slightly smaller set, still denoted $\Omega$, there is a free resolution

$$0 \to \mathscr{O}(E_{N_0}) \xrightarrow{f_{N_0}} \cdots \xrightarrow{f_3} \mathscr{O}(E_2) \xrightarrow{f_2} \mathscr{O}(E_1) \xrightarrow{f_1} \mathscr{O}(E_0) \tag{6.1}$$

of $\mathscr{O}_\Omega/\mathcal{J}$; here $E_k$ are trivial vector bundles over $\Omega$ and $E_0$ is the trivial line bundle. This resolution induces a complex of vector bundles

$$0 \to E_{N_0} \xrightarrow{f_{N_0}} \cdots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \tag{6.2}$$

that is pointwise exact outside $Z$. Let $X_k$ be the set where $f_k$ does not have optimal rank. Then

$$\cdots \subset X_{k+1} \subset X_k \subset \cdots \subset X_{p+1} \subset X_p = \cdots = X_1 = Z;$$

these sets are independent of the choice of resolution and thus invariants of $\mathscr{O}_\Omega/\mathcal{J}$. Since $\mathscr{O}_\Omega/\mathcal{J}$ has *pure* codimension $p$,

$$\operatorname{codim} X_k \geq k+1, \quad \text{for}\, k \geq p+1, \tag{6.3}$$

see [14, Corollary 20.14]. Thus there is a free resolution (6.1) if and only if $X_k = \emptyset$ for $k > N_0$. Unless $n = 0$ (which is not interesting in relation to the $\bar\partial$-equation), we can thus choose the resolution so that $N_0 \leq N-1$. The variety $X$ is Cohen–Macaulay at a point $x$, i.e., the sheaf $\mathscr{O}_\Omega/\mathcal{J}$ is Cohen–Macaulay at $x$, if and only if $x \notin X_{p+1}$. Notice that $Z \backslash (X_{reg})_{red} = Z_{sing} \cup X_{p+1}$. The sets $X_k$ are independent of the choice of embedding, see [9, Lemma 4.2], and are thus intrinsic subvarieties of $Z = X_{red}$, and they reflect the complexity of the singularities of $X$.

Let us now choose Hermitian metrics on the bundles $E_k$. We then refer to (6.1) as a *Hermitian resolution* of $\mathscr{O}_\Omega/\mathcal{J}$ in $\Omega$. In $\Omega\backslash X_k$ we have a well-defined vector bundle morphism $\sigma_{k+1}: E_k \to E_{k+1}$, if we require that $\sigma_{k+1}$ vanishes on $(\operatorname{Im} f_{k+1})^\perp$, takes values in $(\mathcal{K}er\, f_{k+1})^\perp$, and that $f_{k+1}\sigma_{k+1}$ is the identity on $\operatorname{Im} f_{k+1}$. Following [7, Section 2] we define smooth $E_k$-valued forms

$$u_k = (\bar\partial\sigma_k)\cdots(\bar\partial\sigma_2)\sigma_1 = \sigma_k(\bar\partial\sigma_{k-1})\cdots(\bar\partial\sigma_1) \tag{6.4}$$

in $\Omega\backslash X$; for the second equality, see [7, (2.3)]. We have that

$$f_1 u_1 = 1, \quad f_{k+1}u_{k+1} - \bar\partial u_k = 0, \quad k \geq 1,$$

in $\Omega\backslash X$. If $f := \oplus f_k$ and $u := \sum u_k$, then these relations can be written economically as $\nabla_f u = 1$, where $\nabla_f := f - \bar\partial$. To make the algebraic machinery work properly one has to introduce a superstructure on the bundle $E =: \oplus E_k$ so that vectors in $E_{2k}$ are even and vectors in $E_{2k+1}$ are odd; hence $f$, $\sigma := \oplus\sigma_k$, and $u := \sum u_k$ are odd. For details, see [7]. It turns out that $u$ has a (necessarily unique) almost semi-meromorphic extension $U$ to $\Omega$. The residue current $R$ is defined by the relation

$$\nabla_f U = 1 - R. \tag{6.5}$$

It follows directly that $R$ is $\nabla_f$-closed. In addition, $R$ has support on $Z$ and is a sum $\sum R_k$, where $R_k$ is a pseudomeromorphic $E_k$-valued current of bidegree $(0, k)$. It follows from the dimension principle that $R = R_p + R_{p+1} + \cdots + R_N$. If we choose a free resolution that ends at level $N - 1$, then $R_N = 0$. If $X$ is Cohen–Macaulay and $N_0 = p$ in (6.1), then $R = R_p$, and the $\nabla_f$-closedness implies that $R$ is $\bar\partial$-closed.

If $\phi$ is in $\mathcal{J}$ then $\phi R = 0$ and in fact, $\mathcal{J} = \operatorname{ann} R$, see [7, Theorem 1.1].

*Remark 6.1* In case $\mathcal{J}$ is generated by the single non-trivial function $f$, then we have the free resolution $0 \to \mathscr{O}_\Omega \xrightarrow{f} \mathscr{O}_\Omega \to \mathscr{O}_\Omega/(f) \to 0$; thus $U$ is just the principal value current $1/f$ and $R = \bar\partial(1/f)$. More generally, if $f = (f_1, \ldots, f_p)$ is a complete intersection, then

$$R = \bar\partial\frac{1}{f_p} \wedge \cdots \wedge \bar\partial\frac{1}{f_1},$$

where the right hand side is the so-called Coleff–Herrera product of $f$, see for example [1, Corollary 3.5]. □

There are almost semi-meromorphic $\alpha_k$ in $\Omega$, cf., [7, Section 2] and the proof of [6, Proposition 3.3], that are smooth outside $X_k$, such that

$$R_{k+1} = \alpha_{k+1} R_k \tag{6.6}$$

outside $X_{k+1}$ for $k \geq p$. In view of (6.3) and the dimension principle, $\mathbf{1}_{X_{k+1}} R_{k+1} = 0$ and hence (6.6) holds across $X_{k+1}$, i.e., $R_{k+1}$ is indeed equal to the product $\alpha_{k+1} R_k$ in the sense of Proposition 2.1. In particular, it follows that $R_k$ has the SEP with respect to $Z$.

In this section, we let $(z_1, \ldots, z_N)$ denote coordinates on $\mathbb{C}^N$, and let $dz := dz_1 \wedge \cdots \wedge dz_N$.

**Lemma 6.2** *There is a matrix of almost semi-meromorphic currents $b$ such that*

$$R \wedge dz = b\mu, \tag{6.7}$$

*where $\mu$ is a tuple of currents in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$.*

*Proof* As in [6, Section 3], see also [32, Proposition 3.2], one can prove that $R_p = \sigma_F \mu$, where $\mu$ is a tuple of currents in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ and $\sigma_F$ is an almost semi-meromorphic current that is smooth outside $X_{p+1}$.

Let $b_p = \sigma_F$ and $b_k = \alpha_k \cdots \alpha_{p+1}\sigma_F$ for $k \geq p + 1$. Then each $b_k$ is almost semi-meromorphic, cf., [10, Section 4.1]. In view of (6.6) we have that $R_k = b_k\mu$ outside $X_{p+1}$ since $b_k$ is smooth there. It follows by the SEP that it holds across $X_{p+1}$ as well since $R_k$ has the SEP with respect to $Z$. We then take $b = b_p + b_{p+1} + \cdots$. □

By Proposition 2.4 we get

**Corollary 6.3** *The current $R \wedge dz$ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$.*

From Lemma 6.2, Corollary 6.3, (5.1), and (5.3) we get the following analogue to [6, Proposition 3.3]:

**Proposition 6.4** *Let (6.1) be a Hermitian resolution of $\mathscr{O}_\Omega/\mathcal{J}$ in $\Omega$, and let $R$ be the associated residue current. Then there exists a (unique) current $\omega$ in $\mathcal{W}_X^{n,*}$ such that*

$$i_*\omega = R \wedge dz. \tag{6.8}$$

*There is a matrix $b$ of almost semi-meromorphic $(0,*)$-currents in $\Omega$, smooth outside of $X_{p+1}$, and a tuple $\vartheta$ of currents in $\omega_X^n$ such that*

$$\omega = b\vartheta. \tag{6.9}$$

*More precisely, $\omega = \omega_0 + \omega_1 + \cdots + \omega_n$,[1] where $\omega_k \in \mathcal{W}^{n,k}(X, E_{p+k})$, and if $f^j := f_{p+j}$, then*

$$f^0\omega_0 = 0, \quad f^{j+1}\omega_{j+1} - \bar\partial\omega_j = 0, \textit{for } j \geq 0. \tag{6.10}$$

We will also use the short-hand notation $\nabla_f\omega = 0$. As in the reduced case, following [6], we say that $\omega$ is a *structure form* for $X$. The products in (6.9) are defined according to Definition 5.5.

*Remark 6.5* Recall that $X_{p+1} = \emptyset$ if $X$ is Cohen–Macaulay, so in that case $\omega = b\vartheta$, where $b$ is smooth. If we take a free resolution of length $p$, then $\omega = \omega_0$, and $\bar\partial\omega_0 = f^1\omega_1 = 0$, so $\omega$ is in $\omega_X^n$. □

*Remark 6.6* If $X = \{f = 0\}$ is a reduced hypersurface in $\Omega$, then $R = \bar\partial(1/f)$ and $\omega$ is the classical Poincaré residue form on $X$ associated with $f$, which is a meromorphic form on $X$. More generally, if $X$ is reduced, since forms in $\omega_X^n$ are then meromorphic, by (6.9), $\omega$ can be represented by almost semi-meromorphic forms on $X$.

We now consider the case when $X$ is non-reduced. We recall that a differential operator is a Noetherian operator for an ideal $\mathcal{J}$ if $\mathcal{L}\varphi \in \sqrt{\mathcal{J}}$ for all $\varphi \in \mathcal{J}$. It is proved by Björk, [13], see also [32, Theorem 2.2], that if $\mu \in \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$, then there exists a Noetherian operator $\mathcal{L}$ for $\mathcal{J}$ with meromorphic coefficients such that the action of $\mu$ on $\xi$ equals the integral of $\mathcal{L}\xi$ over $Z$. By (5.3), the action of $h$ in $\omega_X^n$ on $\xi$ in $\mathscr{E}_X^{0,*}$ can then be expressed as

$$h.\xi = \int_Z \mathcal{L}\xi.$$

[1] In [6, Proposition 3.3], the sum ends with $\omega_{n-1}$ instead of $\omega_n$, which, as remarked above, one can indeed assume when $n \geq 1$ and the resolution is chosen to be of length $\leq N-1$.

One can then verify using this formula and (6.9) that the action of the structure form $\omega$ on a test form $\xi$ in $\mathscr{E}_X^{0,*}$ equals

$$\omega.\xi = \int_Z \tilde{\mathcal{L}}\xi,$$

where $\tilde{\mathcal{L}}$ is now a tuple of Noetherian operators for $\mathcal{J}$ with almost semi-meromorphic coefficients, cf., [32, Section 4]. $\square$

Notice that (6.1) gives rise to the dual Hermitian complex

$$0 \to \mathscr{O}(E_0^*) \xrightarrow{f_1^*} \cdots \to \mathscr{O}(E_{p-1}^*) \xrightarrow{f_p^*} \mathscr{O}(E_p^*) \xrightarrow{f_{p+1}^*} \cdots . \tag{6.11}$$

Let $\xi = \xi_0 \wedge dz$ be a holomorphic section of the sheaf

$$\mathcal{H}om(E_p, K_\Omega) \simeq \mathscr{O}(E_p^*) \otimes \mathscr{O}(K_\Omega)$$

such that $f_{p+1}^*\xi_0 = 0$. Then $\bar\partial(\xi_0\omega_0) = \pm\xi_0\bar\partial\omega_0 = \pm\xi_0 f_{p+1}\omega_1 = \pm(f_{p+1}^*\xi_0)\omega_1 = 0$, so that $\xi_0\omega_0$ is in $\omega_X^n$. Moreover, if $\xi_0 = f_p^*\eta$ for $\eta$ in $\mathscr{O}(E_{p-1}^*)$, then $\xi_0\omega_0 = f_p^*\eta\omega_0 = \pm\eta f_p\omega_0 = 0$. We thus have a sheaf mapping

$$\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \to \omega_X^n, \quad \xi_0 \wedge dz \mapsto \xi_0\omega_0. \tag{6.12}$$

**Proposition 6.7** *The mapping (6.12) is an isomorphism, which establishes an intrinsic isomorphism*

$$\mathcal{E}xt^{\,p}(\mathscr{O}_\Omega/\mathcal{J}, K_\Omega) \simeq \omega_X^n. \tag{6.13}$$

*Proof* If $h$ is in $\omega_X^n$, then $i_*h$ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$. We have mappings

$$\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \to \omega_X^n \xrightarrow{\simeq} \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z), \tag{6.14}$$

where the first mapping is (6.12), and the second is $h \mapsto i_*h$. In view of (6.8), the composed mapping is $\xi = \xi_0 \wedge dz \mapsto \xi R_p = \xi_0 R_p \wedge dz$.[2] This mapping is an intrinsic isomorphism

$$\mathcal{E}xt^{\,p}(\mathscr{O}_\Omega/\mathcal{J}, K_\Omega) \simeq \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$$

according to [3, Theorem 1.5]. It follows that (6.12) also establishes an intrinsic isomorphism. $\square$

In particular it follows that $\omega_X^n$ is coherent, and we have:
If $\xi^1, \ldots, \xi^m$ are generators of $\mathcal{H}^p(\mathcal{H}om(E_\bullet^*, K_\Omega)))$, where $\xi^\ell = \xi_0^\ell \wedge dz$, then $h^\ell := \xi_0^\ell\omega_0$, $\ell = 1, \ldots, m$, generate the $\mathscr{O}_X$-module $\omega_X^n$, and $\mu^\ell = i_*h^\ell = \xi^\ell R_p$ generate the $\mathscr{O}_\Omega$-module $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$.

[2] There is a superstructure involved, with respect to which $R_p$ has even degree, and therefore $dz \wedge R_p = R_p \wedge dz$, explaining the lack of a sign in the last equality, see [6] or [7].

*Remark 6.8* The isomorphism

$$\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \xrightarrow{\simeq} \mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z) \tag{6.15}$$

was well-known since long ago, the contribution in [3] was the realization $\xi \mapsto \xi R_p$.□

We give here an example where we can explicitly compute generators of $\mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$.

*Example 6.9* Let $\mathcal{J}$ be as in Example 3.4. We claim that $\mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ is generated by

$$\mu_1 := \bar\partial\frac{1}{w_1} \wedge \bar\partial\frac{1}{w_2} \wedge dz \wedge dw \text{ and } \mu_2 := \left(z_1\bar\partial\frac{1}{w_1^2} \wedge \bar\partial\frac{1}{w_2} + z_2\bar\partial\frac{1}{w_1} \wedge \bar\partial\frac{1}{w_2^2}\right) \wedge dz \wedge dw.$$

In order to prove this claim, we use the comparison formula for residue currents from [21], which states that if $\mathcal{O}(F_\bullet)$ and $\mathcal{O}(E_\bullet)$ are free resolutions of $\mathcal{O}_\Omega/\mathcal{I}$ and $\mathcal{O}_\Omega/\mathcal{J}$, respectively, where $\mathcal{I}$ and $\mathcal{J}$ have codimension $\geq p$, and $a : F_\bullet \to E_\bullet$ is a morphism of complexes, then there exists a $\mathcal{H}om(F_0, E_{p+1})$-valued current $M_{p+1}$ such that $R_p^E a_0 = a_p R_p^F + f_{p+1} M_{p+1}$. If $\xi$ is in $\mathcal{K}er\ f_{p+1}^*$, we thus get that

$$\xi R_p^E a_0 = \xi a_p R_p^F. \tag{6.16}$$

We will apply this with $\mathcal{O}_\Omega(E_\bullet)$ as the free resolution

$$0 \to \mathcal{O}_\Omega \xrightarrow{f_3} \mathcal{O}_\Omega^4 \xrightarrow{f_2} \mathcal{O}_\Omega^4 \xrightarrow{f_1} \mathcal{O}_\Omega \to \mathcal{O}_\Omega/\mathcal{J} \to 0,$$

where

$$f_3 = \begin{bmatrix} w_2 \\ -w_1 \\ z_2 \\ -z_1 \end{bmatrix}, \ f_2 = \begin{bmatrix} z_2 & 0 & -w_2 & 0 \\ -z_1 & z_2 & w_1 & -w_2 \\ 0 & -z_1 & 0 & w_1 \\ -w_1 & -w_2 & 0 & 0 \end{bmatrix} \quad \text{and}$$
$$f_1 = \begin{bmatrix} w_1^2 & w_1w_2 & w_2^2 & z_2w_1 - z_1w_2 \end{bmatrix},$$

and the Koszul complex $(F, \delta_{\mathbf{w}^2})$ generated by $\mathbf{w}^2 := (w_1^2, w_2^2)$, which is a free resolution of $\mathcal{O}/(w_1^2, w_2^2)$. We then take the morphism of complexes $a : F_\bullet \to E_\bullet$ given by

$$a_2 = \begin{bmatrix} 0 \\ 0 \\ w_2 \\ w_1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } a_0 = \begin{bmatrix} 1 \end{bmatrix}.$$

Since the current $R_2^F$ is equal to the Coleff–Herrera product $\bar\partial(1/w_1^2) \wedge \bar\partial(1/w_2^2)$, cf., Remark 6.1, we thus get by (6.16) and Remark 6.8 that $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ is generated by

$$(\mathcal{K}er\ f_3^*)a_2\bar\partial\frac{1}{w_1^2} \wedge \bar\partial\frac{1}{w_2^2}.$$

A straightforward calculation gives the generators $\mu_1$ and $\mu_2$ above. $\square$

### 6.1 Proof of Lemma 4.11

Since $T$ is generically injective, it is clearly injective if $n = 0$. We are going to reduce to this case. Fix the point $0 \in Z$ and let $\mathcal{I}$ be the ideal generated by $z = (z_1, \ldots, z_n)$.

Let $\mathscr{O}(E_\bullet)$ be a free Hermitian resolution of $\mathscr{O}_\Omega/\mathcal{J}$ of minimal length $p = N - n$ at 0 and let $R^E$ be the associated residue current. Recall that the canonical isomorphism (6.15) is realized by $\xi \mapsto \xi R_p^E$. Let $F_\bullet$ be the Koszul complex generated by $z$; then $\mathscr{O}(F_\bullet)$ is a free resolution of $\mathscr{O}_\Omega/\mathcal{I}$. Since $\mathcal{J}$ and $\mathcal{I}$ are Cohen–Macaulay and intersect properly in $\Omega$, the complex $\mathscr{O}_\Omega((E \otimes F)_\bullet)$ is a free resolution of $\mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I})$, and the corresponding residue current is

$$R_N^{E\otimes F} = R_p^E \wedge R_n^F$$

according to [2, Theorem 4.2]. From [3, Theorem 1.5] again it follows that the canonical isomorphism

$$\mathcal{H}^N(\mathcal{H}om((E \otimes F)_\bullet, K_\Omega)) \to \mathcal{H}om(\mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I}), \mathcal{CH}_\Omega^{\{0\}})$$

is given by $\eta \mapsto \eta R_N^{E\otimes F}$.

Let $\mu^1, \ldots, \mu^m$ be a minimal set of generators for $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ at 0. Then $\mu^j = \xi^j R_p^E$, where $\xi^j$ is a minimal set of generators for $\mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega))$. Notice that

$$\mathcal{H}^N(\mathcal{H}om((E \otimes F)_\bullet, K_\Omega)) = \mathcal{H}^p(\mathcal{H}om(E_\bullet, K_\Omega)) \otimes_{\mathscr{O}} \mathcal{H}^n(\mathcal{H}om(F_\bullet, \mathscr{O}_\Omega)).$$

Since $\mathcal{H}^n(\mathcal{H}om(F_\bullet, \mathscr{O}_\Omega))$ is generated by 1, it follows that $\mathcal{H}^N(\mathcal{H}om((E\otimes F)_\bullet, K_\Omega))$ is generated by $\xi^j \otimes 1$. We conclude that $\mathcal{H}om(\mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I}), \mathcal{CH}_\Omega^{\{0\}})$ is generated by $\xi^j \otimes 1 \cdot R_p^E \wedge R_n^F = \mu^j \wedge \mu^z$, $j = 1, \ldots, m$, where $R_n^F = \mu^z = \bar\partial(1/z^{\mathbf{1}})$.

If $1, \ldots, w^{\alpha_{\nu-1}}$ is a basis for $\mathscr{O}_\Omega/\mathcal{J}$ as an $\mathscr{O}_Z$-module, then it is also a basis for $\mathscr{O}_{X_0} := \mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I})$ as a module over $\mathscr{O}_{\{0\}} \simeq \mathbb{C}$. Since $\phi\bar\partial(1/z^{\mathbf{1}}) = \phi(0, \cdot)\bar\partial(1/z^{\mathbf{1}})$

we have that

$$
\begin{aligned}
\phi(z,w)\mu^j \wedge \mu^z &= \phi(z,w) \sum a_\ell^j(z) \bar{\partial} \frac{1}{w^{\ell+\mathbf{1}}} \wedge \bar{\partial} \frac{1}{z^{\mathbf{1}}} \\
&= \phi(0,w) \sum a_\ell^j(0) \bar{\partial} \frac{1}{w^{\ell+\mathbf{1}}} \wedge \bar{\partial} \frac{1}{z^{\mathbf{1}}}.
\end{aligned}
$$

The morphism constructed in (4.8) for $X_0$ instead of $X$ is then $T_0 = T(0)$, where $T$ is the morphism (4.8) for $X$. Thus $T(0)$ is injective.

## 7 The intrinsic sheaf $\mathcal{W}_X^{0,*}$ on $X$

Our aim is to find a fine resolution of $\mathscr{O}_X$ and since the complex (1.1) is not exact in general when $X$ is singular we have to consider larger fine sheaves; we first define sheaves $\mathcal{W}_X^{0,*} \supset \mathscr{E}_X^{0,*}$ of $(0,*)$-currents. Given a local embedding $i\colon X \to \Omega$ at a point on $X_{reg}$ and local coordinates $(z,w)$ as before, it is natural, in view of Lemma 4.7, to require that an element in $\mathcal{W}_X^{0,*}$ shall have a unique representation

$$
\phi = \widehat{\phi}_0 \otimes 1 + \widehat{\phi}_1 \otimes w^{\alpha_1} + \cdots + \widehat{\phi}_{\nu-1} \otimes w^{\alpha_{\nu-1}}, \tag{7.1}
$$

where $\widehat{\phi}_j$ are in $\mathcal{W}_Z^{0,*}$. In view of Remark 4.10 we should expect that the same transformation rules hold as for smooth $(0,*)$-forms. In particular it is then necessary that $\mathcal{W}_Z^{0,*}$ is closed under the action of holomorphic differential operators, which in fact is true, see Proposition 7.11 below. We must also define a reasonable extension of these sheaves across $X_{sing}$. Before we present our formal definition we make a preliminary observation.

**Lemma 7.1** *If $\phi$ has the form (7.1) and $\tau$ is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$, expressed in the form (2.11), then*

$$
\phi \wedge \tau := \sum_i \sum_{\gamma \geq \alpha_i} \widehat{\phi}_i \wedge \tau_\gamma \wedge dz \otimes \bar{\partial} \frac{dw}{w^{\gamma - \alpha_i + \mathbf{1}}} \tag{7.2}
$$

*is in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$.*

*Proof* The right hand side defines a current in $\mathcal{W}_\Omega^Z$ since $\widehat{\phi}_i$ are in $\mathcal{W}_Z^{0,*}$ and $\tau_\gamma$ are in $\mathscr{O}_Z$. We have to prove that it is annihilated by $\mathcal{J}$. Take $\xi$ in $\mathcal{J}$. On the subset of $Z$ where $\widehat{\phi}_0, \ldots, \widehat{\phi}_{\nu-1}$ are all smooth, $\phi \wedge \tau$, as defined above, is just multiplication of the smooth form $\phi$ by $\tau$, and thus $\xi\phi \wedge \tau = 0$ there. We have a unique representation

$$
\xi\phi \wedge \tau = \sum_{\ell \geq 0} a_\ell(z) \wedge dz \otimes \bar{\partial} \frac{dw}{w^{\ell+\mathbf{1}}},
$$

with $a_\ell$ in $\mathcal{W}_Z^{0,*}$. Since $a_\ell$ vanish on the set where all $\widehat{\phi}_j$ are smooth, we conclude from Proposition 2.9 that $a_\ell$ vanish identically. It follows that $\xi\phi \wedge \tau = 0$. $\square$

If $\phi$ has the form (7.1) in a neighborhood of some point $x \in X_{reg}$ and $h$ is in $\omega_X^n$, then we get an element $\phi \wedge h$ in $\mathcal{W}_X^{n,*}$ defined by $i_*(\phi \wedge h) = \phi \wedge i_* h$. It follows that $\phi$ in this way defines an element in $\mathcal{H}om_{\mathscr{O}_X}(\omega_X^n, \mathcal{W}_X^{n,*})$. This sheaf is global and invariantly defined and so we can make the following global definition.

**Definition 7.2** $\mathcal{W}_X^{0,*} = \mathcal{H}om_{\mathscr{O}_X}(\omega_X^n, \mathcal{W}_X^{n,*})$.

If $\phi$ is in $\mathcal{W}_X^{0,*}$ and $h$ is in $\omega_X^n$, we consider $\phi(h)$ as the product of $\phi$ and $h$, and sometimes write it as $\phi \wedge h$.

Since $\mathcal{W}_X^{n,*}$ are $\mathscr{E}_X^{0,*}$-modules, $\mathcal{W}_X^{0,*}$ are as well. Before we investigate these sheaves further, we give some motivation for the definition. First notice that we have a natural injection, cf., Proposition 4.1,

$$\mathscr{O}_X \to \mathcal{H}om\left(\omega_X^n, \omega_X^n\right), \quad \phi \mapsto (h \mapsto \phi h). \tag{7.3}$$

**Theorem 7.3** *The mapping (7.3) is an isomorphism in the Zariski-open subset of $X$ where it is $S_2$.*

This is the subset of $X$ where $\operatorname{codim} X_k \geq k+2$, $k \geq p+1$, cf., Sect. 6. Thus it contains all points $x$ such that $\mathscr{O}_{X,x}$ is Cohen–Macaulay. In particular, (7.3) is an isomorphism in $X_{reg}$.

Theorem 7.3 is a consequence of the results in [22]. If $X$ has pure dimension $p$, there is an injective mapping

$$\mathscr{O}_X \to \mathcal{H}om\left(\mathcal{E}xt^{\,p}(\mathscr{O}_X, K_\Omega), \mathcal{CH}_\Omega^Z\right), \tag{7.4}$$

which by [22, Theorem 1.2 and Remark 6.11] is an isomorphism if and only if $\mathscr{O}_X$ is $S_2$. Since the image of such a morphism must be annihilated by $\mathcal{J}$ by linearity, it is indeed a morphism

$$\mathscr{O}_X \to \mathcal{H}om\left(\mathcal{E}xt^{\,p}(\mathscr{O}_X, K_\Omega), \mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)\right). \tag{7.5}$$

In view of (4.2) and (5.3), (7.5) corresponds to a morphism $\mathscr{O}_X \to \mathcal{H}om(\omega_X^n, \omega_X^n)$, and the fact that it is the morphism (7.3) is a rather simple consequence of the definition of the morphism (7.4) in [22, (6.9)].

As mentioned in the introduction, Theorem 7.3 can be seen as a reformulation of a classical result of Roos, [30], which is the same statement about the injection

$$\mathscr{O}_\Omega/\mathcal{J} \to \mathcal{E}xt^{\,p}\left(\mathcal{E}xt^{\,p}(\mathscr{O}_\Omega/\mathcal{J}, K_\Omega), K_\Omega\right); \tag{7.6}$$

here we assume that the ideal has pure dimension. The equivalence of the morphisms (7.4) and (7.6) is discussed in [22, Corollary 1.4].

Let us now consider the case when $X$ is reduced. Since sections of $\omega_X^n$ are meromorphic, see [6, Example 2.8], and thus almost semi-meromorphic and generically smooth, by Proposition 2.4 (with $Z = X = \Omega$) we can extend (7.3) to a morphism

$$\mathcal{W}_X^{0,*} \to \mathcal{H}om\left(\omega_X^n, \mathcal{W}_X^{n,*}\right). \tag{7.7}$$

**Lemma 7.4** *When $X$ is reduced (7.7) is an isomorphism.*

Thus Definition 7.2 is consistent with the previous definition of $\mathcal{W}_X^{0,*}$ when $X$ is reduced.

*Proof* Clearly each $\phi$ in $\mathcal{W}_X^{0,*}$ defines an element $\alpha$ in $\mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$ by $h \mapsto \phi \wedge h$. If we apply this to a generically nonvanishing $h$ we see by the SEP that (7.7) is injective.

For the surjectivity, take $\alpha$ in $\mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$. If $h'$ is nonvanishing at a point on $X_{reg}$, then it generates $\omega_X^n$ and thus $\alpha$ is determined by $\phi := \alpha h'$ there. By [10, Theorem 3.7], $\phi = \psi \wedge h'$ for a unique current $\psi$ in $\mathcal{W}_X^{0,*}$ so by $\mathscr{O}_X$-linearity $\alpha h = \psi \wedge h$ for any $h$. Hence, $\psi$ is well-defined as a current in $\mathcal{W}_X^{0,*}$ on $X_{\mathrm{reg}}$.

We must verify that $\psi$ has an extension in $\mathcal{W}_X^{0,*}$ across $X_{sing}$. Since such an extension must be unique by the SEP, the statement is local on $X$. Thus we may assume that $\alpha$ is defined on the whole of $X$ and that there is a generically nonvanishing holomorphic $n$-form $\gamma$ on $X$. Then $\alpha\gamma$ is a section of $\mathcal{W}^{n,*}(X)$.

Let us choose a smooth modification $\pi : X' \to X$ that is biholomorphic outside $X_{sing}$. Then $\pi^*\gamma$ is a holomorphic $n$-form on $X'$ that is generically non-vanishing. We claim that there is a current $\tau$ in $\mathcal{W}^{n,0}(X')$ such that $\pi_*\tau = \alpha\gamma$. In fact, $\tau$ exists on $\pi^{-1}(X_{reg})$ since $\pi$ is a biholomorphism there. Moreover, by [4, Proposition 1.2], $\alpha h$ is the direct image of some pseudomeromorphic current $\tilde{\tau}$ on $X'$, and is therefore also the image of the (unique) current $\tau = \mathbf{1}_{\pi^{-1}(X_{reg})}\tilde{\tau}$ in $\mathcal{W}^{n,*}(X')$.

By [10, Theorem 3.7] again $\tau$ is locally of the form $\xi \wedge ds$, where $\xi$ is in $\mathcal{W}_{X'}^{0,*}$ and $ds = ds_1 \wedge \cdots \wedge ds_n$ for some local coordinates $s$. Hence, $\tau$ is a $K_{X'}$-valued section of $\mathcal{W}^{0,*}(X')$, so $\tau/\pi^*\gamma$ is a section of $\mathcal{W}^{0,*}(X')$. Now $\Psi := \pi_*(\tau/\pi^*\gamma)$ is a section of $\mathcal{W}^{0,*}(X)$. On $X_{reg} \cap \{\gamma \neq 0\}$ we thus have that $\Psi \wedge \gamma = \pi_*\tau = \alpha\gamma = \psi \wedge \gamma$ and so $\Psi = \psi$ there. By the SEP it follows that $\Psi$ coincides with $\psi$ on $X_{reg}$ and is thus the desired pseudomeromorphic extension to $X$. □

In view of (5.1) and (5.3) we have, given a local embedding $i : X \to \Omega$, the extrinsic representation

$$\mathcal{W}_X^{0,*} \simeq \mathcal{H}om\left(\mathcal{H}om\left(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z\right), \mathcal{H}om\left(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z\right)\right), \phi \mapsto (i_*h \mapsto i_*(\phi \wedge h)). \tag{7.8}$$

**Lemma 7.5** *Assume that $X_{reg} \to \Omega$ is a local embedding and $(z, w)$ coordinates as before. Each section $\phi$ in $\mathcal{W}_X^{0,*}$ has a unique representation (7.1) with $\widehat{\phi}_j$ in $\mathcal{W}_Z^{0,*}$.*

A current with a representation (7.1) is considered as an element of $\mathcal{W}_X^{0,*} = \mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$ in view of the comment after Lemma 7.1.

*Proof* From (4.9) we get an induced sequence

$$0 \to \left(\mathcal{W}_Z^{0,*}\right)^\nu \xrightarrow{T} \left(\mathcal{W}_Z^{0,*}\right)^M \xrightarrow{A} \left(\mathcal{W}_Z^{0,*}\right)^{M'}, \tag{7.9}$$

which is also exact. In fact, $T$ in (7.9) is clearly injective, and by (4.10), if $\xi$ in $(\mathcal{W}_Z^{0,*})^M$ and $A\xi = 0$, then $T\eta = \xi$, if $\eta = S\xi$.

Now take $\phi$ in $\mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*})$. Let us choose a basis $\mu^1, \dots, \mu^m$ for $\omega_X^n$ and let $\tilde{\phi}$ be the element in $(\mathcal{W}_Z^{0,*})^M$ obtained from the coefficients of $\phi\mu^j$ when expressed as in (2.11), cf., Sect. 4.1. We claim that $A\tilde{\phi} = 0$. Taking this for granted, by the exactness of (7.9), $\tilde{\phi}$ is the image of the tuple $\hat{\phi} = S\tilde{\phi}$. Now $\hat{\phi} \wedge \mu^j = \phi\mu^j$ since they are represented by the same tuple in $(\mathcal{W}_Z^{0,*})^M$. Thus $\hat{\phi}$ gives the desired representation of $\phi$.

In view of Proposition 2.9 it is enough to prove the claim where $\tilde{\phi}$ is smooth. Let us therefore fix such a point, say 0, and show that $(A\tilde{\phi})(0) = 0$. From the proof of Lemma 4.11, if we let $\mathcal{I}$ be the ideal generated by $z$, and let $X_0$ be defined by $\mathscr{O}_{X_0} := \mathscr{O}_\Omega/(\mathcal{J} + \mathcal{I})$, then $\mu^1 \wedge \mu^z, \dots, \mu^m \wedge \mu^z$ generate $\omega_{X_0}^0$. If we let $\phi_0$ be the morphism in $\mathcal{H}om(\omega_{X_0}^0, \omega_{X_0}^0)$ given by $\phi_0(\mu^i \wedge \mu^z) := \phi\mu^i \wedge \mu^z$ (which indeed gives a well-defined such morphism), then, as in the proof of Lemma 4.11, $\tilde{\phi}_0 = \tilde{\phi}(0)$. In addition, the sequence (4.9) for $X_0$ is

$$0 \to \mathbb{C}^\nu \stackrel{T(0)}{\to} \mathbb{C}^M \stackrel{A(0)}{\to} \mathbb{C}^{M'}.$$

Since $X_0$ is 0-dimensional, the morphism $\mathscr{O}_{X_0} \to \mathcal{H}om(\omega_{X_0}, \omega_{X_0})$ is an isomorphism by Theorem 7.3, and thus $\phi_0$ is given as multiplication by a function in $\mathscr{O}_{X_0}$, which we also denote by $\phi_0$, i.e., $\tilde{\phi}_0 = T(0)\hat{\phi}_0$. Hence, $A(0)\tilde{\phi}_0 = A(0)T(0)\hat{\phi}_0 = 0$, and thus $(A\tilde{\phi})(0) = 0$. □

*Example 7.6* (Meromorphic functions) Assume that we have a local embedding $X \to \Omega$. Given meromorphic functions $\Phi$, $\Phi'$ in $\Omega$ that are holomorphic generically on $Z$, we say that $\Phi \sim \Phi'$ if and only if $\Phi - \Phi'$ is in $\mathcal{J}$ generically on $Z$. If $\Phi = A/B$ and $\Phi' = A'/B'$, where $B$ and $B'$ are generically non-vanishing on $Z$, the condition is precisely that $AB' - A'B$ is in $\mathcal{J}$. We say that such an equivalence class is a meromorphic function $\phi$ on $X$, i.e., $\phi$ is in $\mathcal{M}_X$. Clearly we have $\mathscr{O}_X \subset \mathcal{M}_X$. We claim that

$$\mathcal{M}_X \subset \mathcal{W}_X^{0,*}.$$

To see this, first notice that if we take a representative $\Phi$ in $\mathcal{M}_\Omega$ of $\phi$, then it can be considered as an almost semi-meromorphic current on $\Omega$ with Zariski-singular support of positive codimension on $Z$, since it is generically holomorphic on $Z$. As in Definition 5.5 we therefore have a current $\Phi \wedge h$ in $\mathcal{W}_X^{n,0}$ for $h$ in $\omega_X^n$. Another representative $\Phi'$ of $\phi$ will give rise to the same current generically and hence everywhere by the SEP. Thus $\phi$ defines a section of $\mathcal{H}om(\omega_X^n, \mathcal{W}_X^{n,*}) = \mathcal{W}_X^{0,*}$. □

By definition, a current $\phi$ in $\mathcal{W}_X^{0,*}$ can be multiplied by a current $h$ in $\omega_X^n$, and the product $\phi \wedge h$ lies in $\mathcal{W}_X^{n,*}$. It will be crucial that we can extend to products by somewhat more general currents. Notice that $\omega_X^n$ is a subsheaf of $\mathcal{C}_X^{n,*}$, which is an $\mathscr{E}_X^{0,*}$-module. Thus, we can consider the subsheaf $\mathscr{E}_X^{0,*}\omega_X^n$ of $\mathcal{C}_X^{n,*}$ which consists of finite sums $\sum \xi_i \wedge h_i$, where $\xi_i$ are in $\mathscr{E}_X^{0,*}$ and $h_i$ are in $\omega_X^n$.

**Lemma 7.7** *Each $\phi$ in $\mathcal{W}_X^{0,*} = \mathcal{H}om_{\mathscr{O}_X}(\omega_X^n, \mathcal{W}_X^{n,*})$ has a unique extension to a morphism in $\mathcal{H}om_{\mathscr{E}_X^{0,*}}(\mathscr{E}_X^{0,*}\omega_X^n, \mathcal{W}_X^{n,*})$.*

*Proof* The uniqueness follows by $\mathscr{E}_X^{0,*}$-linearity, i.e., if $b = \xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r$ is in $\mathscr{E}_X^{0,*}\omega_X^n$, then one must have

$$\phi b = \sum_i (-1)^{(\deg \xi_i)(\deg \phi)} \xi_i \wedge \phi h_i. \tag{7.10}$$

We must check that this is well-defined, i.e., that the right hand side does not depend on the representation $\xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r$ of $b$. By the SEP, it is enough to prove this locally on $X_{\text{reg}}$, and we can then assume that $\phi$ has a representation (7.1). By Proposition 2.9, it is then enough to prove that it is well-defined assuming that $\widehat{\phi}_0, \ldots, \widehat{\phi}_{\nu-1}$ in (7.1) are all smooth. In this case, the right hand side of (7.10) is simply the product of $\xi_1 \wedge h_1 + \cdots + \xi_r \wedge h_r = b$ by the smooth form $\phi$ in $\mathscr{E}_X^{0,*}$, and this product only depends on $b$. □

**Corollary 7.8** *Let $\phi$ be a current in $\mathcal{W}_X^{0,*}$ and let $\alpha$ be a current in $\mathcal{W}_X^{n,*}$ of the form $\alpha = \sum a_i \wedge h_i$, where $a_i$ are almost semi-meromorphic $(0,*)$-currents on $\Omega$ which are generically smooth on $Z$, and $h_i$ are in $\omega_X^n$. Then one has a well-defined product*

$$\phi \wedge \alpha = \sum (-1)^{(\deg a_i)(\deg \phi)} a_i \wedge (\phi \wedge h_i). \tag{7.11}$$

*Proof* The right hand side of (7.11) exists as a current in $\mathcal{W}_X^{n,*}$, and we must prove is that it only depends on the current $\alpha$ and not on the representation $\sum a_i \wedge h_i$. Notice that all the $a_i$ are smooth outside some subvariety $V$ of $Z$ and there the right hand side of (7.11) is the product of $\phi$ and $\alpha$ in $\mathscr{E}_X^{0,*}\omega_X^n$, cf., Lemma 7.7. It follows by the SEP that the right hand side only depends on $\alpha$. □

*Remark 7.9* Recall from (6.9) that $\omega = b\vartheta$. If $\phi$ is in $\mathcal{W}_X^{0,*}$, then we can define the product $\phi \wedge \omega$ by Corollary 7.8.

Expressed extrinsically, if $\mu = i_*\vartheta$, and if we write $R \wedge dz = b\mu$ as in Lemma 6.2, then we can define the product $R \wedge dz \wedge \phi := b\mu \wedge \phi$ as a current in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}_\Omega^Z)$. □

**Lemma 7.10** *Assume that $\phi$ is in $\mathcal{W}_X^{0,*}$, and that $\phi \wedge \omega = 0$ for some structure form $\omega$, where the product is defined by Remark 7.9. Then $\phi = 0$.*

*Proof* Considering the component with values in $E_p$, we get that $\phi \wedge \omega_0 = 0$. By Proposition 6.7, any $h$ in $\omega_X^n$ can be written as $h = \xi\omega_0$, where $\xi$ is a holomorphic section of $E_p^*$, so by $\mathscr{O}$-linearity, $\phi \wedge h = 0$, i.e., $\phi = 0$. □

We end this section with the following result, first part of [10, Theorem 3.7]. We include here a different proof than the one in [10], since we believe the proof here is instructive.

**Proposition 7.11** *If $Z$ is smooth, then $\mathcal{W}_Z$ is closed under holomorphic differential operators.*

*Proof* Let $\tau$ be any current in $\mathcal{W}_Z$. It suffices to prove that if $\zeta$ are local coordinates on $Z$, then $\partial\tau/\partial\zeta_1$ is in $\mathcal{W}_Z$. Consider the current

$$\tau' = \tau \otimes \bar\partial \frac{dw}{2\pi i w^2}$$

on the manifold $Y := Z \times \mathbb{C}_w$. Clearly $\tau'$ has support on $Z$, and it follows from (2.5) that $\tau'$ is in $\mathcal{W}_Y^Z$. Let

$$p : (z, w) \mapsto \zeta = (z_1 + w, z_2, \dots, z_n),$$

which is just a change of variables on $Y$ followed by a projection. It follows from (2.4) that $p_*\tau'$ is in $\mathcal{W}_Z$. Since

$$\bar\partial \frac{dw}{2\pi i w^2} \cdot \xi(w) = \frac{\partial \xi}{\partial w}(0)$$

it is readily verified that $p_*\tau' = \partial\tau/\partial\zeta_1$, so we conclude that $\partial\tau/\partial\zeta_1$ is in $\mathcal{W}_Z$. □

## 8 The $\bar\partial$-operator on $\mathcal{W}_X^{0,*}$

We already know the meaning of $\bar\partial$ on $\mathcal{W}_X^{n,*}$, and we now define $\bar\partial$ on $\mathcal{W}_X^{0,*}$.

**Definition 8.1** Assume that $\phi$, $v$ are in $\mathcal{W}_X^{0,*}$, We say that $\bar\partial v = \phi$ if

$$\bar\partial(v \wedge h) = \phi \wedge h, \quad h \in \omega_X^n. \tag{8.1}$$

If we have an embedding $X \to \Omega$, (8.1) means, cf., (7.8), that

$$\bar\partial(v \wedge \mu) = \phi \wedge \mu, \quad \mu \in \mathcal{H}om\left(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z\right). \tag{8.2}$$

In view of Remark 7.9 we can define the product $\phi \wedge \omega$ for $\phi$ in $\mathcal{W}_X^{0,*}$.

**Definition 8.2** We say that $v$ belongs to $\mathrm{Dom}\,\bar\partial_X$ if $v$ is in $\mathrm{Dom}\,\bar\partial$, i.e., $\bar\partial v = \phi$ for some $\phi$ and in addition $\bar\partial(v \wedge \omega)$, a priori only in $\mathcal{PM}_X^{n,*}$, is in $\mathcal{W}_X^{n,*}$, for each structure form $\omega$ from any possible embedding.

If $X$ is Cohen–Macaulay, then any such $\omega$ is of the form $a_1 h^1 + \cdots + a_m h^m$, where $h^j$ are in $\omega_X^n$ and $a_j$ are smooth, see Remark 6.5, and hence $\mathrm{Dom}\,\bar\partial_X$ coincides with $\mathrm{Dom}\,\bar\partial$ in this case.

*Example 8.3* Assume that $v$ is in $\mathscr{E}_X^{0,*}$ and $\phi = \bar\partial v$ in the sense in Section 4. Then clearly

$$\bar\partial(v \wedge \omega) = \phi \wedge \omega + (-1)^{\deg v} v \wedge \bar\partial\omega.$$

Since $\bar\partial\omega = f\omega$, and $\mathcal{W}_X^{n,*}$ is closed under multiplication with forms in $\mathscr{E}_X^{0,*}$, we get that $\bar\partial(v \wedge \omega)$ is in $\mathcal{W}_X^{n,*}$, so $v$ is in Dom $\bar\partial_X$ and $\bar\partial_X v = \phi$.

If $w$ is in Dom $\bar\partial_X$ and $v$ is in $\mathscr{E}_X^{0,*}$, then

$$\bar\partial(v \wedge w \wedge \omega) = \bar\partial v \wedge w \wedge \omega + (-1)^{\deg v} v \wedge \bar\partial(w \wedge \omega).$$

Thus $v \wedge w$ is in Dom $\bar\partial_X$, and the Leibniz rule $\bar\partial(v \wedge w) = \bar\partial v \wedge w + (-1)^{\deg v} v \wedge \bar\partial w$ holds. $\square$

Let $\chi_\delta = \chi(|h|^2/\delta)$ where $h$ is a tuple of holomorphic functions that cuts out $X_{sing}$.

**Lemma 8.4** *If $v$ is in $\mathcal{W}^{0,*}(X)$, and it is in Dom $\bar\partial_X$ on $X_{\text{reg}}$, then $v$ is in Dom $\bar\partial_X$ on all of $X$ if and only if*

$$\bar\partial\chi_\delta \wedge v \wedge \omega \to 0, \quad \delta \to 0, \tag{8.3}$$

*for all structure forms $\omega$. In this case,*

$$-\nabla_f(v \wedge \omega) = \bar\partial v \wedge \omega. \tag{8.4}$$

*Proof* Since $\mathcal{W}_X^{n,*}$ is closed under multiplication by $f$, $v$ is in Dom $\bar\partial_X$ if and only if $\nabla_f(v \wedge \omega)$ is in $\mathcal{W}_X^{n,*}$ for all structure forms $\omega$. Since $v$ is in Dom $\bar\partial_X$ on $X_{\text{reg}}$, thus $\nabla_f(v \wedge \omega)$ is in $\mathcal{W}_X^{n,*}$ on $X_{\text{reg}}$. By (2.2), $\nabla_f(v \wedge \omega)$ is then in $\mathcal{W}_X^{n,*}$ on all of $X$ if and only if

$$\mathbf{1}_{X_{\text{reg}}}\nabla_f(v \wedge \omega) = \nabla_f(v \wedge \omega). \tag{8.5}$$

By the Leibniz rule,

$$\nabla_f(\chi_\delta v \wedge \omega) = -\bar\partial\chi_\delta \wedge v \wedge \omega + \chi_\delta \nabla_f(v \wedge \omega). \tag{8.6}$$

Since $v$ is in $\mathcal{W}_X^{0,*}$, $v \wedge \omega$ is in $\mathcal{W}_X^{n,*}$, so the left hand side of (8.6) tends to $\nabla_f(v \wedge \omega)$ when $\delta \to 0$, whereas the second term on the right hand side of (8.6) tends to $\mathbf{1}_{X_{\text{reg}}}\nabla_f(v \wedge \omega)$. Thus (8.5) holds if and only if (8.3) does. Thus the first statement in the lemma is proved.

Recall, cf., (6.9), that $\omega = b\vartheta$ where $b$ is smooth on $X_{\text{reg}}$ and $\vartheta$ is in $\omega_X^n$. By the Leibniz rule thus $-\nabla_f(v \wedge \omega) = \bar\partial v \wedge \omega$ on $X_{\text{reg}}$, since $\nabla_f \omega = 0$. Therefore, (8.6) is equivalent to $-\nabla_f(\chi_\delta v \wedge \omega) = \bar\partial\chi_\delta \wedge v \wedge \omega + \chi_\delta \bar\partial v \wedge \omega$. If (8.3) holds, we therefore get (8.4) when $\delta \to 0$. $\square$

*Remark 8.5* In case $X$ is reduced the definition of $\bar\partial_X$ is precisely the same as in [6]. However, the definition of $\bar\partial v = \phi$ given here, for $v, \phi$ in $\mathcal{W}_X^{0,*}$, does *not* coincide with the definition in, e.g., [6]. In fact, that definition means that $\bar\partial(v \wedge h) = \phi \wedge h$ for all *smooth* $h$ in $\omega_X^n$, which in general is a strictly weaker condition. For example, for any weakly holomorphic function $v$, we have $\bar\partial(v\wedge h)=0$ for all smooth $h$ in $\omega_X^n$, while if $X$ is a reduced complete intersection, or more generally Cohen–Macaulay, then $\bar\partial(v\wedge h)=0$ for all $h$ in $\omega_X^n$ is equivalent to $v$ being strongly holomorphic, see [33, p. 124] and [2]. □

We conclude this section with a lemma that shows that $\bar\partial$ means what one should expect when $\phi, v$ are expressed with respect to a local basis $w^{\alpha_j}$ for $\mathscr{O}_X$ over $\mathscr{O}_Z$ as in Lemma 7.5.

**Lemma 8.6** *Assume that we have a local embedding $X_{reg}\to\Omega$ and $\phi, v$ in $\mathcal{W}_X^{0,*}$ represented as in (7.1). Then $\bar\partial v=\phi$ if and only if*

$$\bar\partial\hat v_j=\hat\phi_j,\quad j=0,\dots,\nu-1. \tag{8.7}$$

*Proof* Let us use the notation from the proof of Lemma 7.5. Recall that $\hat v=S\tilde v$. In view of (8.2) and (2.12), $\widetilde{\bar\partial v}=\bar\partial\tilde v$. Since $S$ is holomorphic therefore $\widehat{\bar\partial v}=S\widetilde{\bar\partial v}=S\bar\partial\tilde v=\bar\partial(S\tilde v)=\bar\partial\hat v$. □

## 9 Solving $\bar\partial u=\phi$ on $X$

We will find local solutions to the $\bar\partial$-equation on $X$ by means of integral formulas. We use the notation and machinery from [6, Section 5]. Let $i\colon X\to\Omega\subset\mathbb{C}^N$ be a local embedding such that $\Omega$ is pseudoconvex, let $\Omega'\subset\subset\Omega$ be a relatively compact subdomain of $\Omega$, and let $X'=X\cap\Omega'$.

**Theorem 9.1** *There are integral operators*

$$K\colon\mathscr{E}^{0,*+1}(X)\to\mathcal{W}^{0,*}(X')\cap Dom\,\bar\partial_X,\quad P\colon\mathscr{E}^{0,*}(X)\to\mathscr{E}^{0,*}(X')$$

*such that, for $\phi\in\mathscr{E}^{0,k}(X)$,*

$$\phi=\bar\partial K\phi+K(\bar\partial\phi)+P\phi. \tag{9.1}$$

The operators $K$ and $P$ are described below; they depend on a choice of weight $g$. Since $\Omega$ is Stein one can find such a weight $g$ that is holomorphic in $z$, by which we mean that it depends holomorphically on $z\in\Omega'$ and has no components containing any $d\bar z_i$, cf., Example 5.1 in [6]. In this case, $P\phi$ is holomorphic when $k=0$, and vanishes when $k\ge 1$, i.e.,

$$\phi=\bar\partial K\phi+K(\bar\partial\phi),\quad \phi\in\mathscr{E}^{0,k}(X),\quad k\ge 1. \tag{9.2}$$

If $\bar\partial\phi=0$ in $\Omega$, and $k\ge 1$, then $K\phi$ is a solution to $\bar\partial v=\phi$. If $k=0$, then $\phi=P\phi$ is holomorphic. It follows that a smooth $\bar\partial$-closed function is holomorphic. In the reduced case this is a classical theorem of Malgrange [24]. In Sect. 10 we prove that $K\phi$ is smooth on $X_{reg}$.

We now turn to the definition of $K$ and $P$. For future need, in Sect. 11, we define them acting on currents in $\mathcal{W}^{0,*}(X)$ and not only on smooth forms. Let $\pi : \Omega_\zeta \times \Omega'_z \to \Omega'_z$ be the natural projection. Let us choose a holomorphic Hefer form[3] $H$, a smooth weight $g$ with compact support in $\Omega$ with respect to $z \in \Omega' \subset\subset \Omega$, and let $B$ be the Bochner–Martinelli form. Since we are only are concerned with $(0,*)$-forms, we will here assume that $H$ and $B$ only have holomorphic differentials in $\zeta$, i.e., the factors $d\eta_i = d\zeta_i - dz_i$ in $H$ and $B$ in [6] should be replaced by just $d\zeta_i$.

If $\gamma$ is a current in $\Omega_\zeta \times \Omega'_z$ we let $(\gamma)_N$ be the component of bidegree $(N,*)$ in $\zeta$ and $(0,*)$ in $z$, and let $\vartheta(\gamma)$ be the current such that

$$\vartheta(\gamma) \wedge d\zeta = (\gamma)_N. \tag{9.3}$$

Consider now $\mu$ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}^Z_\Omega)$ and $\phi$ in $\mathcal{W}^{0,*}_X$. We can give meaning to

$$(g \wedge HR(\zeta))_N \wedge \phi(\zeta) \wedge \mu(z) \tag{9.4}$$

as a tensor product of currents in the following way: first of all, by Remark 7.9, we can form the product $R(\zeta) \wedge d\zeta \wedge \phi(\zeta)$ as a current in $\mathcal{W}^Z_\Omega$. In view of [11, Corollary 4.7] the tensor product $R(\zeta) \wedge d\zeta \wedge \phi(\zeta) \wedge \mu(z)$ is in $\mathcal{W}^{Z\times Z'}_{\Omega_\zeta \times \Omega'_z}$, where $Z' = Z \cap \Omega'$. Finally, we multiply this with the smooth form $\vartheta(g \wedge H)$ to obtain (9.4). Similarly, outside of $\Delta$, the diagonal in $\Omega \times \Omega'$, where $B$ is smooth, we can define

$$(B \wedge g \wedge HR(\zeta))_N \wedge \phi(\zeta) \wedge \mu(z) \tag{9.5}$$

as a tensor product of currents.

**Lemma 9.2** *For $\mu$ in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}^{Z'}_{\Omega'})$ and $\phi \in \mathcal{W}^{0,*}(X)$, the current (9.5), a priori defined as a current in $\mathcal{W}^{Z\times Z'\setminus\Delta}_{\Omega_\zeta\times\Omega'_z\setminus\Delta}$ has an extension across $\Delta$. The current (9.4) and the extension of (9.5) depend $\mathscr{O}_\Omega/\mathcal{J}$-bilinearly on $\mu$ and $\phi$, and are such that*

$$K\phi \wedge \mu := \pi_*\big((B \wedge g \wedge HR(\zeta))_N \wedge \phi(\zeta) \wedge \mu(z)\big) \tag{9.6}$$

*and*

$$P\phi \wedge \mu := \pi_*\big((g \wedge HR(\zeta))_N \wedge \phi(\zeta) \wedge \mu(z)\big) \tag{9.7}$$

*are in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}^{Z'}_{\Omega'})$.*

It follows that $K\phi\wedge\mu$ and $P\phi\wedge\mu$ are $\mathbb{C}$-linear in $\phi$ and $\mathscr{O}_{\Omega'}/\mathcal{J}$-linear in $\mu$. In view of (7.8), by considering $\mu$ in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{CH}^{Z'}_{\Omega'})$, we have defined linear operators

$$K\colon \mathcal{W}^{0,*+1}(X) \to \mathcal{W}^{0,*}(X'), \quad P\colon \mathcal{W}^{0,*}(X) \to \mathcal{W}^{0,*}(X'). \tag{9.8}$$

*Proof of Lemma 9.2* In order to define the extension of (9.5) across $\Delta$, we note first that since $B$ is almost semi-meromorphic with Zariski singular support $\Delta$, $\vartheta(B\wedge g\wedge H)$

[3] We are only concerned with the component $H^0$ of this form, so for simplicity we write just $H$.

is an almost semi-meromorphic $(0, *)$-current on $\Omega_\zeta \times \Omega'_z$, which is smooth outside the diagonal. We can thus form the current $\vartheta(B \wedge g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi(\zeta) \wedge \mu(z)$ in $\mathcal{W}^{Z\times Z'}_{\Omega_\zeta \times \Omega'_z}$, cf., Proposition 2.4, and this is the extension of (9.5) across $\Delta$.

From the definitions above, it is clear that (9.4) and the extension of (9.5) are $\mathscr{O}_\Omega$-bilinear in $\phi$ and $\mu$. Both these currents are annihilated by $\mathcal{J}_z$ and $\mathcal{J}_\zeta$, cf., (2.8), so they depend $\mathscr{O}_\Omega/\mathcal{J}$-bilinearly. In view of (2.4) we conclude that (9.6) and (9.7) are in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}^{Z'}_{\Omega'})$. □

**Proposition 9.3** *If $\phi \in \mathcal{W}^{0,k}(X)$, then $P\phi \in \mathscr{E}^{0,k}(X')$, and if in addition $g$ is holomorphic in $z$, then $P\phi \in \mathscr{O}(X')$ if $k = 0$ and vanishes if $k \geq 1$.*

*Proof* Since $\vartheta(g \wedge H)$ is smooth, we get that

$$\begin{aligned}&\pi_*\big(\vartheta(g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi \wedge \mu(z)\big)\\ &\quad = \pi_*\big(\vartheta(g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi\big) \wedge \mu(z) = \pi_*\big((g \wedge HR)_N \wedge \phi\big) \wedge \mu(z),\end{aligned}$$

cf., for example [20, (5.1.2)]. Thus $P\phi(z) = \pi_*\big((g \wedge HR(\zeta))_N \wedge \phi\big)$ which is smooth on $\Omega'$. If $g$ depends holomorphically on $z$, then $P\phi$ is holomorphic in $\Omega'$ if $\phi$ is a $(0, 0)$-current, and vanishes for degree reasons if $\phi$ has positive degree. □

We shall now approximate $K\phi$ by smooth forms. Let $B^\epsilon = \chi(|\zeta - z|^2/\epsilon)B$.

**Proposition 9.4** *For any $\phi \in \mathcal{W}^{0,k}(X)$, $k \geq 1$,*

$$K^\epsilon\phi := \pi_*\big((B^\epsilon \wedge g \wedge HR(\zeta))_N \wedge \phi\big) = \pi_*\big(\vartheta(B^\epsilon \wedge g \wedge H) \wedge R(\zeta) \wedge d\zeta \wedge \phi\big)$$

*is in $\mathscr{E}^{0,k-1}(X')$ and $K^\epsilon\phi \to K\phi$ when $\epsilon \to 0$.*

The last statement means that

$$K^\epsilon\phi \wedge \mu \to K\phi \wedge \mu, \quad \mu \in \mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{CH}^{Z'}_{\Omega'}). \tag{9.9}$$

*Proof* Since $B^\epsilon$ is smooth, the current we push forward is $R(\zeta) \wedge \phi(\zeta)$ times a smooth form of $\zeta$ and $z$. Therefore $K^\epsilon\phi$ is smooth. As in the proof of Proposition 9.3, we obtain since $B^\epsilon$ is smooth that

$$K^\epsilon\phi \wedge \mu = \pi_*\big((B^\epsilon \wedge g \wedge HR(\zeta))_N \wedge \phi \wedge \mu(z)\big). \tag{9.10}$$

By (5.2) applied to $a = B$ we have that

$$(B^\epsilon \wedge g \wedge HR(\zeta))_N \wedge \phi \wedge \mu(z) \to (B \wedge g \wedge HR(\zeta))_N \wedge \phi \wedge \mu(z) \tag{9.11}$$

which implies (9.9). □

### 9.1 Proof of Theorem 9.1

By definition $K\phi$ and $P\phi$ are currents in $\mathcal{W}^{0,*}(X')$ such that (9.6) and (9.7) hold for $\mu$ in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{CH}_{\Omega'}^{Z'})$. We claim that

$$K\phi \wedge R \wedge dz = \pi_*\big((B \wedge g \wedge HR(\zeta))_N \wedge \phi \wedge R(z) \wedge dz\big) \tag{9.12}$$

and

$$P\phi \wedge R \wedge dz = \pi_*\big((g \wedge HR(\zeta))_N \wedge \phi \wedge R(z) \wedge dz\big); \tag{9.13}$$

here the left hand sides are defined in view of Remark 7.9, whereas the right hand sides have meaning by Lemma 9.2 and the fact that $R(z) \wedge dz$ is in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}_{\Omega'}^{Z'})$ by Corollary 6.3.

Recall from Lemma 6.2 that $R \wedge dz = b\mu$, where $\mu$ is a tuple of currents in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{CH}_{\Omega'}^{Z'})$ and $b$ is an almost semi-meromorphic matrix that is smooth generically on $Z'$. Therefore (9.12) and (9.13) hold where $b$ is smooth, in view of Lemma 7.7, and since both sides are in $\mathcal{H}om(\mathscr{O}_{\Omega'}/\mathcal{J}, \mathcal{W}_{\Omega'}^{Z'})$, the equalities hold everywhere by the SEP.

As in [6] we let $R^\lambda = \bar{\partial}|f|^{2\lambda} \wedge U$ for $\operatorname{Re}\lambda \gg 0$. It has an analytic continuation to $\lambda = 0$ and $R = R^\lambda|_{\lambda=0}$. Notice that $R(z) \wedge B$ is well-defined since it is a tensor product with respect to the coordinates $z, \eta = \zeta - z$. Also $R(z) \wedge R^\lambda(\zeta) \wedge B$ admits such an analytic continuation and defines a pseudomeromorphic current[4] when $\lambda = 0$. Let $B_{k,k-1}$ be the component of $B$ of bidegree $(k, k-1)$.

**Lemma 9.5** *For all $k$,*

$$B_{k,k-1} \wedge HR^\lambda(\zeta) \wedge R(z)|_{\lambda=0} = B_{k,k-1} \wedge HR(\zeta) \wedge R(z). \tag{9.14}$$

*Proof of Lemma 9.5* Notice that the equality holds outside $\Delta$. Let $T$ be the left hand side of (9.14). In view of Proposition 2.1 it is therefore enough to check that $\mathbf{1}_\Delta T = 0$. Fix $j, k$ and let

$$T_\ell = B_{k,k-1} \wedge HR_j^\lambda(\zeta) \wedge R_\ell(z)|_{\lambda=0}.$$

Clearly $T_\ell = 0$ if $\ell < p$ so first assume that $\ell = p$. Since $HR_j$ has bidegree $(j, j)$ in $\zeta$, the current vanishes unless $j + k \le N$. Thus the total antiholomorphic degree is $\le N - n + N - 1$. On the other hand, the current has support on $\Delta \cap Z \times Z \simeq Z \times \{pt\}$ which has codimension $N + N - n$. Thus it vanishes by the dimension principle.

We now prove by induction over $\ell \ge p$ that $\mathbf{1}_\Delta T_\ell = 0$. Note that by (6.6), outside of $Z_\ell$, $R_\ell(z) = \alpha_\ell(z) R_{\ell-1}(z)$, where $\alpha_\ell(z)$ is smooth. Thus, outside of $Z_\ell \times \Omega$, $T_\ell$ is a smooth form times $T_{\ell-1}$, and thus, by induction and (2.3), $\mathbf{1}_\Delta T_\ell$ has its support in $\Delta \cap (Z_\ell \times Z) \simeq Z_\ell \times \{pt\}$, which has codimension $\ge N + \ell + 1$, see (6.3). On the other hand, the total antiholomorphic degree is $\le \ell + j + k - 1 \le \ell + N - 1$, so the current vanishes by the dimension principle. We conclude that (9.14) holds. □

[4] One can consider this current as $R(z) \wedge B$ multiplied by the residue of the almost semi-meromorphic current $U$ in (6.5), cf., [10, Section 4.4].

By the same argument[5] as for [6, (5.2)] we have the equality

$$\nabla_{f(z)}\big((B\wedge g\wedge HR^{\lambda}(\zeta))_N\wedge R(z)\wedge dz\big) = [\Delta]'\wedge R(z)\wedge dz-(g\wedge HR^{\lambda})_N\wedge R(z)\wedge dz, \tag{9.15}$$

also for our $R$, where $[\Delta]'$ denotes the part of $[\Delta]$ where $d\eta_i = d\zeta_i - dz_i$ has been replaced[6] by $d\zeta_i$. In view of (9.14) we can put $\lambda = 0$ in (9.15), and then we get

$$\nabla_{f(z)}\big((B\wedge g\wedge HR(\zeta))_N\wedge R(z)\wedge dz\big) = [\Delta]'\wedge R(z)\wedge dz-(HR(\zeta)\wedge g)_N\wedge R(z)\wedge dz. \tag{9.16}$$

Multiplying (9.16) by the smooth form $\phi$, and using (9.12) and (9.13), we get

$$\phi \wedge R \wedge dz = -\nabla_f(K\phi \wedge R \wedge dz) + K(\bar\partial\phi) \wedge R \wedge dz + P\phi \wedge R \wedge dz,$$

or equivalently,

$$\phi \wedge \omega = -\nabla_f(K\phi \wedge \omega) + K(\bar\partial\phi) \wedge \omega + P\phi \wedge \omega. \tag{9.17}$$

Multiplying by suitable holomorphic $\xi_0$ in $E_p^*$ such that $f_{p+1}^*\xi_0 = 0$, cf., Proposition 6.7, we see that $\phi \wedge h = \bar\partial(K\phi \wedge h) + K(\bar\partial\phi) \wedge h + P\phi \wedge h$ for all $h$ in $\omega_X$. Thus by definition (9.1) holds.

Since $\mathcal{W}_X^{0,*}$ is closed under multiplication by $\mathscr{O}_X$, we get that $\psi$ in $\mathcal{W}_X^{0,*}$ is in $\operatorname{Dom}\bar\partial_X$ if and only if $-\nabla_f(\psi \wedge \omega)$ is in $\mathcal{W}_X^{n,*}$. Thus, we conclude from (9.17) that $K\phi$ is in $\operatorname{Dom}\bar\partial_X$ since all the other terms but $-\nabla_f(K\phi \wedge \omega)$ are in $\mathcal{W}_X^{0,*}$.

### 9.2 Intrinsic interpretation of *K* and *P*

So far we have defined $K$ and $P$ by means of currents in ambient space. We used this approach in order to avoid introducing push-forwards on a non-reduced space. However, we will sketch here how this can be done. We must first define the product space $X \times X'$. Given a local embedding $i : X \to \Omega$ as before, we have an embedding $(i \times i)\colon X \times X' \to \Omega \times \Omega'$ such that the structure sheaf is $\mathscr{O}_{\Omega\times\Omega'}/(\mathcal{J}_X + \mathcal{J}_{X'})$. One can check that this sheaf is independent of the chosen embedding, i.e., $\mathscr{O}_{X\times X'}$ is intrinsically defined. Thus we also have definitions of all the various sheaves on $X \times X'$ like $\mathscr{E}^{0,*}_{X\times X'}$. The projection $p\colon X \times X' \to X'$ is determined by $p^*\phi\colon \mathscr{O}_{X'} \to \mathscr{O}_{X\times X'}$, which in turn is defined so that $p^*i^*\Phi = (i \times i)^*\pi^*\Phi$ for $\Phi$ in $\mathscr{O}_{\Omega'}$, where $\pi : \Omega \times \Omega' \to \Omega'$ as before. Again one can check that this definition is independent of the embedding, and also extends to smooth $(0,*)$-forms $\phi$. Therefore, we have the well-defined mapping $p_*\colon \mathcal{C}^{2n,*+n}_{X\times X'} \to \mathcal{C}^{n,*}_{X'}$, and clearly

$$i_*p_* = \pi_*(i \times i)_*. \tag{9.18}$$

[5] There is a sign error in [6, (5.2)] due to $R(z) \wedge dz$ being odd with respect to the super structure. Since we here move $R(z) \wedge dz$ to the right, we get the correct sign.

[6] This change is due to the fact that we do the same change of the differentials in the definition of $H$ and $B$ above.

As before we have the isomorphism

$$(i\times i)_*\colon \mathcal{W}^{2n,*}_{X\times X'} \simeq \mathcal{H}om\left(\mathscr{O}_{\Omega\times\Omega'}/(\mathcal{J}_X+\mathcal{J}_{X'}), \mathcal{W}^{Z\times Z'}_{\Omega\times\Omega'}\right).$$

As in the proof of Lemma 9.2 we see that $\pi_*$ maps a current in $\mathcal{W}^{Z\times Z'}_{\Omega\times\Omega'}$ annihilated by $\mathcal{J}_{X'}$ to a current in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J}, \mathcal{W}^{Z'}_{\Omega'})$. It follows by (9.18) that

$$p_*\colon \mathcal{W}^{2n,*+n}_{X\times X'} \to \mathcal{W}^{n,*}_{X'}.$$

Now, take $h$ in $\omega^n_{X'}$ and let $\mu = i_* h$. Then, cf., the proof of Lemma 9.2,

$$(B\wedge g\wedge HR(\zeta))_N\wedge\phi(\zeta)\wedge\mu(z) = (i\times i)_*\big(\vartheta(B\wedge g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta)\wedge h\big).$$

Thus we can define $K\phi$ intrinsically by

$$K\phi\wedge h = p_*\left(\vartheta(B\wedge g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta)\wedge h(z)\right). \tag{9.19}$$

From above it follows that $K\phi\wedge h$ is in $\mathcal{W}^{n,*}_{X'}$. In the same way we can define $P\phi$ by

$$P\phi\wedge h = p_*\left(\vartheta(g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta)\wedge h(z)\right). \tag{9.20}$$

It is natural to write

$$K\phi(z) = \int_\zeta \vartheta(B\wedge g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta),\quad P\phi(z) = \int_\zeta \vartheta(g\wedge H)\wedge\omega(\zeta)\wedge\phi(\zeta),$$

although the formal meaning is given by (9.19) and (9.20).

## 10 Regularity of solutions on $X_{reg}$

We have already seen, cf., Proposition 9.3, that $P\phi$ is always a smooth form. We shall now prove that $K$ preserves regularity on $X_{reg}$. More precisely,

**Theorem 10.1** *If $\phi$ in $\mathcal{W}^{0,*}_X$ is smooth near a point $x\in X'_{\rm reg}$, then $K\phi$ in Theorem 9.1 is smooth near $x$.*

Throughout this section, let us choose local coordinates $(\zeta,\tau)$ and $(z,w)$ at $x$ corresponding to the variables $\zeta$ and $z$ in the integral formulas, so that $Z=\{(\zeta,\tau);\ \tau=0\}$.

**Lemma 10.2** *Let $B^\epsilon := \chi(|\zeta-z|^2/\epsilon)B$, and assume that $\phi$ has compact support in our coordinate neighborhood. Then $K\phi$ can be approximated by the smooth forms*

$$K^\epsilon\phi := \pi_*\big((B^\epsilon\wedge g\wedge HR)_N\wedge\phi\big).$$

Notice that here we cut away the diagonal $\Delta'$ in $Z \times Z'$ times $\mathbb{C}_\tau \times \mathbb{C}_w$ in contrast to Proposition 9.4, where we only cut away the diagonal $\Delta$ in $\Omega \times \Omega'$.

*Proof* Clearly $B^\epsilon$ is smooth so that each $K^\epsilon\phi$ is smooth in a full neighborhood in $\Omega'$ of $x$. Let $T = \mu(z, w) \wedge (HR(\zeta, \tau) \wedge B \wedge g)_N \wedge \phi$, and let $W = \Delta' \times \mathbb{C}_\tau \times \mathbb{C}_w$. Since $\mu(z, w) \otimes R(\zeta, \tau)$ has support on $\{w = \tau = 0\}$, $T = \mathbf{1}_{\{w=\tau=0\}}T$. Therefore, $\mathbf{1}_W T = \mathbf{1}_W \mathbf{1}_{\{w=\tau=0\}}T = 0$ since $W \cap \{w = \tau = 0\} \subset \Delta$ and $\mathbf{1}_\Delta T = 0$ by definition, cf., Proposition 2.1 (i). Now notice that $\mathbf{1}_W T = 0$ implies (9.11) and in turn (9.9) with our present choice of $B^\epsilon$. □

We first consider a simple but nontrivial example of Theorem 10.1.

*Example 10.3* Let $X = \mathbb{C}_\zeta \subset \mathbb{C}^2_{\zeta,\tau}$ and $\mathcal{J} = (\tau^{m+1})$. Then $R = \bar\partial(1/\tau^{m+1})$. For an arbitrary point $(z, w)$ we can choose the Hefer form

$$H = \frac{1}{2\pi i}\sum_{j=0}^{m} \tau^{m-k} w^k d\tau.$$

From the Bochner–Martinelli form $B$ we only get a contribution from the term

$$B_1 = \frac{1}{2\pi i}\frac{(\bar\zeta - \bar z)d\zeta + (\bar\tau - \bar w)d\tau}{|\zeta - z|^2 + |\tau - w|^2}.$$

Let $\Omega' \subset\subset \Omega$ be open balls with center at the origin, and let $\varphi = \varphi(|\zeta|^2 + |\tau|^2)$ be a smooth cutoff function with support in $\Omega$ that is $\equiv 1$ in a neighborhood of $\overline{\Omega'}$. Then we can choose a holomorphic weight $g = \varphi + \cdots$, see, [6, Example 5.1] with respect to $\Omega'$, and with support in $\Omega$. Now $1, \tau, \ldots, \tau^m$ is a set of generators for $\mathscr{O}_X$ over $\mathscr{O}_Z$. Assume that

$$\phi = (\hat\phi_0(\zeta) \otimes 1 + \cdots + \hat\phi_m(\zeta) \otimes \tau^m)d\bar\zeta$$

is a smooth $(0, 1)$-form. We want to compute $K\phi$. We know that

$$K\phi = a_0(z) \otimes 1 + \cdots + a_m(z) \otimes w^m \tag{10.1}$$

with $a_k(z)$ in $\mathcal{W}_Z^{0,0}$. By Lemma 10.2 and its proof, we have smooth $K^\epsilon\phi(z, w)$ in $\Omega'$ such that

$$K^\epsilon\phi \wedge dz \wedge dw \wedge \bar\partial\frac{1}{w^{m+1}} \to K\phi \wedge dz \wedge dw \wedge \bar\partial\frac{1}{w^{m+1}}. \tag{10.2}$$

It follows that

$$a_k(z) = \lim_{\epsilon\to 0}\frac{1}{k!}\frac{\partial^k}{\partial w^k}K^\epsilon\phi(z, w)\big|_{w=0}.$$

Notice that

$$
\begin{aligned}
(B\wedge g\wedge HR(\tau))_2 &= B_1\wedge g_{0,0}\wedge H\wedge\bar\partial\frac{1}{\tau^{m+1}}\\
&= -\varphi\bar\partial\frac{1}{\tau^{m+1}}\wedge\frac{1}{(2\pi i)^2}\sum_{\ell=0}^{m}\tau^{m-\ell}w^{\ell}d\tau\wedge\frac{(\bar\zeta-\bar z)d\zeta+(\bar\tau-\bar w)d\tau}{|\zeta-z|^2+|\tau-w|^2}\\
&= -\varphi\bar\partial\frac{d\tau}{\tau^{m+1}}\wedge\frac{1}{(2\pi i)^2}\sum_{\ell=0}^{m}\tau^{m-\ell}w^{\ell}\wedge\frac{(\bar\zeta-\bar z)d\zeta}{|\zeta-z|^2+|\tau-w|^2}.
\end{aligned}
$$

For each fixed $\epsilon>0$, $|\zeta-z|>0$ on $\operatorname{supp}\chi_\epsilon$, cf., Lemma 10.2, so we have

$$
\begin{aligned}
&K^\epsilon\phi(z,w)\\
&=\int_{\zeta,\tau}\varphi\frac{1}{(2\pi i)^2}\sum_{\ell=0}^{m}\bar\partial\frac{d\tau}{\tau^{\ell+1}}\wedge w^{\ell}\chi_\epsilon\frac{(\bar\zeta-\bar z)d\bar\zeta\wedge d\zeta}{|\zeta-z|^2+|\tau-w|^2}\wedge\sum_{k=0}^{m}\hat\phi_k(\zeta)\otimes\tau^k.
\end{aligned}
\tag{10.3}
$$

A simple computation yields that

$$
K^\epsilon\phi(z,w)=\sum_{k=0}^{m}a_k^\epsilon(z)\otimes w^k+\mathscr{O}(\bar w), \tag{10.4}
$$

where

$$
a_k^\epsilon(z)=\frac{1}{2\pi i}\int_\zeta\varphi(|\zeta|^2)\chi_\epsilon\frac{\hat\phi_k(\zeta)d\bar\zeta\wedge d\zeta}{\zeta-z}.
$$

Letting $\epsilon$ tend to 0 we get $K\phi$ as in (10.1), where

$$
a_k(z)=\frac{1}{2\pi i}\int_\zeta\varphi(|\zeta|^2)\frac{\hat\phi_k(\zeta)d\bar\zeta\wedge d\zeta}{\zeta-z}.
$$

It is well-known that these Cauchy integrals $a_k(z)$ are smooth solutions to $\bar\partial v=\hat\phi_k d\bar z$ in $Z'=Z\cap\Omega'$. Thus $K\phi$ is smooth. □

*Remark 10.4* The terms $\mathscr{O}(\bar w)$ in the expansion (10.4) of $K^\epsilon\phi(z,w)$ do *not* converge to smooth functions in general when $\epsilon\to 0$. For a simple example, take $\phi=\zeta d\bar\zeta\otimes\tau^m$. Then $K^\epsilon\phi(0,w)$ tends to

$$
w^m\int\varphi(|\zeta|^2)\frac{1}{2\pi i}\frac{|\zeta|^2d\bar\zeta\wedge d\zeta}{|\zeta|^2+|w|^2}
$$

which is a smooth function of $w$ plus (a constant times) $w^m|w|^2\log|w|^2$, and thus not smooth. However, it is certainly in $C^m$. One can check that $K\phi(z,w)=$

$\lim_{\epsilon\to 0^+} K^\epsilon\phi(z,w)$ exists pointwise and defines a function in at least $C^m$ and that our solution can be computed from this limit. In fact, by a more precise computation we get from (10.3) that

$$K^\epsilon\phi(z,w)=\sum_{k=0}^{m}\int_\zeta \varphi(|\zeta|^2)\chi_\epsilon\frac{1}{2\pi i}\frac{(\bar\zeta-\bar z)\hat\phi_k(\zeta)d\bar\zeta\wedge d\zeta}{|\zeta-z|^2+|w|^2}w^k\sum_{j=0}^{m-k}\left(\frac{|w|^2}{|\zeta-z|^2+|w|^2}\right)^j.$$

It is now clear that we can let $\epsilon\to 0$. By a simple computation we then get

$$\begin{aligned}K\phi(z,w)&=\sum_{k=0}^{m}C\hat\phi_k(z)\otimes w^k\\&\quad-\sum_{k=0}^{m}\int_\zeta\varphi(|\zeta|^2)\frac{1}{2\pi i}\frac{\hat\phi_k(\zeta)d\bar\zeta\wedge d\zeta}{\zeta-z}w^k\left(\frac{|w|^2}{|\zeta-z|^2+|w|^2}\right)^{m-k+1}.\end{aligned}$$

Let $\psi=\varphi\hat\phi_k$. Then the $k$th term in the second sum is equal to

$$b(z,w)=\frac{1}{2\pi i}\int_\zeta\frac{\psi(z+\zeta)d\bar\zeta\wedge d\zeta}{\zeta}w^k\left(\frac{|w|^2}{|\zeta|^2+|w|^2}\right)^{m-k+1}.$$

If we integrate outside the unit disk, then we certainly get a smooth function. Thus it is enough to consider the integral over the disk. Moreover, if $\psi(z+\zeta)=\mathscr{O}(|\zeta|^M)$ for a large $M$, then the integral is at least $C^m$. By a Taylor expansion of $\psi(z+\zeta)$ at the point $z$, we are thus reduced to consider

$$\int_{|\zeta|<1}\frac{\zeta^\alpha\bar\zeta^\beta}{\zeta}\left(\frac{|w|^2}{|\zeta|^2+|w|^2}\right)^{m-k+1}.$$

For symmetry reasons, they vanish, except when $\alpha=\beta+1$. Thus we are left with

$$\int_{|\zeta|<1}|\zeta|^{2\beta}\left(\frac{|w|^2}{|\zeta|^2+|w|^2}\right)^{m-k+1}w^k=Cw^k|w|^{2(m-k+1)}\int_0^1\frac{s^\beta ds}{(s+|w|^2)^{m-k+1}}$$

for non-negative integers $\beta$. The right hand side is a smooth function of $w$ if $\beta\le m-k-1$ and a smooth function plus

$$Cw^k|w|^{2(\beta+1)}\log|w|^2$$

if $\beta\ge m-k$. The worst case therefore is when $k=m$ and $\beta=0$; then we have $w^m|w|^2\log|w|^2$ that we encountered above. □

**Proposition 10.5** *Let $z,w$ be coordinates at a point $x\in X_{reg}$ such that $Z=\{w=0\}$ and $x=(0,0)$. If $\phi$ is smooth, and has support where the local coordinates are defined, then*

$$v^\epsilon(z,w)=\int_\zeta \chi(|\zeta-z|^2/\epsilon)(HR\wedge B\wedge g)_N\wedge\phi,$$

*is smooth for* $\epsilon>0$, *and for each multiindex* $\ell$ *there is a smooth form* $v_\ell$ *such that*

$$\partial_w^\ell v^\epsilon|_{w=0}\to v_\ell$$

*as currents on* $Z$.

Taking this proposition for granted we can conclude the proof of Theorem 10.1.

*Proof of Theorem 10.1* If $\phi\equiv 0$ in a neighborhood of $x\in X'_{\rm reg}$, then $K\phi$ is smooth near $x$, cf., the proof of Proposition 9.4. Thus, it is sufficient to prove Theorem 10.1 assuming that $\phi$ is smooth and has support near $x$.

Recall that given a minimal generating set $1, w^{\alpha_1},\dots,w^{\alpha_{\nu-1}}$, one gets the coefficients $\hat v_j^\epsilon$ in the representation

$$v^\epsilon=\hat v_0^\epsilon\otimes 1+\cdots+\hat v_{\nu-1}^\epsilon\otimes w^{\alpha_{\nu-1}}$$

from $\partial_w^\ell v^\epsilon|_{w=0}$, $|\ell|\le M$ by a holomorphic matrix, cf., the proof of Lemma 4.7. It thus follows from Proposition 10.5 that there are smooth $\hat v_j$ such that $\hat v_j^\epsilon\to\hat v_j$ as currents on $Z$. Let $v=\hat v_0\otimes 1+\cdots+\hat v_{\nu-1}\otimes w^{\alpha_{\nu-1}}$. In view of (2.14), $v^\epsilon\wedge\mu\to v\wedge\mu$ for all $\mu$ in $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{CH}_\Omega^Z)$. From Lemma 10.2 we conclude that $v\wedge\mu=K\phi\wedge\mu$ for all such $\mu$. Thus $K\phi=v$ in $\mathcal{W}_X^{0,*}$ and hence $K\phi$ is smooth. $\square$

*Proof of Proposition 10.5* Assume that $X$ is embedded in $\Omega\subset\mathbb{C}^N_{\zeta',\tau'}$. After a suitable rotation we can assume that $Z$ is the graph $\tau'=\psi(\zeta')$. The Bochner–Martinelli kernel in $\Omega$ is rotation invariant, so it is

$$B=\sigma+\sigma\wedge\bar\partial\sigma+\sigma\wedge(\bar\partial\sigma)^2+\cdots,$$

where

$$\sigma=\frac{(\bar\zeta'-\bar z')\cdot d\zeta'+(\bar\tau'-\bar w')\cdot d\tau'}{|\zeta'-z'|^2+|\tau'-w'|^2}.$$

We now choose the new coordinates $\zeta=\zeta'$, $\tau=\tau'-\psi(\zeta')$ in $\Omega$, so that $Z=\{(\zeta,\tau);\ \tau=0\}$.

Recall that on $X_{reg}$ we have that $R\wedge dz$ is a smooth form times $\mu=(\mu_1,\dots,\mu_m)$, where $\mu_j$ is a generating set for $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{CH}_\Omega^Z)$. Thus we are to compute $\partial_w^\ell|_{w=0}$ of integrals like

$$\int_{\zeta,\tau}\bar\partial\frac{d\tau}{\tau^{\alpha+\mathbf{1}}}\wedge B_k^\epsilon\wedge\phi(\zeta,z,w,\tau), \tag{10.5}$$

where $k\le n$ and $\phi$ is smooth with compact support near $x$. It is clear that the symbols $\bar\tau,\bar w, d\bar\tau$ can be omitted in the expression for

$$B^\epsilon=\chi_\epsilon B=\chi(|\zeta-z|^2/\epsilon)B,$$

since $\bar\tau$ and $d\bar\tau$ annihilate $\bar\partial(1/\tau^{\alpha+1})$, and since we only take holomorphic derivatives with respect to $w$ and set $w = 0$.

Let us write $\psi(\zeta) - \psi(z) = A(\zeta, z)\eta$, where $\eta := \zeta - z$ is considered as a column matrix and $A$ is a holomorphic $(N - n) \times n$-matrix. Then

$$\sigma = \frac{\eta^*\nu}{|\zeta - z|^2 + |\tau - w + \psi(\zeta) - \psi(z)|^2},$$

where $\nu$ is the $(1, 0)$-form valued column matrix

$$\nu = d\zeta + A^* d(\tau + \psi(\zeta)).$$

Since $\eta^*\nu$ is a $(1, 0)$-form we have that

$$B_k^\epsilon = \chi_\epsilon \frac{\eta^*\nu \wedge ((d\eta^*)\nu + \eta^*\bar\partial\nu)^{k-1}}{(|\zeta - z|^2 + |\tau - w + \psi(\zeta) - \psi(z)|^2)^k}.$$

**Lemma 10.6** *Let*

$$\xi^i = \xi_1^i \frac{\partial}{\partial\zeta_1} + \cdots + \xi_n^i \frac{\partial}{\partial\zeta_n}$$

*be smooth* $(1, 0)$*-vector fields, and let* $L_i = L_{\xi^i}$ *be the associated Lie derivatives for* $i = 1, \ldots, \rho$*. Let*

$$\gamma_k := \eta^*\nu \wedge ((d\eta^*)\nu + \eta^*\bar\partial\nu)^{k-1}.$$

*If we have a modification* $\pi : \tilde W \to \Omega \times \Omega$ *such that locally* $\pi^*\eta = \eta_0\eta'$*, where* $\eta_0$ *is a holomorphic function, then*

$$\pi^*(L_1 \cdots L_\rho \gamma_k) = \bar\eta_0^k \beta,$$

*where* $\beta$ *is smooth.*

Recall that if $a$ is a form, then $L_\xi a = d(\xi \neg a) + \xi \neg (da)$, and that $L_\xi(\beta \neg a) = [\xi, \beta] \neg a + \beta \neg (L_\xi a)$ if $\beta$ is another vector field.

*Proof* Introduce a nonsense basis $e$ and its dual $e^*$ and consider the exterior algebra spanned by $e_j, e_\ell^*$, and the cotangent bundle. Let

$$c_\ell = \eta^* e \wedge ((d\eta^*)e)^{\ell-1}.$$

Notice that $\gamma_k$ is a sum of terms like

$$(\nu e^* \neg)^\ell c_\ell \wedge (\eta^*\bar\partial\nu)^{k-\ell}.$$

Since $L_i c_\ell = 0$ and $L_i(\eta^* b) = \eta^* L_i b$ it follows after a finite number of applications of $L_i$'s that we get

$$(\nu_1 e^*)\neg \cdots (\nu_\ell e^*)\neg c_\ell(\eta^* b_1)\cdots(\eta^* b_{k-\ell}),$$

where $\nu_j$ and $b_j$ are smooth. Since

$$\pi^* c_\ell = \bar\eta_0^\ell (\eta')^* e \wedge (d(\eta')^* e)^{\ell-1},$$

the lemma now follows. □

We note that $\eta^*(I + A^*A)\eta = |\zeta - z|^2 + |\psi(\zeta) - \psi(z)|^2$. Thus, differentiating (10.5) with respect to $w$, setting $w = 0$, and evaluating the residue with respect to $\tau$ using (2.10), we obtain a sum of integrals like

$$\int_\zeta \chi_\epsilon \frac{(\eta^* a_1)\cdots(\eta^* a_{t+1}) \wedge \gamma_k \wedge \phi}{(\eta^*(I + A^*A)\eta)^{k+t+1}},$$

where $a_1, \dots, a_{t+1}$ are column vectors of smooth functions. We must prove that the limit of such integrals when $\epsilon \to 0$ are smooth in $z$.

**Lemma 10.7** *Let*

$$I_\ell^{r,s} = \int \chi_\epsilon \frac{(\eta^* a_1)\cdots(\eta^* a_r)\mathscr{O}(|\eta|^{2s})\tilde\gamma_k \wedge \phi}{\Phi^{k+\ell}},$$

*where $a_1, \dots, a_r$ are tuples of smooth functions, $\tilde\gamma_k = L_1 \cdots L_\rho \gamma_k$, where $L_i = L_{\xi_i}$ are Lie derivatives with respect to smooth $(1,0)$-vector fields $\xi^i$ as above for $i = 1, \dots, \rho$, $\phi$ is a test form with support close to $z$, and $\Phi := \eta^*(I + A^*A)\eta$. If $r \geq 1$ and $r + s \geq \ell + 1$, then we have the relation*

$$I_{\ell+1}^{r,s} = I_\ell^{r-1,s} + I_{\ell+1}^{r-1,s+1} + I_\ell^{r,s-1} + o(1) \tag{10.6}$$

*when $\epsilon \to 0$.*

*Proof* If

$$\xi = a_r^t (I + A^*A)^{-t} \frac{\partial}{\partial \zeta},$$

and $L = L_\xi$, then using that $\Phi = \eta^t (I + A^*A)^t \bar\eta$, one obtains that

$$L\Phi = \eta^* a_r + \mathscr{O}(|\eta|^2). \tag{10.7}$$

Thus

$$I_{\ell+1}^{r,s} = \int \chi_\epsilon (\eta^* a_1)\cdots(\eta^* a_{r-1})\mathscr{O}(|\eta|^{2s})\tilde\gamma_k \wedge \phi L \frac{1}{\Phi^{k+\ell}} + I_{\ell+1}^{r-1,s+1}$$

in view of (10.7). We now integrate by parts by $L$ in the integral. If a derivative with respect to $\zeta_j$ falls on some $\eta^* a_i$, we get a term $I_\ell^{r-1,s}$. If it falls on $\mathscr{O}(|\eta|^{2s})$ we get either $\mathscr{O}(|\eta|^{2(s-1)})$ times $\eta^* b$, for some tuple $b$ of smooth functions, and this gives rise to the term $I_\ell^{r,s-1}$ or $\mathscr{O}(|\eta|^{2s})$, and this gives rise to another term $I_\ell^{r-1,s}$. If it falls on $\phi$ or $\tilde{\gamma}_k$ we get an additional term $I_\ell^{r-1,s}$. The only possibility left is when the derivative falls on $\chi_\epsilon = \chi(|\eta|^2/\epsilon)$. It remains to show that an integral of the form

$$\int_{\zeta,z} \chi'(|\eta|^2/\epsilon) \frac{(\eta^* a_1)\cdots(\eta^* a_{r-1})(\eta^* b)}{\epsilon} \frac{\mathscr{O}(|\eta|^{2s})\gamma_k \wedge \phi}{\Phi^{k+\ell}}$$

tends to 0, where the factor $\eta^* b$ comes from the derivative of $|\eta|^2$. We now choose a resolution $\widetilde{V} \to \Omega \times \Omega$ such that $\eta = \eta_0 \eta'$ where $\eta'$ is non-vanishing and $\eta_0$ is (locally) a monomial. Notice that $\pi^*\Phi = |\eta_0|^2 \Phi'$ where $\Phi'$ is smooth and strictly positive. In view of Lemma 10.6 we thus obtain integrals of the form

$$\int_{\widetilde{V}} \chi'(|\eta_0|^2 v/\epsilon) \frac{1}{\epsilon} \frac{\bar{\eta}_0^{r+s-\ell}}{\eta_0^{k+\ell-s}} \alpha, \tag{10.8}$$

where $v$ is smooth and strictly positive and $\alpha$ is smooth.

In order to see that the limit of (10.8) tends to 0, we note first that if we let

$$\tilde{\chi}(s) = s\chi'(s) + \chi(s),$$

then just as $\chi$, $\tilde{\chi}$ is also a smooth function on $[0, \infty)$ that is 0 in a neighborhood of 0 and 1 in a neighborhood of $\infty$. By assumption, $r + s - \ell - 1 \geq 0$. Since the principal value current $1/f^m$ acting on a test form $\beta$ can be defined as

$$\lim_{\epsilon \to 0^+} \int \chi(|f|^2 v/\epsilon) \frac{\beta}{f^m}$$

for any cut-off function as above, the principal value current $1/\eta_0^{k+\ell-s}$ acting on $\bar{\eta}_0^{r+s-\ell-1}\alpha$ equals

$$\lim_{\epsilon \to 0^+} \int_{\widetilde{V}} \chi\left(|\eta_0|^2 v/\epsilon\right) \frac{\bar{\eta}_0^{r+s-\ell-1}}{\eta_0^{k+\ell-s}} \alpha = \lim_{\epsilon \to 0^+} \int_{\widetilde{V}} \tilde{\chi}\left(|\eta_0|^2 v/\epsilon\right) \frac{\bar{\eta}_0^{r+s-\ell-1}}{\eta_0^{k+\ell-s}} \alpha.$$

Taking the difference between the left and right hand side, we conclude that (10.8) tends to 0 when $\epsilon \to 0$. □

Now we can conclude the proof of Proposition 10.5. From the beginning we have $I_\ell^{\ell,0}$. After repeated applications of (10.6) we end up with

$$I_\ell^{0,\ell} + I_{\ell-1}^{0,\ell-1} + \cdots + I_0^{0,0} + o(1).$$

However, any of these integrals has an integrable kernel even when $\epsilon = 0$. This means that we are back to the case in [6, Lemma 6.2], and so the limit integral is smooth in $z$. □

## 11 A fine resolution of $\mathscr{O}_X$

We first consider a generalization of Theorem 9.1.

**Lemma 11.1** *Assume that $\phi \in \mathcal{W}^{0,k}(X) \cap \mathscr{E}_X^{0,k}(X_{reg}) \cap Dom\,\bar\partial_X$ and that $K\phi$ is in $Dom\,\bar\partial_X$ (or just in $Dom\,\bar\partial$). Then (9.1) holds on $X'$.*

*Proof* Let $\chi_\delta$ be functions as before that cut away $X_{sing}$. From Koppelman's formula (9.1) for smooth forms we have

$$\chi_\delta\phi\wedge h = \bar\partial(K(\chi_\delta\phi))\wedge h + K(\chi_\delta\bar\partial\phi)\wedge h + P(\chi_\delta\phi)\wedge h + K(\bar\partial\chi_\delta\wedge\phi)\wedge h,\ \ h\in\omega_X^n, \tag{11.1}$$

for $z \in X'_{reg}$. Clearly the left hand side tends to $\phi\wedge h$ when $\delta \to 0$. From Lemma 9.2 it follows that $K(\chi_\delta\phi)\wedge h \to K\phi\wedge h$. Thus the first term on the right hand side of (11.1) tends to $\bar\partial(K\phi)\wedge h$. In the same way the second and third terms on the right hand side tend to $K(\bar\partial\phi)\wedge h$ and $P\phi\wedge h$, respectively. It remains to show that the last term tends to 0. If $z$ belongs to a fixed compact subset of $X'_{\mathrm{reg}}$, then $B$ is smooth in (9.5) when $\zeta$ is in supp $\bar\partial\chi_\delta$ for small $\delta$. Hence it suffices to see that

$$R(\zeta)\wedge d\zeta\wedge\bar\partial\chi_\delta\wedge\phi(\zeta)\wedge i_*h \to 0,$$

and since this is a tensor product of currents, it suffices to see that

$$R(\zeta)\wedge d\zeta\wedge\bar\partial\chi_\delta\wedge\phi(\zeta)\to 0,$$

or equivalently, $\omega(\zeta)\wedge\bar\partial\chi_\delta\wedge\phi(\zeta)\to 0$, which follows by Lemma 8.4 since $\phi$ is in Dom $\bar\partial_X$. We have thus proved that

$$\chi_\delta\phi\wedge h = \chi_\delta\bar\partial(K\phi)\wedge h + \chi_\delta K(\bar\partial\phi)\wedge h + \chi_\delta P\phi\wedge h.$$

The first term on the right hand side is equal to $\bar\partial(\chi_\delta K\phi\wedge h) - \bar\partial\chi_\delta\wedge K\phi\wedge h$, where the latter term tends to 0 if $K\phi$ is in Dom $\bar\partial_X$ or just in Dom $\bar\partial$, cf., Lemma 8.4. Thus we get

$$\phi\wedge h = \bar\partial(K\phi)\wedge h + K(\bar\partial\phi)\wedge h + P\phi\wedge h,\ \ h\in\omega_X^n,$$

which precisely means that (9.1) holds. □

**Definition 11.2** We say that a $(0,q)$-current $\phi$ on an open set $\mathcal{U}\subset X$ is a section of $\mathscr{A}_X^q$ over $\mathcal{U}$, $\phi\in\mathscr{A}^q(\mathcal{U})$, if, for every $x\in\mathcal{U}$, the germ $\phi_x$ can be written as a finite sum of terms

$$\xi_\nu\wedge K_\nu(\cdots\xi_2\wedge K_2(\xi_1\wedge K_1(\xi_0))),$$

where $\xi_j$ are smooth $(0, *)$-forms and $K_j$ are integral operators with kernels $k_j(\zeta, z)$ at $x$, defined as above, and such that $\xi_j$ has compact support in the set where $z \mapsto k_j(\zeta, z)$ is defined.

Clearly $\mathscr{A}_X^*$ is closed under multiplication by $\xi$ in $\mathscr{E}_X^{0,*}$. It follows from (9.8) that $\mathscr{A}_X^*$ is a subsheaf of $\mathcal{W}_X^{0,*}$ and from Theorem 10.1 that $\mathscr{A}_X^k = \mathscr{E}_X^{0,*}$ on $X_{reg}$. Clearly any operator $K$ as above maps $\mathscr{A}_X^{*+1} \to \mathscr{A}_X^*$.

**Lemma 11.3** *If $\phi$ is in $\mathscr{A}_X$, then $\phi$ and $K\phi$ are in Dom $\bar\partial_X$.*

*Proof* Notice that [6, Lemma 6.4] holds in our case by verbatim the same proof, since we have access to the dimension principle for (tensor products of) pseudomeromorphic $(n, *)$-currents, and the computation rule (2.3), cf., the comment after Definition 5.7. Since $\mathscr{A}_X^* = \mathscr{E}_X^{0,*}$ on $X_{\text{reg}}$ it is enough by Lemma 8.4 to check that $\bar\partial\chi_\delta \wedge \omega \wedge \phi \to 0$, and this precisely follows from [6, Lemma 6.4]. □

In view of Lemmas 11.1 and 11.3 we have

**Proposition 11.4** *Let $K$, $P$ be integral operators as in Theorem 9.1. Then*

$$K : \mathscr{A}^{k+1}(X) \to \mathscr{A}^k(X'), \quad P : \mathscr{A}^k(X) \to \mathscr{E}^{0,k}(X'),$$

*and the Koppelman formula (9.1) holds.*

*Proof of Theorem 1.1* By definition, it is clear that $\mathscr{A}_X^k$ are modules over $\mathscr{E}_X^{0,k}$, and by Theorem 10.1, $\mathscr{A}_X^k$ coincides with $\mathscr{E}_X^{0,k}$ on $X_{\text{reg}}$. Since we have access to Koppelman formulas, precisely as in the proof of [6, Theorem 1.2] we can verify that $\bar\partial : \mathscr{A}_X^k \to \mathscr{A}_X^{k+1}$.

It remains to prove that (1.2) is exact. We choose locally a weight $g$ that is holomorphic in $z$, so the term $P\phi$ vanishes if $\phi$ is in $\mathscr{A}_X^k$, $k \geq 1$, and is holomorphic in $z$ when $k = 0$. Assume that $\phi$ is in $\mathscr{A}_X^k$ and $\bar\partial\phi = 0$. If $k \geq 1$, then $\bar\partial K\phi = \phi$, and if $k = 0$, then $\phi = P\phi$. □

### 11.1 Global solvability

Assume that $E \to X$ is a holomorphic vector bundle; this means that the transition matrices have entries in $\mathscr{O}_X$. For instance if we have a global embedding $i : X \to \Omega$ and a holomorphic vector bundle $F \to \Omega$, then $F$ defines a vector bundle $i^*F \to X$. The sheaves $\mathscr{A}_X^*(E)$ give rise to a fine resolution of the sheaf $\mathscr{O}_X(E)$, and by standard homological algebra we have the isomorphisms

$$H^q(X, \mathscr{O}(E)) = \frac{\text{Ker}\,(\mathscr{A}^q(X, E) \overset{\bar\partial}{\to} \mathscr{A}^{q+1}(X, E))}{\text{Im}\,(\mathscr{A}^{q-1}(X, E) \overset{\bar\partial}{\to} \mathscr{A}^q(X, E))}, \quad q \geq 1.$$

Thus, if $\phi \in \mathscr{A}^{q+1}(X, E)$, $\bar\partial\phi = 0$, and its canonical cohomology class vanishes, then the equation $\bar\partial\psi = \phi$ has a global solution in $\mathscr{A}^q(X, E)$. In particular, the equation is always solvable if $X$ is Stein. If for instance $X$ is a pure-dimensional projective variety $i\colon X\to\mathbb{P}^N$, then the $\bar\partial$-equation is solvable, e.g., if $E$ is a sufficiently ample line bundle.

## 12 Locally complete intersections

Let us consider the special case when $X$ locally is a complete intersection, i.e., given a local embedding $i\colon X\to\Omega\subset\mathbb{C}^N$ there are global sections $f_j$ of $\mathscr{O}(d_j)\to\mathbb{P}^N$ such that $\mathcal{J}=(f_1,\dots,f_p)$, where $p=N-n$. In particular, $Z=\{f_1=\cdots=f_p=0\}$. In this case $\mathcal{H}om(\mathscr{O}_\Omega/\mathcal{J},\mathcal{CH}_\Omega)$ is generated by the single current

$$\mu=\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge dz_1\wedge\cdots\wedge dz_N,$$

see, e.g., [3]. Each smooth $(0,q)$-form $\phi$ in $\mathscr{E}_X^{0,q}$ is thus represented by a current $\Phi\wedge\mu$, where $\Phi$ is smooth in a neighborhood of $Z$ and $i^*\Phi=\phi$. Moreover, $X$ is Cohen–Macaulay so $X_{reg}$ coincides with the part of $X$ where $Z$ is regular, and $\bar\partial\phi=\psi$ if and only if $\bar\partial(\phi\wedge\mu)=\psi\wedge\mu$.

Henkin and Polyakov introduced, see [17, Definition 1.3], the notion of *residual currents $\phi$ of bidegree* $(0,q)$ on a locally complete intersection $X\subset\mathbb{P}^N$, and the $\bar\partial$-equation $\bar\partial\psi=\phi$. Their currents $\phi$ correspond to our $\phi$ in $\mathscr{E}_X^{0,q}$ and their $\bar\partial$-operator on such currents coincides with ours.

*Remark 12.1* In [18] Henkin and Polyakov consider a global reduced complete intersection $X\subset\mathbb{P}^N$. They prove, by a global explicit formula, that if $\phi$ is a global $\bar\partial$-closed smooth $(0,q)$-form with values in $\mathscr{O}(\ell)$, $\ell=d_1+\cdots d_p-N-1$, then there is a smooth solution to $\bar\partial\psi=\phi$ at least on $X_{reg}$, if $1\le q\le n-1$. When $q=n$ a necessary obstruction term occurs. However, their meaning of "$\bar\partial$-closed" is that locally there is a representative $\Phi$ of $\phi$ and smooth $g_j$ such that $\bar\partial\Phi=g_1f_1+\cdots+g_pf_p$. If this holds, then clearly $\bar\partial\phi=0$. The converse implication is *not* true, see Example 12.2 below. It is not clear to us whether their formula gives a solution under the weaker assumption that $\bar\partial\phi=0$, neither do we know whether their solution admits some intrinsic extension across $X_{sing}$ as a current on $X$. □

*Example 12.2* Let $X=\{f=0\}\subset\Omega\subset\mathbb{C}^{n+1}$ be a reduced hypersurface, and assume that $df\neq0$ on $X_{reg}$, so that $\mathcal{J}=(f)$. Let $\phi$ be a smooth $(0,q)$-form in a neighborhood of some point $x$ on $X$ such that $\bar\partial\phi=0$. We claim that $\bar\partial u=\phi$ has a smooth solution $u$ if and only if $\phi$ has a smooth representative $\Phi$ in ambient space such that $\bar\partial\Phi=fg$ for some smooth form $g$. In fact, if such a $\Phi$ exists then $0=f\bar\partial g$ and thus $\bar\partial g=0$. Therefore, $g=\bar\partial\gamma$ for some smooth $\gamma$ (in a Stein neighborhood of $x$ in ambient space) and hence $\bar\partial(\Phi-f\gamma)=0$. Thus there is a smooth $U$ such that $\bar\partial U=\Phi-f\gamma$; this means that $u=i^*U$ is a smooth solution to $\bar\partial u=\phi$. Conversely, if $u$ is a smooth solution, then $u=i^*U$ for some smooth $U$ in ambient space, and thus $\Phi=\bar\partial U$ is a representative of $\phi$ in ambient space. Thus $\bar\partial\Phi=fg$ (with $g=0$).

There are examples of hypersurfaces $X$ where there exist smooth $\phi$ with $\bar\partial\phi = 0$ that do not admit smooth solutions to $\bar\partial u = \phi$, see, e.g., [6, Example 1.1]. It follows that such a $\phi$ cannot have a representative $\Phi$ in ambient space as above. □

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