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ORIGINAL PAPER

Norm estimates and asymptotic faithfulness of the quantum $SU(n)$ representations of the mapping class group

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Abstract
We give a direct proof for the asymptotic faithfulness of the quantum $SU(n)$ representations of the mapping class group using peak sections in Kodaira embedding. We give also estimates on the norm of the parallell transport of the projective connection on the Verlinde bundle. The faithfulness has been proved earlier by J. E. Andersen using Toeplitz operators on compact Kähler manifolds and by J. Marché and M. Narimannejad using skein theory.

Keywords Mapping class group · Representation · Faithfulness · Norm estimates · Peak sections.

Mathematics Subject Classification 14D21 · 57R56 · 53C55

1 Introduction

Let Σ be a closed oriented surface of genus $g \geq 2$ and $p \in \Sigma$. We consider the moduli space M of flat $SU(n)$-connections P on $\Sigma\backslash\{p\}$ with fixed holonomy a center element $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{SU(n)}$ of $SU(n)$. We assume that n and d are coprime, in the case of $g = 2$ we also allow $(n, d) = (2, 0)$, namely the $SU(2)$-connections with trivial holonomy.

There is a canonical symplectic form ω on M obtained by integrating wedge product of Lie algebra $\mathfrak{su}(n)$-valued connection forms. The natural action of the mapping class group Γ of Σ on (M, ω) is symplectic. Let $\mathcal{L}$ be the Hermitian line bundle over M and ∇ the compatible connection in $\mathcal{L}$ constructed by Freed [8]. By [8, Proposition 5.27], the curvature

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of ∇ is $\frac{\sqrt{-1}}{2\pi}\omega$. Given any element σ in the Teichmüller space $\mathcal{T}$ the symplectic manifold M can be equipped with a Kähler structure so that $\mathcal{L}$ becomes a holomorphic ample line bundle $\mathcal{L}_\sigma$. The *Verlinde bundle* $\mathcal{V}_k$ is defined by

$$\mathcal{V}_k = H^0(M_\sigma, \mathcal{L}_\sigma^k).$$

It is known by the works of Axelrod et al. [5] and Hitchin [10] that the projective bundle $\mathbb{P}(\mathcal{V}_k)$ is equipped with a natural flat connection. Since there is an action of the mapping class group Γ of Σ on $\mathcal{V}_k$ covering its action on $\mathcal{T}$, which preserves the flat connection in $\mathbb{P}(\mathcal{V}_k)$, we get for each k, a finite dimensional projective representation of Γ. This sequence of projective representations $\pi_k^{n,d}$, $k \in \mathbb{N}_+$, is the *quantum $SU(n)$ representation* of the mapping class group Γ.

Turaev [16] conjectured that there should be no nontrivial element ϕ of the mapping class group in the kernel of $\pi_k^{n,d}$ for all k, keeping (n,d) fixed. This property is called *asymptotic faithfulness* of the quantum $SU(n)$ representations $\pi_k^{n,d}$. In [1, Theorem 1], J. E. Andersen proved Turaev's conjecture, namely the following

Theorem 1.1 ([1], Theorem 1) *Let $\pi_k^{n,d}$ be the projective representation of the mapping class group. Assume that n and d are coprime or that $(n,d) = (2,0)$ when $g = 2$, then*

$$\bigcap_{k=1}^{\infty} Ker(\pi_k^{n,d}) = \begin{cases} \{1, H\}, & g = 2, \quad (n,d) = (2,0) \\ \{1\}, & otherwise, \end{cases} \tag{1.1}$$

where H is the hyperelliptic involution on genus $g = 2$ surfaces.

This theorem is proved in [1] by considering the action of the mapping class group on functions on M as symbols of Toeplitz operators on holomorphic sections of $\mathcal{L}_\sigma^k$ for large k. A different proof using skein theorem is given in [12]; see also [2–4] and references therein for further developments. The existing proofs seem rather involved. We shall give a somewhat more direct and elementary proof using peak sections in the Kodaira embedding.

We describe briefly our approach. Write $\pi_k^{n,d}$ as π_k throughout the rest of the paper. The action of an element $\phi \in \Gamma$ on $\sigma \in \mathcal{T}$ and $p \in M$ will be all denoted by the same, $\phi(\sigma)$ and $\phi(p)$.

Let $\phi \in \Gamma$, $\sigma \in \mathcal{T}$, and $\sigma(t) : [0,1] \to \mathcal{T}$ be a smooth curve connecting $\phi(\sigma)$ and σ. Denote by $P_{\phi(\sigma),\sigma(t)}$ the parallel transport from $\phi(\sigma)$ to $\sigma(t)$ with respect to the projective flat connection (2.1) below. For any $s \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$, set

$$s(t) := P_{\phi(\sigma),\sigma(t)} \circ \phi^*(s) \in H^0\left(M_{\sigma(t)}, \mathcal{L}_{\sigma(t)}^k\right).$$

Here ϕ^* is the induced action of ϕ on the total space of the Verlinde bundle. For any positive smooth function $\rho : M \to (0,1]$ define a Hermitian structure on the trivial bundle $\mathcal{H}_k = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ by

$$\langle s_1, s_2 \rangle_\rho = \int_M \rho \cdot (s_1, s_2) \frac{\omega^m}{m!}, \quad \|s\|_\rho^2 = \langle s, s \rangle_\rho. \tag{1.2}$$

We shall study the variation of $\|s(t)\|_\rho^2$ and obtain

$$e^{-\frac{C_\rho + kC}{k+n}} \|s\|_{\rho\circ\phi^{-1}}^2 \le \|P_{\phi(\sigma),\sigma}\phi^*(s)\|_\rho^2 \le e^{\frac{C_\rho + kC}{k+n}} \|s\|_{\rho\circ\phi^{-1}}^2; \tag{1.3}$$

see Proposition 3.2 below. Here C_ρ and C are positive constants independent of k. We prove that if $\phi \in \bigcap_{k=1}^{\infty} \operatorname{Ker} \pi_k$ then the induced action of ϕ on M is the identity. First of all it follows that the representation $\phi \to P_{\phi(\sigma),\sigma} \circ \phi^*$ is projectively trivial on the space $H^0(M_\sigma, \mathcal{L}_\sigma^k)$,

$$P_{\phi(\sigma),\sigma} \circ \phi^* = \pi_k(\phi) = c_k \mathrm{Id}$$

for some constant $c_k = c_k(\phi) \neq 0$. By taking $\rho = 1$ and using (1.3), we get a lower bound of c_k^2, i.e. $c_k^2 \geq e^{-\frac{C_1+kC}{k+n}}$, which converges to e^{-C} as $k \to \infty$, so $c_k^2 > c$ for some constant $c > 0$. If ϕ on M is not the identity, say $\phi(p) \neq p$ we can construct appropriate weight function ρ and peak section s at p so that the right hand side $e^{\frac{C_\rho+kC}{k+n}} \|s\|^2_{\rho\circ\phi^{-1}}$ is arbitrarily smaller than e^{-C} while as $\|P_{\phi(\sigma),\sigma} \circ \phi^*(s)\|^2_\rho = c_k^2 \|s\|^2_\rho$ has a uniform lower bound e^{-C}, a contradiction to (1.3). Thus ϕ acts as identity on M, and it follows further by standard arguments that ϕ itself is the identity element in Γ under the assumption on $\{g, n, d\}$ or a hyperelliptic involution for genus $g = 2$ surfaces.

We note that even though our proof is simpler than Andersen's proof [1] but the underlying ideas are very much related; indeed Andersen used the result of Bordemann et al. [6] on norm estimates of Teoplitz operators T_f which are based on coherent states, namely specific kinds of peak sections. Finally we mention that constructing representations of the mapping class group and the study of faithfulness of the corresponding representations are of much interests; see [9,12] and references therein.

This article is organized as follows. In Sect. 2 we fix notation and recall some basic facts on the Verlinde bundle, the projective flat connection and peak sections. Theorem 1.1. is proved in Sect. 3.

We would like to thank Jorgen Ellegaard Andersen for some informative explanation of his results.

2 Preliminaries

The results in this section can be found in [1,5,10,11,15] and references therein.

Let Σ be a closed oriented surface of genus $g \geq 2$ and $p_0 \in \Sigma$. Let $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{SU(n)} = \{cI, c^n = 1\}$, the center of $SU(n)$. We assume that n and d are coprime, in the case of $g = 2$ we also allow $(n, d) = (2, 0)$. Let M be the moduli space of flat $SU(n)$-connections P on $\Sigma\backslash\{p_0\}$ with fixed holonomy d around p_0. M is then a compact smooth manifold of dimension $m = (n^2 - 1)(g - 1)$ with tangent vectors given by the Lie algebra $\mathfrak{su}(n)$-valued connection 1-forms.

There is a canonical symplectic form ω on M by taking the trace of the integration of products of 1-forms, the natural action of the mapping class group Γ on M is symplectic. Let $\mathcal{L}$ be the Hermitian line bundle over M and ∇ the compatible connection in $\mathcal{L}$ with curvature $\frac{\sqrt{-1}}{2\pi}\omega$; see [8, Proposition 5.27]. The induced connection in $\mathcal{L}^k$ will also be denoted by ∇.

Let $\mathcal{T}$ be the Teichmüller space of Σ parametrizing all marked complex structures on Σ. By a classical result of Narasimhan and Seshadri [13] each $\sigma \in \mathcal{T}$ induces a Kähler structure on M and thus a Kähler manifold M_σ. By using the $(0, 1)$-part of ∇, the bundle $\mathcal{L}$ is then equipped with a holomorphic structure, which we denote by $\mathcal{L}_\sigma$. Thus the manifold $\mathcal{T}$ also parameterizes Kähler structures $I_\sigma, \sigma \in \mathcal{T}$ on (M, ω) and the holomorphic line bundles $\mathcal{L}_\sigma$.

For any positive integer k the *Verlinde bundle* $\mathcal{V}_k$ is a finite dimensional subbundle of the trivial bundle $\mathcal{H}_k = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ given by

$$\mathcal{V}_k(\sigma) = H^0(M_\sigma, \mathcal{L}_\sigma^k),\ \sigma \in \mathcal{T}.$$

By the results of Axelrod et al. [5] and Hitchin [10], there is a projective flat connection in $\mathcal{V}_k$ given by

$$\hat{\nabla}_v = \hat{\nabla}_v^t - u(v), \quad v \in T(\mathcal{T}), \tag{2.1}$$

where $\hat{\nabla}^t$ is the trivial connection in $\mathcal{H}_k$. The second term $u(v)$ is given by [1, Formula (7)],

$$u(v) = \frac{1}{2(k+n)}\left(\sum_{r=1}^{R} \nabla_{X_r(v)}\nabla_{Y_r(v)} + \nabla_{Z(v)} + nv[F]\right) - \frac{1}{2}v[F], \tag{2.2}$$

where $F : \mathcal{T} \to C^\infty(M)$ is a smooth function such that $F(\sigma)$ is real-valued on M for all $\sigma \in \mathcal{T}$, $\{X_r(v), Y_r(v), Z(v)\} \subset C^\infty(M_\sigma, T)$ are a finite set of vector fields of M_σ taking value in the holomorphic tangent space T of M_σ.

Since $\mathcal{L}_\sigma$ is an ample line bundle over M_σ, one may take a large k such that $\mathcal{L}_\sigma^k$ is a very ample line bundle. Then the Kodaira embedding is given by

$$\Phi_\sigma^k : M \to \mathbb{P}(H^0(M_\sigma, \mathcal{L}_\sigma^k)^*), \quad p \mapsto \Phi_\sigma^k(p) = \{s \in H^0(M_\sigma, \mathcal{L}_\sigma^k), s(p) = 0\}.$$

A *peak section* $s_p^k \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$ of $\mathcal{L}_\sigma^k$ at a point $p \in M$ is a unit norm generator of the orthogonal complement of $\Phi_\sigma^k(p)$ such that

$$|s_p^k(p)|^2 = \sum_{i=1}^{N_k} |s_i(p)|^2,$$

where $N_k = \dim H^0(M_\sigma, \mathcal{L}_\sigma^k)$ and $\{s_i\}_{1\le i\le N_k}$ is an orthonormal basis of $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ with respect to the standard L^2-metric; see [11, Definition 5.1.7]. The existence of peak sections is well-known, and for any sequence $\{r_k\}$ with $r_k \to 0$ and $r_k\sqrt{k} \to \infty$, one has

$$\int_{B(p,r_k)} |s_p^k(x)|^2 \frac{\omega^n}{n!} = 1 - o(1), \quad \text{for} \quad k \to \infty; \tag{2.3}$$

see e. g. [11, Formula (5.1.25)] and [15, Lemma 1.2].

3 A direct approach to the asymptotic faithfulness

In this section, we will present an elementary proof of Theorem 1.1 using peak sections.

Fix $\sigma \in \mathcal{T}$. For any $\phi \in \Gamma$, the mapping class group of Σ, let $\sigma(t) : [0, 1] \to \mathcal{T}$ be a smooth curve with $\sigma(0) = \phi(\sigma)$, $\sigma(1) = \sigma$. For any $s \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$, set

$$s(t) := P_{\phi(\sigma),\sigma(t)} \circ \phi^*(s) \in H^0(M_{\sigma(t)}, \mathcal{L}_{\sigma(t)}^k), \tag{3.1}$$

where $P_{\phi(\sigma),\sigma(t)}$ is the parallel transport from $\phi(\sigma)$ to $\sigma(t)$ with respect to the projective flat connection (2.1). For any positve smooth function $\rho : M \to (0, 1]$ we define a rescaled Hermitian structure on $\mathcal{H}_k$ by

$$\langle s_1, s_2\rangle_\rho = \int_M \rho \cdot (s_1, s_2)\frac{\omega^m}{m!}, \tag{3.2}$$

and denote $\|s\|_\rho^2 = \langle s, s\rangle_\rho$, where $(\cdot,\cdot)$ denotes the pointwise inner product of the Hermitian line bundle $\mathcal{L}^k$. (We note that the question of projectiveness of the norm (3.2) with respect to the connection (2.1) is systematically studied in [14].)

For a $(1,0)$-vector field X on M_σ let X^* denote the dual 1-form of X such that $X^*(X) = |X|_\omega^2$. Denote Λ_t the adjoint of multiplication operator $\omega \wedge \bullet$ by the Kähler metric ω; note that Λ_t and $\bar\partial_t$ all depend on $\sigma(t)$.

Lemma 3.1 *We have the following estimate for the differential operator u along $\sigma(t)$,*

$$\left|\langle u(\sigma'(t))s(t), s(t)\rangle_\rho\right| \le \frac{C_\rho + kC}{2(k+n)}\|s(t)\|_\rho^2,$$

where the constants $C = \max_{[0,1]\times M}\left|\frac{\partial F(\sigma(t))}{\partial t}\right|$ and

$$\begin{aligned} C_\rho = &\max_{[0,1]\times M}|\Lambda_t\bar\partial_t(Z(\sigma'(t))^*\rho)\rho^{-1}| \\ &+\sum_{r=1}^R \max_{[0,1]\times M}\left|\Lambda_t\bar\partial_t\left(Y_r(\sigma'(t))^*\Lambda_t\bar\partial_t(X_r(\sigma'(t))^*\rho)\right)\rho^{-1}\right| \end{aligned}$$

are independent of k.

Proof By (2.2) and (3.2) we have

$$\begin{aligned} \left|\langle u(\sigma'(t))s(t), s(t)\rangle_\rho\right| &= \left|\int_M (\rho u(\sigma'(t))s(t), s(t))\frac{\omega^m}{m!}\right| \\ \le \frac{1}{2(k+n)}&\left|\int_M \left(\nabla_{Z(\sigma'(t))}s(t), \rho s(t)\right)\frac{\omega^m}{m!}\right| \\ + \frac{1}{2(k+n)}&\left|\int_M \left(\sum_{r=1}^R \nabla_{X_r(\sigma'(t))}\nabla_{Y_r(\sigma'(t))}s(t), \rho s(t)\right)\frac{\omega^m}{m!}\right| \\ + \frac{k}{2(k+n)}&\left|\int_M \rho\cdot\left(\frac{\partial F(\sigma(t))}{\partial t}s(t), s(t)\right)\frac{\omega^m}{m!}\right|. \end{aligned} \tag{3.3}$$

The tangent vectors X, Y, Z are $(1,0)$-vectors and ∇ above can all be replaced by $\nabla^{(1,0)}$, which we still denote by ∇. By [7, Chapter VII, Theorem (1.1)], the adjoint of ∇ on forms is $\nabla^{(1,0),*} = \sqrt{-1}[\Lambda_t, \bar\partial_t]$, and so it is $\nabla^{(1,0),*} = \sqrt{-1}\Lambda_t\bar\partial_t$ on $(0,1)$-forms.

The first term in the RHS of (3.3) can be estimated as

$$\begin{aligned} \left|\int_M \left(\nabla_{Z(\sigma'(t))}s(t), \rho s(t)\right)\frac{\omega^m}{m!}\right| &= \left|\langle\nabla_{Z(\sigma'(t))}s(t), \rho s(t)\rangle\right| \\ &= \left|\langle s(t), \nabla^*(Z(\sigma'(t))^*\rho s(t))\rangle\right| \\ &= \left|\langle s(t), \sqrt{-1}\Lambda_t\bar\partial_t(Z(\sigma'(t))^*\rho)\rho^{-1}\cdot\rho s(t)\rangle\right| \\ &\le \max_{[0,1]\times M}|\Lambda_t\bar\partial_t(Z(\sigma'(t))^*\rho)\rho^{-1}|\cdot\|s(t)\|_\rho^2, \end{aligned} \tag{3.4}$$

where the third equality holds since $s(t)$ is a holomorphic section of $\mathcal{L}_{\sigma(t)}^k$, i.e. $\bar\partial_t s(t) = 0$. Similarly the second term is bounded by

$$\left|\int_M \left(\sum_{r=1}^R \nabla_{X_r(\sigma'(t))}\nabla_{Y_r(\sigma'(t))}s(t), \rho s(t)\right)\frac{\omega^m}{m!}\right|$$

$$
\begin{aligned}
&\le \sum_{r=1}^{R} \left|\left\langle \nabla_{X_r(\sigma'(t))} \nabla_{Y_r(\sigma'(t))} s(t), \rho s(t) \right\rangle\right| \\
&= \sum_{r=1}^{R} \left|\left\langle s(t), \nabla^* Y_r(\sigma'(t))^* \nabla^* X_r(\sigma'(t))^* \rho s(t) \right\rangle\right| \qquad (3.5)\\
&= \sum_{r=1}^{R} \left|\langle s(t), -\Lambda_t \bar{\partial}_t \left(Y_r(\sigma'(t))^* \Lambda_t \bar{\partial}_t (X_r(\sigma'(t))^* \rho)\right) \rho^{-1} \cdot \rho s(t) \rangle\right| \\
&\le \sum_{r=1}^{R} \max_{[0,1]\times M} \left|\Lambda_t \bar{\partial}_t \left(Y_r(\sigma'(t))^* \Lambda_t \bar{\partial}_t (X_r(\sigma'(t))^* \rho)\right) \rho^{-1}\right| \cdot \|s(t)\|_\rho^2.
\end{aligned}
$$

For the last term in the RHS of (3.3), we have

$$
\left|\int_M \rho \cdot \left(\frac{\partial F(\sigma(t))}{\partial t} s(t), s(t)\right) \frac{\omega^m}{m!}\right| \le \max_{[0,1]\times M} \left|\frac{\partial F(\sigma(t))}{\partial t}\right| \cdot \|s(t)\|_\rho^2. \qquad (3.6)
$$

Substituting (3.4), (3.5) and (3.6) into (3.3), we obtain

$$
\begin{aligned}
&\left|\langle u(\sigma'(t)) s(t), s(t)\rangle_\rho\right| \\
&\le \frac{1}{2(k+n)} \max_{[0,1]\times M} |\Lambda_t \bar{\partial}_t (Z(\sigma'(t))^* \rho) \rho^{-1}| \cdot \|s(t)\|_\rho^2 \\
&\quad + \frac{1}{2(k+n)} \sum_{r=1}^{R} \max_{[0,1]\times M} \left|\Lambda_t \bar{\partial}_t \left(Y_r(\sigma'(t))^* \Lambda_t \bar{\partial}_t (X_r(\sigma'(t))^* \rho)\right) \rho^{-1}\right| \cdot \|s(t)\|_\rho^2 \\
&\quad + \frac{k}{2(k+n)} \max_{[0,1]\times M} \left|\frac{\partial F(\sigma(t))}{\partial t}\right| \cdot \|s(t)\|_\rho^2 \\
&= \frac{C_\rho + kC}{2(k+n)} \|s(t)\|_\rho^2,
\end{aligned}
$$

completing the proof. □

Proposition 3.2 *We have the following estimate for the norm of the parallel transport $P_{\phi(\sigma),\sigma}$,*

$$
e^{-\frac{C_\rho + kC}{k+n}} \|s\|^2_{\rho\circ\phi^{-1}} \le \|P_{\phi(\sigma),\sigma} \phi^*(s)\|_\rho^2 \le e^{\frac{C_\rho + kC}{k+n}} \|s\|^2_{\rho\circ\phi^{-1}}, \qquad (3.7)
$$

for all $s \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$.

Proof Using the definition of $s(t)$ in (3.1) we have

$$
\hat{\nabla}_{\sigma'(t)} s(t) = 0. \qquad (3.8)
$$

By (2.1) and (3.8) we deduce that

$$
\begin{aligned}
\frac{d}{dt} \|s(t)\|_\rho^2 &= \langle \hat{\nabla}^t_{\sigma'(t)} s(t), s(t)\rangle_\rho + \langle s(t), \hat{\nabla}^t_{\sigma'(t)} s(t)\rangle_\rho \\
&= \int_M (\rho u(\sigma'(t)) s(t), s(t)) + (s(t), \rho u(\sigma'(t)) s(t)) \frac{\omega^m}{m!} \\
&= 2\mathrm{Re}\langle u(\sigma'(t)) s(t), s(t)\rangle_\rho.
\end{aligned}
$$

This is treated in Lemma 3.1 and we find

$$-\frac{C_\rho + kC}{k+n}\|s(t)\|_\rho^2 \le \frac{d}{dt}\|s(t)\|_\rho^2 \le \frac{C_\rho + kC}{k+n}\|s(t)\|_\rho^2.$$

Hence

$$e^{-\frac{C_\rho+kC}{k+n}}\|s(0)\|_\rho^2 \le \|s(1)\|_\rho^2 \le e^{\frac{C_\rho+kC}{k+n}}\|s(0)\|_\rho^2. \tag{3.9}$$

Now $\sigma(t)$ is a curve from $\phi(\sigma)$ to σ, $P_{\phi(\sigma),\sigma(0)} = P_{\phi(\sigma),\phi(\sigma)} = \mathrm{Id}$, $\sigma(1) = \sigma$, and

$$s(0) = \phi^*(s), \quad s(1) = P_{\phi(\sigma),\sigma}\phi^*(s). \tag{3.10}$$

The norm of $s(0)$ is given by

$$\begin{aligned}\|s(0)\|_\rho^2 &= \|\phi^* s\|_\rho^2 = \int_M \rho|\phi^* s|^2 \frac{\omega^m}{m!} = \int_M \rho|s\circ\phi|^2\frac{\omega^m}{m!}\\ &= \int_M \rho\circ\phi^{-1}|s|^2\frac{\omega^m}{m!} = \|s\|^2_{\rho\circ\phi^{-1}}.\end{aligned} \tag{3.11}$$

Here we have used the fact that ϕ induces a symplectomorphism of M, i.e. $\phi^*\omega = \omega$.

Combining (3.10) and (3.9) we find the estimate

$$e^{-\frac{C_\rho+kC}{k+n}}\|s\|^2_{\rho\circ\phi^{-1}} \le \|P_{\phi(\sigma),\sigma}\phi^*(s)\|_\rho^2 \le e^{\frac{C_\rho+kC}{k+n}}\|s\|^2_{\rho\circ\phi^{-1}}.$$

□

We prove now Theorem 1.1.

The proof of Theorem 1.1 We consider first the case of $g \ge 3$, n and d are coprime. Suppose $\phi \in \bigcap_{k=1}^\infty \mathrm{Ker}\,\pi_k$. We prove that ϕ is the identity mapping of Γ.

The projective representation of the mapping class group Γ is defined via the flat connection, in particular Γ acts on the space of covariant constant sections over Teichmüller space, and

$$P_{\phi(\sigma),\sigma}\circ\phi^* = \pi_k(\phi) = c_k\mathrm{Id}, \quad c_k \ne 0, \tag{3.12}$$

when acting on the element of $H^0(M_\sigma, \mathcal{L}_\sigma^k)$.

By taking $\rho = 1$ and using Proposition 3.2, we get

$$e^{-\frac{C_1+kC}{k+n}} \le c_k^2 \le e^{\frac{C_1+kC}{k+n}}. \tag{3.13}$$

We prove first ϕ acts on M as identity. Otherwise suppose $\phi \ne \mathrm{Id}$ as mappings of M. Then there exists a point $p \in M$ such that $p \ne \phi^{-1}(p)$. Let $V_p, U_p \subset M$ be two small neighborhoods of p with

$$p \in V_p \Subset U_p, \quad \phi^{-1}(V_p) \subset M - U_p. \tag{3.14}$$

Let $\rho : M \to (0, 1]$ be a smooth function on M satisfying

$$\rho(x) = \begin{cases} 1, & x \in V_p,\\ \frac{1}{e^{2C}+1}, & x \in M - U_p.\end{cases} \tag{3.15}$$

For each large k we take the initial section s to be the peak section s_p^k of the point p. By (3.12), (3.13), (3.15) and (2.3), we find

$$\begin{aligned} \|P_{\phi(\sigma),\sigma} \circ \phi^*(s_p^k)\|_\rho^2 &= c_k^2 \int_M \rho |s_p^k|^2 \frac{\omega^m}{m!} \\ &\geq e^{-\frac{C_1+kC}{k+n}} \int_{V_p} |s_p^k|^2 \frac{\omega^m}{m!} \\ &\geq e^{-\frac{C_1+kC}{k+n}} (1 - o(1)). \end{aligned} \tag{3.16}$$

On the other hand, by (3.14), (3.15) and (2.3), we have also

$$\begin{aligned} \|s_p^k\|_{\rho\circ\phi^{-1}}^2 &= \int_M \rho \circ \phi^{-1} |s_p^k|^2 \frac{\omega^m}{m!} \\ &= \int_{V_p} \rho \circ \phi^{-1} |s_p^k|^2 \frac{\omega^m}{m!} + \int_{M-V_p} \rho \circ \phi^{-1} |s_p^k|^2 \frac{\omega^m}{m!} \\ &\leq \frac{1}{e^{2C}+1} \int_{V_p} |s_p^k|^2 \frac{\omega^m}{m!} + \int_{M-V_p} |s_p^k|^2 \frac{\omega^m}{m!} \\ &\leq \frac{1}{e^{2C}+1} + o(1). \end{aligned} \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.7) we obtain

$$e^{-\frac{C_1+kC}{k+n}} (1 - o(1)) \leq e^{\frac{C_\rho+kC}{k+n}} \left(\frac{1}{e^{2C}+1} + o(1) \right).$$

As $k \to \infty$ it gives

$$e^{-C} \leq e^C \cdot \frac{1}{e^{2C}+1} = \frac{e^{-C}}{1+e^{-2C}} < e^{-C},$$

which is a contradiction. So ϕ acts on M as the identity. It follows then from the standard argument [1] that ϕ itself is the identity element in Γ (as equivalence class of mappings of Σ).

Now in the case $g = 2$, $(n, d) = (2, 0)$, the same proof above concludes that if $\phi \in \bigcap_{k=1}^{\infty} \mathrm{Ker}(\pi_k^{2,0})$ then it acts trivially on M. It is then either the identity or the hyper-elliptic involution H; see [1]. On the other hand H indeed acts trivially under all $\pi_k^{2,0}$ by its definition. Thus $\bigcap_{k=1}^{\infty} \mathrm{Ker}(\pi_k^{2,0}) = \{1, H\}$. □

References

1. Andersen, J.E.: Asymptotic faithfulness of the quantum $SU(n)$ representations of the mapping class group. Ann. Math. **163**, 347–368 (2006)

2. Andersen, J.E.: Toeplitz Operators and Hitchin's Projectively Flat Connection. The Many Facets of Geometry, pp. 177–209. Oxford Univ. Press, Oxford (2010)
3. Andersen, J.E.: Hitchin's connection. Toeplitz operators, and symmetry invariant deformation quantization. Quantum Topol. **3**(3–4), 293–325 (2012)
4. Andersen, J.E., Gammelgaard, N.L.: Hitchin's projectively flat connection, Toeplitz operators and the asymptotic expansion of TQFT curve operators. In: Grassmannians, Moduli Spaces and Vector Bundles, Clay Mathematics proceedings, vol. 14, American Mathematical Society, Providence, RI, pp. 1–24 (2011)
5. Axelrod, S., Della Pietra, S., Witten, E.: Geometric quantization of Chern–Simon gauge theory. J. Differ. Geom. **33**, 787–902 (1991)
6. Bordemann, M., Meinrenken, E., Schlichenmaier, M.: Toeplitz quantization of Kähler manifolds and $gl(N)$, $N \to \infty$ limits. Commun. Math. Phys. **165**(2), 281–296 (1994)
7. Demailly, J.-P.: Complex analytic and Differential Geometry. https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf (2012)
8. Freed, D.S.: Classical Chern–Simons theory, part 1. Adv. Math. **113**, 237–303 (1995)
9. Freedman, M.H., Walker, K., Wang, Z.: Quantum $SU(2)$ faithfully detects mapping class groups modulo center. Geom. Topol. **6**, 523–539 (2002)
10. Hitchin, N.: Flat connections and geometric quantization. Commun. Math. Phys. **131**, 347–380 (1990)
11. Ma, X., Marinescu, G.: Holomorphic Morse inequalities and Bergman Kernels. Birkhäuser, Boston (2006)
12. Marché, J., Narimannejad, M.: Some asymptotics of topological quantum field theory via skein theory. Duke Math. J. **141**, 573–587 (2008)
13. Narasimhan, M.S., Seshadri, C.S.: Holomorphic vector bundles on a compact Riemann surfaces. Math. Ann. **155**, 69–80 (1964)
14. Ramadas, T.R.: Faltings' construction of the K-Z connection. Commun. Math. Phys. **196**, 133–143 (1998)
15. Tian, G.: On a set of polarized Kähler metrics on algebraic manifolds. J. Differ. Geom. **32**, 99–130 (1990)
16. Turaev, V.G.: Quantum Invariants of Knots and 3-Manifolds, de Gruyter Studies in Mathematics, vol. 18. Walter de Gruyter, Co., Berlin (1994)