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## A few surprising integrals

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### ABSTRACT

Using formulas for certain quantities involving stable vectors, due to I. Molchanov, and in some cases utilizing the so-called divide and color model we prove that certain families of integrals which, ostensibly, depend on a parameter are in fact independent of this parameter.

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### 1. Statement of result and proof

In the pursuit of some other questions, we realized that the following integrals surprisingly had the same value for all  $\alpha$ .

**Theorem 1.1.** *The following two integrals, the first to be taken in the Cauchy principal value sense as it is not Lebesgue integrable, are independent of  $\alpha > 0$ , the first having value  $\pi^2/6$  and the second having value  $\pi^2/4$ .*

$$\int_0^\pi \frac{\log(|\cos \theta|^\alpha + |\sin \theta|^\alpha + |\cos \theta + \sin \theta|^\alpha)}{\alpha \cos \theta \sin \theta} d\theta,$$

$$\int_0^\pi \frac{\log\left(\frac{|\sin \theta|^\alpha}{2} + |\cos \theta + (\frac{1}{2})^{\frac{1}{\alpha}} \sin \theta|^\alpha\right)}{\alpha \cos \theta \sin \theta} d\theta.$$

**Proof.** We will show the independence in  $\alpha$  for  $\alpha \in (0, 2)$  and then appeal to analyticity to draw the conclusion for all  $\alpha > 0$ .

Let  $S, S_1, S_2$  be i.i.d. each having a symmetric stable distribution with stability exponent  $\alpha \in (0, 2)$  and scale one; this means that their common characteristic function is given by  $f(\theta) = e^{-|\theta|^\alpha}$ . Next, let

$$X_1 := \frac{S + S_1}{2^{1/\alpha}}, \quad X_2 := \frac{S + S_2}{2^{1/\alpha}}.$$

One immediately checks (from known theory, e.g. Samorodnitsky and Taqqu (1994), or by computing the characteristic functions) that  $X_1$  and  $X_2$  each also has a symmetric stable distribution with stability exponent  $\alpha$  and scale one.

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(i) For the first integral, we will consider

$$\mathbb{E}[\operatorname{sgn}(X_1) \operatorname{sgn}(X_2)] \quad (1)$$

and compute its value in two different ways. On one hand, Corollary 6.12 in [Molchanov \(2009\)](#) implies, after some work, that (1) is, for a given  $\alpha \in (0, 2)$ ,

$$\frac{2}{\pi^2} \int_0^\pi \frac{\log(|\cos \theta|^\alpha + |\sin \theta|^\alpha + |\cos \theta + \sin \theta|^\alpha)}{\alpha \cos \theta \sin \theta} d\theta,$$

where the integral is to be taken in the Cauchy principal value sense. On the other hand, one can show directly, as we do below, that

$$\mathbb{E}[\operatorname{sgn}(X_1) \operatorname{sgn}(X_2)] = 1/3 \quad (2)$$

for each such value of  $\alpha$ . This implies that this integral is independent of  $\alpha$  with value  $\pi^2/6$ . In order to obtain (2), note first that, by symmetry,

$$\mathbb{E}[\operatorname{sgn}(X_1) \operatorname{sgn}(X_2)] = 4\mathbb{P}[\operatorname{sgn}(X_1) = \operatorname{sgn}(X_2) = 1] - 1$$

and so it suffices to show that

$$\mathbb{P}[\operatorname{sgn}(X_1) = \operatorname{sgn}(X_2) = 1] = 1/3.$$

Here

$$\mathbb{P}[\operatorname{sgn}(X_1) = \operatorname{sgn}(X_2) = 1] = \mathbb{E}[\mathbb{P}[\operatorname{sgn}(X_1) = \operatorname{sgn}(X_2) = 1 \mid S]]$$

which in turn is equal to

$$\mathbb{E}[\mathbb{P}[\operatorname{sgn}(X_1) = 1 \mid S]^2] = \mathbb{E}[\mathbb{P}[S_1 \geq -S \mid S]^2] = \mathbb{E}[\mathbb{P}[S_1 \geq -S]^2].$$

By symmetry of  $S_1$ , this equals

$$\mathbb{E}[\mathbb{P}[S_1 \leq S]^2] = \mathbb{E}[F[S]^2]$$

where  $F$  is the distribution function of  $S$ . For any random variable  $W$  with a continuous distribution function  $G$ , on has that  $G(W)$  has a uniform distribution. It follows that this last expression is

$$\int_0^1 x^2 dx = 1/3.$$

This completes (i).

(ii) For the second integral, we will consider

$$\mathbb{E}[\operatorname{sgn}(X_1) \operatorname{sgn}(S)] \quad (3)$$

and compute its value in two different ways. On one hand, Corollary 6.12 in [Molchanov \(2009\)](#) implies, after some work, that (3) is, for a given  $\alpha \in (0, 2)$ ,

$$\frac{2}{\pi^2} \int_0^\pi \frac{\log\left(\frac{|\sin \theta|^\alpha}{2} + |\cos \theta + (\frac{1}{2})^\frac{1}{\alpha} \sin \theta|^\alpha\right)}{\alpha \cos \theta \sin \theta} d\theta.$$

On the other hand, as we explain below,

$$\mathbb{E}[\operatorname{sgn}(X_1) \operatorname{sgn}(S)] = 1/2 \quad (4)$$

for each such value of  $\alpha$ . This will then imply that this integral is independent of  $\alpha$  with value  $\pi^2/4$ . Similar to (i), symmetry yields

$$\mathbb{E}[\operatorname{sgn}(X_1) \operatorname{sgn}(S)] = 4\mathbb{P}[\operatorname{sgn}(X_1) = \operatorname{sgn}(S) = 1] - 1$$

and so it suffices to show that

$$\mathbb{P}[\operatorname{sgn}(X_1) = \operatorname{sgn}(S) = 1] = 3/8.$$

To this end, note first that

$$\mathbb{P}[\operatorname{sgn}(X_1) = \operatorname{sgn}(S) = 1] = \int_0^\infty \mathbb{P}[S_1 \geq -s \mid S = s] dF(s) = \int_0^\infty \mathbb{P}[S_1 \geq -s] dF(s) = \int_0^\infty F(s) dF(s)$$

by symmetry of  $S_1$ . This becomes after the change of variables  $x = F(s)$ ,

$$\int_{\frac{1}{2}}^1 x dx = 3/8.$$

This completes (ii).  $\square$

**Remark 1.2.**

- (i) After having obtained the above theorem, we asked on Mathematics Stack Exchange if one could more directly obtain the value of  $\pi^2/6$ , independent of  $\alpha$ , in the first integral. This was shown by Jack D'Aurizio, see <https://tinyurl.com/y6fth8vr>.
- (ii) Once we knew that  $\mathbb{E}[\text{sgn}(X_1)\text{sgn}(X_2)]$  was independent of  $\alpha$ , of course any formula for  $\mathbb{E}[\text{sgn}(X_1)\text{sgn}(X_2)]$  would have to be independent of  $\alpha$ , in particular the formula given in Corollary 6.12 in Molchanov (2009) which is the above integral. However we would have guessed that the independence in  $\alpha$  of such a formula would have appeared in a more transparent way in the integral; surprisingly this was not the case.
- (iii) There is an alternative argument of (4) which we very briefly sketch. Consider the vector  $(\text{sgn}(X_1), \text{sgn}(S), \text{sgn}(S_1))$ . It is clear that this vector is  $\pm 1$ -symmetric and has pairwise nonnegative correlations. It follows from Proposition 2.12 in Steif and Tykesson (2019) that this is then a so-called divide and color process. This means that there is a random partition of the set  $\{1, 2, 3\}$  so that if we first randomly partition  $\{1, 2, 3\}$  and then assign the same value to each element of a partition element,  $\pm 1$  each with probability  $1/2$ , independently for different partition elements, then we obtain, in distribution,  $(\text{sgn}(X_1), \text{sgn}(S), \text{sgn}(S_1))$ . What can this random partition look like? Since  $S$  and  $S_1$  are independent, “2” and “3” must always be put in different partition elements. “1” can never be its own partition element, since then the realization  $(-1, 1, 1)$  would have positive probability. However it is clear that for  $(\text{sgn}(X_1), \text{sgn}(S), \text{sgn}(S_1))$ , this has zero probability. Hence the only partitions which can have positive weight are  $\{\{1, 2\}, \{3\}\}$  and  $\{\{1, 3\}, \{2\}\}$  and by symmetry these must each have weight  $1/2$ . It is however clear that the covariance of two variables in a divide and color process is simply the probability that they are in the same partition element, and hence we obtain (4).

One can extend the proof of the independence in  $\alpha$  of the first integral to higher dimensional integrals. Let  $S, S_1, S_2, \dots$  be i.i.d. each having a symmetric stable distribution with stability exponent  $\alpha \in (0, 2)$  and scale one, and let for  $i \geq 1$

$$X_i := \frac{S + S_i}{2^{1/\alpha}}.$$

We now consider  $\mathbb{E}[\text{sgn}(X_1 X_2 \cdots X_n)]$ , the analogue of  $\mathbb{E}[\text{sgn}(X_1)\text{sgn}(X_2)]$ . By symmetry, this is zero for  $n$  odd. The following proposition follows partially from the analysis in Section 3.5 in Steif and Tykesson (2019). The case  $n = 2$  corresponds to (2). The proof is only sketched.

**Proposition 1.3.** For even  $n$  and for all values of  $\alpha \in (0, 2)$ ,  $\mathbb{E}[\text{sgn}(X_1 X_2 \cdots X_n)] = 1/(n+1)$

**Proof.** Clearly  $(\text{sgn}(X_1), \text{sgn}(X_2), \dots)$  is an infinite exchangeable sequence and hence its distribution is given, due to de Finetti's Theorem (Durrett, 2010), by

$$\int_{s=0}^1 \Pi_s d\nu(s), \quad (5)$$

where  $\Pi_s$  denotes product measure on  $\{-1, 1\}^{\mathbb{N}}$  with density  $s$  and  $\nu$  is some (unique) probability measure on  $[0, 1]$ . It is shown in Steif and Tykesson (2019) that for all  $\alpha \in (0, 2)$ ,  $\nu$  is uniform distribution on  $[0, 1]$ .

We now exploit a different representation of this process. Partition the unit interval  $[0, 1]$  into intervals  $I_1, I_2, I_3, \dots$  where  $I_i$  has length  $1/2^i$ . Let  $U_1, U_2, U_3, \dots$  be i.i.d. uniform random variables on  $[0, 1]$  and  $Z_1, Z_2, Z_3, \dots$  be i.i.d. uniform random variables on  $\pm 1$ . Let  $V_1, V_2, V_3, \dots$  be defined by

$$V_i := Z_{j(i)}$$

where  $j(i)$  is chosen so that  $U_i \in I_{j(i)}$ . For people who are familiar with Kingman's theory of exchangeable random partitions of the integers, we are just first choosing an exchangeable random partition of the integers using the paintbox  $(1/2, 1/4, \dots)$  (see Bertoin (2006)) and then assigning the same value  $1$  or  $-1$ , each with probability  $1/2$ , to all elements in a partition element, independently for different partition elements.  $(V_1, V_2, V_3, \dots)$  is clearly exchangeable and its mixing measure  $\nu$  in de Finetti's Theorem is also uniform by Theorem 3.12 in Steif and Tykesson (2019). It follows that for all  $\alpha \in (0, 2)$ ,  $(\text{sgn}(X_1), \text{sgn}(X_2), \dots)$  and  $(V_1, V_2, V_3, \dots)$  have the same distribution. Next it is clear that  $\mathbb{E}[V_1 V_2 \cdots V_n]$  is the probability that in the random  $(1/2, 1/4, \dots)$ -paintbox partition restricted to  $\{1, \dots, n\}$  there are only partitions with an even number of elements. One can show, using induction, conditioning on the number of terms entering the first box and using the scale invariance of this paintbox, that the probability of this latter event is  $1/(n+1)$ , completing the proof.  $\square$

Corollary 6.12 in Molchanov (2009) provides formulas for  $\mathbb{E}[\text{sgn}(X_1 X_2 \cdots X_n)]$  in terms of integrals over the sphere  $\mathbb{S}^{n-1}$  which ostensibly depend on  $\alpha$ . However, a consequence of Proposition 1.3 now is that these higher dimensional integrals do not in fact depend on  $\alpha$ .

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