INVARIANTS OF MODELS OF GENUS ONE CURVES VIA MODULAR FORMS AND DETERMINANTAL REPRESENTATIONS

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ABSTRACT. An invariant of a model of genus one curve is a polynomial in the coefficients of the model that is stable under certain linear transformations. The classical example of an invariant is the discriminant, which characterizes the singularity of models. The ring of invariants of genus one models over a field is generated by two elements. Fisher normalized these invariants for models of degree $n = 2, 3, 4$ in such a way that these invariants are moreover defined over the integers. We will provide an alternative way to express these normalized invariants using modular forms. This method relies on a direct computation for the discriminants based on their own geometric properties. In the case of the discriminant of ternary cubics over the complex numbers, we perform another approach using determinantal representations with a connection to theta functions. Both of these two approaches link a genus one model to a Weierstrass form.

CONTENTS

1. INTRODUCTION

Consider a curve C given by the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{1}$$

There are two classical invariants labeled by c_4 and c_6 defined for instance in [19, p. 42] as

$$c_4 = b_2^2 - 24b_4, c_6 = -b_2^3 + 36b_2b_4 - 216b_6, \tag{2}$$

where $b_2 = a_1^2 + 4a_2, b_4 = 2a_4 + a_1a_3$ and $b_6 = a_3^2 + 4a_6$. They define the discriminant $\Delta = (c_4^3 - c_6^2)/1728$ with the property that $\Delta \neq 0$ if and only if the curve is non-singular. In this case, the curve C is of genus one. An invariant of the model associated to (1) is a polynomial in their coefficients, which remains unchanged under the linear algebraic transformations mentioned in Section 2.

We want to explore this study to other models of genus one curves with two different approaches. On the one hand, it is natural to relate invariants to modular forms as described in Section 4. On the other hand, one can represent a polynomial of certain types as the determinant of a matrix whose elements have linear forms. The study of a curve defined by that polynomial is reduced to the study of the corresponding matrix.

1.1. **Models of genus one and their invariants.** Let C be a smooth curve of genus one over a field K and suppose that D is a K-rational divisor on C of degree n. In case $n = 1$, C has a K-rational point so that it can be given by a Weierstrass equation (1).

If $n \geq 2$, there exists a morphism $C \to \mathbb{P}^{n-1}$ defined by the complete linear system associated to D. This morphism is an embedding if $n \geq 3$. We define models of genus one of degrees $n \leq 5$ as follows

Definition. *A genus one model of degree $n \leq 5$ is a*

a) Weierstrass form if $n = 1$.
b) pair of a binary quadratic and a binary quartic if $n = 2$.
c) ternary cubic if $n = 3$.
d) pair of quadrics in four variables if $n = 4$.
e) 5×5 alternating matrix of linear forms in five variables if $n = 5$.

The equation defined by a genus one model (p, q) of degree $n = 2$ is $y^2 + p(x, z)y = q(x, z)$. In case $n = 5$, the equations defining the model are the 4×4 Pfaffians of the matrix. In general, such models define smooth curves of genus one.

The classical invariants where $n = 2, 3, 4$ were studied in [23] and [1, Section 3]. These invariants were normalized by Fisher [12, Sections 6,7] so that they are usual formulae when restricted to the Weierstrass family.

The aim of this paper is to give an alternative way to express the normalized invariants c_4, c_6 and thus $\Delta = (c_4^3 - c_6^2)/1728$ for genus one models of degrees $n = 2, 3, 4$. To do this, we establish formulae in all characteristics relating the invariants of smooth genus one models of degrees $n \leq 4$ and the corresponding Jacobians in the classical setting as in [1]. More precisely, the authors in [1, Section 3] define a map f_n from a smooth curve C_ϕ, which is defined by a model of genus one ϕ of degree n $(n = 2, 3, 4)$, to the corresponding Jacobian E_ϕ. In addition, they describe explicitly when $\mathrm{char}(K) \neq 2, 3$ the map and the Jacobian E_ϕ given by a Weierstrass equation of the form (3).

The map f_n will be described explicitly in Section 2. We construct from it in Section 3 the map $\varphi_n : X_n \to W$ from the affine space X_n of genus one models of degree n ($n = 2, 3, 4$) to the space W of Weierstrass forms. We first compute the normalized discriminants.

Theorem 1.1. *Let C_ϕ be a curve defined by a genus one model ϕ of degree n ($n = 2, 3, 4$) over a field K and $\Delta_\phi, \Delta_{\varphi_n(\phi)}$ be the discriminants of ϕ and its corresponding Weierstrass form $\varphi_n(\phi)$. We have*

$$\Delta_\phi = \alpha_n^{12} \Delta_{\varphi_n(\phi)},$$

where $\alpha_2 = 1, \alpha_3 = 1/2, \alpha_4 = 2$.

This theorem is established directly using the singularities of genus one models. To obtain the analogous result for any invariant, we will use geometric modular forms defined in [18]. To be precise, we will see in Section 4 that it is possible to associate to a geometric modular form $\mathcal{F}$ an invariant $I_{\mathcal{F}}$ of the same weight. We will prove the following result

Theorem 1.2. *Let C_ϕ be a smooth curve of genus one defined by a model ϕ of degree n ($n = 2, 3, 4$) over a field K with the corresponding Jacobian E_ϕ defined by $\varphi_n(\phi)$. Let k be an integer and $I_{\mathcal{F}}$ be the invariant of weight k associated to a geometric modular form $\mathcal{F}$ of weight k, we have*

$$I_{\mathcal{F}}(\phi) = \alpha_n^k I_{\mathcal{F}}(\varphi_n(\phi)),$$

where $\alpha_2 = 1, \alpha_3 = 1/2, \alpha_4 = 2$.

Recently, Fisher [13, p. 2126] have obtained a formula for the invariants c_4, c_6 and the Jacobian of smooth genus one models of arbitrary degree n in characteristic 0. More details about models of genus one curves and their invariants will be discussed in Section 2.

1.2. **Invariants over $\mathbb{C}$ and determinantal representations.** The second perspective of this paper is to study the invariants of genus one models when $K = \mathbb{C}$ with an emphasis on discriminants of ternary cubics. These invariants have large expressions in general. For instance, the discriminant of a plane cubic curve is a polynomial of degree 12 in coefficients of the cubic with 2040 monomials (see [15, p. 4]). But over $\mathbb{C}$, we have short expressions in terms of theta constants. Consider the classical case where our smooth projective genus one curve C_ϕ is defined by the affine Weierstrass equation:

$$y^2 = 4x^3 - g_2 x - g_3. \tag{3}$$

Using the Weierstrass parametrization, there exists a unique lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with some complex numbers ω_1, ω_2 such that $\mathrm{Im}(\omega_2/\omega_1) > 0$ and $C_\phi(\mathbb{C}) \cong \mathbb{C}/\Lambda$. Let $\tau := \omega_2/\omega_1$ and apply the discriminant formula $\Delta_\phi = 2^{12}(g_2^3 - 27g_3^2)$, we have that

$$\Delta_\phi = 2^{16}\left(\frac{\pi}{\omega_1}\right)^{12}(\theta_2(0,\tau)\theta_3(0,\tau)\theta_4(0,\tau))^8. \tag{4}$$

Here θ_2, θ_3 and θ_4 are the three even Jacobi theta functions.

Remark 1.3. *The normalized discriminant above comes from the formulae in* [2, p. 367-368] *with the normalized invariants $c_4 = 2^6 3 g_2$, $c_6 = 2^9 3^3 g_3$.*

More details on the above will be explained in Section 5. Our purpose is to generalize the formula (4) to other models of genus one. But this is an immediate consequence of Theorem 1.1. We have the following formulae for the discriminants of genus one models of degree $n = 2, 3, 4$. Again, a lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ associated to a Weierstrass form over $\mathbb{C}$ will be understood as the unique one coming from the Weierstrass parametrization.

Corollary 1.4. *Let C_ϕ be a smooth curve of genus one over $\mathbb{C}$ defined by a model ϕ of degree n ($n = 2, 3, 4$) with the Jacobian E_ϕ defined by $\varphi_n(\phi)$. Then $E_\phi(\mathbb{C}) \cong \mathbb{C}/\Lambda$ for the lattice $\Lambda = \omega_1\mathbb{Z}+\omega_2\mathbb{Z}$ with some complex numbers ω_1, ω_2 satisfying $Im(\omega_2/\omega_1) > 0$. Let Δ_ϕ be the discriminant of ϕ, we then have*

$$\Delta_\phi = 2^{16}\alpha_n^{12}\left(\frac{\pi}{\omega_1}\right)^{12}(\theta_2(0,\tau)\theta_3(0,\tau)\theta_4(0,\tau))^8,$$

where $\alpha_2 = 1, \alpha_3 = 1/2$ and $\alpha_4 = 2$.

We want to study the above discriminant formula with a new approach using determinantal representations. For a homogeneous polynomial ϕ, we construct a matrix U whose elements are linear forms such that we can write $\phi = \lambda \det(U)$ for some constant $\lambda \neq 0$. In general, only plane curves, quadratic and cubic surfaces, quadratic threefolds admit a determinantal representation as confirmed in [10]. The study of ϕ has thus been moved to the study of the matrix U. The reader can have a look at [4] for a general discussion of this topic.

Starting with Weierstrass cubics, we find theta functions in their determinantal representations as well as in the discriminants. Let

$$a = \theta_2(0,\tau),\ b = \theta_3(0,\tau),\ c = \theta_4(0,\tau), \tag{5}$$

we will prove the following

Proposition 1.5. *Let C_ϕ be a smooth curve given by the Weierstrass form*

$$\phi(x, y, z) = y^2z - 4x^3 + g_2xz^2 + g_3z^3,$$

where g_2 and g_3 belong to a field K. Then ϕ admits determinantal representations

$$\begin{pmatrix} 2x+tz & y+dz & (3t^2-g_2)z \\ 0 & x-tz & y-dz \\ z & 0 & -2x-tz \end{pmatrix},$$

with $t, d \in \overline{K}$ being arbitrary such that $d^2 = 4t^3 - g_2t - g_3$. When $K = \mathbb{C}$, there is a natural choice for t, d which produces a determinantal representation for ϕ in terms of theta constants as follows

$$\begin{pmatrix} 2x - \frac{\pi^2}{3\omega_1^2}(a^4+b^4)z & y & -(\frac{\pi}{\omega_1})^4 c^8 z \\ 0 & x + \frac{\pi^2}{3\omega_1^2}(a^4+b^4)z & y \\ z & 0 & -2x + \frac{\pi^2}{3\omega_1^2}(a^4+b^4)z \end{pmatrix},$$

where the even theta constants a, b, c were defined as in (5).

The first part of this proposition uses the method in [21, Section 2] where the author established similar representations for other type of Weierstrass equations of the form $y^2z = x(x+\vartheta_1 z)(x+\vartheta_2 z)$ with some constants $\vartheta_1, \vartheta_2 \in K$. The discriminant formula (4) is then a consequence of the second part of this theorem using resultant as in Section 5.

Our goal is to study this phenomena for general smooth cubic curves using determinantal representations. One can actually provide determinantal representations for any non-rational complex plane curve by using a result in [3] as we will see later in Section 6. This deduces in particular the formula of the discriminant of plane cubics by using resultant. Since a cubic curve C_ϕ over $\mathbb{C}$ always has a flex point, it can be transformed to a Weierstrass form after a linear coordinate change M (see [8, Section 4.4]). The resulting Weierstrass form is isomorphic to $\mathbb{C}/\Lambda$ for a unique lattice Λ coming the Weierstrass parametrization. Writing $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ for some $\omega_1, \omega_2 \in \mathbb{C}$ satisfying $\mathrm{Im}(\omega_2/\omega_1) > 0$. Denote by $\tau = \omega_2/\omega_1$, we will prove the following result.

Theorem 1.6. *Let C_ϕ be a smooth plane cubic curve over $\mathbb{C}$ defined by a cubic form ϕ and Δ_ϕ be the discriminant of ϕ, we have*

$$\Delta_\phi = \frac{2^{16}}{\det(M)^{12}} \left(\frac{\pi}{\omega_1}\right)^{12} (abc)^8,$$

where a, b, c were defined as in (5).

This result is known but the above approach with determinantal representations is new. One can compare Theorem 1.6 with Corollary 1.4 for the case when $n = 3$. Here we are using a fixed flex point to give a linear transformation from C_ϕ to a Weierstrass form instead of the explicit map φ_3.

So it is natural to link determinantal representations of a smooth plane cubic to a Weierstrass form as the approach using modular forms in Section 1.1. The discriminant formulae will then be predicted. Thus it might be possible to study determinantal representations in more general case using modular forms.

1.3. **Organization of the paper.** We recall the definitions of genus one models as well as their invariants in Section 2. Then we prove Theorem 1.1 and thus Corollary 1.4 in Section 3 by considering the singularity of genus one models. The theory of modular forms will be discussed in Section 4, which will be used to establish Theorem 1.2. After that, we study determinantal representations of Weierstrass cubics and provide a simple proof to Proposition 1.5 in Section 5. Determinantal representations of complex plane curves is presented in Section 6 and we apply it to establish Theorem 1.6 in Section 7.

2. Models of genus one curves and their invariants

Let X_n be the set of all genus one models of degree n over a field K (see Definition 1.1), we will see later in this section that X_n is an affine space of dimension $5, 8, 10, 20, 50$ for $n = 1, 2, 3, 4, 5$ respectively. In [12, Section 3], the author defined natural linear algebraic groups $\mathcal{G}_n$ acting on X_n $(n \leq 5)$, which preserve the solutions of the models. Let G_n be the commutator subgroups of $\mathcal{G}_n$ and $K[X_n]$ be the coordinate ring of X_n.

Definition 2.1. *The ring of invariants of X_n $(n \leq 5)$ over K is*

$$K[X_n]^{G_n} := \{I \in K[X_n] : I \circ g = I \textit{ for all } g \in G_n(\overline{K})\}.$$

The vector space of invariants of weight k of X_n over K is defined as

$$K[X_n]_k^{G_n} := \{I \in K[X_n] : I \circ g = (\det g)^k I \textit{ for all } g \in \mathcal{G}_n(\overline{K})\}.$$

The character det on $\mathcal{G}_n$ is chosen so that we get appropriate weights for the invariants and moreover (see [12, Lemma 4.3])

$$K[X_n]^{G_n} = \bigoplus_{k \geq 0} K[X_n]_k^{G_n}.$$

We now describe in detail affine spaces X_n for $n \leq 5$, their classical invariants and the map $f_n : C_\phi \to E_\phi$ $(n = 2, 3, 4)$ from a smooth curve defined by a model $\phi \in X_n^0 := \{\phi \in X_n \mid C_\phi \text{ is smooth}\}$ to its corresponding Jacobian E_ϕ. This map is defined by a divisor D of degree n on the curve C_ϕ, which is the intersection of C_ϕ with the hyperplane at infinity (see [1, p. 305]), as $f_n(P) = nP - D$. It is given explicitly when $\mathrm{char}(K) \neq 2, 3$ as below. In addition, we introduce in each degree the natural non-zero regular 1-form ω_ϕ defined in [12, Section 5.4] for a smooth curve of genus one C_ϕ. This 1-form is useful for relating invariants and modular forms as we will see in Section 4.

2.1. **Models of degree** $n = 1$**.** The space X_1 is of dimension 5 where each model corresponding to (1) can be identify with the point $(a_1, a_2, a_3, a_4, a_6)$ in $\mathbb{A}^5$. Their invariants c_4, c_6 are defined in the usual way as in (2). The natural regular 1-form on a smooth curve C_ϕ defined by model ϕ is:

$$\omega_\phi := \frac{dx}{2y + a_1x + a_3}.$$

2.2. **Models of degree** $n = 2$**.** The equation of a genus one model of degree $n = 2$ is written as:

$$y^2 + (\alpha_0x^2 + \alpha_1xz + \alpha_2z^2)y = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4. \tag{6}$$

Each model in X_2 corresponds to a point $(\alpha_0, \alpha_1, \alpha_2, a, b, c, d, e) \in \mathbb{A}^8$ and thus $\dim(X_2) = 8$. Moreover, if $\mathrm{char}(K) \neq 2$ or 3, we can rewrite (6) as:

$$y^2 = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4. \tag{7}$$

As in [1, Section 3.1], the model ϕ defined by (7) has two classical invariants:

$$\begin{aligned} i &= (12ae - 3bd + c^2)/12, \\ j &= (72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3)/432 \end{aligned} \tag{8}$$

and a corresponding Weierstrass equation is:

$$y^2 = 4x^3 - ix - j. \tag{9}$$

The map f_2 from a smooth curve C_ϕ defined by (7) to the Jacobian E_ϕ defined by (9) is given as in [1, (3.3)] by:

$$f_2(x, y, z) = \left(\frac{g(x, z)}{(yz)^2}, \frac{h(x, z)}{(yz)^3}\right),$$

where

$$g(x, z) = \frac{1}{144}(q_{xz}^2 - q_{xx}q_{zz}),\ h(x, z) = \frac{1}{8}\begin{vmatrix} q_x & q_z \\ g_x & g_z \end{vmatrix}$$

with q being the binary quartic on the right hand side of (7).

The corresponding results to the generalized equation (6) can be found in [7, p. 766] by completing the square. We define the regular 1-form for a smooth curve C_ϕ defined by a model ϕ corresponding to (6) as:

$$\omega_\phi := \frac{z^2d(x/z)}{2y + p(x, z)}.$$

Here $p(x, z) = \alpha_0x^2 + \alpha_1xz + \alpha_2z^2$.

2.3. **Models of degree** $n = 3$**.** In case $n = 3$, a genus one model ϕ is a ternary cubic:

$$ax^3 + by^3 + cz^3 + a_2x^2y + a_3x^2x + b_1xy^2 + b_3y^2z + c_1xz^2 + c_2yz^2 + mxyz \tag{10}$$

with two classical invariants S, T defined in [1, p. 309-310] and we can see that $\dim(X_3) = 10$. A corresponding Weierstrass equation is also given there by:

$$y^2 = 4x^3 + 108Sx - 27T. \tag{11}$$

The map f_3 from a smooth curve C_ϕ defined by the model (10) to the Jacobian E_ϕ defined by (11) is given as in [1, (3.9)] by:

$$f_3(x,y,z) = \left(\frac{\Theta(x,y,z)}{H(x,y,z)^2}, \frac{J(x,y,z)}{H(x,y,z)^3} \right),$$

where

$$H = \frac{1}{216} \begin{vmatrix} \phi_{xx} & \phi_{xy} & \phi_{xz} \\ \phi_{yx} & \phi_{yy} & \phi_{yz} \\ \phi_{zx} & \phi_{zy} & \phi_{zz} \end{vmatrix}, \quad J = -\frac{1}{9} \left| \frac{\partial(\phi, H, \Theta)}{\partial(x,y,z)} \right|$$

and Θ is the covariant defined in [1, p. 308]. The regular 1-form on the smooth curve C_ϕ is defined as

$$\omega_\phi := \frac{x^2 d(y/x)}{\partial\phi/\partial z}.$$

2.4. **Models of degree $n = 4$.** We relate to the case $n = 2$ as follows: if the model ϕ is given by a pair of quadrics $q_1, q_2 \in K[x_0, x_1, x_2, x_3]$ (hence $\dim(X_4) = 20$), we can write $q_1 = \overline{x} A \overline{x}^T$ and $q_2 = \overline{x} B \overline{x}^T$ for two symmetric 4×4 matrices A, B with $\overline{x} = (x_0, x_1, x_2, x_3)$. The invariants of ϕ are then defined by the invariants of the quartic:

$$\det(xA + zB) = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4 \tag{12}$$

as in the case $n = 2$. A corresponding Weierstrass equation is thus given in the form (9) with i, j defined by the coefficients of the model (12) as in (8). The explicit map f_4 from a smooth curve defined by (12) to the Jacobian is given in [1, (3.12)] as:

$$f_4(x_0, x_1, x_2, x_3) = \left(\frac{g}{J^2}, \frac{h}{J^3} \right),$$

where g, h, J are defined as in [1, Section 3.3]. The regular 1-form for the smooth curve C_ϕ is defined as:

$$\omega_\phi := \frac{x_0^2 d(x_1/x_0)}{(\partial q_1/\partial x_3)(\partial q_2/\partial x_2) - (\partial q_1/\partial x_2)(\partial q_2/\partial x_3)}.$$

2.5. **Models of degree $n = 5$.** A genus one model of degree $n = 5$ is a 5×5 alternating matrix of linear forms in five variables. The equations defined by this model are the 4×4 Pfaffians of the matrix. Here a square matrix is called alternating if it is skew-symmetric and all of its diagonal entries are zero. The reader can have a look at [12, Section 5.2] and [14] for reference. We can check that $\dim(X_5) = 50$. Fisher [12, p. 770] also defines a regular 1-form to the case $n = 5$ when $\mathrm{char}(K) \neq 2$.

Observe that if I is an invariant of weight k then so is λI for any constant $\lambda \in K^*$. We want to normalize the invariants so that they have appropriate formulae in any characteristic.

2.6. **Normalized invariants.** The author in [12, Theorem 10.2] proves that $K[X_n]^{G_n}$ is isomorphic to $K[X_1]^{G_1}$ ($n \leq 5$) in any characteristic. This extends the invariants $c_4, c_6, \Delta \in \mathbb{Z}[X_1]$ of $K[X_1]^{G_1}$ defined in (2) to the corresponding ones in $K[X_n]^{G_n}$ ($n \leq 5$) denoted again by c_4, c_6, Δ. In fact, we have (see [12, Lemma 4.15 and remark 4.16])

Lemma 2.2. *The invariants c_4, c_6 and Δ are primitive polynomial in $\mathbb{Z}[X_n]$ for any $n \leq 5$.*

Thus it is possible to normalize these invariants up to sign. Furthermore, the two invariants $c_4, c_6 \in K[X_n]^{G_n}$ ($n \leq 5$) constructed above define the suitable discriminant $\Delta = (c_4^3 - c_6^2)/1728$ in the following way:

Definition/Lemma 2.3. *Let R be a unital commutative ring and C_ϕ be a curve over R defined by some genus one model ϕ of degree $n \leq 5$. There exists the discriminant Δ, which is a universal polynomial with integer coefficients, is defined such that $\Delta_\phi \in R^*$ if and only if C_ϕ is non-singular over R. Here R^* is the group of units of R.*

Proof. Fisher [12, Theorem 4.4] shows the properties of the discriminant of genus one models of degree $n \leq 5$ over a field K, but it is indeed equivalent to the Definition/Lemma 2.3. □

He also proves there that if $\mathrm{char}(K) \neq 2$ or 3, then the invariants $c_4(\phi), c_6(\phi)$ provide for the smooth genus one curve C_ϕ defined by a model ϕ of degree $n \leq 5$ the Jacobian

$$y^2 = x^3 - 27c_4(\phi)x - 54c_6(\phi).$$

The main ingredient of the proof of the above result is that, a pair (C_ϕ, ω_ϕ) of a model $\phi \in X_n^0$ is isomorphic to a pair of the form given in Section 2.1. Then the invariants $c_4(\phi), c_6(\phi)$ of ϕ are determined by the ones of that pair as in (2) (see [12, Definition 2.1 and Proposition 5.23]).

Then, in [12, Section 7], he gives the normalized formulae to the invariants c_4, c_6 of genus one models of degrees $n = 2, 3, 4$. More precisely, in the case $n = 2$, the normalized invariants of the model corresponding to (7) are

$$\begin{aligned} c_4 &= 2^4(12ae - 3bd + c^2), \\ c_6 &= 2^5(72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3). \end{aligned} \tag{13}$$

In comparison with the classical case (8), we have $c_4 = 2^6 3i$ and $c_6 = 2^9 3^3 j$. The normalized invariants of the generalized model corresponding to (6) can be found in [7, p. 766] by reducing to the form (7) from a completing square.

When $n = 3$, the normalized invariants of the model (10) are

$$c_4 = -216abcm + 144abc_1c_2 + 144acb_1b_3 + ... - 8a_3b_3m^2 + 16b_1^2c_1^2 - 8b_1c_1m^2 + m^4,$$

$$c_6 = 5832a^2b^2c^2 - 3888a^2bcb_3c_2 + 864a^2bc_2^3 + ... + 64b_1^3c_1^3 - 48b_1^2c_1^2m^2 + 12b_1c_1m^4 - m^6. \quad (14)$$

We have $c_4 = -2^4 3^4 S, c_6 = 2^3 3^6 T$ in comparing with the classical invariants S, T defined in [1, p. 309-310].

When $n = 4$, we relate to the quartic as in (12) and then the invariants are

$$c_4 = 12ae - 3bd + c^2,$$
$$c_6 = \frac{1}{2}(72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3). \quad (15)$$

Thus in comparing with the classical case, we have $c_4 = 2^{10}3i$ and $c_6 = 2^{15}3^3 j$.

In the case of plane cubics, these invariants were normalized before in [2, p. 367-368] where the authors explicitly provide the corresponding Weierstrass equations in any characteristic. Strictly speaking, they associate to a ternary cubic ϕ a Weierstrass form ϕ^* associated to (1) with the coefficients determined by the coefficients of ϕ. Then they observe that $(\phi^*)^* = \phi^*$ and thus naturally define:

$$c_4(\phi) := c_4(\phi^*), c_6(\phi) := c_6(\phi^*), \Delta_\phi := \Delta_{\phi^*}.$$

In Section 4, we will provide a new way for expressing these normalized invariants using modular forms. Before that, however, we will establish the normalized discriminants of genus one models directly by using the singularities of the models in the next section.

3. Discriminant of genus one curves and its Jacobians

The goal of this section is to prove Theorem 1.1. Fix $n \leq 5$, let X_n be the affine space of all genus one models ϕ of degree n, let W be the affine space of Weierstrass forms $y^2 - 4x^3 + g_2x + g_3$ if $\mathrm{char}(K) \neq 2, 3$ and $W = X_1$ otherwise. We define the map $\varphi_n : X_n \to W$ based on the discussion about the map f_n in Section 2 for $n = 2, 3, 4$. More precisely, for any smooth model $\phi \in X_n^0$, we have a map $f_n : C_\phi \to E_\phi$ from the smooth curve C_ϕ to its Jacobian E_ϕ coming from a divisor of degree n. We define the image $\varphi_n(\phi)$ of ϕ to be the model in W defining E_ϕ. This gives us a map $X_n^0 \to W$ which extends uniquely to a map $\varphi_n : X_n \to W$.

The map φ_n is given explicitly in case $\mathrm{char}(K) \neq 2$ or 3, which also applies to singular models as follows. The map φ_2 sends a model ϕ of degree 2 corresponding to (6) to the Weierstrass form defined by (9) of the model corresponding to (7) obtained from ϕ after a completing square. The map φ_3 sends a model ϕ of degree 3 of the form

(10) to the Weierstrass form defined by (11). The map φ_4 is defined by relating to the case $n = 2$. More precisely, φ_4 sends a model ϕ of degree 4 given by a pair of quadrics (q_1, q_2) to the Weierstrass form defined by (9) obtained from the quartic (12) as in the case $n = 2$. We denote by E_ϕ the curve given by the corresponding Weierstrass form $\varphi_n(\phi) \in W$ of a model $\phi \in X_n$.

This map sends non-singular curves to non-singular curves and singular curves to singular curves when $\mathrm{char}(K) \neq 2, 3$. We know that there exist discriminants (see Definition/Lemma 2.3) Δ_{X_n} in X_n and Δ_W in W parametrizing singular curves and they are both geometrically irreducible polynomials (see [12, Proposition 4.5]). The discriminant Δ_ϕ of ϕ is determined by evaluating Δ_{X_n} at coefficients of the model ϕ. i.e., $\Delta_\phi = \Delta_{X_n}(\phi)$.

We denote by $V(p)$ the set of points in $\mathbb{P}^m_K$ vanished at p for any homogeneous polynomial $p \in K[x_0, ..., x_m]$. Observe that Δ_{X_n} and the pull back $\varphi_n^*(\Delta_W)$ of Δ_W have the same vanishing property:

$$V(\Delta_{X_n}) = V(\varphi_n^*(\Delta_W)), \tag{16}$$

where φ_n^* is defined such that $\varphi_n^*(\Delta_W)(\phi) = \Delta_W(\varphi_n(\phi))$ for any model ϕ. We have the following:

Proposition 3.1. *For $n \leq 4$, there exists a constant $c \in K^*$ such that*

$$\Delta_{X_n} = c\varphi_n^*(\Delta_W).$$

Proof. For an ideal $J \subset K[x_0, ..., x_m]$, we denote by $\sqrt{J}$ the radical ideal of J over $\overline{K}$. From (16) and Hilbert's Nullstellensatz, we obtain that $\sqrt{(\Delta_{X_n})} = \sqrt{(\varphi_n^*(\Delta_W))}$. Note that Δ_{X_n} is geometrically irreducible as mentioned above. Then $\sqrt{(\Delta_{X_n})} = (\Delta_{X_n})$ (which is an ideal over $\overline{K}$) and thus there exist constants $k \in \mathbb{Z}$ and $c \in \overline{K}^*$ such that $\Delta_{X_n}^k = c\varphi_n^*(\Delta_W)$. Since both $\Delta_{X_n}^k$ and $\varphi_n^*(\Delta_W)$ are defined over K, we conclude that $c \in K^*$.

We will prove that $k = 1$ by comparing the degrees of Δ_{X_n} and $\varphi_n^*(\Delta_W)$. We first consider the case $n = 3$. By [5, Example 1.8], we know that Δ_{X_n} is a homogeneous polynomial of degree 12 with respect to the coefficients of the plane cubics. Besides, $\varphi_3^*(\Delta_W)$ is also of degree 12 in the coefficients of the cubics so that it has the same degree with Δ_{X_n}.

If $n = 4$, by [5, Example 1.10] we see that Δ_{X_n} is a homogeneous polynomial of degree 24 in the coefficients of the two quadratic forms defining the models. The degree of $\varphi_4^*(\Delta_W)$ is also 24.

The case $n = 2$ is different since the model $y^2 + p(x, z)y = q(x, z)$ is no longer homogeneous. Here p, q are homogeneous polynomials of degrees $2, 4$ respectively. If $\mathrm{char}(K) \neq 2$, this model can be brought to $y^2 = h(x, z)$ by completing square, where

$h = \frac{p^2}{4} + q$. The singular locus of this latter model is $\{2y = h_x = h_z = 0\}$, which is equal to $\{h_x = h_z = 0\}$ if $\mathrm{char}(K) \neq 2$. Consequently, the discriminant of the model (p, q) above is the discriminant of the quartic h from Definition/Lemma 2.3 (up to some power of 2). Hence, by [5, Example 1.8] again, we know that $\Delta_{X_n}(p, q)$ is of degree 6 with respect to the coefficients of h. We can check that $\varphi_2^*(\Delta_W)(p, q)$ is of degree 6 in terms of coefficients of h as well. Thus $k = 1$ for $n = 2, 3, 4$. $\square$

We actually have more information about the constant c by looking at models with integer coefficients.

Proposition 3.2. *For $n = 2, 3, 4$, the constant c in Proposition 3.1 can be expressed as $\pm 2^a$ for some $a \in \mathbb{Z}$. Moreover,*

$$\begin{cases} a = 0, & \text{If } n = 2; \\ a = -12, & \text{If } n = 3; \\ a = 12, & \text{If } n = 4. \end{cases}$$

We will see later in Theorem 4.10 that one can exclude the minus sign of the constant c above and thus obtain Theorem 1.1.

Proof. We consider the curve C_ϕ defined by a model ϕ with integer coefficients so that we can make use of reduction modulo prime numbers. Note that $\Delta_\phi \in \mathbb{Z}$ by Lemma 2.2. Since the map φ_n sends non-singular curves to non-singular curves over characteristics not 2 and 3, the constant c is of the form $\pm 2^a 3^b$ for some integers a, b. To compute the powers of 2 and 3, we need to compare Δ_ϕ and $\Delta_{\varphi_n(\phi)}$ over $\mathbb{Z}_2$ and $\mathbb{Z}_3$ (see Definition/Lemma 2.3). Here $\Delta_{\varphi_n(\phi)} = \Delta_W(\varphi_n(\phi))$ is the discriminant of the Weierstrass form $\varphi_n(\phi)$ of ϕ. Since $\Delta_\phi = c\Delta_{\varphi_n(\phi)}$, we get the following p-adic valuation identity for a prime number p:

$$v_p(\Delta_\phi) = v_p(c) + v_p(\Delta_{\varphi_n(\phi)}). \tag{17}$$

Suppose that C_ϕ is non-singular over $\mathbb{F}_2$ and $\mathbb{F}_3$, then $v_2(\Delta_\phi) = v_3(\Delta_\phi) = 0$ and we get from (17) that $v_2(c) = -v_2(\Delta_{\varphi_n(\phi)}), v_3(c) = -v_3(\Delta_{\varphi_n(\phi)})$. So to find c, we just need to compute $\Delta_{\varphi_n(\phi)}$ in some special case in which C_ϕ is non-singular over $\mathbb{F}_2$ and $\mathbb{F}_3$.

In the case $n = 2$, we consider the genus one model ϕ with equation

$$y^2 + yz^2 = x^4 + x^3 z + x^2 z^2.$$

The corresponding Weierstrass equation of $\varphi_2(\phi)$ is:

$$y^2 = 4x^3 - \frac{1}{3}x - \frac{37}{1728}$$

and $\Delta_{\varphi_2(\phi)} = 101$. The model ϕ is non-singular over both $\mathbb{F}_2$ and $\mathbb{F}_3$. Hence $v_2(c) = 0, v_3(c) = 0$ and thus $a = 0, b = 0$.

If $n = 3$, we consider the curve C_ϕ given by $y^2z + yz^2 - x^3 = 0$ with the corresponding equation of $\varphi_3(\phi) : y^2 = 4x^3 + 1$ and we have $\Delta_{\varphi_3(\phi)} = -2^{12}3^3$. The curve C_ϕ is non-singular over $\mathbb{F}_2$ and thus $v_2(c) = -12$ or $a = -12$. To compute the power of 3, we consider the following curve $y^2z - x^3 - xz^2 = 0$ which is non-singular over $\mathbb{F}_3$ with the corresponding equation of $\varphi_3(\phi) : y^2 = 4x^3 + 4x$ and $\Delta_{\varphi_3(\phi)} = -2^{18}$. Therefore, we have $v_3(c) = 0$ or $b = 0$.

When $n = 4$, we consider the curve C_ϕ given by the following complete intersection of two quadratic forms

$$\begin{cases} x_0x_1 + x_0x_2 + x_2x_3 = 0 \\ x_0x_3 + x_1x_2 + x_1x_3 = 0 \end{cases}$$

and the corresponding Weierstrass equation

$$y^2 = 4x^3 - \frac{1}{2^{10}3}x + \frac{161}{2^{15}3^3}$$

computed by φ_4 with $\Delta_{\varphi_4(\phi)} = -3.5/2^{12}$. It is possible to check that C_ϕ is non-singular over $\mathbb{F}_2$. This implies that $v_2(c) = 12$ and hence $a = 12$. Observe that this curve is singular over $\mathbb{F}_3$ since $(1, 1, 1, 1)$ is a singular point modulo 3. So to compute the power of 3, we need to look at another example. For instance, we consider the following complete intersection which is non-singular over $\mathbb{F}_3$

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 + 3x_3^2 = 0 \\ x_0^2 + 2x_1^2 + 3x_2^2 + 5x_3^2 = 0 \end{cases}$$

with the corresponding equation of $\varphi_4(\phi) : y^2 = 4x^3 - x$ and we obtain in this case that $\Delta_{\varphi_4(\phi)} = 2^{12}$. This means that $v_3(c) = 0$ and thus $b = 0$. This completes the proof of Proposition 3.2. $\square$

4. Invariants and modular forms

We now in this section study all invariants of genus one models. To do this, we first give a brief introduction to the theory of modular forms and the connection to invariants. The goal is to establish Theorem 1.2 by proving Theorem 4.10, which is the main result of this section.

4.1. Weakly holomorphic and geometric modular forms.

A weakly holomorphic modular form F of weight $k \in \mathbb{Z}$ is a holomorphic function on the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$, that is meromorphic at ∞ and satisfies the equation

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k F(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

where

$$SL(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad-bc = 1 \right\}.$$

Denote by $M_k^!(\mathbb{C})$ the space of weakly holomorphic modular forms of weight k and $M^!(\mathbb{C})$ the graded algebra

$$M^!(\mathbb{C}) := \bigoplus_{k\in\mathbb{Z}} M_k^!(\mathbb{C}).$$

F is called holomorphic if it is holomorphic at ∞, i.e., F has a Fourier expansion

$$F(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau}$$

which is absolutely convergent for each $\tau \in \mathbb{H}$. We denote by $M_k(\mathbb{C})$ the space of holomorphic modular forms of weight k and $M(\mathbb{C})$ the graded algebra

$$M(\mathbb{C}) := \bigoplus_{k\geq 0} M_k(\mathbb{C}).$$

One of the most important examples of holomorphic modular forms is the Eisenstein series G_{2k}, which is of weight $2k$, is defined for an integer $k \geq 2$ as

$$G_{2k} = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m+n\tau)^{2k}}. \tag{18}$$

We usually use the following normalized notation of the Eisenstein series

$$E_{2k} := \frac{G_{2k}}{2\zeta(2k)} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{+\infty} \sigma_{2k-1}(n) q^n, \tag{19}$$

where ζ is the Riemann zeta function, B_{2k} are the Bernoulli numbers, σ is the divisor sum function and $q = e^{2\pi i\tau}$.

Next step is to follow [18, p. 9,10] to introduce the notion of geometric modular forms. Here an elliptic curve E over a scheme S is a proper smooth morphism $\pi : E \to S$ whose generic fibers are connected smooth curves of genus one together with a section $e : S \to E$.

Definition 4.1. *A geometric modular form of weight $k \in \mathbb{Z}$ over a scheme S is a rule $\mathcal{F}$ which assigns to every pair $(E/R,\omega)$ of an elliptic curve $\pi : E \to R$ over S and a basis ω of $\pi_*\Omega^1_{E/R}$ an element $\mathcal{F}(E/R,\omega) \in R$ such that*

a) $\mathcal{F}(E/R,\omega)$ depends only on the R-isomorphism class of the pair $(E/R,\omega)$.
b) For any $\lambda \in R^*$ we have $\mathcal{F}(E,\lambda\omega) = \lambda^{-k}\mathcal{F}(E,\omega)$.
c) $\mathcal{F}(E'/R',\omega_{R'}) = \psi(\mathcal{F}(E/R,\omega))$ for any morphism $\psi : R \to R'$, i.e., $\mathcal{F}$ commutes with arbitrary base change. Here $(E'/R',\omega'_R)$ is the base change of $(E/R,\omega)$ along ψ.

We adopt the same definition if we only assume that E/R is a smooth genus one curve over R by the following lemma, which is surely known to the experts but the author was unable to locate it in the literature.

Lemma 4.2. *There is a natural correspondence between:*

- *Geometric modular forms of weight k for elliptic curves over a scheme S.*
- *Geometric modular forms of weight k for smooth genus one curves over a scheme S.*

Proof. Suppose first that we are given a geometric modular form for curves of genus one, $\mathcal{F}$. An elliptic curve E/R is a pair (C, e) where C is a smooth genus one curve C over R and a section of $e : \mathrm{Spec}(R) \to C$. We can simply forget the section and set $\mathcal{F}(E/R, \omega) := \mathcal{F}(C/R, \omega)$. This will satisfy all the properties in the definition because $\mathcal{F}(C/R, \omega)$ does.

We have to show the converse, and suppose we are given a geometric modular form $\mathcal{F}$ for elliptic curves. Locally for the étale topology (see [16, 17.16.3 (ii)]), there are étale ring extensions $\psi_i : R \to R_i$ such that $C_i := C \otimes_R R_i$ admits a section, e, over R_i, giving an elliptic curve E_i/R_i. We argue that $\mathcal{F}(E_i/R_i, \omega_i)$ is independent of the choice of section. One can compare e with another choice of section $P_i : \mathrm{Spec}(R_i) \to C_i$ by the R_i-isomorphism given by translation by P_i, $\tau_{P_i} : (C_i, e_i) \simeq (C_i, P_i)$. This is an isomorphism of elliptic curves. Since the differential ω_i is invariant under τ_{P_i}, the property a) in Definition 4.1 shows that $\mathcal{F}((C_i, e), \omega_i) = \mathcal{F}((C_i, P_i), \omega_i)$.

We then set $\alpha_i := \mathcal{F}(E_i/R_i, \omega_i) \in R_i$, independent of any choice of section. Let R_j be another (local) ring extension such that C_j/R_j admits a section, and denote by $\psi_{ij} : R_i \to R_{ij} = R_i \otimes_R R_j$ the natural map. Then $\psi_{ji}(\alpha_j) = \alpha_{ij} = \psi_{ij}(\alpha_i)$ by the property c) of Definition 4.1. By étale descent we obtain an element $\alpha \in R$ such that for all the maps $\psi_i : R \to R_i$, $\psi_i(\alpha) = \alpha_i$. It is a straightforward verification that $\mathcal{F}(C/R, \omega) = \alpha$ satisfies the definition of a geometric modular form of a genus one curve. $\square$

Denote by $\mathcal{M}^!_k(R)$ the R-module of geometric modular forms of weight k over a ring R and $\mathcal{M}^!(R)$ the graded algebra

$$\mathcal{M}^!(R) := \bigoplus_{k \in \mathbb{Z}} \mathcal{M}^!_k(R).$$

As in [18, p. 10], a geometric modular form $\mathcal{F}$ has its q-expansion as an element of $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} R$ obtained by evaluating on the pair $(\mathrm{Tate}(q), \omega)_R$ consisting of the Tate curve and its canonical differential. $\mathcal{F}$ is called holomorphic if it is holomorphic at ∞, i.e., its q-expansion lies in $\mathbb{Z}[[q]] \otimes_{\mathbb{Z}} R$.

We denote by $\mathcal{M}_k(R)$ the R-module of holomorphic geometric modular forms of weight k over a ring R and $\mathcal{M}(R)$ the graded algebra

$$\mathcal{M}(R) := \bigoplus_{k\geq 0} \mathcal{M}_k(R).$$

We have in addition the relation

$$\mathcal{M}^!(R) = \mathcal{M}(R)[\mathcal{D}^{-1}], \tag{20}$$

where $\mathcal{D}$ is the cusp form of weight 12 defined below.

It turns out that we can identify weakly holomorphic and geometric modular forms over $\mathbb{C}$ from the following discussion in [18, p. 91]. Let $C_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for any $\tau \in \mathbb{H}$. For any geometric modular form $\mathcal{F} \in \mathcal{M}^!_k(\mathbb{C})$, we can define the corresponding weakly holomorphic modular form of the same weight

$$F(\tau) = \mathcal{F}(C_\tau, 2\pi i\ dz) \tag{21}$$

with dz being the canonical differential on $\mathbb{C}$. Then the map $\mathcal{F} \mapsto F$ is an isomorphism $\mathcal{M}^!_k(\mathbb{C}) \cong M^!_k(\mathbb{C})$.

Observe that for $k = 2, 3$ the quotient $4k/B_{2k} \in \mathbb{Z}$ and thus the Eisenstein series E_4, E_6 in (19) are defined over $\mathbb{Z}$. By the q-expansion principle as in [18, Corollary 1.9.1], the corresponding geometric modular forms $\mathcal{E}_{2k}$ of E_{2k} of weight $2k$ ($k = 2, 3$) are also defined over $\mathbb{Z}$.

It is known that the ring of holomorphic geometric modular forms $\mathcal{M}(\mathbb{Z})$ over $\mathbb{Z}$ is generated by $\mathcal{E}_4, \mathcal{E}_6$ and the cusp form $\mathcal{D}$ satisfying $1728\mathcal{D} = \mathcal{E}_4^3 - \mathcal{E}_6^2$. More precisely, we have (see [9, Proposition 6.1])

$$\mathcal{M}(\mathbb{Z}) \cong \mathbb{Z}[\mathcal{E}_4, \mathcal{E}_6, \mathcal{D}]/(\mathcal{E}_4^3 - \mathcal{E}_6^2 - 1728\mathcal{D}).$$

If $\mathrm{char}(K) \neq 2$ or 3, then 1728 is invertible over K and thus $\mathcal{M}(K) = K[\mathcal{E}_4, \mathcal{E}_6]$. In this case, $\mathcal{E}_4$ and $\mathcal{E}_6$ are algebraically independent since so are E_4 and E_6.

We have a type of geometric modular forms in any characteristic p called the Hasse invariants defined for instance in [18] (p. 29). For any prime number p, the Hasse invariant $\mathcal{A}_p$ is a geometric modular form over $\mathbb{F}_p$ of weight $p - 1$, which satisfy a certain property. The Hasse invariant $\mathcal{A}_p$ has q-expansion equal to 1 in $\mathbb{F}_p[[q]]$. For any prime $p > 3$, we have $\mathcal{A}_p = \mathcal{E}_{p-1} \pmod p$ since they are both geometric modular forms of the same weight $p - 1$ with the same q-expansions (see [18] (p. 30)).

The structure of the graded ring of holomorphic geometric modular forms can be summarized from Propositions 6.1, 6.2, Remark 6.3 and the formula (8.4) in [9] as follows:

Proposition 4.3. *The graded ring $\mathcal{M}(K)$ of holomorphic geometric modular forms over a field K is*

$$\begin{cases} K[\mathcal{E}_4, \mathcal{E}_6], & \text{if } char(K) \neq 2, 3; \\ K[\mathcal{A}_2, \mathcal{D}], \mathcal{E}_4 = \mathcal{A}_2^4 \text{ and } \mathcal{E}_6 = -\mathcal{A}_2^6, & \text{if } char(K) = 2; \\ K[\mathcal{A}_3, \mathcal{D}], \mathcal{E}_4 = \mathcal{A}_3^2 \text{ and } \mathcal{E}_6 = -\mathcal{A}_3^3, & \text{if } char(K) = 3. \end{cases}$$

4.2. **Invariants and modular forms.** Similarly, the structure of the graded ring of invariants of X_n can be summarized from [12, Theorem 4.4, Lemma 10.1, Theorem 10.2] as below. Here c_4, c_6 are the usual invariants defined in Section 2 and a_1, b_2 are the invariants of weight 1,2 respectively defined as in [12, Theorem 10.2].

Proposition 4.4. *The ring of invariants $K[X_n]^{G_n}$ of X_n $(n \leq 5)$ over a field K is*

$$\begin{cases} K[c_4, c_6], & \text{if } char(K) \neq 2, 3; \\ K[a_1, \Delta], & \text{if } char(K) = 2; \\ K[b_2, \Delta], & \text{if } char(K) = 3. \end{cases}$$

The algebraic independence of the invariants c_4 and c_6 if $\text{char}(K) \neq 2$ or 3, a_1 and Δ if $\text{char}(K) = 2$, b_2 and Δ if $\text{char}(K) = 3$ is clear in case $n = 1$. Thus they are algebraically independent for all $n \leq 5$.

We can link modular forms to invariants. To see this, we first recall a result from [12, Proposition 5.19]. Here the regular 1-form ω_ϕ of a model $\phi \in X_n^0$ is defined as in Section 2.

Lemma 4.5. *Let $C_\phi, C_{\phi'}$ be smooth curves of genus one over a field K corresponding to the models $\phi, \phi' \in X_n^0$ respectively $(n \leq 4)$. Suppose $\phi' = g\phi$ for some $g \in \mathcal{G}_n$, then the isomorphism $\varphi : C_{\phi'} \to C_\phi$ determined by g satisfies $\varphi^*\omega_\phi = (\det g)\omega_{\phi'}$. This statement holds to the case $n = 5$ providing that char$(K) \neq 2$.*

The author in [12] provides an explicit proof for $n \leq 5$ corresponding to the explicit linear algebraic groups $\mathcal{G}_n$ acting on X_n. We observe from this lemma that for any $g \in \mathcal{G}_n$ and any genus one curves $C_\phi, C_{\phi'}$ defined by models $\phi, \phi' = g\phi$ in X_n^0, we have for a geometric modular form $\mathcal{F}$ of weight k

$$\mathcal{F}(C_\phi, \omega_\phi) = \mathcal{F}(C_{\phi'}, \varphi^*\omega_\phi) = \mathcal{F}(C_{\phi'}, (\det g)\omega_{\phi'}) = (\det g)^{-k}\mathcal{F}(C_{\phi'}, \omega_{\phi'})$$

or

$$\mathcal{F}(C_{\phi'}, \omega_{\phi'}) = (\det g)^k \mathcal{F}(C_\phi, \omega_\phi).$$

We have thus proved

Proposition 4.6. *For $n \leq 4$, a geometric modular form $\mathcal{F}$ over a field K defines an invariant of the same weight $I_\mathcal{F}$ over K of X_n^0 such that $I_\mathcal{F}(\phi) = \mathcal{F}(C_\phi, \omega_\phi)$ for any $\phi \in X_n^0$. This statement holds to the case $n = 5$ providing that char$(K) \neq 2$.*

This fact over the complex numbers can also be seen directly from holomorphic modular forms. For a smooth curve C given in the Weierstrass form ϕ such that $C(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$. Then ϕ is of the form $y^2 - 4x^3 + g_2 x + g_3$, whose the two coefficients g_2, g_3 is written as

$$g_2 = 60G_4, g_3 = 140G_6 \tag{22}$$

with G_4, G_6 defined in (18). The normalized invariants of the model ϕ are (see Remark 1.3)

$$c_4(\phi) = 2^6 3 g_2 \ , \ c_6(\phi) = 2^9 3^3 g_3. \tag{23}$$

The following identities are then a consequence of (19), (22), (23) and the special values of zeta function $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$

$$c_4(\phi) = (4\pi)^4 E_4(\tau),\ c_6(\phi) = (4\pi)^6 E_6(\tau). \tag{24}$$

It is then possible to compare the invariant $I_{\mathcal{F}}$ in case $\mathcal{F} = \mathcal{E}_4$ and $\mathcal{E}_6$ with the normalized ones c_4, c_6 as follows

Lemma 4.7. *We have the following identities for any smooth model $\phi \in X_n^0$ $(n \leq 4)$ over a field K*

$$I_{\mathcal{E}_4}(\phi) = c_4(\phi), I_{\mathcal{E}_6}(\phi) = -c_6(\phi).$$

This statement holds to the case $n = 5$ providing that char$(K) \neq 2$.

Proof. Since both $I_{\mathcal{E}_{2k}}(\phi)$ and $c_{2k}(\phi)$ $(k = 2, 3)$ depend only on the K-isomorphism class of the pair (C_ϕ, ω_ϕ) (see Definition 4.1 and [12, Definition 2.1 and Proposition 5.23]), it is enough to consider the case $n = 1$. We know that both $I_{\mathcal{E}_4}$ and c_4 are defined over $\mathbb{Z}$. Moreover, c_4 is a primitive polynomial. There exists thus a constant $\alpha \in \mathbb{Z}$ such that $I_{\mathcal{E}_4}(\phi) = \alpha c_4(\phi)$ for any $\phi \in X_1^0$. Consider a model ϕ with coefficients in $\mathbb{Z} \subset \mathbb{C}$, we obtain from (21) and (24) that $I_{\mathcal{E}_4}(\phi) = c_4(\phi)$ and hence $\alpha = 1$. Here we are using the regular 1-form $\omega_\phi = dx/2y$ while the differential dz in (21) is equal to dx/y. Similarly, we have $I_{\mathcal{E}_6}(\phi) = -c_6(\phi)$ for any $\phi \in X_1^0$. $\square$

One can look closer at the connection between invariants and geometric modular forms. The author in [9, Propositions 6.1, 6.2 and Remark 6.3] proved that $\mathcal{M}(K)$ is isomorphic to $K[X_1]^{G_1}$ over any field K. We want to get a similar phenomena when replacing X_1 by X_n for any $n \leq 5$. Denote by $K[X_n^0]^{G_n}$ the ring of invariants of X_n^0 defined in the same way as in Definition 2.1, i.e.,

$$K[X_n^0]^{G_n} := \{I \in K[X_n^0] : I \circ g = I \text{ for all } g \in G_n(\overline{K})\}.$$

Since $K[X_n^0] = K[X_n][\Delta^{-1}]$, we conclude that

$$K[X_n^0]^{G_n} = (K[X_n][\Delta^{-1}])^{G_n} = K[X_n]^{G_n}[\Delta^{-1}],$$

where the latter identity holds since Δ is an invariant in $K[X_n]^{G_n}$. The structure of $K[X_n^0]^{G_n}$ is deduced from Proposition 4.4 with notations from there as follows.

Proposition 4.8. *The ring of invariants $K[X_n^0]^{G_n}$ of X_n^0 $(n \leq 5)$ over a field K is*

$$\begin{cases} K[c_4, c_6][\Delta^{-1}], & \text{if } char(K) \neq 2, 3; \\ K[a_1, \Delta][\Delta^{-1}], & \text{if } char(K) = 2; \\ K[b_2, \Delta][\Delta^{-1}], & \text{if } char(K) = 3. \end{cases}$$

Proposition 4.6 yields a ring homomorphism $I : \mathcal{M}^!(K) \to K[X_n^0]^{G_n}$ defined by $\mathcal{F} \mapsto I_{\mathcal{F}}$. There are, however, even more to this.

Theorem 4.9. *Let K be a field, the map $I : \mathcal{M}^!(K) \to K[X_n^0]^{G_n}$ $(n \leq 4)$ is an isomorphism. The statement also holds to the case $n = 5$ if $char(K) \neq 2$.*

Proof. When $\mathrm{char}(K) \neq 2$ or 3, we know from Lemma 4.7 that on X_n^0: $I_{\mathcal{E}_4} = c_4, I_{\mathcal{E}_6} = -c_6$ and $I_{\mathcal{D}} = \Delta$. Thus the ring homomorphism I is bijective from Propositions 4.3, 4.8, formula (20) and the algebraic independence of $\mathcal{E}_4$ and $\mathcal{E}_6$, c_4 and c_6. Hence I is an isomorphism.

In the case $\mathrm{char}(K) = 2$, I sends $\mathcal{A}_2$ to αa_1 for some constant $\alpha \in \mathbb{F}_2^*$, i.e., $\alpha = 1$. This comes from the fact that a_1 is (up to constants) the only invariant of weight one. Moreover, I sends $\mathcal{D}$ to Δ by Lemma 4.7 and is thus an isomorphism from the algebraic independence of $\mathcal{A}_2$ and $\mathcal{D}$, a_1 and Δ. The independence of $\mathcal{A}_2$ and $\mathcal{D}$ is deduced from the independence of a_1 and Δ since $I_{\mathcal{A}_2} = \alpha a_1$ and $I_{\mathcal{D}} = \Delta$ on X_n^0. The case $\mathrm{char}(K) = 3$ is treated similarly with the identities $I_{\mathcal{A}_3} = \beta b_2$ and $I_{\mathcal{D}} = \Delta$ on X_n^0. Here β is some constant in $\mathbb{F}_3^*$. $\square$

Back to our purpose, the key result in this section (which is Theorem 1.2 in Section 1) is the following:

Theorem 4.10. *Let $\mathcal{F}$ be a geometric modular form of weight k, there exists a constant $\alpha_n \in K^*$ $(n \leq 4)$ such that*

$$I_{\mathcal{F}}(\phi) = \alpha_n^k I_{\mathcal{F}}(\varphi_n(\phi))$$

for any smooth genus one model (C_ϕ, ω_ϕ) of degree n over a field K with the Jacobian $(E_\phi, \omega_{\varphi_n(\phi)})$ constructed by the map φ_n. Moreover, $\alpha_2 = \pm 1, \alpha_3 = \pm 1/2, \alpha_4 = \pm 2$.

From Proposition 4.8 we know that the ring of invariants of X_n^0 is generated by elements of even weights except in the case of characteristic 2 in which $1 = -1$. Therefore, we can in any case forget the about the sign of α_n and this gives a proof for Theorem 1.2.

Proof. We consider the map $\varphi_n : X_n \to W$ defined in Section 3, there exists $\alpha_n = \alpha_n(\phi) \in K^*$ depending on φ_n and ϕ such that $\varphi_n^*(\omega_{\varphi_n(\phi)}) = \alpha_n \omega_\phi$ for any $\phi \in X_n^0$. We have for any $\phi \in X_n^0$

$$I_{\mathcal{F}}(\varphi_n(\phi)) = \mathcal{F}(E_\phi, \omega_{\varphi_n(\phi)}) = \mathcal{F}(C_\phi, \varphi_n^* \omega_{\varphi_n(\phi)})$$
$$= \mathcal{F}(C_\phi, \alpha_n \omega_\phi) = \alpha_n^{-k} \mathcal{F}(C_\phi, \omega_\phi) = \alpha_n^{-k} I_{\mathcal{F}}(\phi)$$

and hence

$$I_{\mathcal{F}}(\phi) = \alpha_n^k I_{\mathcal{F}}(\varphi_n(\phi)). \tag{25}$$

Since α_n does not depend on k, we can consider the case in which $k = 12$ and $\mathcal{F} = \mathcal{D}$ to get the identities $\Delta_\phi = I_{\mathcal{D}}(\phi)$, $\Delta_{\varphi_n(\phi)} = I_{\mathcal{D}}(\varphi_n(\phi))$ from Lemma 4.7 for the corresponding geometric modular form $\mathcal{D} = (\mathcal{E}_4^3 - \mathcal{E}_6^2)/1728$. This deduces

$$\Delta_\phi = \alpha_n^{12} \Delta_{\varphi_n(\phi)}.$$

As proved in Proposition 3.2, $\Delta_\phi = c\Delta_{\varphi_n(\phi)}$ for some constant c when $n \leq 4$. More precisely,

$$\begin{cases} c = \pm 1, & \text{If } n = 2; \\ c = \pm 2^{-12}, & \text{If } n = 3; \\ c = \pm 2^{12}, & \text{If } n = 4. \end{cases}$$

We now need to compute α_n from $\alpha_n^{12} = c$. Look again at (25) to the case $\mathcal{F} = \mathcal{E}_4$ and $\mathcal{E}_6$, we have $c_4(\phi) = \alpha_n^4 c_4(\varphi_n(\phi))$ and $c_6(\phi) = \alpha_n^6 c_6(\varphi_n(\phi))$ for any $\phi \in X_n^0$. Consider a model ϕ with integer coefficients, we conclude from Lemma 2.2 that $\alpha_n^4, \alpha_n^6 \in \mathbb{Q}$ and thus $\alpha_n^2 \in \mathbb{Q}$. This enables us to exclude the minus sign of the constant c above and deduce the discussion at the end of Section 3. We have in addition

$$\alpha_2^2 = 1, \alpha_3^2 = 1/4 \text{ and } \alpha_4^2 = 4$$

or

$$\alpha_2 = \pm 1, \alpha_3 = \pm 1/2 \text{ and } \alpha_4 = \pm 2.$$

□

Observe that the formulae of c_4, c_6 obtained from Theorem 1.2 for $k = 4, 6$ respectively in cases $n = 2, 3, 4$ are the same with the normalized ones given by Fisher as in (13), (14) and (15).

5. Determinantal representations of Weierstrass cubics

We will in this section study discriminants of smooth curves in Weierstrass form and provide a proof to Theorem 1.5. Consider a smooth curve C_ϕ given by

$$\phi(x, y, z) = y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3, \tag{26}$$

where g_2 and g_3 are elements in a field K. We want to find the 3×3 square matrices L, M, N such that

$$\det(xL + yM + zN) = \phi(x, y, z).$$

We obtain from [21, Section 2] the following determinantal representations of ϕ

$$\begin{pmatrix} 2x + tz & y + dz & (3t^2 - g_2)z \\ 0 & x - tz & y - dz \\ z & 0 & -2x - tz \end{pmatrix}, \tag{27}$$

with $t, d \in \overline{K}$ be such that $d^2 = 4t^3 - g_2 t - g_3$. It can be checked that the determinant of (27) is equal to ϕ.

Now we move to the theory of theta functions to study the case when $K = \mathbb{C}$. The following discussion bases on Wang and Guo [22]. In this case, there exists a unique lattice Λ coming from the Weierstrass parametrization such that $C_\phi(\mathbb{C}) \cong \mathbb{C}/\Lambda$. Here $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ for some $\omega_1, \omega_2 \in \mathbb{C}$ with $\tau = \omega_2/\omega_1 \in \mathbb{H}$. The two coefficients g_2 and g_3 of the curve given by ϕ can be determined by (see [22, p. 509])

$$g_2 = \frac{2}{3}\left(\frac{\pi}{\omega_1}\right)^4 (a^8 + b^8 + c^8),$$

$$g_3 = \frac{4}{27}\left(\frac{\pi}{\omega_1}\right)^6 (a^4 + b^4)(b^4 + c^4)(c^4 - a^4),$$

where $a = \theta_2(0, \tau) = e^{\frac{\pi i \tau}{4}}\theta(\frac{1}{2}\tau, \tau)$, $b = \theta_3(0, \tau) = \theta(0, \tau)$ and $c = \theta_4(0, \tau) = \theta(\frac{1}{2}, \tau)$ with the even Jacobi theta functions:

$$\theta(z, \tau) = \theta_3(z, \tau) := \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z),$$

$$\theta_2(z, \tau) = \exp(\pi i \tau/4 + \pi i z)\theta(z + \tau/2, \tau),$$

$$\theta_4(z, \tau) = \theta(z + \tau/2, \tau).$$

The above a, b, c are called even theta constants.

Since (t, d) is a point on the affine curve associated to C_ϕ defined by $\{z \neq 0\}$, it is determined by theta constants via Weierstrass $\mathcal{P}$-function and so are all the coefficients in the linear matrix (27). To be precise, we consider the Weierstrass $\mathcal{P}$-function associated to the lattice Λ defined for all $s \notin \Lambda$ as

$$\mathcal{P}(s) = \mathcal{P}(s; \omega_1, \omega_2) := \frac{1}{s^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \left(\frac{1}{(s + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2}\right).$$

As in [22, p. 469], it satisfies the differential equation

$$\mathcal{P}'(s)^2 = 4\mathcal{P}(s)^3 - g_2\mathcal{P}(s) - g_3.$$

We can parametrize the point (t, d) on the curve as $t = \mathcal{P}(s)$ and $d = \mathcal{P}'(s)$ for some $s \notin \Lambda$. It is known that the discriminant of the cubic (26) is given by the formula (see Remark 1.3)

$$\Delta_\phi = 2^{12}(g_2^3 - 27g_3^2) = 2^{16}\left(\frac{\pi}{\omega_1}\right)^{12}(abc)^8. \tag{28}$$

We will give another proof for the formula (28) using resultant and the determinantal representation (27). From [15, p. 434], the discriminant of a homogeneous cubic polynomial $\phi(x, y, z)$ can be computed by resultant defined there as

$$\Delta_\phi = -\mathrm{Res}(\phi_x, \phi_y, \phi_z)/27. \tag{29}$$

The reader can have a look at [15, Chapter 13] for a general discussion about resultants. We choose the minus sign here so that the sign of the discriminant is compatible to other sections of the paper. To simplify the computation, we choose a special value for the Weierstrass function $\mathcal{P}(s)$, namely, we choose the 2-torsion point $s = \omega_2/2$. In this case $\mathcal{P}(\omega_2/2) = -\frac{\pi^2}{3\omega_1^2}(a^4 + b^4)$ and $\mathcal{P}'(\omega_2/2) = 0$ by [22, p. 470,509]. Then $d = 0$ and $t = -\frac{\pi^2}{3\omega_1^2}(a^4 + b^4)$. Besides, using the Jacobi's identity $a^4 + c^4 = b^4$ (see [22, p. 504]), the matrix (27) above can be written in the form

$$\begin{pmatrix} 2x - \frac{\pi^2}{3\omega_1^2}(a^4 + b^4)z & y & -(\frac{\pi}{\omega_1})^4 c^8 z \\ 0 & x + \frac{\pi^2}{3\omega_1^2}(a^4 + b^4)z & y \\ z & 0 & -2x + \frac{\pi^2}{3\omega_1^2}(a^4 + b^4)z \end{pmatrix}. \tag{30}$$

We have thus proved Theorem 1.5. From the representation $\phi = \det(U)$, where U is given by (30), we get that

$$\phi_x = -12x^2 + \frac{2}{3}\left(\frac{\pi}{\omega_1}\right)^4 (a^8 + b^8 + c^8)z^2,$$

$$\phi_y = 2yz, \text{ and}$$

$$\phi_z = y^2 + \frac{4}{3}\left(\frac{\pi}{\omega_1}\right)^4 (a^8 + b^8 + c^8)xz + \frac{4}{9}\left(\frac{\pi}{\omega_1}\right)^6 (a^4 + b^4)(b^4 + c^4)(c^4 - a^4)z^2.$$

The discriminant Δ_ϕ of the cubic ϕ is then obtained via (29)

$$\Delta_\phi = 2^{16}\left(\frac{\pi}{\omega_1}\right)^{12}(abc)^8.$$

In fact, we can directly use (29) to the curve (26). But this approach of determinantal representations might be applied to more general cases. We will explain in more detail in Section 7.

6. Determinantal representations of complex plane curves

In Section 5, we have already seen that one can compute the discriminant of smooth curves over $\mathbb{C}$ in Weierstrass form by using determinantal representations. Our goal is to generalize to plane curves of arbitrary degrees based on Theorem 5.1 in [3]. In other words, we will in this section prove Theorem 6.3. Let us first introduce some notations.

Let X be a compact Riemann surface, let $\mathcal{L}$ is a line bundle of half differentials on X (a theta characteristic), i.e., $\mathcal{L}^{\otimes 2}$ is the canonical bundle ω_X on X and let χ be a flat line bundle over X such that $h^0(\chi \otimes \mathcal{L}) = 0$. We associate to χ the Cauchy kernel $K(\chi; \cdot, \cdot)$ as defined in Section 2 of [3]. Let λ_1, λ_2 be two scalar meromorphic functions on X, which generate the whole field of meromorphic functions. Assume that all poles of λ_1, λ_2 are simple and labeled as $P_1, ..., P_d \in X$. We write the Laurent expansion of λ_k at P_i $(1 \leq i \leq d, k = 1, 2)$ with some fixed local coordinate $t_i = t_i(P)$ centered at $P = P_i$

$$\lambda_k(P) = -\frac{c_{ik}}{t_i} - d_{ik} + O(|t_i|).$$

Then we define the $d \times d$ matrices L, M, N by

$$L = \mathrm{diag}_{1 \leq i \leq m}(c_{i2}), \;\; M = \mathrm{diag}_{1 \leq i \leq m}(-c_{i1}), \;\; N = (n_{ij})_{i,j},$$

where

$$n_{ij} = \begin{cases} d_{i1}c_{i2} - d_{i2}c_{i1}, & i = j; \\ (c_{i1}c_{j2} - c_{j1}c_{i2})\dfrac{K(\chi; P_i, P_j)}{dt_j(P_j)}, & i \neq j. \end{cases}$$

The result mentioned in [3] is the following

Proposition 6.1. *The map $\pi_0 : X \to \mathbb{C}^2$ given by $\pi_0(P) = (\lambda_1(P), \lambda_2(P))$ maps $X \backslash \{P_1, ..., P_d\}$ onto the affine part C^0 of an algebraic curve $C \subset \mathbb{P}^2$ and extends to a proper birational map $\pi : X \to C$ of X in $\mathbb{P}^2$. The defining irreducible homogeneous polynomial $\phi(x, y, z)$ of C is such that (up to multiplying by some constant)*

$$\phi(x, y, z) = \det(xL + yM + zN).$$

Here the affine part C^0 of C is defined by $\{z \neq 0\}$.

The authors in [3] prove a more general version of the above proposition where they consider χ to be any flat vector bundle. We restrict here to the case of line bundle since it is enough for our purpose.

Suppose in this case that χ is defined by a unitary representation of the fundamental group of X given by

$$\chi(\alpha_i) = \exp(-2\pi i a_i) \text{ and } \chi(\beta_i) = \exp(2\pi i b_i), \;\; i = 1, ..., g,$$

where $a_i, b_i \in \mathbb{R}$, g is the genus of X and $\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g$ form a symplectic basis of $H_1(X, \mathbb{Z})$. Let $(\eta_1, ..., \eta_g)$ be a basis of holomorphic 1-forms on X, we form the period matrix with respect to these bases which is the $g \times 2g$-matrix $(\Omega_1 \mid \Omega_2)$ whose entries are

$$(\Omega_1)_{ij} = \int_{\alpha_j} \eta_i \text{ and } (\Omega_2)_{ij} = \int_{\beta_j} \eta_i, \text{ for } i,j = 1,...,g.$$

We choose the canonical basis $(\eta_1, ..., \eta_g)$ of holomorphic 1-forms in the sense that $\int_{\alpha_i} \eta_j = \delta_{ij}$, then the corresponding period matrix will be of the form $(I_g \mid \Omega)$. The matrix Ω lies in the Siegel upper half space $\mathbb{H}^g$ and it is called the Riemann period matrix of X with respect to the homology basis $\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g$. We fix such a symplectic homology basis and the resulting period matrix Ω. Let $J(X) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ be the Jacobian of X and $\varphi : X \to J(X)$ be the Abel-Jacobi map with any fixed base point. Then we have an explicit formula for the Cauchy kernel (see [3, Theorem 4.1]).

$$K(\chi; P, Q) = \frac{\theta[\delta](\varphi(Q) - \varphi(P))}{\theta[\delta](0)E(Q, P)},$$

where $\theta[\delta]$ is the associated theta function with characteristic $\delta = b + \Omega a = \varphi(\chi)$ ($a = (a_j)_j$ and $b = (b_j)_j$) and $E(\cdot, \cdot)$ is the prime form on $X \times X$. Recall from [11, Chapter II] that the prime form E is a bi-half-differential with simple poles along the diagonal of $X \times X$.

Here the theta characteristic $\mathcal{L}$ is chosen such that $\varphi(\mathcal{L}) = -\mathcal{K}$, where $\mathcal{K}$ is the vector of Riemann constants. Note that a consequence of the Riemann singularity theorem is that $\theta(b + \Omega a) \neq 0$ if and only if $h^0(\chi \otimes \mathcal{L}) = 0$. Hence $\theta[\delta](0) \neq 0$ and the formula above makes sense.

From this proposition, one can provide explicitly determinantal representations for complex plane curves using theta functions and the Abel-Jacobi map. The reader can have a look at [17, Section 4], [20, Theorem 6] or [6, Theorem 2.2] for reference. Note that results in the reference above only apply to the family of hyperbolic curves with a normalization, but it can be written in the following general form.

Theorem 6.2. *Let $C_\phi \subset \mathbb{P}^2$ be a non-rational irreducible complex plane curve defined by $\phi = 0$, where $\phi(x, y, z)$ is an irreducible homogeneous polynomial of degree d. Suppose the d intersection points of C_ϕ with the line $\{y = 0\}$ are distinct non-singular points $P_1, ..., P_d$ with coordinates $P_i = (1, 0, \beta_i)$, $\beta_i \neq 0$. Then*

$$\phi(x, y, z) = \lambda \det(xM + yN + zI),$$

where $\lambda = \phi(0, 0, 1)$, $M = \mathrm{diag}(-\beta_1, ..., -\beta_d)$ and $N = (n_{ij})_{i,j}$ with

$$n_{ii} = -\beta_i \frac{\phi_y(1, 0, \beta_i)}{\phi_x(1, 0, \beta_i)}$$

and for $i \neq j$

$$n_{ij} = \frac{\beta_i - \beta_j}{\theta[\delta](0)} . \frac{\theta[\delta](\varphi(P_j) - \varphi(P_i))}{E(P_j, P_i)} . \frac{1}{\sqrt{d(-y/x)(P_i)}\sqrt{d(-y/x)(P_j)}}.$$

Here δ is an even theta characteristic such that $\theta[\delta](0) \neq 0$, $\varphi : X \to J(X)$ is the Abel-Jacobi map from the desingularizing Riemann surface X of C_ϕ to its Jacobian and $E(.,.)$ is the prime form on $X \times X$.

We want to generalize Theorem 6.2 such that we replace the line $\{y = 0\}$ by a general line passing through distinct points of C_ϕ. Let l be a line defined by $\alpha x + \beta y + \gamma z = 0$ so that its affine part l^0 defined by $\alpha x + \beta y + \gamma = 0$ intersects the affine part C^0_ϕ of C_ϕ at d distinct non-singular points P^0_i, $i = 1, ..., d$. Since α and β can not be both zero, we can suppose w.l.o.g that $\beta \neq 0$ (the case $\alpha \neq 0$ can be treated similarly). In this case, we can suppose further that $\beta = -1$. Therefore, the line l can be rewritten as $y = \alpha x + \gamma z$. Assume that the intersections points P^0_i of l^0 and C^0_ϕ have non-zero x-coordinates so that we can write $P^0_i = (1/\beta_i, \alpha/\beta_i + \gamma)$ with $\beta_i \neq \beta_j$ if $i \neq j$. Thus the intersection points of l and C_ϕ are $P_i = (1, \alpha + \gamma\beta_i, \beta_i)$. We now prove the following

Theorem 6.3. *Let $C_\phi \subset \mathbb{P}^2$ be a non-rational irreducible complex plane curve defined by $\phi = 0$, where $\phi(x, y, z)$ is an irreducible homogeneous polynomial of degree d. Suppose the d intersection points of C_ϕ with the line $\{y = \alpha x + \gamma z\}$ are distinct non-singular points $P_1, ..., P_d$ with coordinates $P_i = (1, \alpha + \gamma\beta_i, \beta_i)$, $\beta_i \neq 0$. Then up to multiplying by some constant*

$$\phi(x, y, z) = \det((M - \alpha N)x + Ny + (I - \gamma N)z),$$

where $M = \mathrm{diag}(-\beta_1, ..., -\beta_d)$ and $N = (n_{ij})_{i,j}$ with

$$n_{ii} = -\frac{\beta_i \phi_y(P_i)}{(\phi_x + \alpha\phi_y)(P_i)}$$

and for $i \neq j$

$$n_{ij} = \frac{\theta[\delta](\varphi(P_j) - \varphi(P_i))}{\theta[\delta](0)E(P_j, P_i)} \frac{\beta_i - \beta_j}{\sqrt{\beta_i(\alpha dx - dy)(P_i)}\sqrt{\beta_j(\alpha dx - dy)(P_j)}}.$$

Here δ is an even theta characteristic such that $\theta[\delta](0) \neq 0$, $\varphi : X \to J(X)$ is the Abel-Jacobi map from the desingularizing Riemann surface X of C_ϕ to its Jacobian and $E(.,.)$ is the prime form on $X \times X$.

Proof. Apply Proposition 6.1 with the pair of meromorphic functions on the desingularizing Riemann surface X of C_ϕ:

$$\lambda_1 = \frac{1}{y - \alpha x - \gamma}, \ \lambda_2 = \frac{x}{y - \alpha x - \gamma}$$

and $t = \frac{\alpha x - y + \gamma}{x}$ be the local coordinates at the poles P_i (zeros of $\alpha x - y + \gamma$). The next step is to write down Laurent expansions of λ_1, λ_2 at P_i. We have

$$\lambda_2 = -1/t \Rightarrow c_{i2} = 1, \ d_{i2} = 0 \ \forall i.$$

At P_i we have $\lambda_1 = -\frac{1}{t}(\frac{1}{x})$. Since

$$\frac{1}{x} = \beta_i + \frac{d(\frac{1}{x})}{d(\frac{\alpha x - y + \gamma}{x})}(P_i)t + O(|t|^2) = \beta_i + \beta_i \frac{dx}{d(y-\alpha x)}(P_i)t + O(|t|^2),$$

we deduce that

$$c_{i1} = \beta_i,\ d_{i1} = \beta_i \frac{dx}{d(y-\alpha x)}(P_i).$$

We then obtain from Proposition 6.1 that (up to some constant)

$$\phi = \det((M-\alpha N)x + Ny + (I-\gamma N)z)$$

where $M = \operatorname{diag}(-\beta_1, ..., -\beta_d)$ and $N = (n_{ij})$ with

$$n_{ii} = \beta_i \frac{dx}{d(y-\alpha x)}(P_i)$$

and for $i \neq j$

$$n_{ij} = \frac{\theta[\delta](\varphi(P_j) - \varphi(P_i))}{\theta[\delta](0)E(P_j, P_i)} \frac{\beta_i - \beta_j}{d(\frac{\alpha x - y + \gamma}{x})(P_j)}.$$

Here δ is an even theta characteristic with $\theta[\delta](0) \neq 0$. Note that the affine part C_ϕ^0 of C_ϕ is defined by

$$\{(\lambda_1(P), \lambda_2(P)) \mid P \in X \setminus \{P_1, ..., P_d\}\}.$$

Furthermore, if we replace N by the matrix N' which has the same diagonal elements with N but different off-diagonal elements

$$n'_{ij} = \frac{\theta[\delta](\varphi(P_j) - \varphi(P_i))}{\theta[\delta](0)E(P_j, P_i)} \frac{\beta_i - \beta_j}{\sqrt{d(\frac{\alpha x - y + \gamma}{x})(P_i)}\sqrt{d(\frac{\alpha x - y + \gamma}{x})(P_j)}},$$

then the determinantal representation does not change. Indeed, let

$$U = (M-\alpha N)x + Ny + (I-\gamma N)z$$

and

$$U' = (M-\alpha N')x + N'y + (I-\gamma N')z,$$

if we multiply the i^{th}-column of U and the i^{th}-row of U' (for $i = 1, ..., d$) with the term $\sqrt{d(\frac{\alpha x - y + \gamma}{x})(P_i)}$ then both of them will become the same matrix U^*. Consequently,

$$\det(U) = \det(U') = \frac{\det(U^*)}{\prod_{i=1}^d \sqrt{d(\frac{\alpha x - y + \gamma}{x})(P_i)}}.$$

Observe that $d(\frac{\alpha x - y + \gamma}{x})(P_i) = \beta_i(\alpha dx - dy)(P_i)$ and

$$\frac{dx}{d(y-\alpha x)}(P_i) = -\frac{\phi_y(P_i)}{(\phi_x + \alpha\phi_y)(P_i)}$$

by implicit function theorem with the fact that $(\phi_x + \alpha\phi_y)(P_i) \neq 0$. Indeed, since the polynomial $f(x) := \phi(x, \alpha x + \gamma, 1)$ has distinct roots $(1/\beta_i)$ we conclude that $f'(1/\beta_i) \neq 0$ and hence $(\phi_x + \alpha\phi_y)(P_i) \neq 0$. We have thus proved Theorem 6.3. □

Theorem 6.2 is then established by reducing to the case $\alpha = \gamma = 0$. We will apply Theorem 6.3 in the next section to get a formula for the discriminant of plane cubic curves.

Remark 6.4. *We can also reformulate the analogous statement to the Theorem 6.3 if the line $y = \alpha x + \gamma z$ is replaced by $x = \alpha y + \gamma z$.*

7. Discriminant of plane cubic curves

We now study the main object of interest in which we consider a smooth plane curve C_ϕ over $\mathbb{C}$ defined by the cubic form $\phi = 0$. The affine part of the curve C_ϕ is parametrized as

$$\{(x, y, 1) = (R_1(\mathcal{P}(s), \mathcal{P}'(s)), R_2(\mathcal{P}(s), \mathcal{P}'(s)), 1)\},$$

where $\mathcal{P}(s; \omega_1, \omega_2)$ is the Weierstrass $\mathcal{P}$-function associated to some $\omega_1, \omega_2 \in \mathbb{C}$ satisfying $\mathrm{Im}(\omega_2/\omega_1) > 0$.

In this section, we use the standard notation τ of genus one case instead of Ω for the period matrix. Moreover, we use the general Jacobian $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ in place of the normalized one $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for $\tau = \omega_2/\omega_1$ in order to use the properties of the function $\mathcal{P}$. By this change, an extra factor $1/\omega_1$ appears in the below elements n_{ij} $(i \neq j)$ in comparing with Theorem 6.3. This idea was mentioned in [6, Theorem 2.4]. In addition, we use the notation θ_δ $(\delta = 1, 2, 3, 4)$ for theta functions as in Section 5 instead of $\theta[\delta]$.

The prime form $E(P, Q)$ in genus one case is better understood so that we obtain a consequence of Theorem 6.3 as follows.

Corollary 7.1. *Let $C_\phi \subset \mathbb{P}^2$ be a smooth plane cubic curve defined by $\phi = 0$, where $\phi(x, y, z)$ is a non-singular homogeneous cubic polynomial. Suppose the line $y = \alpha x + \gamma z$ intersects C_ϕ at 3 distinct points P_1, P_2, P_3 with coordinates $P_i = (1, \alpha + \gamma\beta_i, \beta_i)$, $\beta_i \neq 0$. Then up to multiplying by some constant*

$$\phi(x, y, z) = \det((M - \alpha N)x + Ny + (I - \gamma N)z),$$

where $M = \mathrm{diag}(-\beta_1, -\beta_2, -\beta_3)$ and $N = (n_{ij})_{i,j}$ with

$$n_{ii} = -\frac{\beta_i \phi_y(P_i)}{(\phi_x + \alpha\phi_y)(P_i)}$$

and for $i \neq j$

$$n_{ij} = \frac{\theta_1'(0)\theta_\delta((Q_j - Q_i)/\omega_1)}{\omega_1\theta_\delta(0)\theta_1((Q_j - Q_i)/\omega_1)} \frac{\beta_i - \beta_j}{\sqrt{\beta_i(\alpha R_1' - R_2')(Q_i)}\sqrt{\beta_j(\alpha R_1' - R_2')(Q_j)}}$$

Here δ is any even theta characteristic, i.e., $\delta = 2, 3$ or 4 and $Q_i = \varphi(P_i)$. Note that we also have an analogous statement of this corollary by Remark 6.4.

The field of meromorphic functions on a general genus one curve is generated by $\mathcal{P}, \mathcal{P}'$ associated to some periods ω_1, ω_2. Thus R_1 and R_2 are rational functions on $\mathcal{P}, \mathcal{P}'$. In general, R_1 and R_2 have complicated expressions. But we have better interpretations in the case of plane cubic curves. In this case, C_ϕ always has a flex point and hence can be transformed to a Weierstrass equation after a linear coordinate change (see [8, Section 4.4]). Thus, we are able to present rational functions R_1, R_2 as:

$$\begin{aligned} R_1(s) &= \lambda_{11}\mathcal{P}(s) + \lambda_{12}\mathcal{P}'(s) + \lambda_{13}, \\ R_2(s) &= \lambda_{21}\mathcal{P}(s) + \lambda_{22}\mathcal{P}'(s) + \lambda_{23}. \end{aligned} \tag{31}$$

The constants $\lambda_{ij} \in \mathbb{C}$ satisfy $\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}$ and depend on the coefficients of ϕ. Here we fix any flex point and the corresponding periods ω_1, ω_2 coming from the Weierstrass parametrization of the Weierstrass equation.

To shorten the determinantal representation, we should look at 2-torsion points to simplify θ and $\mathcal{P}$. More precisely, we consider the line l which intersects C_ϕ at the points P_i such that the corresponding points Q_i on the torus $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ are $\omega_1/2, (\omega_1 + \omega_2)/2$ and $\omega_2/2$ respectively. Suppose that the x-coordinates of P_i are all non-zero. We will treat the case l to have the form $y = \alpha x + \gamma z$ and then make use of Corollary 7.1. The other case can be treated similarly using Remark 6.4. The choice of 2-torsion points gives us the convenience to work with some computations below. Let $a = \theta_2(0, \tau), b = \theta_3(0, \tau), c = \theta_4(0, \tau)$, where $\tau = \omega_2/\omega_1$, we will prove the following

Proposition 7.2. *Let $C_\phi \subset \mathbb{P}^2$ be a smooth plane curve defined by $\phi = 0$, where $\phi(x, y, z)$ is a non-singular homogeneous cubic polynomial. Suppose the line $y = \alpha x + \gamma z$ intersects C_ϕ at 3 distinct points P_1, P_2, P_3 with coordinates $P_i = (1, \alpha + \gamma\beta_i, \beta_i)$, $\beta_i \neq 0$ so that the corresponding points $Q_i = \varphi(P_i)$ of P_i on the torus $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ are $\omega_1/2, (\omega_1 + \omega_2)/2$ and $\omega_2/2$ respectively. Denote by $k = \alpha\lambda_{12} - \lambda_{22}$, then we have the following expressions (up to some constant) for the discriminant Δ_ϕ of ϕ*

(1)

$$\Delta_\phi = \frac{\lambda_{11}^6\omega_1^{24}}{2^8k^{12}\pi^{24}(abc)^{16}}(\beta_1 - \beta_2)^6(\beta_1 - \beta_3)^6(\beta_2 - \beta_3)^6,$$

(2)

$$\Delta_\phi = 16\left(\frac{\lambda_{11}^2\pi\beta_1\beta_2\beta_3}{2k\omega_1}\right)^{12}(abc)^8.$$

Proof. By [22, p. 470, 509], we have $\mathcal{P}'(Q_i) = 0$ for all i and

$$\mathcal{P}(Q_1) = \frac{\pi^2}{3\omega_1^2}(b^4 + c^4),\ \mathcal{P}(Q_2) = \frac{\pi^2}{3\omega_1^2}(a^4 - c^4),\ \mathcal{P}(Q_3) = -\frac{\pi^2}{3\omega_1^2}(a^4 + b^4).$$

Besides, $\mathcal{P}''(s) = 6(\mathcal{P}(s))^2 - g_2/2$ with $g_2 = \frac{2}{3}(\frac{\pi}{\omega_1})^4(a^8+b^8+c^8)$ as in [22, p. 469]. Thus

$$\mathcal{P}''(Q_1) = \frac{2\pi^4 b^4 c^4}{\omega_1^4},\ \mathcal{P}''(Q_2) = -\frac{2\pi^4 a^4 c^4}{\omega_1^4},\ \mathcal{P}''(Q_3) = \frac{2\pi^4 a^4 b^4}{\omega_1^4}.$$

We also have for each i

$$-\frac{\phi_y(P_i)}{(\phi_x + \alpha\phi_y)(P_i)} = \frac{dx}{d(y-\alpha x)}(P_i) = \frac{R_1'}{(R_2' - \alpha R_1')}(Q_i) = \frac{\lambda_{12}}{\lambda_{22} - \alpha\lambda_{12}}.$$

Choosing $\delta = 3$, we now simplify the matrix N in Corollary 7.1. Let $k_1 = -\lambda_{12}/k$ and note that $\theta_1'(0) = \pi abc$ as in [22, p. 507], we have $n_{ii} = k_1\beta_i$ and $n_{13} = n_{31} = 0$ as $\theta((1+\tau)/2) = 0$. Moreover,

$$n_{12}^2 = n_{21}^2 = \frac{\pi^2(abc)^2\theta^2\left(\frac{\tau}{2}\right)(\beta_1-\beta_2)^2}{\omega_1^2 k^2 b^2 \theta_1^2\left(\frac{\tau}{2}\right)\beta_1\beta_2\mathcal{P}''(Q_1)\mathcal{P}''(Q_2)} = \frac{\omega_1^6(\beta_1-\beta_2)^2}{4k^2\pi^6\beta_1\beta_2 b^4 c^8},$$

$$n_{23}^2 = n_{32}^2 = \frac{\pi^2(abc)^2\theta^2\left(\frac{1}{2}\right)(\beta_2-\beta_3)^2}{\omega_1^2 k^2 b^2 \theta_1^2\left(\frac{1}{2}\right)\beta_2\beta_3\mathcal{P}''(Q_2)\mathcal{P}''(Q_3)} = -\frac{\omega_1^6(\beta_2-\beta_3)^2}{4k^2\pi^6\beta_2\beta_3 a^8 b^4}.$$

Here we use the fact that (see [22, p. 502])

$$\theta\left(\frac{\tau}{2}\right) = q^{-\frac{1}{8}}a, \theta_1\left(\frac{\tau}{2}\right) = iq^{-\frac{1}{8}}c, \theta\left(\frac{1}{2}\right) = c, \theta_1\left(\frac{1}{2}\right) = a$$

with $q = e^{2\pi i\tau}$. We have $1/\beta_i = \lambda_{11}\mathcal{P}(Q_i) + \lambda_{13}$ from (31) and the fact $R_1(Q_i) = 1/\beta_i$. Therefore,

$$\begin{aligned}
\frac{\beta_1-\beta_2}{\beta_1\beta_2} &= \lambda_{11}(\mathcal{P}(Q_2) - \mathcal{P}(Q_1)) = -\lambda_{11}\frac{\pi^2 c^4}{\omega_1^2},\\
\frac{\beta_1-\beta_3}{\beta_1\beta_3} &= \lambda_{11}(\mathcal{P}(Q_3) - \mathcal{P}(Q_1)) = -\lambda_{11}\frac{\pi^2 b^4}{\omega_1^2},\\
\frac{\beta_2-\beta_3}{\beta_2\beta_3} &= \lambda_{11}(\mathcal{P}(Q_3) - \mathcal{P}(Q_2)) = -\lambda_{11}\frac{\pi^2 a^4}{\omega_1^2}.
\end{aligned} \tag{32}$$

It can be seen from (32) that $\lambda_{11} \neq 0$. Similarly we have $\lambda_{21} = \alpha\lambda_{11}$ from the identities $R_2(Q_i) = \alpha/\beta_i + \gamma$. Breaking out the determinant, one get the following expression for ϕ (up to some constant λ)

$$\begin{aligned}
&-\beta_1\beta_2\beta_3 x^3 + 3\beta_1\beta_2\beta_3 k_1 x^2 y + (\beta_3 n_{12}^2 + \beta_1 n_{23}^2 - 3\beta_1\beta_2\beta_3 k_1^2)xy^2 +\\
&k_1(\beta_1\beta_2\beta_3 k_1^2 - \beta_3 n_{12}^2 - \beta_1 n_{23}^2)y^3 + (\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)x^2 z - 2k_1(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)xyz\\
&+(k_1^2(\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3) - n_{12}^2 - n_{23}^2)y^2 z - (\beta_1+\beta_2+\beta_3)xz^2 + k_1(\beta_1+\beta_2+\beta_3)yz^2 + z^3.
\end{aligned} \tag{33}$$

Consequently, $\mathrm{Res}(\phi_x/\lambda, \phi_y/\lambda, \phi_z/\lambda) =$

$$(-432)(\beta_2-\beta_3)^2(\beta_1-\beta_2)^2 n_{23}^4 n_{12}^4(\beta_1-\beta_3)^6(\beta_2 n_{12}^2 - \beta_3 n_{12}^2 - \beta_1 n_{23}^2 + \beta_2 n_{23}^2)^2.$$

The term $\beta_2 n_{12}^2 - \beta_3 n_{12}^2 - \beta_1 n_{23}^2 + \beta_2 n_{23}^2$ is equal to

$$\begin{aligned}\frac{(\beta_1-\beta_2)(\beta_2-\beta_3)\omega_1^6}{4k^2\pi^6 b^4}&\left(\frac{\beta_1-\beta_2}{c^8\beta_1\beta_2}+\frac{\beta_2-\beta_3}{a^8\beta_2\beta_3}\right)\\ &= -\frac{\lambda_{11}\omega_1^4(\beta_1-\beta_2)(\beta_2-\beta_3)}{4k^2\pi^4 a^4 c^4},\end{aligned}$$

where the later equality comes from (32). Furthermore,

$$(n_{12}n_{23})^4 = \frac{\omega_1^{24}(\beta_1-\beta_2)^4(\beta_2-\beta_3)^4}{2^8k^8\pi^{24}(abc)^{16}(\beta_1\beta_2)^2(\beta_2\beta_3)^2} = \frac{\lambda_{11}^4\omega_1^{16}(\beta_1-\beta_2)^2(\beta_2-\beta_3)^2}{2^8k^8\pi^{16}a^8b^{16}c^8}.$$

The later equality again comes from (32). Hence

$$\Delta_\phi = -\frac{1}{27}\mathrm{Res}(\phi_x,\phi_y,\phi_z) = \frac{\lambda^{12}\lambda_{11}^6\omega_1^{24}}{2^8k^{12}\pi^{24}(abc)^{16}}(\beta_1-\beta_2)^6(\beta_1-\beta_3)^6(\beta_2-\beta_3)^6.$$

We also obtain an alternative form of the discriminant by using (32):

$$\Delta_\phi = 16\left(\frac{\lambda\lambda_{11}^2\pi\beta_1\beta_2\beta_3}{2k\omega_1}\right)^{12}(abc)^8. \tag{34}$$

This completes the proof of Proposition 7.2. □

We now simplify the formula (34) by looking at the relationships between λ, λ_{11}, k and $\beta_1\beta_2\beta_3$. It can be seen from (33) that $\lambda\beta_1\beta_2\beta_3 = -\phi(1,0,0)$. The transformation (31) means that if we write

$$x = \lambda_{11}X + \lambda_{12}Y + \lambda_{13},$$

$$y = \lambda_{21}X + \lambda_{22}Y + \lambda_{23}$$

then the affine curve $\phi(x,y,1) = 0$ will be transformed to a Weierstrass form $-Y^2 + 4X^3 - g_2X - g_3 = 0$. In addition, the inverse transformation

$$X = l_{11}x + l_{12}y + l_{13},$$

$$Y = l_{21}x + l_{22}y + l_{23}$$

would transform the Weierstrass equation $-Y^2 + 4X^3 - g_2X - g_3 = 0$ to:

$$\begin{gathered}4l_{11}^3x^3 + 12l_{11}^2l_{12}x^2y + 12l_{11}l_{12}^2xy^2 + 4l_{12}^3y^3 + (12l_{11}^2l_{13} - l_{21}^2)x^2 + \\ (24l_{11}l_{12}l_{13} - 2l_{21}l_{22})xy + (12l_{12}^2l_{13} - l_{22}^2)y^2 + (12l_{11}l_{13}^2 - 2l_{21}l_{23} - l_{11}g_2)x + \\ (12l_{12}l_{13}^2 - 2l_{22}l_{23} - l_{12}g_2)y + 4l_{13}^3 - l_{23}^2 - l_{13}g_2 - g_3.\end{gathered} \tag{35}$$

One can check that $l_{11} = \lambda_{22}/D, l_{12} = -\lambda_{12}/D, l_{21} = -\lambda_{21}/D$ and $l_{22} = \lambda_{11}/D$ with $D = \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}$. Compare the coefficients of x^3 and x^2y in (33) and (35), we have

$$\begin{cases} 4l_{11}^3 = -\lambda\beta_1\beta_2\beta_3, \\ 12l_{11}^2l_{12} = 3\lambda\beta_1\beta_2\beta_3k_1. \end{cases} \Leftrightarrow \begin{cases} 4\lambda_{22}^3 = -\lambda\beta_1\beta_2\beta_3D^3, \\ 12\lambda_{22}^2\lambda_{12} = -3\lambda\beta_1\beta_2\beta_3k_1D^3. \end{cases}$$

The second identity shows that $\lambda_{11} = -4\lambda_{22}^2/(\lambda\beta_1\beta_2\beta_3 D^2)$. Hence $\lambda\lambda_{11}^3\beta_1\beta_2\beta_3 = -4$ from the first identity. We have thus proved from (34) the following result

Theorem 7.3. *Let C_ϕ be a smooth plane cubic curve as in Proposition 7.2. Then the discriminant Δ_ϕ of ϕ satisfies*

$$\Delta_\phi = \frac{2^{16}}{(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})^{12}} \left(\frac{\pi}{\omega_1}\right)^{12} (abc)^8.$$

Let us look at the example when ϕ is given in the Weierstrass form $-y^2+4x^3-g_2x-g_3$. In this case, $\lambda_{11} = \lambda_{22} = 1$ and $\lambda_{12} = \lambda_{21} = 0$. We thus recover the classical formula $\Delta_\phi = 2^{16}(\frac{\pi}{\omega_1})^{12}(abc)^8$. Since the discriminant of plane cubics is an invariant of weight 12, the factor $(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})$ in Theorem 7.3 naturally appears as the determinant of the linear transformation (31) which transform a cubic to a Weierstrass form.

From Remark 6.4, we can also treat the other case where the line l passes through 2-torsion points of C_ϕ. Furthermore, the set of cubics ϕ in the above theorem forms an open dense subset of the space of all ternary cubics and we have thus obtained Theorem 1.6.

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