The role of singletons in S ^{7} compactifications

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The role of singletons in S^7 compactifications

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ABSTRACT: We derive the isometry irrep content of squashed seven-sphere compactifications of eleven-dimensional supergravity, i.e., the left-squashed (LS^7) with $\mathcal{N} = 1$ and right-squashed (RS^7) with $\mathcal{N} = 0$ supersymmetry, in a manner completely independent of the round sphere. Then we compare this result with the spectrum obtained by Higgsing the round sphere spectrum. This way we discover features of the spectra which makes it possible to argue that the only way the round spectrum can be related by a Higgs mechanism to the one of LS^7 is if the singletons are included in the round sphere spectrum. For this to work also in the RS^7 case it seems that the gravitino of the LS^7 spectrum must be replaced by a fermionic singleton present in the RS^7 spectrum.

Contents

1 Introduction

The role of singletons [1] in the Freund-Rubin [2] compactification of eleven-dimensional supergravity on $AdS_4 \times S^7$ has been unclear ever since the spectrum was first obtained. When deriving the spectrum [3–6], one finds that the round seven-sphere has special symmetries that can be used to gauge away precisely the irreps $D(E_0, s)$ that correspond to the supersingleton irrep of SO(3,2), that is $D(\frac{1}{2}, 0) \oplus D(1, \frac{1}{2})$, see, e.g., [7].

    Here we make the natural assumption that the AdS_4 supergravity theory obtained by compactifying eleven-dimensional supergravity on the squashed seven-sphere [8], either left- or right-squashed (LS^7 or RS^7), is a Higgsed version [9] of the one obtained from the round sphere. As we will see below this makes it possible to argue that the supersingleton has a key role to play in this context. It should be emphasised that the singletons that appear in the round seven-sphere spectrum are gauged away in the bulk of AdS_4 by symmetries that arise precisely when the sphere is round. A common interpretation of this fact is that the supersingleton is not part of the compactification spectrum on the round sphere. However, one could instead take the point of view that the supersingleton should be kept in the spectrum but confined to the boundary of AdS_4 in accordance with the standard picture we have of singletons, see, e.g., [7, 10–12].

    In order to back up this point of view we take advantage of a particular property of the relation between the round and left-squashed spectra demonstrated in the following sections to show that the supersingleton must be kept in the round sphere spectrum in order for the Higgsing/deHiggsing to give the correct spectrum on the supersymmetric squashed sphere. As will be clear later this requires the singletons to undergo some kind of Higgsing, i.e., eating some ordinary field of the same spin so that it can become an

ordinary scalar/spin-$\frac{1}{2}$ bulk field itself. We mention here that we, however, are not aware of any known field theory realisation of such a phenomenon in the literature and we have nothing to add to this question. To reach this conclusion we have constructed the complete isometry representation content of the squashed sphere spectrum[1] in a manner that is entirely independent from the round sphere. As it turns out, using the same kind of reasoning, also the spectrum on the RS^7 must contain a fermionic singleton.

In section 2 we first review the spectrum on the round sphere and then derive the complete spectrum of isometry irreps on LS^7 and RS^7 independently of the round case. These spectra are then compared and some conclusions drawn concerning the role of singletons. In the final section we summarise our conclusions and make some additional comments. Tables of squashed sphere harmonics are collected in the appendix.

2 Comparison of the spectra on the round and squashed seven-spheres

In the first subsection of this section we give a very brief account of Freund-Rubin compactifications of eleven-dimensional supergravity and their general features relevant for our discussion of the spectrum, see [7] for a more detailed discussion. This includes tables over unitary irreps of $SO(3,2)$ giving E_0 as a function of mass (fermions) or mass squared (bosons) (table 1) and expressions for the mass operators in terms of invariant operators (Δ_L etc) on the seven-sphere (table 2).

Then, in the second subsection, we recall the AdS_4 spectrum of irreps $D(E_0, s)$ of $SO(3,2)$ appearing in the compactification on the round seven-sphere including the $SO(8)$ irrep each of these transform under. This is presented in table 3 which also contains the round sphere eigenvalues of the relevant operators. In subsection 3 we turn to the squashed sphere spectrum and introduce a Young Tableau method by means of which we can derive the complete spectrum of isometry irreps for each field appearing in the AdS_4 supergravity theory here choosing the orientation that leads to an $\mathcal{N}=1$ supersymmetric theory. This case is referred to as the left-squashed (LS^7) case while its orientation-flipped (or skew-whiffed) right-squashed (RS^7) cousin has no supersymmetries. This latter case will also play a role in this paper and we will comment on it both here in the final subsection and in the Conclusions. The fourth and final subsection is devoted to a comparison between the Higgsed/deHiggsed version of the spectrum on the round sphere and the spectrum obtained directly on the squashed sphere in subsection 3. Tables of squashed sphere harmonics for all the relevant operators are collected in the appendix.[2] The two crucial tables for the arguments presented here are the ones for Δ_L and $\not{D}_{3/2}$.

2.1 General features of Freund-Rubin compactifications

The theory under consideration in this paper is eleven-dimensional supergravity compactified on AdS_4 times either the round or squashed seven-sphere. This latter factor can have

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[1]Note that we have no new information on the eigenvalue spectrum of the mass operators on the squashed seven-sphere some of which are known, see [7] and references therein. As will be clear below, the eigenvalues are not needed to reach our conclusions about the singletons.

[2]Note that the irrep spectrum, contrary to the eigenvalue spectrum, is the same for the left and right squashed spheres.

two different orientations, left or right, denoted LS^7 and RS^7, having $\mathcal{N} = 1$ and $\mathcal{N} = 0$ supersymmetry, respectively [13, 14]. The bosonic field equations in eleven dimensions are [15]

$$R_{MN} - \frac{1}{2} G_{MN} R = \frac{1}{3} \left(F_M{}^{PQR} F_{NPQR} - \frac{1}{8} G_{MN} F_{PQRS} F^{PQRS} \right), \qquad (2.1)$$

$$\nabla_M F^{MNPQ} = -\frac{1}{576} \epsilon^{NPQM_1 \dots M_8} F_{M_1 \dots M_4} F_{M_5 \dots M_8}, \qquad (2.2)$$

and the Bianchi identities read

$$\partial_{[M} F_{NPQR]} = 0. \qquad (2.3)$$

Using a product metric ansatz for the background splitting eleven dimensions into $4+7$ ($M = (\mu, m)$ etc), and a non-zero background value only for the spacetime components of the four-form field strength, that is [2]

$$\langle G_{MN} \rangle = \begin{pmatrix} \bar{g}_{\mu\nu} & 0 \\ 0 & \bar{g}_{mn} \end{pmatrix}, \ \langle F_{\mu\nu\rho\sigma} \rangle = 3m \bar{\epsilon}_{\mu\nu\rho\sigma}, \qquad (2.4)$$

where m is a positive parameter with dimension mass, we find that the Ricci tensors in the external and internal directions become, respectively,

$$\langle R_{\mu\nu} \rangle = -12m^2 \bar{g}_{\mu\nu}, \ \langle R_{mn} \rangle = 6m^2 \bar{g}_{mn}, \qquad (2.5)$$

where background values are indicated by a bar over the field in question.

The seven-sphere compactifications (round and squashed) of eleven-dimensional supergravity is a very well-studied subject, see, e.g., [7]. The spectrum is obtained by linearising the field equations in eleven dimensions followed by a diagonalisation of the coupled equations. This leads to a number of relations between the mass operators (M^2 for bosonic fields in AdS_4 and M for fermionic ones) and operators on the seven-sphere as given below in table 2. From table 1 one can then read off which irreps $D(E_0, s)$ of $SO(3,2)$ the eigenvalues of the mass operators correspond to up to the sign ambiguity for scalars and spin 1/2 fermions.[3]

The spectra obtained in such compactifications are therefore given in terms of towers of irreps of the isometry groups $SO(3,2) \times SO(8)$ and $SO(3,2) \times (Sp(2) \times Sp(1))$ for the cases of interest here. The AdS_4 fields transform in $SO(3,2)$ irreps $D(E_0, s)$ for spins $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ and some parity (see table 2) with E_0 values constrained by unitarity as $E_0 \geq s + \frac{1}{2}$ for matter fields and $E_0 \geq s + 1$ for gauge fields. In addition scalar fields have masses restricted by the Breitenlohner-Freedman condition $M^2 \geq -m^2$ [17]. For scalars the minus sign in E_0 is therefore only relevant for masses in the range $3m^2 \geq M^2 \geq -m^2$. This is summarised in table 1 [5, 6, 18, 19].

The singletons that are of interest in this paper are the two irreps $D(\frac{1}{2}, 0)$ and $D(1, \frac{1}{2})$ that saturate the unitary bounds for the two matter fields of spin zero and one half, respectively. Both of them rely for their existence on the possibility to choose the minus sign in the above expressions for E_0. For the gauge fields the unitary bounds correspond to massless fields.

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[3]Imposing supersymmetry may in some cases remove this ambiguity [16].

$$\begin{array}{|ll|}
\hline
E_0 = \frac{3}{2} \pm \frac{1}{2}\sqrt{(M/m)^2 + 1} \geq \frac{1}{2}, & s = 0, \\
E_0 = \frac{3}{2} \pm \frac{1}{2}|M/m| \geq 1, & s = \frac{1}{2}, \\
E_0 = \frac{3}{2} + \frac{1}{2}\sqrt{(M/m)^2 + 1} \geq 2, & s = 1, \\
E_0 = \frac{3}{2} + \frac{1}{2}|M/m - 2| \geq \frac{5}{2}, & s = \frac{3}{2}, \\
E_0 = \frac{3}{2} + \frac{1}{2}\sqrt{(M/m)^2 + 9} \geq 3, & s = 2. \\
\hline
\end{array}$$

Table 1. E_0 for AdS_4 fields of given mass M and spin s (in $SO(3,2)$ irreps $D(E_0, s)$) and the corresponding unitarity bounds.

s	Mass operator
2^+	Δ_0
$\frac{3}{2}^{(1),(2)}$	$\not{D}_{1/2} + \frac{7m}{2}$
$1^{-(1),(2)}$	$\Delta_1 + 12m^2 \pm 6m\sqrt{\Delta_1 + 4m^2}$
1^+	Δ_2
$\frac{1}{2}^{(4),(1)}$	$\not{D}_{1/2} - \frac{9m}{2}$
$\frac{1}{2}^{(3),(2)}$	$\frac{3m}{2} - \not{D}_{3/2}$
$0^{+(1),(3)}$	$\Delta_0 + 44m^2 \pm 12m\sqrt{\Delta_0 + 9m^2}$
$0^{+(2)}$	$\Delta_L - 4m^2$
$0^{-(1),(2)}$	$Q^2 + 6mQ + 8m^2$

Table 2. Mass operators appearing in the Freund-Rubin compactifications. The superscript signs on bosonic fields specify its parity. For spins 1^- and 0^+ the minus sign in M^2 should be selected for the superscript (1) and the plus sign for the second superscript. For spins $\frac{3}{2}, \frac{1}{2}$ and 0^- the first superscript corresponds to the negative part of the spectrum of the linear operator in question (i.e., $\not{D}_{1/2}, \not{D}_{3/2}$ and Q) while the second label corresponds to the positive part of the spectrum.

The AdS_4 irreps were combined into $\mathcal{N} = 1$ multiplets by Heidenreich [19] and into $\mathcal{N} = 8$ multiplets by Freedman and Nicolai [4] (see also the review by Nicolai [20]). These results are implicit in the discussions in the following subsections.

2.2 The spectrum on the round seven-sphere

The spectrum on the round seven-sphere has been derived by several different methods, e.g., using the supergroup $OSp(4/N)$ or Young Tableaux, giving the result in table 3. By inserting the eigenvalue spectra of the operators in table 3 into the mass operators of table 2 one arrives at the $\mathcal{N} = 8$ supermultiplet spectra presented in table 4 where $n \geq 0$ refers to the level with $n = 0$ giving $\mathcal{N} = 8$ supergravity and $n \geq 1$ massive supermultiplets.

Since singletons play a key role in this paper we emphasise here that, as discussed in [7], spin-0 singletons arise in the $0^{(1)}$ tower (see table 2) when the scalar Laplacian Δ_0 on the 7-manifold has modes with eigenvalue $7m^2$. These can occur only for the round S^7 (see, for example, [14]), and so only the round S^7 vacuum can have spin-0 singletons.

SO(7)	Operator	SO(8) Dynkin label	Eigenvalues
1	Δ_0	$(p,0,0,0)$	$p(p+6)m^2$
7	Δ_1	$(p-1,1,0,0)$	$[p(p+6)+5]m^2$
21	Δ_2	$(p-1,0,1,1)$	$[p(p+6)+8]m^2$
35	Q	$(p-1,0,2,0)$	$-(p+3)m$
		$(p-1,0,0,2)$	$+(p+3)m$
27	Δ_L	$(p-2,2,0,0)$	$[p(p+6)+12]m^2$
8	$\not{D}_{1/2}$	$(p,0,1,0)$	$+(p+\frac{7}{2})m$
		$(p,0,0,1)$	$-(p+\frac{7}{2})m$
48	$\not{D}_{3/2}$	$(p-1,1,1,0)$	$+(p+\frac{7}{2})m$
		$(p-1,1,0,1)$	$-(p+\frac{7}{2})m$

Table 3. The round S^7 eigenvalues of the differential operators acting on SO(8) harmonics. The SO(7) representations are given in terms of their dimension.

Spin	SO(8) rep	E_0	(Mass)^2	Operator
2	$(n,0,0,0)$	$(n+6)/2$	$(n+3)-9$	Δ_0
$\frac{3}{2}^{(1)}$	$(n,0,0,1)$	$(n+5)/2$	n^2	$\not{D}_{\frac{1}{2}} < 0$
$\frac{3}{2}^{(2)}$	$(n-1,0,1,0)$	$(n+7)/2$	$(n+6)^2$	$\not{D}_{\frac{1}{2}} > 0$
$1^{-(1)}$	$(n,1,0,0)$	$(n+4)/2$	$(n+1)^2-1$	$\Delta_1 \geq 12$
$1^{-(2)}$	$(n-2,1,0,0)$	$(n+8)/2$	$(n+5)^2-1$	$\Delta_1 \geq 12$
1^+	$(n-1,0,1,1)$	$(n+6)/2$	$(n+3)^2-1$	Δ_2
$\frac{1}{2}^{(4)}$	$(n-2,0,0,1)$	$(n+9)/2$	$(n+6)^2$	$\not{D}_{\frac{1}{2}} < 0$
$\frac{1}{2}^{(1)}$	$(n+1,0,1,0)$	$(n+3)/2$	n^2	$\not{D}_{\frac{1}{2}} > 0$
$\frac{1}{2}^{(2)}$	$(n-1,1,1,0)$	$(n+5)/2$	$(n+2)^2$	$\not{D}_{\frac{3}{2}} > 0$
$\frac{1}{2}^{(3)}$	$(n-2,1,0,1)$	$(n+7)/2$	$(n+4)^2$	$\not{D}_{\frac{3}{2}} < 0$
$0^{+(1)}$	$(n+2,0,0,0)$	$(n+2)/2$	$(n-1)^2-1$	Δ_0
$0^{+(3)}$	$(n-2,0,0,0)$	$(n+10)/2$	$(n+7)^2-1$	Δ_0
$0^{+(2)}$	$(n-2,2,0,0)$	$(n+6)/2$	$(n+3)^2-1$	Δ_L
$0^{-(1)}$	$(n,0,2,0)$	$(n+4)/2$	$(n+1)^2-1$	$Q < 0$
$0^{-(2)}$	$(n-2,0,0,2)$	$(n+8)/2$	$(n+5)^2-1$	$Q > 0$

Table 4. The complete spectrum of particles on the round sphere compactification of eleven-dimensional supergravity. Each integer n gives an entire supermultiplet of particles. For the linear operators the positive and negative part of the spectrum is associated with different towers as indicated by the inequality signs.

Type A: Wess-Zumino multiplets for $E_0 > 1/2$	
$D(E_0, 0) \oplus D(E_0 + 1/2, 1/2) \oplus D(E_0 + 1, 0)$	
Type B: Massive higher spin multiplets for $E_0 > s + 1$, $s \geq 1/2$	
$D(E_0, s) \oplus D(E_0 + 1/2, s + 1/2) \oplus D(E_0 + 1/2, s - 1/2) \oplus D(E_0 + 1, s)$	
Type C: Massless higher spin multiplets for $s \geq 1/2$	
$D(s + 1, s) \oplus D(s + 3/2, s + 1/2)$	
Type D: Dirac singleton	
$D(1/2, 0) \oplus D(1, 1/2)$	

Table 5. $\mathcal{N} = 1$ supermultiplets.

In anticipation of the discussion of the squashed sphere spectrum in the next subsection we note that spin-$\frac{1}{2}$ singletons arise in the $\frac{1}{2}^{(1)}$ tower (see table 2) when the Dirac operator on the 7-manifold has modes with eigenvalue $+\frac{7m}{2}$. On the round sphere there is an 8_c of such modes, together with an 8_s of modes with eigenvalue $-\frac{7m}{2}$ that give the 8 supersymmetries of the round vacuum. With the exception of the round S^7, Dirac modes with eigenvalues $+\frac{7m}{2}$ and $-\frac{7m}{2}$ cannot co-exist on any manifold (see [14]). So at most, for any other manifold one can have either supersymmetry but no spin-$\frac{1}{2}$ singletons or else spin-$\frac{1}{2}$ singletons but no supersymmetry which is precisely what we will argue happens for the left and right squashed spheres, respectively.

2.3 The spectrum of irreps on the squashed seven-sphere

For the left-squashed sphere the structure of the various isometry towers must be compatible with the $\mathcal{N} = 1$ supersymmetry present in AdS_4 in this case. The unitary $\mathcal{N} = 1$ supermultiplets constructed by Heidereich [19] are given in table 5.

Since the supersymmetry parameter (as well as the gravitino) is an isometry singlet all member fields in a supermultiplet must transform under the same isometry irrep. That the isometry irreps of all towers fit exactly into such $\mathcal{N} = 1$ supermultiplets has been verified in full detail in this work.

The method employed here to get the full spectrum directly on the squashed sphere is based on an application of the rules of the game spelt out by Salam and Strathdee in [21]. Consider again each operator in table 2 and their harmonics (eigenfunctions) but now on the coset $Sp_2 \times Sp_1^C / Sp_1^A \times Sp_1^{B+C}$. Here we have used the splitting $Sp_2 \to Sp_1^A \times Sp_1^B$ in order to be able to define the diagonal subgroup Sp_1^{B+C} in the denominator subgroup of the coset. We will refer to this method as the Young Tableau Method (YTM) which is essentially just a realisation of the Fourier analysis for coset spaces G/H. It has, e.g., been used by D'Auria and Fré in connection with compactification on spaces like M^{pqr}, see [22] and references therein (see also [7]).

Thus we first split the tangent space SO(7) irrep of the squashed S^7 tensor/spinor field in question into $H = Sp_1^A \times Sp_1^{B+C}$ irreps (m, n) and then, for each such irrep, we tabulate all $G = Sp_2 \times Sp_1^C$ irreps that in their decompositions under H contain the H irrep (m, n)

we consider. Once this is done we collect all the G irreps for the S^7 field and remove the irreps that correspond to longitudinal states (and possible other similar states) so that the purely transverse spectrum is arrived at at the end. We will illustrate the procedure in detail for the Lichnerowitz operator Δ_L below and give the result for all other operators in the appendix. As it turns out, the spectrum of Δ_L together with that of $\slashed{D}_{3/2}$ contain all the crucial features that will be used in the final argument for the need to incorporate singletons in the round sphere spectrum as well as in the one for RS^7. We will now explain the procedure in a number of separate steps.

Step 1: To start with we need to decompose the various tangent space tensors into irreps of the coset denominator group $Sp_1^A \oplus Sp_1^{B+C}$. As it happens, by using the McKay and Patera tables [23], this is most easily done by a two-step decomposition via G_2 as follows: $SO(7) \to G_2 \to Sp_1^A \times Sp_1^{B+C}$. We find

$$\text{scalar}(\mathbf{1}): \quad (000) \to (00) \to (0,0), \tag{2.6}$$

$$\text{1-form}(\mathbf{7}): \quad (100) \to (01) \to (1,1) \oplus (0,2), \tag{2.7}$$

$$\text{Dirac}(\mathbf{8}): \quad (001) \to (01) \oplus (00) \to (1,1) \oplus (0,2) \oplus (0,0), \tag{2.8}$$

$$\text{2-form}(\mathbf{21}): \quad (010) \to (01) \oplus (10) \to (1,1) \oplus (0,2) \oplus (0,2) \oplus (1,3) \oplus (2,0), \tag{2.9}$$

$$\text{metric}(\mathbf{27}): \quad (200) \to (02) \to (2,2) \oplus (1,3) \oplus (0,4) \oplus (1,1) \oplus (0,0), \tag{2.10}$$

$$\text{3-form}(\mathbf{35}): \quad (002) \to (02) \oplus (01) \oplus (00) \to \text{see above}, \tag{2.11}$$

$$\text{Rarita-Schw.}(\mathbf{48}): \quad (101) \to (02) \oplus (10) \oplus (01) \to \text{see above}. \tag{2.12}$$

Here the irreps are given in terms of Dynkin labels except for the numbers in bold which refer to the dimension of the $SO(7)$ irrep the corresponding field on S^7 belongs to.

We now concentrate on the harmonics of the operator Δ_L, that is metric harmonics.

Step 2: The traceless metric on the squashed seven-sphere is in the $\mathbf{27}$ of $SO(7)$ which splits as just mentioned into $H = Sp_1^A \times Sp_1^{B+C}$ irreps (m,n) (via G_2) as follows:

$$\mathbf{27}: \quad (200) \to (02) \to (2,2) \oplus (1,3) \oplus (0,4) \oplus (1,1) \oplus (0,0). \tag{2.13}$$

Step 3: For each of the H irreps (m,n) on the right hand side in the last equation we now look for all G irreps that have the H irrep in its decomposition. This is best done using Young Tableaux (YTs). A general irrep of $G = Sp_2 \times Sp_1^C$ can be parametrised by three non-negative integers $(p,q;r)$ related to a Sp_2 YT with q columns with two boxes and p columns of single boxes. The dimension of such Sp_2 irreps is given by $d(p,q) = \frac{1}{6}(p+1)(q+1)(p+q+2)(p+2q+3)$. The one-box YT is thus denoted $(1,0)$ and has dimension four while one two-box column is $(0,1)$ with dimension five. Each Sp_2 irrep must then be combined with the irrep (r) of the second group factor of G, i.e., Sp_1^C having one row YTs with r boxes. A general $(p,q;r)$ YT of this kind thus has the following form:

$$\tag{2.14}$$

Consider for example[4] the H irrep $(m, n) = (1, 3)$ which we will think of as a tensor with one undotted and three symmetrised dotted indices (all two-dimensional)[5] where the dotted indices arise from forming the diagonal subgroup of Sp_1^B and Sp_1^C. To see if this H irrep is part of the decomposition of a particular G irrep we first check if the H irrep can be obtained by filling the $Sp_2 \times Sp_1^C$ YT with undotted and dotted indices using the fact that dotted indices represent the diagonal subgroup Sp_1^{B+C} irrep. Here we must keep in mind that each column filled with two indices of the same type is a singlet and hence does not contribute to the final H irrep.

As an illustration we choose the $G = Sp_2 \times Sp_1^C$ irrep $(p, q; r) = (5, 2; 3)$. Then the single undotted index of $(1, 3)$ can arise only in two ways, namely either by filling the single box columns with one undotted index and the remaining boxes with dotted ones (and the same kind of index in all double-box columns) or by filling one of the boxes in one double-box column together with all single box columns with dotted indices. This gives the undotted content of this H irrep. The dotted index content is then determined by forming the diagonal subgroup with the (3) irrep of Sp_1^C (which of course is also filled with dotted indices). This leads to the following two cases:

$YT1:$

$YT2:$

where empty boxes are considered to be filled with undotted indices and those with a dot with dotted indices.

It is clear that the irrep (3) of Sp_1^{B+C} (i.e., three dotted indices) is present in the Sp_1 tensor product in both cases (since $(4) \otimes (3) = (1) \oplus (3) \oplus (5) \oplus (7)$ and similarly for the second case).

The crucial next step is to consider general G irreps $(p, q; r)$ and decide which will contain the looked for H irrep. In the example considered above, namely the H irrep $(1, 3)$, we find that the first Sp_2 YT can be extended to any $q \geq 0$ while for the second YT we get the restriction $q \geq 1$. Finally we must determine the relation between the two integers p and r so that their tensor product contains the irrep (3). Clearly this gives the following eight cases (equal to the dimension of the H irrep $(1, 3)$):

YT1: $p \geq 1, q \geq 0$, and $r = p + 2$, $r = p$, $r = p - 2$, or $r = p - 4$,
YT2: $p \geq 0, q \geq 1$ and $r = p + 4$, $r = p + 2$, $r = p$, or $r = p - 2$.

The final step to determine the possible values for the three integers p, q, r is to check if the low integer cases really occur. This is not the case in general and we find that the

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[4]Note that in the list of H irreps (m, n) appearing in this context $m + n$ is always an even integer.

[5]This stems from the decomposition of the index A for the irrep $\mathbf{4}$ of Sp_2 into $Sp_1 \times Sp_1$ as $A = (a, \dot{a})$.

eight cases must be given individual lower bounds on p. The final result is

YT1: $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (2.15)

$q \geq 0$ with $(r = p + 2, p \geq 1)$, $(r = p, p \geq 2)$, $(r = p - 2, p \geq 3)$ or $(r = p - 4, p \geq 4)$,

YT2: $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (2.16)

$q \geq 1$ with $(r = p + 4, p \geq 0)$, $(r = p + 2, p \geq 0)$, $(r = p, p \geq 1)$ or $(r = p - 2, p \geq 2)$.

To facilitate the presentation of these eight infinite sets (towers) of G irreps for the H irrep $(1, 3)$, as well as all the other cases appearing in the above decomposition of the tangent space irreps, we will use "tower diagrams". In the example discussed above the eight cases look like:[6]

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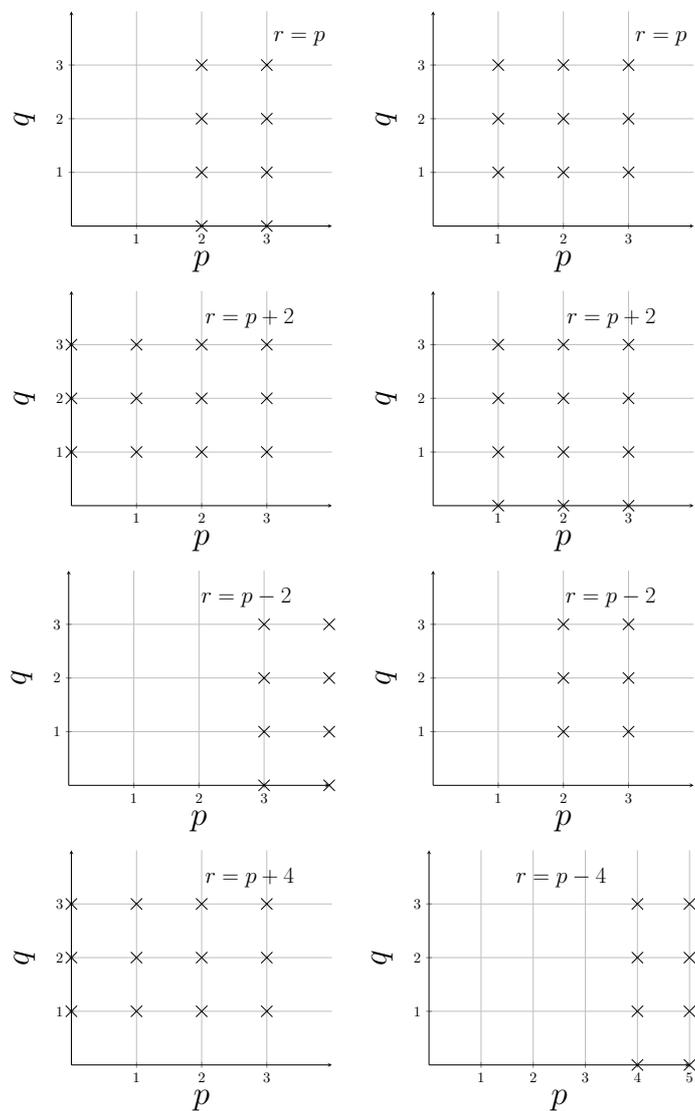
[6]Note that the diagrams are supposed to be extended to infinity in the p and q directions as suggested by the crosses displayed.

We end this subsection by presenting the full supermultiplet content on LS^7, the squashed seven-sphere with $\mathcal{N} = 1$ supersymmetry. Thus we put all the AdS_4 fields obtained in the compactification on the left-squashed seven-sphere into $\mathcal{N} = 1$ supermultiplets as given by Heidenreich [19]. The information needed for this is provided in appendix B of this paper. Each supermultiplet is specified by the spin and parity of the field in AdS_4 with highest spin together with its isometry irrep (which is the same for all fields in the supermultiplet). Note that we have here rearranged the Heidenreich supermultiplets by ordering the irreps from largest to lowest spin s as follows

$$ D(E_0, s) \oplus D\left(E_0 + \frac{1}{2}, s - \frac{1}{2}\right) \oplus D\left(E_0 - \frac{1}{2}, s - \frac{1}{2}\right) \oplus D(E_0, s - 1). \qquad (2.17) $$

For instance, the massive graviton supermultiplet in isometry irrep $(1, 0; 1) = (\mathbf{4}, \mathbf{2})$ then reads, with $E_0 = \frac{3}{2} + \frac{1}{3}\sqrt{11}$,

$$ D(E_0, 2^+) \oplus D\left(E_0 + \frac{1}{2}, \frac{3}{2}\right) \oplus D\left(E_0 - \frac{1}{2}, \frac{3}{2}\right) \oplus D(E_0, 1^+). \qquad (2.18) $$

The total multiplet content of the LS^7 theory is (with massless multiplets being special cases in the relevant tower diagrams)

$$ 1 \times \left(D(E_0, 2^+) \oplus D\left(E_0 + \frac{1}{2}, \frac{3}{2}\right) \oplus D\left(E_0 - \frac{1}{2}, \frac{3}{2}\right) \oplus D(E_0, 1^+) \right), \qquad (2.19) $$

$$ 6 \times \left(D\left(E_0, \frac{3}{2}\right) \oplus D\left(E_0 + \frac{1}{2}, 1^\pm\right) \oplus D\left(E_0 - \frac{1}{2}, 1^\mp\right) \oplus D\left(E_0, \frac{1}{2}\right) \right), \qquad (2.20) $$

$$ 6 \times \left(D(E_0, 1^-) \oplus D\left(E_0 + \frac{1}{2}, \frac{1}{2}\right) \oplus D\left(E_0 - \frac{1}{2}, \frac{1}{2}\right) \oplus D(E_0, 0^-) \right), \qquad (2.21) $$

$$ 8 \times \left(D(E_0, 1^+) \oplus D\left(E_0 + \frac{1}{2}, \frac{1}{2}\right) \oplus D\left(E_0 - \frac{1}{2}, \frac{1}{2}\right) \oplus D(E_0, 0^+) \right), \qquad (2.22) $$

$$ 14 \times \left(D\left(E_0, \frac{1}{2}\right) \oplus D\left(E_0 + \frac{1}{2}, 0^\pm\right) \oplus D\left(E_0 - \frac{1}{2}, 0^\mp\right) \right), \qquad (2.23) $$

where the multiplicity (number in front) refers to the number of tower diagrams that appear for the given supermultiplet. It should be emphasised that not only are all tower diagrams accounted for by the list above but in fact each individual irrep (i.e., cross) is given a place in a supermultiplet.

Note that in the last case, the Wess-Zumino multiplets, we have not committed ourselves to the parity assignment of the scalar fields. This can, however, be done for the spin 1 fields in the six spin $3/2$ multiplets since we know exactly their E_0 values.[7] Although we have full knowledge of the operator spectra on the squashed sphere for some operators $(\Delta_0, \Delta_1$ and $\slashed{D}_{1/2}$, see [7]) we lack this for the other operators (although partial results exist) which means that some supermultiplets among the last two categories in the above list cannot unambiguously be assigned values of E_0.

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[7]There are three of each choice of signs.

The known massless supermultiplets are the graviton supermultiplet

$$\text{Spin } 2^+(0,0;0) = (\mathbf{1},\mathbf{1}) : \ D(3,2) \oplus D\left(\frac{5}{2},\frac{3}{2}\right) \tag{2.24}$$

and the two gauge supermultiplets

$$\text{Spin } 1^-(0,0;2) = (\mathbf{1},\mathbf{3}) : \ D(2,1) \oplus D\left(\frac{3}{2},\frac{1}{2}\right) \tag{2.25}$$

$$\text{Spin } 1^-(2,0;0) = (\mathbf{10},\mathbf{1}) : \ D(2,1) \oplus D\left(\frac{3}{2},\frac{1}{2}\right) \tag{2.26}$$

The only other kind of massless supermultiplets that can appear are Wess-Zumino ones. However, we need more detailed information about the more complicated operators (Δ_L, Q and $\slashed{D}_{3/2}$) to determine whether or not they occur.

2.4 Comparison of the irrep spectra on the round and squashed seven-spheres

The supergravity theory obtained by compactifying on the left-squashed seven-sphere has one (four-dimensional) supersymmetry, i.e., $\mathcal{N} = 1$, and is believed to arise as a spontaneously broken version of the round sphere supergravity theory with $\mathcal{N} = 8$. In this spontaneous symmetry breaking (SSB) the isometry is broken as $SO(8) \to Sp_2 \times Sp_1$ with two possible results depending how the broken isometry group is embedded into the unbroken one. The two options are defined by stating how the three eight-dimensional irreps of SO(8) break: for the left-squashed case LS^7

$$S^7 \to LS^7 : \ \mathbf{8}_v \to (\mathbf{4},\mathbf{2}), \ \mathbf{8}_s \to (\mathbf{4},\mathbf{2}), \ \mathbf{8}_c \to (\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}), \tag{2.27}$$

and for the right-squashed case RS^7

$$S^7 \to RS^7 : \ \mathbf{8}_v \to (\mathbf{4},\mathbf{2}), \ \mathbf{8}_s \to (\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}), \ \mathbf{8}_c \to (\mathbf{4},\mathbf{2}). \tag{2.28}$$

In [7] this is discussed in some detail in particular in connection with the so called *space invader scenario*. This refers to the fact that the eight massless gravitini on the round sphere are replaced by a single gravitino on the left-squashed sphere despite the fact that the symmetry breaking $SO(8) \to Sp(2) \times Sp(1)$ tells us that all the eight gravitini on the round sphere become massive after the breaking. This follows immediately if we recall that they belong to the SO(8) irrep $\mathbf{8}_s$ on the round sphere and hence belong to the irrep $(\mathbf{4},\mathbf{2})$ after the SSB. This seemingly strange fact is explained by noting that there is after the breaking a singlet mode coming from the massive part of the round sphere spectrum which zooms down and becomes massless in the left-squashed sphere spectrum.

Thus the space invader scenario involves two Higgs phenomena, one ordinary one in which the eight round sphere gravitini become massive by absorbing a set of spin $\frac{1}{2}$ fields in the same isometry representation which is $(\mathbf{4},\mathbf{2})$ after the spontaneous symmetry breaking (SSB). The second kind of Higgsing is perhaps more appropriately called deHiggsing since in this case a singlet massive gravitino becomes massless by spitting out a spin $\frac{1}{2}$ fermion field. In this paper we have performed a complete analysis of the irrep content of the

spectrum on the LS^7 using a method that is completely independent of the round sphere theory and its connection to the squashed sphere via the SSB described above. This is quite interesting in its own right but it is by trying to connect it to the SSB of the round sphere that certain special features are discovered. As we will see below some of these features seem to tell us that the singleton representations that normally are gauged away and thus discarded on the round sphere (at least in the bulk) must be kept as part of the round S^7 theory although they are strictly speaking confined to the boundary of AdS_4.

The G irreps that do not match in the comparison between the round and squashed irrep spectra are

$$\text{for } \Delta_L : \ (\mathbf{4}, \mathbf{2}), (\mathbf{5}, \mathbf{3}), \tag{2.29}$$

$$\text{for } \not{D}_{3/2} : \ (\mathbf{1}, \mathbf{1})^*, (\mathbf{4}, \mathbf{2}), (\mathbf{5}, \mathbf{1}), (\mathbf{1}, \mathbf{3}). \tag{2.30}$$

Here we emphasise that all of these irreps except $(\mathbf{1}, \mathbf{1})$ (thus the *) arise from the SSB of the round spectrum but do not occur in the squashed spectrum when it is derived directly by our YT methods as explained above. The natural way to understand this is through a Higgs mechanism as done in [7]. However, there this was only applied to the Δ_L mode $(\mathbf{5},\mathbf{3})$ and the $\not{D}_{3/2}$ mode $(\mathbf{4},\mathbf{2})$:[8] The $(\mathbf{5},\mathbf{3})$ bosonic scalar modes are eaten so that the gauge fields in this irrep can become massive on the squashed sphere, for both left and right squashing, while the fermionic $(\mathbf{4},\mathbf{2})$ is eaten by the Rarita-Schwinger fields so that they can become massive, again for both left and right squashing. The bosonic $(\mathbf{4},\mathbf{2})$ and the fermionic modes $(\mathbf{5},\mathbf{1}), (\mathbf{1},\mathbf{3})$, on the other hand, do not get any obvious SSB explanation in [7] since they were, in fact, not identified as modes in this category.

This brings us to the novel aspect of this paper namely the proposition that the remaining modes should be viewed as some kind of Goldstone modes for the singletons that we now incorporate into the round sphere spectrum. Since they break to $(\mathbf{4},\mathbf{2})$ for the spin zero singleton and to $(\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ for the spin half one in the left-squashed case, their presence explains why the remaining modes do not appear in the squashed spectrum namely because they are eaten by these singleton fields which then become ordinary bulk fields. We should emphasise that these bulk fields do appear in the left-squashed spectrum.

Having provided a plausible SSB explanation for the left-squashed excess modes (encircled in the Δ_L and $\not{D}_{3/2}$ tower diagrams in appendix B) we now turn to the $\not{D}_{3/2}$ singlet mode $(\mathbf{1},\mathbf{1})$ which is lacking in the SSB spectrum although it does appear in the direct YT construction of the squashed spectrum (indicated by a box with a cross in the $\not{D}_{3/2}$ tower diagram in appendix B). On the left-squashed sphere this is naturally explained by a reversed Higgsing (or deHiggsing) of the singlet (after SSB) massive Rarita-Schwinger field on the round sphere that becomes massless in the left-squashed case and hence must relieve itself of its spin $\frac{1}{2}$ states which then appear by themselves in the squashed spectrum as we have found in this paper.

This gives a complete picture for the SSB relation between the round and left-squashed spectra. Turning to the corresponding situation in the right-squashed case we note that the only difference is the fact that the deHiggsing of the massive Rarita-Schwinger field

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[8]This was done without knowing to which operators they belong on the squashed sphere.

on LS^7 does not take place on RS^7 since in this latter case there are no supersymmetries. However, there is still the singlet fermionic mode in the SSB of the round $\not{D}_{3/2}$ spectrum that also appears in the direct YT construction of the spectrum.

In the spirit of this paper we therefore propose that also the right-squashed supergravity theory on AdS_4 must contain a singleton, but this time only a fermionic one in the irrep $(\mathbf{1},\mathbf{1})$, which, together with the mode we are looking for, is the result of a deHiggsing of an ordinary bulk field. This is also supported by the fact that this singleton is indeed present (see the mass operators in table 2) since it corresponds to the $\not{D}_{1/2}$ Killing spinor mode that gives rise to the massless Rarita-Schwinger field on the left-squashed sphere. On the left-squashed sphere this mode satisfies $\not{D}_{1/2} = -\frac{7m}{2}$ which after skew-whiffing to the right-squashed case flips sign[9] to $\not{D}_{1/2} = +\frac{7m}{2}$ and hence gives rise to a $M = -m$ spin $\frac{1}{2}$ fermionic field in AdS_4 which is a singleton irrep of SO(3, 2) if the minus sign is chosen for $s = \frac{1}{2}$ in table 2. It is perhaps interesting to note in this context that the origin on the round sphere of this singlet fermionic mode is $\mathbf{56}_s$ (which in the RS^7 context breaks as $\mathbf{56}_c$ in LS^7) which happens to have zero mass. Hence this mode has $E_0 = \frac{3}{2}$ and the sign in table 2 is irrelevant. Once supersymmetry is lost in the SSB to the right-squashed S^7 the properties of the fermion[10] becomes independent of the other fields and may pick the minus sign in the formula for E_0. This way we have described a possible scenario based on SSB and Higgsing/deHiggsing that explains all the modes listed above.[11]

3 Conclusions

The role of singletons in the context of seven-sphere compactification of eleven-dimensional supergravity has been an intriguing subject for a long time. In this paper we have found indications that the $\mathcal{N} = 8$ supersingleton must be kept in the AdS_4 spectrum arising from compactification on the round S^7 in order that its spectrum can be Higgsed/deHiggsed to produce the spectrum obtained from compactification on the $\mathcal{N} = 1$ left-squashed sphere LS^7.

The assumption about a Higgs relation between these two compactifications seems to explain all modes present in the two cases including their precise relation. The new ingredient that is needed for this picture to work is, however, a novel kind of Higgsing of singleton that turn a singleton which in some sense lives on the boundary of AdS_4 into an ordinary bulk field (of the same spin) by "eating" another bulk field of the same spin. As explained in section 2, by adopting this point of view a mismatch between the spectra of the left-squashed and right-squashed RS^7 (with no supersymmetry) can also be corrected giving further support for this singleton Higgsing picture. As it turns out, for this to work a fermionic singleton is required in the spectrum of RS^7 whose existence can, in fact, be directly verified.

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[9]All operators linear in derivatives flip sign under skew-whiffing which just corresponds to a reversal of the orientation of the squashed seven-sphere [7, 13].

[10]The properties relevant here are discussed in detail in [24].

[11]See [25] for more details on the singlet sector of the round and squashed sphere spectra.

Although a singleton Higgs effect may be novel, one could interpret the different state diagrams for SO(3, 2) irreps $D(E_0, s)$ in, e.g., the appendix of the review [20] as indicating a singleton Higgs effect similar to the one hinted at in that review for spin 1 gauge fields. If this can be made explicit in a field theory for singletons it would be quite interesting but as far as we are aware nothing in this direction has been attempted so far.

If the scenario presented here is correct it might have implications for how we view for instance the connection between AdS bulk theories and their CFT duals as stated in the AdS/CFT correspondence [26]. This in particular could mean that singletons have two different roles to play in the AdS/CFT context, being present both as the CFT and as part of the AdS bulk theory.

We may also mention that the incorporation of a fermionic singleton in the RS^7 spectrum does not affect its established stability properties. Despite the fact that it has no supersymmetry, it is Breitenlohner-Freedman stable since it can be obtained as a skew-whiffed version of the LS^7 theory, see, e.g., [7]. Furthermore, the question if there is a an instability due to marginal bound states operators in the CFT of the RS^7 theory (corresponding to tadpoles in the bulk) [27] is not affected either by the fermionic singleton (which has conformal dimension equal to one).

Acknowledgments

A   H-tower diagrams

(a) (0,0) tower.
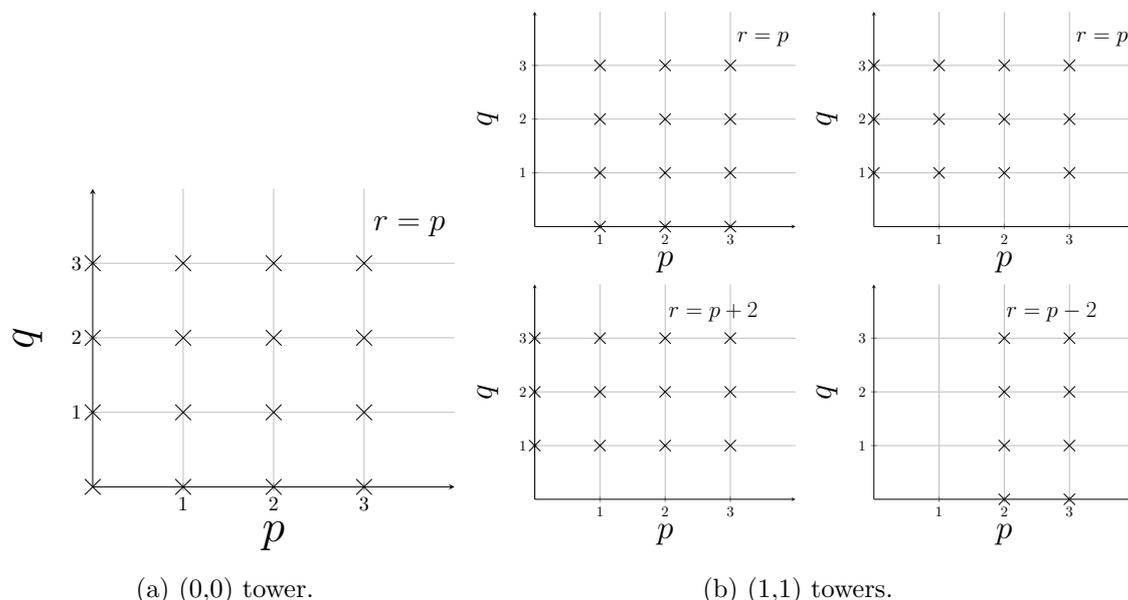
(b) (1,1) towers.

Figure 1. (0,0) and (1,1) towers.
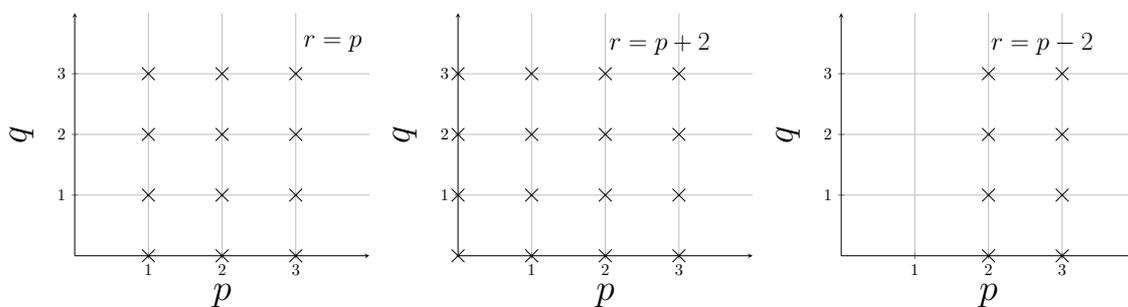
Figure 2. (0,2) towers.
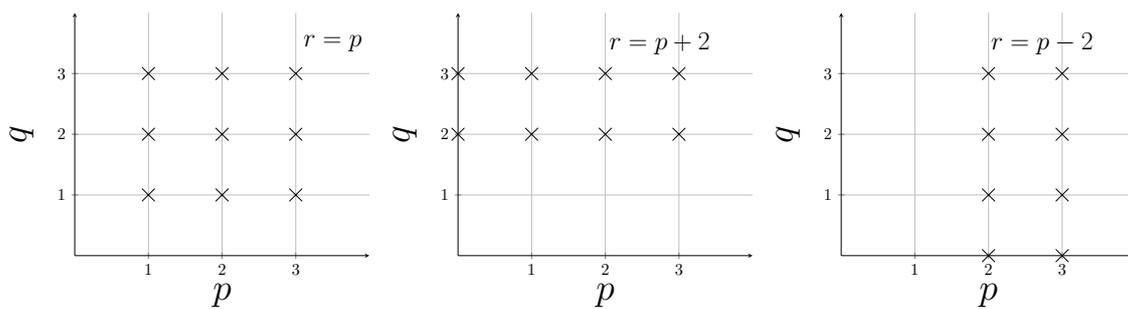
**Figure 3.** (2,0) towers.
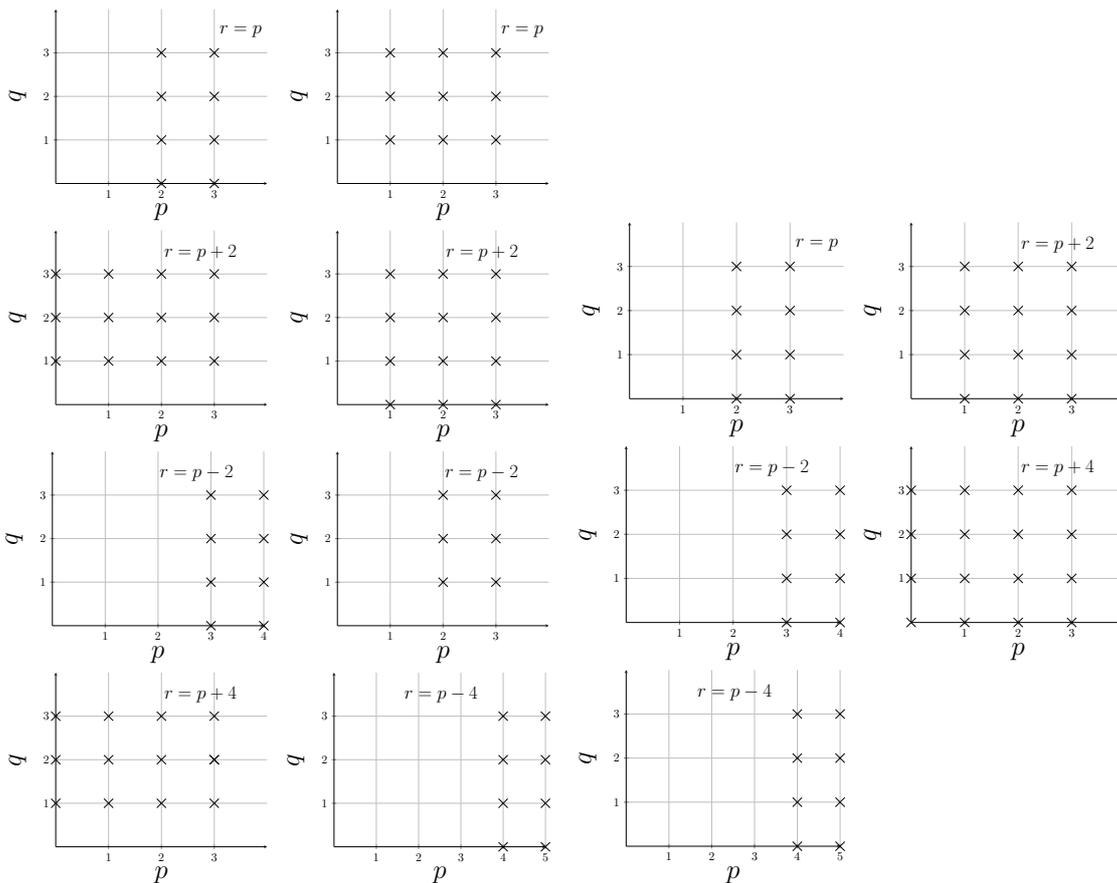
(a) (1,3) towers.　　　　　(b) (0,4) towers.

Figure 4. (1,3) and (0,4) towers.
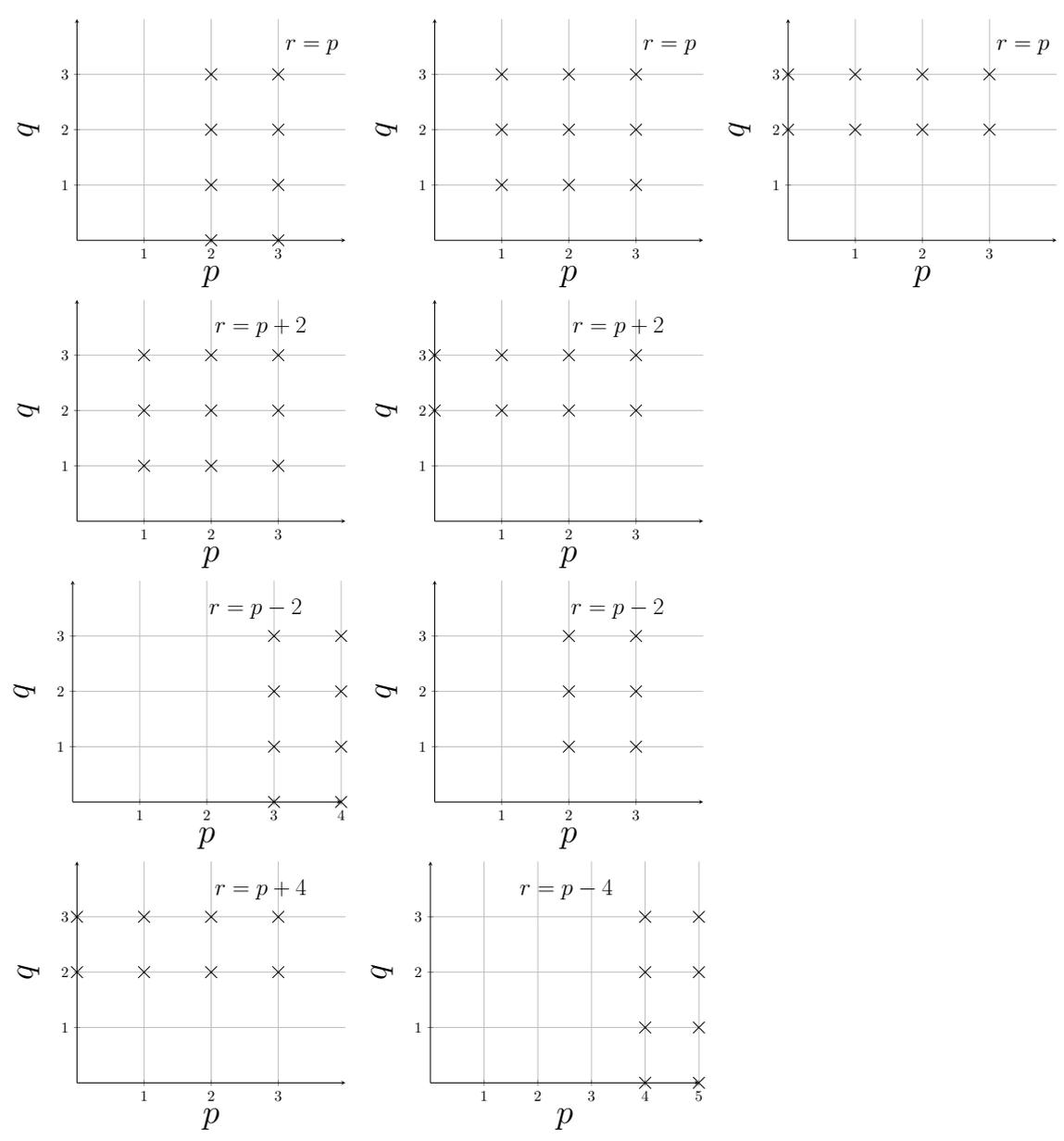
Figure 5. (2,2) towers.

B SO(7) tensor tower diagrams
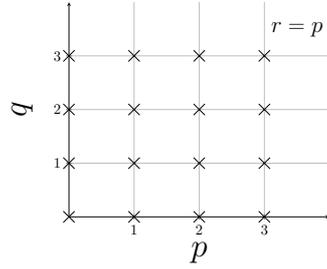
Figure 6. Scalar tower.
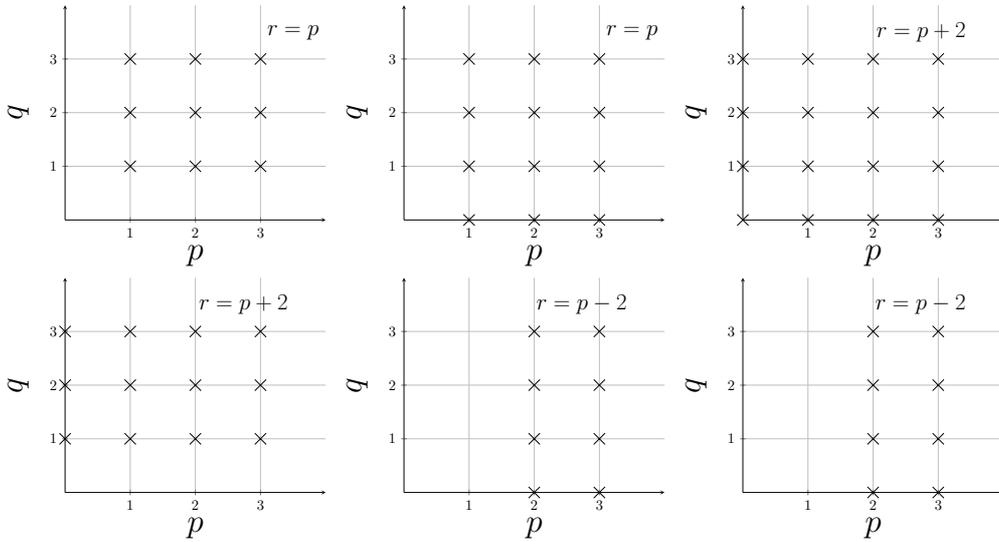
**Figure 7.** One-form towers.
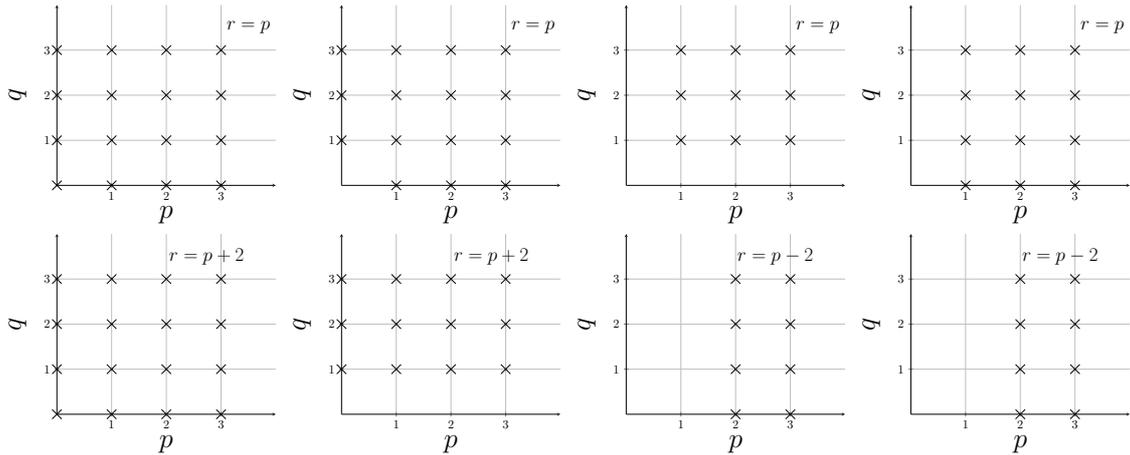

**Figure 8.** $D\!\!\!\!/_{1/2}$ towers.
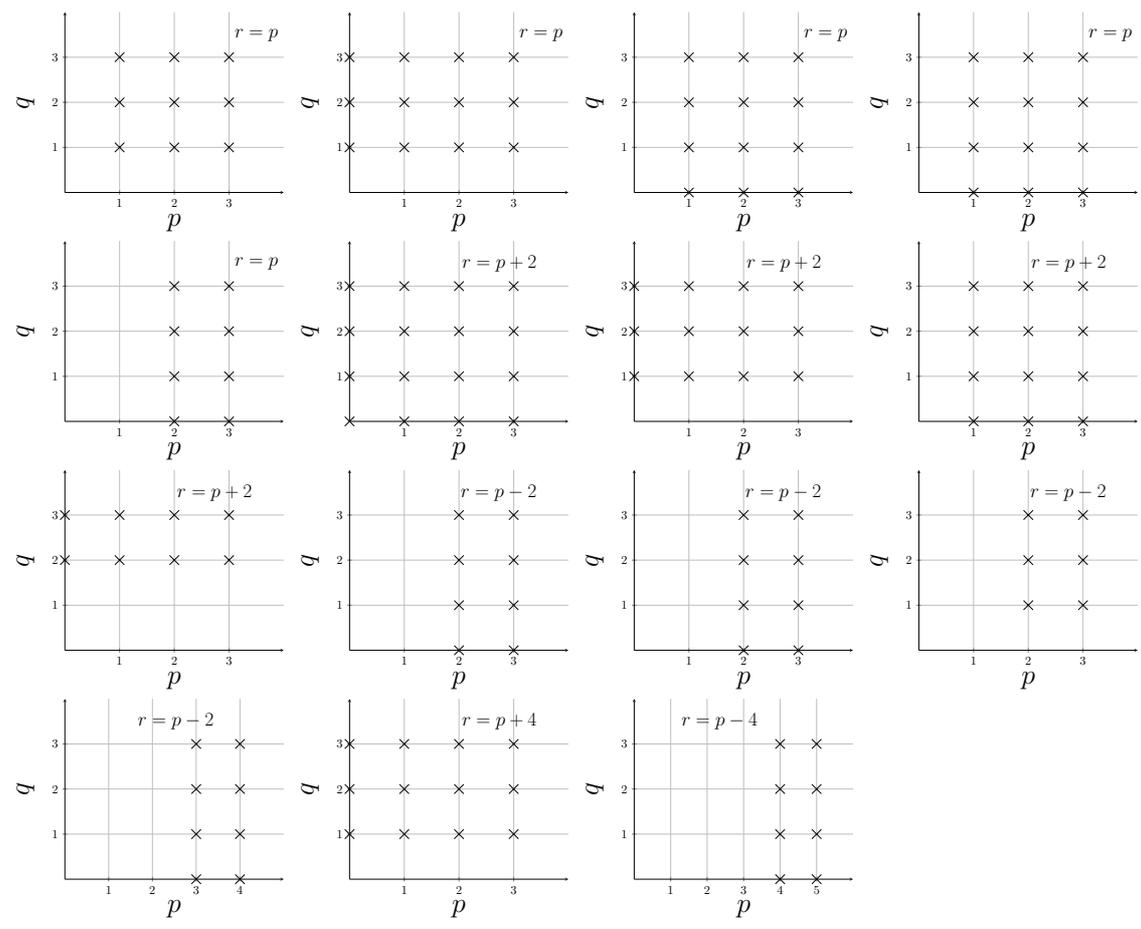
Figure 9. Two-form towers.
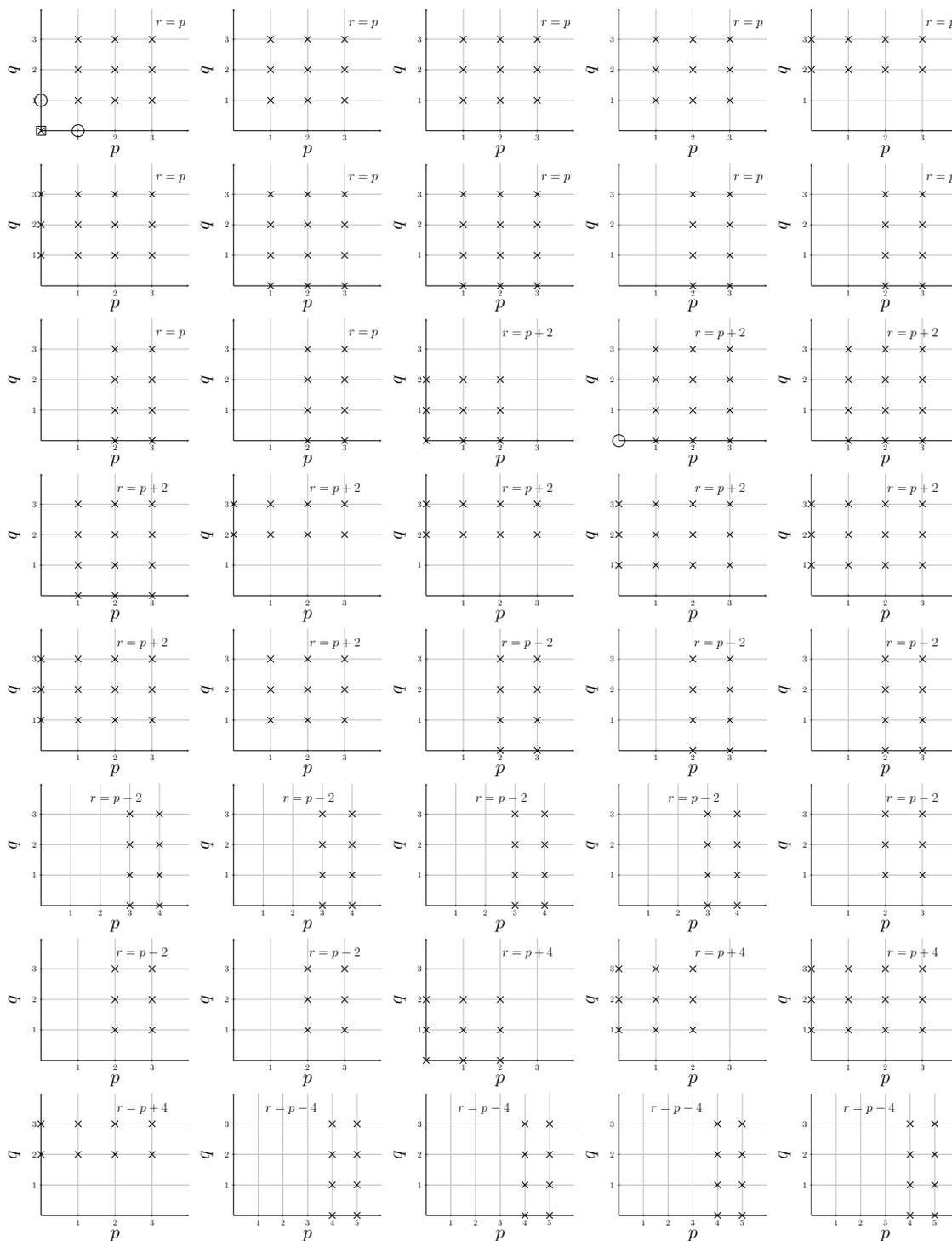
Figure 10. $\not{D}_{3/2}$ towers. The empty circles mark modes which appear from the decomposition of representations of SO(8) on the round sphere, but do not exist when derived in the conventional method. The excess modes are $(1,0;1) = (\mathbf{4},\mathbf{2})$, $(0,1;0) = (\mathbf{5},\mathbf{1})$ and $(0,0;2) = (\mathbf{1},\mathbf{3})$. The square with a cross, on the other hand, emphasises the fact that the G irrep $(0,0;0) = (\mathbf{1},\mathbf{1})$ appears in the squashed spectrum but is not produced in the spontaneous symmetry breaking of the round sphere spectrum.
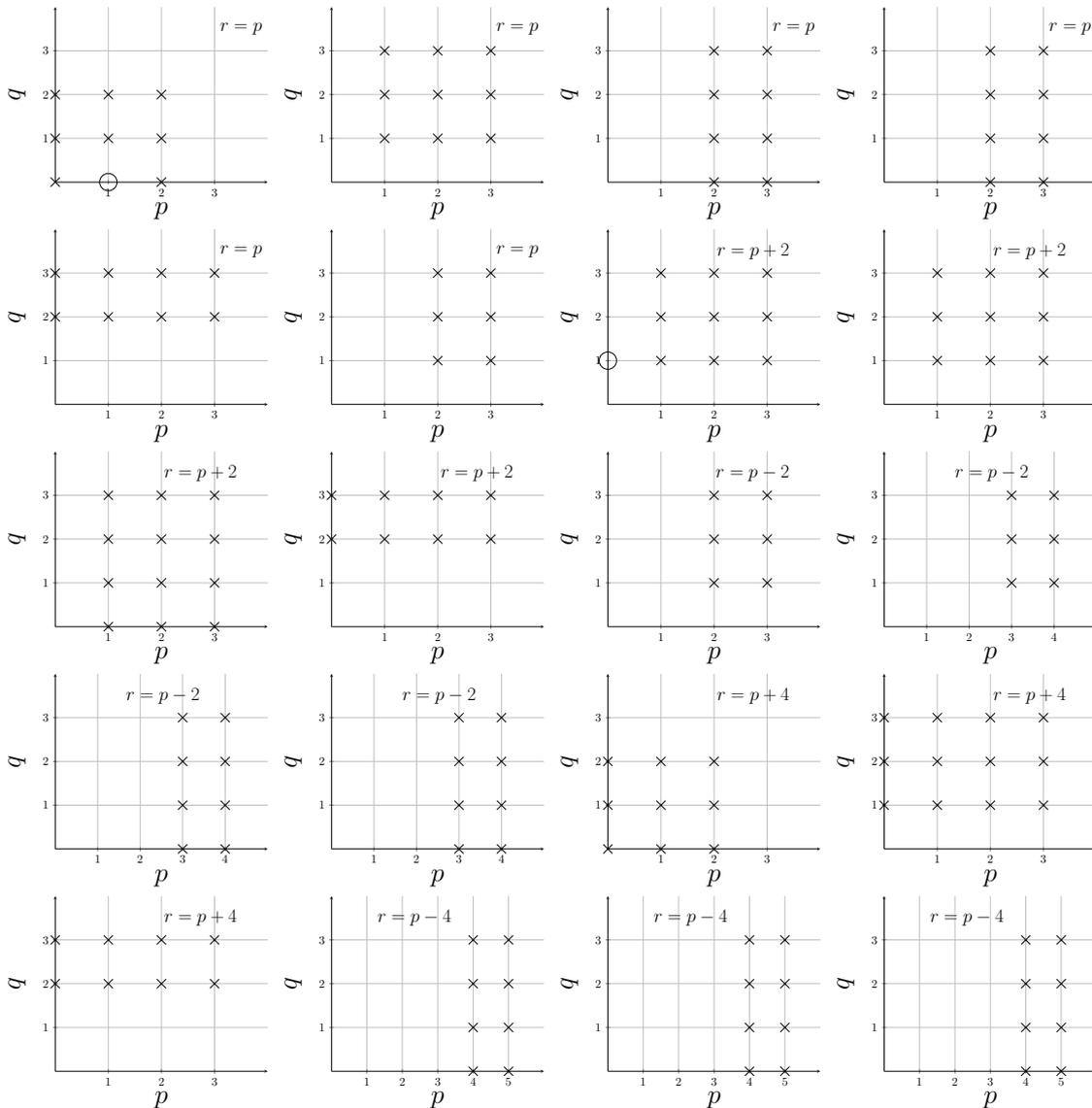
Figure 11. Δ_L (symmetric traceless and transverse rank 2 tensors) towers. The empty circles mark modes which appear from the decomposition of representations of SO(8) on the round sphere, but do not exist when derived in the conventional method. The excess modes are $(0, 1; 2) = (\mathbf{5}, \mathbf{3})$ and $(1, 0; 1) = (\mathbf{4}, \mathbf{2})$.
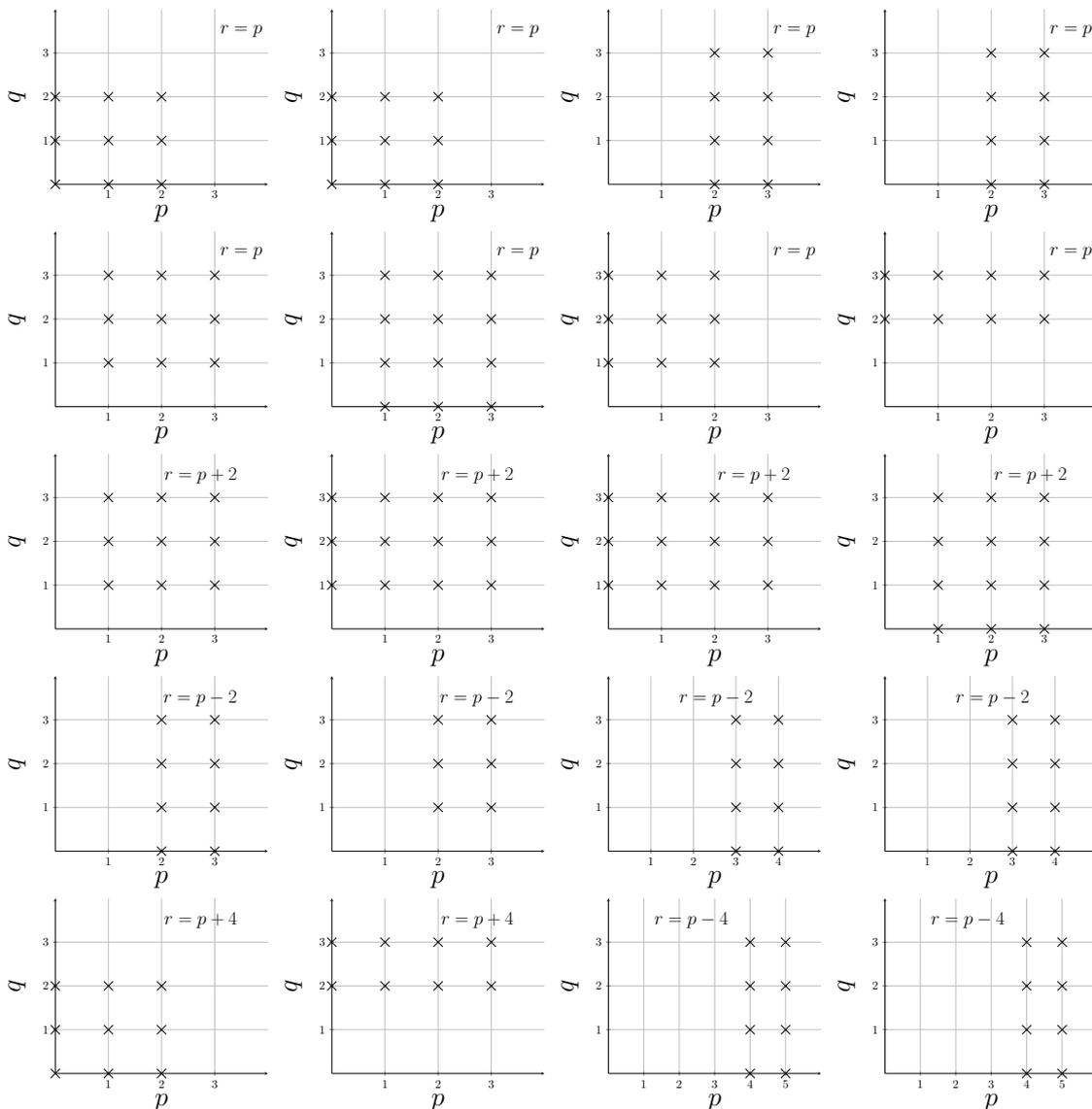
Figure 12. Q towers.

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