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Generators and relations for (generalised) Cartan type superalgebras

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Abstract. In Kac's classification of finite-dimensional Lie superalgebras, the contragredient ones can be constructed from Dynkin diagrams similar to those of the simple finite-dimensional Lie algebras, but with additional types of nodes. For example, $A(n-1,0) = \mathfrak{sl}(1|n)$ can be constructed by adding a "gray" node to the Dynkin diagram of $A_{n-1} = \mathfrak{sl}(n)$, corresponding to an odd null root. The Cartan superalgebras constitute a different class, where the simplest example is $W(n)$, the derivation algebra of the Grassmann algebra on n generators. Here we present a novel construction of $W(n)$, from the same Dynkin diagram as $A(n-1,0)$, but with additional generators and relations.

This talk, given by JP at "The 32nd International Colloquium on Group Theoretical Methods in Physics (Group32)" in Prague, 9–13 July, 2018, is based on [1], where more details and references can be found.

In attempts to understand the origin of the duality symmetries appearing in supergravity theories, the Lie algebras $\mathfrak{g}$ describing the symmetries have been extended to infinite-dimensional Lie superalgebras. These extensions include Borcherds superalgebras [2], here denoted by $\mathscr{B}(\mathfrak{g})$, as well as tensor hierarchy algebras [3], here denoted by $W(\mathfrak{g})$ and $S(\mathfrak{g})$.

The construction of tensor hierarchy algebras in [3] was only applicable for finite-dimensional $\mathfrak{g}$ and thus in particular not to the cases $\mathfrak{g} = E_r$ for $r \geqslant 9$. Here we solve this problem by a new construction with generators and relations. We focus on the case $\mathfrak{g} = A_{n-1}$ since $W(\mathfrak{g})$ and $S(\mathfrak{g})$ then turn out to be finite-dimensional and well known as *Cartan type* superalgebras. In this sense, the general tensor hierarchy algebras $W(\mathfrak{g})$ and $S(\mathfrak{g})$ are *generalised* Cartan type superalgebras.

We consider algebras over an algebraically closed field $\mathbb{K}$ of characteristic zero. A *superalgebra* G is an algebra with a $\mathbb{Z}_2$-grading, which means that it can be decomposed into a direct sum $G = G_{(0)} \oplus G_{(1)}$ of an even subalgebra $G_{(0)}$ and an odd subspace $G_{(1)}$, such that $G_{(i)}G_{(j)} = G_{(i+j)}$ where $i, j \in \mathbb{Z}_2$. (This means that $G_{(1)}$ does not close under the product and is thus not a subalgebra.) In a *Lie* superalgebra, the product is a bracket that satisfies the identities

$$[x,y] = -(-1)^{|x||y|}[y,x]\,, \tag{1}$$

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]\,, \tag{2}$$

where $|x| = 0$ if $x \in G_{(0)}$ and $|x| = 1$ if $x \in G_{(1)}$.

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A $\mathbb{Z}$-*grading* of the Lie superalgebra G is a decomposition of G into a direct sum of subspaces G_i for all integers i, called *levels*, such that $[G_i, G_j] \subseteq G_{i+j}$. The $\mathbb{Z}$-grading is said to be *consistent* if $G_i \subseteq G_{(i)}$ for all levels $i \in \mathbb{Z}$, that is, if odd elements appear at odd levels and even elements at even levels.

One way to obtain a Lie superalgebra if we only have a superalgebra G to start with (not necessarily a Lie superalgebra) is to consider all even and odd derivations of it. Let $|d|$ be equal to either 1 or 0. A linear map $d : G \to G$ satisfying

$$d(xy) = d(x)y + (-1)^{|d||x|}xd(y) \tag{3}$$

is an odd derivation if $|d| = 1$ and an even derivation if $|d| = 0$. All even and odd derivations of G span a vector space which, together with the commutator as the bracket,

$$[c, d] = c \circ d - (-1)^{|c||d|}d \circ c, \tag{4}$$

forms a Lie superalgebra. This is the *derivation algebra* of G, denoted $\operatorname{der} G$.

The Grassmann algebra $\Lambda(n)$ is a basic example of a superalgebra that is not a Lie superalgebra. It is the associative superalgebra generated by n odd elements $\theta^0, \theta^1, \ldots, \theta^{n-1}$, modulo the relations $\theta^a\theta^b = -\theta^b\theta^a$, where $a, b = 0, 1, \ldots, n-1$. It is spanned by monomials $\theta^{a_1} \cdots \theta^{a_p}$, where $0 \leqslant p \leqslant n$, which are fully antisymmetric in the upper indices,

$$\theta^{a_1} \cdots \theta^{a_p} = \theta^{[a_1} \cdots \theta^{a_p]}\,. \tag{5}$$

The derivation algebra of $\Lambda(n)$ has a basis consisting of elements

$$K^{a_1\cdots a_p}{}_b = \theta^{a_1} \cdots \theta^{a_p}\frac{\partial}{\partial\theta^b} \tag{6}$$

acting on a monomial $\theta^{c_1} \cdots \theta^{c_q}$ by a contraction,

$$K^{a_1\cdots a_p}{}_b \quad : \quad \theta^{c_1} \cdots \theta^{c_q} \mapsto q\,\delta_b{}^{[c_1}\theta^{|a_1} \cdots \theta^{a_p|}\theta^{c_2} \cdots \theta^{c_q]}\,. \tag{7}$$

This Lie superalgebra $\operatorname{der}\Lambda(n)$ is also denoted by $W(n)$. It is easy to see that it has a consistent $\mathbb{Z}$-grading where the subspace at level $-p+1$ has a basis of elements $K^{a_1\cdots a_p}{}_b$ which are fully antisymmetric in the upper indices, and thus there are no elements at level $-p+1$ for $p > n$ (or $p < 0$). Negative and positive levels are here reversed compared to the usual conventions.

level	basis
1	K_a
0	$K^a{}_b$
-1	$K^{ab}{}_c$
$\vdots$	$\vdots$
$-n+1$	$K^{a_1\cdots a_n}{}_b$

(8)

In the classification of simple finite-dimensional Lie superalgebras, $W(n)$ appear as Lie superalgebras of *Cartan type*. Such Lie superalgebras are distinguished from the *classical* Lie superalgebras, which are further divided into *basic* and *strange* ones [4].

The basic Lie superalgebras are finite-dimensional cases of *contragredient* Lie superalgebras [4], which means that they can be constructed from a (generalised) Cartan matrix, or from a

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Figure 1: The Dynkin diagram of $\mathscr{B}(A_{n-1}) = A(n-1,0)$.

Dynkin diagram encoding the same information as the matrix. For a general contragredient Lie superalgebra, the only condition on the Cartan matrix is that it be a square matrix B_{ab} with values in $\mathbb{K}$, where the index set labelling rows and columns is $\mathbb{Z}_2$-graded, that is, the disjoint union of an odd and an even subset. Here we restrict to the cases where B_{ab} $(a,b = 0,1,2\ldots,r)$ is obtained by adding a row and a column to the Cartan matrix A_{ij} $(i,j = 1,2\ldots,r)$ of a Kac–Moody algebra $\mathfrak{g}$ of rank r, such that

$$B_{ij} = A_{ij}\,, \qquad\qquad B_{0a} = B_{a0} = \begin{cases} -1 & (a=1)\,, \\ 0 & (a \neq 1)\,, \end{cases} \tag{9}$$

and such that $\{0\}$ and $\{1,2,\ldots,r\}$ are the odd and even subsets, respectively, of the index set $\{0,1,2,\ldots,r\}$. Furthermore, for simplicity we assume that A_{ij} is symmetric (implying that B_{ab} is symmetric as well) and that both A_{ij} and B_{ab} are non-degenerate. To this Cartan matrix we associate a $\mathbb{Z}_2$-graded set $M = M_{(0)} \cup M_{(1)}$ of generators,

$$M_{(0)} = \{e_i, f_i, h_a\} \qquad\qquad M_{(1)} = \{e_0, f_0\}\,, \tag{10}$$

where $i = 1,2,\ldots,r$ and $a = 0,1,2,\ldots,r$. Let $\mathscr{B}(\mathfrak{g})$ be the Lie superalgebra generated by the set M modulo the Chevalley–Serre relations

$$[h_a, e_b] = B_{ab}e_b\,, \qquad [h_a, f_b] = -B_{ab}f_b\,, \qquad [e_a, f_b] = \delta_{ab}h_b\,,$$

$$(\text{ad } e_a)^{1-B_{ab}}(e_b) = (\text{ad } f_a)^{1-B_{ab}}(f_b) = 0\,. \tag{11}$$

Then $\mathscr{B}(\mathfrak{g})$ is the contragredient Lie superalgebra constructed from the Cartan matrix B_{ab}. With the restrictions on B_{ab} here, $\mathscr{B}(\mathfrak{g})$ is not only a contragredient Lie superalgebra but also a *Borcherds* superalgebra [5, 6].

Let us now apply the above construction to the case of the (finite-dimensional) Kac–Moody algebra $\mathfrak{g} = A_{n-1}$ (thus $r = n-1$) with a Cartan matrix A_{ij} where row and column 1 correspond to one of the end nodes in the Dynkin diagram. Then we get the Cartan matrix B_{ab} of $\mathscr{B}(\mathfrak{g})$ given in (12). We can associate a Dynkin diagram to it, given in Figure 1, where row and column 0 correspond to the "gray" node.

$$B_{ab} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \tag{12}$$

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The resulting Lie superalgebra $\mathscr{B}(A_{n-1})$ is $A(n-1,0)$, one of the basic Lie superalgebras in the classification [4].

If we put e_0 and f_0 at level 1 and level -1, respectively, and all the other generators at level 0, then we get a consistent $\mathbb{Z}$-grading of $\mathscr{B}(\mathfrak{g})$. In the case $\mathfrak{g} = A_{n-1}$ there are no more levels, this $\mathbb{Z}$-grading of $\mathscr{B}(A_{n-1})$ is a 3-grading:

$$A(n-1,0) = \mathscr{B}_{-1} \oplus \mathscr{B}_0 \oplus \mathscr{B}_1 \,. \tag{13}$$

The subalgebra $\mathscr{B}_0$ is $\mathfrak{sl}(n) \oplus \mathbb{K} = \mathfrak{gl}(n)$, and the basis elements can thus be written as $\mathfrak{gl}(n)$ tensors:

generators	level	basis
e_0	1	E_a
e_i, f_i, h_a	0	${G^a}_b$
f_0	-1	F^a

(14)

The commutation relations are

$$[{G^a}_b, {G^c}_d] = {\delta_b}^c {G^a}_d - {\delta_d}^a {G^c}_b \,, \qquad [E_a, F^b] = -{G^b}_a + {\delta_a}^b G \,,$$

$$[{G^a}_b, F^c] = {\delta_b}^c F^a \,, \qquad [{G^a}_b, E_c] = -{\delta_c}^a E_b \,, \qquad [E_a, E_b] = [F^a, F^b] = 0 \,, \tag{15}$$

where $G = \sum_{a=0}^{n-1} {G^a}_a$. With the identifications

$$e_0 = E_0 \,, \qquad f_0 = F^0 \,, \qquad h_0 = {G^1}_1 + {G^2}_2 + \cdots + {G^{n-1}}_{n-1} = G - {G^0}_0 \,,$$

$$e_i = {G^{i-1}}_i \,, \qquad f_i = {G^i}_{i-1} \,, \qquad h_i = {G^{i-1}}_{i-1} - {G^i}_i \,, \tag{16}$$

the commutation relations (15) follow from the Chevalley–Serre relations (11).

Let us now compare the basis (14) of $A(n-1,0)$ to the basis (8) of $W(n)$. Level 1 and 0 have the same index structure in $W(n)$ as in $A(n-1,0)$. If we consider level -1 in $W(n)$, the tensors can be decomposed into a traceless part and the trace, obtained by contracting the lower index with one of the upper indices. If we take the subalgebra of $W(n)$ generated by level 1 and only the traceless part of level -1, then we get another Lie superalgebra of Cartan type, denoted $S(n)$, with traceless tensors all the way down to level $-n+2$. If we instead take the subalgebra generated by level 1 and only the trace part of level -1, then there will be no lower levels, and we get $A(n-1,0)$. Thus $A(n-1,0)$ is a subalgebra of $W(n)$, which can be constructed from the generators (10) and the relations (11). The question arises whether we can obtain not only this subalgebra, but the whole of $W(n)$ by extending the set of generators and relations.

We extend the set $M = M_{(0)} \cup M_{(1)}$ of generators to $M' = M_{(0)} \cup {M'}_{(1)}$, where f_0 is replaced by r generators f_{0a}.

$$M_{(0)} = \{e_i, f_i, h_a\} \,, \qquad {M'}_{(1)} = \{e_0, f_{0a} \,|\, a \neq 1\} \,. \tag{17}$$

Henceforth, whenever f_{0a} appears we assume $a = 0, 2, 3, \ldots, r$ (with $r = n-1$ in the case of $\mathfrak{g} = A_{n-1}$), and whenever f_a appears we assume $a \neq 0$. As we will see, the new generator f_{00} corresponds to the old f_0. Identifying them with each other, M' is indeed an extension of M.

We let $\widetilde{W}(\mathfrak{g})$ be the Lie superalgebra generated by the set M' modulo the relations (11) and the additional relations

$$[e_0, f_{0a}] = h_a\,, \qquad [h_a, f_{0b}] = -B_{a0} f_{0b}\,, \qquad [e_1, f_{0a}] = 0\,,$$

$$[e_a, [e_a, e_{0b}]] = [f_a, [f_a, f_{0b}]] = 0\,,$$

$$i, j = 2, 3, \ldots, r \quad \Rightarrow \quad [e_i, [f_j, f_{0a}]] = \delta_{ij} B_{aj} f_{0j}\,, \tag{18}$$

where $a = 0, 1, 2, \ldots, r$.

In the same way as for $\mathscr{B}(\mathfrak{g})$ we get a consistent $\mathbb{Z}$-grading of $\widetilde{W}(\mathfrak{g})$ if we put e_0 at level 1, all f_{0a} at level -1, and all the other generators at level 0. The sum of the subspaces at level ± 1 and 0 constitute the *local part* of $\widetilde{W}(\mathfrak{g})$. We then define $W(\mathfrak{g})$ as $W(\mathfrak{g}) = \widetilde{W}(\mathfrak{g})/J$, where J is the maximal ideal of $\widetilde{W}(\mathfrak{g})$ intersecting the local part trivially.

The following theorem summarises the main results of [1] (where the proof can be found).

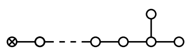
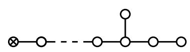
Theorem. *The Lie superalgebra $W(A_{n-1}) = \widetilde{W}(A_{n-1})/J$ is isomorphic to $W(n)$. The ideal J of $\widetilde{W}(A_{n-1})$ is generated by the relations*

$$[f_{0a}, f_{0b}] = [f_{0i}, [f_{0j}, f_1]] = [(f_{02} - f_{00}), [f_{0j}, f_1]] = 0\,, \tag{19}$$

where $i, j = 3, \ldots, n-1$. Thus $W(n)$ has generators (17) and relations (11), (18) and (19).

By removing h_0 and f_{00} from the set M' of generators we get a subalgebra $S(\mathfrak{g})$ of $W(\mathfrak{g})$. In the case $\mathfrak{g} = A_{n-1}$ we have $S(\mathfrak{g}) = S(n)$. In general, for finite-dimensional $\mathfrak{g}$ this definition of $S(\mathfrak{g})$ agrees with the definition of the corresponding tensor hierarchy algebra in [3]. In cases other than $\mathfrak{g} = A_{n-1}$ we do not know whether the relations (19) generate the whole ideal J or if additional relations are needed.

We conclude this talk with an overview of the cases where $\mathfrak{g}$ belongs to the A, D or E series of Kac–Moody algebras, with node 1 being the node to which another node is connected when going to the next algebra in the series. Another Lie superalgebra of Cartan type, $H(2r)$, appears as $S(D_r)$.

			
$\mathfrak{g}$	$A_{n-1} = \mathfrak{sl}(n)$	$D_r = \mathfrak{so}(2r)$	E_r
$\mathscr{B}(\mathfrak{g})$	$A(n-1,0) = \mathfrak{sl}(n\,\|\,1)$	$D(d,1) = \mathfrak{osp}(2r\,\|\,2)$	infinite-dimensional
$S(\mathfrak{g})$	$S(n)$	$H(2r)$	infinite-dimensional
$W(\mathfrak{g})$	$W(n)$	infinite-dimensional	infinite-dimensional

In applications to extended geometry, the A, D and E cases above correspond to ordinary, double and exceptional geometry, respectively [7]. In cases where so-called ancillary transformations are absent, the Borcherds superalgebras can be used to derive expressions for the generalised diffeomorphisms [8] and furthermore an L_∞ algebra encoding their gauge structure [9, 10]. When ancillary transformations are present it seems that the Borcherds superalgebra needs to be replaced by a tensor hierarchy algebra [11, 12].

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